Applications and Applied Mathematics: An International Journal (AAM)

Volume 15
Issue 3 *Special Issue 6 (April 2020)*

Article 8

4-2020

Ruscheweyh-Goyal Derivative of Fractional Order, its Properties Pertaining to Pre-starlike Type Functions and Applications

Ritu Agarwal
Malaviya National Institute of Technology

G. S. Paliwal
JECRC University

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Special Issue No. 6 (April 2020), pp. 103 – 121

Ruscheweyh-Goyal Derivative of Fractional Order, its Properties Pertaining to Pre-starlike Type Functions and Applications

[1]Ritu Agarwal and [2,3,*]G. S. Paliwal

[1]Department of Mathematics
Malaviya National Institute of Technology
Jaipur-302017, Rajasthan, INDIA
ragarwal.maths@mnit.ac.in

[2]Department of Mathematics
JECRC University
Jaipur-303905, Rajasthan, INDIA
gaurishankar.paliwal@jecrcu.edu.in

[3]Department of Mathematics
JK Lakshmipat University
Jaipur-302026, Rajasthan, INDIA
gaurishankarpaliwal@gmail.com

*Corresponding Author

Received: February 29, 2020; Accepted: March 15, 2020

Abstract

The study of the operators possessing convolution form and their properties is considered advantageous in geometric function theory. In 1975 Ruscheweyh defined operator for analytic functions using the technique of convolution. In 2005, Goyal and Goyal generalized the Ruscheweyh operator to fractional order (which we call here Ruscheweyh-Goyal differential operator) using Srivastava-Saigo fractional differential operator involving hypergeometric function. Inspired by these earlier efforts, we discuss the properties of the Ruscheweyh-Goyal derivative of arbitrary order. We define a class of pre-starlike type functions involving the Ruscheweyh-Goyal fractional derivative and obtain the inclusion relation. Further, we prove that Ruscheweyh-Goyal derivative operator preserve the convexity and starlikeness for an analytic function. The majorization results for fractional Ruscheweyh-Goyal derivative has been discussed using a newly defined subclass.

Keywords: Starlike functions; Convex functions; Subordination; Majorization; Fractional derivative; Ruscheweyh derivative; Pre-starlike functions

MSC 2010 No.: Primary: 26A33, Secondary: 44A10

1. Introduction

In a paper "New Criteria for Univalent Functions, 1975" that appeared in the Proceedings of American Mathematical Society, Ruscheweyh (1975) defined a function class and showed that the functions of this class are univalent in open unit disc. For various values of parameter, this class reduced to the function class of starlike functions of order $1/2$ and class of convex functions. This result is an addition of the result given by Strohäcker in 1933 (see Gupta and Jain (1976)). Ruscheweyh (1975) proved many important properties for this class.

In their papers, Al Amiri (1980) and Kumar and Shukla (1984) mentioned the operator D^m as Ruscheweyh derivatives for the first time, which lead to many other useful results. Owa (1985a) (see also Owa (1985b)) generalized the Ruscheweyh derivative to arbitrary order β. Ruscheweyh (1975) considered the subclasses of R_β, β being a non-negative integer. Pre-starlike functions were studied by Ruscheweyh (1977) and Suffridge (1976), in different parametrization. Note that $R_{\beta+1} \subset R_\beta$ as proved by Ruscheweyh. Also, R_0 is known class of univalent and starlike functions which are of order $1/2$ and R_1 is the class of univalent and convex functions (see Al-Amiri (1979)). Ruscheweyh derivative was generalized and studied by many authors, Goel and Sohi in 1980, Owa in 1985, and others.

The strength of using convolution operators lies in their ability to unify a number of diverse results. The proofs using convolution operators are clearer and more concise and point out how the unifying linear structure that is common to so many of the problems can be used to solve them via convolution operator techniques (refer to Barnard and Kellogg (1978)). Ruscheweyh derivative and its various generalizations have been defined using convolution. Inspired by the work of Ruscheweyh, in this paper, we have studied various properties of the fractional generalization of Ruscheweyh derivative involving Saigo operator defined by Goyal and Goyal (2005).

2. Preliminaries

The functions f such that

$$f(z) = z + \sum_{s=j}^{\infty} a_s z^s, \quad j \in \mathbb{N} = \{1, 2, 3, ...\}, \tag{1}$$

which are analytic in the disk of unit radius $\Delta = \{z : \ z \in \mathbb{C}, \ |z| < 1\}$. The class of such functions is denoted by A. Thus, f is univalent and analytic in Δ.

A function $f \in S(\delta)$ is called *univalent starlike of order* δ. The function $f \in A$ is said to be in class $S(\delta)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta, \quad z \in \Delta, \tag{2}$$

for some δ $(0 \leq \delta < 1)$.

A function $f \in$ A is said to be in the class of univalent convex functions $K(\delta)$ of order δ, if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right) > \delta, \quad z \in \Delta,\, 0 \le \delta < 1. \tag{3}$$

It can be easily shown that

$$f \in K(\delta) \Leftrightarrow zf' \in S(\delta), \quad (0 \le \delta < 1). \tag{4}$$

Let f is analytic univalent function (1) and g is also analytic univalent function given as:

$$g(z) = z + \sum_{s=j}^{\infty} b_s z^s, \quad j \in \mathbb{N}. \tag{5}$$

Then, the Hadamard product $f * g$ of the functions f and g is given by

$$(f * g)(z) = z + \sum_{s=j}^{\infty} a_s b_s z^s = (g * f)(z). \tag{6}$$

The Hadamard product also known as convolution.

Convolution is a binary operation in the set A of analytic functions. The identity element l under convolution is the geometric series

$$l(z) = \frac{1}{1-z} = \sum_{0}^{\infty} z^n, \tag{7}$$

since $f * l = f$ for $f \in A$. For details, one can refer Robertson (1962).

2.1. Ruscheweyh derivative

Here we recall the origin and definition of the Ruscheweyh derivatives. Ruscheweyh (1975) defined a function class K_m as $f \in K_m$, if

$$\Re\frac{(z^m f)^{(m+1)}}{(z^{m-1} f)^{(m)}} > \frac{m+1}{2}, z \in \Delta,\; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{8}$$

and showed that f is univalent in Δ.

Clearly, $K_0 = S^*_{1/2}$, the function class of starlike functions of order $1/2$ and $K_1 = K$, the function class of convex functions. This result is an addition of the result $K \subset S^*_{1/2}$ given by Strohäcker in 1933 (see Gupta and Jain (1976)).

Ruscheweyh (1975) proved the following important property of the class K_n.

Theorem 2.1.

$K_{m+1} \subset K_m$ holds for $m \in N_0 = \mathbb{N} \cup \{0\}$.

One more significant relation between K and $S^*_{1/2}$ is $f \in K \Leftrightarrow z\sqrt{f'(z)} \in S^*_{1/2}$ is also extended to K_m.

Owa, Fukui et al. (1986) proved the following properties of Ruscheweyh derivative $D^\alpha f(z)$ of a function f.

Theorem 2.2.

Let $f \in A$ be the element of the class S^* of starlike functions and the condition $D^\alpha f(z) \neq 0$, $0 < |z| < 1$ for $\alpha \geq -1$, is satisfied by the function $f(z)$. Then, $D^\alpha f(z)$ is also in the class S^*.

Theorem 2.3.

Let $f \in A$ be element of the class K of convex functions and the condition $D^\alpha(zf'(z)) \neq 0$, $0 < |z| < 1$ for $\alpha \geq -1$, satisfied by the function $f(z)$. Then, $D^\alpha f(z)$ is also in K.

Writing

$$D^m f \equiv \frac{z(z^{m-1}f)^{(m)}}{m!}, \tag{9}$$

in terms of Hadamard product, (9) can be expressed as

$$D^m f \equiv \frac{z}{(1-z)^{m+1}} * f, \quad m \in \mathbb{N}. \tag{10}$$

With this notation, the various subclasses of class A of analytic functions can be written as

$$f \in S^*_{1/2} \Leftrightarrow \Re\left(\frac{D^1 f}{D^0 f}\right) > 1/2, \quad z \in \Delta, \tag{11}$$

and

$$f \in K \Leftrightarrow \Re\left(\frac{D^2 f}{D^1 f}\right) > 1/2, \quad z \in \Delta. \tag{12}$$

In general,

$$f \in K_m \Leftrightarrow \Re\left(\frac{D^{m+1} f}{D^m f}\right) > 1/2, \quad z \in \Delta. \tag{13}$$

In this notation, K_{-1} can also be defined and it contains the functions $f \in A$ with $\Re(f(z)/z) > 1/2$, $z \in \Delta$ as $D^{-1}f(z) = z$ from (10).

Owa (1985a) (see also Owa (1985b)) generalized the Ruscheweyh derivative to arbitrary order β as

$$D^{\beta} f \equiv \frac{z}{(1-z)^{\beta+1}} * f, \quad \beta > -1. \tag{14}$$

Definition 2.4.

The subclass of function class $f \in A$ of analytic functions, is called pre-starlike class of order β, if and only if

$$\Re\left(\frac{D^{\beta+1} f(z)}{D^{\beta} f(z)}\right) > \frac{1}{2},\ \beta \geq -1,$$

where, $D^{\beta} f(z) = \frac{z}{(1-z)^{\beta+1}} * f(z)$. The collection of **pre-starlike** functions of order β is represented by R_{β} (see Al-Amiri (1979)).

The extremal function (see, e.g. Owa and Uralegaddi (1984)) for the class of the pre-starlike functions is

$$S_{\beta}(z) = \frac{z}{(1-z)^{2(1-\beta)}}. \tag{15}$$

$S_{\beta}(z)$ can also be written as,

$$S_{\beta}(z) = z + \sum_{s=2}^{\infty} C(\beta, s) z^{s}, \tag{16}$$

where

$$C(\beta, s) = \frac{\prod_{k=2}^{s}(k-2\beta)}{(s-1)!}, \quad (s = 2, 3, 4, ...). \tag{17}$$

Clearly, the coefficient $C(\beta, s)$ is decreasing in β and satisfies

$$\lim_{s \to \infty} C(\beta, s) = \begin{cases} \infty, & (\beta < 1/2), \\ 1, & (\beta = 1/2), \\ 0, & (\beta > 1/2). \end{cases} \tag{18}$$

Singh and Singh (1979) defined the class $R_m \subset K_m$ (the class in (13) studied by Ruscheweyh) as,

$$f \in R_m \Leftrightarrow \Re\left(\frac{D^{m+1} f}{D^{m} f}\right) > \frac{m}{m+1}, \quad z \in \Delta. \tag{19}$$

3. Ruscheweyh-Goyal derivative operator

In this section, we discuss fractional generalization of Ruscheweyh derivative which was introduced by Goyal and Goyal (2005). The fractional operators have gained much attention by many authors these days because of its increasing applications (e.g., see Singh, Kumar, and Baleanu (2019), Goswami et al. (2019), and Singh, Kumar, Baleanu et al. (2019)).

Fractional Ruscheweyh−Goyal derivative operator involving Saigo fractional differential operator (see Owa, Saigo et al. (1989)), established by Goyal and Goyal (2005) (see also, Agarwal and Paliwal (2014), Parihar and Agarwal (2011)), is described as follows.

Definition 3.1. (Ruscheweyh-Goyal derivative operator)

Let $f \in A$ be an analytic univalent function defined on unit disk Δ.

$$\mathbb{J}^{\lambda,\mu} f(z) := \frac{\Gamma(\mu-\lambda+\eta+2)}{\Gamma(\mu+1)\Gamma(\eta+2)} z J_{0,z}^{\lambda,\mu,\eta}(z^{\mu-1}f(z)) = z + \sum_{s=n}^{\infty} a_s B^{\lambda,\mu}(s) z^s, \tag{20}$$

where

$$B^{\lambda,\mu}(s) := \frac{\Gamma(s+\mu)\Gamma(s+\eta+1)\Gamma(\eta+2+\mu-\lambda)}{\Gamma(1+\mu)\Gamma(s)\Gamma(\eta+2)\Gamma(s+\eta+1+\mu-\lambda)},$$

where Saigo fractional derivative of order λ (for details, see Owa, Saigo et al. (1989), Owa and Srivatsava (1990) [Eq. 1.15]) of the function $f(z)$ is defined as:

$$J_{0,z}^{\lambda,\mu,\eta} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z (z-\zeta)^{-\lambda} {}_2F_1\left(\mu-\lambda, -\eta; 1-\lambda; 1-\frac{\zeta}{z}\right) f(\zeta) d\zeta \right\}, \\ (1 > \lambda \geq 0) \\ \frac{d^m}{dz^m} J_{0,z}^{\lambda-m,\mu,\eta} f(z), \qquad (m \leq \lambda < m+1,\ m \in N) \end{cases} \tag{21}$$

$f(z) = O(|z|^s), (z \to 0, \max\{0, \mu-\eta-1\} - 1 < s)$.

Taking $\mu = \lambda$ in the above definition, it follows that

$$J_{0,z}^{\lambda,\lambda,\eta} f(z) := D_z^{\lambda} f(z), (1 > \lambda \geq 0).$$

Hence, for $\mu = \lambda$, the Ruscheweyh-Goyal derivative $\mathbb{J}^{\lambda,\mu}$ reduces into the Ruscheweyh derivative D^{λ}.

Furthermore, the fractional derivative of power function, in terms of Gamma function, is given by

$$J_{0,z}^{\lambda,\mu,\eta} z^{\nu} := \frac{\Gamma(\nu+1)\Gamma(\nu-\mu+\eta+2)}{\Gamma(\nu-\mu+1)\Gamma(\nu-\lambda+\eta+2)} z^{\nu-\mu}, \quad 1 > \lambda \geq 0;\ \nu > (\max\{0, \mu-\eta-1\} - 1). \tag{22}$$

The recurrence relation for the operator $J_{0,z}^{\lambda,\mu,\eta}$ is

$$z(J_{0,z}^{\lambda,\mu,\eta} f(z))' = J_{0,z}^{\lambda+1,\mu,\eta} f(z) - (\mu-\lambda+\eta+1) J_{0,z}^{\lambda,\mu,\eta} f(z), \tag{23}$$

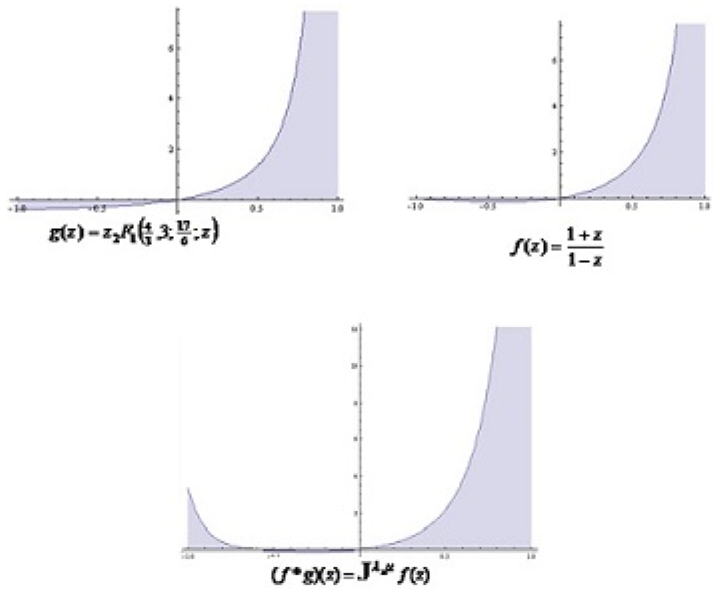

Figure 1. Graph illustrating the application of Ruscheweyh-Goyal derivative on $f(z) = \frac{1+z}{1-z}$

$(1 > \lambda \geq 0;\ \nu > \max\{0, \mu - \eta - 1\} - 1)$.

The convolution form of the Ruscheweyh-Goyal derivative, defined in (20), of an analytic function $f(z)$ of the form (1), can be characterized as:

$$\mathbb{J}^{\lambda,\mu} f(z) = z \cdot\ {}_2F_1(\mu + 1, \eta + 2; \eta + 2 + \mu - \lambda; z) * f(z). \tag{24}$$

The trailing result is about the recurrence relation for the operator defined in (20).

Theorem 3.2.

The recurrence relation for the fractional Ruscheweyh-Goyal derivative operator $\mathbb{J}^{\lambda,\mu} f(z)$ defined in (20) is given by

$$z\left[\mathbb{J}^{\lambda,\mu} f(z)\right]' = (\mu - \lambda + \eta + 1)\mathbb{J}^{\lambda+1,\mu} f(z) - (\mu - \lambda + \eta)\mathbb{J}^{\lambda,\mu} f(z). \tag{25}$$

Proof:

Differentiating (20) with respect to z, we get

$$z\{\mathbb{J}^{\lambda,\mu} f(z)\}' = \frac{\Gamma(\mu - \lambda + \eta + 2)}{\Gamma(\mu + 1)\Gamma(\eta + 2)}\{z J_{0,z}^{\lambda,\mu,\eta}(z^{\mu-1} f(z)) + z^2[J_{0,z}^{\lambda,\mu,\eta}(z^{\mu-1} f(z))]'\}. \tag{26}$$

Using the relation (23) therein, we obtain

$$
\begin{aligned}
z\{\mathbb{J}^{\lambda,\mu}f(z)\}' =& \frac{\Gamma(\mu-\lambda+\eta+2)}{\Gamma(\mu+1)\Gamma(\eta+2)}\left\{zJ_{0,z}^{\lambda,\mu,\eta}(z^{\mu-1}f(z))\right. \\
&\left.+z[J_{0,z}^{\lambda+1,\mu,\eta}(z^{\mu-1}f(z))-(\mu-\lambda+\eta+1)J_{0,z}^{\lambda,\mu,\eta}(z^{\mu-1}f(z))]\right\}.
\end{aligned} \tag{27}
$$

Rewriting the above result in terms of fractional Ruscheweyh-Goyal operator (20), we obtain the relation (25). ■

For $\mu=\lambda$, we get the corresponding result for the Ruscheweyh derivative.

4. Class $S^{\lambda,\mu}$ of pre-starlike type functions

Authors have defined various classes using fractional Ruscheweyh-Goyal derivative operator and studied their geometric properties (see Agarwal and Paliwal (2014), Agarwal and Paliwal (2015), Agarwal et al. (2019), Goyal and Goyal (2005)). Motivated by the work of Ruscheweyh (1975), in the current section, a class of pre-starlike type functions involving fractional Ruscheweyh-Goyal derivative is introduced and the inclusion relation for this class is obtained.

Definition 4.1.

The function $f \in A$, which is analytic, is belongs to the class $S^{\lambda,\mu}$ if the undermentioned relation is fulfilled:

$$
\Re\left[\frac{\mathbb{J}^{\lambda+1,\mu}f(z)}{\mathbb{J}^{\lambda,\mu}f(z)}\right] > \frac{1}{2}, \;\; z\in\Delta,
$$

where $\lambda,\ \mu\in\mathbb{R}$ and $0\leq\lambda<1$.

To prove our results, the following lemma by Jack (1971) is needed.

Lemma 4.2.

If w is regular in $|z|<R,\ w(0)=0$ and w be non-constant, then, $|w|$ reaches its threshold value on the circle $|z|=\rho<R,\ 0\leq\rho<1$ at a point z_0. Thus, $\frac{z_0w'(z_0)}{w(z_0)}\geq 1$.

Further, the following result given by Ruscheweyh and Sheil-Small (1973) is needed as well.

Lemma 4.3.

Let $\phi(z)$ and $g(z)$ are analytic in the disk of unit radius Δ and satisfy $g(0)=0=\phi(0)$, $g'(0)\neq 0$, $\phi'(0)\neq 0$. Assume that for each σ, $|\sigma|=1$ and δ, $|\delta|=1$, we have

$$
\phi(z)*\frac{1+\delta\sigma z}{1-\sigma z}g(z)\neq 0, \quad 0<|z|<1. \tag{28}
$$

Then, for $\Re\{F(z)\} > 0,\ z \in \Delta$ where $F(z)$ is analytic in the unit disk Δ,

$$\Re\frac{\phi(z) * G(z)}{\phi(z) * g(z)} > 0, \quad z \in \Delta, \quad G(z) = F(g(z)). \tag{29}$$

Theorem 4.4.

$S^{\lambda+1,\mu} \subset S^{\lambda,\mu}$ holds for, $\mu \in \mathbb{R},\ 0 \leq \lambda < 1$.

Proof:

Assume $f \in S^{\lambda+1,\mu} f(z)$.

Define $g(z) = z\left[\frac{\mathbb{J}^{\lambda,\mu}f(z)}{z}\right]^{\frac{2}{\mu-\lambda+\eta+1}}$ and $R = \sup\{\rho\,|g(z)| \neq 0,\ 0 < |z| < \rho\}$.

Thereupon, $g(z)$ is single valued and $u(z) = zg'(z)/g(z)$ is regular within the circle $|z| < R$.

Since

$$g(z) = z\left[\frac{\mathbb{J}^{\lambda,\mu}f(z)}{z}\right]^{\frac{2}{\mu-\lambda+\eta+1}},$$

differentiating it logarithmically wth respect to z, we attain

$$\frac{zg^{'}(z)}{g(z)} = 1 + \frac{2}{(\mu-\lambda+\eta+1)}\left[\frac{z\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{'}}{\mathbb{J}^{\lambda,\mu}f(z)} - 1\right].$$

The recurrence relation for fractional Ruscheweyh-Goyal derivative is

$$z\left[\mathbb{J}^{\lambda,\mu}f(z)\right]' = (\mu-\lambda+\eta+1)\mathbb{J}^{\lambda+1,\mu}f(z) - (\mu-\lambda+\eta)\mathbb{J}^{\lambda,\mu}f(z).$$

So that,

$$\frac{\mathbb{J}^{\lambda+1,\mu}f(z)}{\mathbb{J}^{\lambda,\mu}f(z)} - \frac{1}{2} = \frac{u}{2}. \tag{30}$$

Again, differentiating (30) logarithmically with respect to z and using above recurrence relation, we arrive at

$$\frac{\mathbb{J}^{\lambda+2,\mu}f(z)}{\mathbb{J}^{\lambda+1,\mu}f(z)} - \frac{1}{2} = \frac{1}{(\mu-\lambda+\eta)}\left[\frac{(\mu-\lambda+\eta+1)u}{2} + \frac{z\,u^{'}}{u+1} - \frac{1}{2}\right].$$

The condition

$$\Re\left[\frac{(\mu-\lambda+\eta+1)u}{2} + \frac{z\,u^{'}}{u+1}\right] > \frac{1}{2}, \quad |z| \in R,$$

implies $\Re(u) > 0, \ \ |z| < R$. In fact, let $u = \frac{1+w}{1-w}$, so that,

$$\frac{(\mu-\lambda+\eta+1)u}{2} + \frac{z\,u^{'}}{u+1} = \frac{w^{'}z}{1-w} + \frac{(\mu-\lambda+\eta+1)}{2}\frac{(1+w)}{(1-w)}, \tag{31}$$

$$\frac{(\mu-\lambda+\eta+1)u}{2} + \frac{z\,u^{'}}{u+1} = \frac{w^{'}z}{w}\frac{w}{(1-w)} + \frac{(\mu-\lambda+\eta+1)}{2}\frac{(1+w)}{(1-w)}.$$

If $\Re(u(z_0)) = 0$, $|z_0| < R$ and for $|z| \leq |z_0|$, $\Re(u(z)) \geq 0$ for a certain z_0, then, $|w(z)|$ $\leq |w(z_0)| = 1$, $w(z_0) \neq 1$.

Lemma 4.2 and Equation (31) give

$$\Re\left[\frac{z\,u^{'}(z_0)}{u(z_0)+1} + \frac{(\mu-\lambda+\eta+1)u(z_0)}{2}\right] = -\frac{1}{2}\frac{z_o w^{'}(z_0)}{w(z_0)} \leq -\frac{1}{2},$$

a falsity to our assumption. $\Re(u(z_0)) > 0$ implies the univalence of $g(z)$ in the circle $|z| < R$. Thus, $g(z)$ cannot elapse on $|z| = R < 1$ and we draw to close that $R = 1$. ■

Corollary 4.5.

For the functions $f \in A$, that satisfy the following condition:

$$\Re\left[\frac{\mathbb{J}^{\lambda+1,\mu}f(z)}{\mathbb{J}^{\lambda,\mu}f(z)}\right] > 1, \ \ z \in \Delta,$$

where $\lambda, \mu \in \mathbb{R}$, $0 \leq \lambda < 1$ and $\lambda = \mu$, the class $S^{\lambda,\mu}$ reduces to the class K_n studied by Ruscheweyh (1975).

Lemma 4.6.

The Ruscheweyh-Goyal derivative $\mathbb{J}^{\lambda,\mu}f$ of $f \in A$ defined by (1), satisfies the following relation:

$$z\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{'} = \mathbb{J}^{\lambda,\mu}\left(zf^{'}(z)\right). \tag{32}$$

Proof:

Since $f(z)$ is of the form (1), so that

$$zf^{'}(z) = z + \sum_{k=s+1}^{\infty} ka_k z^k, \ \ (s \in \mathbb{N} = \{1, 2, 3, ...\}),$$

thus,

$$\mathbb{J}^{\lambda,\mu}\left(zf^{'}(z)\right) = z + \sum_{k=s+1}^{\infty} ka_k B^{\lambda,\mu} z^k,$$

which is same as

$$z\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{'} = z + \sum_{k=s+1}^{\infty} k a_k B^{\lambda,\mu} z^k.$$

■

Theorem 4.7.

Let f is in the class S^* which is analytic and starlike and fulfill the requirement $\mathbb{J}^{\lambda,\mu}f(z) \neq 0, \ \ (0 < |z| < 1)$. Then, $\mathbb{J}^{\lambda,\mu}f(z)$ is also starlike.

Proof:

We can see that, by using Lemma 4.6

$$\frac{z\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{'}}{\mathbb{J}^{\lambda,\mu}f(z)} = \frac{\mathbb{J}^{\lambda,\mu}\left(zf^{'}(z)\right)}{\mathbb{J}^{\lambda,\mu}f(z)} = \frac{z\,{}_2F_1(\mu+1,\eta+2;\mu-\lambda+\eta+2;z) * (zf^{'}(z))}{z\,{}_2F_1(\mu+1,\eta+2;\mu-\lambda+\eta+2;z) * f(z)}.$$

Setting $\delta = -1, \ \phi(z) = z\,{}_2F_1(\mu+1,\eta+2;\mu-\lambda+\eta+2;z), \ F(z) = zf^{'}(z)/f(z)$ and $f(z) = g(z)$, and in Lemma 4.3, we acquire

$$\Re\left[\frac{z\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{'}}{\mathbb{J}^{\lambda,\mu}f(z)}\right] > 0, \quad (z \in \Delta),$$

which signify $\mathbb{J}^{\lambda,\mu}f(z) \in S^*$. ■

Theorem 4.8.

Let $f(z)$ is in the class K which is analytic convex function and meet the condition $\mathbb{J}^{\lambda,\mu}(zf^{'}(z)) \neq 0 \ \ (1 > |z| > 0)$. Then, $\mathbb{J}^{\lambda,\mu}f(z)$ is also in class K.

Proof:

As $f(z) \in K$ if and only if $zf^{'}(z) \in S^*$, above theorem derives $z\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{'} = \mathbb{J}^{\lambda,\mu}\left(zf^{'}(z)\right) \in S^*$. Hence we have $\mathbb{J}^{\lambda,\mu}f(z) \in K$. ■

5. Majorization results for fractional Ruscheweyh-Goyal derivative and the subclass $S_q^{\lambda,\mu}(A, B; \gamma)$

Motivated by the class $S_{\alpha,\beta}^{p,q}[A,\ B;\ \gamma]$ studied by Goyal et al. (2010), a subclass of analytic functions of complex order $\gamma \neq 0$, is introduced here.

Definition 5.1.

A function $g(z) \in A$ is belongs to the class $S_q^{\lambda,\mu}(A, B; \gamma)$ of complex order $\gamma \neq 0$ of univalent functions in Δ, if and only if

$$\left[1+\frac{1}{\gamma}\left\{\frac{z\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q+1)}}{\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)}}-1+q\right\}\right]\prec\frac{1+Az}{1+Bz}, \tag{33}$$

$(z\in\Delta,\,\gamma\in\mathbb{C}\setminus\{0\},\ -1\le B<A\le 1,\ 0\le\lambda<1,\ \mu>0,\ q\in N_0=N\cup\{0\}).$

For $\mu=\lambda=0$, the aforementioned class reduces to class $S^*(A,B;\gamma,q)$ introduced earlier by Polatoğlu et al. (2006). Further, for $q=0$, it changes in the class given by Polatoğlu and Özkan (2006), which easily get reduced to class given by Nasr and Aouf (1985), i.e., $S(\gamma)$ of starlike function of complex order $\gamma\neq 0$. For $\gamma=1-\alpha,\ 0\le\alpha<1$, the class $S(1-\alpha)=S^*(\alpha)$ of the starlike functions of order α is obtained.

Altinas et al. (2001) has obtained the majorization results for the class $S(\gamma)$. Also, Majorization problems have been investigated by MacGregor (1967) for the class $S^*=S^*(0)$.

In the present investigation, we begin with following majorization problems.

Theorem 5.2.

Suppose that $g\in S_q^{\lambda,\mu}(A,B;\gamma)$ and let the function $f\in A$. If $\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{(q)}$ majorized through $\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)}$ in Δ. Then,

$$\left|\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{(q+1)}\right|\le\left|\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q+1)}\right|,$$

for $|z|\le\rho_0$, where $\rho_0=\rho_0(\gamma,q,A,B)$ is the smallest positive real root of the following equation,

$$|(1-q)B+\gamma(A-B)|\,\rho^3-\{2|B|+|1-q|\}\,\rho^2-\{|(1-q)B+\gamma(A-B)|+2\}\rho+|1-q|=0. \tag{34}$$

Proof:

Since $\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)}\in S_q^{\lambda,\mu}(A,B;\gamma)$, from (33), we have

$$\left[1+\frac{1}{\gamma}\left\{\frac{z\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q+1)}}{\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)}}+q-1\right\}\right]=\frac{1+Aw(z)}{1+Bw(z)}, \tag{35}$$

$(z\in\Delta,\ -1\le B<A\le 1,\ 0\le\lambda<1,\ \mu>0,\ q\in N_0=N\cup\{0\},\ \gamma\in\mathbb{C}\setminus\{0\}),$

where $w(z)=c_1z+c_2z^2+,...,\ w\in P,\ P$ represents class of bounded functions which are analytic in Δ and fulfill the conditions

$$|w(z)|<|z|,\quad w(0)=0,\ \text{ and }\ (z\in\Delta). \tag{36}$$

From (35), we have

$$\frac{z\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q+1)}}{\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)}}=\frac{(1-q)+\{(1-q)B+\gamma(A-B)\}\,w(z)}{1+Bw(z)} \tag{37}$$

$$\implies z\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q+1)} = \frac{(1-q)+\{(1-q)B+\gamma(A-B)\}\,w(z)}{1+Bw(z)}\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)}$$

$$\implies \left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)} = \frac{(1+Bw(z))\,z}{(1-q)+\{(1-q)B+\gamma(A-B)\}\,w(z)}\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q+1)}. \tag{38}$$

Using (36) in (38), making simple calculations, we obtain

$$\left|\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)}\right| \leq \frac{(1+|B||z|)\,|z|}{|1-q|-|\gamma(A-B)+(1-q)B|\,|z|}\left|\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q+1)}\right|. \tag{39}$$

Now, since $\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{(q)}$ is majorized by $\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)}$ in unit disk Δ, we have

$$\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{(q)} = \phi(z)\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)}. \tag{40}$$

Differentiating (40) with respect to z, we get

$$\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{(q+1)} = \phi^{'}(z)\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q)} + \phi(z)\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q+1)},\quad (z\in\Delta). \tag{41}$$

Thus, noting that $\phi \in P$ satisfies the inequality (e.g., see Nehari (1952))

$$\frac{1-|\phi(z)|^2}{1-|z|^2} \geq |\phi^{'}(z)|,\quad (z\in\Delta), \tag{42}$$

and by applying (39) and (42) in (41), we get

$$\begin{aligned}&\left|\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{(q+1)}\right| \\ &\leq \left[|\phi(z)| + \frac{(|B||z|+1)\,|z|}{|1-q|-|(1-q)B+\gamma(A-B)|\,|z|}.\frac{1-|\phi(z)|^2}{1-|z|^2}\right]\left|\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q+1)}\right|,\end{aligned} \tag{43}$$

which, by setting $|z|=\rho$ and $|\phi(z)|=\nu,\ (0\leq\nu\leq 1)$ leads to the inequality

$$\left|\left(\mathbb{J}^{\lambda,\mu}f(z)\right)^{(q+1)}\right| \leq \Psi(r,\nu)\left|\left(\mathbb{J}^{\lambda,\mu}g(z)\right)^{(q+1)}\right|, \tag{44}$$

where

$$\Psi(\rho,\nu) = \frac{(1-\nu^2)\,(1+|B|\rho)\,\rho}{(1-\rho^2)\,(|1-q|-|(1-q)B+\gamma(A-B)|\,\rho)} + \nu, \tag{45}$$

$$\implies \Psi(\rho,\nu) = \frac{(1-\nu^2)\,(1+|B|\rho)\,\rho+\nu(1-\rho^2)\,(|1-q|-|(1-q)B+\gamma(A-B)|\,\rho)}{(|1-q|-|\gamma(A-B)+(1-q)B|\,\rho)\,(1-\rho^2)}. \tag{46}$$

In order to determine ρ_0, we note that

$$\max\{\rho\in[0,1]:\Psi(\rho,\nu)\leq 1,\forall\nu\in[0,1]\} = \rho_0,$$

and

$$\max\{\rho\in[0,1]:\Theta(\rho,\nu)\geq 0,\forall\nu\in[0,1]\} = \rho_0,$$

where

$$\begin{aligned}\Theta(\rho,\nu) = (1-\rho^2)\,(|1-q|-|(1-q)B+\gamma(A-B)|\,\rho)-(1+|B|\rho)\,(1-\nu^2)\rho \\ -\nu(1-\rho^2)\,(|1-q|-|(1-q)B+\gamma(A-B)|\,\rho).\end{aligned} \tag{47}$$

After simple calculations, the inequality $\Theta(\rho,\nu) \geq 0$ is comparable to

$$u(\rho,\nu) = (1-\rho^2)\left(|1-q| - |(1-q)B + \gamma(A-B)|\,\rho\right) - (1+|B|\rho)(1+\nu)\rho \geq 0.$$

Obviously, the function $u(\rho,\nu)$ obtains its least value at $\nu = 1$, i.e.,

$$u(\rho,1) = \min\{u(\rho,\nu) : \nu \in [0,1]\} = v(\rho),$$

where

$$\begin{aligned} v(\rho) = |(1-q)B + \gamma(A-B)|\,\rho^3 - \{|1-q| + 2|B|\}\,\rho^2 \\ - \{|(1-q)B + \gamma(A-B)| + 2\}\rho + |1-q|. \end{aligned} \tag{48}$$

It pursues that $0 \leq v(\rho)$ for all $\rho \in [0,\rho_0]$, where $\rho_0(\gamma,q,A,B)$ is the least positive root of Equation (34). ∎

Setting $B = -1$, and $A = 1$, we get the following.

Corollary 5.3.

Suppose that $g \in S_q^{\lambda,\mu}(1,-1;\gamma)$ and let the function $f \in A$. If $\left(\mathbb{J}^{\lambda,\mu} f(z)\right)^{(q)}$ majorized by $\left(\mathbb{J}^{\lambda,\mu} g(z)\right)^{(q)}$ in Δ. Then,

$$\left|\left(\mathbb{J}^{\lambda,\mu} f(z)\right)^{(q+1)}\right| \leq \left|\left(\mathbb{J}^{\lambda,\mu} g(z)\right)^{(q+1)}\right|, \text{ for } \rho_1 \geq |z|,$$

where $\rho_1 = \rho_1(\gamma,q,1,-1)$ is the least positive real root of the equation

$$|2\gamma - (1-q)|\,\rho^3 - \{2 + |1-q|\}\,\rho^2 - \{2 + |2\gamma - (1-q)|\}\rho + |1-q| = 0. \tag{49}$$

Setting $q = 0$ in Corollary 5.3, we get the following result.

Corollary 5.4.

Suppose that $g \in S_0^{\lambda,\mu}(1,-1;\gamma)$ and let the function $f \in A$. If $\mathbb{J}^{\lambda,\mu} f(z)$ majorized by $\mathbb{J}^{\lambda,\mu} g(z)$ in Δ. Then,

$$\left|\left(\mathbb{J}^{\lambda,\mu} f(z)\right)'\right| \leq \left|\left(\mathbb{J}^{\lambda,\mu} g(z)\right)'\right|, \text{ for } |z| \leq \rho_1,$$

where $\rho_1 = \rho_1(\gamma,0,1,-1)$ is the smallest positive real root of the equation

$$|2\gamma - 1|\,\rho^3 - 3\rho^2 - 2\{|\gamma - 1| + 1\}\rho + 1 = 0. \tag{50}$$

Setting $\lambda = \mu = 0$ in Corollary (5.4), the consequence is the following.

Corollary 5.5.

Suppose that $g \in S_0^{0,0}(1,-1;\gamma)\, i.e.\, g \in S(1,-1;\gamma) := S(\gamma)$ and let the function $f \in A$. If $f(z)$ majorized by $g(z)$ in Δ. Then,

$$|g^{'}(z)| \geq |f^{'}(z)| \ \text{ for } \ |z| \leq \rho_1,$$

where $\rho_1 = \rho_1(\gamma, 0, 1, -1)$ is the least positive real solution of the equation

$$|2\gamma - 1|\, \rho^3 - 3\rho^2 - 2\{|\gamma - 1| + 1\}\rho + 1 = 0. \tag{51}$$

This is a known result obtained by Altinas et al. (2001), which contains an additional result given for $\gamma = 1$ by MacGregor (1967).

Theorem 5.6.

Suppose that $g \in S_q^{\lambda,\mu}(A,B;\gamma)$ and let the function $f \in A$. If $\left(\mathbb{J}^{\lambda,\mu} f(z)\right)^{(q)}$ majorized by $\left(\mathbb{J}^{\lambda,\mu} g(z)\right)^{(q)}$ in Δ. Then,

$$\left|\left(\mathbb{J}^{\lambda+1,\mu} f(z)\right)^{(q)}\right| \leq \left|\left(\mathbb{J}^{\lambda+1,\mu} g(z)\right)^{(q)}\right|,$$

for $|z| \leq \rho_0$, where $\rho_0 = \rho_0(\gamma, q, A, B, \lambda, \mu)$ is the smallest positive real root of the equation,

$$\begin{aligned}
&\{|\mu - \lambda + \eta + q||B| + |\gamma(A - B) + (1 - q)B|\}\, \rho^3 \\
&- \{|\mu - \lambda + \eta + q| - 2|B| - |1 - q|\}\, \rho^2 \\
&- \{|(1 - q)B + \gamma(A - B)| + |\mu - \lambda + \eta + q||B| + 2\}\rho \\
&+ \{|\mu - \lambda + \eta + q| + |1 - q|\} = 0.
\end{aligned} \tag{52}$$

Proof:

Now use the following easily verifiable identity

$$z\left(\mathbb{J}^{\lambda,\mu} f(z)\right)^{(q+1)} = (\mu - \lambda + \eta + 1)\left(\mathbb{J}^{\lambda+1,\mu} f(z)\right)^{(q)} - (\mu - \lambda + \eta + q)\left(\mathbb{J}^{\lambda,\mu} f(z)\right)^{(q)}. \tag{53}$$

Using (53) and (36) in (37), simple calculations leads to

$$\begin{aligned}
&\left|\left(\mathbb{J}^{\lambda,\mu} g(z)\right)^{(q)}\right| \\
&\leq \frac{|1 + \mu - \lambda + \eta|\,(|z||B| + 1)}{|1 - q| - |(1 - q)B + \gamma(A - B)|\,|z| - |\mu - \lambda + \eta + q|\,(|B||z| - 1)} \left|\left(\mathbb{J}^{\lambda+1,\mu} g(z)\right)^{(q)}\right|.
\end{aligned} \tag{54}$$

Multiplying (41) by z, we get

$$z\,\left(\mathbb{J}^{\lambda,\mu} f(z)\right)^{(q+1)} = z\,\phi^{'}(z)\left(\mathbb{J}^{\lambda,\mu} g(z)\right)^{(q)} + z\,\phi(z)\left(\mathbb{J}^{\lambda,\mu} g(z)\right)^{(q+1)}, \ \ (z \in \Delta). \tag{55}$$

Using (53) in the above equation, it yields

$$\left(\mathbb{J}^{\lambda+1,\mu} f(z)\right)^{(q)} = \frac{z}{(\mu - \lambda + \eta + 1)}\, \phi^{'}(z)\left(\mathbb{J}^{\lambda,\mu} g(z)\right)^{(q)} + \phi(z)\ \left(\mathbb{J}^{\lambda+1,\mu} g(z)\right)^{(q)}, \tag{56}$$

and by applying (54) and (42) in (56), we get

$$\begin{aligned}
&\left|\left(\mathbb{J}^{\lambda+1,\mu}f(z)\right)^{(q)}\right| \\
&\leq \left[\begin{aligned}&|\phi(z)| \\ &+\frac{(|B||z|+1)\,|z|}{|1-q|-|\gamma(A-B)+(1-q)B|\,|z|-|\mu-\lambda+\eta+q|\,(|B||z|-1)}.\frac{1-|\phi(z)|^2}{1-|z|^2}\end{aligned}\right] \\
&\times\left|\left(\mathbb{J}^{\lambda+1,\mu}g(z)\right)^{(q)}\right|,
\end{aligned} \tag{57}$$

which upon setting $|\phi(z)|=\nu,\ (0\leq\nu\leq1)$ and $|z|=\rho$ results in inequality

$$\left|\left(\mathbb{J}^{\lambda+1,\mu}f(z)\right)^{(q)}\right|\leq\Psi(\rho,\nu)\left|\left(\mathbb{J}^{\lambda+1,\mu}g(z)\right)^{(q)}\right|, \tag{58}$$

where

$$\begin{aligned}
&\Psi(\rho,\nu) \\
&=\frac{\left[\begin{aligned}&(1-\nu^2)\,(1+|B|\rho)\,\rho \\ &+\nu(1-\rho^2)\left\{|1-q|-|(1-q)B+\gamma(A-B)|\,\rho-|\eta+\mu-\lambda+q|\,(|B|\rho-1)\right\}\end{aligned}\right]}{(1-\rho^2)\left\{|1-q|-|(1-q)B+\gamma(A-B)|\,\rho-|\mu-\lambda+\eta+q|\,(|B||z|-1)\right\}}.
\end{aligned} \tag{59}$$

In order to determine ρ_0, we note that

$$\max\left\{\rho\in[0,1]:\Psi(\rho,\nu)\leq1,\forall\nu\in[0,1]\right\}=\rho_0,$$

$$\max\left\{\rho\in[0,1]:\Theta(\rho,\nu)\geq0,\ \forall\nu\in[0,1]\right\}=\rho_0,$$

where

$$\begin{aligned}
\Theta(\rho,\nu)=&\,(1-\rho^2)\left\{|1-q|-|(1-q)B+\gamma(A-B)|\,\rho-|\eta+\mu-\lambda+q|\,(|B||z|-1)\right\} \\
&-(1-\nu^2)\,(|B|\rho+1)\,\rho-\nu(1-\rho^2)\times \\
&\left\{|1-q|-|(1-q)B+\gamma(A-B)|\,\rho-|\mu-\lambda+\eta+q|\,(|B||z|-1)\right\}.
\end{aligned} \tag{60}$$

After simple calculations the inequality $\Theta(\rho,\nu)\geq0$ is comparable to

$$\begin{aligned}
&u(\rho,\nu)= \\
&(1-\rho^2)\left\{|1-q|-|\gamma(A-B)+(1-q)B|\,\rho-|\mu-\lambda+\eta+q|\,(|B||z|-1)\right\} \\
&-(1+\nu)\rho(1+|B|\rho)\geq0.
\end{aligned} \tag{61}$$

Obviously, the function $u(\rho,\nu)$ attains its least value at $\nu=1$, i.e.,

$$u(\rho,1)=\min\left\{u(\rho,\nu):\nu\in[0,1]\right\}=v(\rho),$$

where

$$\begin{aligned}
v(\rho)=&\left\{|(1-q)B+\gamma(A-B)|+|\mu-\lambda+\eta+q||B|\right\}\rho^3 \\
&-\left\{|\eta+\mu-\lambda+q|-2|B|-|1-q|\right\}\rho^2 \\
&-\left\{|(1-q)B+\gamma(A-B)|+|\mu-\lambda+\eta+q||B|+2\right\}\rho \\
&+\left\{|\mu-\lambda+\eta+q|+|1-q|\right\}.
\end{aligned} \tag{62}$$

It pursues that $v(\rho)\geq0$ for all $\rho\in[0,\rho_0]$, where $\rho_0(\gamma,q,A,B,\lambda,\mu)$ is the least positive solution of Equation (52), which completes our proof. ■

Setting $B = -1$ and $A = 1$ in Theorem 5.6, we get the following.

Corollary 5.7.

Suppose that, $g \in S_q^{\lambda,\mu}(1,-1;\gamma)$ and let the function $f \in A$. If $\left(\mathbb{J}^{\lambda,\mu} f(z)\right)^{(q)}$ majorized by $\left(\mathbb{J}^{\lambda,\mu} g(z)\right)^{(q)}$ in Δ. Then,

$$\left|\left(\mathbb{J}^{\lambda+1,\mu} f(z)\right)^{(q)}\right| \leq \left|\left(\mathbb{J}^{\lambda+1,\mu} g(z)\right)^{(q)}\right|,$$

for $|z| \leq \rho_1$, where $\rho_1 = \rho_1(\gamma, q, 1, -1, \lambda, \mu)$ is the smallest positive real root of the equation,

$$\begin{aligned} &\{|2\gamma - (1-q)| + |\eta + \mu - \lambda + q|\}\,\rho^3 + \{|\mu - \lambda + \eta + q| + 2 - |1-q|\}\,\rho^2 \\ &- \{2 + |2\gamma - (1-q)| + |\eta + \mu - \lambda + q|\}\rho + \{|\eta + \mu - \lambda + q| + |1-q|\} = 0. \end{aligned} \tag{63}$$

Setting $q = 0$ in Corollary 5.7, we get the result contained in the following corollary.

Corollary 5.8.

Suppose that $g \in S_0^{\lambda,\mu}(1,-1;\gamma)$ and let the function $f \in A$. If $\mathbb{J}^{\lambda,\mu} f(z)$ majorized by $\mathbb{J}^{\lambda,\mu} g(z)$ in Δ. Then,

$$\left|\mathbb{J}^{\lambda+1,\mu} f(z)\right| \leq \left|\mathbb{J}^{\lambda+1,\mu} g(z)\right|,$$

for $|z| \leq \rho_1$, where $\rho_1 = \rho_1(\gamma, 0, 1, -1, \lambda, \mu)$ is the smallest positive real root of the equation,

$$\begin{aligned} &\{|2\gamma - 1| + |\mu - \lambda + \eta|\}\,\rho^3 + \{|\mu - \lambda + \eta| + 1\}\,\rho^2 - \{|2\gamma - 1| + |\mu - \lambda + \eta| + 2\}\rho \\ &+ \{|\mu - \lambda + \eta| + 1\} = 0. \end{aligned} \tag{64}$$

6. Conclusion

The history of pre-starlike functions goes back to 1977, when this class was studied by Ruscheweyh. We defined a class $S^{\lambda,\mu}$ of pre-starlike type functions involving the Ruscheweyh-Goyal fractional derivative. The inclusion relationship for this class is obtained. It is also shown that the Ruscheweyh-Goyal derivative operator preserves not only analyticity but the convexity and starlikeness also. Majorization results for the subclass of analytic functions, introduced by means of fractional Ruscheweyh-Goyal derivative, are also found. The scope is open for finding many other geometric properties of this class.

Acknowledgment:

Authors would like to thank reviewers and the Editor-in-Chief Professor Aliakbar Montazer Haghighi for careful reading of the manuscript and their valuable comments and suggestions for improving quality of the present paper.

REFERENCES

Agarwal, R. and Paliwal, G.S. (2014). On the Fekete-Szegö problem for certain subclasses of analytic function. In Babu, B. V., editor, Proceedings of the Second International Conference on Soft Computing for Problem Solving (SocProS 2012), Vol. 236 of Advances in Intelligent Systems and Computing, pp. 353-361.

Agarwal, R. and Paliwal, G.S. (2015). Some results on differential subordinations for a class of functions defined using generalized Ruscheweyh derivative operator, International Bulletin of Mathematical Research, Vol. 2, No. 1, pp. 16-26.

Agarwal, R., Paliwal, G.S. and Goswami, P. (2019). Results of differential subordination for a unified subclass of analytic functions defined using generalized Ruscheweyh derivative operator, Asian-European Journal of Mathematics, Vol. 12, No. 1, 1950035 (17 pages). https://doi.org/10.1142/S1793557119500359

Al-Amiri, H.S. (1979). Certain generalizations of prestarlike functions, J. Austral. Math. Soc. (Series A), Vol. 28, pp. 325-334.

Altinas, O., Özkan, O. and Srivastava, H.M. (2001). Majorization by starlike functions of complex order, Complex Var., Vol. 46, pp. 207-218.

Barnard, R.W. and Kellogg, C. (1978). Applications of convolution operators to problems in univalent function theory, Michigan Math J., Vol. 27, pp. 81-94.

Goswami, A., Singh, J., Kumar, D. and Sushila (2019). An efficient analytical approach for fractional equal width equations describing hydro-magnetic waves in cold plasma, Physica A: Statistical Mechanics and its Applications, Vol. 524, pp. 563-575.

Goyal, S.P., Bansal, S. K. and Goswami, P. (2010). Majorization for the subclass of analytic functions defined by linear operator using differential subordination, Journal of Applied Mathematics, Statistics and Informatics (JAMSI), Vol. 6, No. 2, pp. 45-50.

Goyal, S.P. and Goyal, R. (2005). On a class of multivalent functions defined by generalized Ruscheweyh derivatives involving a general fractional derivative operator, J. Indian Acad. Math., Vol. 27, No. 2, pp. 439-456.

Gupta, V.P. and Jain, P.K. (1976). Certain classes of univalent functions with negative coefficients ii, Bull. Austral. Math. Soc., Vol. 15, pp. 467-473.

Jack, I.S. (1971). Functions starlike and convex of order α, J. London Math. Soc., Vol. 3, pp. 469-474.

MacGregor, T.H. (1967). Majorization by univalent functions, Duke Math. J., Vol, 34, pp. 95-102.

Nasr, M.A. and Aouf, M.K. (1985). Starlike function of complex order, J. Natur. Sci. Math., Vol. 25, No. 1, pp. 1-12.

Nehari, Z. (1952). *Conformal Mappings*, McGraw-Hill, New York.

Owa, S. (1985a). An application of the Ruscheweyh derivatives ii, Publications del′Institut Mathématique, Vol. 38, pp. 99-110.

Owa, S. (1985b). On a certain class of functions defined by using the Ruscheweyh derivatives, Math. Japonicae, Vol. 30, pp. 301-306.

Owa, S., Fukui, S., Sakaguchi, X. and Ogawa, S. (1986). An application of the Ruscheweyh derivatives, Internat. J. Math. and Math. Sci., Vol. 9, No. 4, pp.721-730.

Owa, S., Saigo, M. and Srivastava, H.M. (1989). Some characterization theorems for starlike and convex functions involving a certain fractional integral operator, J. Math. Anal. Appl., Vol. 140, pp. 419-426.

Owa, S. and Srivatsava, H.M. (1990). An application of a certain fractional derivative operator, Proc. Japan Acad., Vol. 66, Ser. A, pp. 307-311.

Owa, S. and Uralegaddi, B.A. (1984). A class of functions $\alpha-$prestarlike of order β, Bull. Korean Math. Soc., Vol. 21, No. 4, pp. 77-85.

Parihar, H.S. and Agarwal, R. (2011). Application of generalized Ruscheweyh derivatives on p-valent functions, Journal of Mathematics and Applications, Vol. 34, pp. 75-86.

Polatoğlu, Y., Bolcal, M., Sen, A. and Esra Özkan, H. (2006). The radius of starlikeness $p-$valently analytic functions in the unit disc, Turk. J. Math., Vol. 30, pp, 277-284.

Polatoğlu, Y. and Esra Özkan, H. (2006). New subclass of starlike functions of complex order, J. Prime Research in Mathematics, Vol. 2, pp. 157-169.

Robertson, M.S. (1962). Convolutions of schlicht functions, Proceedings of the American Mathematical Society, Vol. 13, No. 4, pp. 585-589.

Ruscheweyh, S. (1975). New criteria for univalent functions, Proc. Amer. Math. Soc., Vol. 49, pp. 109-115.

Ruscheweyh, S. (1977). Linear operators between classes of prestarlike functions, Comm. Math. Helv., Vol. 52, pp. 497-509.

Ruscheweyh, S. and Sheil-Small, T. (1973). Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture, Commentarii Mathematici Helvetici, Vol. 48, pp. 119-136.

Singh, J., Kumar, D. and Baleanu, D. (2019). New aspects of fractional Biswas-Milovic model with Mittag-Leffler law, Math. Model. Nat. Phenom., Vol. 14, No. 3, pp. 303.

Singh, J., Kumar, D., Baleanu, D. and Rathore, S. (2019). On the local fractional wave equation in fractal strings, Mathematical Methods in the Applied Sciences, Vol. 42, No. 5, pp. 1588-1595.

Singh, R. and Singh, S. (1979). Integrals of certain univalent functions, Proceedings of the American Mathematical Society, Vol. 77, No. 3, pp. 336-340.

Suffridge, T.J. (1976). Starlike functions as limits of polynomials, Advances in Complex Function Theory, Springer-Verlag, Berlin, pp. 164-202. (Lecture Notes in Mathematics 505).