Applications and Applied Mathematics: An International Journal (AAM)

Volume 5 | Issue 1 Article 9

6-2010

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Pardeep Kumar
Himachal Pradesh University

Gursharn Jit Singh
SCD Government College

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Vol. 5, Issue 1 (June 2010) pp. 110 - 119
(Previously, Vol. 5, No. 1)

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On the Stability of Superposed Viscous-Viscoelastic Fluids Through Porous Medium

Pardeep Kumar
Department of Mathematics
ICDEOL
Himachal Pradesh University
Shimla-171005, India
drpardeep@sancharnet.in

Gursharn Jit Singh
SCD Government College
Ludhiana, Punjab
sandhugjs@yahoo.co.in

Received: January 2, 2010; Accepted: March 25, 2010

Abstract

Rayleigh-Taylor instability of a Newtonian viscous fluid overlying Walters's B′ viscoelastic fluid through porous medium is considered. For the stable configuration the system is found to be stable or unstable. However, the system is found to be unstable for the unstable configuration. The effects of a uniform horizontal magnetic field and a uniform rotation are also considered. For the stable configuration, in the hydro magnetic case also, the system is found to be stable or unstable. However, for the unstable configuration, the magnetic field and viscoelasticity have got stabilizing effects. The system is found to be unstable for the potentially unstable case, for highly viscous fluids, in the presence of a uniform rotation.

Keywords: Rayleigh-Taylor instability, Newtonian fluid, Walters B' viscoelastic fluid, Porous medium, Horizontal magnetic field, Uniform rotation

MSC (2000) No.: 76A10, 76E25, 76E07, 76S05

1. Introduction

The instability of the plane interface separating two fluids when one is accelerated towards the other or when one is superposed over the other has been studied by several authors and Chandrasekhar (1981) has given a detailed account of these investigations. The influence of viscosity on the stability of the plane interface separating two electrically conducting, incompressible superposed fluids of uniform densities, when the whole system is immersed in a uniform horizontal magnetic field, has been studied by Bhatia (1974). He has carried out the stability analysis for two fluids of high viscosities and different uniform densities. The stability of superposed fluids in the presence of a variable horizontal magnetic field has been studied by Sharma and Thakur (1982).

With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. Oldroyd (1958) proposed a theoretical model for a class of viscoelastic fluids. Toms and Strawbridge (1953) showed that a dilute solution of methyl methacrylate in n-butyl acetate agrees well with the theoretical model of Oldroyd fluid. Sharma and Sharma (1978) have studied the stability of the plane interface separating two Oldroydian viscoelastic superposed fluids of uniform densities. There are many elastico-viscous fluids that cannot be characterized by Oldroyd's constitutive relations. One such class of elastico-viscous fluids is Walters B' fluid having relevance and importance in geophysical fluid dynamics, chemical technology and petroleum industry. Walters (1962) reported that the mixture of polymethyl methacrylate and pyridine at 25^0C containing 30.5g of polymer per litre with density 0.98g per litre behaves very nearly as the Walters B' elastico-viscous fluid. Polymers are used in the manufacture of spacecrafts, aero planes, tyres, belt conveyers, ropes, cushions, seats, foams, plastic engineering equipments, contact lens etc. Walters B' elastico-viscous fluid forms the basis for the manufacture of many such important and useful products. Sharma and Kumar (1997) have studied the stability of the plane interface separating two viscoelastic (Walters B') superposed fluids of uniform densities.

In all the above studies, the medium has been considered to be non-porous. The flow through porous media is of considerable interest for petroleum engineers, for geophysical fluid dynamicists and has importance in chemical technology and industry. An example in the geophysical context is the recovery of crude oil from the pores of reservoir rocks. When the fluid permeates a porous material, the gross effect is represented by Darcy's law. As a result of this macroscopic law, the usual viscous term in the equations of motion of Walters B' fluid is replaced by the resistance term $\left[-\frac{1}{k_1}\left(\mu-\mu'\frac{\partial}{\partial t}\right)\vec{v}\right]$, where μ and μ' are the viscosity and viscoelasticity of the Walters fluid, k_1 is the medium permeability and $\vec{v}$ is the Darcian (filter) velocity of the fluid. Chakraborty and Sengupta (1994) have studied the flow of unsteady viscoelastic (Walters liquid B') conducting fluid through two porous concentric non-conducting infinite circular cylinders rotating with different angular velocities in the presence of a uniform axial magnetic field. In another study, Sharma and Kumar (1995) have studied the steady flow and heat transfers of Walters fluid (Model B') through a porous pipe of uniform circular cross-section with small suction.

The instability of the plane interfaces between viscous and viscoelastic fluids through porous medium may find applications in geophysics, chemical technology and bio-mechanics. It is therefore, the motivation of the present paper to study the Rayleigh-Taylor instability of viscous-viscoelastic fluids in porous medium. The effects of uniform rotation and uniform magnetic field, having relevance and importance in geophysics, are also considered. These aspects form the subject matter of the present paper.

2. Formulation of the Problem and Perturbation Equations

Let T_{ij}, τ_{ij}, e_{ij}, δ_{ij}, v_i, x_i, p, μ and μ' denote the stress tensor, shear stress tensor, rate-of-strain tensor, kronecker delta, velocity vector, position vector, isotropic pressure, viscosity and viscoelasticity, respectively. The constitutive relations for the Walters B′ elastico-viscous fluid are:

$$\begin{cases} T_{ij} = -p\delta_{ij} + \tau_{ij}\,, \\ \tau_{ij} = 2\left[\mu + \mu'\dfrac{\partial}{\partial t}\right]e_{ij}\,, \\ e_{ij} = \dfrac{1}{2}\left[\dfrac{\partial v_i}{\partial x_j} + \dfrac{\partial v_j}{\partial x_i}\right]. \end{cases} \tag{1}$$

Here we consider a static state in which an incompressible Walters B′ elastico-viscous fluid is arranged in horizontal strata in porous medium and the pressure p and the density ρ are functions of the vertical co-ordinate z only. The character of the equilibrium of this initial static state is determined, as usual, by supposing that the system is slightly disturbed and then by following its further evolution.

Let $\vec{v}(u,v,w), \rho, p, \in$ and k_1 denote the velocity of fluid, density, pressure, medium porosity and medium permeability, respectively. Then the equations of motion and continuity for Walters B′ incompressible viscoelastic fluid in a porous medium are:

$$\frac{\rho}{\in}\left[\frac{\partial \vec{v}}{\partial t} + \frac{1}{\in}(\vec{v}\cdot\nabla)\vec{v}\right] = \left[-\nabla p + \rho\vec{g}\right] - \frac{\rho}{k_1}\left[\upsilon - \upsilon'\frac{\partial}{\partial t}\right]\vec{v}\,, \tag{2}$$

$$\nabla\cdot\vec{v} = 0\,, \tag{3}$$

where $\vec{g}(0,0,-g)$ is the acceleration due to gravity $\upsilon\left(=\dfrac{\mu}{\rho}\right)$ is kinematic viscosity of the fluid and $\upsilon'\left(=\dfrac{\mu'}{\rho}\right)$ is kinematic viscoelasticity of the fluid.

Since the density of the moving fluid remains unchanged, we have

$$\in \frac{\partial \rho}{\partial t} + \left(\vec{v}\cdot\nabla\right)\ \rho = 0\,. \tag{4}$$

Let $\vec{v}\left(u, v, w\right), \delta\rho$ and δp denote respectively the perturbations in fluid velocity (0, 0, 0), density ρ and pressure p. Then the linearized perturbation equations Chandrasekhar (1981), Sharma and Kumar (1997) relevant for the porous medium are:

$$\frac{\rho}{\in}\frac{\partial \vec{v}}{\partial t} = -\nabla\delta p + \vec{g}\delta\rho - \frac{\rho}{k_1}\left(\upsilon - \upsilon'\frac{\partial}{\partial t}\right)\vec{v}, \tag{5}$$

$$\nabla\cdot\vec{v} = 0, \tag{6}$$

$$\frac{\rho}{\in}\frac{\partial}{\partial t}\left(\delta\rho\right) = -wD\rho, \tag{7}$$

where $D = \dfrac{d}{dz}$.

Analyzing the disturbances into normal modes, we seek solutions whose dependence on x, y and t is given by

$$\exp(ik_x x + ik_y y + nt), \tag{8}$$

where k_x, k_y are horizontal wave numbers, $k^2 = k_x^2 + k_y^2$ and n is the rate at which the system departs from the equilibrium.

For perturbations of the form (8), equations (5)-(7) give

$$\frac{1}{\in}\rho n u = -ik_x\delta p - \frac{\rho}{k_1}\left(\upsilon - \upsilon' n\right)u, \tag{9}$$

$$\frac{1}{\in}\rho n v = -ik_y\delta p - \frac{\rho}{k_1}\left(\upsilon - \upsilon' n\right)v, \tag{10}$$

$$\frac{1}{\in}\rho n w = -D\delta p - g\delta\rho - \frac{\rho}{k_1}\left(\upsilon - \upsilon' n\right)w, \tag{11}$$

$$ik_x u + ik_y v + Dw = 0\,, \tag{12}$$

$$\in n\delta\rho = -wD\rho\,. \tag{13}$$

Eliminating δp between equations (9)-(11) with the help of equations (12) and (13), we obtain

$$\frac{n}{\in}\left[D(\rho Dw)-k^2\rho w\right]+\frac{1}{k_1}\left[D\{\rho(\upsilon-\upsilon' n)Dw\}-k^2\rho(\upsilon-\upsilon' n)w\right]+\frac{gk^2}{\in n}(D\rho)w=0\,. \tag{14}$$

3. Two Uniform Viscous and Viscoelastic (Walters B') Fluids Separated by a Horizontal Boundary

Consider the case of two uniform fluids of densities, viscosities, ρ_2, μ_2 (upper, Newtonian fluid) and ρ_1, μ_1 (lower, Walters B′ viscoelastic fluid) separated by a horizontal boundary at z = 0. Then in each region of constant ρ, constant μ and constant μ', equation (14) reduces to

$$\left(D^2-k^2\right)w=0\,. \tag{15}$$

The general solution of equation (15) is

$$w = Ae^{+kz}+Be^{-kz}\,, \tag{16}$$

where A and B are arbitrary constants.

The boundary conditions to be satisfied in the present problem are:

(i) The velocity w should vanish when $z\to+\infty$ (for the upper fluid) and $z\to-\infty$ (for the lower fluid).
(ii) w (z) is continuous at z = 0.
(iii) The jump condition at the interface z = 0 between the fluids.
(iv) The jump condition mentioned in (iii) is obtained by integrating equation (14) over an infinitesimal element of z including 0, and is

$$\frac{n}{\in}\left[\rho_2 Dw_2-\rho_1 Dw_1\right]_{z=0}+\frac{1}{k_1}\left[\mu_2 Dw_2-(\mu_1-\mu_1' n)Dw_1\right]_{z=0}=-\frac{gk^2}{\in n}\left[\rho_2-\rho_1\right]w_0\,, \tag{17}$$

remembering the configuration that upper fluid is Newtonian and lower fluid is Walters B′ viscoelastic. Here w_0 is the common value of w at z = 0.

Applying the boundary conditions (i) and (ii), we can write

$$w_1 = Ae^{+kz}, \qquad (z<0), \tag{18}$$

$$w_2 = Ae^{-kz}, \qquad (z>0), \tag{19}$$

where the same constant A has been chosen to ensure the continuity of w at z = 0.

Applying the condition (17) to the solutions (18) and (19), we obtain

$$\left[1-\frac{\in}{k_1}\alpha_1\upsilon_1'\right]n^2+\frac{\in}{k_1}\left[\alpha_2\upsilon_2+\alpha_1\upsilon_1\right]n-gk\left[\alpha_2-\alpha_1\right]=0, \tag{20}$$

where $\alpha_{1,2}=\frac{\rho_{1,2}}{\rho_1+\rho_2}$, $\upsilon_1=\frac{\mu_1}{\rho_1}, \upsilon_2=\frac{\mu_2}{\rho_2}, \upsilon_1'=\frac{\mu_1'}{\rho_1}$.

Discussion

(a) For the **potentially stable case** $(\alpha_2<\alpha_1)$, if $1>\frac{\in}{k_1}\alpha_1\upsilon_1'$, equation (20) does not admit of any change of sign and so has no positive root. The system is therefore stable. But, if $1<\frac{\in}{k_1}\alpha_1\upsilon_1'$, the coefficient of n^2 in equation (20) is negative. Therefore, equation (20) allows one change of sign and so has one positive root. The occurrence of positive root implies that the system is unstable. Therefore, for the potentially stable arrangement $(\alpha_1>\alpha_2)$, the system is stable or unstable according as

$$\upsilon_1'<\ or\ \ >\frac{k_1}{\in\alpha_1}. \tag{21}$$

(b) For the **potentially unstable case** $(\alpha_2>\alpha_1)$, the constant term in equation (20) is negative. Equation (20), therefore, allows one change of sign and so has one positive root and hence the system is unstable. Therefore, the system is unstable for unstable configuration.

4. Effect of a Horizontal Magnetic Field

Consider the motion of an incompressible, infinitely conducting Newtonian and Walters B′ viscoelastic fluids in porous medium in the presence of a uniform horizontal magnetic field $\vec{H}(H,0,0)$. Let $\vec{h}(h_x,h_y,h_z)$ denote the perturbation in the magnetic field, then the linearized perturbation equations are:

$$\frac{\rho}{\in}\frac{\partial\vec{v}}{\partial t}=-\nabla\delta p+\vec{g}\delta\rho+\frac{\mu_e}{4\pi}\left(\nabla\times\vec{h}\right)\times\vec{H}-\frac{\rho}{k_1}\left(\upsilon-\upsilon'\frac{\partial}{\partial t}\right)\vec{v}, \tag{22}$$

$$\nabla\cdot\vec{h}=0, \tag{23}$$

$$\in \frac{\partial \vec{h}}{\partial t} = \nabla \times \left(\vec{v} \times \vec{H} \right), \tag{24}$$

together with equations (6) and (7). Assume that the perturbation $\vec{h}\left(h_x, h_y, h_z\right)$ in the magnetic field has also a space and time dependence of the form (8). Here μ_e stands for the magnetic permeability. Following the procedure as in Section 3, we obtain

$$\left[1 - \frac{\in}{k_1} \upsilon_1' \alpha_1\right] n^2 + \frac{\in}{k_1}\left[\upsilon_2 \alpha_2 + \upsilon_1 \alpha_1\right] n + \left[2k_x^2 V_A^2 - gk\left(\alpha_2 - \alpha_1\right)\right] = 0, \tag{25}$$

where $V_A = \sqrt{\frac{\mu_e H^2}{4\pi\left(\rho_1 + \rho_2\right)}}$ is the Alfvén velocity.

Discussion

For the **potentially stable case** $(\alpha_2 < \alpha_1)$, it is evident from equation (25) that the system is stable or unstable according as $1 >$ or $< \frac{\in}{k_1} \upsilon_1' \alpha_1$, i.e., $\upsilon_1' < or > \frac{k_1}{\in \alpha_1}$.

For the **potentially unstable configuration** $\left(\alpha_2 > \alpha_1\right)$, if $1 > \frac{\in}{k_1} \upsilon_1' \alpha_1$ and $2k_x^2 V_A^2 > gk\left(\alpha_2 - \alpha_1\right)$, i.e., if $\upsilon_1' < \frac{k_1}{\in \alpha_1}$ and $2k_x^2 V_A^2 > gk\left(\alpha_2 - \alpha_1\right)$, equation (25) does not admit of any change of sign and so has no positive root. The system is therefore stable.

But, if $2k_x^2 V_A^2 < gk\left(\alpha_2 - \alpha_1\right)$, the constant term in equation (25), is negative. Equation (25), therefore, allows at least one change of sign and so has at least one positive root. The occurrence of a positive root implies that the system is unstable.

Thus, for the stable configuration, the system is found to be stable or unstable under certain condition. However, for the potentially unstable configuration, the presence of magnetic field and viscoelasticity have got stabilizing effects and completely stabilize the system for all wave numbers which satisfy the inequalities

$$\upsilon_1' < \frac{k_1}{\in \alpha_1} \text{ and } 2k_x^2 V_A^2 > gk\left(\alpha_2 - \alpha_1\right) .$$

5. Effect of Uniform Rotation

Here, we consider the motion of an incompressible Walters B' elastico-viscous fluid in porous medium in the presence of a uniform rotation $\vec{\Omega}(0,0,\Omega)$. Then the linearized perturbation equations are:

$$\frac{\rho}{\in}\frac{\partial\vec{v}}{\partial t}=-\nabla\delta p+\vec{g}\delta\rho+\frac{2\rho}{\in}\left(\vec{v}\times\vec{\Omega}\right)-\frac{\rho}{k_1}\left(\upsilon-\upsilon'\frac{\partial}{\partial t}\right)\vec{v}, \tag{26}$$

together with equations (6) and (7).

Following the same procedure as in Section 3 and Chandrasekhar (1981, p.453), we obtain

$$1+\frac{4\Omega^2}{\left[n+\frac{\in}{k_1}\left(\upsilon-\upsilon'n\right)\right]^2}+\frac{gk^2\left(\alpha_1-\alpha_2\right)}{n\left[n+\frac{\in}{k_1}\left(\upsilon-\upsilon'n\right)\right]\kappa}=0, \tag{27}$$

where

$$\kappa=\frac{k}{1+\frac{2\Omega^2}{\left[n+\frac{\in}{k_1}\left(\upsilon-\upsilon'n\right)\right]^2}}, \tag{28}$$

for highly viscous fluid and viscoelastic fluids. Here we assumed the kinematic viscosities and kinematic viscoelasticities of both fluids to be equal i.e. $\upsilon_1=\upsilon_2=\upsilon$ Chandrasekhar (1981), p.443, $\upsilon_1'=\upsilon'$, as these simplifying assumptions do not obscure any of the essential features of the problem. Equation (27), after substituting the value of κ from (28) and simplification, yields

$$\begin{aligned}&\left[\left(1-\frac{\in\upsilon'}{k_1}\right)^3\right]n^4+\left[\frac{3\in\upsilon}{k_1}\left(1-\frac{\in\upsilon'}{k_1}\right)^2\right]n^3+\left[1-\frac{\in\upsilon'}{k_1}\right]\\&\left[\left(\frac{3\in^2\upsilon'^2}{k_1^2}+4\Omega^2\right)+\left(1-\frac{\in\upsilon'}{k_1}\right)gk\left(\alpha_1-\alpha_2\right)\right]n^2\\&+\left[\frac{\in^3\upsilon^3}{k_1^3}+4\Omega^2\frac{\in\upsilon}{k_1}+2\frac{\in\upsilon}{k_1}\left(1-\frac{\in\upsilon'}{k_1}\right)gk\left(\alpha_1-\alpha_2\right)\right]n+\left[\left(\frac{\in^2\upsilon^2}{k_1^2}+2\Omega^2\right)gk\left(\alpha_1-\alpha_2\right)\right]=0.\end{aligned} \tag{29}$$

Discussion

For the **potentially stable arrangement** $(\alpha_2 < \alpha_1)$, if $1 > \frac{\in \upsilon'}{k_1}$, equation (29) does not allow any positive root as there is no change of sign. The system is therefore stable. Thus when the ordinary (Newtonian) viscous fluid overlies Walters B' viscoelastic fluid in a porous medium in the presence of a uniform rotation, the system is stable for the potentially stable configuration if $1 > \frac{\in \upsilon'}{k_1}$. Otherwise if $1 < \frac{\in \upsilon'}{k_1}$, the system is unstable for potentially stable configuration.

For the **potentially unstable arrangement** $(\alpha_2 > \alpha_1)$, the constant term in equation (29) is negative and so there is at least one change of sign in equation (29). Therefore, equation (29) allows at least one positive root of n, meaning thereby instability of the system.

Acknowledgements

The authors are grateful to the referees for their technical comments and valuable suggestions, which led to a significant improvement of the paper.

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