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A New Method for Fuzzy Critical Path Analysis in Project Networks with a New Representation of Triangular Fuzzy Numbers

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Abstract

The method for finding fuzzy optimal solution of fully fuzzy critical path (FFCP) problems i.e., critical path problems in which all the parameters are represented by fuzzy numbers, is at best scant; possibly non-existent. In this paper, a method is proposed to find the fuzzy optimal solution of FFCP problems, together with a new representation of triangular fuzzy numbers. This paper will show the advantages of using, the proposed representation over the existing representations of triangular fuzzy numbers and will present with great clarity the proposed method and illustrate its application to FFCP problems occurring in real life situations.

Keywords: Fully fuzzy critical path problem, Ranking function, Triangular fuzzy number

MSC (2010) No: 03E72, 90C70, 90C05, 90C08

1. Introduction

In today's highly competitive business environment, project management's ability to schedule activities and monitor progress within strict cost, time and performance guidelines is becoming increasingly important to obtain competitive priorities such as on-time delivery and customization. In many situations, projects can be complicated and challenging to manage.

When the activity times in the project are deterministic and known, critical path method (CPM) has been demonstrated to be a useful tool in managing projects in an efficient manner to meet this challenge. The purpose of CPM is to identify critical activities on the critical path so that resources may be concentrated on these activities in order to reduce the project length time.

The successful implementation of CPM requires the availability of clear determined time duration for each activity. However, in practical situations this requirement is usually hard to fulfill, since many of activities will be executed for the first time. To deal with such real life situations, Zadeh (1965) introduced the concept of fuzzy set. Since there is always uncertainty about the time duration of activities in the network planning, due to which fuzzy critical path method (FCPM) was proposed since the late 1970s.

For finding the fuzzy critical path, several approaches are proposed over the past years. The first method called FPERT, was proposed by Chanas and Kamburowski (1981). They presented the project completion time in the form of fuzzy set in the time space. Gazdik (1983) developed a fuzzy network of unknown project to estimate the activity durations and used fuzzy algebraic operators to calculate the duration of the project and its critical path. Kaufmann and Gupta (1988) devoted a chapter of their book to the critical path method in which activity times are represented by triangular fuzzy numbers. McCahon and Lee (1988) presented a new methodology to calculate the fuzzy completion project time.

Nasution (1994) proposed how to compute total floats and find critical paths in a project network. Yao and Lin (2000) proposed a method for ranking fuzzy numbers without the need for any assumptions and have used both positive and negative values to define ordering which then is applied to CPM. Dubois et al. (2003) extended the fuzzy arithmetic operational model to compute the latest starting time of each activity in a project network. Lin and Yao (2003) introduced a fuzzy CPM based on statistical confidence-interval estimates and a signed distance ranking for $(1-\alpha)$ fuzzy number levels. Liang and Han (2004) presented an algorithm to perform fuzzy critical path analysis for project network problem. Zielinski (2005) extended some results for interval numbers to the fuzzy case for determining the possibility distributions describing latest starting time for activities.

Chen (2007) proposed an approach based on the extension principle and linear programming (LP) formulation to critical path analysis in networks with fuzzy activity durations. Chen and Hsueh (2008) presented a simple approach to solve the CPM problems with fuzzy activity times (being fuzzy numbers) on the basis of the linear programming formulation and the fuzzy number ranking method that are more realistic than crisp ones. Yakhchali and Ghodsypour (2010) introduced the problems of determining possible values of earliest and latest starting times of an activity in networks with minimal time lags and imprecise durations that are represented by means of interval or fuzzy numbers.

In this paper, a new method is proposed to find the fuzzy optimal solution of FFCP problems. Also, a new representation of triangular fuzzy numbers is proposed. It is shown that it is better to use the proposed representation instead of existing representations of triangular fuzzy numbers, to find the fuzzy optimal solution of FFCP problems. To illustrate the proposed method and to show the advantages of the proposed representation of triangular fuzzy numbers, a numerical example is solved by representing all the parameters as existing and proposed type of triangular fuzzy numbers. The proposed method is very easy to understand and to apply for finding the fuzzy optimal solution of FFCP problems occurring in real life situations.

This paper is organized as follows: In the Section 2, some basic definitions, existing representation of triangular fuzzy numbers, arithmetic operations and ranking function are reviewed. In the Section 3, linear programming formulation of crisp critical path (CCP) problems is reviewed and also the linear programming formulation of FFCP problems is proposed. In the Section 4, a new method is proposed to find the fuzzy optimal solution of FFCP problems. In the Section 5, a new representation of triangular fuzzy number is proposed. Advantages of new representation of triangular fuzzy numbers over existing representation of triangular fuzzy numbers are discussed in the Section 6. To illustrate the proposed method, numerical example is solved in the Section 7. Conclusions are discussed in the Section 8.

2. Preliminaries

In this section some basic definitions, existing representation of triangular fuzzy numbers, arithmetic operations between triangular fuzzy numbers and ranking function are reviewed.

2.1. Basic definitions

In this section, some basic definitions are reviewed.

Definition 1. (Kaufmann and Gupta, 1985). The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X . This function can be generalized to a function $\mu_{\tilde{A}}$ such that the value assigned to the element of the universal set X fall within a specified range i.e., $\mu_{\tilde{A}} : X \to [0,1]$. The assigned value indicate the membership grade of the element in the set A. The function $\mu_{\tilde{A}}$ is called the membership function and the set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)); x \in X\}$ defined by $\mu_{\tilde{A}}(x)$ for each $x \in X$ is called a fuzzy set.

Definition 2. (Kaufmann and Gupta, 1985). A fuzzy set $\tilde{A}$, defined on the universal set of real numbers R, is said to be a fuzzy number if its membership function has the following characteristics:

(i) $\tilde{A}$ is convex i.e., $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda) x_2) \geq \text{minimum}(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)) \forall x_1, x_2 \in R, \forall \lambda \in [0,1]$.

(ii) $\tilde{A}$ is normal i.e., $\exists\ x_0 \in R$ such that $\mu_{\tilde{A}}(x_0) = 1$.

(iii) $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Definition 3. (Dubois and Prade, 1980) A fuzzy number $\tilde{A}$ is called non-negative fuzzy number if $\mu_{\tilde{A}}(x) = 0\ \forall x < 0$.

2.2. Existing Representation of Triangular Fuzzy Numbers

In the literature, triangular fuzzy numbers are represented as follows:

2.2.1. (a, b, c) Representation of Triangular Fuzzy Numbers

Definition 4. (Kaufmann and Gupta, 1985). A fuzzy number $\tilde{A} = (a, b, c)$ is said to a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{for } -\infty < x \le a, \\ \dfrac{x-a}{b-a}, & \text{for } a \le x < b, \\ \dfrac{c-x}{c-b}, & \text{for } b \le x < c, \\ 0, & \text{for } c \le x < \infty. \end{cases}$$

Definition 5. (Kaufmann and Gupta, 1985). A triangular fuzzy number $\tilde{A} = (a, b, c)$ is said to be zero triangular fuzzy number iff $a = 0, b = 0, c = 0$.

Definition 6. (Kaufmann and Gupta, 1985). A triangular fuzzy number $\tilde{A} = (a, b, c)$ is said to be non-negative triangular fuzzy number iff $a \ge 0$.

Definition 7. (Kaufmann and Gupta, 1985). Two triangular fuzzy numbers $\tilde{A} = (a_1, b_1, c_1)$ and $\tilde{B} = (a_2, b_2, c_2)$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ iff $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$.

2.2.2. (m, α, β) Representation of Triangular Fuzzy Numbers

A triangular fuzzy number $\tilde{A} = (a, b, c)$, described in the Section 2.2.1, may also be represented as $\tilde{A} = (m, \alpha, \beta)$, where $m = b, \alpha = b - a \ge 0, \beta = c - b \ge 0$.

Definition 8. (Dubois and Prade, 1980). A fuzzy number $\tilde{A} = (m, \alpha, \beta)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & \text{for } -\infty < x \le m-\alpha, \\ 1 - \dfrac{m-x}{\alpha}, & \text{for } m-\alpha \le x < m, \\ 1 - \dfrac{x-m}{\beta}, & \text{for } m \le x < m+\beta, \\ 0, & \text{for } m+\beta \le x < \infty, \end{cases}$$

where

$\alpha, \beta \ge 0$.

Definition 9. (Dubois and Prade, 1980). A triangular fuzzy number $\tilde{A} = (m, \alpha, \beta)$ is said to be zero triangular fuzzy number iff $m = 0, \alpha = 0, \beta = 0.$

Definition 10. (Dubois and Prade, 1980). A triangular fuzzy number $\tilde{A} = (m, \alpha, \beta)$ is said to be non-negative triangular fuzzy number iff $m - \alpha \ge 0.$

Definition 11. (Dubois and Prade, 1980). Two triangular fuzzy numbers $\tilde{A} = (m_1, \alpha_1, \beta_1)$ and $\tilde{B} = (m_2, \alpha_2, \beta_2)$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ iff $m_1 = m_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2.$

2.3. Arithmetic Operations

In this section addition and multiplication operations between two triangular fuzzy numbers are reviewed.

2.3.1. Arithmetic Operations between (a, b, c) Type Triangular Fuzzy Numbers (Kaufmann and Gupta, 1985)

Let $\tilde{A}_1 = (a_1, b_1, c_1)$ and $\tilde{A}_2 = (a_2, b_2, c_2)$ be two triangular fuzzy numbers, then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$, and

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 = (a', b', c')$, where a' = minimum $(a_1 a_2, a_1 c_2, c_1 a_2, c_1 c_2)$, $b' = b_1 b_2$, c' = maximum $(a_1 a_2, a_1 c_2, c_1 a_2, c_1 c_2)$

2.3.2. Arithmetic operations between (m, α, β) type triangular fuzzy numbers (Dubois and Prade, 1980)

Let $\tilde{A}_1 = (m_1, \alpha_1, \beta_1)$ and $\tilde{A}_2 = (m_2, \alpha_2, \beta_2)$ be two triangular fuzzy numbers, then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)$

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 = (m^{''}, \alpha^{''}, \beta^{''})$, where $m^{''} = m_1 m_2$, $\alpha^{''} = m^{''} -$ minimum(d),
$\beta^{''} =$ maximum$(d) - m^{''}$ and
$d = (m_1 m_2 - m_1\alpha_2 - m_2\alpha_1 + \alpha_1\alpha_2, m_1m_2 + m_1\beta_2 - m_2\alpha_1 - \alpha_1\beta_2, m_1m_2 - m_1\alpha_2 + m_2\beta_1 - \beta_1\alpha_2, m_1m_2 + m_1\beta_2 + m_2\beta_1 + \beta_1\beta_2)$

2.4. Ranking Function

An efficient approach for comparing the fuzzy numbers is by the use of a ranking function (Liou and Wang, 1992) $\Re : F(R) \to R$, where $F(R)$ is a set of fuzzy numbers defined on set of real numbers, which maps each fuzzy number into the real line, where a natural order exists.

Let (a,b,c) be a triangular fuzzy number then $\Re(a,b,c) = \dfrac{a+2b+c}{4}$. Also, let (m,α,β) be a triangular fuzzy number then $\Re(m,\alpha,\beta) = m + \dfrac{\beta-\alpha}{4}$.

3. LP Formulation of CCP and FFCP Problems

The CPM is a network-based method designed to assist in the planning, scheduling and control of the project. Its objective is to construct the time scheduling for the project. Two basic results provided by CPM are the total duration time needed to complete the project and the critical path. One of the efficient approaches for finding the critical paths and total duration time of project networks is LP. The LP formulation assumes that a unit flow enters the project network at the start node and leaves at the finish node.

In this section the LP formulation of CCP problems is reviewed and also the LP formulation of FFCP problems is proposed.

3.1. LP Formulation of CCP Problems (Taha, 2003)

Consider a project network $G = (N, A)$ consisting of a finite set $N = \{1, 2, ..., n\}$ of n nodes (events) and A is the set of activities (i, j). Denote t_{ij} as the time period of activity (i, j). The LP formulation of CCP problems is as follows:

Maximize

$$\sum_{(i,j)\in A} t_{ij}\ x_{ij}$$

subject to

$$\sum_{j:(1,\, j)\, \in A} x_{1j} = 1,$$
$$\sum_{i:(i,\, j) \in A} x_{ij} = \sum_{j:(j,k) \in A} x_{jk}, \quad i \neq 1, k \neq n,$$
$$\sum_{i:(i,n)\, \in A} x_{in} = 1,$$
x_{ij} is a non-negative real number $\forall\, (i,\ j) \in A$,

where x_{ij} is the decision variable denoting the amount of flow in activity $\forall\, (i,\ j) \in A$, t_{ij} is the time duration of activity $(i,\ j)$ and the constraints represent the conservation of flow at each node.

3.2. Proposed LP Formulation of FFCP Problems

There are several real life problems in which a decision maker may be uncertain about the precise values of activity time. Suppose time parameters t_{ij} and x_{ij}, $\forall\, (i,\ j) \in A$ are imprecise and are represented by fuzzy numbers $\tilde{t}_{ij}$ and $\tilde{x}_{ij}$ $\forall\, (i,\ j) \in A$ respectively. Then the FFCP problems may be formulated into the following fuzzy linear programming (FLP) problem:

Maximize

$$\sum_{(i,\ j) \in A} \tilde{t}_{ij} \otimes \tilde{x}_{ij}$$

subject to

$$\sum_{j:(1,\, j)\, \in A} \tilde{x}_{1j} = \tilde{1},$$
$$\sum_{i:(i,\, j) \in A} \tilde{x}_{ij} = \sum_{j:(j,k) \in A} \tilde{x}_{jk}, \quad i \neq 1, k \neq n,$$
$$\sum_{i:(i,n)\, \in A} \tilde{x}_{in} = \tilde{1},$$
$\tilde{x}_{ij}$ is a non-negative real number $\forall\, (i,\ j) \in A.$

Remark 3.1. In this paper, at all places $\sum_{(i,\, j) \in A} \tilde{\lambda}_{ij} \otimes \tilde{A}_{ij}$ and $\sum_{(i,\, j) \in A} \lambda_{ij}\, A_{ij}$ represents the fuzzy and crisp addition respectively.

3.3. Application of Ranking Function for Solving FFCP Problems

The fuzzy optimal solution of FLP problems of FFCP problems will be a fuzzy number $\tilde{x}_{ij}$ which will satisfy the following characteristics:

(i) $\tilde{x}_{ij}$ is a non-negative fuzzy number.

(ii) $\sum_{j:(1,j)\in A}\tilde{x}_{1j} = \tilde{1}, \quad \sum_{i:(i,j)\in A}\tilde{x}_{ij} = \sum_{j:(j,k)\in A}\tilde{x}_{jk}, \quad i \neq 1, k \neq n, \quad \sum_{i:(i,n)\in A}\tilde{x}_{in} = \tilde{1}$

(iii) If there exist any non-negative triangular fuzzy number $\tilde{x}'_{ij}$ such that

$$\sum_{j:(1,j)\in A}\tilde{x}'_{1j} = \tilde{1}, \quad \sum_{i:(i,j)\in A}\tilde{x}'_{ij} = \sum_{j:(j,k)\in A}\tilde{x}'_{jk}, \quad i \neq 1, k \neq n, \quad \sum_{i:(i,n)\in A}\tilde{x}'_{in} = \tilde{1},$$

then

$$\Re\left(\sum_{(i,j)\in A}\tilde{t}_{ij}\otimes\tilde{x}_{ij}\right) \geq \Re\left(\sum_{(i,j)\in A}\tilde{t}_{ij}\otimes\tilde{x}'_{ij}\right).$$

Remark 3.2. Let $\tilde{x}_{ij}$ be a fuzzy optimal solution of FLP problem and there exist one or more fuzzy numbers $\tilde{x}''_{ij}$ such that

(i) $\tilde{x}''_{ij}$ is a non-negative fuzzy number.

(ii) $\sum_{j:(1,j)\in A}\tilde{x}''_{1j} = \tilde{1}, \quad \sum_{i:(i,j)\in A}\tilde{x}''_{ij} = \sum_{j:(j,k)\in A}\tilde{x}''_{jk}, \quad i \neq 1, k \neq n, \quad \sum_{i:(i,n)\in A}\tilde{x}''_{in} = \tilde{1},$

(iii) $\Re\left(\tilde{x}_{ij}\right) = \Re\left(\tilde{x}''_{ij}\right),$

then $\tilde{x}''_{ij}$ is said to be an alternative fuzzy optimal solution of FFCP problem.

4. Proposed Method

To the best of our knowledge, there is no method in the literature to find the fuzzy optimal solution of FFCP problems. In this section, a new method is proposed to find the fuzzy optimal solution of FFCP problems. The steps of the proposed method are as follows:

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Step 1.
Represent all the parameters of FFCP problem by a particular type of triangular fuzzy number and formulate the given problem, as proposed in the Section 3.2.

Step 2.
Convert the fuzzy objective function into the crisp objective function form by using appropriate ranking formula.

Step 3.
Convert all the fuzzy constraints and restrictions into the crisp constraints and restrictions by using the arithmetic operations.

Step 4.
Find the optimal solution of obtained crisp linear programming (CLP) problem by using software (LINGO or LINDO etc.).

Step 5.
Find the fuzzy optimal solution using the crisp optimal solution obtained in Step 4.

Step 6.
Find the fuzzy critical path and the corresponding maximum total completion fuzzy time using the fuzzy optimal solution obtained from Step 5.

4.1. Proposed Method with (a,b,c) Representation of Triangular Fuzzy Numbers

If all the parameters of FFCP problems are represented by (a,b,c) type triangular fuzzy numbers then the steps of the proposed method are as follows:

Step 1.
Suppose all the parameters $\tilde{t}_{ij}$ and $\tilde{x}_{ij}$ are represented by (a,b,c) type triangular fuzzy numbers $(t_{ij}, t'_{ij}, t''_{ij})$ and (x_{ij}, y_{ij}, z_{ij}) respectively then the LP formulation of FFCP problems, proposed in the Section 3.2, may be written as:

Maximize

$$\sum_{(i,\,j)\in A}(t_{ij}, t'_{ij}, t''_{ij}) \otimes (x_{ij}, y_{ij}, z_{ij})$$

subject to

$$\sum_{j:(1,\,j)\in A}(x_{1j}, y_{1j}, z_{1j}) = (1,1,1),$$
$$\sum_{i:(i,\,j)\in A}(x_{ij}, y_{ij}, z_{ij}) = \sum_{j:(j,\,k)\in A}(x_{jk}, y_{jk}, z_{jk}), i \neq 1, k \neq n,$$

$$\sum_{i:(i,n)\in A}(x_{in}, y_{in}, z_{in}) = (1,1,1),$$

(x_{ij}, y_{ij}, z_{ij}) is a non-negative triangular fuzzy number $\forall (i, j) \in A.$

Step 2.
Using ranking formula, presented in the Section 2.4, the LP formulation of FFCP problems may be written as:

Maximize

$$\Re\Big[\sum_{(i,j)\in A}(t_{ij}, t'_{ij}, t''_{ij}) \otimes (x_{ij}, y_{ij}, z_{ij})\Big]$$

subject to

$$\sum_{j:(1,j)\in A}(x_{1j}, y_{1j}, z_{1j}) = (1,1,1),$$
$$\sum_{i:(i,j)\in A}(x_{ij}, y_{ij}, z_{ij}) = \sum_{j:(j,k)\in A}(x_{jk}, y_{jk}, z_{jk}), i \neq 1, k \neq n,$$
$$\sum_{i:(i,n)\in A}(x_{in}, y_{in}, z_{in}) = (1,1,1),$$

(x_{ij}, y_{ij}, z_{ij}) is a non-negative triangular fuzzy number $\forall (i, j) \in A.$

Step 3.
Using the arithmetic operations, described in the Section 2.3.1 and Definitions 6, 7, FLP problem, obtained in Step 2, is converted into the following CLP problem:

Maximize

$$\Re\Big[\sum_{(i,j)\in A}(t_{ij}, t'_{ij}, t''_{ij}) \otimes (x_{ij}, y_{ij}, z_{ij})\Big]$$

subject to

$$\sum_{j:(1,j)\in A} x_{1j} = 1, \quad \sum_{j:(1,j)\in A} y_{1j} = 1, \quad \sum_{j:(1,j)\in A} z_{1j} = 1,$$
$$\sum_{i:(i,j)\in A} x_{ij} = \sum_{j:(j,k)\in A} x_{jk}, i \neq 1, k \neq n,$$
$$\sum_{i:(i,j)\in A} y_{ij} = \sum_{j:(j,k)\in A} y_{jk}, i \neq 1, k \neq n,$$
$$\sum_{i:(i,j)\in A} z_{ij} = \sum_{j:(j,k)\in A} z_{jk}, i \neq 1, k \neq n,$$

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$$\sum_{i:(i,n)\in A} x_{in} = 1, \quad \sum_{i:(i,n)\in A} y_{in} = 1, \quad \sum_{i:(i,n)\in A} z_{in} = 1,$$
$$y_{ij} - x_{ij} \geq 0, \quad z_{ij} - y_{ij} \geq 0,$$
$$x_{ij}, y_{ij}, z_{ij} \geq 0 \ \forall (i, j) \in A.$$

Step 4.
Find the optimal solution x_{ij}, y_{ij}, z_{ij} by solving the CLP problem, obtained in Step 3.

Step 5.
Find the fuzzy optimal solution $\tilde{x}_{ij}$ by putting the values of x_{ij}, y_{ij} and z_{ij} in $\tilde{x}_{ij} = (x_{ij}, y_{ij}, z_{ij})$.

Step 6.
Find the maximum total completion fuzzy time by putting the values of $\tilde{x}_{ij}$ in $\sum_{(i,j)\in A} \tilde{t}_{ij} \otimes \tilde{x}_{ij}$.

Step 7.
Find the fuzzy critical path by combining all the activities (i, j) such that $\tilde{x}_{ij} = (1, 1, 1)$.

4.2. Proposed Method with (m, α, β) Representation of Triangular Fuzzy Numbers

If all the parameters of FFCP problems are represented by (m, α, β) type triangular fuzzy numbers then the steps of the proposed method are as follows:

Step 1.
Suppose all the parameters $\tilde{t}_{ij}$ and $\tilde{x}_{ij}$ are represented by (m, α, β) type triangular fuzzy numbers $(t'_{ij}, \gamma_{ij}, \delta_{ij})$ and $(y_{ij}, \alpha_{ij}, \beta_{ij})$ respectively then the LP formulation of FFCP problems, proposed in the Section 3.2, may be written as:

Maximize

$$\sum_{(i,j)\in A} (t'_{ij}, \gamma_{ij}, \delta_{ij}) \otimes (y_{ij}, \alpha_{ij}, \beta_{ij})$$

subject to

$$\sum_{j:(1,j)\in A} (y_{1j}, \alpha_{1j}, \beta_{1j}) = (1, 0, 0),$$
$$\sum_{i:(i,j)\in A} (y_{ij}, \alpha_{ij}, \beta_{ij}) = \sum_{j:(j,k)\in A} (y_{jk}, \alpha_{jk}, \beta_{jk}), i \neq 1, k \neq n,$$
$$\sum_{i:(i,n)\in A} (y_{in}, \alpha_{in}, \beta_{in}) = (1, 0, 0),$$
$(y_{ij}, \alpha_{ij}, \beta_{ij})$ is a non-negative triangular fuzzy number $\forall (i, j) \in A$.

Step 2.
Using ranking formula, presented in the Section 2.4, the LP formulation of FFCP problems may be written as:

Maximize

$$\Re[\sum_{(i,j)\in A}(t'_{ij},\gamma_{ij},\delta_{ij})\otimes(y_{ij},\alpha_{ij},\beta_{ij})]$$

subject to

$$\sum_{j:(1,j)\in A}(y_{1j},\alpha_{1j},\beta_{1j}) = (1,0,0),$$
$$\sum_{i:(i,j)\in A}(y_{ij},\alpha_{ij},\beta_{ij}) = \sum_{j:(j,k)\in A}(y_{jk},\alpha_{jk},\beta_{jk}), i\neq 1, k\neq n,$$

$$\sum_{i:(i,n)\in A}(y_{in},\alpha_{in},\beta_{in}) = (1,0,0),$$

$(y_{ij},\alpha_{ij},\beta_{ij})$ is a non-negative triangular fuzzy number $\forall (i,j)\in A.$

Step 3.
Using the arithmetic operations, described in the Section 2.3.2 and Definitions 10, 11, the FLP problem, obtained in Step 2, is converted into the following CLP problem:

Maximize

$$\Re[\sum_{(i,j)\in A}(t'_{ij},\gamma_{ij},\delta_{ij})\otimes(y_{ij},\alpha_{ij},\beta_{ij})]$$

subject to

$$\sum_{j:(1,j)\in A}y_{1j} = 1, \quad \sum_{j:(1,j)\in A}\alpha_{1j} = 0, \quad \sum_{j:(1,j)\in A}\beta_{1j} = 0,$$
$$\sum_{i:(i,j)\in A}y_{ij} = \sum_{j:(j,k)\in A}y_{jk}, i\neq 1, k\neq n,$$
$$\sum_{i:(i,j)\in A}\alpha_{ij} = \sum_{j:(j,k)\in A}\alpha_{jk}, i\neq 1, k\neq n,$$
$$\sum_{i:(i,j)\in A}\beta_{ij} = \sum_{j:(j,k)\in A}\beta_{jk}, i\neq 1, k\neq n,$$
$$\sum_{i:(i,n)\in A}\alpha_{in} = 0, \quad \sum_{i:(i,n)\in A}\beta_{in} = 0,$$
$$y_{ij}-\alpha_{ij}\geq 0,$$
$$y_{ij},\alpha_{ij},\beta_{ij}\geq 0 \ \forall (i,j)\in A.$$

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Step 4.
Find the optimal solution $y_{ij}, \alpha_{ij}, \beta_{ij}$ by solving the CLP problem, obtained in Step 3.

Step 5.
Find the fuzzy optimal solution $\tilde{y}_{ij}$ by putting the values of y_{ij}, α_{ij} and β_{ij} in $\tilde{y}_{ij} = (y_{ij}, \alpha_{ij}, \beta_{ij})$.

Step 6.
Find the maximum total completion fuzzy time by putting the values of $\tilde{y}_{ij}$ in $\sum_{(i,\ j) \in A} \tilde{t}_{ij} \otimes \tilde{y}_{ij}$.

Step 7.
Find the fuzzy critical path by combining all the activities $(i,\ j)$ such that $\tilde{y}_{ij} = (1, 0, 0)$.

5. *JMD* Representation of Triangular Fuzzy Numbers

In this section, a new representation of triangular fuzzy numbers, named as *JMD* representation of triangular fuzzy numbers, is proposed. It is shown that if all the parameters are represented by *JMD* representation instead of existing representation of triangular fuzzy numbers, and proposed method is applied to find the fuzzy optimal solution of FFCP problems then the fuzzy optimal solution is same but the total number of constraints, in converted CLP problem, is less than the number of constraints, obtained by using the existing representation of triangular fuzzy numbers.

Definition 12. Let (a,b,c) be a triangular fuzzy number then its *JMD* representation is $(x, \alpha, \beta)_{JMD}$, where $x = a, \alpha = b - a \geq 0$ and $\beta = c - b \geq 0$.

Definition 13. Let (m, α, β) be a triangular fuzzy number then its *JMD* representation is $(x, \alpha, \beta)_{JMD}$, where $x = m - \alpha$.

Definition 14. A triangular fuzzy number $\tilde{A} = (x, \alpha, \beta)_{JMD}$ is said to be zero triangular fuzzy number iff $x = 0, \alpha = 0, \beta = 0$.

Definition 15. A triangular fuzzy number $\tilde{A} = (x, \alpha, \beta)_{JMD}$ is said to be non-negative triangular fuzzy number iff $x \geq 0$.

Definition16. Two triangular fuzzy numbers $\tilde{A} = (x_1, \alpha_1, \beta_1)_{JMD}$ and $\tilde{B} = (x_2, \alpha_2, \beta_2)_{JMD}$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ iff $x_1 = x_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2$.

5.1. Arithmetic Operations between *JMD* Type Triangular Fuzzy Numbers

Let $(a_1, b_1, c_1), (a_2, b_2, c_2)$ be two triangular fuzzy numbers and $(x_1, \alpha_1, \beta_1)_{JMD}, (x_2, \alpha_2, \beta_2)_{JMD}$ be their *JMD* representation then the addition and multiplication operations, presented in the Section 2.3.1, are converted into the following arithmetic operations:

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (x_1 + x_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{JMD}$.

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 = (x_3, \alpha_3, \beta_3)_{JMD}$, where x_3 = minimum (d_1), $\alpha_3 = d_2 - x_3$, $\beta_3 =$ maximum $(d_1) - d_2$, $d_1 = (x_1x_2, x_1x_2 + x_1\alpha_2 + x_1\beta_2, x_1x_2 + x_2\alpha_1 + x_2\beta_1, x_1x_2 + x_1\alpha_2 + x_1\beta_2 + x_2\alpha_1 + \alpha_1\alpha_2 + \alpha_1\beta_2 + x_2\beta_1 + \alpha_2\beta_1 + \beta_1\beta_2)$, $d_2 = x_1x_2 + x_1\alpha_2 + x_2\alpha_1 + \alpha_1\alpha_2$.

5.2. Ranking Formula for *JMD* Triangular Fuzzy Numbers

The ranking formula, presented in the Section 2.4, is converted into the following ranking formula:

Let $(x, \alpha, \beta)_{JMD}$ be a triangular fuzzy number then $\Re(x, \alpha, \beta)_{JMD} = \dfrac{4(x) + 3(\alpha) + \beta}{4}$.

5.3. Proposed Method with $(x, \alpha, \beta)_{JMD}$ Representation of Triangular Fuzzy Numbers

If all the parameters of FFCP problems are represented by $(x, \alpha, \beta)_{JMD}$ type triangular fuzzy numbers then the steps of the proposed method are as follows:

Step 1.
Suppose all the parameters $\tilde{t}_{ij}$ and $\tilde{x}_{ij}$ are represented by triangular fuzzy numbers $(t_{ij}, \gamma_{ij}, \delta_{ij})_{JMD}$ and $(x_{ij}, \alpha_{ij}, \beta_{ij})_{JMD}$ respectively then the LP formulation of FFCP problems, proposed in the Section 3.2, may be written as:

Maximize

$$\sum_{(i,\, j) \in A} (t_{ij}, \gamma_{ij}, \delta_{ij})_{JMD} \otimes (x_{ij}, \alpha_{ij}, \beta_{ij})_{JMD}$$

subject to

$$\sum_{j:(1,\,j)\in A}(x_{1j},\alpha_{1j},\beta_{1j})_{JMD} = (1,0,0)_{JMD},$$
$$\sum_{i:(i,\,j)\in A}(x_{ij},\alpha_{ij},\beta_{ij})_{JMD} = \sum_{j:(j,k)\in A}(x_{jk},\alpha_{jk},\beta_{jk})_{JMD}, i\neq 1, k\neq n,$$
$$\sum_{i:(i,n)\in A}(x_{in},\alpha_{in},\beta_{in})_{JMD} = (1,0,0)_{JMD}, \text{ and}$$
$(x_{ij},\alpha_{ij},\beta_{ij})_{JMD}$ is a non-negative triangular fuzzy number $\forall\,(i,\,j)\in A.$

Step 2.
Using ranking formula, presented in the Section 5.2, the LP formulation of FFCP problems may be written as:

Maximize

$$\Re[\sum_{(i,\,j)\in A}(t_{ij},\gamma_{ij},\delta_{ij})_{JMD}\otimes(x_{ij},\alpha_{ij},\beta_{ij})_{JMD}]$$

subject to

$$\sum_{j:(1,\,j)\in A}(x_{1j},\alpha_{1j},\beta_{1j})_{JMD} = (1,0,0)_{JMD},$$
$$\sum_{i:(i,\,j)\in A}(x_{ij},\alpha_{ij},\beta_{ij})_{JMD} = \sum_{j:(j,k)\in A}(x_{jk},\alpha_{jk},\beta_{jk})_{JMD}, i\neq 1, k\neq n,$$
$$\sum_{i:(i,n)\in A}(x_{in},\alpha_{in},\beta_{in})_{JMD} = (1,0,0)_{JMD}, \text{ and}$$
$(x_{ij},\alpha_{ij},\beta_{ij})_{JMD}$ is a non-negative triangular fuzzy number $\forall\,(i,\,j)\in A.$

Step 3.
Using the arithmetic operations, described in the Section 5.1 and Definitions 15, 16, FLP problem, obtained in Step 2, is converted into the following CLP problem:

Maximize

$$\Re[\sum_{(i,\,j)\in A}(t_{ij},\gamma_{ij},\delta_{ij})_{JMD}\otimes(x_{ij},\alpha_{ij},\beta_{ij})_{JMD}]$$

subject to

$$\sum_{j:(1,\,j)\in A}x_{1j} = 1,\quad \sum_{j:(1,\,j)\in A}\alpha_{1j} = 0,\quad \sum_{j:(1,\,j)\in A}\beta_{1j} = 0,$$
$$\sum_{i:(i,\,j)\in A}x_{ij} = \sum_{j:(j,k)\in A}x_{jk}, i\neq 1, k\neq n,$$
$$\sum_{i:(i,\,j)\in A}\alpha_{ij} = \sum_{j:(j,k)\in A}\alpha_{jk}, i\neq 1, k\neq n,$$

$$\sum_{i:(i,\,j)\in A}\beta_{ij} = \sum_{j:(j,k)\in A}\beta_{jk},\ i\neq 1, k\neq n,$$
$$\sum_{i:(i,n)\in A}x_{in} = 1,\quad \sum_{i:(i,n)\in A}\alpha_{in} = 0,\quad \sum_{i:(i,n)\in A}\beta_{in} = 0,$$
$$x_{ij},\alpha_{ij},\beta_{ij}\geq 0\ \ \forall\,(i,\,j)\in A.$$

Step 4.
Find the optimal solution $x_{ij},\alpha_{ij},\beta_{ij}$ by solving the CLP problem, obtained in Step 3.

Step 5.
Find the fuzzy optimal solution $\tilde{x}_{ij}$ by putting the values of x_{ij},α_{ij} and β_{ij} in $\tilde{x}_{ij} = (x_{ij},\alpha_{ij},\beta_{ij})_{JMD}$.

Step 6.
Find the maximum total completion fuzzy time by putting the values of $\tilde{x}_{ij}$ in $\sum_{(i,\,j)\in A}\tilde{t}_{ij}\otimes\tilde{x}_{ij}$.

Step 7.
Find the fuzzy critical path by combining all the activities $(i,\,j)$ such that $\tilde{x}_{ij} = (1, 0, 0)$.

Remark 5.1. The proposed *JMD* representation of triangular fuzzy number may be called as "JAI MATA DI" or "JAI MEHAR DI". Mehar is a lovely daughter of Parmpreet Kaur.

6. Advantages of *JMD* Representation over the Existing Representation of Triangular Fuzzy Numbers

In this section, it is shown that it is better to use the *JMD* representation of triangular fuzzy numbers, instead of existing representation of triangular fuzzy numbers, for finding the fuzzy optimal solution of FFCP problems.

It is obvious from the Section 4.1, 4.2 and 5.3 that

(i) If all the parameters of FFCP problems are represented by (a,b,c) type triangular fuzzy numbers and FLP problem is converted into the corresponding CLP problem by using the proposed method, presented in the Section 4.1, then number of constraints in CLP problem = 3 × number of constraints in FLP problem + 2 × number of fuzzy variables.

(ii) If all the parameters of FFCP problems are represented by (m,α,β) type triangular fuzzy numbers and FLP problem is converted into the corresponding CLP problem by using the proposed method then number of constraints in CLP problem = 3 × number of constraints in FLP problem + 1 × number of fuzzy variables.

(iii) If all the parameters of FFCP problems are represented by $(x, \alpha, \beta)_{JMD}$ type triangular fuzzy numbers and FLP problem is converted into the corresponding CLP problem by using the proposed method then number of constraints in CLP problem = 3 $\times$ number of constraints in FLP problem.

On the basis of above results it can be concluded that if all the parameters are represented by existing type triangular fuzzy numbers then total number of constraints, in the obtained CLP problem, will be more than as compared to number of constraints in CLP problem, obtained by representing all the parameters as *JMD* type triangular fuzzy numbers. So it is better to use *JMD* representation of triangular fuzzy numbers for finding the fuzzy optimal solution of FFCP problems as compared to the existing representation of triangular fuzzy numbers.

7. Numerical Example

To show the advantages of *JMD* representation over existing representation of fuzzy numbers, the same numerical example is solved by using all three representations of fuzzy numbers. The problem is to find the fuzzy critical path and maximum total completion fuzzy time of the project network, shown in Figure 1, in which the fuzzy time duration of each activity is represented by the following (a, b, c) type triangular fuzzy numbers

$\tilde{t}_{12} = (3, 4, 5)$, $\tilde{t}_{23} = (2.8, 3, 3.2)$, $\tilde{t}_{24} = (4, 5, 6)$, $\tilde{t}_{34} = (1.8, 2, 2.2)$

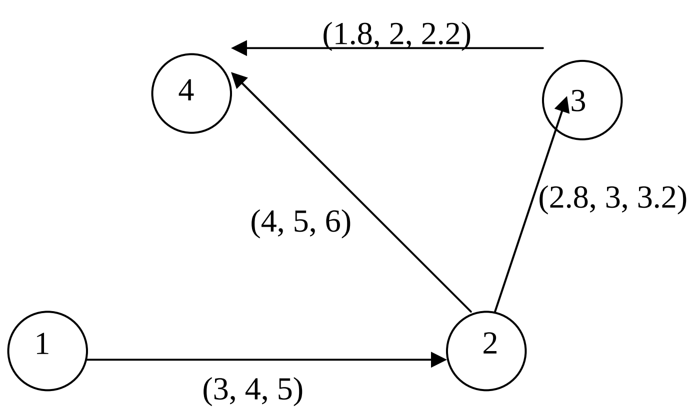

Figure 1. Project network with fuzzy time duration of each activity as (a, b, c) type triangular fuzzy numbers

7.1. Fuzzy Optimal Solution Using (a, b, c) Representation of Triangular Fuzzy Numbers

Step 1.
Using the Section 4.1, the given problem may be formulated as follows:

Maximize$((3,4,5)\otimes(x_{12},y_{12},z_{12})\oplus(2.8,3,3.2)\otimes(x_{23},y_{23},z_{23})\oplus(4,5,6)\otimes$
$(x_{24},y_{24},z_{24})\oplus(1.8,2,2.2)\otimes(x_{34},y_{34},z_{34}))$

subject to $(x_{12},y_{12},z_{12})=(1,1,1),\ (x_{23},y_{23},z_{23})\oplus(x_{24},y_{24},z_{24})=(x_{12},y_{12},z_{12}),$
$(x_{34},y_{34},z_{34})=(x_{23},y_{23},z_{23}),\ (x_{24},y_{24},z_{24})\oplus(x_{34},y_{34},z_{34})=(1,1,1)$
$(x_{12},y_{12},z_{12}),(x_{23},y_{23},z_{23}),(x_{24},y_{24},z_{24}),(x_{34},y_{34},z_{34})$ are non-negative triangular fuzzy numbers.

Step 2.
Using ranking formula the FLP problem, formulated in Step 1, may be written as

Maximize $\Re[(3,4,5)\otimes(x_{12},y_{12},z_{12})\oplus(2.8,3,3.2)\otimes(x_{23},y_{23},z_{23})\oplus(4,5,6)\otimes$
$(x_{24},y_{24},z_{24})\oplus(1.8,2,2.2)\otimes(x_{34},y_{34},z_{34})]$

subject to $(x_{12},y_{12},z_{12})=(1,1,1),\ (x_{23},y_{23},z_{23})\oplus(x_{24},y_{24},z_{24})=(x_{12},y_{12},z_{12}),$
$(x_{34},y_{34},z_{34})=(x_{23},y_{23},z_{23}),\ (x_{24},y_{24},z_{24})\oplus(x_{34},y_{34},z_{34})=(1,1,1)$
$(x_{12},y_{12},z_{12}),(x_{23},y_{23},z_{23}),(x_{24},y_{24},z_{24}),(x_{34},y_{34},z_{34})$ are non-negative triangular fuzzy numbers.

Step 3.
Using the arithmetic operations, described in the Section 2.3.1, the FLP problem, obtained in Step 2, is converted into the following CLP problem:

Maximize $(0.75\,x_{12} + 2\,y_{12} + 1.25\,z_{12} + 0.7\,x_{23} + 1.5\,y_{23} + 0.8\,z_{23} + x_{24} + 2.5\,y_{24} +$
$1.5\,z_{24} + 0.45\,x_{34} + y_{34} + 0.55\,z_{34})$

subject to $x_{12}=1,\ y_{12}=1,\ z_{12}=1,\ x_{23}+x_{24}=x_{12}, y_{23}+y_{24}=y_{12}, z_{23}+z_{24}=z_{12},$
$x_{23}=x_{34}, y_{23}=y_{34}, z_{23}=z_{34}, x_{24}+x_{34}=1,\ y_{24}+y_{34}=1,\ z_{24}+z_{34}=1,$
$y_{12}-x_{12}\ge 0,\ z_{12}-y_{12}\ge 0,\ y_{23}-x_{23}\ge 0,\ z_{23}-y_{23}\ge 0,\ y_{24}-x_{24}\ge 0,$
$z_{24}-y_{24}\ge 0,\ y_{34}-x_{34}\ge 0,\ z_{34}-y_{34}\ge 0,$
$x_{12}, y_{12}, z_{12}, x_{23}, y_{23}, z_{23}, x_{24}, y_{24}, z_{24}, x_{34}, y_{34}, z_{34} \ge 0.$

Step 4.
On solving CLP problem, obtained in Step 3, an optimal solution is
$x_{12}=x_{23}=x_{34}=y_{12}=y_{23}=y_{34}=z_{12}=z_{23}=z_{34}=1$ and $x_{24}=y_{24}=z_{24}=0$.

In this problem an alternative optimal solution is also obtained and that alternative optimal solution is $x_{12}=x_{24}=y_{12}=y_{24}=z_{12}=z_{24}=1$ and $x_{34}=y_{34}=z_{34}=x_{23}=y_{23}=z_{23}=0$.

Step 5.
Putting the values of x_{ij}, y_{ij} and z_{ij} in $\tilde{x}_{ij}$ = (x_{ij}, y_{ij}, z_{ij}), the fuzzy optimal solution is $\tilde{x}_{12}$ = (1, 1, 1), $\tilde{x}_{23}$ = (1, 1, 1), $\tilde{x}_{34}$ = (1, 1, 1), $\tilde{x}_{24}$ = (0, 0, 0) and the alternative fuzzy optimal solution is $\tilde{x}_{12}$ = (1, 1, 1), $\tilde{x}_{23}$ = (0, 0, 0), $\tilde{x}_{34}$ = (0, 0, 0), $\tilde{x}_{24}$ = (1, 1, 1).

Step 6.
Using the fuzzy optimal solution, the fuzzy critical path is $1 \to 2 \to 3 \to 4$.

Putting the values of x_{ij}, y_{ij} and z_{ij} in $(3, 4, 5) \otimes (x_{12}, y_{12}, z_{12}) \oplus (2.8, 3, 3.2) \otimes (x_{23}, y_{23}, z_{23}) \oplus (4, 5, 6) \otimes (x_{24}, y_{24}, z_{24}) \oplus (1.8, 2, 2.2) \otimes (x_{34}, y_{34}, z_{34})$, the maximum total completion fuzzy time is (7.6, 9, 10.4).

Using the alternative fuzzy optimal solution, the fuzzy critical path is $1 \to 2 \to 4$ and the maximum total completion fuzzy time is (7, 9, 11).

Hence, in this problem, the fuzzy critical paths are $1 \to 2 \to 3 \to 4$ and $1 \to 2 \to 4$ and the corresponding maximum total completion fuzzy time are (7.6, 9, 10.4) and (7, 9, 11) respectively.

7.2. Fuzzy Optimal Solution Using (m, α, β) Representation of Triangular Fuzzy Numbers

Using the Section 2.2.2, the (m, α, β) representation of $\tilde{t}_{12}$ = (3, 4, 5), $\tilde{t}_{23}$ = (2.8, 3, 3.2), $\tilde{t}_{24}$ = (4, 5, 6), $\tilde{t}_{34}$ = (1.8, 2, 2.2) are $\tilde{t}_{12}$ = (4, 1, 1), $\tilde{t}_{23}$ = (3, .2, .2), $\tilde{t}_{24}$ = (5, 1, 1), $\tilde{t}_{34}$ = (2, .2, .2) respectively. The network, shown in Figure 1, with (m, α, β) representation of fuzzy time duration of each activity is shown in Figure 2.

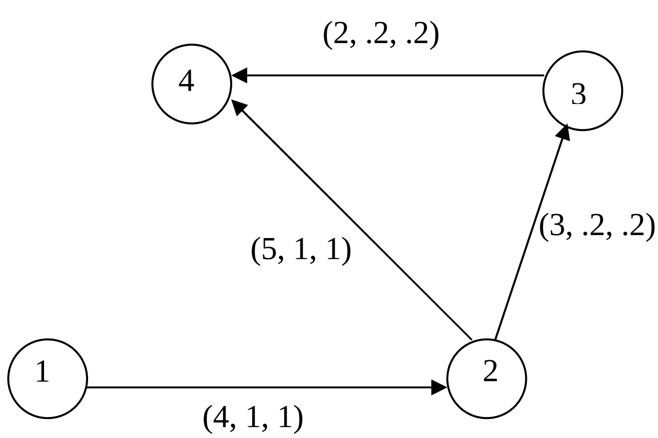

Figure 2. Project network with fuzzy time duration of each activity as (m, α, β) type triangular fuzzy numbers

Step 1.
Using the Section 4.2, the given problem may be formulated as follows:

$$\text{Maximize } ((4,1,1)\otimes(y_{12},\alpha_{12},\beta_{12})\oplus(3,.2,.2)\otimes(y_{23},\alpha_{23},\beta_{23})\oplus(5,1,1)\otimes (y_{24},\alpha_{24},\beta_{24})\oplus(2,.2,.2)\otimes(y_{34},\alpha_{34},\beta_{34}))$$

subject to $(y_{12},\alpha_{12},\beta_{12})=(1,0,0)$, $(y_{23},\alpha_{23},\beta_{23})\oplus(y_{24},\alpha_{24},\beta_{24})=(y_{12},\alpha_{12},\beta_{12})$, $(y_{34},\alpha_{34},\beta_{34})=(y_{23},\alpha_{23},\beta_{23})$, $(y_{24},\alpha_{24},\beta_{24})\oplus(y_{34},\alpha_{34},\beta_{34})=(1,0,0)$
$(y_{12},\alpha_{12},\beta_{12}),(y_{23},\alpha_{23},\beta_{23}),(y_{24},\alpha_{24},\beta_{24}),(y_{34},\alpha_{34},\beta_{34})$ are non-negative triangular fuzzy numbers.

Step 2.
Using ranking formula the FLP problem, formulated in Step 1, may be written as

$$\text{Maximize } \Re[(4,1,1)\otimes(y_{12},\alpha_{12},\beta_{12})\oplus(3,.2,.2)\otimes(y_{23},\alpha_{23},\beta_{23})\oplus(5,1,1)\otimes (y_{24},\alpha_{24},\beta_{24})\oplus(2,.2,.2)\otimes(y_{34},\alpha_{34},\beta_{34})]$$

subject to $(y_{12},\alpha_{12},\beta_{12})=(1,0,0)$, $(y_{23},\alpha_{23},\beta_{23})\oplus(y_{24},\alpha_{24},\beta_{24})=(y_{12},\alpha_{12},\beta_{12})$, $(y_{34},\alpha_{34},\beta_{34})=(y_{23},\alpha_{23},\beta_{23})$, $(y_{24},\alpha_{24},\beta_{24})\oplus(y_{34},\alpha_{34},\beta_{34})=(1,0,0)$
$(y_{12},\alpha_{12},\beta_{12}),(y_{23},\alpha_{23},\beta_{23}),(y_{24},\alpha_{24},\beta_{24}),(y_{34},\alpha_{34},\beta_{34})$ are non-negative triangular fuzzy numbers.

Step 3.
Using the arithmetic operations, described in the Section 2.3.2, the FLP problem, obtained in Step 2, is converted into the following CLP problem:

$$\text{Maximize}(4y_{12}-0.75\alpha_{12}+1.25\beta_{12}+3y_{23}-0.7\alpha_{23}+0.8\beta_{23}+2y_{34}-0.45\alpha_{34}+0.55\beta_{34} +5y_{24}-\alpha_{24}+1.5\beta_{24})$$

subject to

$$y_{12}=1,\ \alpha_{12}=0,\ \beta_{12}=0,\ y_{23}+y_{24}=y_{12},\ \alpha_{23}+\alpha_{24}=\alpha_{12},\ \beta_{23}+\beta_{24}=\beta_{12},$$
$$y_{23}=y_{34},\ \alpha_{23}=\alpha_{34},\ \beta_{23}=\beta_{34},\ y_{24}+y_{34}=1,\ \alpha_{24}+\alpha_{34}=0,\ \beta_{24}+\beta_{34}=0,$$
$$y_{12}-\alpha_{12}\ge 0,\ y_{23}-\alpha_{23}\ge 0,\ y_{24}-\alpha_{24}\ge 0,\ y_{34}-\alpha_{34}\ge 0,$$
$$y_{12},\alpha_{12},\beta_{12},y_{23},\alpha_{23},\beta_{23},y_{24},\alpha_{24},\beta_{24},y_{34},\alpha_{34},\beta_{34}\ge 0,$$

Step 4.
On solving CLP problem, obtained in Step 3, an optimal solution is $y_{12}=y_{23}=y_{34}=1$ and $y_{24}=\alpha_{12}=\alpha_{23}=\alpha_{24}=\alpha_{34}=\beta_{12}=\beta_{23}=\beta_{24}=\beta_{34}=0.$

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In this problem an alternative optimal solution is also obtained and the alternative optimal solution is $y_{12} = y_{24} = 1$ and $y_{23} = y_{34} = \alpha_{12} = \alpha_{23} = \alpha_{24} = \alpha_{34} = \beta_{12} = \beta_{23} = \beta_{24} = \beta_{34} = 0$.

Step 5.
Putting the values of y_{ij}, α_{ij} and β_{ij} in $\tilde{y}_{ij}$ = $(y_{ij}, \alpha_{ij}, \beta_{ij})$, the fuzzy optimal solution is $\tilde{y}_{12}$ = (1, 0, 0), $\tilde{y}_{23}$ = (1, 0, 0), $\tilde{y}_{34}$ = (1, 0, 0), $\tilde{y}_{24}$ = (0, 0, 0) and the alternative fuzzy optimal solution is $\tilde{y}_{12}$ = (1, 0, 0), $\tilde{y}_{23}$ = (0, 0, 0), $\tilde{y}_{34}$ = (0, 0, 0), $\tilde{y}_{24}$ = (1, 0, 0).

Step 6.
Using the fuzzy optimal solution, the fuzzy critical path is $1 \to 2 \to 3 \to 4$.

Putting the values of $(y_{ij}, \alpha_{ij}, \beta_{ij})$ in $(4,1,1) \otimes (y_{12}, \alpha_{12}, \beta_{12}) \oplus (3,.2,.2) \otimes (y_{23}, \alpha_{23}, \beta_{23}) \oplus (5,1,1) \otimes (y_{24}, \alpha_{24}, \beta_{24}) \oplus (2,.2,.2) \otimes (y_{34}, \alpha_{34}, \beta_{34})$, the maximum total completion fuzzy time is (9, 1.4, 1.4).

Using the alternative fuzzy optimal solution, the fuzzy critical path is $1 \to 2 \to 4$ and the maximum total completion fuzzy time is (9, 2, 2).

Hence, in this problem, the fuzzy critical paths are $1 \to 2 \to 3 \to 4$ and $1 \to 2 \to 4$ and the corresponding maximum total completion fuzzy time are (9, 1.4, 1.4) and (9, 2, 2) respectively.

7.3. Fuzzy Optimal Solution Using *JMD* Representation of Triangular Fuzzy Numbers

Using Definition 12, the $(x, \alpha, \beta)_{JMD}$ representation of $\tilde{t}_{12}$ = (3, 4, 5), $\tilde{t}_{23}$ = (2.8, 3, 3.2), $\tilde{t}_{24}$ = (4, 5, 6), $\tilde{t}_{34}$ = (1.8, 2, 2.2) are $\tilde{t}_{12} = (3,1,1)_{JMD}$, $\tilde{t}_{23} = (2.8,.2,.2)_{JMD}$, $\tilde{t}_{24} = (4,1,1)_{JMD}$, $\tilde{t}_{34} = (1.8,.2,.2)_{JMD}$, respectively. The network, shown in Figure 1 with *JM*D representation of fuzzy time duration of each activity is shown in Figure 3.

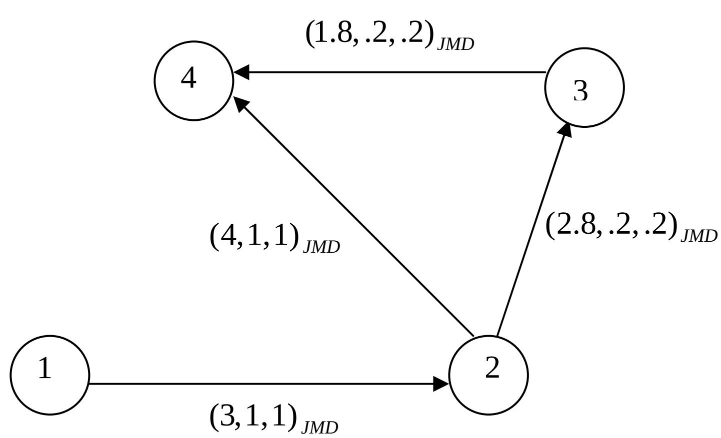

Figure 3. Project network with fuzzy time duration of each activity as $(x, \alpha, \beta)_{JMD}$ type triangular fuzzy numbers

Step 1.
Using the Section 5.3, the given problem may be formulated as follows:

$$\text{Maximize } ((3,1,1)_{JMD} \otimes (x_{12},\alpha_{12},\beta_{12})_{JMD} \oplus (2.8,.2,.2)_{JMD} \otimes (x_{23},\alpha_{23},\beta_{23})_{JMD} \oplus (4,1,1)_{JMD} \otimes (x_{24},\alpha_{24},\beta_{24})_{JMD} \oplus (1.8,.2,.2)_{JMD} \otimes (x_{34},\alpha_{34},\beta_{34})_{JMD})$$

subject to

$$(x_{12},\alpha_{12},\beta_{12})_{JMD} = (1,0,0)_{JMD},$$
$$(x_{23},\alpha_{23},\beta_{23})_{JMD} \oplus (x_{24},\alpha_{24},\beta_{24})_{JMD} = (x_{12},\alpha_{12},\beta_{12})_{JMD},$$
$$(x_{34},\alpha_{34},\beta_{34})_{JMD} = (x_{23},\alpha_{23},\beta_{23})_{JMD},$$
$$(x_{24},\alpha_{24},\beta_{24})_{JMD} \oplus (x_{34},\alpha_{34},\beta_{34})_{JMD} = (1,0,0)_{JMD}$$

$(x_{12},\alpha_{12},\beta_{12})_{JMD}, (x_{23},\alpha_{23},\beta_{23})_{JMD}, (x_{24},\alpha_{24},\beta_{24})_{JMD}, (x_{34},\alpha_{34},\beta_{34})_{JMD}$ are non-negative triangular fuzzy numbers.

Step 2.
Using ranking formula the FLP problem, formulated in Step 1, may be written as

$$\text{Maximize } \Re[(3,1,1)_{JMD} \otimes (x_{12},\alpha_{12},\beta_{12})_{JMD} \oplus (2.8,.2,.2)_{JMD} \otimes (x_{23},\alpha_{23},\beta_{23})_{JMD} \oplus (4,1,1)_{JMD} \otimes (x_{24},\alpha_{24},\beta_{24})_{JMD} \oplus (1.8,.2,.2)_{JMD} \otimes (x_{34},\alpha_{34},\beta_{34})_{JMD}]$$

subject to

$$(x_{12},\alpha_{12},\beta_{12})_{JMD} = (1,0,0)_{JMD},$$
$$(x_{23},\alpha_{23},\beta_{23})_{JMD} \oplus (x_{24},\alpha_{24},\beta_{24})_{JMD} = (x_{12},\alpha_{12},\beta_{12})_{JMD},$$
$$(x_{34},\alpha_{34},\beta_{34})_{JMD} = (x_{23},\alpha_{23},\beta_{23})_{JMD},$$
$$(x_{24},\alpha_{24},\beta_{24})_{JMD} \oplus (x_{34},\alpha_{34},\beta_{34})_{JMD} = (1,0,0)_{JMD}$$

$(x_{12},\alpha_{12},\beta_{12})_{JMD}, (x_{23},\alpha_{23},\beta_{23})_{JMD}, (x_{24},\alpha_{24},\beta_{24})_{JMD}, (x_{34},\alpha_{34},\beta_{34})_{JMD}$ are non-negative triangular fuzzy numbers.

Step 3.
Using the arithmetic operations, described in the Section 5.1, the FLP problem, obtained in Step 2, is converted into the following CLP problem:

$$\text{Maximize } (4x_{12} + 3.25\alpha_{12} + 1.25\beta_{12} + 3x_{23} + 2.3\alpha_{23} + 0.8\beta_{23} + 2x_{34} + 1.55\alpha_{34} + 0.55\beta_{34} + 5x_{24} + 4\alpha_{24} + 1.5\beta_{24})$$

subject to

$$x_{12} = 1,\ \ \alpha_{12} = 0,\ \ \beta_{12} = 0,\ x_{23} + x_{24} = x_{12},\ \alpha_{23} + \alpha_{24} = \alpha_{12},\ \beta_{23} + \beta_{24} = \beta_{12},$$

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$$x_{23} = x_{34},\ \alpha_{23} = \alpha_{34},\ \beta_{23} = \beta_{34},\ x_{24} + x_{34} = 1,\ \alpha_{24} + \alpha_{34} = 0,\ \beta_{24} + \beta_{34} = 0,$$
$$x_{12}, \alpha_{12}, \beta_{12}, x_{23}, \alpha_{23}, \beta_{23}, x_{24}, \alpha_{24}, \beta_{24}, x_{34}, \alpha_{34}, \beta_{34} \geq 0.$$

Step 4.
On solving CLP problem, obtained in Step 3, an optimal solution is $x_{12} = x_{23} = x_{34} = 1$ and $x_{24} = \alpha_{12} = \alpha_{23} = \alpha_{24} = \alpha_{34} = \beta_{12} = \beta_{23} = \beta_{24} = \beta_{34} = 0.$

In this problem an alternative optimal solution is also obtained and the alternative optimal solution is $x_{12} = x_{24} = 1$ and $x_{23} = x_{34} = \alpha_{12} = \alpha_{23} = \alpha_{24} = \alpha_{34} = \beta_{12} = \beta_{23} = \beta_{24} = \beta_{34} = 0.$

Step 5.
Putting the values of x_{ij}, α_{ij} and β_{ij} in $\tilde{x}_{ij} = (x_{ij}, \alpha_{ij}, \beta_{ij})_{JMD}$, the fuzzy optimal solution is $\tilde{x}_{12} = (1, 0, 0)_{JMD}$, $\tilde{x}_{23} = (1, 0, 0)_{JMD}$, $\tilde{x}_{34} = (1, 0, 0)_{JMD}$, $\tilde{x}_{34} = (0, 0, 0)_{JMD}$ and the alternative fuzzy optimal solution is $\tilde{x}_{12} = (1, 0, 0)_{JMD}$, $\tilde{x}_{23} = (0, 0, 0)_{JMD}$, $\tilde{x}_{34} = (0, 0, 0)_{JMD}$, $\tilde{x}_{24} = (1, 0, 0)_{JMD}$.

Step 6.
Using the fuzzy optimal solution, the fuzzy critical path is $1 \to 2 \to 3 \to 4$.

Putting the values of $(x_{ij}, \alpha_{ij}, \beta_{ij})_{JMD}$ in $(3,1,1)_{JMD} \otimes (x_{12}, \alpha_{12}, \beta_{12})_{JMD} \oplus (2.8,.2,.2)_{JMD} \otimes (x_{23}, \alpha_{23}, \beta_{23})_{JMD} \oplus (4,1,1)_{JMD} \otimes (x_{24}, \alpha_{24}, \beta_{24})_{JMD} \oplus (1.8,.2,.2)_{JMD} \otimes (x_{34}, \alpha_{34}, \beta_{34})_{JMD}$ and the maximum total completion fuzzy time is $(7.6, 1.4, 1.4)_{JMD}$.
Using the alternative fuzzy optimal solution, the fuzzy critical path is $1 \to 2 \to 4$ and the maximum total completion fuzzy time is $(7, 2, 2)_{JMD}$.

Hence, in this problem, the fuzzy critical paths are $1 \to 2 \to 3 \to 4$ and $1 \to 2 \to 4$ and the corresponding maximum total completion fuzzy time are $(7.6, 1.4, 1.4)_{JMD}$ and $(7, 2, 2)_{JMD}$, respectively.

7.4. Results and Discussion

The results of the numerical example, obtained from the Section 7.1, 7.2 and 7.3, are shown in Table 1. On the basis of these results it can be easily seen that if all the parameters are represented by *JMD* representation of triangular fuzzy numbers, instead of existing representation of triangular fuzzy numbers, and the proposed method is applied to find the fuzzy optimal solution of FFCP problems then the fuzzy optimal solution is the same but with fewer constraints, in converted CLP problem, than those, obtained using the existing representation of triangular fuzzy numbers. Hence, it is better to use *JMD* representation instead of existing representation of triangular fuzzy numbers to find the fuzzy optimal solution of FFCP problems.

Table 1. Results using existing and proposed representation of triangular fuzzy numbers

Type of triangular fuzzy numbers	Number of constraints in FLP problem	Number of constraints in CLP problem	Fuzzy critical path	Maximum total completion fuzzy time
(a,b,c)	8	$(3\times 8)+(2\times 4)$ = 24 + 8 = 32	$1\rightarrow 2\rightarrow 3\rightarrow 4$ and $1\rightarrow 2\rightarrow 4$	(7.6, 9, 10.4) and (7, 9, 11)
(m,α,β)	8	$(3\times 8)+(1\times 4)$ = 24 + 4 = 28	$1\rightarrow 2\rightarrow 3\rightarrow 4$ and $1\rightarrow 2\rightarrow 4$	(9, 1.4, 1.4) and (9, 2, 2)
$(x,\alpha,\beta)_{JMD}$	8	(3×8) = 24	$1\rightarrow 2\rightarrow 3\rightarrow 4$ and $1\rightarrow 2\rightarrow 4$	$(7.6, 1.4, 1.4)_{JMD}$ and $(7, 2, 2)_{JMD}$

8. Conclusions

A new method is proposed to find the fuzzy optimal solution of fully fuzzy critical path problems. Also a new representation of triangular fuzzy numbers is proposed and it is shown that it is better to use the proposed representation of triangular fuzzy numbers instead of existing representations, to find the fuzzy optimal solution of fully fuzzy critical path problems.

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