Applications and Applied Mathematics: An International Journal (AAM)

Volume 6 | Issue 2 | Article 9

12-2011

Robust ?1 and ?∞ Solutions of Linear Inequalities

Maziar Salahi
University of Guilan

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 6, Issue 2 (December 2011), pp. 522 - 528

Robust l_1 and l_∞ Solutions of Linear Inequalities

Maziar Salahi

Department of Applied Mathematics,
Faculty of Mathematical Sciences
University of Guilan,
salahim@guilan.ac.ir

Received: March 28, 2011; Accepted: November 15, 2011

Abstract

Infeasible linear inequalities appear in many disciplines. In this paper we investigate the l_1 and l_∞ solutions of such systems in the presence of uncertainties in the problem data. We give equivalent linear programming formulations for the robust problems. Finally, several illustrative numerical examples using the cvx software package are solved showing the importance of the robust model in the presence of uncertainties in the problem data.

Keywords: Linear inequalities, Infeasibility, Uncertainty, Robust model, Linear programs

MSC 2010 No.: 15A39, 90C05, 90C25

1. Introduction

Robust optimization is an effective approach to many real world optimization problems (Ben-Tal, 2001 and Soyster, 1973). In this paper we consider the following form of linear inequalities:

$$Ax \geq b, \tag{1}$$

where $A \in R^{m\times n}$ and $b \in R^m$. The solution of such linear inequalities is a fundamental problem that arises in several applications. Some important applications arise in medical image reconstruction from projections, and in inverse problems in radiation therapy (Censor, 1988, 1982, 1997, 2008 and Herman, 1975, 1978).

In practice, it often happens that the feasible region of (1) is empty due to measurements errors in the data vector b. In such a case it is desired to find the smallest correction of b that recovers feasibility (Dax, 2006, Ketabchi, 2009 and Salahi, 2010). More precisely, the following two problems should be solved in l_1 and l_∞ norms:

$$min_x ||(b - Ax)_+||_1 \tag{2}$$

$$min_x ||(b - Ax)_+||_\infty \tag{3}$$

where $(b_i - a_i^T x)_+ = max(b_i - a_i^T x, 0)$.

Obviously both (2) and (3) are equivalent to LPs that can be efficiently solved using simplex or interior point methods (Grant, 2010, 2008). Our goal in this paper is to consider (2) and (3) when there are uncertainties in both A and b. In such a case, we can not get the solution by solving problems in (2) and (3). We give new LP formulations of both problems in (2) and (3) in the presence of uncertainties. Finally several randomly generated test problems are presented showing the importance of the robust approach using cvx software package (Grant, 2010, 2008).

2. The l_1 Case

In this section we consider the robust solution of linear inequalities in the l_1 norm sense i.e.,

$$min_{x,\ ||[\Delta A,\ \Delta b]||_1 \leq \rho} ||((b + \Delta b) - (A + \Delta A)x)_+||_1 \tag{4}$$

where ΔA and Δb are uncertainties in the matrix A and vector b, respectively and ρ is a given positive parameter. We may write (4) as follows

$$min_x\, max_{\ ||[\Delta A,\ \Delta b]||_1 \leq \rho} ||(b - Ax + \Delta b - \Delta A x)_+||_1 \tag{5}$$

which minimizes the residual in the worst case.

Now for a given $x \in R^n$, let us consider the inner problem in (5), then we have

$$||(b - Ax + [\Delta A\ \ \Delta b][-x^T\ 1]^T)_+||_1 \leq ||(b - Ax)_+||_1 + \rho||[-x^T\ 1]^T||_1 \tag{6}$$

In the sequel we show that for a specific choice of ΔA and Δb we have inequality as equality in (6). Let us consider

$$[\Delta A \;\; \Delta b] = \rho \frac{(b-Ax)_+}{||(b-Ax)_+||_1}(sgn([-x^T \; 1]^T))^T \tag{7}$$

where

$$sgn(a) = \begin{cases} 1, & if \; a > 0, \\ -1, & if \; a \leq 0. \end{cases}$$

Obviously $||[\Delta A \;\; \Delta b]||_1 = \rho$ and

$$\begin{aligned}(b - Ax + [\Delta A \;\; \Delta b][-x^T \; 1]^T)_+ \\ &= (b - Ax + \rho \frac{(b-Ax)_+}{||(b-Ax)_+||_1}(sgn([-x^T \; 1]^T))^T[-x^T \; 1]^T)_+ \\ &= (b - Ax + \rho \frac{(b-Ax)_+}{||(b-Ax)_+||_1}||[-x^T \; 1]^T||_1)_+ \\ &= (b - Ax)_+ + \rho \frac{(b-Ax)_+}{||(b-Ax)_+||_1}||[-x^T \; 1]^T||_1.\end{aligned}$$

This implies the equality in (6). Therefore, (5) is equivalent to the following problem:

$$min_x ||(b + \Delta b)_+||_1 \; + \; \rho||[-x^T \; 1]^T||_1 \tag{8}$$

Obviously, this is equivalent to an LP as follows:

$$\begin{aligned} &min \; e^T z \; + \rho e^T s + \rho \\ &Ax + z \geq b \\ &x \leq s \\ &x \geq -s \\ &s, z \geq 0 \end{aligned} \tag{9}$$

where two vectors e are all one vector of appropriate dimensions.

3. The l_∞ Case

In this section we consider the robust solution of linear inequalities in the l_∞ norm sense i.e.,

$$min_{x, \; ||[\Delta A, \; \Delta b]||_\infty \leq \rho} ||((b + \Delta b) - (A + \Delta A)x)_+||_\infty \tag{10}$$

where ΔA and Δb are uncertainties in the matrix A and vector b, respectively and ρ is a given positive parameter. We may write (10) as follows

$$min_x \; max_{\,||[\Delta A,\ \Delta b]||_\infty \leq \rho} ||(b - Ax + \Delta b - \Delta Ax)_+||_\infty. \tag{11}$$

Now for a given $x \in R^n$, let us consider the inner problem in (11), then we have

$$||(b - Ax + [\Delta A \;\; \Delta b][-x^T \;\; 1]^T)_+||_\infty \leq ||(b - Ax)_+||_\infty + \rho||[-x^T \;\; 1]^T||_\infty \;. \tag{12}$$

Let $\bar{x} = [x^T \;\; 1]^T$ and $r = arg\, max_{i=1,\dots,n+1}|\bar{x}_i|$. Now we show that for the following specific choice of ΔA and Δb we have the inequality (12) as equality. Let

$$[\Delta A \;\; \Delta b] = \rho\, sgn(\bar{x}_r) \frac{(b-Ax)_+}{||(b-Ax)_+||_\infty} {e_r}^T \tag{13}$$

where e_r is the rth column of the identity matrix and

$$sgn(a) = \begin{cases} 1, & if\ a > 0, \\ 0, & if\ a \leq 0. \end{cases}$$

Obviously $||[A \;\; b]||_\infty = \rho$ and

$$\begin{aligned}(b - Ax + [\Delta A \;\; \Delta b][-x^T \;\; 1]^T)_+ &= (b - Ax + \rho\, sgn(\bar{x}_r) \frac{(b-Ax)_+}{||(b-Ax)_+||_\infty} e_r^T \bar{x})_+ \\ &= (b - Ax + \rho\, sgn(\bar{x}_r) \frac{(b-Ax)_+}{||(b-Ax)_+||_\infty} \bar{x})_+ \\ &= (b - Ax)_+ + \rho \frac{(b-Ax)_+}{||(b-Ax)_+||_\infty} ||\bar{x}||_\infty.\end{aligned}$$

This implies the equality in (12). Therefore, (10) is equivalent to the following problem:

$$min_x ||(b + \Delta b)_+||_\infty \; + \; \rho||[x^T \;\; 1]^T||_\infty \tag{14}$$

This itself is equivalent to the following LP:

$$\begin{aligned} &min\; z \; + \rho t \\ &b - Ax \leq y \\ &y_i \leq z, \quad \forall i = 1, \dots, m \\ &-t \leq x_i \leq t, \quad \forall i = 1, \dots, n \\ &t \geq 1, y \geq 0. \end{aligned} \tag{15}$$

4. Numerical Results

In this section we present numerical results for some randomly generated test problems with different dimensions. In both tables, the first three test problems are generated using the following simple MATLAB code:

```
clear all
clc
seed=0;
randn('state',seed);
m=enter('number of rwos');
n=enter('number of columns');
A=randn(m,n);
b=randn(m,1);
```

For problems generated by this code, matrix A is not necessarily ill-condition, so we use MATLAB's **hilb()** command to generate the well-know ill-condition Hilbert matrix of the given dimension and make the system infeasible. The last two test problems in both tables are generated by the following code:

```
A=hilb(m);
b=randn(m,1);
A=[A;-A(m,:)];
b=[b;b(m)+1];
```

We solve LPs using cvx (Grant, 2010, 2008) software package. Results for l_1 and l_∞ norms are summarized in Tables 1 and 2, respectively. We have used three different values for uncertainty parameter, namely 0.1, 1, 5 and numbers in the parenthesis of both tables are also for these three values, respectively.

As our results show, the robust solution is different than the original problem although their objective values might be closer. Moreover, when the coefficient matrix is ill-condition, even a small uncertainty might significantly change the solution, see the last two rows of both tables. Therefore, in the presence of uncertainties, it is better to use the robust model rather than the original one.

Table 1: Comparison of Problem (2) and its Robust version (9)

m, n	Method	$\|\|(b - Ax^*)_+\|\|_1$	$\|\|x^*\|\|_1$
50,10	Problem (2)	12.0456	3.9991
	Problem (9)	(12.0456,12.415,16.796)	(3.97,3.28,0.4620)
500,100	Problem (2)	143.24	7.398
	Problem (9)	(143.24,143.62,151.007)	(7.398,6.735,3.75)
700,300	Problem (2)	152.9	15.52
	Problem (9)	(152.9,153.85,171.25)	(14.97,13.25,7.18)
100,30	Problem (2)	0.469	3.4252e4
	Problem (9)	(3.41,25.44,38.13)	(62.27,8.71,8.7e-9)
500,300	Problem (2)	0.871	3.6729e5
	Problem (9)	(20.28,126.36,174.22)	(377.78,30.56,4.3e-8)

Table 2: Comparison of Problem (3) and its Robust version (15)

m, n	Method	$\|\|(b - Ax^*)_+\|\|_\infty$	$\|\|x^*\|\|_\infty$
50,10	Problem (3)	1.0893	0.5045
	Problem (15)	(1.0893,1.0893,1.0893)	(0.5045, 0.5045, 0.5045)
500,100	Problem (3)	1.243	0.222
	Problem (15)	(1.243,1.243,1.243)	(0.222,0.222,0.222)
700,300	Problem (3)	0.937	0.3340
	Problem (15)	(0.937,0.937,0.937)	(0.3340,0.3340.0.3340)
100,30	Problem (3)	0.234	3.407e4
	Problem (15)	(1.091,1.091,1.091)	(1,1,1)
500,300	Problem (3)	0.436	2.1894e9
	Problem (15)	(1.5851,1.5851,1.5851)	(1,1,1)

5. Conclusions

In this paper, we have studied the robust l_1 and l_∞ solutions of linear inequalities in the presence of uncertainties in both A and b. Equivalent LP formulations of both robust problems are given. Finally, several numerical results are presented showing the importance of the robust framework.

REFERENCES

Ben-Tal, A. and Nemirovski, A. (2001). Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications, *MPS-SIAM Series on Optimization*, SIAM, Philadelphia.

Censor, Y., Altschuler, M.D. and Powlis, W.D. (1988). A computational solution of the inverse problem in radiation therapy treatment planning, Appl. Math. Comput. 25, pp. 57-87.

Censor, Y. and Elfving, T. (1982). New methods for linear inequalities, Linear Algebra Appl., 42, pp. 199-211.

Censor, Y. and Zenios, S.A. (1997). Parallel Optimization, Theory Algorithms, and Applications, Oxford University Press, Oxford.

Censor, Y., Ben-Israel, A., Xiao, Y. and Galvin, J.M. (2008). On linear infeasibility arising in intensity-modulated radiation therapy inverse planning, Linear Algebra and Its Applications, 428, pp. 1406–1420.

Dax, A. (2006). The l_1 solutions of linear inequalities, Computational Statistics & Data Analysis, 50, pp. 40–60.

Grant, M. and Boyd, S. (2010). CVX: Matlab software for disciplined convex programming, version 1.21. http://cvxr.com/cvx.

Grant, M. and Boyd, S. (2008). Graph implementations for nonsmooth convex programs, Recent Advances in Learning and Control (a tribute to M. Vidyasagar), V. Blondel, S. Boyd, and H. Kimura, editors, pp. 95–110, Lecture Notes in Control and Information Sciences, Springer, http://stanford.edu/~boyd/graph_dcp.html.

Herman, G.T. (1975). A relaxation method for reconstructing object from noisy x-rays. Math. Prog. 8, pp. 1-19.

Herman, G.T. and Lent, A. (1978). A family of iterative quadratic optimization algorithms for pairs of inequalities, with applications in diagnostic radiology. Math. Prog. Study, 9, pp. 15-29.

Ketabchi, S. and Salahi, M. (2009). Correcting an inconsistent set of linear inequalities by the generalized Newton method, Comput. Sci. J. Moldova, 17, 2(50), pp. 179–192.

Salahi, M. and Ketabchi, S. (2010). Correcting an inconsistent set of linear inequalities by the generalized Newton method, Optimization Methods and Software, 25(3), pp. 457–465.

Soyster, A.L. (1973). Convex programming with set-inclusive constraints and applications to inexact linear programming, *Oper. Res.*, 21, pp. 1154-1157.