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Improved G'/G-Expansion Method and Comparing with Tanh-Coth Method

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Abstract

In this paper, improved G'/G-expansion and tanh-coth methods for solving partial differential equations are compared. It has been shown that the tanh-coth method is a special case of the improved G'/G-expansion method. For illustration and more explanation of the idea, exact solutions of the Burgers and Boussinesq equations are obtained by improved G'/G-expansion and the results obtained compared with those of tanh-coth method.

Keywords: Improved G'/G-expansion method; tanh-coth method; partial Differential equation; Burgers equation; Boussinesq equation

MSC 2010 No.: 65M70, 65Q20

1. Introduction

In recent years, seeking exact solutions of nonlinear partial differential equations (NLPDEs) is of great significance as it appears to be mathematical the models of complex phenomena arising in physics, mechanics, biology, chemistry and ring. Various methods have been presented to obtain solutions of NLPDEs. These include the tanh method [Wazwaz (2005), Malfliet- Hereman (1996)], sine–cosine method [Wazwaz (2006)], tanh-coth method [Wazwaz (2007)], Exp function method [He (2006), Biazar-Ayati (2008)], G'/G-expansion method [Wang-Zhang (2008), Zhang (2008)], and many others [He (2005), (1999)].

In this paper, tanh-coth and Improved G'/G-expansion methods are investigated. In the pioneer work of Malfliet (1992), the powerful tanh method widely used by researchers for a reliable treatment of the nonlinear wave equations was introduced. The tanh–coth method, developed by Wazwaz (2007), is also a direct and effective algebraic method for handling nonlinear equations. The second method, G'/G-expansion method, was proposed by Wang et al. (2008) for the first time, to look for travelling wave solutions of nonlinear evolution equations. This method is based on the assumptions that the travelling wave solutions can be expressed as a polynomial in G'/G and that $G = G(\xi)$ satisfies a second order linear ordinary differential equation (ODE).

In this paper, we show that improved G'/G-expansion method [Zhang (2010)] is a general form of the tanh-coth method.

2. The Improved G'/G-Expansion Method

Consider a nonlinear partial differential equation, in two independent variables, say x and t, in the form

$$p(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0, \tag{1}$$

where $u = u(x,t)$ is an unknown function, and p is non-linear equation in $u = u(x,t)$ and its partial derivatives. To apply the method, we proceed as follows:

Step 1.

The transformation

$$\xi = x - wt, \tag{2}$$

where w is a constant , converts the PDE (equation (1)) to an ODE:

$$Q(u, u', u'', u''', \ldots) = 0, \tag{3}$$

where the superscripts stands for the derivatives with respect to ξ.

Step 2.

We suppose that the solution of equation (3) can be expressed as the following polynomial in G'/G:

$$u(\xi) = \sum_{i=-m}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \tag{4}$$

where $G = G(\xi)$ satisfies a second order LODE in the form

$$G'' + \lambda G' + \mu G = 0. \tag{5}$$

with $\alpha_i, \lambda,$ and μ being constants to be determined later ($\alpha_m \neq 0$). The parameter m is a positive integer, in most cases, and can be determined by considering the homogeneous balance between the highest order derivatives and the highest order nonlinear terms, appearing in ODE (3).

Step 3.

Substituting equation (4) into equation (3) and using the second order LODE, Eq. (5), yields an algebraic equation involving powers of G'/G. Equating the coefficient of each power of G'/G to zero leads to a system of algebraic equations for determining $\alpha_i, w, \lambda,$ *and* μ.

Step 4.

Having the values of $\alpha_i, w, \lambda,$ and $\mu,$ from Step 3, and the solutions of LODE (5), which can be obtained easily, we are closed to the solutions of the nonlinear evolution equation (1).

3. The Tanh-Coth Method

The tanh-coth technique will be simply reviewed as follows:

Step 1.

Use the wave variable $\xi = x - wt$, to change the PDE into an ODE.

Step 2.

Introduce a new independent variable

$$Y = \tanh(\eta\xi), \tag{6}$$

that leads to the change of derivatives:

$$\frac{d}{d\xi} = \eta(1-Y^2)\frac{d}{dY}, \tag{7}$$

$$\frac{d^2}{d\xi^2} = -2\eta^2 Y(1-Y^2)\frac{d}{dY} + \eta^2(1-Y^2)^2\frac{d^2}{dY^2}. \tag{8}$$

Other derivatives can be derived in a similar manner.

Step 3.

We then propose the following finite series expansion

$$u(\mu\xi) = s(Y) = \sum_{k=0}^{m}\beta_k Y^k + \sum_{k=1}^{m}\beta_{-k} Y^{-k} = \sum_{k=0}^{m}\beta_k \tanh^k(\eta\xi) + \sum_{k=1}^{m}\beta_{-k} \tanh^{-k}(\eta\xi), \tag{9}$$

in which in most cases m is a positive integer. To determine the parameter m, we usually balance the linear terms of highest order in the equation (3) with the highest order nonlinear terms. Substituting (7), (8) and (9) into the ODE yields an equation in powers of Y.

Step 4.

With m determined, we collect all coefficients of powers of Y in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters $\beta_k\,(k=-m,\dots,m), \mu,$ and w. Having determined these parameters and using (9) we obtain an analytic solution $u = u(x,t)$, in a closed form.

The main characteristic of these two techniques is that they reduce partial differential equations to a system of algebraic equations.

4. Comparison of Improved *G'*/*G*-Expansion and Tanh-Coth Methods

Theorem:

The tanh-coth method is a special case of the improved *G'*/*G*-expansion method.

Proof:

For $\lambda^2 - 4\mu > 0,$ the general solution of equation (6) will be obtained as following form

$$\frac{G'(\xi)}{G(\xi)} = \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{A\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right) + B\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)}{A\cosh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right) + B\sinh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi\right)}\right) - \frac{\lambda}{2}, \tag{10}$$

where A and B are arbitrary constants. If we take $B=0$ and $\lambda=0$, equation (10) turns to

$$\frac{G'(\xi)}{G(\xi)}=\sqrt{-\mu}\tanh\left(\sqrt{-\mu}\,\xi\right). \tag{11}$$

By substituting equation (11) in equation (4), we get

$$u(\xi)=\sum_{i=-m}^{m}\alpha_i\left(\sqrt{-\mu}\right)^i\tanh^i\left(\sqrt{-\mu}\,\xi\right). \tag{12}$$

By comparing equation (9) and equation (12), we conclude that if $B=0$ and $\lambda=0$, then the results of tanh-coth method can be obtained directly by improved G'/G-expansion and we will have

$$\sqrt{-\mu}=\eta,\quad \alpha_i\eta^i=\beta_i,\quad i=-m,\ldots,m. \tag{13}$$

So, the tanh-coth method is a special case of the improved G'/G-expansion method.

5. Examples

In this section, the improved G'/G-expansion method will be applied to the two equations and these solutions will be compared with the obtained results of the tanh-coth method.

Example 1.

Consider one-dimensional Burgers' equation in the following form

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}. \tag{14}$$

In order to apply these methods, using the transformation

$$u=U(\xi)\ \text{and}\ \xi=x-wt, \tag{15}$$

where w is a constant to be determined, equation (14) becomes an ordinary differential equation as follows:

$$-wu'+uu'-\nu u''=0. \tag{16}$$

Integrating equation (16) yields

$$-wu+\frac{1}{2}u^2-\nu u'=0. \tag{17}$$

For applying G'/G-expansion method, by considering the homogeneous balance between u' and u^2 in equation (17), we get

$$m=1. \tag{18}$$

So, we can write (4) in the following simple form

$$u(\xi)=\alpha_1\left(\frac{G'}{G}\right)+\alpha_0+\alpha_{-1}\left(\frac{G'}{G}\right)^{-1},\ \ \alpha_1\neq 0. \tag{19}$$

By substituting (19) into equation (17) and collecting all terms with the same power of G'/G together, the left-hand side of equation (17) is converted into another polynomial in G'/G. Equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for determining $\alpha_1,\alpha_0,\alpha_{-1},w,\lambda,$ and μ as follows:

$$\begin{aligned}
&\left(\frac{G'}{G}\right)^{-2}: \frac{1}{2}\alpha_{-1}^2\text{-}\nu\alpha_{-1}\mu=0,\\
&\left(\frac{G'}{G}\right)^{-1}: \text{-w}\alpha_{-1}+\alpha_0\alpha_{-1}\text{-}\nu\alpha_{-1}\lambda=0,\\
&\left(\frac{G'}{G}\right)^{0}: \nu\alpha_1\mu\text{-w}\alpha_0+\alpha_1\alpha_{-1}\text{-}\nu\alpha_{-1}+\frac{1}{2}\alpha_0^2=0,\\
&\left(\frac{G'}{G}\right)^{1}: \nu\alpha_1\lambda\text{-w}\alpha_1+\alpha_0\alpha_1=0,\\
&\left(\frac{G'}{G}\right)^{2}: \frac{1}{2}\alpha_1^2+\nu\alpha_1=0.
\end{aligned} \tag{20}$$

Solving these algebraic equations gives three sets of solutions:

The first solutions set:

$$\alpha_1=\text{-}2\nu,\ \alpha_0=\text{-}\nu\lambda+\text{w},\ \alpha_{-1}=0,\ \mu=\text{-}\frac{1}{4}\frac{(\text{w}^2\text{-}\nu^2\lambda^2)}{\nu^2}. \tag{21}$$

The second solutions set:

$$\alpha_1=0,\ \mu=\text{-}\frac{1}{4}\frac{(\text{w}^2\text{-}\nu^2\lambda^2)}{\nu^2},\ \alpha_{-1}=\text{-}\frac{1}{2}\frac{(\text{w}^2\text{-}\nu^2\lambda^2)}{\nu},\ \alpha_0=\text{w}+\nu\lambda. \tag{22}$$

The third solutions set:

$$\alpha_1 = -2\nu,\ \lambda = 0,\ \mu = -\frac{1}{16}\frac{\mathrm{w}^2}{\nu^2},\ \alpha_{-1} = -\frac{1}{8}\frac{\mathrm{w}^2}{\nu},\ \alpha_0 = \mathrm{w}. \tag{23}$$

Substituting equations (10), (21) - (23) into equation (19), turns to the following exact solutions, respectively,

$$u(\xi) = w\left(1 - \frac{A\sinh\left(\frac{w}{2\nu}\xi\right) + B\cosh\left(\frac{w}{2\nu}\xi\right)}{A\cosh\left(\frac{w}{2\nu}\xi\right) + B\sinh\left(\frac{w}{2\nu}\xi\right)}\right), \tag{24}$$

$$u(\xi) = \mathrm{w} + \nu\lambda - \frac{1}{2}\frac{(\mathrm{w}^2 - \nu^2\lambda^2)}{\nu}\left(\frac{w}{2\nu}\left(\frac{A\sinh\left(\frac{w}{2\nu}\xi\right) + B\cosh\left(\frac{w}{2\nu}\xi\right)}{A\cosh\left(\frac{w}{2\nu}\xi\right) + B\sinh\left(\frac{w}{2\nu}\xi\right)}\right) - \frac{\lambda}{2}\right)^{-1}, \tag{25}$$

and

$$u(\xi) = \frac{w}{2}\left(2 - \left(\frac{A\sinh\left(\frac{w}{4\nu}\xi\right) + B\cosh\left(\frac{w}{4\nu}\xi\right)}{A\cosh\left(\frac{w}{4\nu}\xi\right) + B\sinh\left(\frac{w}{4\nu}\xi\right)}\right) - \left(\frac{A\sinh\left(\frac{w}{4\nu}\xi\right) + B\cosh\left(\frac{w}{4\nu}\xi\right)}{A\cosh\left(\frac{w}{4\nu}\xi\right) + B\sinh\left(\frac{w}{4\nu}\xi\right)}\right)^{-1}\right). \tag{26}$$

In particular case of $B = 0$ and $\lambda = 0$, equation (24)- (26) turn to

$$u(x,t) = w\left(1 - \tanh\left(\frac{w}{2\nu}(x - wt)\right)\right), \tag{27}$$

$$u(\xi) = \mathrm{w}\left(1 - \coth\left(\frac{w}{2\nu}(x - wt)\right)\right), \tag{28}$$

and

$$u(\xi) = \frac{w}{2}\left(2 - \left(\tanh\left(\frac{w}{4\nu}(x - wt)\right)\right) - \coth\left(\frac{w}{4\nu}(x - wt)\right)\right), \tag{29}$$

where equations (27) and (29) are the same as the result obtained by tanh-coth method [Wazwaz (2008)].

Example 2.

Let's consider the fourth-order nonlinear Boussinesq equation

$$u_{tt} - u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0. \tag{30}$$

We first use $u(x,t) = u(\xi)$ that will carry out the equation (24) into the following form

$$(w^2 - 1)u'' - 3(u^2)'' - u^{(iv)} = 0. \tag{31}$$

By taking the twofold integral from the equation (31), and taking the integral constant zero for the sake of simplicity, we obtain

$$(w^2 - 1)u - 3u^2 - u'' = 0. \tag{32}$$

To apply G'/G-expansion method, we consider the homogeneous balance between u'' and u^2 in equation (32). So we drive $m = 2$, and equation (4) turns to the following simple form

$$v(\xi) = \alpha_2\left(\frac{G'}{G}\right)^2 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_0 + \alpha_{-1}\left(\frac{G'}{G}\right)^{-1} + \alpha_{-2}\left(\frac{G'}{G}\right)^{-2}. \tag{33}$$

By substituting (33) into equation (32) and collecting all terms with the same power of G'/G together, the left-hand side of equation (32) is converted into another polynomial in G'/G. Equating each coefficient of this polynomial to zero yields to a set of simultaneous algebraic equations for determining $\alpha_2, \alpha_1, \alpha_0, \alpha_{-1}, \alpha_{-2}, w, \lambda,$ and μ as follows:

$$\left(\frac{G'}{G}\right)^{-4} : -6\alpha_{-2}\mu^2 - 3\alpha_{-2}{}^2 = 0,$$

$$\left(\frac{G'}{G}\right)^{-3} : -6\alpha_{-1}\alpha_{-2} = 0,$$

$$\left(\frac{G'}{G}\right)^{-2} : -8\alpha_{-2}\mu - 3\alpha_{-1}\lambda\mu - 4\alpha_{-2}\lambda^2 + \mathrm{w}^2\alpha_{-2} - 6\alpha_0\alpha_{-2} - \alpha_{-2} - 3\alpha_{-1}{}^2 = 0,$$

$$\left(\frac{G'}{G}\right)^{-1} : -\alpha_{-1}\lambda^2 - 6\alpha_1\alpha_{-2} + \mathrm{w}^2\alpha_{-1} - 6\alpha_{-2}\lambda - 2\alpha_{-1}\mu - \alpha_{-1} - 6\alpha_0\alpha_{-1} = 0,$$

$$\left(\frac{G'}{G}\right)^{0} : -\alpha_1\lambda\mu - 2\alpha_2\mu^2 - \alpha_{-1}\lambda + \mathrm{w}^2\alpha_0 - 2\alpha_{-2} - \alpha_0 - 3\alpha_0{}^2 - 6\alpha_1\alpha_{-1} - 6\alpha_{-2}\alpha_{-2} = 0,$$

$$\left(\frac{G'}{G}\right)^{1} : -6\alpha_2\alpha_{-1}1 - \alpha_1\lambda^2 + + \mathrm{w}^2\alpha_1 - \alpha_1 - 2\alpha_1\mu - 6\alpha_2\lambda\mu - 6\alpha_1\alpha_0 = 0,$$

$$\left(\frac{G'}{G}\right)^{2} : -4\alpha_2\lambda^2 - 3\alpha_1\lambda\mu - 3\alpha_1^2 - \alpha_2 - 8\alpha_2\mu - 6\alpha_0\alpha_2 + \mathrm{w}^2\alpha_2 = 0,$$

$$\left(\frac{G'}{G}\right)^3 : -2\alpha_1 - 2\alpha_{-1}\mu^2 - 6\alpha_2\alpha_1 - 10\alpha_{-2}\lambda\mu - 10\alpha_2\lambda = 0,$$

$$\left(\frac{G'}{G}\right)^4 : -6\alpha_2 - 3\alpha_2^{\ 2} = 0.$$

For the sake of simplicity, we suppose that $\alpha_1 = \alpha_{-1} = 0$. Then, by solving this system of algebraic equations we find the following sets of solutions, in which w is an arbitrary constant:

First set:

$$\alpha_0 = \frac{w^2 - 1}{2},\ \alpha_2 = -2,\ \alpha_{-2} = 0,\ \mu = \frac{1 - w^2}{4}, \lambda = 0. \tag{34}$$

Second set:

$$\alpha_0 = \frac{1 - w^2}{6},\ \alpha_2 = -2,\ \alpha_{-2} = 0,\ \mu = \frac{w^2 - 1}{4}, \lambda = 0. \tag{35}$$

Third set:

$$\alpha_0 = \frac{w^2 - 1}{2},\ \alpha_2 = 0,\ \alpha_{-2} = -\frac{(w^2 - 1)^2}{8},\ \mu = \frac{1 - w^2}{4}, \lambda = 0. \tag{36}$$

Forth set:

$$\alpha_0 = \frac{1 - w^2}{6},\ \alpha_2 = 0,\ \alpha_{-2} = -\frac{(w^2 - 1)^2}{8},\ \mu = \frac{w^2 - 1}{4}, \lambda = 0. \tag{37}$$

Fifth set:

$$\alpha_0 = \frac{w^2 - 1}{4},\ \alpha_2 = -2,\ \alpha_{-2} = -\frac{(w^2 - 1)^2}{128},\ \mu = \frac{1 - w^2}{16}, \lambda = 0. \tag{38}$$

Sixth set:

$$\alpha_0 = \frac{w^2 - 1}{12},\ \alpha_2 = -2,\ \alpha_{-2} = -\frac{(w^2 - 1)^2}{128},\ \mu = \frac{w^2 - 1}{16}, \lambda = 0. \tag{39}$$

By substituting equation (34), and the general solutions of equation (5) into equation (33), we have the following exact solutions:

for

$w^2 > 1$

$$u(\xi)=\frac{w^2-1}{2}\left[1-\left(\frac{A\sinh\left(\frac{\sqrt{w^2-1}}{2}\xi\right)+B\cosh\left(\frac{\sqrt{w^2-1}}{2}\xi\right)}{A\cosh\left(\frac{\sqrt{w^2-1}}{2}\xi\right)+B\sinh\left(\frac{\sqrt{w^2-1}}{2}\xi\right)}\right)^2\right], \tag{40}$$

while

$w^2 < 1$

$$u(\xi)=\frac{w^2-1}{2}\left[1+\left(\frac{-A\sin\left(\frac{\sqrt{1-w^2}}{2}\xi\right)+B\cos\left(\frac{\sqrt{1-w^2}}{2}\xi\right)}{A\cos\left(\frac{\sqrt{1-w^2}}{2}\xi\right)+B\sin\left(\frac{\sqrt{1-w^2}}{2}\xi\right)}\right)^2\right]. \tag{41}$$

In particular case $A \neq 0, \text{and} B = 0,$ equations (40) and (41) turn to

$$u(x,t)=\frac{w^2-1}{2}\sec h^2\left(\frac{1}{2}\sqrt{w^2-1}(x-wt)\right), \quad w^2>1, \tag{42}$$

and

$$u(x,t)=\frac{w^2-1}{2}\sec^2\left(\frac{1}{2}\sqrt{1-w^2}\,(x-wt)\right), \quad w^2<1, \tag{43}$$

where equations (42) and (43) are the same as the result obtained by tanh-coth method [Wazwaz (2007)]. In the same way, by substituting equations (35)- (37) and the general solutions of equation (5) into equation (33), and also by considering $A \neq 0, \text{and} B = 0,$ we obtain all result obtained by tanh-coth method.

6. Conclusion

This paper seeks a comparison between improved *G'/G*-expansion and tanh-coth methods. We have succeeded in identifying the equivalence of the two methods under special conditions in two examples presented as solid evidence. Consequently, the solution of the equations via improved *G'*/*G*-expansion is exactly the same as the solution of tanh-coth method if the conditions $B=0$ are $\lambda=0$ are satisfied. In fact, we proved that the tanh-coth method is a special case of the improved *G'*/*G*-expansion method. (The computations associated in this work were performed by using Maple 13.)

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