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Approximating Solutions for Ginzburg – Landau Equation by HPM and ADM

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Abstract

In this paper, an analytical approximation to the solution of Ginzburg-Landauis discussed. A Homotopy perturbation method introduced by He is employed to derive the analytic approximation solution and results compared with those of the Adomian decomposition method. Two examples are presented to show the capability of the methods. The results reveal that the methods are almost equally effective and promising.

Keywords: Ginzburg-Landau equation; Homotopy perturbation method; Adomian decomposition method.

MSC (2010) No.: 35c10, 65M70

1. Introduction

In this paper we consider the Ginzburg-Landau equation in the form

$$iu_t + \alpha\, u_{xx} + \beta |u|^2 u - bu - iau = 0, \tag{1}$$

where u is a complex value function.

Originally discovered by Ginzburg and Landau (1965), for a phase transition in superconductivity, the equation has been extended to various fields such as chemical reactions, fluid mechanics and pattern formation, to mention just a few. For a detailed information on this equation, see, [Bechouche and Jungel (2000)] and references therein. In the case $a = b = 0$, the equation reduces to the famous non-linear Schrödinger equation, [Biazar and Ghazvini (2007)].

The homotopy perturbation method was introduced by He [(1999), (2004), (2006)]. In this method the solution assumed to be the summation of an infinite series which converges to the solution. Using a technique of topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ considered a ‘‘small parameter’’.

Considerable research has been conducted on the application of this method to a class of linear and non-linear equations, [Abbasbandy (2007), Sadighi and Ganji (2008)]. This method has also been used to solve hyperbolic differential equations [Biazar and Ghazvini (2008)], and other equations**,** [Biazar and Ghazvini (2009, 2008)]**.** Here we extend the method to solve the Ginzburg-landau equation.

2. Basic Idea of Homotopy Perturbation Method

To illustrate the basic ideas of the method, we consider the following non-linear differential equation

$$A(u) - f(r) = 0, \qquad r \in \Omega, \tag{2}$$

with the following boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \qquad r \in \Gamma, \tag{3}$$

Where A is a general functional operator, B is a boundary operator, $f(r)$ is a known analytical function, and Γ is the boundary of the domain Ω.

The operator A can be decomposed into two operators, L and N, where L is a linear, and N is a non-linear operator. Hence, Equation (2) can be rewritten as follows

$$L(u) + N(u) - f(r) = 0. \tag{4}$$

We construct a homotopy $U(r,p) : \Omega \times [0,1] \to \mathbb{R}$, which satisfies

$$H(U,p) = (1-p)[L(U) - L(u_0)] + p[A(U) - f(r)] = 0, \quad p \in [0,1], \quad r \in \Omega, \tag{5}$$

or

$$H(U,p)=L(U)-L(u_0)+pL(u_0)+p[N(U)-f(r)]=0, \tag{6}$$

where $p\in[0,1]$, is an embedding parameter, u_0 is an initial approximation for the solution of Equation (2), which satisfies the boundary conditions. Obviously, from Equations (5) and (6) we will have

$$H(U,0)=L(U)-L(u_0)=0, \qquad H(U,1)=A(U)-f(r)=0. \tag{7}$$

The changing process of p from zero to unity is just that of $U(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called homotopy. According to the HPM, we can first use the embedding parameter p as a small parameter, and assume that the solution of Equation (5) and (6) can be written as a power series in p

$$U=U_0+pU_1+p^2U_2+\cdots. \tag{8}$$

Setting $p=1$, results in the approximate solution of Equation (2)

$$u=\lim_{p\to 1}U=U_0+U_1+U_2+\cdots. \tag{9}$$

It is worth mentioning that if k, u_k's are all zero an exact solution is $u\approx\sum_{k=0}^{n}U_k$. The series (9) is convergent under some established criteria [He (1992)] for most other cases.

3. Methods of Solution

3.1. The Homotopy perturbation Method Applied to Ginzburg-Landau equation

Consider the following Ginzburg-landau equation with the following initial condition

$$i\,\frac{\partial u(x,t)}{\partial t}+\alpha\,\frac{\partial^2 u(x,t)}{\partial x^2}+\beta|u|^2u-bu-iau=0,\quad u(x,0)=u_0(x),\qquad x\in\mathbb{R}, \tag{10}$$

where in α and β are two real constants.

To solve Equation (10) by homotopy perturbation method, we construct the following homotopy

$$(1-p)(\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t})+p(\frac{\partial U}{\partial t}-i(\alpha\frac{\partial^2 U}{\partial x^2}+\beta|u|^2u-bu-iau))=0,$$

or

$$\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t}=p(-\frac{\partial u_0}{\partial t}+i(\alpha\frac{\partial^2 U}{\partial x^2}+\beta|u|^2u-bu-iau)). \tag{11}$$

Suppose the solution of Equation (10) to be in the following form

$$U=U_0+pU_1+p^2U_2+\cdots. \tag{12}$$

Substituting (12) into (11), and equating the coefficients of the terms with the identical powers of p, leads to the following

$$\begin{aligned}
&p^0: \frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t}=0,\\
&p^1: \frac{\partial U_1}{\partial t}+\frac{\partial u_0}{\partial t}-i(\alpha\frac{\partial^2 U_0}{\partial x^2}+\beta\overline{U_0}U_0^2-bU_0-iaU_0)=0, \qquad U_1(x,0)=0,\\
&p^2: \frac{\partial U_2}{\partial t}-i(\alpha\frac{\partial^2 U_1}{\partial x^2}+\beta(2\overline{U_0}U_0U_1+\overline{U_1}U_0^2)-bU_1-iaU_1)=0, \qquad U_2(x,0)=0,\\
&p^3: \frac{\partial U_3}{\partial t}-i(\alpha\frac{\partial^2 U_2}{\partial x^2}+\beta(\overline{U_2}U_0^2+2\overline{U_0}U_0U_2+2U_0U_1\overline{U_1}+U_1U_1\overline{U_0})-bU_2-iaU_2)=0,\\
&U_3(x,0)=0,\\
&p^j: \frac{\partial U_j}{\partial t}-i(\alpha\frac{\partial^2 U_{j-1}}{\partial x^2}+\beta(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}U_iU_k\overline{U}_{j-k-i-1})-bU_{j-1}-iaU_{j-1})=0,\\
&\qquad U_j(x,0)=0,
\end{aligned} \tag{13}$$

where in p^j, there are the multiplication of two series $|U|^2$ and U.

For the sake of the simplicity we take

$$U(x,t)=u_0(x,t)=u_0(x). \tag{14}$$

Having this assumption we get the following iterative equation

$$U_j=i\int_0^t(\alpha\frac{\partial^2 U_{j-1}}{\partial x^2}+\beta(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}U_iU_k\overline{U}_{j-k-i-1})-bU_{j-1}-iaU_{j-1})dt, \quad j=1,2,3,\ldots \tag{15}$$

The approximate solution of (10) can be obtained by setting $p=1$,

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$$u = \lim_{p\to 1} U = U_0 + U_1 + U_2 + \cdots.$$

The results of the following examples are compared with the results of the original ADM, They seem to be in good agreement, as expected, [Biazar, Ayati and Ebrahimi (2008)]. It is worth to noting that it was Wazwaz that introduced a reliable modification of the ADM, which accelerates the convergence of the series solution.

3.2. The Adomian Decomposition Method Applied to Ginzburg-Landau Equation

Consider the following Ginzburg-landau equation with the following initial condition

$$iu_t + \alpha u_{xx} + \beta |u|^2 u - bu - iau = 0,$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R},$$

where α and β are two real constants.

Pay attention to initial conditions operator $L_t = \frac{\partial}{\partial t}$. Therefore, we have

$$u_t = i(\alpha u_{xx} + \beta |u|^2 u - bu - iau). \tag{16}$$

The inverse operator of L_t is $L_t^{-1} = \int_0^t (.)dt$. Applying the inverse operator L_t^{-1} to both sides of (16), we get

$$u(x,t) = u(x,0) + i\int_0^t (\alpha \frac{\partial^2 u}{\partial x^2} + \beta |u|^2 u - bu - iau)dt,$$

or

$$u(x,t) = u(x,0) + i\int_0^t (\alpha \frac{\partial^2 u}{\partial x^2} + \beta u^2 \bar{u} - bu - iau)dt. \tag{17}$$

To solve Equation (17) by Adomian decomposition method we consider, as usual in this method, the series solution $u = \sum_{n=0}^{\infty} u_n$. So that the components u_n can be determined recursively. The integrand on the right side is the sum of a series.

$$u^2\bar{u} = \sum_{n=0}^{\infty} A_n, \tag{18}$$

where $A_n(u_0, u_1, \ldots, u_n)$ called the Adomian polynomials are computed using methods introduced in [Wazwaz (2000)]. We have

$$\sum_{n=0}^{\infty} u_n = u(x,0) + i\int_0^t (\alpha \frac{\partial^2 \sum_{n=0}^{\infty} u_n}{\partial x^2} + \beta \sum_{n=0}^{\infty} A_n - b\sum_{n=0}^{\infty} u_n - ia\sum_{n=0}^{\infty} u_n)\, dt, \tag{19}$$

which in turn, yields:

$$\begin{cases} u_0(x,t) = u(x,0) \\ u_{n+1}(x,t) = i\int_0^t (\alpha \frac{\partial^2 u_n}{\partial x^2} + \beta A_n - bu_n - iau_n)\,dt, \qquad n = 0,1,2,\ldots. \end{cases} \tag{20}$$

Resulting in the following approximations for the Adomian polynomials,

$$\begin{aligned} A_0 &= u_0^2 \overline{u_0}, \\ A_1 &= u_0^2 \overline{u_1} + 2u_0 \overline{u_0} u_1, \\ A_2 &= u_0^2 \overline{u_2} + 2u_0 \overline{u_0} u_2 + 2u_1 \overline{u_1} u_0 + \overline{u_0} u_1^2, \\ A_3 &= u_0^2 \overline{u_3} + 2u_0 \overline{u_0} u_3 + 2\overline{u_0} u_1 u_2 + 2u_0 \overline{u_1} u_2 + 2u_0 u_1 \overline{u_2} + u_1^2 \overline{u_1}, \\ A_4 &= \overline{u_0} u_2^2 + 2u_0 \overline{u_0} u_4 + 2\overline{u_0} u_1 u_3 + 2u_0 \overline{u_1} u_3 + 2u_1 \overline{u_1} u_2 + 2u_0 \overline{u_2} u_2 \\ &\quad + u_1^2 \overline{u_2} + 2u_0 \overline{u_1} u_3 + u_0^2 \overline{u_4}, \\ &\ \vdots \end{aligned}$$

From (20) we have

$$\begin{aligned} u_0(x,t) &= u(x,0) \\ u_1(x,t) &= i\int_0^t (\alpha \frac{\partial^2 u_0}{\partial x^2} + \beta\, u_0^2 \overline{u} - b u_0 - iau_0)\,dt, \\ u_2(x,t) &= i\int_0^t (\alpha \frac{\partial^2 u_1}{\partial x^2} + \beta\, (u_0^2 \overline{u_1} + 2u_0 \overline{u_0} u_1) - b u_1 - iau_1)\,dt, \\ u_3(x,t) &= i\int_0^t (\alpha \frac{\partial^2 u_2}{\partial x^2} + \beta\, (u_0^2 \overline{u_2} + 2u_0 \overline{u_0} u_2 + 2u_1 \overline{u_1} u_0 + \overline{u_0} u_1^2) - b u_2 - iau_2)\,dt, \\ u_4(x,t) &= i\int_0^t (\alpha \frac{\partial^2 u_3}{\partial x^2} + \beta\, (u_0^2 \overline{u_3} + 2u_0 \overline{u_0} u_3 + 2\overline{u_0} u_1 u_2 + 2u_0 \overline{u_1} u_2 + 2u_0 u_1 \overline{u_2} + u_1^2 \overline{u_1}) \\ &\qquad - b u_3 - iau_3)\,dt, \end{aligned}$$

$$u_5(x,t)=i\int_0^t (\alpha\frac{\partial^2 u_4}{\partial x^2}+\beta\ (\overline{u_0}u_2^2+2u_0\overline{u_0}u_4+2\overline{u_0}u_1u_3+2u_0\overline{u_1}u_3+2u_1\overline{u_1}u_2+2u_0\overline{u_2}u_2+u_1^2\overline{u_2}$$
$$+2u_0\overline{u_1}u_3+u_0^2\overline{u_4})-bu_4-iau_4)dt,$$
$$\vdots$$

We can determine the components u_n as far as we like to enhance the accuracy of the approximation.

4. Examples

To illustrate the methods and to demonstrate their capability, two examples are presented.

Example 1. Consider the following partial differential equation

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+2|u|^2u-u+iu=0,\quad t\geq 0,$$
$$u(x,0)=e^{ix}. \tag{21}$$

We construct a homotopy $\Omega\times[0,1]\to$ which satisfies

$$(1-p)(\frac{\partial U}{\partial t}-\frac{\partial u_0}{\partial t})+p(\frac{\partial U}{\partial t}-i(\frac{\partial^2 U}{\partial x^2}+2|u|^2u-u+iu))=0.$$

From (14), (15) we have the following scheme

$$u_0=u(x,0)=e^{ix},$$
$$u_j=i\int_0^t(\frac{\partial^2 U_{j-1}}{\partial x^2}+2(\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1}U_iU_k\overline{U}_{j-k-i-1})-U_{j-1}+iU_{j-1})dt,\quad j=1,2,3,\ldots.$$

For the first few j, we derive

$$u_1(x,t)=-\frac{1}{1!}te^{ix},$$
$$u_2(x,t)=\frac{1}{2!}t^2e^{ix}-\frac{12}{6}it^2e^{ix},$$

$$u_3(x,t)=-\frac{1}{3!}t^3e^{ix}+\frac{30}{9}it^3e^{ix},$$

$$u_4(x,t)=\frac{1}{4!}t^4e^{ix}-\frac{36}{12}it^4e^{ix}-\frac{24}{12}t^4e^{ix},$$

$$u_5(x,t)=-\frac{1}{5!}t^5e^{ix}+\frac{29}{15}it^5e^{ix}+\frac{70}{15}t^5e^{ix},$$

$$\vdots$$

These approximations are presented as follows

$$u(x,t)=\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{n!}e^{ix}+ie^{ix}(-\frac{12}{6}t^2+\frac{30}{9}t^3-\frac{36}{12}t^4+\frac{29}{15}t^5-\cdots)-2t^4e^{ix}+4t^5e^{ix}-\cdots.$$

Example 2. Consider the following partial differential equation

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+2|u|^2u-u+iu=0,\quad t\geq 0,$$
$$u(x,0)=e^{ix}.$$

From (20) we obtain

$$\begin{cases} u_0=e^{ix} \\ u_{n+1}(x,t)=i\int_0^t(\frac{\partial^2 u_n}{\partial x^2}+2A_n-u_n+iu_n)dt, \qquad n=0,1,2,\ldots. \end{cases}$$

For the first few n, we have

$$u_1(x,t)=-\frac{1}{1!}te^{ix},$$

$$u_2(x,t)=\frac{1}{2!}t^2e^{ix}-\frac{12}{6}it^2e^{ix},$$

$$u_3(x,t)=-\frac{1}{3!}t^3e^{ix}+\frac{30}{9}it^3e^{ix},$$

$$u_4(x,t)=\frac{1}{4!}t^4e^{ix}-\frac{36}{12}it^4e^{ix}-\frac{24}{12}t^4e^{ix},$$

$$u_5(x,t)=-\frac{1}{5!}t^5e^{ix}+\frac{29}{15}it^5e^{ix}+\frac{70}{15}t^5e^{ix},$$

$$\vdots$$

These approximations are presented as follows

$$u(x,t)=\sum_{n=0}^{\infty}(-1)^n \frac{t^n}{n!}e^{ix}+ie^{ix}(-\frac{12}{6}t^2+\frac{30}{9}t^3-\frac{36}{12}t^4+\frac{29}{15}t^5-\cdots)-2t^4e^{ix}+4t^5e^{ix}-\cdots.$$

5. Conclusions

In this paper, the homotopy perturbation method is proposed for solving non-linear Ginzburg-Landau equations. This method reveals that solution are exactly the same as those obtained by Adomian decomposition method, which has to overcome the difficulties in the calculation of the Adomian 's polynomials, as demonstrated in [Biazar, et al (2008)]. As was pointed out, there is a reliable modification of the ADM, which accelerates the convergence. Comparison of known methods, such as HPM, HAM, VIM, with MADM [Wazwaz (1999)] is on-going. The analytical approximation to the solution is clearly more reliable and confirms the power and capability of He' s homotopy perturbation method as an easie procedure for obtaining the solution of non-linear equations. Computations in this work were performed by using Maple 11.

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