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Latest Developments in Nonlinear Sciences

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Abstract

This paper outlines a detailed study of some latest trends and developments in nonlinear sciences. The major focus of our study will be variational iteration (VIM) and its modifications, homotopy perturbation (HPM), parameter expansion and exp-function methods. The above mentioned schemes are highly accurate, extraordinary efficient, capable to cope with the versatility of the physical problems and are being used to solve a wide class of nonlinear problems. Several examples are given which reveal the justification of our claim.

Keywords: Variational iteration method, Homotopy perturbation method, nonlinear problems, He's polynomials, Adomian's polynomials, exp-function method.

MSC (2000) No.: 65 N 10

1. Introduction

The nonlinear problems (see Abbasbandy (2007), Abdou and Soliman (2005), Abassy et al. (2007), Batiha et al. (2007), Biazar and Ghazvini (2007), Adomian (1998), Chan and Hon (1987), Ganji et al. (2007), El-Wakil et al. (2007), Ghorbani and Nadjfi (2007), Golbabai and Javidi (2007), He (2003, 2004, 2006, 2007, 2008), Inokuti et al. (1978), Kaya (1999, 2003), Ma (2004), Momani and Odibat (2006), Mohyud-Din et al. (2007, 2008, 2009, 2010), Sweilam (2007), Tatari and Dehghan (2007), Wazwaz (1999), Wu and He (2007), Yusufoglu (2008), Zhou et al. (2008), Zhang (2007), Zhu (2007)) arise very frequently in the mathematical modeling of diversified physical problems related to engineering and applied sciences. The applications of these problems involve physics, astrophysics, astronomy, experimental and mathematical physics, nuclear charge in heavy atoms, thermal behavior of a spherical cloud of gas, thermodynamics, population models, chemical kinetics and fluid mechanics.

Several techniques including decomposition, modified decomposition, variational iteration, finite element, finite difference, polynomial and non-polynomial splines, differential transform, homotopy analysis, exp-function, variation of parameters, Sink-Glariken, parameter expansion, homotopy analysis, Runge-Kutta, and homotopy perturbation have been developed to tackle the nonlinearity of such problems. Most of these developed techniques have their inbuilt deficiencies like limited convergence, divergent results, linearization, discretization, unrealistic assumptions, absence of a small parameter and non-compatibility with the physical nature of nonlinear problems. He (2003, 2004, 2006, 2007, 2008) developed a number of efficient and reliable techniques to solve a wide class of nonlinear problems. These relatively new methods proved to be fully synchronized with the complexities of the physical problems, (see Abbasbandy (2007), Abdou and Soliman (2005), Abassy et al. (2007), Batiha et al. (2007), Biazar and Ghazvini (2007), Ganji et al. (2007), El-Wakil et al. (2007), Ghorbani and Nadjfi (2007), Golbabai and Javidi (2007), He (2003, 2004, 2006, 2007, 2008), Inokuti et al. (1978), Kaya (1999, 2003), Momani and Odibat (2006), Mohyud-Din et al. (2007, 2008, 2009, 2010), Sweilam (2007), Tatari and Dehghan (2007), Wu and He (2007), Yusufoglu (2008), Zhou et al. (2008), Zhang (2007), Zhu (2007) and the references therein.

In the present study, we will focus our attention on He's variational iteration (VIM), homotopy perturbation (HPM), modified variational iteration (MVIMS), parameter expansion and exp-function methods. The VIM (He (2003, 2004, 2006, 2007, 2008)) was developed by He in its present iterative form though its origin may be traced back to Inokuti, Sekine and Mura but the real potential of the method was realized and exploited by He. Moreover, He (2003, 2004, 2006, 2007, 2008) introduced another wonderful technique namely homotopy perturbation (HPM) by merging the standard homotopy and perturbation. The HPM is independent of the drawbacks of the coupled techniques and absorbs all their positive features. Recently, He and Wu (2008) formulated the exp-function method to solve nonlinear problems of versatile nature. It is to be highlighted that the present study would also outline the implementation of He's parameter-expansion techniques which comprise the book keeping parameter and modified Lindstedt-Poincare methods using parameter-expansion for finding the frequency of nonlinear oscillators. These techniques have been applied to a wide class of nonlinear problems and the references therein. With the passage of time some modifications in He's variational iteration method (VIM) has been introduced by various authors.

Abbasbandy (2007) made the elegant coupling of Adomian's polynomials with the correction functional (VIMAP) of the VIM and applied this reliable version to solve Riccati differential and Klein Gordon equations. In a later work, Noor and Mohyud-Din (2007, 2008, 2009, 2010) exploited this concept for solving various singular and non singular boundary and initial value problems. Recently, Ghorbani et al. (2007) introduced He's polynomials by splitting the nonlinear term and also proved that He's polynomials are fully compatible with Adomian's polynomials but are easier to calculate and are more user friendly. More recently, Noor and Mohyud-Din (2007, 2008, 2009, 2010) combined He's polynomials and correction functional (VIMHP) of the VIM and applied this reliable version to a number of physical problems. It has been observed that the modification based on He's polynomials (VIMHP) is much easier to implement as compare to the one (VIMAP) where the so-called Adomian's polynomials along with their complexities are used.

The basic motivation of the present study is the review of these reliable techniques which are being used very frequently for the solution of nonlinear initial and boundary value problems of diversified physical nature. Several examples are given to reveal the efficiency and reliability of these relatively new techniques.

2. Exp-function Method

Consider the general nonlinear partial differential equation of the type

$$P u, u_t, u_x, u_{tt}, u_{xx}, u_{xxx}, \dots = 0. \quad (1)$$

Using a transformation

$$\eta = kx + \omega t, \quad (2)$$

where k and ω are constants, we can rewrite equation (1) in the following nonlinear ODE;

$$Q u, u', u'', u''', u^{(iv)}, \dots = 0, \quad (3)$$

According to the exp-function method, which was developed by He and Wu (2008), we assume that the wave solutions can be expressed in the following form

$$u(\eta) = \frac{\sum_{n=-c}^d a_n \exp[n\eta]}{\sum_{m=-p}^q b_m \exp[m\eta]}, \quad (4)$$

where p, q, c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite equation (4) in the following equivalent form.

$$u(\eta) = \frac{a_c \exp[c\eta] + \dots + a_{-d} \exp[-d\eta]}{b_p \exp[p\eta] + \dots + b_{-q} \exp[-q\eta]} \tag{5}$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems [He and Wu (2008)]. To determine the value of c and p , we balance the linear term of highest order of equation (4) with the highest order nonlinear term. Similarly, to determine the value of d and q , we balance the linear term of lowest order of equation (3) with lowest order non linear term.

3. Variational Iteration Method (VIM)

To illustrate the basic concept of the He’s VIM, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{6}$$

where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term.

According to variational iteration method [He (2003, 2004, 2006, 2007, 2008)], we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + N\tilde{u}_n(s) - g(s)) ds, \tag{7}$$

where λ is a Lagrange multiplier [He (2003, 2004, 2006, 2007, 2008)], which can be identified optimally via variational iteration method. The subscripts n denote the n th approximation, \tilde{u}_n is considered as a restricted variation. i.e., $\delta\tilde{u}_n = 0$; (7) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [He (2003, 2004, 2006, 2007, 2008)]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1} , $n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0 , consequently, the solution is given by $u = \lim_{n \rightarrow \infty} u_n$. We summarize some useful iteration formulae [He (2003, 2004, 2006, 2007, 2008)] which would be used in the subsequent section:

$$\begin{cases} u' + f u, u' = 0, \\ u_{n+1}(t) = u_n(t) - \int_0^t u'_n(s) + f u_n, u'_n ds. \end{cases} \tag{3a}$$

$$\begin{cases} u'' + f(u, u', u'') = 0, \\ u_{n+1}(t) = u_n(t) + \int_0^t (s-t) u_n''(s) + f(u_n, u_n', u_n'') ds. \end{cases} \quad 3b$$

$$\begin{cases} u''' + f(u, u', u'', u''') = 0, \\ u_{n+1}(t) = u_n(t) - \int_0^t \frac{1}{2!} (s-t)^2 u_n'''(s) + f(u_n, u_n', u_n'', u_n''') ds. \end{cases} \quad 3c$$

$$\begin{cases} u^{(iv)} + f(u, u', u'', u''', u^{(iv)}) = 0, \\ u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{3!} (s-t)^3 u_n^{(iv)}(s) + f(u_n, u_n', u_n'', u_n''', u_n^{(iv)}) ds. \end{cases} \quad 3d$$

$$\begin{cases} u^{(n)} + f(u, u', u'', u''', u^{(iv)}, \dots, u^{(n)}) = 0, \\ u_{n+1}(t) = u_n(t) + (-1)^n \int_0^t \frac{1}{(n-1)!} (s-t)^{n-1} u_n^{(n)}(s) + f(u_n, u_n', u_n'', u_n''', u_n^{(iv)}, \dots, u_n^{(n)}) ds. \end{cases} \quad 3e$$

4. Homotopy Perturbation Method (HPM) and He's Polynomials

To explain the He's homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0, \tag{8}$$

where L is any integral or differential operator. We define a convex homotopy $H(u, p)$ by

$$H(u, p) = (1-p)F(u) + pL(u), \tag{9}$$

where $F(u)$ is a functional operator with known solutions v_0 , which can be obtained easily.

It is clear that, for

$$H(u, p) = 0, \tag{10}$$

we have

$$H(u,0) = F(u), H(u,1) = L(u).$$

This shows that $H(u, p)$ continuously traces an implicitly defined curve from a starting point $H(u, 0)$ to a solution function $H(u, 1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u)=0$, is continuously deforms the original problem $L(u)=0$. The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter [He (2003, 2004, 2006, 2007, 2008), Mohyud-Din et al. (2007, 2008, 2009, 2010)]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + p u_1 + p^2 u_2 + p^3 u_3 + \dots, \quad (11)$$

if $p \rightarrow 1$, then (11) corresponds to (9) and becomes the approximate solution of the form,

$$f = \lim_{p \rightarrow 1} u = \sum_{i=0}^{\infty} u_i. \quad (12)$$

It is well known that series (12) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$. We assume that (12) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, He's HPM considers the nonlinear term $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} p^i H_i = H_0 + p H_1 + p^2 H_2 + \dots,$$

where H_n 's are the so-called He's polynomials, which can be calculated by using the formula

$$H_n(u_0, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left(N \left(\sum_{i=0}^n p^i u_i \right) \right)_{p=0}, n = 0, 1, 2, \dots$$

5. Adomian's Decomposition Method (ADM)

Consider the differential equation (Wazwaz (1999))

$$Lu + Ru + Nu = g, \quad (13)$$

where L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of order lesser order than L , Nu represents the nonlinear terms and g is

the source term. Applying the inverse operator L^{-1} to both sides of (13) and using the given conditions, we obtain

$$u = f - L^{-1}(Ru) - L^{-1}(Nu),$$

where the function f represents the terms arising from integrating the source term g and by using the given conditions. Adomian's decomposition method [Wazwaz (1999)] defines the solution $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} u_n(x),$$

where the components $u_n(x)$ are usually determined recurrently by using the relation

$$u_0 = f, \quad u_{k+1} = L^{-1}(Ru_k) - L^{-1}(Nu_k), \quad k \geq 0.$$

The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials $F(u) = \sum_{n=0}^{\infty} A_n$, where A_n are the so-called Adomian's polynomials that can be generated for various classes of nonlinearities according to the specific algorithm developed in Wazwaz (1999), which yields

$$A_n = \left(\frac{1}{n!}\right) \left(\frac{d^n}{d\lambda^n}\right) N \left(\sum_{i=0}^n \lambda^i u_i \right)_{\lambda=0}, \quad n = 0, 1, 2, \dots$$

6. Modified Variational Iteration Methods (MVIMS)

The modified variational iteration techniques (MVIMS) are obtained by the elegant coupling of correction functional of VIM with He's and Adomian's polynomials.

6.1. Variational Iteration Method Using He's Polynomials (VIMHP)

This modified version of variational iteration method [Mohyud-Din et al. (2007, 2008, 2009, 2010)] is obtained by the elegant coupling of correction functional (7) of variational iteration method (VIM) with He's polynomials and is given by

$$\sum_{n=0}^{\infty} p^{(n)} u_n = u_0(x) + p \int_0^x \lambda(s) \left(\sum_{n=0}^{\infty} p^{(n)} L(u_n) + \sum_{n=0}^{\infty} p^{(n)} N(\tilde{u}_n) \right) ds - \int_0^x \lambda(s) g(s) ds. \quad (14)$$

Comparisons of like powers of p give solutions of various orders.

6.2. Variational Iteration Method Using Adomian's Polynomials (VIMAP)

This modified version of VIM is obtained by the coupling of correction functional (3) of VIM with Adomian's polynomials Mohyud-Din et al. (2007, 2008, 2009, 2010) and is given by

$$u_{n+1}(x) = u_n(x) + \int_0^t \lambda (L u_n(x) + \sum_{n=0}^{\infty} A_n - g(x)) dx, \quad (15)$$

where A_n are the so-called Adomian's polynomials and are calculated for various classes of nonlinearities by using the specific algorithm developed in (Wazwaz (1999)).

7. Numerical Applications

In this section, we apply the VIM, HPM, MVIMS, exp-function method and expansion of parameter for solving various nonlinear initial and boundary value problems.

Example 7.1. Consider the following "good" Boussinesq equation

$$u_{tt} = -u_{xxxx} + u_{xx} + u^2_{xx}, \quad (6b)$$

Introducing a transformation as $\eta = kx + \omega t$, we can convert equation (6b) into ordinary differential equations

$$\omega^2 u'' + k^4 u^{iv} - k^2 u'' - k^2 u^2'' = 0, \quad (7b)$$

The solution of the equation (7b) can be expressed in the form, equation (6b) as

$$u(\eta) = \frac{a_c \exp[c\eta] + \dots + a_{-d} \exp[-d\eta]}{b_p \exp[p\eta] + \dots + b_{-q} \exp[-q\eta]}.$$

To determine the value of c and p , we balance the linear term of highest order of equation (7b) with the highest order nonlinear term. Proceeding as before, we obtain

$$p = c \text{ and } d = q.$$

Case 7.1. We can freely choose the values of p, c, d , but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d . For simplicity, we set $p = c = 1$ and $q = d = 1$. Hence, equation (6b) reduces to the following form:

$$u \eta = \frac{a_1 \exp \eta + a_0 + a_{-1} \exp -\eta}{b_1 \exp \eta + a_0 + b_{-1} \exp -\eta}.$$

Substituting equation (7b), we have

$$\frac{1}{A} [c_4 \exp 4\eta + c_3 \exp 3\eta + c_2 \exp 2\eta + c_1 \exp \eta + c_0 + c_{-1} \exp -\eta + c_{-2} \exp -2\eta + c_{-3} \exp -3\eta + c_{-4} \exp -4\eta] = 0 \tag{8b}$$

where $A = b_1 \exp \eta + b_0 + b_{-1} \exp -\eta$, and $c_i \ i = -4, -3, \dots, 3, 4$ are constants obtained by Maple 11.

Equating the coefficients of $\exp n\eta$ to be zero, we obtain:

$$\{ c_{-4} = 0, c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0 \} \tag{9b}$$

Solution of (9b) will yield

$$b_1 = \frac{1}{4} \frac{b_0^2}{b_{-1}}, \quad b_{-1} = b_{-1}, \quad a_0 = -\frac{1}{2} \frac{b_0 (5k^4 + k^2 - \omega^2)}{k^2}, \quad \omega = \omega, \tag{10b}$$

$$a_{-1} = \frac{1}{2} \frac{b_{-1} (\omega^2 + k^4 - k^2)}{k^2}, \quad a_1 = \frac{1}{8} \frac{b_0^2 (\omega^2 + k^4 - k^2)}{k^2 b_{-1}}, \quad b_0 = b_0.$$

We, therefore, obtained the following generalized solitary solution $u(x, t)$ of equation (6b):

$$u(x, t) = \frac{\frac{1}{8} \frac{b_0^2 (\omega^2 - k^2 + k^4)}{k^2 b_{-1}} e^{2kx+2\omega t} - \frac{1}{2} \frac{b_0 (5k^4 + k^2 - \omega^2)}{k^2} e^{kx+\omega t} + \frac{1}{2} \frac{b_{-1} (\omega^2 - k^2 + k^4)}{k^2}}{\frac{1}{4} \frac{b_0^2 e^{2kx+2\omega t}}{b_{-1}} + b_0 e^{kx+\omega t} + b_{-1}},$$

or simply, we have

$$u(x, t) = \frac{\omega^2 + k^4 - k^2}{2k^2} - \frac{3b_0 k^2}{\frac{1}{4} \frac{b_0^2 e^{2kx+2\omega t}}{b_{-1}} + b_0 e^{kx+\omega t} + b_{-1}}, \tag{11b}$$

where b_0, b_{-1}, ω and k are real numbers.

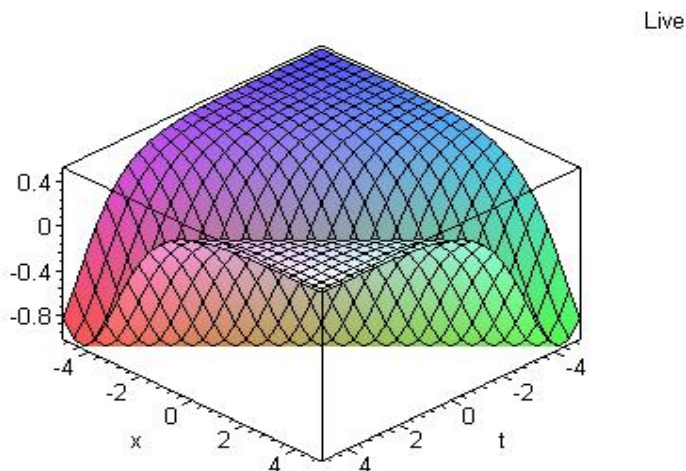


Figure 3.1 (a)

Figure 3.1 (a) depicts the soliton solutions of equation (6b), when $a_0 = a_{-1} = \omega = k = 1$. In case k is an imaginary number, the obtained soliton solutions can be converted into periodic or compact-like solutions. Therefore, we write $k = iK$, consequently, equation (11b) becomes

$$u_{x,t} = \frac{\omega^2 + K^4 + K^2}{-2K^2} + \frac{3b_0K^2}{\frac{1}{4} \frac{b_0^2}{b_{-1}} e^{iKx+\omega t} + b_0 + b_{-1} e^{-iKx-\omega t}}, \quad (12b)$$

$$u_{x,t} = \frac{\omega^2 + K^4 + K^2}{-2K^2} + \frac{3b_0K^2}{\frac{1}{4} \frac{b_0^2}{b_{-1}} e^{\omega t} \cos Kx + I \sin Kx + b_0 + b_{-1} e^{-\omega t} \cos Kx + I \sin Kx}. \quad (13c)$$

Now, to obtain periodic solutions from soliton solutions, we put imaginary part of equation (13b), which is $\sin kx$, equal to zero. Hence, equation (13b) becomes

$$u_{x,t} = \frac{\omega^2 + K^4 + K^2}{-2K^2} + \frac{3b_0K^2}{\frac{1}{4} \frac{b_0^2}{b_{-1}} e^{\omega t} \cos Kx + b_0 + b_{-1} e^{-\omega t} \cos Kx}. \quad (14b)$$

Live

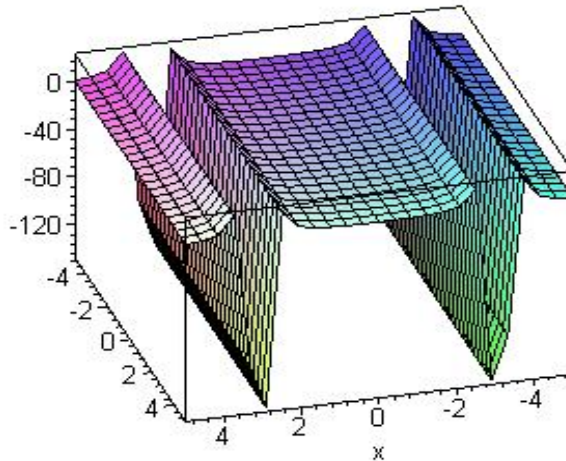


Figure 3.1 (b)

Figure 3.1 (b) depicts the periodic solutions of equation (6b) drawn on the whole domain when $b_{-1}=b_0=k=\omega=1$.

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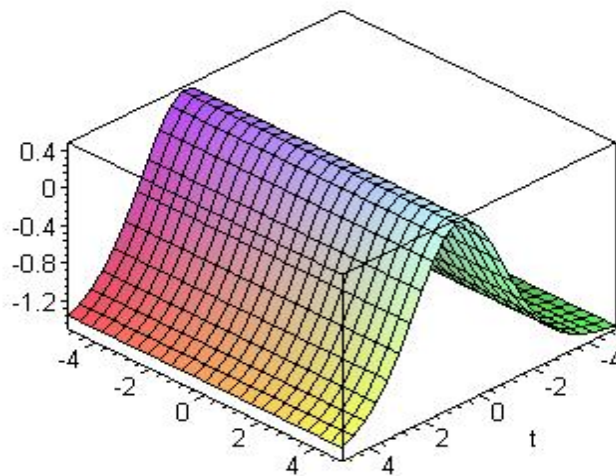


Figure 3.1 (c)

Figure 3.1 (c) depicts the periodic solutions of equation (6b) drawn on the whole domain when $b_{-1}=b_0=k=\omega=1$.

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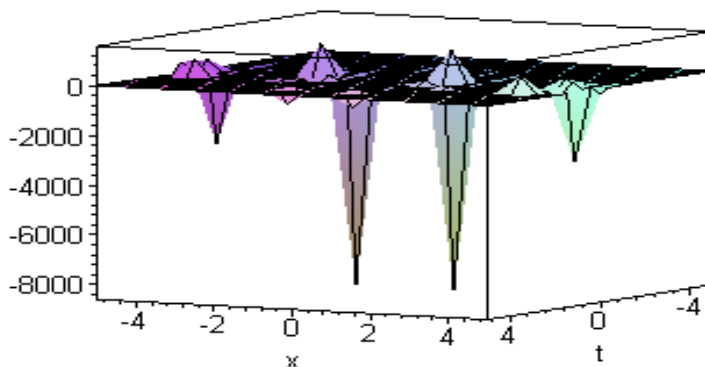


Figure 3.1 (d)

Figure 3.1 (d) depicts the periodic solutions of equation (6b) when $b_{-1} = b_0 = \omega = 1$, and $K = 2$.

Case 7. 2. If $p = c = 2$, and $q = d = 1$, then equation (6b) reduces to

$$u \eta = \frac{a_2 \exp 2\eta + a_1 \exp \eta + a_0 + a_{-1} \exp -\eta}{b_2 \exp 2\eta + b_1 \exp \eta + b_0 + b_{-1} \exp -\eta}. \quad (15b)$$

Proceeding as before, we obtain

$$\begin{aligned} b_2 &= \frac{1}{4} \frac{b_1^2}{b_0}, \quad b_1 = b_1, \quad b_0 = b_0, \quad a_1 = -\frac{1}{2} \frac{b_1 (5k^4 + k^2 - \omega^2)}{k^2}, \quad \omega = \omega, \quad b_{-1} = 0, \\ a_0 &= \frac{1}{2} \frac{b_0 (\omega^2 + k^4 - k^2)}{k^2}, \quad a_2 = \frac{1}{8} \frac{b_1^2 (\omega^2 + k^4 - k^2)}{k^2 b_0}, \quad b_0 = b_0, \quad a_{-1} = 0. \end{aligned} \quad (16b)$$

Hence, we get the generalized solitary solution $u(x, t)$ of equation (6b) as follows

$$u(x, t) = \frac{\frac{1}{8} \frac{b_1^2 (\omega^2 - k^2 + k^4)}{k^2 b_0} e^{2kx+2\omega t} - \frac{1}{2} \frac{b_1 (5k^4 + k^2 - \omega^2)}{k^2} e^{kx+\omega t} + \frac{1}{2} \frac{b_0 (\omega^2 - k^2 + k^4)}{k^2}}{\frac{1}{4} \frac{b_1^2 e^{2kx+2\omega t}}{b_0} + b_1 e^{kx+\omega t} + b_0}$$

or simply, we have

$$u(x,t) = \frac{\omega^2 + k^4 - k^2}{2k^2} - \frac{3b_1k^2}{\frac{1}{4} \frac{b_1^2}{b_0} e^{2kx+2\omega t} + b_1 e^{kx+\omega t} + b_0}, \tag{17b}$$

where b_0, b_1, ω and k are real numbers.

Example 7.2. Consider the following nonlinear differential equation which governs the unsteady flow of gas through a porous medium.

$$y''(x) + \frac{2x}{\sqrt{1-\alpha y}} y'(x) = 0, \quad 0 < \alpha < 1,$$

with the following typical boundary conditions imposed by the physical properties

$$y(0) = 1, \quad \lim_{x \rightarrow \infty} y(x) = 0.$$

The correction functional is given as

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left(y_n''(s) + \frac{2s}{\sqrt{1-\alpha y_n(s)}} y_n'(s) \right) ds, \quad 0 < \alpha < 1.$$

Making the correction functional stationary, the Lagrange multipliers can be identified as $\lambda(s) = -x - s$, we get the following iterative formula

$$y_{n+1}(x) = y_n(x) + \int_0^x (s-x) \left(y_n''(s) + \frac{2s}{\sqrt{1-\alpha y_n(s)}} y_n'(s) \right) ds, \quad 0 < \alpha < 1.,$$

where $A = y'(0)$. Consequently, following approximants are obtained

$$\begin{aligned} y_0(x) &= 1, \\ y_1(x) &= 1 + Ax, \\ y_2(x) &= 1 + Ax - \frac{A}{3\sqrt{1-\alpha}} x^3, \\ y_3(x) &= 1 + Ax - \frac{A}{3\sqrt{1-\alpha}} x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}} x^4 + \frac{A}{10(1-\alpha)} x^5, \\ y_4(x) &= 1 + Ax - \frac{A}{3\sqrt{1-\alpha}} x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}} x^4 + \frac{A}{10(1-\alpha)} x^5 - \frac{3\alpha^2 A^3}{80(1-\alpha)^{5/2}} x^5 + \frac{\alpha A^2}{15(1-\alpha)^2} x^6 + \dots \end{aligned}$$

The series solution is given as

$$y(x) = 1 + Ax - \frac{A}{3\sqrt{1-\alpha}}x^3 - \frac{\alpha A^2}{12(1-\alpha)^{3/2}}x^4 + \left(\frac{A}{10(1-\alpha)} - \frac{3\alpha^2 A^3}{80(1-\alpha)^{5/2}} \right)x^5 + \left(\frac{\alpha A^2}{15(1-\alpha)^2} - \frac{\alpha^3 A^4}{48(1-\alpha)^{7/2}} \right)x^6 + O(x^7),$$

The diagonal Pade' approximants [55] can be applied to analyze the physical behavior. Based on this, the [2/2] Pade' approximants produced the slope A to be

$$A = -\frac{2(1-\alpha)^{1/4}}{\sqrt{3\alpha}}, \tag{16}$$

and using [3/3] Pade' approximants we find

$$A = -\frac{\sqrt{(-4674\alpha + 8664)\sqrt{1-\alpha} - 144\gamma}}{57\alpha}, \tag{17}$$

where

$$\gamma = \sqrt{5(1-\alpha)(1309\alpha^2 - 2280\alpha + 1216)}. \tag{18}$$

Using (16-18) gives the values of the initial slope $A = y'(0)$ listed in the table 5. The formulas (16) and (17) suggest that the initial slope $A = y'(0)$ depends mainly on the parameter α , where $0 < \alpha < 1$.

Table 7.2 (a). Table 7.2 (a) exhibits the initial slopes $A = y'(0)$ for various values of α .

α	$B_{[2/2]} = y'(0)$	$B_{[3/3]} = y'(0)$
0.1	-3.556558821	-1.957208953
0.2	-2.441894334	-1.786475516
0.3	-1.928338405	-1.478270843
0.4	-1.606856838	-1.231801809
0.5	-1.373178096	-1.025529704
0.6	-1.185519607	-0.8400346085
0.7	-1.021411309	-0.6612047893
0.8	-0.8633400217	-0.4776697286
0.9	-0.6844600642	-0.2772628386

Table 7.2 (b). Table 7.2 (b) exhibits the values of $y(x)$ for $\alpha = 0.5$ for $x = 0.1$ to 1.0 .

x	y kidder	$Y_{[2/2]}$	$Y_{[3/3]}$
0.1	0.8816588283	0.8633060641	0.8979167028
0.2	0.7663076781	0.7301262261	0.7985228199
0.3	0.6565379995	0.6033054140	0.7041129703
0.4	0.5544024032	0.4848898717	0.6165037901
0.5	0.4613650295	0.3761603869	0.5370533796
0.6	0.3783109315	0.2777311628	0.4665625669
0.7	0.3055976546	0.1896843371	0.4062426033
0.8	0.2431325473	0.1117105165	0.3560801699
0.9	0.1904623681	0.04323673236	0.3179966614
1.0	0.1587689826	0.01646750847	0.2900255005

Example 7.3. Consider the following problem formulated in section 2 and is related to the free-convective boundary layer flow

$$f'''(\eta) + \theta(\eta) - f'(\eta)^2 = 0, \tag{19}$$

$$\theta''(\eta) - 3\sigma f'(\eta)\theta(\eta) = 0, \tag{20}$$

subject to the boundary conditions

$$f(0) = 0, \quad f'(0) = 0, \quad f'(+\infty) = 0, \tag{21}$$

$$\theta(0) = 1, \quad \theta(+\infty) = 0, \tag{22}$$

where the primes denote differentiation with respect to η and σ is the Prandtl number.

The correction functional is given by

$$\begin{cases} f_{n+1}(\eta) = f_n(\eta) + \int_0^x \lambda_1(s) \left(\frac{d^3 f_n}{ds^3} + \tilde{\theta}_n(\eta) - \left(\frac{d f_n}{d\eta} \right)^2 \right) ds. \\ \theta_{n+1}(\eta) = \theta_n(\eta) + \int_0^x \lambda_2(s) \left(\frac{d^2 \theta_n}{ds^2} - 3\sigma \left(\frac{d f_n}{d\eta} \right) \tilde{\theta}_n \right) ds. \end{cases}$$

Making the correction functional stationary, the Lagrange multipliers can easily be identified as

$$\lambda_1(s) = -\frac{1}{2!} \delta(s-\eta), \quad \lambda_2(s) = (s-\eta). \text{ Consequently,}$$

$$\begin{cases} f_{n+1}(\eta) = f_n(\eta) - \int_0^x \frac{1}{2!} \left(-\eta \right) \left(\frac{d^3 f_n}{d s^3} + \theta_n(\eta) - \left(\frac{d f_n}{d \eta} \right)^2 \right) ds. \\ \theta_{n+1}(\eta) = \theta_n(\eta) + \int_0^x (s - \eta) \left(\frac{d^2 \theta_n}{d s^2} - 3 \sigma \left(\frac{d f_n}{d \eta} \right) \theta_n \right) ds. \end{cases}$$

Applying the modified variational iteration method (MVIM), we get

$$\begin{cases} f_0 + p f_1 + \dots = f_0(\eta) \\ - p \int_0^x \frac{1}{2!} \left(-\eta \right) \left(\left(\frac{d^3 f_0}{d s^3} + p \frac{d^3 f_1}{d s^3} + \dots \right) + \theta_0 + p \theta_1 + \dots - \left(\frac{d f_0}{d \eta} + p \frac{d f_1}{d \eta} + \dots \right)^2 \right) ds. \\ \theta_0 + p \theta_1 + \dots = \theta_0(\eta) \\ + p \int_0^x (s - \eta) \left(\left(\frac{d^2 \theta_0}{d s^2} + p \frac{d^2 \theta_1}{d s^2} + \dots \right) - 3 \sigma \left(\frac{d f_0}{d \eta} + p \frac{d f_1}{d \eta} + \dots \right) \theta_0 + p \theta_1 + \dots \right) ds. \end{cases}$$

Comparing the coefficients of like powers of p , we get

$$p^0 : f_0(\eta) = \left(\frac{\alpha_1}{2} \right) \eta^2, \tag{23}$$

$$p^1 : f_1(\eta) = \left(\frac{\alpha_1}{2} \right) \eta^2 - \left(\frac{1}{6} \right) \eta^3 + \left(\frac{\alpha_2}{24} \right) \eta^4 + \left(\frac{\alpha_1^2}{60} \right) \eta^5, \tag{24}$$

$$\begin{aligned} p^2 : f_2(\eta) = & \left(\frac{\alpha_1}{2} \right) \eta^2 - \left(\frac{1}{6} \right) \eta^3 + \left(\frac{\alpha_2}{24} \right) \eta^4 + \left(\frac{\alpha_1^2}{60} \right) \eta^5 \\ & - \left(\frac{\sigma \alpha_1}{240} + \frac{\alpha_1}{120} \right) \eta^6 + \left(\frac{\alpha_1 \alpha_2}{630} + \frac{\sigma \alpha_1 \alpha_2}{120} \right) \eta^7 + \left(\frac{\alpha_1^3}{2016} \right) \eta^8, \end{aligned} \tag{25}$$

$$\begin{aligned} p^3 : f_3(\eta) = & \left(\frac{\alpha_1}{2} \right) \eta^2 - \left(\frac{1}{6} \right) \eta^3 + \left(\frac{\alpha_2}{24} \right) \eta^4 + \left(\frac{\alpha_1^2}{60} \right) \eta^5 - \left(\frac{\sigma \alpha_1}{240} + \frac{\alpha_1}{120} \right) \eta^6 \\ & + \left(\frac{\alpha_1 \alpha_2}{630} + \frac{\sigma \alpha_1 \alpha_2}{120} + \frac{\sigma}{1680} + \frac{1}{840} \right) \eta^7 + \left(\frac{\alpha_1^3}{2016} + \frac{\sigma \alpha_2}{3360} + \frac{\alpha_2}{2016} \right) \eta^8 \\ & + \left(-\frac{\alpha_1^2 \sigma^2}{10080} + \frac{\alpha_1^2 \sigma}{8640} + \frac{\sigma \alpha_2^2}{30240} - \frac{11 \alpha_1^2}{30240} + \frac{\alpha_2^2}{18144} \right) \eta^9 \\ & + \left(-\frac{\alpha_1^2 \alpha_2}{14400} - \frac{19 \alpha_1^2 \alpha_2 \sigma}{604800} - \frac{\alpha_1^2 \sigma^2 \alpha_2}{40320} \right) \eta^{10} + \left(\frac{\alpha_1^{24}}{66528} \right) \eta^{11}, \\ & \vdots \end{aligned} \tag{26}$$

and

$$p^0 : \theta_0(\eta) = 1 + \alpha_2 \eta, \tag{27}$$

$$p^1 : \theta_1(\eta) = 1 + \alpha_2 \eta + \left(\frac{\sigma \alpha_1 \alpha_2}{2}\right) \eta^3 + \left(\frac{\sigma \alpha_1 \alpha_2}{4}\right) \eta^4, \tag{28}$$

$$p^2 : \theta_2(\eta) = 1 + \alpha_2 \eta + \left(\frac{\sigma \alpha_1 \alpha_2}{2}\right) \eta^3 + \left(\frac{\sigma \alpha_1 \alpha_2}{4} + \frac{\sigma}{8}\right) \eta^4 - \left(\frac{\alpha_1 \alpha_2}{10}\right) \eta^5 + \left(\frac{\alpha_1^2 \sigma^2}{20} + \frac{\alpha_1^2 \sigma}{120} - \frac{\alpha_2^2 \sigma}{60}\right) \eta^6 + \left(\frac{\alpha_1^2 \sigma \alpha_2}{168} + \frac{\alpha_1^2 \alpha_2 \sigma^2}{56}\right) \eta^7, \tag{29}$$

$$p^3 : \theta_3(\eta) = 1 + \alpha_2 \eta + \left(\frac{\sigma \alpha_1 \alpha_2}{2}\right) \eta^3 + \left(\frac{\sigma \alpha_1 \alpha_2}{4} + \frac{\sigma}{8}\right) \eta^4 - \left(\frac{\alpha_1 \alpha_2}{10}\right) \eta^5 + \left(\frac{\alpha_1^2 \sigma^2}{20} + \frac{\alpha_1^2 \sigma}{120} - \frac{\alpha_2^2 \sigma}{60}\right) \eta^6 + \left(\frac{\alpha_1^2 \sigma \alpha_2}{168} + \frac{\alpha_1^2 \alpha_2 \sigma^2}{56} - \frac{\sigma \alpha_1}{280} - \frac{\alpha_1 \sigma^2}{35}\right) \eta^7 + \left(-\frac{11 \alpha_1 \sigma \alpha_2}{3360} - \frac{41 \alpha_1 \alpha_2 \sigma^2}{2240}\right) \eta^8 + \left(-\frac{\alpha_1 \sigma^2 \alpha_2^2}{360} + \frac{\alpha_1^3 \sigma}{6048} + \frac{\alpha_1^3 \sigma^2}{480} - \frac{\alpha_1 \alpha_2^2 \sigma}{2160} + \frac{\alpha_1^3 \sigma^3}{480}\right) \eta^9 + \left(\frac{\alpha_1^3 \alpha_2 \sigma \alpha_2^2}{7560} + \frac{\alpha_1^3 \sigma^2 \alpha_2}{1120} + \frac{\alpha_1^3 \alpha_2 \sigma^3}{1680}\right) \eta^{10}, \tag{30}$$

⋮

The series solution is given by

$$f(\eta) = \left(\frac{\alpha_1}{2}\right) \eta^2 - \left(\frac{1}{6}\right) \eta^3 + \left(\frac{\alpha_2}{24}\right) \eta^4 + \left(\frac{\alpha_1^2}{60}\right) \eta^5 - \left(\frac{\sigma \alpha_1}{240} + \frac{\alpha_1}{120}\right) \eta^6 + \left(\frac{\alpha_1 \alpha_2}{630} + \frac{\sigma \alpha_1 \alpha_2}{120} + \frac{\sigma}{1680} + \frac{1}{840}\right) \eta^7 + \left(\frac{\alpha_1^3}{2016} + \frac{\sigma \alpha_2}{3360} + \frac{\alpha_2}{2016}\right) \eta^8 + \left(-\frac{\alpha_1^2 \sigma^2}{10080} + \frac{\alpha_1^2 \sigma}{8640} + \frac{\sigma \alpha_2^2}{30240} - \frac{11 \alpha_1^2}{30240} + \frac{\alpha_2^2}{18144}\right) \eta^9 + \left(-\frac{\alpha_1^2 \alpha_2}{14400} - \frac{19 \alpha_1^2 \alpha_2 \sigma}{604800} - \frac{\alpha_1^2 \sigma^2 \alpha_2}{40320}\right) \eta^{10} + \left(\frac{\alpha_1^{24}}{66528}\right) \eta^{11} + \dots$$

$$\begin{aligned} \theta(\eta) = & 1 + \alpha_2 \eta + \left(\frac{\sigma \alpha_1 \alpha_2}{2} \right) \eta^3 + \left(\frac{\sigma \alpha_1 \alpha_2}{4} + \frac{\sigma}{8} \right) \eta^4 - \left(\frac{\alpha_1 \alpha_2}{10} \right) \eta^5 \\ & + \left(\frac{\alpha_1^2 \sigma^2}{20} + \frac{\alpha_1^2 \sigma}{120} - \frac{\alpha_2^2 \sigma}{60} \right) \eta^6 + \left(\frac{\alpha_1^2 \sigma \alpha_2}{168} + \frac{\alpha_1^2 \alpha_2 \sigma^2}{56} - \frac{\sigma \alpha_1}{280} - \frac{\alpha_1 \sigma^2}{35} \right) \eta^7 \\ & + \left(-\frac{11 \alpha_1 \sigma \alpha_2}{3360} - \frac{41 \alpha_1 \alpha_2 \sigma^2}{2240} \right) \eta^8 + \left(-\frac{\alpha_1 \sigma^2 \alpha_2^2}{360} + \frac{\alpha_1^3 \sigma}{6048} + \frac{\alpha_1^3 \sigma^2}{480} - \frac{\alpha_1 \alpha_2^2 \sigma}{2160} + \frac{\alpha_1^3 \sigma^3}{480} \right) \eta^9 \\ & + \left(\frac{\alpha_1^3 \alpha_2 \sigma \alpha_2^2}{7560} + \frac{\alpha_1^3 \sigma^2 \alpha_2}{1120} + \frac{\alpha_1^3 \alpha_2 \sigma^3}{1680} \right) \eta^{10} + \dots \end{aligned}$$

Table 7.3 (a). Numerical values of $\alpha_1 = f''(0)$

σ	[4,4]	[5,5]	[6,6]	α_1
0.001	1.1135529418	1.1272760416	1.1252849854	1.1231381347
0.01	1.0631737963	1.0741895683	1.0638385351	1.0633808585
0.1	0.9128082210	0.9238226280	0.9242158493	0.9240830397
1	0.6941230861	0.6929598014	0.6932195158	0.6932116298
10	0.4511240728	0.4502429544	0.4476712316	0.4471165250
100	0.2679197151	0.2681474363	0.2641295627	0.2645235434
1000	0.2204061432	0.1524783266	0.1500456755	0.1512901971
10000	0.0858587180	0.0858519249	0.0844775473	0.0855408524

Table 7.3 (b). Numerical values of $\alpha_2 = \theta'(0)$

σ	[4,4]	[5,5]	[6,6]	α_2
0.001	-0.0371141028	-0.0415417739	-0.0436188230	-0.0468074648
0.01	-0.1274922800	-0.1221616907	-0.1351353865	-0.1357607439
0.1	-0.3621215470	-0.3505589981	-0.3499273453	-0.3500596733
1	-0.7694165843	-0.7695971295	-0.7698955992	-0.7698611967
10	-1.5028543431	-1.5007437650	-1.4985484075	-1.4970992078
100	-2.7627624234	-2.7637067330	-2.7445541894	-2.7468855016
1000	-5.7787858408	-4.9468469883	-4.9104728566	-4.9349476252
10000	-8.8057265644	-8.8032691004	-8.7384279086	-8.8044492660

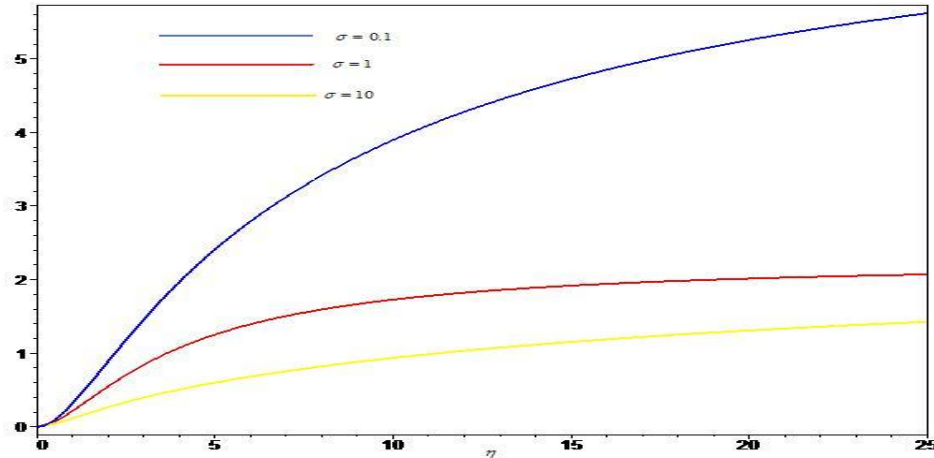


Figure 7.3 (a). Variation of $f(\eta)$ using $\phi_{6[6,6]}$ for $\sigma = 0.1$, $\phi_{6[5,5]}$ for $\sigma = 1$ and $\phi_{6[4,4]}$ for $\sigma = 10$.

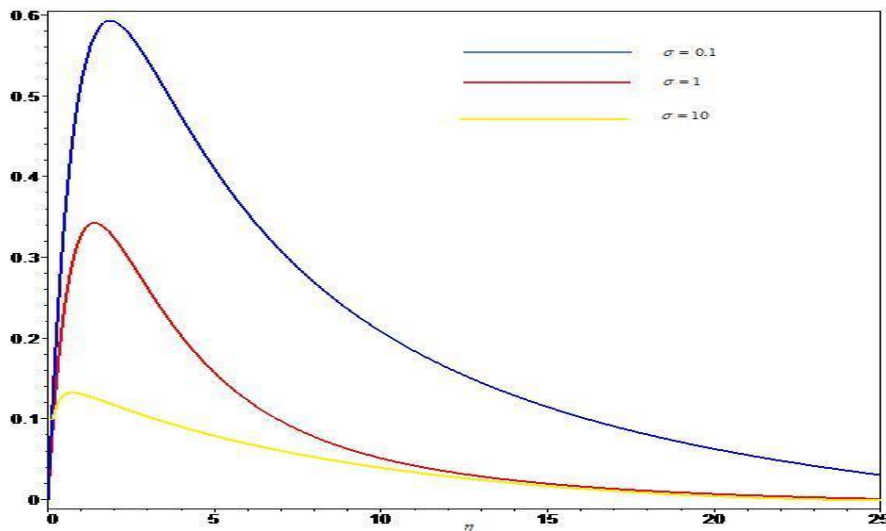


Figure 7.3 (b). Variation of $f'(\eta)$ using $\phi'_{6[6,6]}$ for $\sigma = 0.1$, $\phi'_{6[5,5]}$ for $\sigma = 1$ and $\phi'_{6[4,4]}$ for $\sigma = 10$.

It is observed in Figs.7.3 (a) -7.3 (b) that the flow has a boundary layer structure and the thickness of this boundary layer decreases with increase in the Prandtl number, σ as expected. This is due to the inhibiting influence of the viscous forces.

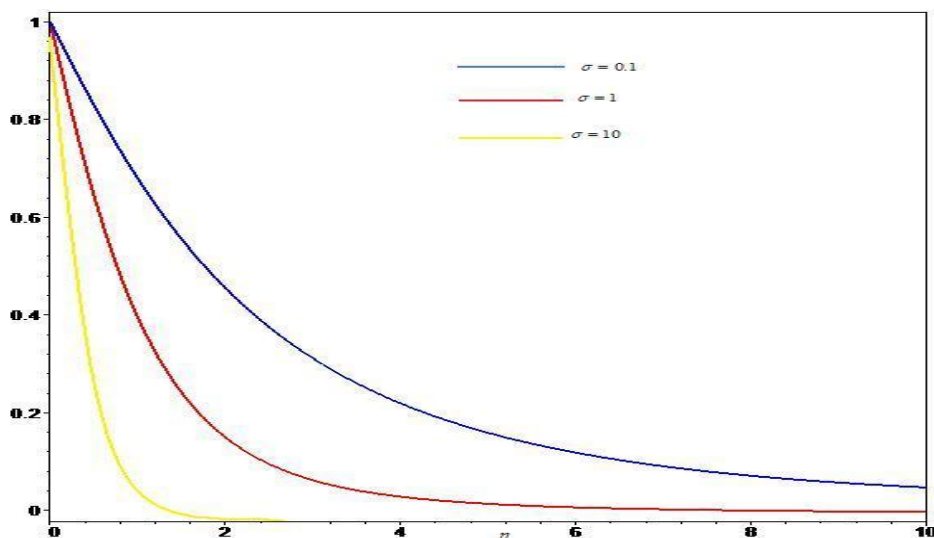


Figure 7.3 (c). Variation of $\theta(\eta)$ using $\varpi_{7[6,6]}$ for $\sigma = 0.1$, $\varpi_{7[5,5]}$ for $\sigma = 1$ and $\phi_{7[4,4]}$ for $\sigma = 10$.

Figure 7.3 (c) shows that the increase of the Prandtl number, σ , and results in the decrease, as expected, of temperature distribution at a particular point of the flow region, i.e. there would be a decrease of the thermal boundary layer thickness with the increase of values of σ implying a slow rate of thermal diffusion. Thus higher Prandtl number σ leads to faster cooling of the plane sheet.

Example 7.4 . Consider the following nonlinear inhomogeneous Goursat problem

$$u_{xt} = -u^2 + e^{2x} + e^{2t}t + 2e^{x+t},$$

$$u(x,0) = 1 + e^x, \quad u(0,t) = 1 + e^t, \quad u(0,0) = 2.$$

The correction functional for the above problem is given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(s) \left(\frac{\partial^2 u_n}{\partial x \partial s} + \tilde{u}_n^2 - e^{2x} - e^{2s}s - 2e^{x+s} \right) ds, \quad n \geq 0.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = -1$. Consequently,

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial^2 u_n}{\partial x \partial s} + u_n^2 - e^{2x} - e^{2s}s - 2e^{x+s} \right) ds, \quad n \geq 0.$$

Applying the modified variational iteration method

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial^2 u_n}{\partial x \partial s} + \left(\sum_{n=0}^{\infty} A_n \right) - e^{2x} - e^{2s} - 2e^{x+s} \right) ds, \quad n \geq 0,$$

where A_n are the so-called Adomian's polynomials and can be generated for all type of nonlinearities according to the specific algorithm defined in Mohyud-Din and Noor (2009), Mohyud-Din et al. (2009). Choosing $u_0(x,t) = Ae^x + Be^t$, as the initial value, the following approximants are obtained

$$\begin{aligned} u_0(x,t) &= Ae^x + Be^t, \\ u_1(x,t) &= Ae^x + Be^t - 2ABe^{x+t} - \frac{1}{2}B^2e^{2t} - A^2te^{2x} + \frac{1}{2}e^{2t} + te^{2x} + 2e^{x+t} \\ &\quad + 2ABe^x - 2e^x + \frac{1}{2}(B^2 - 1), \\ &\vdots \end{aligned}$$

The series solution is given by

$$u(x,t) = Ae^x + Be^t - 2ABe^{x+t} - \frac{1}{2}B^2e^{2t} - A^2te^{2x} + \frac{1}{2}e^{2t} + te^{2x} + 2e^{x+t} + 2ABe^x - 2e^x + \frac{1}{2}(B^2 - 1).$$

Imposing the boundary conditions $u(0,0) = 2$, $u(x,0) = 1 + e^x$, to find the constants A and B will yield $A=1$, and $B=1$, consequently

$$u(x,t) = e^x + e^t.$$

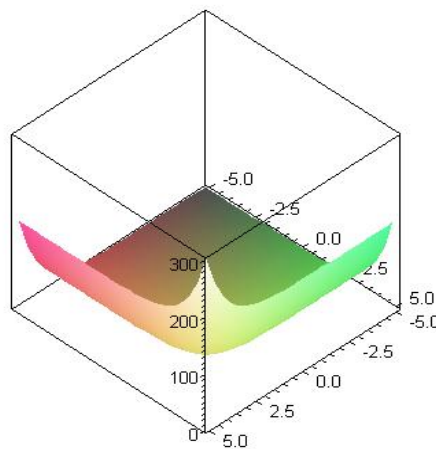


Figure 7.4. Figure 7.4 depicts closed form solution.

Example 7.5. Consider the Thomas-Fermi equation

$$y''(x) = \frac{y^{3/2}}{x^{1/2}}, \tag{31}$$

with boundary conditions

$$y(0) = 1, \quad \lim_{x \rightarrow \infty} y(x) = 0. \tag{32}$$

Applying the homotopy perturbation method

$$y_0 + y_1 + y_2 + \dots = y_0(x) + p \int_0^x \int_0^x x^{-1/2} (y_0 + p y_1 + p^2 y_2 + \dots)^{3/2} dx dx. \tag{33}$$

The given initial values admits the use of $y_0(x) = 1 + Bx$, where $B = y'(0)$, but we use the modified approach and take $y_0(x) = 1$. Comparing the co-efficient of like powers of p

$$\begin{aligned} p^{(0)} : y_0(x) &= 1, \\ p^{(1)} : y_1(x) &= Bx + \frac{4}{3}x^{3/2}, \\ p^{(2)} : y_2(x) &= \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3, \\ p^{(3)} : y_3(x) &= \frac{3}{70}B^2x^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} + \frac{1}{3}x^3, \\ p^{(4)} : y_4(x) &= -\frac{1}{252}B^3x^{9/2} + \frac{1}{175}B^2x^5 + \frac{2}{27}x^{9/2} + \frac{3}{70}B^2x^{7/2}, \\ p^{(5)} : y_5(x) &= \frac{1}{1056}B^4x^{11/2} + \frac{4}{1575}B^3x^6 + \frac{557}{100100}B^2x^{13/2} + \frac{4}{693}Bx^7 + \frac{101}{52650}x^{15/2}, \\ p^{(6)} : y_6(x) &= -\frac{3}{9152}B^5x^{13/2} - \frac{29}{24255}B^4x^7 - \frac{512}{351000}B^3x^{15/2} \\ &\quad - \frac{46}{45045}B^2x^8 - \frac{113}{1178100}Bx^{17/2} + \frac{23}{473850}x^9, \\ &\quad \vdots \end{aligned}$$

The series solution is given as

$$\begin{aligned}
 y(x) = & 1 + Bx + Bx + \frac{4}{3}x^{3/2} + \frac{2}{5}Bx^{5/2} + \frac{1}{3}x^3 + \frac{3}{70}B^2x^{7/2} + \frac{2}{15}Bx^4 + \frac{2}{27}x^{9/2} + \frac{3}{70}B^2x^{7/2} \\
 & - \frac{1}{252}B^3x^{9/2} + \frac{1}{175}B^2x^5 + \frac{2}{27}x^{9/2} + \frac{3}{70}B^2x^{7/2} + \frac{1}{1056}B^4x^{11/2} + \frac{4}{1575}B^3x^6 \\
 & + \frac{557}{100100}B^2x^{13/2} + \frac{4}{693}Bx^7 + \frac{101}{52650}x^{15/2} - \frac{3}{9152}B^5x^{13/2} - \frac{29}{24255}B^4x^7 \\
 & - \frac{512}{351000}B^3x^{15/2} - \frac{46}{45045}B^2x^8 - \frac{113}{1178100}Bx^{17/2} + \frac{23}{473850}x^9 \dots, \\
 & \vdots
 \end{aligned}$$

Setting $x^{1/2} = t$, the series solution is obtained as

$$\begin{aligned}
 y(t) = & 1 + Bt^2 + \frac{4}{3}t^3 + \frac{2}{5}Bt^5 + \frac{1}{3}t^6 + \frac{3}{70}B^2t^7 + \frac{2}{15}Bt^8 + \left(-\frac{1}{252}B^3 + \frac{2}{27}\right)t^9 + \frac{1}{175}B^2t^{10} \\
 & + \left(\frac{1}{1056}B^4 + \frac{31}{1485}B\right)t^{11} + \left(\frac{4}{1575}B^3 + \frac{4}{405}\right)t^{12} + \left(-\frac{3}{9152}B^5 + \frac{557}{100100}B^2\right)t^{13} \\
 & + \left(-\frac{29}{24255}B^4 + \frac{4}{693}B\right)t^{14} + \left(\frac{7}{499}B^6 - \frac{623}{351000}B^3 + \frac{101}{52650}\right)t^{15} \\
 & + \left(\frac{68}{105105}B^4 - \frac{46}{45045}B^2\right)t^{16} + \left(-\frac{3}{43520}B^7 + \frac{153173}{116424000}B^4 - \frac{113}{1178100}B\right)t^{17} \\
 & + \left(-\frac{4}{10395}B^6 + \frac{1046}{675675}b^3 + \frac{23}{473850}\right)t^{18} + \left(-\frac{1232941}{1278076800}B^5 + \frac{799399}{698377680}B^2\right)t^{19} \\
 & + \left(-\frac{99856}{70945875}B^4 + \frac{51356}{103378275}B\right)t^{20} + \dots.
 \end{aligned}$$

Table 7.5. Pade' approximants and initial slopes $y'(0)$.

Pade approximants	Initial slope $y'(0)$	Error (%)
[2/2]	-1.211413729	23.71
[4/4]	-1.550525919	2.36
[7/7]	-1.586021037	12.9×10^{-2}
[8/8]	-1.588076820	3.66×10^{-4}
[10/10]	-1.588076779	3.64×10^{-4}

Example 7.6. Consider the following nonlinear oscillator with discontinuous term

$$u_{tt} = -u|u|, \quad u(0) = A, \quad u_t(0) = 0. \tag{34}$$

We rewrite (37) in the form

$$u_{tt} + \omega u + 1u|u| = 0. \quad (35)$$

Applying the parameter expansion technique, we get

$$u_0'' + \omega^2 u_0 = 0, \quad u_0(0) = A, \quad u_0'(0) = 0. \quad (36)$$

$$u_1'' + \omega^2 u_1 + a_1 u_0 + b_1 u_0 |u_0| = 0, \quad u_1(0) = 0, \quad u_1'(0) = 0. \quad (37)$$

Using the Fourier series expansion $|A \cos \omega t| = A \left(\frac{2}{\pi} + \frac{4}{3\pi} \cos 2\omega t + \sum_{k=2}^{\infty} a_{2k} \cos 2k\omega t \right)$ would yield

$$u_1'' + \omega^2 u_1 + A \cos \omega t \left(a_1 + \frac{8}{3\pi} b_1 A \right) + \frac{2}{3\pi} b_1 A^2 \cos 3\omega t + A^2 b_1 \cos \omega t \sum_{k=2}^{\infty} \cos 2k\omega t = 0 \quad (38)$$

Elimination of the secular term requires $a_1 = -\frac{8}{3\pi} b_1 A$. If only the first-order approximation is searched for, then $a_1 = -\omega^2 b_1$, which leads to the result $\omega^2 = \frac{8A}{3\pi}$.

6. Conclusion

In this paper, we made a detailed study of some relatively new techniques along with some of their modifications. In particular, we focused on He's VIM, HPM, MVIMS, exp-function and expansion of parameters methods and discussed in length their respective applications to solve various diversified initial and boundary value problems. These proposed methods and their modifications are employed without using linearization, discretization, transformation or restrictive assumptions, absorb the positive features of the coupled techniques and hence are very much compatible with the diversified and versatile nature of the physical problems. Moreover, the modification of VIM based upon He's polynomials (VIMHP) is easier to implement and is more user friendly as compare to the one where Adomian's polynomials (VIMAP) along with their complexities are used. It is also observed that the coupling of He's or Adomian's polynomials with the correction functional of VIM makes the solution procedure simpler and hence the evaluation of nonlinear term becomes easier. It may be concluded that these relatively new techniques can be treated as alternatives for solving a wide class of nonlinear problems.

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