Applications and Applied Mathematics: An International Journal (AAM)

Volume 5
Issue 3 *Special Issue 1 – August 2010*

Article 9

8-2010

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Special Issue No. 1 (August 2010) pp. 101 – 109

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An Approximate Analytical Solution of the Fractional Diffusion Equation with External Force and Different Type of Absorbent Term – Revisited

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Received: May 15, 2010; Accepted: July 5, 2010

Abstract

In this article Homotopy Perturbation Method (HPM) is applied to obtain an approximate analytical solution of a fractional diffusion equation with an external force and a reaction term different from the reaction term used by Das and Gupta (2010). The anomalous behavior of diffusivity in presence or absence of linear external force due to the presence of this force of reaction term are obtained and presented graphically.

Keywords: Fractional diffusion equation; External force; Reaction term; Absorbent term; Homotopy Perturbation Method.

MSC (2010) No: 26A33; 34A25; 35R11; 60G22.

1. Introduction

We focus our attention to find the solution of the equation

$$\frac{\partial^{\beta}}{\partial t^{\beta}}u(x,t)=D\frac{\partial^{2}}{\partial x^{2}}u(x,t)-\frac{\partial}{\partial x}[F(x)u(x,t)]-\int_{0}^{t}\alpha(t-\xi)\,\frac{\partial u(x,\xi)}{\partial x}d\xi\ ,\ 0<\beta\leq 1, \qquad (1)$$

where, D is a diffusion coefficient, $F(x)$ is an external force, $\alpha(t)$ is a time-dependent absorbent term which may be related to a reaction diffusion process. Here the reaction term $\int_0^t \alpha(t-\xi)\,\frac{\partial u(x,\xi)}{\partial x}\,d\xi$ is different from that used by Schot et al. (2007) and Das and Gupta (2010). Here we have used the fractional Riemann-Liouville fractional integral operator of order $0<\alpha\le 1,\ t>0$.

Also, the Caputo fractional derivative, applied to the time variable is defined by

$$D^{\alpha}U(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{U^{(m)}(\xi)\,d\xi}{(t-\xi)^{\alpha+1-m}},\quad m-1<\alpha\le m,\ m\in N,\ t>0.$$

In this article the Homotopy Perturbation Method He (1999, 2000) is used to solve the fractional diffusion equation problem in the presence of both linear external force and an absorbent term. Using the initial condition, the approximate analytical expressions of $u(x,t)$ for different Brownian motions are obtained. The objective of the study is to show the effect of reaction term on the fractional diffusion equation with or without the presence of the linear external force.

2. Solution of the problem

Our aim is to solve the analytical fractional diffusion equation (1) for $D=1$ and $F(x)=-k\,x$. The equation (1) now becomes

$$\frac{\partial^{\beta}u(x,t)}{\partial t^{\beta}}=\frac{\partial^{2}u(x,t)}{\partial x^{2}}+k\frac{\partial}{\partial x}(x\,u(x,t))-\int_0^t \alpha(t-\xi).\frac{\partial u(x,\xi)}{\partial x}\,d\xi \tag{2}$$

with initial condition

$$u(x,0)=f(x). \tag{3}$$

Equation (2) can be written in operator form as

$$D_t^{\beta}\,u(x,t)=D_{xx}\,u(x,t)+k\,x\,D_x\,(u(x,t))+k\,u(x,t)-\int_0^t \alpha(t-\xi).\frac{\partial u(x,\xi)}{\partial x}\,d\xi \tag{4}$$

where, $D_t^{\beta}\equiv\dfrac{\partial^{\beta}}{\partial t^{\beta}}$.

According to the homotopy perturbation method, we construct the following homotopy

$$D_t^{\beta} u(x,t) = p[D_{xx} u(x,t) + k\, x D_x (u(x,t)) + k\, u(x,t) - \int_0^t \alpha(t-\xi).\frac{\partial u(x,\xi)}{\partial x} d\xi], \quad (5)$$

where the homotopy parameter p is considered as a small parameter ($p \in [0,1]$). In case $p = 0$, equation (5) becomes a linear equation, $D_t^{\beta} u = 0$, which is easy to solve Belendez et al. (2008), Darvishi and Khani (2008), Mausa and Ragab (2008). Now applying the classical perturbation technique, we can assume that the solution of equation (2) can be expressed as a power series in p as given below:

$$u(x,t) = u_0(x,t) + p u_1(x,t) + p^2 u_2(x,t) + p^3 u_3(x,t) + p^4 u_4(x,t) + \cdots. \quad (6)$$

When $p = 1$, equation (5) corresponds to equation (4) and (6) becomes the approximate solution of (4) i.e., of equation (2). The convergence of the method has been proved in He (2000). Substituting equation (6) for equation (5), and equating the terms with the identical powers of p, we can obtain a series of equations:

$$p^0: \; D_t^{\beta} u_0(\text{x, t}) = 0 \quad (7)$$

$$p^1: \; D_t^{\beta} u_1(\text{x, t}) = D_{xx} u_0(\text{x, t}) + k\, x\, D_x (u_0(\text{x, t})) + \text{k} u_0(\text{x, t}) - \int_0^t \alpha(t-\xi).\frac{\partial u_0(x,\xi)}{\partial x} d\xi \quad (8)$$

$$p^2: \; D_t^{\beta} u_2(\text{x, t}) = D_{xx} u_1(\text{x, t}) + k\, x\, D_x (u_1(\text{x, t})) + \text{k} u_1(\text{x, t}) - \int_0^t \alpha(t-\xi).\frac{\partial u_1(x,\xi)}{\partial x} d\xi \quad (9)$$

$$p^3: \; D_t^{\beta} u_3(\text{x, t}) = D_{xx} u_2(\text{x, t}) + k\, x\, D_x (u_2(\text{x, t})) + \text{k} u_2(\text{x, t}) - \int_0^t \alpha(t-\xi).\frac{\partial u_2(x,\xi)}{\partial x} d\xi \quad (10)$$

and so on.

Applying the operator J_t^{β} (the inverse of Caputo operator D_t^{β}) on both sides of the equations (7) – (10), we obtain the solutions of $u_i(\text{x, t})$, $\text{i} \geq 0$ for $\alpha(t) = \dfrac{\alpha\, t^{\beta-1}}{\Gamma(\beta)}$, $\alpha > 0 . 0 < \beta \leq 1$ (Schot et al. (2007)) as

$$u_0(x,t) = f(x) \quad (11)$$

$$u_1(x,t) = (f''(x) + k\, x\, f'(x) + k\, f(x)) \frac{t^{\beta}}{\Gamma(\beta+1)} - \alpha\, f'(x) \frac{t^{2\beta}}{\Gamma(2\beta+1)} \quad (12)$$

$$u_2(x,t) = (f^{(4)}(x) + 2k\,x\,f^{(3)}(x) + (k^2x^2 + 4k)f^{(2)}(x) + 3k^2 x\,f\,'(x) + k^2\;f(x))\frac{t^{2\beta}}{\Gamma(2\beta+1)}$$

$$-\alpha\,(2\,f^{(3)}(x) + 2k\,x\,f^{(2)}(x) + 3kf\,'(x))\frac{t^{3\beta}}{\Gamma(3\beta+1)} + \alpha^2\;f\,''(x)\frac{t^{4\beta}}{\Gamma(4\beta+1)} \tag{13}$$

$$u_3(x,t) = (f^{(6)}(x) + 3k\,x\,f^{(5)}(x) + (3k^2x^2 + 9k)f^{(4)}(x) + (k^3x^3 + 15k^2x)\;f^{(3)}(x)$$

$$+(6k^3x^2 + 13\;k^2)\;f^{(2)}(x) + 7k^3x\,f\,'(x) + k^3\;f(x))\frac{t^{3\beta}}{\Gamma(3\beta+1)} - \alpha\,(3\,f^{(5)}(x) + 6k\,x\,f^{(4)}(x)$$

$$+(3k^2x^2 + 15\;k)\;f^{(3)}(x) + 12k^2x\,f^{(2)}(x) + 7k^2\;f\,'(x))\frac{t^{4\beta}}{\Gamma(4\beta+1)} + 3\alpha^2(\,f^{(4)}(x)$$

$$+k\,x\,f^{(3)}(x) + 2k\;f^{(2)}(x))\frac{t^{5\beta}}{\Gamma(5\beta+1)} - \alpha^3\;f^{(3)}(x)\frac{t^{6\beta}}{\Gamma(6\beta+1)} \tag{14}$$

where,

$$f^{(r)}(x) = \frac{\partial^r}{\partial x^r}(\,f(x)), \quad r \geq 1.$$

Proceeding in this manner the rest of the components of $u_n, n \geq 0$ can be completely obtained, and the series solutions are thus entirely determined.

Finally, we approximate the analytical solution of $u(x,t)$ by the truncated series

$$u(x,t) = \lim_{N\to\infty} \Phi_N(x,t), \tag{15}$$

where

$$\Phi_N(x,t) = \sum_{n=0}^{N-1} u_n(x,t).$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault (1995).

3. Particular Cases

Case I. If $f(x)=x,\ \alpha=0,\ k=1$ i.e., in the presence of only external force, the expression of the displacement becomes

$$u(x,t)=x\left[1+\frac{2t^{\beta}}{\Gamma(\beta+1)}+\frac{4t^{2\beta}}{\Gamma(2\beta+1)}+\frac{8t^{3\beta}}{\Gamma(3\beta+1)}+\cdots\cdots\right]$$

$$=x\sum_{r=0}^{\infty}\frac{2^{r}t^{r\beta}}{\Gamma(r\beta+1)}$$

$$=xE_{\beta}(2t^{\beta}), \tag{16}$$

where $E_{\eta}(t)=\sum_{r=0}^{\infty}\frac{t^{r}}{\Gamma(\eta r+1)}$, ($\eta>0$) is the Mittag-Leffler function in one parameter. This result is same as the results given in Das and Gupta (2010), Saha Ray and Bera (2006) and Das (2009).

Case II. If $f(x)=x,\ \alpha=1,\ k=1$ i.e., in the presence of both the linear external force and absorbent term,

$$u(x,t)=x\left[1+\frac{2t^{\beta}}{\Gamma(\beta+1)}+\frac{4t^{2\beta}}{\Gamma(2\beta+1)}+\frac{8t^{3\beta}}{\Gamma(3\beta+1)}+\cdots\right]-\left[\frac{t^{2\beta}}{\Gamma(2\beta+1)}+\frac{3t^{3\beta}}{\Gamma(3\beta+1)}+\frac{7t^{\beta}}{\Gamma(4\beta+1)}\cdots\right]$$

$$=x\sum_{r=0}^{\infty}\frac{2^{r}t^{r\beta}}{\Gamma(r\beta+1)}-t^{\beta}\sum_{r=0}^{\infty}\frac{(2^{r}-1)t^{(r+1)\beta}}{\Gamma((r+1)\beta+1)}$$

$$=xE_{\beta}(2t^{\beta})-t^{\beta}E_{\beta,\beta+1}(Kt^{\beta}), \tag{17}$$

where, $K^{r}=(2^{r}-1)$.

Case III. If $f(x)=x,\ \alpha=1,\ k=0$ i.e., in the presence of the absorbent term,

$$u(x,t)=x-\frac{t^{2\beta}}{\Gamma(2\beta+1)}. \tag{18}$$

4. Numerical Results and Discussion

In this article, the displacement $u(x,t)$ for different fractional Brownian motions $\beta = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and also for the standard case, the numerical values motion $\beta = 1$ are calculated Case II and Case III, for various values of time t at $x = 1$. The results thus obtained are depicted through Figure 1 for Case II and Figure 2 for Case III.

It is observed from Figure 1 that the displacement increases with time in the presence of both the external force and the reaction term but converges rapidly with the increase of β. For Brownian motion the rate of convergence is more and hence the effective damping is better than in standard motion. The rate of convergence is very high in comparison to the treatment carried out by Das and Gupta (2010).

Figure 2 shows the convergence of displacement $u(x,t)$ in the presence of only the reaction term. It is obvious that the displacement converges to zero unlike the earlier case studies carried out by Das and Gupta (2010). It is seen that for the response to converge to zero, more time is required with the increase of β. Therefore the effective damping is less as it goes to the Brownian motions from the standard motion.

5. Conclusion

There are three important goals that we have achieved through this study. First one is the usage of extremely simple, concise and highly efficient mathematical tool like HPM to solve the general fractional diffusion equation. Secondly, the effect of reaction term on the fractional diffusion equation with or without the presence of drift term (external force) has been analysed.

The most important part of this study is to compare the effect of reaction term on the fractional diffusion equation for different Brownian motions with the existing result of Das and Gupta (2010) while using different reaction term and the same absorbent term. Hence it is seen that the choice of this reaction term helps to exercise better damping on the dynamic response of the system. The results of the study have been clearly exhibited through graphs.

Plots reveal the rapid convergence of displacement $u(x,t)$ with increase in β in the presence of both the external force and the reaction term. So, it can be concluded that dynamic performance and relative stability characteristics of a forced system are better under the influence of the new absorbent term. On the other hand dynamic performance and relative stability characteristics of an unforced system are better for Brownian motion than for standard motion. So we can infer that β should be chosen carefully keeping in consideration the performance requirements.

The authors strongly believe that the detailed study of the stability analysis will be beneficial and appealing to the scientists and engineers working in this field of research and they would be motivated to choose the proposed reaction term during their study.

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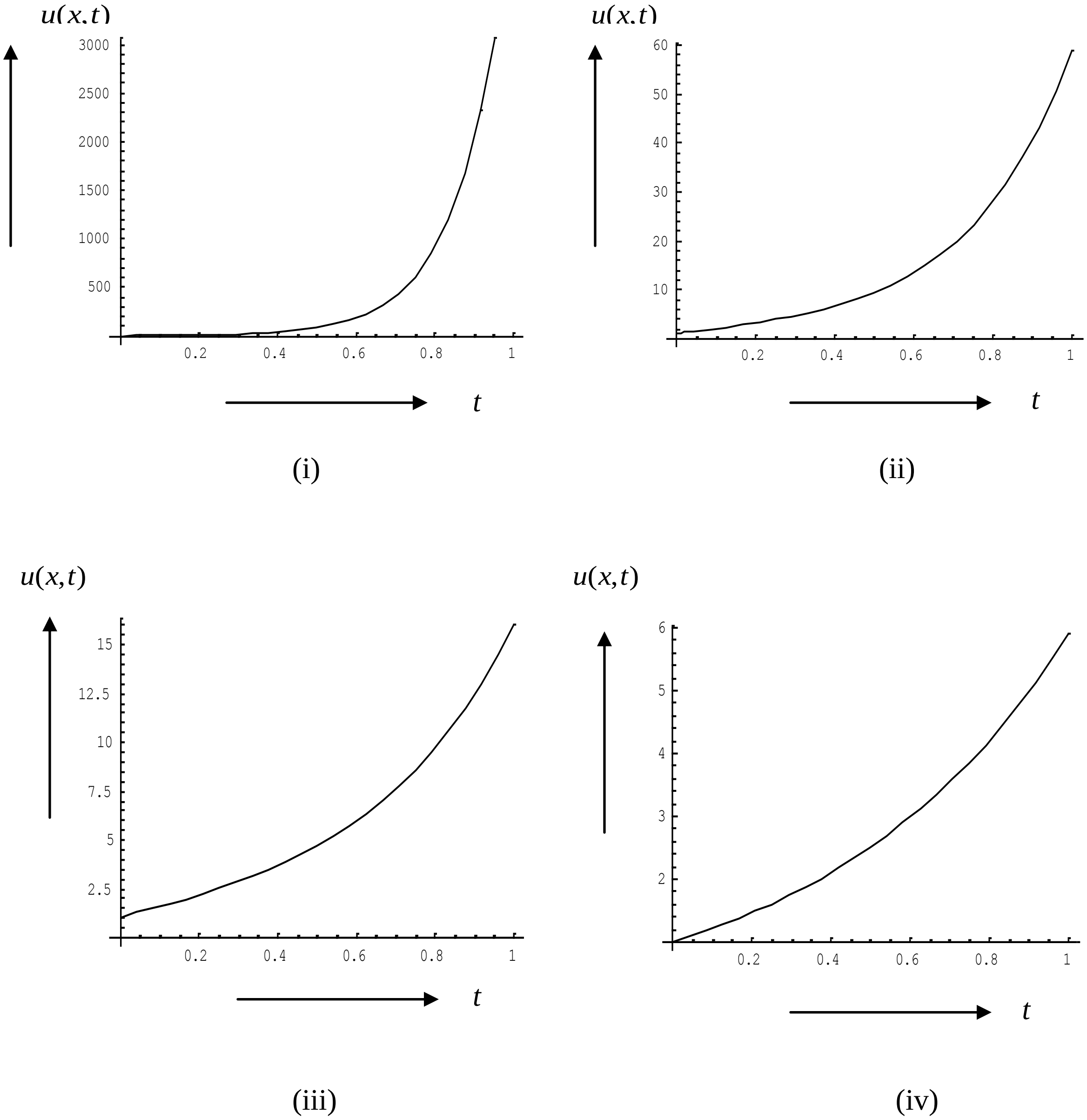

Figure 1. (i) Plot of $u(x,t)$ vs. t at $x=1$ for $\alpha=1, k=1, \beta=\frac{1}{3}$,

(ii) Plot of $u(x,t)$ vs. t at $x=1$ for $\alpha=1, k=1, \beta=\frac{1}{2}$,

(iii) Plot of $u(x,t)$ vs. t at $x=1$ for $\alpha=1, k=1, \beta=\frac{2}{3}$,

(iv) Plot of $u(x,t)$ vs. t at $x=1$ for $\alpha=1, k=1, \beta=1$.

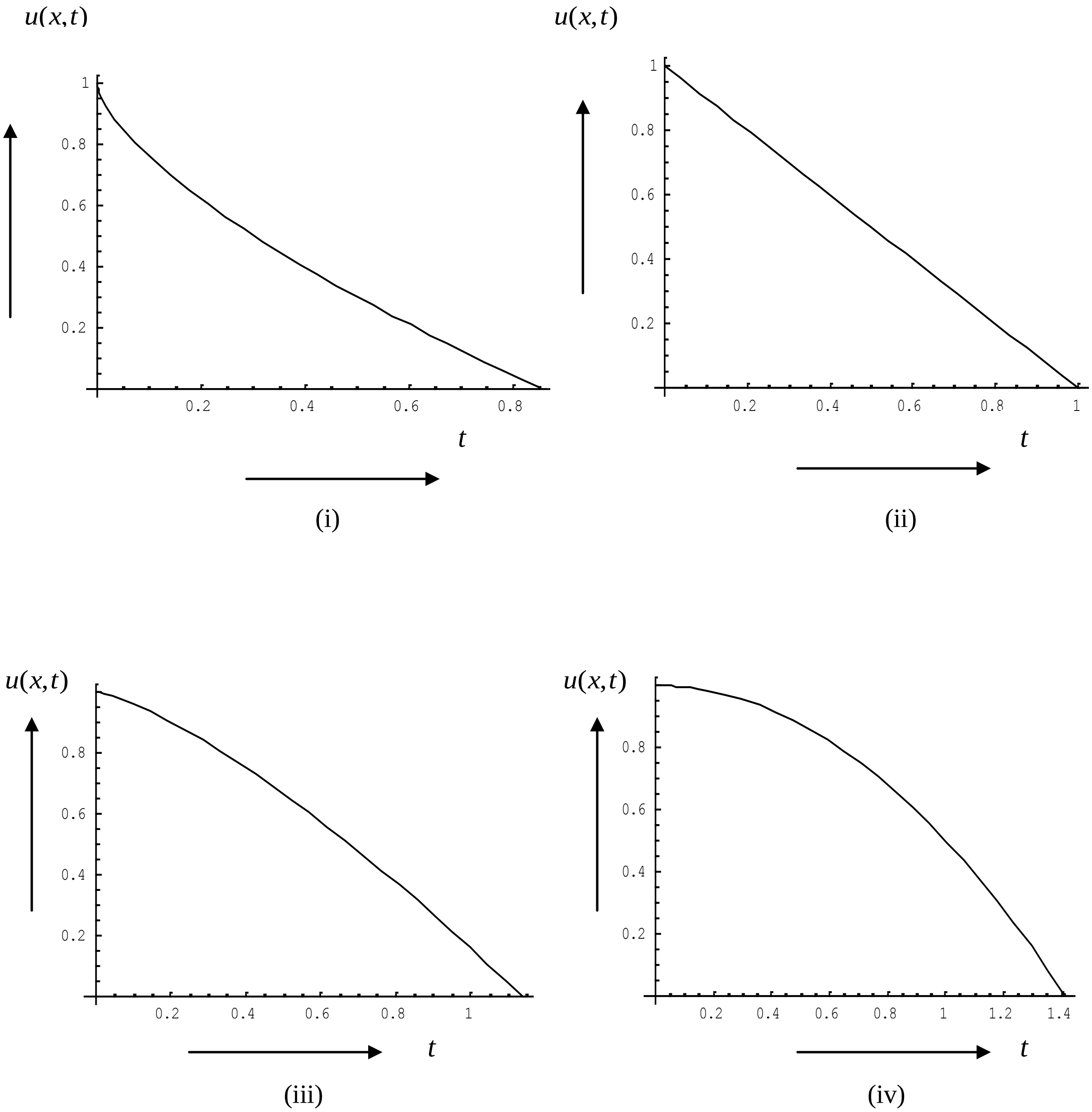

Figure 2. (i) Plot of $u(x,t)$ vs. t at $x=1$ for $\alpha=1, k=0, \beta=\frac{1}{3}$,

(ii) Plot of $u(x,t)$ vs. t at $x=1$ for $\alpha=1,\ k=0, \beta=\frac{1}{2}$,

(iii) Plot of $u(x,t)$ vs. t at $x=1$ for $\alpha=1,\ k=0, \beta=\frac{2}{3}$,

(iv) Plot of $u(x,t)$ vs. t at $x=1$ for $\alpha=1,\ k=0, \beta=1$