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Two Temperature Dual-phase-lag Fractional Thermal Investigation of Heat Flow Inside a Uniform Rod

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Abstract

A non-classical, coupled, fractionally ordered, dual-phase-lag (DPL) heat conduction model has been presented in the framework of the two-temperature theory in the bounded Cartesian domain. Due to the application of two-temperature theory, the governing heat conduction equation is well-posed and satisfying the required stability criterion prescribed for a DPL model. The mathematical formulation has been applied to a uniform rod of finite length with traction free ends considered in a perfectly thermoelastic homogeneous isotropic medium. The initial end of the rod has been exposed to the convective heat flux and energy dissipated by convection into the surrounding medium through the last end. The State-space approach has been employed to solve the corresponding boundary value problem to obtain the conductive and thermomechanical temperature along with thermal displacement and stresses in the Laplace domain. The role of the time-fractional order and delay time in the heat flux and temperature gradient has been investigated through numerical results representing graphically along the length of the rod. The classical, fractional and generalized thermoelasticity theory have been recovered and the finite speed of thermal wave has been attained.

Keywords: Fractional thermoelasticity; Two-temperature theory; Phase-lags; Laplace transform; Dual-Phase-Lag (DPL)

MSC 2010 No.: 26A33, 44A10, 65R10, 74F05

1. Introduction

Solid heat conduction models came into existence after the development of the Fourier law of heat conduction, which states that during the heat conduction process in a perfectly homogeneous and isotropic medium, the heat flux vector and temperature gradient appears instantly at the same time and consequently implies that the thermal signal propagates with an infinite speed. The drawback of infinite speed in Fourier law becomes unacceptable. This motivates scientists and physicists to search for non-classical theories required to frame the new constitutive relations.

Following the Fourier law of heat conduction, Biot (1956) has derived the coupled heat conduction equation by coupling the thermal and mechanical forces and hence introduced the classical coupled thermoelasticity. It has been found that the thermal and mechanical forces are not independent. However, the proposed theory was failed to achieve the finite speed of thermal wave propagation. Following Ignaczak and Ostoja-Starzewki (2010) it has been seen that Cattaneo and Vernotte have reconstructed the Fourier law in terms of the thermal relaxation parameter called the relaxation time. The resulting heat conduction equation was found to be hyperbolic that characterizes the combined diffusion and wave-like behaviour of heat conduction and predicts the finite speed of thermal wave propagation. Later, Lord and Shulman (1967) have applied Cattaneo-Vernotte law and hence derived the generalized thermoelastic heat conduction model with one relaxation time. Apart from the conductive temperature Chen and Gurtin (1968) have introduced the thermodynamical temperature and then established the mathematical relationship between these temperatures. The said theory is well known as the two-temperature theory.

Following generalized coupled thermoelasticity, Green and Lindsay (1972) have proposed a thermoelastic model where not only the heat conduction equation but also the equations of motion and stresses were associated with the relaxation time. Hereafter, Green and Naghdi (1993) have proposed a non-classical, generalized, coupled thermoelastic model without energy dissipation. It was observed that during the process of heat conduction governed by the Fourier law, the heat energy does not dissipate. Youssef (2006) introduced a generalized coupled thermoelastic model and studied to recover various special cases in the context of two temperature theory.

To study the lagging behaviour of the heat conduction, Tzou (1995) introduced phase-lags to the heat flux vector and temperature gradient into the Fourier law and deduced a DPL law of heat conduction. Later, Quintanilla (2008) claimed that whenever a DPL heat conduction law is coupled with the Biot's (1956) energy equation, then the resulting problem may have a sequence of eigenvalues such that its real part is positive and tends to infinite. Consequently, the resulting coupled DPL heat conduction equation may not be stable and hence the proposed model is not well-posed. Moreover, if a dual-phase lag heat conduction equation is derived in the context of two-temperature theory introduced by Chen and Gurtin (1968), then the model established must be well-posed. Hamilton et al. (2008) have developed a thermal model of friction stir welding that utilizes a new slip factor based on the energy per unit length of the weld. The slip factor is derived from an empirical linear relationship observed between the ratio of the maximum welding temperature to the solidus temperature and the welding energy. The thermal model successfully predicts the maximum welding temperature over a wide range of energy levels.

The non-local properties of fractional derivatives indicate that the next state of a dynamical system depends not only on its current state but also on its historical states. Scientifically, due to the non-local properties inherited by fractional order derivatives and integrals, fractional calculus is frequently applied to realistic problems. In particular, the theorems on Laplace transforms derived by Liang et al. (2013) have been studied to find the Laplace transforms of the fractional-order ordinary and partial differential equations most frequently nowadays. Povstenko (2004) updated the Fourier law of heat conduction using time-fractional derivatives that gave rise to the theory of fractional thermoelasticity. Later, formal theory of thermoelasticity associated with one relaxation time was given by Sherief et al. (2010) along with the derivation of corresponding uniqueness theorem, reciprocal theorem and the variational principle. A few years later, the theory of thermoelasticity with two relaxation times was derived by Hamza et al. (2014).

Ezzat (2011) proposed a one-dimensional application for a conducting half-space of thermoelectric elastic material, which is thermally shocked in the presence of a magnetic field. The problem was solved using Laplace transform and state-space techniques and some conclusions about the newly developed theory of magneto-thermoelasticity were discussed. Ezzat and Karamany (2009 and 2011) applied two temperature theory to fractional order thermoelasticity and proposed a couple of new models in the field of thermal sciences.

Kulkarni and Parab (2018) developed the general analytical solution of the most generalized thermal bending problem in the Cartesian domain in the context of non-homogeneous transient heat equation subjected to Robin's boundary conditions. The well-posedness of the problem has been discussed by the existence, uniqueness, and stability of series solutions obtained analytically. The convergence of infinite series solutions was discussed. Bhatta (2018) examined the effect of the vertical rate of change in thermal diffusivity due to a hydrothermal convective flow in a horizontal porous medium. Considering the vertically varying basic state the corresponding linear system was designed through this, and hence, the critical Rayleigh and wave number were evaluated. The marginal stability curves and linear solutions were further investigated to examine the solution pattern for different diffusivity parameters.

Mittal and Kulkarni (2019) designed a fractionally ordered, coupled dual-phase-lag model in the context of two-temperature theory derived by Chen and Gurtin (1968). The governing fractional-order dual-phase-lag heat conduction equation has been derived for a couple of delay time translations called intrinsic properties of the medium and well-identified as phase-lags which are responsible to capture the phonon-electron interactions. It has been found that subjected to various combinations of time-fractional order lying between $(0, 1)$, various existing thermoelastic heat conduction models proposed by Biot (1956), Lord and Shulman (1967), Sherief et al. (2010), Ezzat and Karamany (2009 and 2011), Mittal and Kulkarni (2018) were recovered. As a special case, the formulation was applied to examine a spherical cavity having traction free inner and outer boundaries which were subjected to constant thermal loading. The thermal investigations have been done by considering various numerical values of fractional order and phase-lag values.

The present manuscript is an attempt to investigate properties of heat flow and related thermal variation across a finite-dimensional uniform rod. The initial end of the rod has been exposed to the time-dependent convective heat flux whereas the heat energy dissipates due to convection into the surrounding medium through the other end. Following Mittal and Kulkarni (2019), the dimensionless form of fractional order heat conduction equation in the context of two temperature theory has been used for the mathematical formulation of the problem. The main aim of the mathematical analysis is to establish the scientific role of time-fractional order and delay time in the heat flux and temperature gradient to classify a solid conducting material as per its conductive ability. Analytical results are obtained in the Laplace domain and corresponding inversions are computed numerically following the Gaver (1966) and Stehfest (1970) algorithm and satisfying the Kuznetsov (2013) convergence theorem. The existing theories of classical, fractional and generalized thermoelasticity have been recovered and the finite speed of thermal wave has been achieved. The mathematical model presented has a wide scope of applications in the field of material and structural designing.

No one has presented and examined the fractionally ordered dual-phase-lag thermal stress analysis in the framework of two temperatures in the finite Cartesian coordinate system. This is a new and novel contribution to the field.

2. Mathematical Formulation of the Problem

The following dimensionless physical and geometrical parameters have been used to develop a mathematical formulation of the problem,

$$(u^*, x^*, t^*, \tau_1^*, \tau_2^*) = c_l\zeta(u, x, c_l t, c_l\tau_1, c_l\tau_2),\ \Theta = \frac{(\theta-\theta_0)\gamma}{\lambda+2\mu},\ \Phi = \frac{\phi\gamma}{\lambda+2\mu},$$

$$c_l = \sqrt{\frac{\lambda+2\mu}{\rho_m}},\ \sigma_{xx}^* = \frac{\sigma_{xx}}{\lambda+2\mu},\ \zeta = \frac{\rho_m c_m}{k},\ \xi = bc_l^2\zeta^2,\ \epsilon = \frac{\gamma^2\theta_0}{\rho_m c_m(\lambda+2\mu)}.$$

Consider a uniform rod of finite length L in the bounded Cartesian domain satisfying dimensionless heat conduction equation and thermal stress formulation as (neglecting the $*$ sign)

$$\begin{aligned}&\left[\frac{\partial}{\partial t} + \frac{\tau_1^\nu}{\Gamma(\nu+1)}\frac{\partial^{\nu+1}}{\partial t^{\nu+1}} + \frac{\tau_1^{2\nu}}{\Gamma(2\nu+1)}\frac{\partial^{2\nu+1}}{\partial t^{2\nu+1}}\right](\Phi+\epsilon e)\\ &= \left[1 + \frac{\tau_2^\nu}{\Gamma(\nu+1)}\frac{\partial^\nu}{\partial t^\nu} + \frac{\tau_2^{2\nu}}{\Gamma(2\nu+1)}\frac{\partial^{2\nu}}{\partial t^{2\nu}}\right]\nabla^2\Theta,\end{aligned} \tag{1}$$

$$\Theta - \Phi = \xi\nabla^2\Theta, \tag{2}$$

$$\frac{\partial e}{\partial x} - \frac{\partial \Theta}{\partial x} = \ddot{u}, \tag{3}$$

$$\sigma_{xx} = e - \Theta. \tag{4}$$

where θ and ϕ denote the conductive and thermodynamical temperatures, the delay time translations τ_1 and τ_2 are called the intrinsic properties of the medium and well-identified as phase-lags which are responsible to capture the phonon-electron interactions. Here, θ_0 denotes the reference temperature of the medium, e is the cubical dilatation factor, and t is the time. The constant ρ_m, is the density and c_m is specific heat of the solid material, α_t is the coefficient of linear thermal expansion, and λ, μ are Lamé constants of elasticity connected by the fixed term $\gamma = \alpha_t(3\lambda + 2\mu)$. The symbol $\triangledown^2$ represents one-dimensional Laplacian operator in the Cartesian domain.

Equations (1)-(4) describe the governing dimensionless equations of the fractional dual-phase-lag heat conduction model presented in this manuscript.

2.1. Initial and boundary conditions

Assume that the initial end of the rod at $x = 0$ has been exposed to the time dependent convective heat flux expressed by the function $F : (0, \infty) \longrightarrow (0, \infty)$ in the context of error function given by $F(t) = erf\sqrt{(t\varepsilon)}$ whereas the extreme end at $x = L$ has kept at zero temperature. Then,

$$\frac{\partial \Theta}{\partial x} + k_1 \Theta = \varOmega e^{-\omega t} F(t)|_{x=0}, \tag{5}$$

$$\frac{\partial \Theta}{\partial x} + k_2 \Theta = 0|_{x=L}, \tag{6}$$

$$\Theta(x,\, 0) = 0, \tag{7}$$

where $erf\sqrt{\cdot}$ denotes the error function that describes the area under the Gaussian curve for the time interval $(0, \sqrt{t\epsilon}),\ t > 0$. Physically, the heat flux function here indicates that for any fixed value of time $t > 0$, the strength of energy received at the initial end $x = 0$ of the rod must be strictly less than the strength $\varOmega > 0$ and approaches to zero over a long period, or how far the heat flux could be provided from the initial end of the rod. The constants $\omega,\ \varepsilon$ are the positive real numbers. The terms k_1, k_2 denotes the heat transfer coefficients.

The traction-free boundary conditions are given below:

$$\sigma_{xx}(x,\, t) = 0|_{x=0,\, x=L}. \tag{8}$$

Apart from these boundary conditions, it is presumed that all the initial conditions are homogeneous.

Equations (5) - (8) describe the heat conduction model proposed in this manuscript.

3. The Solution

In the view of zero initial conditions, application of the Liang et al. (2013) theorem to the governing dimensionless equations $(1) - (4)$ converts these in the Laplace domain as

$$\begin{aligned} &\left[p + p^{\nu+1}\frac{\tau_1^{\nu}}{\Gamma(\nu+1)} + p^{2\nu+1}\frac{\tau_1^{2\nu}}{\Gamma(2\nu+1)}\right](\bar{\Phi} + \epsilon \bar{e}) \\ &= \left[1 + p^{\nu}\frac{\tau_2^{\nu}}{\Gamma(\nu+1)} + p^{2\nu}\frac{\tau_2^{2\nu}}{\Gamma(2\nu+1)}\right]\triangledown^2 \bar{\Theta}, \end{aligned} \tag{9}$$

$$\bar{\Theta} - \bar{\Phi} = \xi \nabla^2 \bar{\Theta}, \tag{10}$$

$$\frac{d\bar{e}}{dx} - \frac{d\bar{\Theta}}{dx} = p^2 \bar{u}, \tag{11}$$

$$\bar{\sigma}_{xx} = \bar{e} - \bar{\Theta}. \tag{12}$$

Assuming the following replacements in Equation (9) for simplicity,

$$\alpha_1 = \left[p + p^{\nu+1} \frac{\tau_1^{\nu}}{\Gamma(\nu+1)} + p^{2\nu+1} \frac{\tau_1^{2\nu}}{\Gamma(2\nu+1)} \right], \tag{13}$$

$$\alpha_2 = \left[1 + p^{\nu} \frac{\tau_2^{\nu}}{\Gamma(\nu+1)} + p^{2\nu} \frac{\tau_2^{2\nu}}{\Gamma(2\nu+1)} \right], \tag{14}$$

using the above replacements, Equation (9) reduces to

$$\alpha_1(\bar{\Phi} + \epsilon \bar{e}) = \alpha_2 \nabla^2 \bar{\Theta}. \tag{15}$$

differentiating Equation (11) with respect to variable x, one gets

$$\nabla^2(\bar{e} - \bar{\Theta}) = p^2 \bar{e}. \tag{16}$$

Transposing $\bar{e}$, one gets

$$(\nabla^2 - p^2)\bar{e} = \nabla^2 \Theta. \tag{17}$$

Expressing $\bar{\Phi}$ from Equation (10) in terms of $\bar{\Theta}$, one gets

$$\bar{\Phi} = (1 - \xi \nabla^2)\bar{\Theta}. \tag{18}$$

Eliminating $\bar{\Theta}$ and $\bar{\Phi}$ from Equations $(15) - (18)$, one gets the following equation in $\bar{e}$ as

$$\{\nabla^4(\alpha_2 + \alpha_1 \xi) - \nabla^2(p^2\alpha_2 + \alpha_1 + \alpha_1 \xi p^2 + \alpha_1 \epsilon) + p^2 \alpha_1\}\bar{e} = 0. \tag{19}$$

Equation (19) can be factorized in ∇^2 as

$$(\nabla^2 - g_1^2)(\nabla^2 - g_2^2)\bar{e} = 0, \tag{20}$$

where g_1^2 and g_2^2 are the real characteristic values of Equation (20) given below:

$$g_1^2,\, g_2^2 = \frac{(p^2\alpha_2 + \alpha_1 + \alpha_1 \xi p^2 + \alpha_1 \epsilon) \pm \sqrt{((p^2\alpha_2 + \alpha_1 + \alpha_1 \xi p^2 + \alpha_1 \epsilon)^2 - 4p^2 \alpha_1)}}{2(\alpha_2 + \alpha_1 \xi)}. \tag{21}$$

The boundary conditions in the transformed domain have been expressed by following equations as

$$\frac{d\bar{\Theta}}{dx} + k_1 \bar{\Theta} = \frac{\Omega \sqrt{\varepsilon}}{(p+\omega)\sqrt{(p+\omega+\varepsilon)}}|_{x=0}, \tag{22}$$

$$\frac{d\bar{\Theta}}{dx} + k_2 \bar{\Theta} = 0|_{x=L}, \tag{23}$$

$$\bar{\sigma}_{xx}(x,\, p) = 0|_{x=0,x=L}. \tag{24}$$

Equations $(9) - (24)$ describes the governing equations of the problem in the Laplace domain.

3.1. The analytical solutions in the Laplace domain

The generalized solution of Equation (20) is given by

$$\bar{e}(x,\, p) = \sum_{i=1}^{2} A_i(p) g_i^2 \cosh g_i x. \tag{25}$$

Substituting $\bar{e}(x,\, p)$ into Equation (17), one gets the conductive temperature $\Theta(x,\, p)$ given by

$$\bar{\Theta}(x,\, p) = \sum_{i=1}^{2} A_i(p)(g_i^2 - p^2) \cosh g_i x. \tag{26}$$

Replacing $\Theta(\bar{x},\, p)$ from Equation (26) to (18), one gets the thermomechanical temperature $\bar{\Phi}(x,\, p)$ given by

$$\bar{\Phi}(x,\, p) = \sum_{i=1}^{2} A_i(p)(1 - \xi g_i^2)(g_i^2 - p^2) \cosh g_i x. \tag{27}$$

Employing $\bar{e}(x,\, p)$ and $\bar{\Theta}(x,\, p)$, obtained in Equations $(25) - (26)$ to Equation (11), one gets the displacement function $\bar{u}(x,\, p)$ as

$$\bar{u}(x,\, p) = \sum_{i=1}^{2} A_i(p) g_i \sinh g_i x. \tag{28}$$

Substituting results of dilatation and conductive temperature obtained in the Laplace domain to Equation (12), the thermal stress function $\bar{\sigma}_{xx}(x,\, p)$ is obtained as

$$\bar{\sigma}_{xx}(x,\, p) = \sum_{i=1}^{2} A_i(p) p^2 \cosh g_i x. \tag{29}$$

The constants A_1 and A_2 could be obtained through the application of the thermal boundary conditions expressed by couple of simultaneous equations given below:

$$A_1 k_1 (g_1^2 - p^2) + A_2 k_1 (g_2^2 - p^2)] = \frac{\Omega\sqrt{\varepsilon}}{(p+\omega)\sqrt{(p+\omega+\varepsilon)}}, \tag{30}$$

$$\begin{aligned} &A_1[k_2(g_1^2 - p^2)\cosh g_1 L + g_1(g_1^2 - p^2)\sinh g_1 L] \\ &+ A_2[k_2(g_2^2 - p^2)\cosh g_2 L + g_2(g_2^2 - p^2)\sinh g_2 L] = 0, \end{aligned} \tag{31}$$

$$A_1 = -\frac{\Omega\sqrt{\varepsilon}(\cosh g_2 L + k_2 g_2 \sinh g_2 L)}{k_1(g_1^2 - p^2)\Lambda}, \tag{32}$$

$$A_2 = \frac{\Omega\sqrt{\varepsilon}(\cosh g_1 L + k_2 g_1 \sinh g_1 L)}{k_1(g_2^2 - p^2)\Lambda}, \tag{33}$$

where the term Λ is considered as follows:

$$\Lambda = [k_2(\cosh g_2 L - \cosh g_1 L) + (g_2 \sinh g_2 L - g_1 \sinh g_1 L)](p+\omega)\sqrt{(p+\omega+\varepsilon)}\,. \tag{34}$$

The analytical results for desired thermal unknowns in the Laplace domain along with the unknown constants A_1 and A_2 have been shown by Equations $(25) - (34)$.

3.2. Transformation of the analytical results in the time domain

The inversion of analytical results obtained here in the manuscript presented by the Equations (25)-(34) are computed through Gaver (1966) and Stehfest (1970) numerical algorithm. The finite trial value of algorithm parameter J has been considered to implement the algorithm through MATLAB programming. Considering the fixed Aluminium metal properties, the starting value for the Laplace parameter p has been computed by taking fixed time value $t = 0.25s$ to obtain the initial values of unknown thermal parameters. The above computation procedure has been repeatedly applied to add on the values of the desired time-domain solutions for all values of assumed parameter ranging from 0 to 8. Hereby, the required thermal results have been computed for the fixed time domain. Following Kuznetsov (2013), authors have noticed that for parameter values of $J > 8$, numerically inverted time-domain solutions of unknown thermal values are bounded by fixed real values.

4. Numerical Computations

Following Ignaczak and Ostoja-Starzewki (2010), the unknown thermal parameters have been computed pair wise for phase-lag and fractional order variations. In case of phase-lag variations the numerical computations have been done by considering $\nu = 0,\ 0.5,\ 0.75$ and $\nu = 1.0$. However, for the later case subjected to fixed values of phase-lags $\tau_1 = 0.8ps,\ \tau_2 = 0.4ps$ the fractional order changes were assumed as $\nu = 0,\ 0.5,\ 0.95$ and $\nu = 1.0$. Moreover, for both the cases are determined for the fixed time $t = 0.25s$ to recover the classical, fractional and generalized thermoelasticity theories.

4.1. Material properties and dimension

To study the rate of heat flow and thermal variations within the uniform rod, as a special case aluminium material with following well-known properties referring Hamilton et al. (2008) has been considered.

$$\alpha_t = 6.9 \times 10^{-5}\ K^{-1}, \qquad \kappa = 167\ Wm^{-1}K^{-1}, \qquad \lambda = 5.20 \times 10^{10} Nm^{-2},$$

$$\epsilon = 0.0709 NmJ^{-1}, \qquad \mu = 2.60 \times 10^{10} Nm^{-2}, \qquad \rho_m = 8965\ kgm^{-3},$$

$$c_l = 4.158 \times 10^3 ms^{-1}, \qquad c_m = 896\, JKg^{-1}K^{-1}, \qquad b = 0.075,\ \theta_0 = 293K,$$

$$L = 10m\ \omega = 20; \qquad \Omega = 400K, \qquad \varepsilon = 0.08$$

5. Results and Discussion

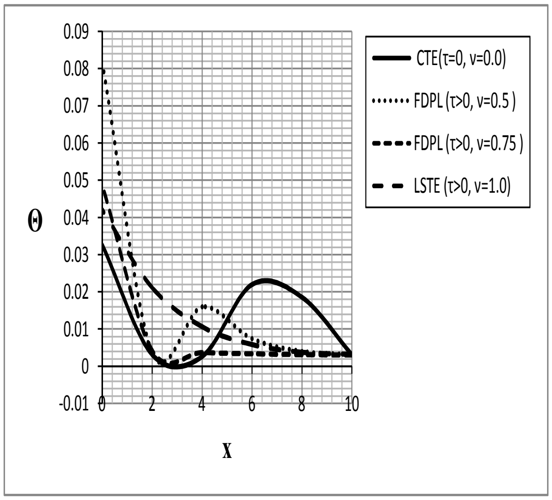

Figure 1. Conductive temperature $\Theta(x,\, t)$ within rod for phase-lag variations

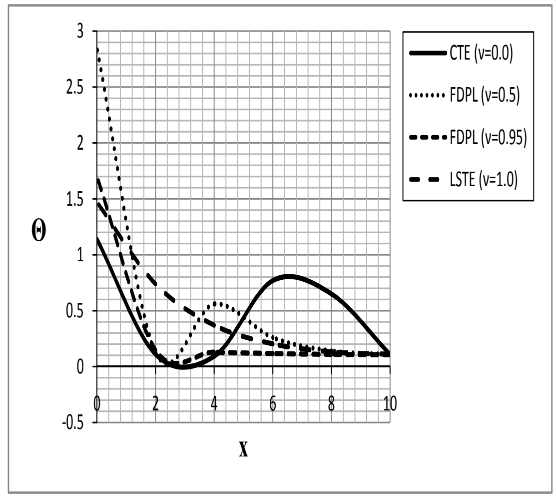

Figure 2. Conductive temperature $\Theta(x,\, t)$ within rod for fractional order ν

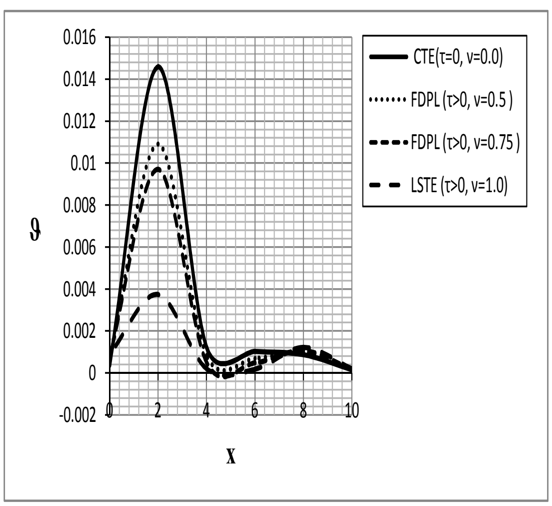

Figure 3. Thermodynamical temperature $\vartheta(x,\, t)$, within rod for phase lag variations

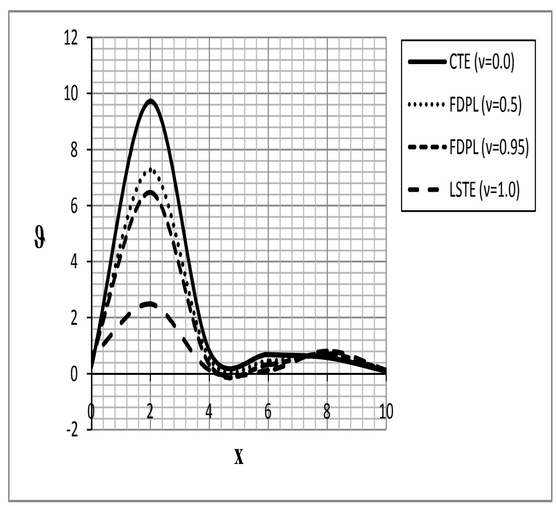

Figure 4. Thermodynamical temperature $\vartheta(x,\, t)$, within rod for fractional order ν

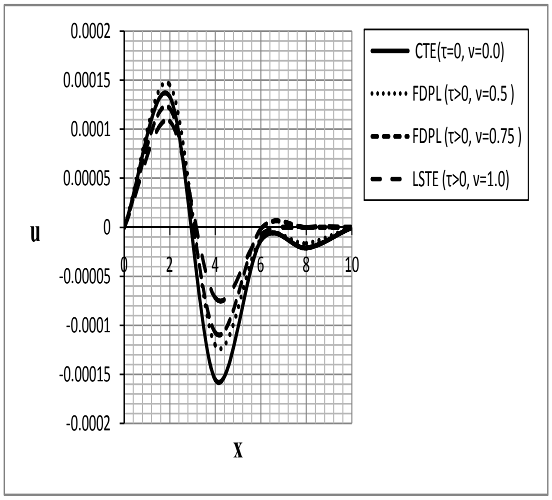

Figure 5. Thermal displacement $u(x,\ t)$, within rod for phase-lag variations

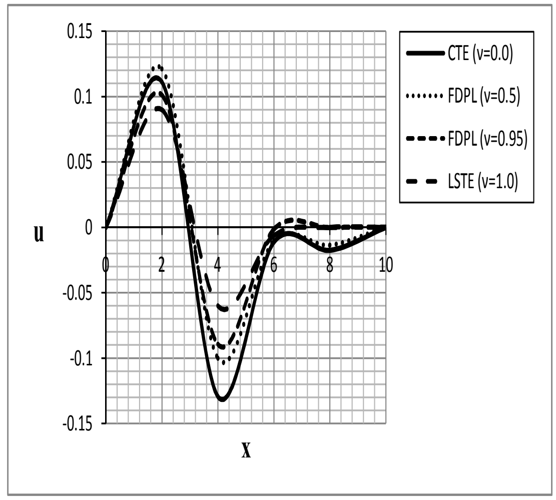

Figure 6. Thermal displacement $u(x,\ t)$, within rod for fractional order ν

Figures 1 and 2 illustrate the conductive temperature $\Theta(x,\ t)$, computed for several combinations of $\tau_1,\ \tau_2$ and ν, respectively. It could be seen collectively that the conductive temperature variations are comparatively high for the fractionally ordered DPL model rather than the Biot's classical (1956) and Lord-Shulman (1967) thermoelastic case. It has been observed in Figure 2 that the conductive temperature values are maximum for fractional-order $\nu = 0.5$, and minimum for $\nu = 0.0$, whenever $0 \leq x \leq 2$. However, the thermal wavefront for CTE is dominating while $4 < x < 10$, and graphically it has been found that results are similar to phase-lag comparisons with comparatively high values considering various fractional orders under consideration. One may also verify that as and when heat equation approaches to Lord-Shulman (1967) theory, then the heat signals traverse with finite speed given by $\sqrt{\dfrac{\Gamma(\nu+1)}{\zeta\tau_1^{\nu}}}$.

Figures 3 and 4 show the thermodynamical temperature $\vartheta(x,\ t)$ gradually increases in the range of $x \in [0,\ 2]$. Hereafter, it reduces continuously from $x \in (2,\ 10)$ for various phase-lags τ_1, τ_2. The temperature variations obtained subjected to the coupled, generalized and fractional theory of thermoelasticity are found to be significantly different within the range $0 \leq x \leq 4$. Moreover, one can note that the thermal fluctuations shown here appears inversely related to the applied fractional order. It has been noted that the thermodynamical temperature computed for different phase-lag variations are lower in magnitude but showing similar pattern to those of obtained for the fractional orders under consideration.

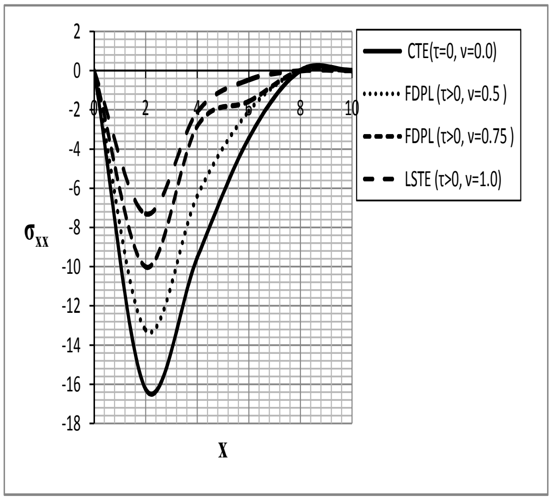

Figure 7. Thermal stress $\sigma_{xx}(x,\, t)$, within rod for phase-lag variations

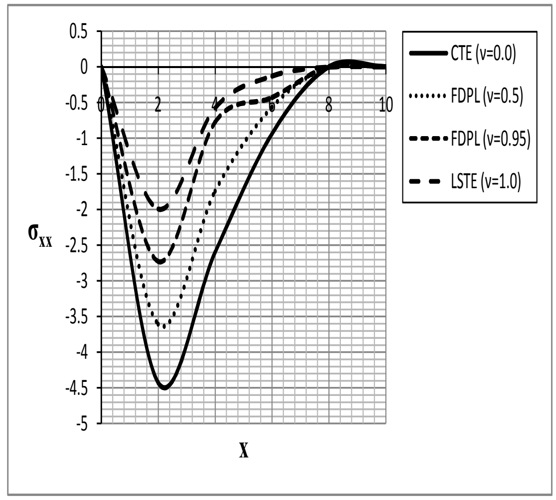

Figure 8. Thermal stress $\sigma_{xx}(x,\, t)$, within rod for fractional order ν

Figures 5 and 6, exhibit the thermal displacement $u(x,\, t)$ that begins from rest is seen to have expansive behaviour around $0 < x < 3$, then compressive along $3 < x < 6$, and finally approaches to zero at the outer end $x = 10$ of the rod for different phase-lags and fractional-order variations under study.

Figures 7 and 8 describe the thermal stress $\sigma_{xx}(x,\, t)$ along the length of the rod. It has been found that the stress variation satisfies the condition of traction free ends. Initially, the stress variations are compressive and attain minima at $x = 2$. Later, it keeps on increasing and finally rests at the other end for a couple of cases considered in this manuscript. Mathematically, connecting stress variations to fractional orders, one can claim that the thermal stress is precisely comparable with the applied fractional-order ν.

6. Conclusion

The main outcomes of the fractional-order dual-phase-lag heat conduction model presented in this manuscript are as follows.

The presented heat conduction model designed in the framework of the two-temperature relationship between thermodynamical and conductive temperature is well-defined and fulfills the prescribed stability conditions for a generalized dual-phase-lag model.

The results for thermal variations of two distinct temperatures, thermal displacement and thermal stress have been computed and satisfying the imposed physical restrictions for several fractional-order and phase-lags values. Thermal parameters computed corresponding to hyperbolic and parabolic cases of fractional DPL heat conduction equation have been illustrated by Figures 1-8.

Whenever the time-fractional order approaches unity and $\tau_1 > \tau_2$, then the resulting fractional-order dual-phase-lag heat conduction equation derived for two temperatures becomes hyperbolic. Consequently, the thermal signals move in the respective conductive medium with finite speed.

Examining the derived two-temperature dual-phase-lag heat conduction equation for various phase-lags and distinct fractional orders, it could be concluded that the presented model is compatible with the existing theories of coupled thermoelasticity, and hence, could be used scientifically for various physical heat transport problems. For example, the time-fractional order and applied phase-lags terms can be used to differentiate various semiconductors and composite materials according to their heat transport capacity.

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Nomenclature

σ_{ij} (Pa)	:	Thermal stresses;	ν	:	Fractional order;
Θ	:	Non-dimensional conductive temperature;	α_t (K^{-1})	:	Coefficient of linear thermal expansion;
T (K)	:	Conductive temperature;	b	:	Temperature discrepancy;
T_0 (K)	:	Reference temperature;	ζ $(m^2 \cdot s)$	:	Reciprocal of thermal diffusivity;
τ_1, τ_2 (ps)	:	Phase-Lags;			
c_l (m/s)	:	Speed of iso-thermal elastic wave;	ϵ	:	Dimensionless coupling constant;
c_m $(J \cdot K^{-1} \cdot kg^{-1})$	:	Specific heat capacity;	k $(W/(m \cdot K))$	:	Thermal conductivity;
			λ, μ (Pa)	:	Lamé constants ;
e	:	Cubical dilatation;	$\gamma = \alpha_t(3\lambda + 2\mu)$ $(Pa \cdot K^{-1})$	:	Material constant;
u (m)	:	Thermal Displacement;			
t (s)	:	Time;			
∇^2	:	The Laplacian operator;	φ (K)	:	Thermodynamical temperature;
p	:	The Laplace parameter;			
x	:	The Cartesian coordinate;	Φ	:	Non-dimensional thermodynamical temperature.
ρ_m $(Kg \cdot m^{-3})$	:	Material density;			