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Abstract

In this paper we give some local estimates for the Koornwinder Jacobi-type polynomials by using asymptotic properties of Jacobi orthogonal polynomials.

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1. Introduction

Let

$$\omega^{(\alpha,\beta)}(x) = (1-x)^{\alpha} \cdot (1+x)^{\beta}, x \in [-1,1]$$

be a Jacobi weight with $\alpha, \beta > -1$. Let also

$$p_n(x) = p_n^{(\alpha,\beta)}(x) = \gamma_n^{(\alpha,\beta)} x^n + \ldots, n \in \mathbb{N}_0$$

denote the unique Jacobi polynomials of precise degree n, with leading coefficients $\gamma_n^{(\alpha,\beta)} > 0$, fulfilling the orthogonal conditions

$$\int_{-1}^{1} p_n(x) p_m(x) \omega^{(\alpha,\beta)}(x) = \delta_{m,n}, n, m \in \mathbb{N} .$$

Felten (2007), introduced modified Jacobi weights as

$$\omega_n^{(\alpha,\beta)}(x) := \left(\sqrt{1-x} + \frac{1}{n}\right)^{2\alpha} \left(\sqrt{1+x} + \frac{1}{n}\right)^{2\beta}, x \in [-1,1], n \in \mathbb{N} . \tag{1}$$

He proved the following theorem [see Felten (2007)]:

Theorem 1.1:

Let $\alpha, \beta > -1$ and $n \in \mathbb{N}$. Then,

$$| p_n^{(\alpha,\beta)}(x) | \leq C \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4}\right)}(x)}, \tag{2}$$

for all $x \in [-1,1]$ with a positive constant $C = C(\alpha, \beta)$ being independent of n and x.

The above estimation first appeared in Lubinski and Totik (1994). Then for $\alpha, \beta \geq -\frac{1}{2}$, Felten (2004) extended the previous results as follows:

Theorem 1.2:

Let $\alpha, \beta \geq -\frac{1}{2}$ and $n \in \mathbb{N}$. Then,

$$| p_n^{(\alpha,\beta)}(t) | \leq C \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4}\right)}(x)}, \tag{3}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$, where

$$U_n(x) := \left\{ t \in [-1,1] : |t-x| \le \frac{\varphi_n(x)}{n} \right\} = \left[x - \frac{\varphi_n(x)}{n}, x + \frac{\varphi_n(x)}{n} \right], \tag{4}$$

for $n \in \mathbb{N}$ and $x \in [-1,1]$ with $\varphi_n(x) := \sqrt{1-x^2} + \frac{1}{n}$.

Koornwinder (1984), introduced the polynomials $\left(P_n^{(\alpha,\beta,M,N)}(x) \right)_{n=0}^{\infty}$ defined as follows:

Definition 1.3.

Fix $M, N \ge 0$ and $\alpha, \beta > -1$. For $n = 0,1,2,\cdots$ define

$$P_n^{(\alpha,\beta,M,N)}(x) = \left(\frac{(\alpha+\beta+1)_n}{n!} \right)^2 \cdot \left[(\alpha+\beta+1)^{-1} (B_n M (1-x) - A_n N (1+x) \frac{d}{dx} + A_n B_n) \right] p_n^{(\alpha,\beta)}(x),$$

where

$$A_n = \frac{(\alpha+1)_n n!}{(\beta+1)_n (\alpha+\beta+1)_n} + \frac{n(n+\alpha\beta+1)M}{(\beta+1)(\alpha+\beta+1)}, \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{5}$$

and

$$B_n = \frac{(\beta+1)_n n!}{(\alpha+1)_n (\alpha+\beta+1)_n} + \frac{n(n+\alpha\beta+1)N}{(\alpha+1)(\alpha+\beta+1)}. \tag{6}$$

We call these polynomials the Koornwinder's Jacobi-type polynomials.

The above defined polynomials are orthogonal on the interval $[-1,1]$ with respect to the measure μ defined by

$$\int_{-1}^{1} f(x) d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} f(x)(1-x)^{\alpha}(1+x)^{\beta} dx + Mf(-1) + Nf(1), \tag{7}$$

where $f \in C([-1,1])$ and $M, N \ge 0, \alpha, \beta > -1$.

Clearly, for $M = N = 0$ one has

$$P_n^{(\alpha,\beta,0,0)}(x) = P_n^{(\alpha,\beta)}(x). \tag{8}$$

Also

$$P_n^{(\alpha,\beta,M,N)}(-x)=(-1)^n P_n^{(\beta,\alpha,N,M)}(x). \tag{9}$$

Some basic properties of $P_n^{(\alpha,\beta,M,N)}(x)$ are given as below [Varona (1989), chapter IV)].

$$P_n^{(\alpha,\beta,M,N)}(1)\sim\begin{cases} n^{-\alpha-\frac{3}{2}}, \text{ if } N>0 \\ n^{\alpha+\frac{1}{2}}, \text{if } N=0\end{cases} \tag{10}$$

and

$$\left|P_n^{(\alpha,\beta,M,N)}(-1)\right|\sim\begin{cases} n^{-\beta-\frac{3}{2}}, \text{ if } M>0 \\ n^{\beta+\frac{1}{2}}, \text{if } M=0.\end{cases} \tag{11}$$

Theorem 1.4 [Varona (1989)]**:**

Let $\alpha,\beta>-1, M, N>0.$ For every $x\in[-1,1]$, there exists a unique constant C such that the following relation holds for each $n\in\mathbb{N}$:

$$\left(h_n^{\alpha,\beta,M,M}\right)^{-\frac{1}{2}}\left|P_n^{(\alpha,\beta,M,N)}(x)\right|\leq C\left(1-x+\frac{1}{n^2}\right)^{-\frac{\alpha}{2}-\frac{1}{4}}\left(1+x+\frac{1}{n^2}\right)^{-\frac{\beta}{2}-\frac{1}{4}},$$

where

$$h_n^{(\alpha,\beta,M,N)}=\int_{-1}^{1}(P^{(\alpha,\beta,M,N)}(x))^2\,d\mu.$$

Based on Theorem 1.4 and properties of Jacobi polynomials [see Lubinski and Totik (1994) and Szego (1975)], we get the following estimation for the Koornwinder Jacobi-type polynomials:

$$\left|P_n^{(\alpha,\beta,M,N)}(\cos\theta)\right|=\begin{cases} 0(\theta^{-\alpha-\frac{1}{2}}), \text{ if } \frac{c}{n}\leq\theta\leq\frac{\pi}{2} \\ 0(n^{\alpha+\frac{1}{2}}), \text{ if } 0\leq\theta\leq\frac{c}{n},\end{cases} \tag{12}$$

for

$\alpha \geq -1, \beta \geq -1$ and $n \geq 1$.

The aim of this paper is to prove similar results as those given in Theorem 1.1 and Theorem 1.2, for Koornwinder Jacobi-type polynomials, when $\alpha, \beta \geq -1$, respectively, for $\alpha, \beta \geq -\frac{1}{2}$.

2. Results

The following Theorem is the main result of this note.

Theorem 2.1:

Let $\alpha, \beta > -1$ and $n \in \mathbb{N}$. Then,

$$| P_n^{(\alpha,\beta,M,N)}(x) | \leq D \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)}, \tag{13}$$

for all $x \in [-1,1]$ with a positive constant $D = D(\alpha, \beta)$ being independent of n and x.

Proof:

Proof of the Theorem is similar to Theorem 2.1 in Felten (2007). Let $x \in [0,1]$, and let $\theta \in \left[0, \frac{\pi}{2}\right]$ such that $x = \cos\theta$. From (12), one has the following estimation

$$\left|P_n^{(\alpha,\beta,M,N)}(\cos\theta)\right| \leq C \begin{cases} \theta^{-\alpha-\frac{1}{2}}, \; if \; \frac{c}{n} \leq \theta \leq \frac{\pi}{2} \\ n^{\alpha+\frac{1}{2}}, \; if \; 0 \leq \theta \leq \frac{c}{n} \end{cases}. \tag{14}$$

If in the last relation, we substitute $x = \cos\theta$, then we will have

$$\left|P_n^{(\alpha,\beta,M,N)}(x)\right| \leq C \begin{cases} n^{\alpha+\frac{1}{2}}, \; if \; 0 \leq \arccos x \leq \frac{c}{n} \\ (\arccos x)^{-\left(\alpha+\frac{1}{2}\right)}, \; if \; \frac{c}{n} \leq \arccos x \leq \frac{\pi}{2}, \end{cases} \tag{15}$$

where C is fixed positive constant being independent of n and θ.

In what follows we will make use of the following estimates

$$\frac{\pi}{2}\sqrt{1-x}=\frac{\pi}{\sqrt{2}}\sqrt{\frac{1-x}{2}}=\frac{\pi}{\sqrt{2}}\sin\frac{t}{2}\geq\frac{\pi}{\sqrt{2}}\left(\frac{2}{\pi}\cdot\frac{t}{\sqrt{2}}\right)=t=\arccos x \tag{16}$$

and

$$\sqrt{2}\sqrt{1-x}=2\sqrt{\frac{1-x}{2}}=2\sin\frac{t}{2}\leq 2\cdot\frac{t}{2}=t=\arccos x\,. \tag{17}$$

We differ two cases:

Case 1. $-1<\alpha\leq-\frac{1}{2}$. In this case, $-\left(\alpha+\frac{1}{2}\right)\geq 0$.

If $0\leq\arccos x\leq\frac{c}{n}$, then from (17) we obtain $\frac{c}{n}\geq\sqrt{2}\sqrt{1-x}$ and from (15) we get the following relation

$$|P_n^{(\alpha,\beta,M,N)}|\leq Cn^{\alpha+\frac{1}{2}}=C\left(\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}\leq C_1\left(\sqrt{1-x}\right)^{-\left(\alpha+\frac{1}{2}\right)}\leq C_2\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

If $\frac{c}{n}\leq\arccos x\leq\frac{\pi}{2}$, then from relations (15) and (17) we get

$$|P_n^{(\alpha,\beta,M,N)}|\leq C_3(\arccos x)^{-\left(\alpha+\frac{1}{2}\right)}\leq C_4\left(\sqrt{1-x}\right)^{-\left(\alpha+\frac{1}{2}\right)}\leq C_5\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

Case 2. $\alpha>-\frac{1}{2}$. In this case $-\left(\alpha+\frac{1}{2}\right)<0$.

If $0\leq\arccos x\leq\frac{c}{n}$, then from relations (15) and (17) we obtain

$$|P_n^{(\alpha,\beta,M,N)}|\leq C_6 n^{\alpha+\frac{1}{2}}=C_6\left(\frac{c}{n}+\frac{\sqrt{2}}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}\leq C_7\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

If $\frac{c}{n}\leq\arccos x\leq\frac{\pi}{2}$, again according to relations (15) and (17) we have

$$| P_n^{(\alpha,\beta,M,N)} |\leq C_8(\arccos x)^{-\left(\alpha+\frac{1}{2}\right)} = C_9(\arccos x+\arccos x)^{-\left(\alpha+\frac{1}{2}\right)} \leq C_{10}\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}.$$

From previous cases we have proved that

$$| P_n^{(\alpha,\beta,M,N)}(x) |\leq C_{11}(\alpha,\beta)\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)}\cdot\left(\sqrt{1+x}+\frac{1}{n}\right)^{-\left(\beta+\frac{1}{2}\right)},$$

for all $x\in[0,1], n\in\mathbb{N}$ and $\alpha,\beta\geq -1$.

From (10) we obtain

$$| P_n^{(\alpha,\beta,M,N)}(x) |\leq C_{12}(\beta,\alpha)\left(\sqrt{1+x}+\frac{1}{n}\right)^{-\left(\beta+\frac{1}{2}\right)}\times\left(\sqrt{1-x}+\frac{1}{n}\right)^{-\left(\alpha+\frac{1}{2}\right)},$$

for all $x\in[-1,0), n\in\mathbb{N}$ and $\alpha,\beta\geq -1$.

The proof is completed.

Next, we will show that the local estimates of previous theorem can be further extended. We will prove that $| P_n^{(\alpha,\beta,M,N)}(x) |$ in (14) can be replaced by $| P_n^{(\alpha,\beta,M,N)}(t) |$, whenever t is in the interval $U_n(x)=\left[x-\frac{\varphi_n(x)}{n}, x+\frac{\varphi_n(x)}{n}\right]\cap[-1,1]$. In order to do that we will make use of the following Lemma [see Felten (2007)].

Lemma 2.2:

Let $a,b\leq 0, n\in\mathbb{N}$ and $x\in[-1,1]$. Then,

$$\omega_n^{(a,b)}(t)\leq 16^{-(a+b)}\omega_n^{(a,b)}(x), \tag{18}$$

for all $t\in U_n(x)$.

Theorem 2.3:

Let $\alpha,\beta\geq -\frac{1}{2}$ and $n\in\mathbb{N}$. Then,

$$| P_n^{(\alpha,\beta,M,N)}(t) | \leq D \frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)}, \tag{19}$$

for all $t \in U_n(x)$ and each $x \in [-1,1]$, where $D = D(\alpha,\beta)$ is a positive constant independent of n, t and x.

Proof:

Since $\alpha, \beta \geq -\frac{1}{2}$, it follows that $\frac{\alpha}{2}+\frac{1}{4}, \frac{\beta}{2}+\frac{1}{4} \geq 0$. Therefore, by Lemma 2.2 with $a = -\frac{\alpha}{2}-\frac{1}{4}$ and $\beta = -\frac{\alpha}{2}-\frac{1}{4}$, we obtain

$$\frac{1}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)} = \omega_n^{\left(-\frac{\alpha}{2}-\frac{1}{4},-\frac{\beta}{2}-\frac{1}{4}\right)}(x) \leq \frac{4^{\alpha+\beta+1}}{\omega_n^{\left(\frac{\alpha}{2}+\frac{1}{4},\frac{\beta}{2}+\frac{1}{4}\right)}(x)},$$

for all $t \in U_n(x)$. Applying Theorem 2.1 yields inequality (14) for all $t \in U_n(x)$, as claimed.

Corollary 2.4:

Let $n \in \mathbb{N}$ and $\alpha, \beta \geq -\frac{1}{2}, x \in [-1,1]$. Then,

$$\int_{U_n(x)} | P_n^{(\alpha,\beta,M,N)}(t) |^2 \, \omega_n^{(\alpha,\beta)}(t) dt \leq D(\alpha,\beta) \cdot \frac{1}{n}.$$

Proof:

Applying Theorem 2.3 we obtain

$$\int_{U_n(x)} | P_n^{(\alpha,\beta,M,N)}(t) |^2 \, \omega_n^{(\alpha,\beta)}(t) dt \leq D \cdot \frac{1}{\omega_n^{\left(\alpha+\frac{1}{2},\beta+\frac{1}{2}\right)}(x)} \cdot \int_{U_n(x)} \omega_n^{(\alpha,\beta)}(t) dt.$$

Using the following result from Felten (2008), we obtain

$$\int_{U_n(x)} \omega_n^{(\alpha,\beta)}(t)dt \le \frac{D}{n}\cdot\omega_n^{\left(\alpha+\frac{1}{2},\beta+\frac{1}{2}\right)}(x)$$

and, thus, the proof is completed.

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