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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Special Issue No. 5 (August 2019), pp. 43 – 55

On Integrability of Semi-invariant Submanifolds of Trans-Sasakian Finsler Manifolds

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Received: February 28, 2019; Accepted: May 28, 2019

Abstract

We introduce the notion of trans-Sasakian Finsler manifold, then study semi-invariant submanifold $F^m = (\mathcal{Y}, \mathcal{Y}', F)$ of a trans-Sasakian Finsler manifold $\bar{F}^{2n+1} = (\bar{\mathcal{Y}}, \bar{\mathcal{Y}}', \bar{F})$ and we discuss the integrability conditions of the distrubitions of the semi-invariant submanifolds of the trans-Sasakian Finsler manifold.

Keywords: Almost contact Finsler metric structure; Distribution; Integrability; Finsler connection; Gauss formula; Semi-invariant submanifold; Trans-Sasakian Finsler manifold; $\mathfrak{v}$-shape operator; $\mathfrak{v}$-second fundamental form

MSC 2010 No.: 58B20, 53C25, 58A30

1. Introduction

Finsler submanifolds have been studied after Finsler manifolds were discussed as generalized metric spaces in Finsler's doctoral dissertation (Finsler (1951)). Then, Shen (1998) introduced the notions of mean and normal curvature tensors for submanifolds and found some results without using any connection in Finsler geometry. Bejancu and Farran (2000) analysed the vertical vector bundles to study Finsler submanifolds and more generally focused on induced Cartan, Berwald and Rund connections, curves and surfaces (also see Bejancu (2000)) as a well-established submanifold theory for Finsler spaces. Besides, Bejancu and Farran (2010) discussed Riemannian metrics on the tangent bundle of Finsler submanifolds.

Oubina (1986) introduced trans-Sasakian manifolds reduced to α-Sasakian and β-Kenmotsu manifolds. Subsequently, trans-Sasakian manifolds are discussed in differential geometric sense by many mathematicians efficiently (see Gherge (2000)) as well as f-Kenmotsu manifolds (Hui, Yadav and Chaubey (2018)). Bejancu (1986) generalized CR-structures to Sasakian manifolds using invariant and anti-invariant submanifolds of almost contact-metric manifolds and used the notion of semi-invariant submanifold. Afterwards, semi-invariant submanifolds of several structures are studied by many geometers such as nearly trans-Sasakian and nearly Kenmotsu manifolds by Kim, Lin and Tripathi (2004) and by Ahmad (2009). Additionally, Shahid (1993) got some fundamental results on almost semi-invariant submanifolds of trans-Sasakian manifolds and Shahid et al. (2013) discussed submersion and cohomology class of semi-invariant submanifolds of trans-Sasakian manifolds.

Sinha and Yadav (1991) introduced almost Sasakian Finsler manifolds and determined the set of all almost Sasakian Finsler h-connection on almost Sasakian Finsler manifold. In addition, Yaliniz and Caliskan (2013) examined Sasakian Finsler manifolds on horizontal and vertical vector bundles with curvature and connection properties. In this paper, we structured trans-Sasakian Finsler manifolds and discussed semi-invariant submanifolds of trans-Sasakian Finsler manifolds. In this regard, in the second section, a brief introduction of Finsler manifold $\bar{F}^{m+p}(\bar{\mathcal{Y}}, \bar{\mathcal{Y}}', \bar{F})$ and its submanifold $F^m = (\mathcal{Y}, \mathcal{Y}^{'}, F)$ is given. We use the orthogonal decomposition with respect to Finsler metric $\bar{G}^v$ on $V\bar{\mathcal{Y}}'$,

$$V\bar{\mathcal{Y}}' \mid_{\mathcal{Y}'} = V\mathcal{Y}^{'} \oplus V\mathcal{Y}'^{\perp}, \tag{1}$$

where $\bar{\mathcal{Y}}^{m+p}$ is an $(2n+1)$-dimensional Finsler manifold. In the third section, we construct the trans-Sasakian structure $\left(\phi^V, \xi^V, \eta^V, G^V\right)$ on the vertical vector bundle $V_v\bar{\mathcal{Y}}_x^{'}$ of $\bar{F}^{2n+1}$. In the fourth section, semi-invariant submanifolds of trans-Sasakian Finsler manifolds are discussed. In the fifth section, integrability conditions of the distributions on semi-invariant submanifolds of trans-Sasakian Finsler manifolds are obtained.

2. Finsler Submanifolds

Assume that $\mathcal{Y}^m$ and $\bar{\mathcal{Y}}^{m+p}$ are manifolds where $f : \mathcal{Y} \to \bar{\mathcal{Y}}$ is a C^{∞}-differentiable with the local coordinates $x^i = x^i(u^1, ..., u^m),\ \ 1 \leq i \leq m+p$. Followingly, the differential map $f_* : T\mathcal{Y} \to T\bar{\mathcal{Y}}$

is defined with the coordinates $(u^\alpha, v^\alpha) \mapsto (x^i(u))\,, (y^i(u,v))\,$, satisfying $(y^i(u,v)) = A_i^\alpha v^\alpha$ and $A_i^\alpha = \frac{\partial x^i}{\partial u^\alpha}$. Then, f is called an immersion of $\mathcal{Y}$ into $\bar{\mathcal{Y}}$ if $(f_*)_u : T_u\mathcal{Y} \to T_{f(u)}\bar{\mathcal{Y}}$ is injective for all $u\in \mathcal{Y}$. Additionally, if f is an injective immersion, it said to be an imbedding. So, $\mathcal{Y}$ is called an imbedded submanifold of $\bar{\mathcal{Y}}$. Obviously, each $(m+p)-$dimensional submanifold of $\bar{\mathcal{Y}}$ is called an open submanifold of $\bar{\mathcal{Y}}$.

Let $\bar{\mathcal{Y}}'$ be a non empty open submanifold of $T\bar{\mathcal{Y}}$ such that $\pi(\bar{\mathcal{Y}}') = \bar{\mathcal{Y}}$ and $\theta(\bar{\mathcal{Y}}) \cap \bar{\mathcal{Y}}' = \varnothing$, where θ is the zero section of $T\bar{\mathcal{Y}}$. Suppose that $\bar{\mathcal{Y}}'_x = T_x\bar{\mathcal{Y}} \cap \bar{\mathcal{Y}}'$ is a positive conic set, i.e. for any $k > 0$ and $y \in \bar{\mathcal{Y}}'_x$, then we have $k_y \in \mathcal{Y}'_x$.

Now, consider an $(m+p)$-dimensional Finsler manifold $\bar{F}^{m+p} = (\bar{\mathcal{Y}}, \bar{\mathcal{Y}}', \bar{F})$ and m-dimensional submanifold $\mathcal{Y}$ of $\bar{\mathcal{Y}}$. Suppose $\mathcal{Y}'_u = (f_*)^{-1}\left(\bar{\mathcal{Y}}'_{f(u)}\right)$ is nonempty for any $u \in \mathcal{Y}$. Then $\mathcal{Y}'_u$ is a positive conic set in $T_u\mathcal{Y}$, since $\bar{\mathcal{Y}}'_{f(u)}$ is so in $T_{f(u)}\bar{\mathcal{Y}}$. Moreover $\mathcal{Y}' = (f_*)^{-1}(\bar{\mathcal{Y}}')$ is an open submanifold of $T\mathcal{Y}$ such that $\pi(\mathcal{Y}') = \mathcal{Y}$ and $\theta(\mathcal{Y}) \cap \mathcal{Y}' = \emptyset$, where π and θ are the canonical projection of $T\mathcal{Y}$ on $\mathcal{Y}$ and the zero section of $T\mathcal{Y}$, respectively. Also $\bar{F}$ induces on $\mathcal{Y}'$ the function F locally given by $F(u,v) = \bar{F}(x(u), y(u,v))$ where

$$g_{\alpha\beta} = \frac{1}{2}\frac{\partial^2 F^2}{\partial v^\alpha \partial v^\beta} = g_{ij}(x(u), y(u,v))A^{ij}_{\alpha\beta}. \tag{2}$$

On condition that $\bar{F}^{m+p}$ is a Finsler manifold, then $g_{\alpha\beta}$ define a positive definite quadratic form on each $U' \subset \mathcal{Y}'$. So, F^m is a Finsler submanifold of $\bar{F}^{m+p}$.

Assume that $F^m = (\mathcal{Y}, \mathcal{Y}', F)$ is the Finsler submanifold of the Finsler manifold $\bar{F}^{m+p} = (\bar{\mathcal{Y}}, \bar{\mathcal{Y}}', \bar{F})$ with the frames $\left\{\bar{U}; x^i, y^i\right\}$ of $\bar{\mathcal{Y}}'$ and $\{U, u^\alpha, v^\alpha\}$ of $\mathcal{Y}'$, with the coordinate neighbourhood $\bar{U}$ on $\bar{\mathcal{Y}}'$ satisfying $U = \bar{U} \cap f_*(\mathcal{Y}')$. So, their bases $\left\{\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial v^\alpha}\right\}$ and $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right\}$ on $\mathcal{Y}'$ and $\bar{\mathcal{Y}}'$ are related on U with the following equations $\frac{\partial}{\partial u^\alpha} = A^i_\alpha \frac{\partial}{\partial x^i} + A^i_\alpha \frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial v^\alpha} = A_i^\alpha \frac{\partial}{\partial y^i}$. By (2) the vertical vector bundle $V\mathcal{Y}'$ is obtained as a vector subbundle of $V\bar{\mathcal{Y}}' \mid_{\mathcal{Y}'}$. So, the Riemannian metric $\bar{G^V}$ on $V\mathcal{Y}'$ induces a Riemannian metric G^V on $V\mathcal{Y}'$. Actually, the Finsler metric of F^m is $G^V_{\alpha\beta} = G^V\left(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta}\right)$. Additionally, $V\mathcal{Y}'^{\perp}$ denotes Finsler normal bundle of the Finsler submanifold F^m in $\bar{F}^{m+p}$ where it is the ortogonal complement of the vector bundle $V\mathcal{Y}'$ in $V\bar{\mathcal{Y}}' \mid_{\mathcal{Y}'}$ with respect to $\bar{G^V}$. So, we get the ortogonal decomposition (1). (For further information please see Bejancu and Farran (2000).)

3. Trans-Sasakian Finsler manifolds

Definition 3.1.

Consider the tensor field ϕ, the 1-form η and the vector field ξ on $\bar{\mathcal{Y}}'$, by the decompositions:

$$\begin{aligned}
&\phi = \phi^H + \phi^V = \phi^i_j(x,y)\frac{\delta}{\delta x^i} \otimes dx^j + \tilde{\phi}^i_j(x,y)\frac{\partial}{\partial y^i} \otimes \delta y^j,\\
&\eta = \eta^H + \eta^V = \eta_i(x,y)dx^i + \tilde{\eta}_i(x,y)\delta y^i,\\
&\xi = \xi^H + \xi^V = \xi^i(x,y)\frac{\delta}{\delta x^i} + \tilde{\xi}_i(x,y)\frac{\partial}{\partial y^i}.
\end{aligned}$$

If the following relations hold,

$$(\phi^H)^2 = -I^H + \eta^H \otimes \xi^H, (\phi^V)^2 = -I^V + \eta^V \otimes \xi^V,$$

$$\eta^H(\xi^H) = \eta^V(\xi^V) = 1,$$

$$\phi^H(\xi^H) = \phi^V(\xi^V) = 0, \eta^H \circ \phi^H = \eta^V \circ \phi^V = 0,$$

then (ϕ^H, ξ^H, η^H) and (ϕ^V, ξ^V, η^V) are called an almost contact Finsler structure on $(\bar{\mathcal{Y}}')^h$ and $(\bar{\mathcal{Y}}')^v$, respectively, where $\bar{\mathcal{Y}}' = (\bar{\mathcal{Y}}')^h \oplus (\bar{\mathcal{Y}}')^v$ is a Finsler vector bundle (**?**).

Let $F^{m+p} = (\bar{\mathcal{Y}}, \bar{\mathcal{Y}}', \bar{F})$ be a Finsler manifold. We define the metric structure

$$g^{\bar{F}} : \Gamma(H\bar{\mathcal{Y}}') \times \Gamma(H\bar{\mathcal{Y}}') \to \Im(\bar{\mathcal{Y}}'),$$

$$g^{\bar{F}}(v, w)(x, y) = g^{\bar{F}}_{ij}(x, y)v^i(x, y)w^j(x, y),$$

where

$$g^{\bar{F}}_{ij}(x, y) = g^{\bar{F}}\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right)(x, y).$$

Hence, g is said to be a Finsler metric of F^{m+p}. Similarly, it possible to think that $g^{\bar{F}}$ is a Riemannian metric on the Finsler vector bundle $(H\bar{\mathcal{Y}}')$. We define

$$G : \Gamma(T\bar{\mathcal{Y}}') \times \Gamma(T\bar{\mathcal{Y}}') \to \Im(\bar{\mathcal{Y}}'),$$

$$G(X, Y) = G^H(X, Y) + G^V(X, Y), \forall X, Y \in \Gamma(T\bar{\mathcal{Y}}').$$

Thus, the symmetric tensor field G on $\bar{\mathcal{Y}}'$ is said to be the Sasaki Finsler metric on $\bar{\mathcal{Y}}'$. Then we state

$$G = g^{\bar{F}}_{ij} dx^i \otimes dx^i + g^{\bar{F}}_{ij} \delta y^i \otimes \delta y^j = G^H + G^V.$$

Definition 3.2.

Suppose that (ϕ^H, ξ^H, η^H) and (ϕ^V, ξ^V, η^V) are almost contact structures on horizontal and vertical vector bundles $(\bar{\mathcal{Y}}')^h$ and $(\bar{\mathcal{Y}}')^v$. If the metric structures G^H and G^V satisfy the following equations;

$$G(\phi X, \phi Y) = G^H(\phi X, \phi Y) + G^V(\phi X, \phi Y), \tag{3}$$

$$\begin{aligned} G^H(\phi X^H, \phi Y^H) &= G^H(X^H, Y^H) - \eta^H(X^H)\eta^H(Y^H), \\ G^V(\phi X^V, \phi Y^V) &= G^V(X^V, Y^V) - \eta^V(X^V)\eta^V(Y^V), \end{aligned} \tag{4}$$

$$\eta^H(X^H) = G^H(X^H, \xi^H), \quad \eta^V(X^V) = G^V(X^V, \xi^V), \tag{5}$$

then, $(\phi^H, \xi^H, \eta^H, G^H)$ is called almost contact metric Finsler structure on $(\bar{\mathcal{Y}}')^h$ and also $(\phi^V, \xi^V, \eta^V, G^V)$ is called almost contact metric Finsler structure on $(\bar{\mathcal{Y}}')^v$.

Let $(\phi^H, \xi^H, \eta^H, G^H)$ and $(\phi^V, \xi^V, \eta^V, G^V)$ are the almost contact metric Finsler structures on $(\bar{\mathcal{Y}}')^h$ and $(\bar{\mathcal{Y}}')^v$, respectively. Then from (3), (4) and (5), we get

$$G^H(\phi X^H, Y^H) = -G(X^H, \phi Y^H), \quad G^V(\phi X^V, Y^V) = -G^V(X^V, \phi Y^V),$$
$$G^H(\phi X^H, Y^H) = -G^H(\phi^2 X^H, Y^H), \quad G^V(\phi X^V, Y^V) = -G^V(\phi^2 X^V, Y^V).$$

Now, we define the fundamental 2-form:

$$\Omega(X^H, Y^H) = G^H(X^H, \phi Y^H), \quad \Omega(X^V, Y^V) = G^V(X^V, \phi Y^V),$$

where the following relation holds:

$$\Omega(\phi X^H, \phi X^H) = \Omega(X^H, Y^H), \quad \Omega(\phi X^V, \phi X^V) = \Omega(X^V, Y^V),$$
$$\Omega(X^H, Y^H) = -\Omega(Y^H, X^H), \quad \Omega(X^V, Y^V) = -\Omega(Y^V, X^V).$$

Definition 3.3.

Assume that, for the almost contact Finsler structures on horizontal and vertical Finsler vector bundles $(\bar{\mathcal{Y}}')^h$ and $(\bar{\mathcal{Y}}')^v$, we have the following relations:

$$\Omega(X, Y) = d\eta(X, Y),$$
$$\Omega(X^H, Y^H) = (\nabla^H_X \eta)(Y)^H - (\nabla^H_Y \eta)(X)^H + \eta(T(X^H, Y^H)),$$
$$\Omega(X^V, Y^V) = (\nabla^V_X \eta)(Y)^V - (\nabla^V_Y \eta)(X)^V + \eta(T(X^V, Y^V)),$$

where $\bar{\nabla}$ is a Finsler connection on $\bar{\mathcal{Y}}'$ and Ω is the fundamental 2-form. Then, the structures are called almost Sasakian Finsler structures. Suppose that the almost Sasakian Finsler connection $\bar{\nabla}$ on $\bar{\mathcal{Y}}'$ is torsion free. Hence, the below relations are satisfied:

$$\Omega(X^H, Y^H) = (\nabla^H_X \eta)Y^H - (\nabla^H_Y \eta)X^H,$$
$$\Omega(X^V, Y^V) = (\nabla^V_X \eta)Y^V - (\nabla^V_Y \eta)X^V.$$

Definition 3.4.

Almost Sasakian Finsler structures $\left(\phi^H, \eta^H, \xi^H, G^H\right)$ on $(\bar{\mathcal{Y}}')^h$ and $\left(\phi^V, \eta^V, \xi^V, G^V\right)$ on $(\bar{\mathcal{Y}}')^v$ are called Sasakian Finsler structures if the following relations hold:

$$(\nabla^H_X \eta)Y^H + (\nabla^H_Y \eta)X^H = 0, \quad (\nabla^V_X \eta)Y^V + (\nabla^V_Y \eta)X^V = 0,$$

where Sasakian Finsler connection $\bar{\nabla}$ on $\bar{\mathcal{Y}}'$ is torsion free.

Definition 3.5.

Let $\bar{\mathcal{Y}}$ be an $(2n+1)$-dimensional Finsler manifold. Then, the almost contact metric structure $\left(\phi^V, \eta^V, \xi^V, G^V\right)$ on $\bar{\mathcal{Y}}'_x$ is called trans-Sasakian Finsler structure of type (α, β) if the following relations hold:

$$\begin{aligned} 2(\bar{\nabla}^V_X \phi)Y^V = &\alpha \left\{G^V(X^V, Y^V)\xi^V - \eta^V\left(Y^V\right)X^V\right\} \\ &+ \beta\left\{G^V(\phi X^V, Y^V)\xi^V - \eta^V\left(Y^V\right)\phi X^V\right\}, \end{aligned} \tag{6}$$

for the functions α and β on $\bar{\mathcal{Y}}'_x$, and the Finsler connection $\bar{\nabla}$ with respect to G^V. From Formula (6), one can easily obtain

$$2\bar{\nabla}^V_X \xi^V = -\alpha\left(\phi X^V\right) + \beta\left(X^V - \eta^V\left(X^V\right)\xi^V\right). \tag{7}$$

4. Semi-Invariant Submanifolds of trans-Sasakian Finsler Manifolds

Definition 4.1.

Let $\mathcal{Y}'_u$ be an m-dimensional Finsler submanifold of a trans-Sasakian Finsler manifold $\mathcal{Y}'_x$. Then, $\mathcal{Y}'_u$ is said to be a semi-invariant submanifold if following relations hold:

$$(i)\ V\mathcal{Y}'_u = D \oplus D^{\perp} \oplus \left\{\xi^V\right\},$$
$$(ii)\ \text{The distribution } D \text{ is invariant under } \phi,\ \text{i.e.,} \phi D_v = D_v, \forall v \in \mathcal{Y}'_u,$$
$$(iii)\ \text{The distribution } D^{\perp} \text{ is anti-invariant under } \phi,\ \text{i.e.,} \phi\left(D_v^{\perp}\right) \subset \left(V_v\mathcal{Y}'\right)^{\perp}, \tag{8}$$

for all $v \in \mathcal{Y}'_u,\ \xi^V \in V\mathcal{Y}'_u$ where $(D, D^{\perp})$ are orthogonal distributions on $\mathcal{Y}'_u$ and also $V_v\mathcal{Y}'_u$ and $V_v{\mathcal{Y}'_u}^{\perp}$ are the tangent space and the normal space of $\mathcal{Y}'_u$ at v.

The distributions D and $D^{\perp}$ are called the horizontal and vertical distributions, respectively. A semi-invariant Finsler submanifold $\mathcal{Y}'_u$ is called an invariant and anti-invariant submanifold if $D_v^{\perp} = \{0\}$ and $D_v = \{0\}$ for each $v \in \mathcal{Y}'_u$, respectively. Besides, $\mathcal{Y}'_u$ is said proper if D and $D^{\perp}$ are not null spaces. In addition, each hypersurface of $\mathcal{Y}'_u$ which is tangent to ξ^V has a semi-invariant Finsler structure on a submanifold of $\mathcal{Y}'_u$.

The tensor field G of type $(0,2)$ of $(\bar{\mathcal{Y}}')^v$ is an induced metric on $(\mathcal{Y}')^v$. Then, the induced Finsler connection on $F^m = (\mathcal{Y}, \mathcal{Y}', F)$ is denoted by ∇, where $\bar{\nabla}$ is a Finsler connection on $\bar{F}^{2n+1} = (\bar{\mathcal{Y}}, \bar{\mathcal{Y}}', \bar{F})$. Besides, the second fundamental form of F^m is an $\Im(\mathcal{Y}')$-bilinear mapping on $\Gamma(V\mathcal{Y}') \times \Gamma(V\mathcal{Y}')$ and denoted by B.

By taking $h(X^V, Y^V) = B(X^V, Y^V)$, we have the following $\Im(\mathcal{Y}')$-bilinear mapping:

$$h^V : \Gamma(V\mathcal{Y}') \times \Gamma(V\mathcal{Y}') \to \Gamma(V\mathcal{Y}')^{\perp},$$

where $X, Y \in \Gamma(T\mathcal{Y}')$ and h^V is a $\mathfrak{v}$-second fundamental form of $F^m = (\mathcal{Y}, \mathcal{Y}', F)$. In the light of Gauss formula we get the below relation:

$$\bar{\nabla}^V_X Y^V = \nabla^V_X Y^V + h^V(X^V, Y^V), \tag{9}$$

for any $X^V, Y^V \in \Gamma(V\mathcal{Y}')$.

So, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{10}$$

for any $N \in \Gamma V{\mathcal{Y}'}^{\perp}$, $X \in \Gamma(T\mathcal{Y}')$ where $A_N X \in \Gamma(V\mathcal{Y}')$ and $\nabla^{\perp}_X N \in \Gamma(V\mathcal{Y}') \perp$.

We define the $\mathfrak{v}$-shape operator $A^V_N : \Gamma(V\mathcal{Y}') \to \Gamma(V\mathcal{Y}')$ with the following relation:

$$A^V_N X^V = A_{N^V} X^V, \tag{11}$$

where $N^V \in \Gamma(V\mathcal{Y}'^{\perp})$.

Thus, from the Weingarten formula we deduce the following equations:

$$\bar{\nabla}_{X^V} N^V = -A_N^V X^V + \nabla^{\perp}_{X^V} N^V, \tag{12}$$

$$G(h^V(X^V, Y^V), N^V) = G(A_N^V X^V, Y^V), \tag{13}$$

$$X^V = PX^V + QX^V + \eta^V\left(X^V\right)\xi^V, \tag{14}$$

where $X^V \in \Gamma(V\mathcal{Y}')$, $N^V \in \Gamma(V\mathcal{Y}'^{\perp})$, and $PX^V \in D$ and $QX^V \in D^{\perp}$.

Lemma 4.2.

Let $\mathcal{Y}'_u$ be a semi-invariant submanifold of a trans-Sasakian Finsler manifold $\mathcal{Y}'_x$. Then

$$P\nabla_X^V \phi PY^V - PA^V_{\phi QY^V} X^V = \phi P \nabla_X^V Y^V - \frac{\alpha}{2}\eta^V(Y^V)PX^V - \frac{\beta}{2}\eta^V(Y^V)\phi PX^V,$$

$$Q\nabla_X^V \phi PY^V - QA^V_{\phi QY^V} X^V = fh^V(X^V, Y^V) - \frac{\alpha}{2}\eta^V(Y^V)QX^V,$$

$$G^V(\xi^V, \nabla_X^V \phi PY^V) - G^V(\xi^V, A^V_{\phi QY^V} X^V) = \frac{\alpha}{2}\left\{G(X^V, Y^V) - \eta^V(X^V)\eta^V(Y^V)\right\} + \frac{\beta}{2}G(\phi X^V, Y^V),$$

$$\nabla^{\perp}_{X^V}(\phi QY^V) + h^V(X^V, \phi PY^V) = \phi Q \nabla_X^V Y^V + qh^V(X^V, Y^V) - \frac{\beta}{2}\eta^V(Y^V)\phi QX^V, \tag{15}$$

for any vector field $X^V, Y^V \in V_v\mathcal{Y}'$.

Proof:

In the view of the relations (9), (10), (13) and (14), we obtain

$$\begin{aligned}\left(\bar{\nabla}_X^V \phi\right) Y^V = & P\nabla_X^V(\phi PY^V) + Q\nabla_X^V(\phi PY^V) + \eta^V(\nabla_X^V(\phi PY^V))\xi^V + h(X^V, \phi PY^V) \\ & - PA^V_{\phi QY} X^V - QA^V_{\phi QY} X^V - \eta(A^V_{\phi QY} X^V)\xi^V + \nabla^{\perp}_{X^V}(\phi QY^V) \\ & - \phi P\nabla_X^V Y^V - \phi Q \nabla_X^V Y^V - fh(X^V, Y^V) - qh(X^V, Y^V),\end{aligned} \tag{16}$$

for all $X^V, Y^V \in V_v\mathcal{Y}'$, where $\nabla_X^V(\phi PY)^V \in V_v\mathcal{Y}'$, $A^V_{\phi QY}X \in V_v\mathcal{Y}'$, $\phi\nabla_{X^V}Y^V \in V_v\mathcal{Y}'$, and $N \in (V_v\mathcal{Y}')^{\perp}$.

By the use of (6), we have

$$\begin{aligned}\left(\bar{\nabla}_X^V \phi\right) Y^V = & -\left\{\frac{\alpha}{2}\eta^V(Y^V)PX^V + \frac{\beta}{2}\eta^V(Y^V)\phi PX^V\right\} - \frac{\alpha}{2}\eta^V(Y^V)QX^V \\ & - \frac{\beta}{2}\eta^V(Y^V)\left(\phi QX^V\right) - \frac{\alpha}{2}\eta^V(Y^V)\eta^V(X^V)\xi^V \\ & + \frac{\alpha}{2}G(X^V, Y^V)\xi^V + \frac{\beta}{2}G(\phi X^V, Y^V)\xi^V.\end{aligned} \tag{17}$$

From (16) and (17), we get the tangential and normal parts in the following way:

$$P\nabla_X^V(\phi PY^V) - PA^V_{\phi QY}X^V = \phi P\nabla_X^V Y^V - \frac{\alpha}{2}\eta^V(Y^V)PX^V - \frac{\beta}{2}\eta^V(Y^V)\left(\phi PX^V\right),$$

$$Q\nabla^V_X(\phi PY^V) - QA^V_{\phi QY}X^V = fh(X^V,Y^V) - \frac{\alpha}{2}\eta^V(Y^V)QX^V,$$

$$\eta^V\left(\nabla^V_X\left(\phi PY^V\right)\right) - \eta^V\left(A^V_{\phi QY}X^V\right) = -\frac{\alpha}{2}\eta^V(X^V)\eta^V(Y^V) + \frac{\alpha}{2}G(X^V,Y^V) + \frac{\beta}{2}G(\phi X^V,Y^V),$$

then, we have (15). ■

Lemma 4.3.

Let $\mathcal{Y}'_u$ be a semi-invariant submanifold of a trans-Sasakian Finsler manifold $\mathcal{Y}'_x$. Then,

$$\nabla^V_X\xi^V = -\frac{\alpha}{2}\phi X^V + \frac{\beta}{2}X^V, \;\; h^V(X^V,\xi^V) = 0, \tag{18}$$

$$\nabla^V_Y\xi^V = \frac{\beta}{2}Y^V, \quad h^V(Y^V,\xi^V) = -\frac{\alpha}{2}\phi Y^V. \tag{19}$$

Proof:

From (7) and (9), we have

$$\bar{\nabla}_{X^V}\xi^V = -\frac{\alpha}{2}\phi X^V + \frac{\beta}{2}\left(X^V - \eta^V(X^V)\xi^V\right) = \nabla^V_X\xi^V + h(X^V,\xi^V).$$

Since $V\mathcal{Y}' = D \oplus D^\perp \oplus \left\{\xi^V\right\}$, taking $\eta^V(X^V) = 0$, we get

$$\nabla^V_X\xi^V = -\frac{\alpha}{2}\phi X^V + \frac{\beta}{2}X^V \text{ and } h^V(X^V,\xi^V) = 0 \text{ for } X^V \in D.$$

From (7), if we put $\eta^V(Y^V) = 0$ we get

$$\bar{\nabla}^V_Y\xi^V = -\frac{\alpha}{2}\phi Y^V + \frac{\beta}{2}Y^V = \nabla^V_Y\xi^V + h^V(Y^V,\xi^V) \text{ for } Y^V \in D^\perp.$$

Thus we get the tangential and normal parts in (18) and (19). ■

5. Integrability of Distributions of Semi-Invariant Submanifolds of trans-Sasakian Finsler Manifolds

Theorem 5.1.

Let $\mathcal{Y}'_u$ be a semi-invariant submanifold of a trans-Sasakian Finsler manifold $\mathcal{Y}'_x$. So, the distribution D is not integrable.

Proof:

Assume that the distrubution D is integrable. Hence, $\left[X^V,Y^V\right] \in D$ for all $X^V,Y^V \in D$. From Lemma 4.2, we get

$$\nabla^\perp_{X^V}(\phi QY)^V + h^V(X^V,\phi PY^V) = \phi Q\nabla^V_XY^V + qh^V(X^V,Y^V) - \frac{\beta}{2}\eta^V(Y^V)\phi QX^V.$$

Since $PY^V = Y^V$ and $QY^V = 0$, we have

$$h^V(X^V, \phi Y^V) = \phi Q \nabla^V_X Y^V + q h^V(X^V, Y^V),$$

and

$$h^V(Y^V, \phi X^V) = \phi Q \nabla^V_Y X^V + q h^V(Y^V, X^V).$$

Thus,

$$h^V(X^V, \phi Y^V) - h^V(Y^V, \phi X^V) = \phi Q \left[X^V, Y^V\right].$$

Since $\left[X^V, Y^V\right] \in D$, we get $Q\left[X^V, Y^V\right] = 0$. From this we have

$$h^V(X^V, \phi Y^V) = h^V(Y^V, \phi X^V).$$

On the other hand,

$$G(\left[X^V, Y^V\right], \xi^V) = G(X^V, \nabla^V_Y \xi^V) - G(Y^V, \nabla^V_Y \xi^V).$$

From (17) we have

$$\frac{\alpha}{2} G(Y^V, \phi X^V) + \frac{\alpha}{2} G(\phi X^V, Y^V) = 0,$$

or

$$\alpha G(Y^V, \phi X^V) = 0.$$

Since $G(\phi X^V, Y^V) \neq 0$, we obtain $\alpha = 0$. But it is a contradiction. That is, the distrubution D is not integrable. ∎

Corollary 5.2.

Let $\mathcal{Y}'_u$ be a semi-invariant submanifold of a trans-Sasakian Finsler manifold $\mathcal{Y}'_x$. Then, the distribution $D \oplus D^\perp$ is not integrable.

Corollary 5.3.

Let $\mathcal{Y}'_u$ be a semi-invariant submanifold of a trans-Sasakian Finsler manifold $\mathcal{Y}'_x$. Then, the distribution $D \oplus \left\{\xi^V\right\}$ is integrable if and only if

$$h^V(X^V, \phi Y^V) = h^V(\phi X^V, Y^V). \tag{20}$$

Lemma 5.4.

Let $\mathcal{Y}'_u$ be a semi-invariant submanifold of a trans-Sasakian Finsler manifold $\mathcal{Y}'_x$. Then $\forall X^V \in D$, $\left[X^V, \xi^V\right] \in D \oplus \left\{\xi^V\right\}$.

Proof:

For $X^V \in D$ and $Y^V \in D^\perp$, we get

$$G(\left[X^V, \xi^V\right], Y^V) = G(\nabla_{X^V}\xi^V, Y^V) - G(\nabla_{\xi^V}X^V, Y^V).$$

Now, from (17) we have

$$G(\left[X^V, \xi^V\right], Y^V) = -G(\bar{\nabla}_\xi^V X^V, Y^V).$$

Then, taking ϕX^V instead of X^V, we obtain the following relation

$$G(\left[\phi X^V, \xi^V\right], Y^V) = -G(X^V, (\bar{\nabla}_\xi^V(\phi Y^V))).$$

Since $Y^V \in D^\perp$ and $\phi Y^V \in (V_v\mathcal{Y}'^\perp)$ by using (10), we have

$$G(\left[\phi X^V, \xi^V\right], Y^V) = G(X^V, A_{\phi Y}^V \xi^V) - G(X^V, \nabla_{\xi^V}^\perp \phi Y^V),$$

where $X^V \in V\mathcal{Y}'_u$ and $\nabla_\xi^\perp \phi Y \in (V_v\mathcal{Y}'^\perp)$. That is $G(X^V, \nabla_{\xi^V}^\perp \phi Y^V) = 0$. Thus, we get

$$G(\left[\phi X^V, \xi^V\right], Y^V) = G(h^V(\xi^V, X^V), \phi Y^V) = 0.$$

By the use of $\phi X^V = X^V$, we have

$$G(\left[\phi X^V, \xi^V\right], Y^V) = 0.$$

So, $\forall X^V \in D$, $Y^V \in D^\perp$, and $\left[X^V, \xi^V\right] \in D \oplus \left\{\xi^V\right\}$. ■

Theorem 5.5.

Let $\mathcal{Y}'_u$ be a semi-invariant submanifold of a trans-Sasakian Finsler manifold $\mathcal{Y}'_x$. Then the distribution $D^\perp$ is integrable.

Proof:

By using (9) and (12), we get

$$G(A_{\phi X}^V Y^V, Z^V) = G(\bar{\nabla}_Z^V Y^V, \phi X^V) - G(\nabla_Z^V Y^V, \phi X^V),$$

for X^V,$Y^V \in D^\perp$ and $Z^V \in V_v\mathcal{Y}'$. (Due to $\phi X^V, \phi Y^V \in (V_v\mathcal{Y}'^\perp)$). Since $G(\nabla_Z^V Y^V, \phi X^V) = 0$, we have

$$\begin{aligned} G(A_{\phi X}^V Y^V, Z^V) &= -G(\phi\bar{\nabla}_Z^V Y^V, X^V) \\ &= G((\bar{\nabla}_Z^V\phi)Y^V, X^V) - G(\nabla_Z^V(\phi Y^V), X^V). \end{aligned}$$

From (6) we have $G((\bar{\nabla}_Z^V \phi Y^V), X^V) = 0$. Thus,

$$G(A_{\phi X}^V Y^V, Z^V) = -G(\bar{\nabla}_Z^V(\phi Y^V), X^V).$$

Because of $\phi Y^V \in (V_v\mathcal{Y}'^\perp)$, we get

$$G(A_{\phi X}^V Y^V, Z^V) = G(A_{\phi Y}^V Z^V, X^V) - G(\nabla_Z^{V\perp}(\phi Y^V), X^V).$$

Since $\nabla_Z^\perp(\phi Y^V) \in (V_v\mathcal{Y}'^\perp)$ and $X^V \in V_v\mathcal{Y}'$, we get $G(A_{\phi X}^V Y^V, Z^V) = G(A_{\phi Y}^V X^V, Z^V)$. That is,

$$A_{\phi Y}^V X^V = A_{\phi X}^V Y^V \ \text{ for } \ Z^V \in V_v\mathcal{Y}'.$$

On the other hand, from (6) following relations hold:

$$\bar{\nabla}^V{}_X(\phi Y^V) = \frac{\alpha}{2}G(X^V, Y^V)\xi^V + \phi\nabla^V_X Y^V + \phi h^V(X^V, Y^V), \tag{21}$$

$$\phi\nabla^V_X Y^V = \phi P\nabla^V_X Y^V + \phi Q\nabla^V_X Y^V. \tag{22}$$

Due to $h^V(X^V, Y^V) \in (V\mathcal{Y}'^{\perp}_u)$, and via the distributions in (14), we have

$$\phi h^V(X^V, Y^V) = fh^V(X^V, Y^V) + qh^V(X^V, Y^V). \tag{23}$$

Using (22) and (23) in (21), we get the following relation:

$$\bar{\nabla}^V_X(\phi Y^V) = \frac{\alpha}{2}G(X^V, Y^V)\xi^V + \phi P\nabla^V_X Y^V + \phi Q\nabla^V_X Y^V + fh^V(X^V, Y^V) + qh^V(X^V, Y^V).$$

Moreover, because of $\phi Y^V \in (V_v\mathcal{Y}'^{\perp})$ and by the use of the relation (10), we have the below relation:

$$\begin{aligned} -A^V_{\phi Y}X^V + \nabla^{V\perp}_X(\phi Y^V) = {} & \frac{\alpha}{2}G(X^V, Y^V)\xi^V + \phi P\nabla^V_X Y^V \\ & + \phi Q\nabla^V_X Y^V + fh^V(X^V, Y^V) + qh^V(X^V, Y^V). \end{aligned} \tag{24}$$

From (24), we get the tangential and normal parts:

$$-A^V_{\phi Y}X^V = \frac{\alpha}{2}G(X^V, Y^V)\xi^V + \phi P\nabla^V_X Y^V + fh^V(X^V, Y^V),$$

$$\nabla^{V\perp}_X(\phi Y^V) = \phi Q\nabla^V_X Y + qh^V(X^V, Y^V).$$

Thus, we get

$$A^V_{\phi X}Y^V - A^V_{\phi Y}X^V = \phi P\left[X^V, Y^V\right].$$

Since $A^V_{\phi X}Y^V = A^V_{\phi Y}X^V$, we have $\phi P\left[X^V, Y^V\right] = 0$, which means $P\left[X^V, Y^V\right] = 0$, and $\left[X^V, Y^V\right] \in D^{\perp} \oplus \left\{\xi^V\right\}$. Since $X^V, Y^V \in D^{\perp}$, from (18) we can write $\nabla^V_Y\xi^V = \frac{\beta}{2}Y^V$ and $\nabla^V_X\xi^V = \frac{\beta}{2}X^V$. Thus, we have the below relation:

$$G(\left[X^V, Y^V\right], \xi) = \frac{\beta}{2}(G(X^V, Y^V) - G(Y^V, X^V)) = 0.$$

So, for all $X^V, Y^V \in D^{\perp}$, we get $[X, Y] \in D^{\perp}$. ■

6. Conclusion

The trans-Sasakian Finsler metric structure $\left(\phi^V, \eta^V, \xi^V, G^V\right)$ of type (α, β) is described by the relations (6) and (7) on $\bar{\mathcal{Y}}'_x$.

The semi-invariant submanifold $\mathcal{Y}'_u$ of the trans-Sasakian Finsler manifold $\mathcal{Y}'_x$ is defined by the relation (8). So, Gauss and Weingarten formulas and $\mathfrak{v}$-shape operator are given by the relations (8), (9), (10), (11), (12), (13), (14) of these structures and their tangential and normal parts are clarified and classified by Lemma 4.2 and Lemma 4.3.

For the tangent bundle $V\mathcal{Y}'_u$ of an m-dimensional semi-invariant submanifold $\mathcal{Y}'_u$ of a trans-Sasakian Finsler manifold $\mathcal{Y}'_x$, we get following distributions: $V\mathcal{Y}'_u = D \oplus D^{\perp} \oplus \left\{\xi^V\right\}$ for all $v \in (\mathcal{Y}')^v$, $\xi^V \in V_v\mathcal{Y}'$. Furthermore, the integrability properties of these distrubitions are as follows: the vertical distribution $D^{\perp}$ and the distribution $D \oplus \left\{\xi^V\right\}$ are integrable. The horizontal distribution D and the distribution $D \oplus D^{\perp}$ are not integrable. The distribution $D \oplus \left\{\xi^V\right\}$ is integrable if and only if (20) holds.

REFERENCES

Ahmad, M. (2009). Semi-invariant submanifolds of nearly Kenmotsu manifold with the canonical semi-symmetric semi-metric connection, Mat Vesnik, Vol. 62, pp. 189–198.

Bejancu, A. (1986). *Geometry of CR-submanifolds*, Volume 23 of Mathematics and its Applications East European Series, Reidel Publishing Co., Dordrecht, Holland.

Bejancu, A. (2000). *On The Theory of Finsler Submanifolds* in *Finslerian Geometries*, Springer, Dordrecht, Holland, pp. 111–129.

Bejancu, A. and Farran, H.R. (2000). *Pseudo-Finsler Submanifolds* in *Geometry of Pseudo-Finsler Submanifolds*, Springer, Dordrecht, Holland, pp. 70–124.

Bejancu, A. and Farran, H.R. (2010). Riemannian metrics on the tangent bundle of a Finsler Submanifold, Annals of the Alexandru Ioan Cuza University-Mathematics, Vol. 56, No. 1, pp. 429–436.

Finsler, P. (1951). *Uber Kurven und Flächen in Allgemeinen Räumen*, Verlag Birkhäuser; Basel.

Gherge, C. (2000). Harmonicity on nearly trans-Sasaki manifolds, Demostratio Math, Vol. 33, pp. 151–157.

Hui, S.K., Yadav, S.K. and Chaubey, S.K. (2018). η-Ricci solution on 3- dimensional f-Kenmotsu manifolds. Appl Appl Math, Vol. 13, No. 1, pp. 933–951.

Kim, S.J., Lin, X. and Tripathi, M.M. (2004). On semi-invariant submanifolds of nearly trans-Sasakian manifolds, Int J Pure Appl Math, Vol. 1, No. 1, pp. 15–34.

Kobayashi, M. (1986). Semi-invariant submanifolds of a certain class of almost contact manifolds, Tensor, Vol. 43, pp. 28–36.

Oubina, J.A. (1986). New class of almost contact metric structures, Publ Math Debrecen, Vol. 32, pp. 187–193.

Shahid, M.H. (1993). Almost semi-invariant submanifold of trans-Sasakian manifolds, Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, Vol. 37, No. 3/4, pp. 133–147.

Shahid, M.H., Al-Solamy, F.R., Jun, J.B. and Ahmad, M. (2013). Submersion of Semi-Invariant Submanifolds of Trans-Sasakian Manifold, Bull Malays Math Sci Soc, Vol. 9, No. 1, pp. 63–71.

Shen, Z. (1998). On Finsler geometry of submanifolds, Math Ann, Vol. 311, No. 3, pp. 549–576.

Sinha, B.B. and Yadav, R.K. (1991). On almost trans-Sasakian Finsler Manifolds, Indian J Pure Appl Math, Vol. 22, No. 3, pp. 185–192.

Yaliniz, A.F. and Caliskan, N. (2013). Sasakian Finsler manifolds, Turk J Math, Vol. 37, No. 2, pp. 319–339.