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An Efficient Technique for Solving Special Integral Equations

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Abstract

In this paper, we apply a new technique for solving two-dimensional integral equations of mixed type. Comparisons are made between the homotopy perturbation method and the new technique. The results reveal that the new technique is effective and convenient.

Keywords: Homotopy perturbation method, New technique, Two-dimensional integral equations

MSC (2000) No.: 45A05, 45F99

1. Introduction

In recent years there has been a growing interest in the integral equations. An integral equation is an important branch of modern mathematics. It arises frequently in many applied areas which include engineering, mechanics potential theory, electrostatics, etc.

Consider the following linear integral equation of mixed type

$$u(x, y) = f(x, y) + \int_c^y \int_a^b k(x, y, s, t) u(s, t) ds dt, \quad (1)$$

where $u(x, y)$ is an unknown function, the known functions $f(x, y), k(x, y, s, t)$ are defined $D = [a, b] \times [c, d]$ and $s = \{(x, y, s, t) : a \leq x, s \leq b, c \leq t \leq y \leq d\}$, respectively.

In this paper, we apply He's homotopy perturbation method and a new technique for solving two-dimensional integral equations of mixed type. The homotopy perturbation method was introduced by He (1999, 2000, 2004, 2003, 2004, 2005, 2006, 2005, 2008) in 1998. In recent years the application of homotopy perturbation theory has appeared in many researches such as Biazar et al. (2007), Biazar et al. (2009), Cveticanin (2006), Bildik and Konuralp (2006), Abbasbandy (2006), Siddiqui and Ghorri (2006), Ariel et al. (2006), Ganji (2006), Wang (2008), and Sajid et al. (2007).

We consider (1) as

$$L(u) = u(x, y) - f(x, y) - \int_c^y \int_a^b k(x, y, s, t) u(s, t) ds dt = 0, \quad (2)$$

with solution $u(x, y) = \varphi(x, y)$. We define the homotopy by

$$H(u, p) = (1 - p)F(u) + pL(u) = 0, \quad (3)$$

where $F(u)$ is a functional operator, and let's u_0 , be the solution of $F(u) = 0$, which can be obtained easily. Obviously, from equation (3) we will have

$$H(u, 0) = F(u), \quad H(u, 1) = L(u),$$

and continuously trace an implicitly defined curve from a starting point $H(u_0, 0)$ to a solution $H(\varphi, 1)$. The embedding parameter p monotonically increases from zero to unity as the trivial problem $F(u) = 0$ continuously deformed to the original problem $L(\varphi) = 0$.

The homotopy perturbation method considers the solution as a power series, say

$$u = u_0 + pu_1 + p^2u_2 + \dots, \quad (4)$$

where p is called an embedding parameters. Substituting (4) into (3) and equating the terms with identical power of p , we obtain

$$\begin{aligned}
 p^0 : u_0(x, y) &= f(x, y), \\
 p^1 : u_1(x, y) &= \int_c^y \int_a^b k(x, y, s, t) u_0(s, t) ds dt, \\
 p^2 : u_2(x, y) &= \int_c^y \int_a^b k(x, y, s, t) u_1(s, t) ds dt, \\
 &\vdots \\
 p^j : u_j(x, y) &= \int_c^y \int_a^b k(x, y, s, t) u_{j-1}(s, t) ds dt, \\
 &\vdots
 \end{aligned}$$

These terms can be used to construct the solution, for $p = 1$, given in (4).

2. The New Technique

To accelerate the convergence of homotopy perturbation method, when it used for Volterra-Fredholm integral equations, $k(x, y, s, t)$ or $f(x, y)$ can be replaced by a series of finite components. These terms be expressed in series

$$f(x, y) = \sum_{l=1}^k f_l(x, y), \quad k(x, y, s, t) = \sum_{l=1}^k k_l(x, y, s, t).$$

In case of separable $k_l(x, y, s, t), f_l(x, y), l = 1, \dots, k$, say $k_l(x, y, s, t) = k_{l1}(x)k_{l2}(y)k_{l3}(s)k_{l4}(t), f_l(x, y) = f_{l1}(x)f_{l2}(y)$, and functions $k_{l1}(x), k_{l2}(y), k_{l3}(s), k_{l4}(t)$ and $f_{l1}(x), f_{l2}(y)$ are analytic suggested to be replaced by their Taylor series form

$$\begin{aligned}
 f_l(x, y) &= \sum_{i=0}^{\infty} f_{l1i}(x) \sum_{i=0}^{\infty} f_{l2i}(y), \tag{5} \\
 k_l(x, y, s, t) &= k_{l1}(x)k_{l2}(y)k_{l3}(s)k_{l4}(t) = \sum_{i=0}^{\infty} k_{l1i}(x) \sum_{i=0}^{\infty} k_{l2i}(y) \sum_{i=0}^{\infty} k_{l3i}(s) \sum_{i=0}^{\infty} k_{l4i}(t).
 \end{aligned}$$

Substitution Equation (5) into Equation (2) results in

$$L(u) = u(x, y) - \sum_{l=0}^k \left(\sum_{i=0}^{\infty} f_{l1i}(x) \sum_{i=0}^{\infty} f_{l2i}(y) \right) - \int_c^y \int_a^b \sum_{l=0}^k \left(\sum_{i=0}^{\infty} k_{l1i}(x) \sum_{i=0}^{\infty} k_{l2i}(y) \sum_{i=0}^{\infty} k_{l3i}(s) \sum_{i=0}^{\infty} k_{l4i}(t) \right) u(s, t) ds dt = 0.$$

The following homotopy can be constructed

$$\begin{aligned}
 H(u, p) &= u(x, y) - \sum_{l=0}^k \left(\sum_{i=0}^{\infty} f_{l1i}(x) p^i \sum_{i=0}^{\infty} f_{l2i}(y) p^i \right) \\
 &\quad - p \int_c^y \int_a^b \sum_{l=0}^k \left(\sum_{i=0}^{\infty} k_{l1i}(x) p^i \sum_{i=0}^{\infty} k_{l2i}(y) p^i \sum_{i=0}^{\infty} k_{l3i}(s) p^i \sum_{i=0}^{\infty} k_{l4i}(t) p^i \right) u(s, t) ds dt = 0. \tag{6}
 \end{aligned}$$

Substituting (4) into (6) and equating the coefficients of the terms with identical powers of p , components of series solution (4) will be obtained. This technique is simple and very effective tool which usually leads to the exact solutions.

3. Numerical Example

To illustrate the ability and reliability of the method, two examples are presented. See Hadizadeh and Asgary (2005) and Guoqiang and Liqinq (1994).

Example 1:

Consider the following Volterra-Fredholm integral equation

$$u(x, y) = f(x, y) + \int_0^y \int_{-1}^1 x^2 e^{-s} u(s, t) ds dt, \quad (7)$$

where

$$f(x, y) = y^2 e^x - \frac{2}{3} x^2 y^3,$$

with the exact solution as $u(x, y) = y^2 e^x$.

Homotopy Perturbation Method (HPM):

Using HPM, we have

$$u(x, y) = f(x, y) + p \int_0^y \int_{-1}^1 x^2 e^{-s} u(s, t) ds dt. \quad (8)$$

Substituting (4) into (8) and equating the coefficients of the terms with the identical powers of p , leads to

$$\begin{aligned} p^0 : u_0(x, y) &= f(x, y) = y^2 e^x - \frac{2}{3} x^2 y^3, \\ p^1 : u_1(x, y) &= \int_0^y \int_{-1}^1 x^2 e^{-s} u_0(s, t) ds dt \Rightarrow u_1(x, y) = -\frac{1}{6} y^4 x^2 e + \frac{5}{6} y^4 x^2 e^{-1} + \frac{2}{3} x^2 y^3, \\ p^2 : u_2(x, y) &= \int_0^y \int_{-1}^1 x^2 e^{-s} u_1(s, t) ds dt \Rightarrow u_2(x, y) = \frac{1}{3} y^5 x^2 - \frac{5}{6} y^5 x^2 e^{-2} - \frac{1}{30} y^5 x^2 e^2 - \frac{5}{6} y^4 x^2 e^{-1} + \frac{1}{6} y^4 x^2 e, \\ &\vdots \\ p^{i+1} : u_{i+1}(x, y) &= \int_0^y \int_{-1}^1 x^2 e^{-s} u_i(s, t) ds dt, \\ &\vdots \end{aligned}$$

Then, the series solution, by the homotopy perturbation method, is as follows

$$u(x, y) = \sum_{i=0}^{\infty} u_i(x, y) = y^2 e^x - \frac{2}{3} x^2 y^3 - \frac{1}{6} y^4 x^2 e + \frac{5}{6} y^4 x^2 e^{-1} + \frac{2}{3} x^2 y^3 + \frac{1}{3} y^5 x^2 - \frac{5}{6} y^5 x^2 e^{-2} - \frac{1}{30} y^5 x^2 e^2 - \frac{5}{6} y^4 x^2 e^{-1} + \frac{1}{6} y^4 x^2 e + \dots$$

The New Technique:

The Taylor series of $f(x, y)$ and $k(x, y, s, t)$ are as follows

$$f(x, y) = y^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} - \frac{2}{3} x^2 y^3,$$

$$k(x, y, s, t) = x^2 \sum_{n=0}^{\infty} \frac{(-s)^n}{n!}.$$

By substitution of these series into the equation (7), the following homotopy can be constructed

$$u(x, y) = y^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} p^n - \frac{2}{3} x^2 y^3 + p \int_0^y \int_{-1}^1 x^2 \sum_{n=0}^{\infty} \frac{(-s)^n}{n!} p^n u(s, t) ds dt. \tag{9}$$

Substituting (4) in to (9), and equating the coefficients of the terms with the identical powers of p , leads to

$$\begin{aligned} p^0 : u_0(x, y) &= y^2 - \frac{2}{3} x^2 y^3, \\ p^1 : u_1(x, y) &= y^2 x + \int_0^y \int_{-1}^1 x^2 \left(-\frac{2}{3} s^2 t^3 + t^2 \right) ds dt \Rightarrow u_1(x, y) = y^2 x + \frac{2}{3} x^2 y^3 - \frac{1}{9} x^2 y^4, \\ p^2 : u_2(x, y) &= y^2 \frac{x^2}{2} + \int_0^y \int_{-1}^1 x^2 \left(-\frac{1}{9} s^2 t^4 + \frac{2}{3} s^2 t^3 + t^2 s - s \left(-\frac{2}{3} s^2 t^3 + t^2 \right) \right) ds dt \\ &\qquad \qquad \qquad \Rightarrow u_2(x, y) = \frac{1}{2} y^2 x^2 + \frac{1}{9} x^2 y^4 - \frac{2}{135} x^2 y^5, \\ p^{j+1} : u_{j+1}(x, y) &= y^2 \frac{x^{(j+1)}}{(j+1)!} + \int_0^y \int_{-1}^1 x^2 \sum_{k=0}^j \frac{(-s)^k}{k!} u_{j-k}(s, t) ds dt, \\ &\vdots \end{aligned}$$

Therefore, the solution of Example 2 can be readily obtained by

$$u(x, y) = \sum_{i=0}^{\infty} u_i(x, y) = y^2 \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \dots \right),$$

which is the Taylor series of $u(x, y) = y^2 e^x$, i.e., the exact solution.

Example 2.

Let's solve the Volterra-Fredholm integral equation

$$u(x, y) = f(x, y) + \int_0^y \int_{-1}^1 s e^t u(s, t) ds dt, \quad (10)$$

where

$$f(x, y) = x + \sin y - \frac{2}{3} e^y + \frac{2}{3},$$

with the exact solution

$$u(x, y) = x + \sin y.$$

Let's use the new technique to solve this equation:

Taylor series of $f(x, y)$ and $k(x, y, s, t)$ will be used as

$$f(x, y) = x + \sum_{n=0}^{\infty} \frac{(-1)^n (y)^{2n+1}}{(2n+1)!} - \frac{2}{3} \sum_{n=0}^{\infty} \frac{y^n}{n!} + \frac{2}{3},$$

And

$$k(x, y, s, t) = s \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

Substitution of these series in equation (10), the following homotopy can be constructed

$$u(x, y) = x + \sum_{n=0}^{\infty} \frac{(-1)^n (y)^{2n+1}}{(2n+1)!} p^n - \frac{2}{3} \sum_{n=0}^{\infty} \frac{y^n}{n!} p^n + \frac{2}{3} + p \int_0^y \int_{-1}^1 s \sum_{n=0}^{\infty} \frac{t^n}{n!} p^n u(s, t) ds dt. \quad (11)$$

Substituting (4) into (11) and equating the terms with identical powers of p , gives

$$\begin{aligned} p^0 : u_0(x, y) &= x + y, \\ p^1 : u_1(x, y) &= -\frac{y^3}{6} - \frac{2}{3} y + \int_0^y \int_{-1}^1 s(s+t) ds dt \Rightarrow u_1(x, y) = -\frac{y^3}{3!}, \end{aligned}$$

$$\begin{aligned}
 p^2 : u_2(x, y) &= \frac{y^5}{5!} - \frac{1}{3}y^2 + \int_0^y \int_{-1}^1 s \left(t(t+s) - \frac{t^3}{6} \right) ds dt \Rightarrow u_2(x, y) = \frac{y^5}{5!}, \\
 &\vdots \\
 p^{j+1} : u_{j+1}(x, y) &= \frac{(-1)^{j+1} y^{2j+3}}{(2j+3)!} - \frac{2}{3} \frac{y^{j+1}}{(j+1)!} + \int_0^y \int_{-1}^1 s \sum_{k=0}^j \frac{t^k}{k!} u_{j-k}(s, t) ds dt, \\
 &\vdots
 \end{aligned}$$

Therefore, the solution of Example 2 can be readily presented by

$$u(x, y) = \sum_{i=0}^{\infty} u_i(x, y) = x + y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \dots$$

With the summation, $u(x, y) = x + \sin y$, which is the exact solution.

4. Conclusion

In this paper we present an efficient technique for solving special integral equations of mixed type. Comparisons are made between the homotopy perturbation method and the new technique. From the computational viewpoint, the new technique is more efficient and easy to use.

REFERENCES

- Abbasbandy, S. (2007). Application of He's Homotopy Perturbation Method to Functional Integral Equations. *Chaos, Solitons & Fractals*, **31**, pp. 1243-1247.
- Biazar, J., Eslami, M. Ghazvini, H. (2007). He's Homotopy Perturbation Method for Solving System of Volterra Integral Equations of the Second Kind, *International Journal of Nonlinear Sciences and Numerical Simulation*, 8 (3) pp.411-416.
- Biazar, J., Eslami, M. and Aminikhah, H. (2009). Application of homotopy perturbation method for systems of Volterra integral equations of the first kind, *Chaos, Solitons and Fractals* 42 pp. 3020–3026.
- Cveticanin, L. (2006). Homotopy Perturbation Method for Pure Nonlinear Differential Equation, *Chaos. Solitons and Fractals*, **30**, pp. 1221–1230.
- Ganji, D. D. (2006). The Application of He's Homotopy Perturbation Method to Nonlinear Equations Arising in Heat Transfer, *Physics Letters A*, **355**, pp. 337–341.
- Guoqiang, H., Liqinq, Z. (1994). Asymptotic error expansion for the trapezoidal Nystrom method of linear Volterra-Fredholm integral equations, *Journal of Computational and Applied Mathematics*, 51 (3) pp. 339-348.

- Hadizadeh, M., Asgary, M. (2005). An efficient numerical approximation for the linear class of mixed integral equations, *Applied Mathematics and Computation*, 167 pp. 1090-1100.
- He, J.H. (1999). Homotopy Perturbation Technique, *Computer Methods in Applied Mechanics and Engineering*, **178**, pp. 257–262.
- He, J. H. (2000). A Coupling Method of Homotopy Technique and Perturbation Technique for Nonlinear Problems. *International Journal of Non-Linear Mechanics*, **35 (1)**, pp. 37–43.
- He, J. H. (2004). Comparison of Homotopy Perturbation Method and Homotopy Analysis Method, *Applied Mathematics and Computation*, **156**, pp. 527-539.
- He, J. H. (2003). Homotopy Perturbation Method: A New Nonlinear Analytical Technique. *Applied Mathematics and Computation*, **135**, pp. 73-79.
- He, J. H. (2004). The Homotopy Perturbation Method for Nonlinear Oscillators with Discontinuities. *Applied Mathematics and Computation*, **151**, pp. 287-292.
- He, J. H. (2005). Application of Homotopy Perturbation Method to Nonlinear Wave Equations. *Chaos. Solitons and Fractals*, **26**, pp. 695–700.
- He, J. H. (2006). Homotopy Perturbation Method for Solving Boundary Value Problems. *Physics Letters A*, **350**, pp. 87-88.
- He, J. H. (2005). Limit cycle and bifurcation of nonlinear problems, *Chaos, Solitons and Fractals*, **26 (3)**, pp. 827-833.
- He, J. H. (2008). Recent Development of the Homotopy Perturbation Method, *Topological Methods in Nonlinear Analysis*, **31**, pp. 205-209.
- He, J. H. (2006). New Interpretation of Homotopy Perturbation Method, *International Journal of Modern Physics B*, **20**, pp. 2561-2568.
- Sajid, M., Hayat, T. and Asghar, S. (2007). Comparison Between the HAM and HPM Solutions of Thin Film Flows of Non-Newtonian Fluids on a Moving Belt, *Nonlinear Dynam.*, **50**, pp. 27-35.