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Exact Optimal Solution of Fuzzy Critical Path Problems

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Abstract

In this paper, a fuzzy critical path problem is chosen to show that the results, obtained by using the existing method [Liu, S.T.: Fuzzy activity times in critical path and project crashing problems. *Cybernetics and Systems* 34 (2), 161-172 (2003)], could be improved to reflect, more appropriate real life situations. To obtain more accurate results of fuzzy critical path problems, a new method that modifies the existing one is proposed here. To demonstrate the advantages of the proposed method it is used to solve a specific fuzzy critical path problem.

Keywords: Fuzzy critical path problem, Linear Programming, Triangular fuzzy number

MSC 2000 No.: 03E72, 90C05, 90C70

1. Introduction

In today's highly competitive business environment, project management's ability to schedule activities and monitor progress within strict cost, time and performance guidelines is becoming

increasingly important to obtain competitive priorities such as on-time delivery and customization. In many situations, projects can be complicated and challenging to manage. When the activity times in the project are deterministic and known, critical path method (CPM) has been demonstrated to be a useful tool in managing projects in an efficient manner to meet this challenge. The purpose of CPM is to identify critical activities on the critical path so that resources may be concentrated on these activities in order to reduce the project length time.

The successful implementation of CPM requires the availability of clear determined time duration for each activity. However, in practical situations this requirement is usually hard to fulfill, since many of activities will be executed for the first time. To deal with such real life situations, Zadeh (1965) introduced the concept of fuzzy set. Since there is always uncertainty about the time duration of activities in the network planning, due to which fuzzy critical path method (FCPM) was proposed since the late 1970s.

For finding the fuzzy critical path, several approaches are proposed over the past years. The first method called FPERT was proposed by Chanas and Kamburowski (1981). They presented the project completion time in the form of fuzzy set in the time space. Gazdik (1983) developed a fuzzy network of unknown project to estimate the activity durations and used fuzzy algebraic operators to calculate the duration of the project and its critical path. Kaufmann and Gupta (1988) devoted a chapter of their book to the critical path method in which activity times are represented by triangular fuzzy numbers. McCahon and Lee (1988) presented a new methodology to calculate the fuzzy completion project time.

Nasution (1994) proposed how to compute total floats and find critical paths in a project network. Yao and Lin (2000) proposed a method for ranking fuzzy numbers without the need for any assumptions and have used both positive and negative values to define ordering which then is applied to CPM. Dubois et al. (2003) extended the fuzzy arithmetic operational model to compute the latest starting time of each activity in a project network. Lin and Yao (2003) introduced a fuzzy CPM based on statistical confidence-interval estimates and a signed distance ranking for $(1 - \alpha)$ fuzzy number levels. Liu (2003) developed solution procedures for the critical path and the project crashing problems with fuzzy activity times in project planning. Liang and Han (2004) presented an algorithm to perform fuzzy critical path analysis for project network problem.

Zielinski (2005) extended some results for interval numbers to the fuzzy case for determining the possibility distributions describing latest starting time for activities. Chen (2007) proposed an approach based on the extension principle and linear programming (LP) formulation to critical path analysis in networks with fuzzy activity durations. Chen and Hsueh (2008) presented a simple approach to solve the CPM problems with fuzzy activity times (being fuzzy numbers) on the basis of the linear programming formulation and the fuzzy number ranking method that are more realistic than crisp ones. Yakhchali and Ghodspour (2010) introduced the problems of determining possible values of earliest and latest starting times of an activity in networks with minimal time lags and imprecise durations that are represented by means of interval or fuzzy numbers. Shankar et al. (2010) proposed a new approach for finding the total float of each activity, critical activities and critical path in a fuzzy project network. Kumar and Kaur (2010) proposed a new method to find the fuzzy optimal solution of fully fuzzy critical path problems.

In this paper, a fuzzy critical path problem is chosen to show that the results, obtained by using the existing method [Liu (2003)], are not appropriate according to the real life situations. To obtain the appropriate results of fuzzy critical path problems, a new method is proposed by modifying the existing method. To show the advantages of the proposed method over existing method the chosen fuzzy critical path problem is solved by using the proposed method and it is shown that the appropriate results are obtained by using the proposed method.

This paper is organized as follows: In Section 2, some basic definitions, an existing approach for comparing fuzzy numbers and arithmetic operations between two triangular fuzzy numbers are presented. In Section 3, linear programming formulation of fuzzy critical path problems is presented. In Section 4, an existing method for solving fuzzy critical path problems is presented. Shortcoming of an existing method [Liu (2003)] for solving a fuzzy critical path problem is discussed in Section 5. In Section 6, a method to find the maximum and minimum of two triangular fuzzy numbers is presented. In Section 7, by modifying an existing method [Liu (2003)], a new method is proposed to find the exact solution of fuzzy critical path problems. Advantages of the proposed method over existing method are discussed in Section 8. In Section 9, the obtained results are discussed. Conclusion and future work is presented in Section 10.

2. Preliminaries

In this section, some basic definitions, Yager's ranking approach for comparing fuzzy numbers and arithmetic operations between triangular fuzzy numbers are presented.

2.1. Basic Definitions

In this section, some basic definitions are presented [Kaufmann and Gupta (1985)].

Definition 1.

The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X . This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e. $\mu_{\tilde{A}} : X \rightarrow [0,1]$. The assigned value indicate the membership grade of the element in the set A . The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); x \in X\}$ defined by $\mu_{\tilde{A}}(x)$ for each $x \in X$ is called a fuzzy set.

Definition 2.

A fuzzy number $\tilde{A} = (a, b, c)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & -\infty < x \leq a, \\ \frac{(x-a)}{(b-a)}, & a \leq x < b, \\ \frac{(x-c)}{(b-c)}, & b \leq x < c, \\ 0, & c \leq x < \infty. \end{cases}$$

Definition 3.

Two triangular fuzzy numbers $\tilde{A} = (a_1, b_1, c_1)$ and $\tilde{B} = (a_2, b_2, c_2)$ are said to be equal, i.e., $\tilde{A} = \tilde{B}$ iff $a_1 = a_2, b_1 = b_2, c_1 = c_2$.

Definition 4.

Let $\tilde{A} = (a, b, c)$ be a triangular fuzzy number and λ be a real number in the interval $[0,1]$ then the crisp set $A_\lambda = \{x \in X : \mu_{\tilde{A}}(x) \geq \lambda\} = [a + (b-a)\lambda, c - (c-b)\lambda]$, is said to be λ -cut of \tilde{A} .

2.2. Yager's Ranking Approach

Yager (1981) proposed a procedure for ordering fuzzy sets in which a ranking index $\mathfrak{R}(\tilde{A})$ is calculated for the fuzzy number $\tilde{A} = (a, b, c)$ from its λ -cut $A_\lambda = [a + (b-a)\lambda, c - (c-b)\lambda]$ according to the following formula:

$$\mathfrak{R}(\tilde{A}) = \frac{1}{2} \left(\int_0^1 (a + (b-a)\lambda) d\lambda + \int_0^1 (c - (c-b)\lambda) d\lambda \right) = \frac{a + 2b + c}{4}.$$

Let \tilde{A} and \tilde{B} be two fuzzy numbers, then

- (i) $\tilde{A} \succ \tilde{B}$ if $\mathfrak{R}(\tilde{A}) > \mathfrak{R}(\tilde{B})$,
- (ii) $\tilde{A} \approx \tilde{B}$ if $\mathfrak{R}(\tilde{A}) = \mathfrak{R}(\tilde{B})$, and
- (iii) $\tilde{A} \prec \tilde{B}$ if $\mathfrak{R}(\tilde{A}) < \mathfrak{R}(\tilde{B})$.

Since $\mathfrak{R}(\tilde{A})$ is calculated from the extreme values of λ -cut of \tilde{A} i.e., $a + (b-a)\lambda$ and $c - (c-b)\lambda$, rather than its membership function, it is not required knowing the explicit form of the membership functions of the fuzzy numbers to be ranked. That is, unlike most of the ranking

methods that require the knowledge the membership functions of all fuzzy numbers to be ranked, the Yager's ranking index is still applicable even if the explicit form the membership function of the fuzzy number is unknown.

Remark 2.1:

Yager's ranking index (1981) satisfies the linearity property

$$\mathfrak{R}(\lambda_1 \tilde{A} \oplus \lambda_2 \tilde{B}) = \lambda_1 \mathfrak{R}(\tilde{A}) + \lambda_2 \mathfrak{R}(\tilde{B}) \quad \forall \lambda_1, \lambda_2 \in R \quad (R \text{ is a set of real numbers}).$$

2.3. Arithmetic Operations

In this section, addition and multiplication operations between two triangular fuzzy numbers are presented [Kaufmann and Gupta (1985)]. Let $\tilde{A}_1 = (a_1, b_1, c_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2)$ be two triangular fuzzy numbers, then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2),$

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 \approx (a', b', c'),$ where $a' = \text{minimum}(a_1 a_2, a_1 c_2, c_1 a_2, c_1 c_2),$ $b' = b_1 b_2,$ $c' = \text{maximum}(a_1 a_2, a_1 c_2, c_1 a_2, c_1 c_2),$ and

(iii) $\lambda \tilde{A}_1 = \begin{cases} (\lambda a_1, \lambda b_1, \lambda c_1) & \lambda \geq 0 \\ (\lambda c_1, \lambda b_1, \lambda a_1) & \lambda \leq 0. \end{cases}$

3. Linear Programming Formulation of Fuzzy Critical Path Problems

In this section, the fuzzy linear programming (FLP) formulation of fuzzy critical path problems is presented [Liu (2003)].

Maximize

$$\sum_{j:(i,j) \in A} \tilde{t}_{ij} x_{ij}$$

subject to

$$\sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} = \begin{cases} 1 & i = 1 \\ -1 & i = n \\ 0 & i \in N - \{1, n\} \end{cases}$$

$$x_{ij} \geq 0 \quad \forall (i, j) \in A,$$

where, A : Set of all activities (i, j) ,
 \tilde{t}_{ij} : Fuzzy time duration of the activity (i, j) ,
 N : Set of nodes, 1: Source node, n : Destination node.
 x_{ij} : time of the event occurring corresponding to the activity (i, j) .

4. Existing Method

Liu (2003) proposed a new method to solve the fuzzy critical path problems by representing all the fuzzy activity times as triangular fuzzy numbers. In this section, a brief review of an existing method (Liu, 2003) for solving the fuzzy critical path problems is presented.

The steps of the existing method are as follows:

Step 1.

Formulate the chosen fuzzy critical path problem into the FLP problem (P_1):

Maximize

$$\sum_{j:(i,j) \in A} \tilde{t}_{ij} x_{ij}$$

subject to

$$\sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} = \begin{cases} 1 & i = 1 \\ -1 & i = n \\ 0 & i \in N - \{1, n\} \end{cases} \quad (P_1)$$

$$x_{ij} \geq 0 \quad \forall (i, j) \in A.$$

Step 2.

Convert the FLP problem (P_1) into the following crisp linear programming (CLP) problem (P_2):

Maximize

$$\sum_{j:(i,j) \in A} \Re(\tilde{t}_{ij}) x_{ij}$$

subject to

$$\sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} = \begin{cases} 1 & i = 1 \\ -1 & i = n \\ 0 & i \in N - \{1, n\} \end{cases} \quad (P_2)$$

$$x_{ij} \geq 0 \quad \forall (i, j) \in A.$$

Step 3.

Solve the CLP problem (P_2) to find the optimal solution $\{x_{ij}\}$.

Step 4.

Use the optimal solution $\{x_{ij}\}$, obtained in Step 3, to find the fuzzy critical path and also put the values of x_{ij} in $\sum_{j:(i,j) \in A} \tilde{t}_{ij} x_{ij}$ to find the maximum total fuzzy completion time of the project.

5. Shortcomings of the Existing Method

In Step 2 of the existing method, to convert the FLP problem (P_1) into the CLP problem (P_2), only the ranking function is used i.e., to find the maximum of fuzzy numbers only rank of fuzzy numbers are compared but in literature [Kaufmann and Gupta (1988)], it is pointed out that it is not possible to find the maximum value of different fuzzy numbers by using rank only and for this purpose in the literature mode and divergence are also used.

In this section, a fuzzy critical path problem, chosen in Example 5.1, is solved by using the existing method and it is shown that the obtained results are not appropriate according to the real life situations.

Example 5.1.

The problem is to find the fuzzy critical path and maximum total fuzzy project completion time of the project, shown in Figure 1, in which the fuzzy time duration of each activity is represented by the following (a,b,c) type triangular fuzzy numbers: $\tilde{t}_{12} = (2, 4, 6)$, $\tilde{t}_{13} = (9, 13, 17)$, $\tilde{t}_{23} = (7, 9, 11)$, $\tilde{t}_{24} = (12, 19, 26)$, and $\tilde{t}_{34} = (6, 10, 14)$

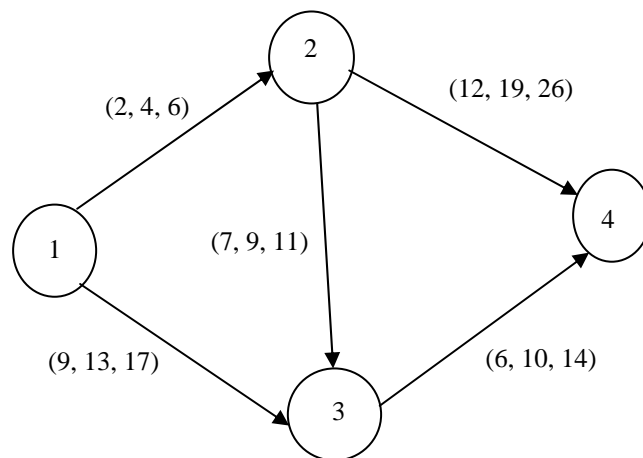


Figure 1. Project network of the illustrated Example 5.1

Solution:

The fuzzy critical path problem, chosen in Example 5.1, can be solved by using the following steps of the existing method [Liu (2003)].

Step 1.

Using Section 3, the fuzzy critical path problem, chosen in Example 5.1, can be formulated as follows:

Maximize

$$((2, 4, 6) x_{12} \oplus (9, 13, 17) x_{13} \oplus (7, 9, 11) x_{23} \oplus (12, 19, 26) x_{24} \oplus (6, 10, 14) x_{34})$$

subject to

$$x_{12} + x_{13} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{13} + x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$$

$$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0.$$

Step 2.

Using ranking formula, presented in Section 2.2, the FLP problem, obtained in Step 1, can be written as:

Maximize

$$(\mathfrak{R}(2,4,6) x_{12} + \mathfrak{R}(9,13,17) x_{13} + \mathfrak{R}(7,9,11) x_{23} + \mathfrak{R}(12,19,26) x_{24} + \mathfrak{R}(6,10,14) x_{34})$$

subject to

$$x_{12} + x_{13} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{13} + x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$$

$$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0,$$

i.e., Maximize

$$(4 x_{12} + 13 x_{13} + 9 x_{23} + 19 x_{24} + 10 x_{34})$$

subject to

$$x_{12} + x_{13} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{13} + x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$$

$$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0.$$

Step 3.

On solving CLP problem, obtained in Step 2, the following three optimal solutions are obtained:

- (i) $x_{12} = x_{24} = 1$ and $x_{13} = x_{23} = x_{34} = 0$,
- (ii) $x_{12} = x_{23} = x_{34} = 1$ and $x_{13} = x_{24} = 0$,
- (iii) $x_{13} = x_{34} = 1$ and $x_{12} = x_{23} = x_{24} = 0$.

Step 4.

Using the values of x_{ij} , obtained from Step 3, the following three fuzzy critical paths are obtained :

- (i) $1 \Rightarrow 2 \Rightarrow 4$,
- (ii) $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$,
- (iii) $1 \Rightarrow 3 \Rightarrow 4$.

Putting the values of x_{ij} , obtained from Step 3, in $[(2, 4, 6) x_{12} \oplus (9, 13, 17) x_{13} \oplus (7, 9, 11) x_{23} \oplus (12, 19, 26) x_{24} \oplus (6, 10, 14) x_{34}]$, the maximum total fuzzy project completion times corresponding to paths $1 \Rightarrow 2 \Rightarrow 4$, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$ are (14, 23, 32), (15, 23, 31) and (15, 23, 31), respectively. Let the obtained maximum total fuzzy project completion time (14, 23, 32), corresponding to path $1 \Rightarrow 2 \Rightarrow 4$, is represented by \tilde{T}_1 and the maximum total fuzzy project completion time (15, 23, 31), corresponding to both the paths $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$, be represented by \tilde{T}_2 , i.e., $\tilde{T}_1 = (14, 23, 32)$ and $\tilde{T}_2 = (15, 23, 31)$.

If there exist more than one fuzzy critical paths for a project network problem then the maximum total fuzzy completion time of the project should be same corresponding to all the fuzzy critical paths but it can be seen from the obtained solution that the maximum total fuzzy project completion time corresponding to path $1 \Rightarrow 2 \Rightarrow 4$ is different from the maximum total fuzzy project completion time corresponding to paths $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$ i.e., $\tilde{T}_1 \neq \tilde{T}_2$, only $\Re(\tilde{T}_1) = \Re(\tilde{T}_2) = 23$.

Since \tilde{T}_1 and \tilde{T}_2 are two different fuzzy numbers so their physical interpretation will also be different i.e., for the maximum total fuzzy completion time of a same project, two different interpretations will be required which may not be acceptable for real life problems.

6. Method for the Ordering of Two Triangular Fuzzy Numbers

In this section, the existing method [Kaufmann and Gupta (1988)] to find order of two triangular fuzzy numbers, which will be used in the proposed method, is presented.

Let $\tilde{A} = (a_1, b_1, c_1)$ and $\tilde{B} = (a_2, b_2, c_2)$ be two triangular fuzzy numbers then use the following steps to compare \tilde{A} and \tilde{B} .

Step 1.

Find $\Re(\tilde{A}) = \frac{a_1 + 2b_1 + c_1}{4}$ and $\Re(\tilde{B}) = \frac{a_2 + 2b_2 + c_2}{4}$.

Case (i) If $\Re(\tilde{A}) > \Re(\tilde{B})$, then $\tilde{A} \succ \tilde{B}$, i.e.,

maximum $\{\tilde{A}, \tilde{B}\} = \tilde{A}$ and minimum $\{\tilde{A}, \tilde{B}\} = \tilde{B}$.

Case (ii) If $\Re(\tilde{A}) < \Re(\tilde{B})$, then $\tilde{A} < \tilde{B}$, i.e.,
 maximum $\{\tilde{A}, \tilde{B}\} = \tilde{B}$ and minimum $\{\tilde{A}, \tilde{B}\} = \tilde{A}$.

Case (iii) If $\Re(\tilde{A}) = \Re(\tilde{B})$ then go to Step 2.

Step 2.

Find $mode(\tilde{A}) = b_1$ and $mode(\tilde{B}) = b_2$.

Case (i) If $mode(\tilde{A}) > mode(\tilde{B})$ then $\tilde{A} > \tilde{B}$, i.e.,
 maximum $\{\tilde{A}, \tilde{B}\} = \tilde{A}$ and minimum $\{\tilde{A}, \tilde{B}\} = \tilde{B}$.

Case (ii) If $mode(\tilde{A}) < mode(\tilde{B})$ then $\tilde{A} < \tilde{B}$, i.e.,
 maximum $\{\tilde{A}, \tilde{B}\} = \tilde{B}$ and minimum $\{\tilde{A}, \tilde{B}\} = \tilde{A}$.

Case (iii) If $mode(\tilde{A}) = mode(\tilde{B})$ then go to Step 3.

Step 3.

Find $divergence(\tilde{A}) = c_1 - a_1$ and $divergence(\tilde{B}) = c_2 - a_2$.

Case (i) If $divergence(\tilde{A}) > divergence(\tilde{B})$ then $\tilde{A} > \tilde{B}$, i.e.,
 maximum $\{\tilde{A}, \tilde{B}\} = \tilde{A}$ and minimum $\{\tilde{A}, \tilde{B}\} = \tilde{B}$.

Case (ii) If $divergence(\tilde{A}) < divergence(\tilde{B})$ then $\tilde{A} < \tilde{B}$, i.e.,
 maximum $\{\tilde{A}, \tilde{B}\} = \tilde{B}$ and minimum $\{\tilde{A}, \tilde{B}\} = \tilde{A}$.

Case (iii) If $divergence(\tilde{A}) = divergence(\tilde{B})$ then $\tilde{A} = \tilde{B}$.

7. Proposed Method

In this section, to overcome the shortcomings, discussed in Section 5, a new method is proposed to find the exact optimal solution of fuzzy critical path problems by modifying the existing method (Liu, 2003). The steps of the proposed method are as follow:

Step 1.

Find the fuzzy critical path and maximum total fuzzy completion time of the chosen problem by using the existing method, discussed in Section 5. There can be two cases:

Case (i): If unique fuzzy critical path and hence unique fuzzy number, representing maximum total fuzzy project completion time, is obtained then the obtained maximum total fuzzy project completion time is the maximum total fuzzy completion time of the project and the obtained fuzzy critical path is the only fuzzy critical path of the project.

Case (ii): If more than one fuzzy critical paths are obtained then go to Step 2.

Step 2.

Check that the fuzzy numbers, representing the maximum total fuzzy project completion time, corresponding to all the fuzzy critical paths are same or not.

Case (i): If a unique triangular fuzzy number, representing maximum total fuzzy project completion time, is obtained then all the fuzzy critical paths, obtained in Step 1, corresponding to which the obtained fuzzy number was obtained, are fuzzy critical paths of the project and the obtained triangular fuzzy number will represent the maximum total fuzzy completion time of the project.

Case (ii): If more than one triangular fuzzy numbers, representing maximum total fuzzy completion time of the project, are obtained then go to Step 3.

Step 3.

Let using the previous steps p different triangular fuzzy numbers $\tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_p$, representing total fuzzy completion time, are obtained i.e., $\tilde{T}_i \neq \tilde{T}_j$ and $\Re(\tilde{T}_i) = \Re(\tilde{T}_j) \forall i \neq j, i = 1, 2, \dots, p, j = 1, 2, \dots, p$ then find the optimal solution of the following CLP problem:

Maximize

$$\sum_{j:(i,j) \in A} (\text{mode}(a_{ij}, b_{ij}, c_{ij})) x_{ij}$$

subject to

$$\sum_{j:(i,j) \in A} \Re(a_{ij}, b_{ij}, c_{ij}) x_{ij} = \Re(\tilde{T}_i), \quad i = 1 \text{ or } 2 \text{ or } \dots \text{ or } p,$$

$$\sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} = \begin{cases} 1 & i = 1 \\ -1 & i = n \\ 0 & i \in N - \{1, n\} \end{cases}$$

$$x_{ij} \geq 0 \quad \forall (i, j) \in A.$$

Case (i): If by putting the obtained values of x_{ij} in $(\sum_{j:(i,j) \in A} (a_{ij}, b_{ij}, c_{ij}) x_{ij})$ a unique triangular fuzzy number, representing maximum total fuzzy project completion time, is obtained then the obtained triangular fuzzy number will represent the maximum total fuzzy completion time of the project and all the fuzzy critical paths may be obtained by using the obtained values of x_{ij} .

Case (ii): If more than one triangular fuzzy numbers, representing maximum total fuzzy completion time of the project, are obtained then go to Step 4.

Step 4.

Let using the previous steps l triangular fuzzy numbers $\tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_l$, where $l \leq p$, representing maximum total fuzzy project completion time, are obtained, then find the optimal solution of the following CLP problem:

Maximize

$$\sum_{j:(i,j) \in A} (\text{divergence}(a_{ij}, b_{ij}, c_{ij})) x_{ij}$$

subject to

$$\sum_{j:(i,j) \in A} \Re(a_{ij}, b_{ij}, c_{ij}) x_{ij} = \Re(\tilde{T}_i), \quad i = 1 \text{ or } 2 \text{ or } \dots \text{ or } p,$$

$$\sum_{j:(i,j) \in A} \text{mode}(a_{ij}, b_{ij}, c_{ij}) x_{ij} = \text{mode}(\tilde{T}_i), \quad i = 1 \text{ or } 2 \text{ or } \dots \text{ or } l,$$

$$\sum_{j:(i,j) \in A} x_{ij} - \sum_{j:(j,i) \in A} x_{ji} = \begin{cases} 1 & i = 1 \\ -1 & i = n \\ 0 & i \in N - \{1, n\} \end{cases} \quad x_{ij} \geq 0 \quad \forall (i, j) \in A.$$

Now, putting the obtained values of x_{ij} in $(\sum_{j:(i,j) \in A} (a_{ij}, b_{ij}, c_{ij}) x_{ij})$ a unique triangular fuzzy number, representing maximum total fuzzy project completion time, will be obtained and the fuzzy critical path will be obtained by using the obtained values of x_{ij} .

8. Advantages of the Proposed Method Over the Existing Method

The main advantage of the proposed method over the existing method is that on solving the fuzzy critical path problems by using the existing method, the maximum total fuzzy completion time corresponding to two different fuzzy critical paths of a project may be different and due to which there will be different interpretations for the maximum total fuzzy project completion time of a same project which is not appropriate according to real life situations while by using the proposed method the maximum total fuzzy project completion time corresponding to all fuzzy

critical paths will be same. So, there will be a unique interpretation of maximum total fuzzy completion time of project.

To show the advantages of the proposed method over existing method the fuzzy critical path problem, chosen in Example 5.1, for which the maximum total fuzzy project completion time corresponding to two different fuzzy critical paths are different, is solved by the proposed method and it is shown that by using the proposed method a unique fuzzy number, representing the maximum total fuzzy project completion time, is obtained corresponding to different fuzzy critical paths.

8.1. Exact Optimal Solution of the Chosen Fuzzy Critical Path Problem

The fuzzy critical path problem, chosen in Example 5.1, can be obtained by using the following steps of the proposed method:

Step 1.

Using the results of Example 5.1, the fuzzy critical paths for the chosen problem are $1 \Rightarrow 2 \Rightarrow 4$, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$ respectively.

Since more than one fuzzy critical paths are obtained, i.e., Case (ii) of Step 1 of the proposed method is satisfied, so go to Step 2 of the proposed method.

Step 2.

Using the results of the Example 5.1, the maximum total fuzzy completion time of the project corresponding to the fuzzy critical paths $1 \Rightarrow 2 \Rightarrow 4$, $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$ are (14, 23, 32), (15, 23, 31) and (15, 23, 31) respectively. Since the fuzzy numbers (14, 23, 32) and (15, 23, 31), representing the maximum total fuzzy completion time of the project, are different, i.e., Case (ii) of Step 2 of the proposed method is satisfied, so go to Step 3 of the proposed method.

Step 3.

Let the obtained maximum total fuzzy project completion time (14, 23, 32), corresponding to path $1 \Rightarrow 2 \Rightarrow 4$, be represented by \tilde{T}_1 and the maximum total fuzzy project completion time (15, 23, 31), corresponding to both the paths $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$, be represented by \tilde{T}_2 i.e., $\tilde{T}_1 = (14, 23, 32)$ and $\tilde{T}_2 = (15, 23, 31)$.

Solving the CLP problem:

$$\text{Maximize } (\text{mode}(2,4,6) x_{12} + \text{mode}(9,13,17) x_{13} + \text{mode}(7,9,11) x_{23} + \text{mode}(12,19,26) x_{24} + \text{mode}(6,10,14) x_{34})$$

subject to

$$\Re(2,4,6) x_{12} + \Re(9,13,17) x_{13} + \Re(7,9,11) x_{23} + \Re(12,19,26) x_{24} + \Re(6,10,14) x_{34} =$$

$\mathfrak{R}(14,23,32)$ or $\mathfrak{R}(15,23,31)$,

$$x_{12} + x_{13} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{13} + x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$$

$$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0.$$

The following three optimal solutions are obtained:

- (i) $x_{12} = x_{24} = 1$ and $x_{13} = x_{23} = x_{34} = 0$,
- (ii) $x_{12} = x_{23} = x_{34} = 1$ and $x_{13} = x_{24} = 0$,
- (iii) $x_{13} = x_{34} = 1$ and $x_{12} = x_{23} = x_{24} = 0$.

Putting the obtained values of x_{ij} in $((2, 4, 6) x_{12} \oplus (9, 13, 17) x_{13} \oplus (7, 9, 11) x_{23} \oplus (12, 19, 26) x_{24} \oplus (6, 10, 14) x_{34})$, the obtained maximum total fuzzy project completion times are $(14, 23, 32)$, $(15, 23, 31)$ and $(15, 23, 31)$ respectively. Since $(14, 23, 32) \neq (15, 23, 31)$ i.e., Case (ii) of Step 3 of the proposed method is satisfied, so go to Step 4 of the proposed method.

Step 4.

Let the obtained maximum total fuzzy project completion time $(14, 23, 32)$, corresponding to path $1 \Rightarrow 2 \Rightarrow 4$, be represented by \tilde{T}_1' and the maximum total fuzzy project completion time $(15, 23, 31)$, corresponding to both the paths $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$, be represented by \tilde{T}_2' , i.e., $\tilde{T}_1' = (14, 23, 32)$ and $\tilde{T}_2' = (15, 23, 31)$.

Solving the CLP problem:

Maximize $(divergence(2,4,6) x_{12} + divergence(9,13,17) x_{13} + divergence(7,9,11) x_{23} + divergence(12,19,26) x_{24} + divergence(6,10,14) x_{34})$
 subject to

$$\mathfrak{R}(2,4,6) x_{12} + \mathfrak{R}(9,13,17) x_{13} + \mathfrak{R}(7,9,11) x_{23} + \mathfrak{R}(12,19,26) x_{24} + \mathfrak{R}(6,10,14) x_{34} = \mathfrak{R}(14,23,32) \text{ or } \mathfrak{R}(15,23,31),$$

$$mode(2,4,6) x_{12} + mode(9,13,17) x_{13} + mode(7,9,11) x_{23} + mode(12,19,26) x_{24} + mode(6,10,14) x_{34} = mode(14,23,32) \text{ or } mode(15,23,31),$$

$$x_{12} + x_{13} = 1, \quad x_{12} - x_{24} - x_{23} = 0, \quad x_{13} + x_{23} - x_{34} = 0, \quad x_{24} + x_{34} = 1$$

$$x_{12}, x_{13}, x_{23}, x_{24}, x_{34} \geq 0$$

The optimal solution is $x_{12} = x_{24} = 1$ and $x_{13} = x_{23} = x_{34} = 0$. Putting the obtained values of x_{ij} in $((2, 4, 6) x_{12} \oplus (9, 13, 17) x_{13} \oplus (7, 9, 11) x_{23} \oplus (12, 19, 26) x_{24} \oplus (6, 10, 14) x_{34})$ a unique triangular fuzzy number $(14, 23, 32)$, representing maximum total fuzzy project completion time, is obtained and using the same values of x_{ij} the obtained fuzzy critical path is $1 \Rightarrow 2 \Rightarrow 4$.

9. Results and Discussion

To show the advantage of the proposed over existing method the results of fuzzy critical path problem, chosen in Example 5.1, obtained by using the existing and proposed methods is shown in Table 1.

Table 1. Results of the existing and proposed method

Example	Existing method (Liu, 2003)		Proposed method	
	Fuzzy critical path	Maximum total fuzzy completion time	Fuzzy critical path	Maximum total fuzzy completion time
5.1	$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$, $1 \Rightarrow 2 \Rightarrow 4$ and $1 \Rightarrow 3 \Rightarrow 4$	$(15, 23, 31)$, $(14, 23, 32)$ and $(15, 23, 31)$	$1 \Rightarrow 2 \Rightarrow 4$	$(14, 23, 32)$

It is obvious from the results shown in Table 1 that the maximum total fuzzy project completion time, obtained by using the existing method, is different corresponding to different fuzzy critical paths which is not appropriate according to real life situations, while on solving the same problem by using the proposed method a unique fuzzy number, representing the maximum total fuzzy project completion time is obtained.

On the basis of these results it may be suggested that it is better to use the proposed method instead of existing method for solving the fuzzy critical path problems.

10. Conclusion and Future Work

By choosing some fuzzy critical path problems it is shown that it is not better to use the existing method [Liu (2003)] for solving the fuzzy critical path problems and a new method is proposed for solving the fuzzy critical path problems. To show the advantage of the proposed method over existing method same fuzzy critical path problem is solved by using the existing and the proposed method and it is shown that the results obtained by using the proposed method are better than the results obtained by using the existing method.

In the proposed method, the existing method [Kaufmann and Gupta (1988)] is used for the ordering of fuzzy numbers but since the existing method [Kaufmann and Gupta (1988)] is applicable only for the ordering of triangular fuzzy numbers. So the proposed method with existing method [Kaufmann and Gupta (1988)] can not be used for solving such fuzzy critical

path problems in which the fuzzy activity times are represented by other types of fuzzy numbers. In future the existing method [Kaufmann and Gupta (1988)] may be modified to find the exact ordering of other types of fuzzy numbers and then the proposed method with modified method can be used for finding the exact solution of such fuzzy critical path problems in which fuzzy activity times are represented by other types of fuzzy numbers.

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