Exact Solutions of Two Nonlinear Space-time Fractional Differential Equations by Application of Exp-function Method

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Abstract

In this paper, we discuss on the exact solutions of the nonlinear space-time fractional Burger-like equation and also the nonlinear fractional fifth-order Sawada-Kotera equation with the exp-function method. We use the functional derivatives in the sense of Riemann-Jumarie derivative and fractional convenient variable transformation in this study. Further, we obtain some exact analytical solutions including hyperbolic function.

1. Introduction

Obvious, fractional differential equations (FDEs) have been the focus of many studies due to their frequent appearance in several applications in physics, biology, engineering, signal processing, systems identification, control theory, finance, fractional dynamics, and other different sciences (see Kilbas et al. (2006), Mille and Ross (1993), and Podlubny (1999)). The exact solution of the fractional differential equations have received tremendous attention of great, several investigators have successfully used reliable and algebraic methods for this purpose (see Taghizadeh et al. (2015, 2016)). However, there are several methods to obtain exact solutions of fractional

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differential Burger-like equation and fractional differential Sawada-Kotera equation; for example: the Sub-equation method, see Wang et al. (2017), the (G'/G) method, see Cerdid et al. (2019), the F-expansion method, see Inan et al.(2017), the $\exp(-\phi(\xi))$ method see Ali et al. (2016) and other methods, see Bulut et al. (2013), but we apply the exp-function method to nonlinear FDES's, space-time fractional Burger-like equation, see Cerdid et al. (2019) and Inan et al. (2017) and time-fractional fifth-order Sawada-Kotera equation, see Ali et al. (2016) and Guner et al. (2017).

The organization of the paper follows in the second and third sections of the paper we define the Riemann-Jumarie Derivative and we describe the exp-function method respectively. In Section 4 and Section 5 application of the exp-function method for Burger-like equation and Sawada-Kotera equation respectively. In the end, the general conclusion is given in Section 6.

2.   The Riemann-Jumarie Derivative

Assume that $f : R \longrightarrow R, x \longrightarrow f(x)$ denotes a continuous (but not nesseciary differentiable) function, then Riemann-Jumarie derivative (see Jumarie(2006, 2009)),

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha}(f(\xi) - f(0))d\xi, & \text{if} \quad 0 < \alpha \leq 1, \\ (f^{(n)}(t))^{(\alpha-n)}, & \text{if} \quad n \leq \alpha < n+1, n \geq 1. \end{cases} \tag{1}$$

Some important properties for the Riemann-Jumarie derivative of order α are listed below,

$$D_t^\alpha t^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{\beta-\alpha}, \tag{2}$$

$$D_t^\alpha(f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{3}$$

$$D_t^\alpha f[g(t)] = f_g'[g(t)]D_t^\alpha g(t) = D_t^\alpha f[g(t)](g'(t))^\alpha. \tag{4}$$

We consider the fractional partial differential equation, with independent variable $t, x_1, x_2, ..., x_n$ and dependent variable u :

$$P(u, D_t^\alpha u, D_{x_1}^\alpha u, ..., D_t^{2\alpha} U, ...) = 0 \qquad 0 < \alpha \leq 1, t > 0. \tag{5}$$

By using the fractional variable transformation

$$u(t, x_1, ..., x_n) = u(\xi) \quad \text{and} \quad \xi = \frac{ct^\alpha}{\Gamma(1+\alpha)} + \frac{k_1 x_1^\alpha}{\Gamma(1+\alpha)} + ... + \frac{k_n x_n^\alpha}{\Gamma(1+\alpha)}, \tag{6}$$

where c and $k_i, (i = 1, ..., n)$ are nonzero arbitrary constants. The fractional differential equation (5) reduced to a nonlinear ordinary differential equation (ODE)

$$F(u, u', u'', ...) = 0, \tag{7}$$

where the prime denotes the derivation with respect to ξ and $u' = \frac{du}{d\xi}$.

3.   The Exp-function method

In the exp-function method, we assume that the solution is a function,

$$u(\xi) = \frac{\sum_{p=-m_1}^{m_2} a_p \exp[p\xi]}{\sum_{q=-n_1}^{n_2} a_q \exp[q\xi]}, \tag{8}$$

where m_1, m_2, n_1 and n_2 are unknown positive integers that will be determined by using balance method, for constructing the relations between m_2, m_1, n_2 and n_1, first we consider the highest degree and the highest derivative order of function $u(\xi)$ in the nonlinear (ODE) (7). For relation between m_2 and n_2 by creating a balance between the first sentence of the highest degree and the first sentence of the highest derivative order. Similarly, the relation between m_1 and n_1 is obtained by balancing the last sentence of the highest degree and the last sentence of the highest derivative order. Ebaid (2012) proved in Theorem 1 and Theorem 2 that $m_2 = n_2$ and $m_1 = n_1$ are the only relations that can be obtained by applying the balancing method in the nonlinear (ODE) (7). In the exp-function method, additional calculations of balancing the highest order linear term with the highest order nonlinear term are not longer required in future because we only use their first and last sentences. Hence, the method becomes more straightforward.

4.   The space-time fractional Burger-like equation

Burger's equation is related to applications in acoustic phenomena and has been utilized to model turbulence and certain steady-state viscous flows. We consider conformable space-time fractional Burger-like equation (see Cerdid et al. (2019) and Inan et al. (2017)),

$$D_t^\alpha u + D_x^\alpha u + u D_x^\alpha u + \frac{1}{2} D_x^{2\alpha} u = 0, \qquad t > 0, 0 < \alpha \le 1, \tag{9}$$

where α is a parameter that describes the order of the fractional time derivative. By using variable transformations

$$u(x,t) = u(\xi), \qquad \text{and} \qquad \xi = \frac{ct^\alpha}{\Gamma(1+\alpha)} + \frac{kx^\alpha}{\Gamma(1+\alpha)}, \tag{10}$$

where c and k are constants, Equation (9) is reduced to an ordinary defferential equation

$$(c+k)u' + kuu' + \frac{k^2}{2}u'' = 0, \tag{11}$$

where $u' = \frac{du}{d\xi}$ .Integrating equation (11) with respect to ξ yields

$$(c+k)u + \frac{1}{2}ku^2 + \frac{k^2}{2}u' = 0. \tag{12}$$

By using Equation (8) and Equation (12), we have

$$u'(\xi) = \frac{A\exp[(n_2+m_2)\xi] + ... + B\exp[-(n_1+m_1)\xi]}{C\exp[2n_2\xi] + ... + D\exp[-2n_1\xi]}, \tag{13}$$

$$u^2(\xi) = \frac{E\exp[2m_2\xi] + ... + F\exp[-2m_1\xi]}{G\exp[2n_2\xi] + ... + H\exp[-2n_1\xi]}, \tag{14}$$

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where A, B, C, D, E, F, G and H are coefficients determined by $a_{m_2}, a_{-m_1}, b_{n_2}$ and b_{-n_1}.

By balancing the highest order Exp-function in u' and u^2 in Equation (12), we have $n_2 + m_2 = 2m_2$, which implies $m_2 = n_2$. Similarly, balancing the lowest order exp-function in u' and u^2, we have $n_1 + m_1 = 2m_1$, which implies $m_1 = n_1$. For simplicity, we set $m_2 = n_2 = 1$ and $m_1 = n_1 = 1$ so Equation (8) reduces to

$$u(\xi) = \frac{a_1 \exp[\xi] + a_0 + a_{-1} \exp[-\xi]}{b_1 \exp[\xi] + b_0 + b_{-1} \exp[-\xi]}. \tag{15}$$

Substitating Equation (15) into Equation (12) and the help of Maple, we have

$$\frac{1}{A}(R_1 \exp[2\xi] + R_2 \exp[\xi] + R_3 + R_4 \exp[-\xi] + R_5 \exp[-2\xi]) = 0, \tag{16}$$

where

$$A = (b_1 \exp[\xi] + b_0 + b_{-1} \exp[-\xi])^2,$$

$$R_1 = (c + k)a_1 b_1 + \frac{1}{2}ka_1^2,$$

$$R_2 = (c + k)[a_0 b_1 + a_1 b_0] + ka_1 a_0 + \frac{1}{2}k^2 a_1 b_0 - \frac{1}{2}k^2 a_0 b_1,$$

$$R_3 = (c + k)[a_0 b_0 + 2a_1 b_{-1} + a_{-1} b_1] + k[a_1 a_{-1} + a_{-1} b_1 + \frac{1}{2}a_0^2] + k^2[a_1 b_{-1} - a_{-1} b_1],$$

$$R_4 = (c + k)[a_0 b_{-1} + a_{-1} b_0] + kb_{-1} a_0 - \frac{1}{2}k^2 a_{-1} b_0 + \frac{1}{2}k^2 a_0 b_{-1},$$

$$R_5 = (c + k)a_{-1} b_{-1} + \frac{1}{2}ka_{-1}^2. \tag{17}$$

Solving this system of algebraic equation by using symbolic computation, we obtain the following results.

If $a_1 = 0$, $a_0 = 0$, $a_{-1} = -2kb_{-1}$, $b_1 = b_1$, $b_0 = 0$, $b_{-1} = b_{-1}$, $c = k^2 - k$ and $k = k$.

If we set $b_1 = \frac{1}{2}$, $b_{-1} = -\frac{1}{2}$ and $k = 2$, then we have $a_{-1} = -2$ and $c = 2$, we obtain the following hyperbolic function solitary solution,

$$u(x, t) = \frac{2 \exp[-2(\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)})]}{\sinh(2(\frac{-x^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)}))}. \tag{18}$$

5. The space-time fractional fifth-order Sawada-Kotera equation

We consider the space-time fractional fifth-order Sawada-Kotera equation (see Ali et al. (2016) and Guner et al. (2017)),

$$D_t^\alpha u + D_x^{5\alpha} u + 45U^2 D_x^\alpha u + 15(D_x^\alpha u D_x^{2\alpha} u + u D_x^{3\alpha} u) = 0, \qquad t > 0, 0 < \alpha \le 1, \tag{19}$$

where α is a parameter describing the order of the fractional time-derivative. The function $u(x,t)$ is assumed to be a causal function of time. As in the previous section, we use variable transformations

$$u(x,t) = u(\xi), \qquad \text{and} \qquad \xi = \frac{ct^\alpha}{\Gamma(1+\alpha)} + \frac{kx^\alpha}{\Gamma(1+\alpha)}, \qquad (20)$$

where c and k are constants, Equation (19) is reduced to an ordinary defferential equation

$$cu' + k^5 u^{(5)} + 45ku^2 u' + 15(k^3 u'u'' + k^3 uu^{(3)}) = 0, \qquad (21)$$

where $u' = \frac{du}{d\xi}$. Integrating Equation (21) with respect to ξ yields.

$$cu + k^5 u^{(4)} + 15ku^3 + 15k^3(uu'') = 0. \qquad (22)$$

We have

$$(u(\xi))^{(4)} = \frac{I\exp[(15n_2 + m_2)\xi] + ... + J\exp[-(15n_1 + m_1)\xi]}{K\exp[16n_2\xi] + ... + L\exp[-16n_1\xi]}, \qquad (23)$$

$$uu''(\xi) = \frac{M\exp[(3n_2 + 2m_2)\xi] + ... + N\exp[-(3n_1 + 2m_1)\xi]}{P\exp[5n_2\xi] + ... + Q\exp[-5n_1\xi]}, \qquad (24)$$

where I, J, K, L, M, N, P and Q are coefficients determined by $a_{m_2}, a_{-m_1}, b_{n_2}$ and b_{-n_1}.

By the same approach as in before this section, balancing the terms $u^{(4)}$ and uu'' in Equation (22) we have $m_2 = n_2, m_1 = n_1$. To simplicity, we set $m_2 = n_2 = 1$ and $m_1 = n_1 = 1$ so equation (8) degrades to

$$u(\xi) = \frac{a_1\exp[\xi] + a_0 + a_{-1}\exp[-\xi]}{b_1\exp[\xi] + b_0 + b_{-1}\exp[-\xi]}. \qquad (25)$$

Substiuting Equation (25) into Equation (22), and by the help of Maple, we have

$$\begin{aligned}
\frac{1}{A}(&R_1\exp[5\xi] + R_2\exp[4\xi] + R_3\exp[3\xi] + R_4\exp[2\xi] + R_5\exp[\xi] + R_6 \\
&+ R_7\exp[-\xi] + R_8\exp[-2\xi] + R_9\exp[-3\xi] + R_{10}\exp[-4\xi] + R_{11}\exp[-5\xi]) = 0,
\end{aligned} \qquad (26)$$

where

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$$A = (b_1 \exp[\xi] + b_0 + b_{-1} \exp[-\xi])^5,$$

$$R_1 = ca_1 b_1^4 + 15 k a_1^3 b_1^2,$$

$$R_2 = 15k^3 a_1 a_0 b_1^3 + c a_0 b_1^4 + 4 c a_1 b_1^3 b_0 - 15 k^3 a_1^2 b_1^2 b_0 + 45 k a_1^2 a_0 b_1^2 + k_5 a_0 b_1^4 - k^5 a_1 b_1^3 b_0 + 30 k a_1^3 b_1 b_0,$$

$$R_3 = 30 k a_1^3 b_1 b_{-1} + 15 k a_1^3 b_0^2 + c a_{-1} b_1^4 + 45 k a_1 a_0^2 b_1^2 - 16 k^5 a_1 b_1^3 b_{-1} + 15 k^3 a_0^2 b_1^3 - 60 k^3 a_1^2 b_1^2 b_{-1}$$
$$+ 60 k^3 a_1 a_{-1} b_1^3 + 16 k^5 a_{-1} b_1^4 - 11 k^5 a_0 b_1^3 b_0 + 4 c a_0 b_1^3 b_0 + 90 k a_1^2 a_0 b_1 b_0 + 6 c a_1 b_1^2 b_0^2 + 45 k a_1^2 a_{-1} b_1^2$$
$$+ 4 c a_1 b_1^3 b_{-1} - 15 k^3 a_1 a_0 b_1^2 b_0 + 11 k^5 a_1 b_1^2 b_0^2,$$

$$R_4 = -135 k^3 a_1 a_0 b_1^2 b_{-1} + 77 k^5 a_1 b_1^2 b_0 b_{-1} + 30 k a_1^3 b_0 b_{-1} - 76 k^5 a_0 b_1^3 b_{-1} + 4 c a_1 b_1 b_0^3 + 45 k a_1^2 a_0 b_0^2$$
$$- 11 k^5 a_1 b_1 b_0^3 + 15 k^3 a_1^2 b_0^3 + 75 k^3 a_0 a_{-1} b_1^3 + 6 c a_0 b_1^2 b_0^2 - k^5 a_{-1} b_1^3 b_0 - 30 k^3 a_1^2 b_1 b_{-1} b_0$$
$$+ 90 k^3 a_1 a_{-1} b_1^2 b_0 + 12 c a_1 b_1^2 b_0 b_{-1} + 11 k^5 a_0 b_1^2 b_0^2 + 15 k a_0^3 b_1^2 - 15 k^3 a_1 a_0 b_1 b_0^2 + 4 c a_{-1} b_1^3 b_0$$
$$+ 90 k a_1^2 a_{-1} b_1 b_0 + 4 c a_0 b_1^3 b_{-1} + 90 k a_1^2 a_0 b_1 b_{-1} + 90 k a_1 a_0 a_{-1} b_1^2 + 90 k a_1 a_0^2 b_1 b_0,$$

$$R_5 = 4 c a_0 b_1 b_0^3 + k^5 a_1 b_0^4 - 60 k^3 a_1 a_{-1} b_1^2 b_{-1} + 60 k^3 a_1 a_{-1} b_1 b_0^2 + 105 k^3 a_0 a_{-1} b_1^2 b_0 - 58 k^5 a_1 b_1 b_0^2 b_{-1}$$
$$+ 12 c a_0 b_1^2 b_{-1} b_0 + 12 c a_1 b_1 b_0^2 b_{-1} + 47 k^5 a_0 b_1^2 b_{-1} b_0 + 90 k a_1^2 a_0 b_0 b_{-1} - k^5 a_0 b_1 b_0^3 + 45 k a_1 a_0^2 b_0^2$$
$$+ 30 k a_0^3 b_1 b_0 - 15 k^3 a_0^2 b_1 b_0^2 + 15 k^3 a_0 a_1 b_0^3 + c a_1 b_0^4 + 15 k a_1^3 b_{-1}^2 + 60 k^3 a_{-1}^2 b_1^3$$
$$+ 180 k a_1 a_0 a_{-1} b_1 b_0 - 150 k^3 a_1 a_0 b_1 b_{-1} b_0 + 6 c a_1 b_1^2 b_{-1}^2 + 4 c a_{-1} b_1^3 b_{-1} + 6 c a_{-1} b_1^2 b_0^2$$
$$+ 176 k^5 a_1 b_1^2 b_{-1}^2 - 176 k^5 a_{-1} b_1^3 b_{-1} + 11 k^5 a_{-1} b_1^2 b_0^2 + 45 k a_1^2 a_{-1} b_0^2 + 45 k a_1 a_{-1}^2 b_1^2 + 45 k a_0^2 a_{-1} b_1^2$$
$$+ 60 k^3 a_1^2 b_0^2 b_{-1} - 75 k^3 a_0^2 b_1^2 b_{-1} + 90 k a_1^2 a_{-1} b_1 b_{-1} + 90 k a_1 a_0^2 b_1 b_{-1},$$

$$R_6 = 30 k^3 a_1 a_{-1} b_0^3 + 230 k^5 a_0 b_1^2 b_{-1}^2 + 45 k a_0 a_{-1}^2 b_1^2 + 105 k^3 a_1^2 b_0 b_{-1}^2 + 30 k a_0^3 b_1 b_{-1} + 90 k a_1^2 a_{-1} b_0 b_{-1}$$
$$+ 90 k a_1 a_0^2 b_0 b_{-1} + 90 k a_1 a_0 a_{-1} b_0^2 + 90 k a_1 a_{-1}^2 b_1 b_0 + 90 k a_0^2 a_{-1} b_1 b_0 - 75 k^3 a_1 a_0 b_1 b_{-1}^2$$
$$+ 180 k a_1 a_0 a_{-1} b_1 b_{-1} - 60 k^3 a_1 a_{-1} b_1 b_{-1} b_0 - 115 k^5 a_1 b_1 b_0 b_{-1}^2 - 115 k^5 a_{-1} b_1^2 b_0 b_{-1} + 12 c a_{-1} b_1^2 b_0 b_{-1}$$
$$+ 12 c a_1 b_1 b_0 b_{-1}^2 - 10 k^5 a_0 b_1 b_{-1} b_0^2 + 4 c a_1 b_0^3 b_{-1} - 120 k^3 a_0^2 b_1 b_{-1} b_0 + 12 c a_0 b_1 b_{-1} b_0^2 + 45 k a_1^2 a_0 b_{-1}^2$$
$$+ 4 c a_{-1} b_1 b_0^3 + 5 k^5 a_{-1} b_1 b_0^3 + 5 k^5 a_1 b_0^3 b_{-1} + 6 c a_0 b_1^2 b_{-1}^2 + c a_0 b_0^4 + 15 k^3 a_0^3 b_0^2 + 45 k^3 a_1 a_0 b_{-1} b_0^2$$
$$- 75 k^3 a_0 a_{-1} b_1^2 b_{-1} + 45 k^3 a_0 a_{-1} b_1 b_0^2 + 105 k^3 a_{-1}^2 b_1^2 b_0,$$

$$R_7 = -150 k^3 a_0 a_{-1} b_1 b_{-1} b_0 + 180 k a_1 a_0 a_{-1} b_0 b_{-1} + 4 c a_0 b_{-1} b_0^3 - k^5 a_0 b_{-1} b_0^3 + 30 k a_0^3 b_0 b_{-1} + 45 k a_0^2 a_{-1} b_0^2$$
$$+ 15 k^3 a_0 a_{-1} b_0^3 - 15 k^3 a_0^2 b_{-1} b_0^2 + 4 c a_1 b_1 b_{-1}^3 + 6 c a_1 b_0^2 b_{-1}^2 + 6 c a_{-1} b_1^2 b_{-1}^2 - 176 k^5 a_1 b_1 b_{-1}^3 + 11 k^5 a_1 b_0^2 b_{-1}^2$$
$$+ 176 k^5 a_{-1} b_1^2 b_{-1}^2 + 45 k a_1^2 a_{-1} b_{-1}^2 + 45 k a_1 a_0^2 b_{-1}^2 + 45 k a_1 a_{-1}^2 b_0^2 - 75 k^3 a_0^2 b_1 b_{-1}^2 + 60 k^3 a_{-1}^2 b_1 b_0^2$$
$$+ c a_{-1} b_0^4 + k^5 a_{-1} b_0^4 + 15 k a_{-1}^3 b_1^2 + 60 k^3 a_1^2 b_{-1}^3 - 58 k^5 a_{-1} b_1 b_0^2 b_{-1} + 12 c a_0 b_1 b_{-1}^2 b_0 + 12 c a_{-1} b_1 b_0^2 b_{-1}$$
$$+ 47 k^5 a_0 b_1 b_{-1}^2 b_0 + 90 k a_1 a_{-1}^2 b_1 b_{-1} + 90 k a_0^2 a_{-1} b_1 b_{-1} + 90 k a_0 a_{-1}^2 b_1 b_0 - 60 k^3 a_1 a_{-1} b_1 b_{-1}^2$$
$$+ 105 k^3 a_1 a_0 b_{-1}^2 b_0 + 60 k^3 a_1 a_{-1} b_0^2 b_{-1},$$

$$R_8 = 4 c a_0 b_1 b_{-1}^3 - 76 k^5 a_0 b_1 b_{-1}^3 + 90 k^3 a_1 a_{-1} b_0 b_{-1}^2 + 90 k a_0 a_{-1}^2 b_1 b_{-1} - k^5 a_1 b_0 b_{-1}^3 + 12 c a_{-1} b_1 b_0 b_{-1}^2$$
$$+ 90 k a_0^2 a_{-1} b_0 b_{-1} + 77 k^5 a_{-1} b_1 b_0 b_{-1}^2 - 15 k^3 a_0 a_{-1} b_0^2 b_{-1} + 6 c a_0 b_{-1}^2 b_0^2 + 30 k a_{-1}^3 b_1 b_0 - 30 k^3 a_{-1}^2 b_1 b_{-1} b_0$$
$$+ 45 k a_0 a_{-1}^2 b_0^2 - 11 k^5 a_{-1} b_0^3 b_{-1} + 15 k^3 a_{-1}^2 b_0^3 + 4 c a_1 b_0 b_{-1}^3 + 75 k^3 a_1 a_0 b_{-1}^3 - 135 k^3 a_0 a_{-1} b_1 b_{-1}^2$$
$$+ 90 k a_1 a_0 a_{-1} b_{-1}^2 + 11 k^5 a_0 b_{-1}^2 b_0^2 + 90 k a_1 a_{-1}^2 b_0 b_{-1} + 15 k a_0^3 b_{-1}^2 + 4 c a_{-1} b_0^3 b_{-1},$$

$$R_9 = 4 c a_{-1} b_1 b_{-1}^3 + 30 k a_{-1}^3 b_1 b_{-1} + 6 c a_{-1} b_0^2 b_{-1}^2 - 15 k^3 a_0 a_{-1} b_0 b_{-1}^2 + 90 k a_0 a_{-1}^2 b_0 b_{-1} + 15 k a_{-1}^3 b_0^2$$
$$+ 60 k^3 a_{-1} a_1 b_{-1}^3 + 45 k a_0^2 a_{-1} b_{-1}^2 - 16 k^5 a_{-1} b_1 b_{-1}^3 + 15 k^3 a_0^2 b_{-1}^3 - 60 k^3 a_{-1}^2 b_1 b_{-1}^2 + 11 k^5 a_{-1} b_0^2 b_{-1}^2$$
$$+ c a_1 b_{-1}^4 + 4 c a_0 b_{-1}^3 b_0 - 11 k^5 a_0 b_{-1}^3 b_0 + 16 k^5 a_1 b_{-1}^4 + 45 k a_1 a_{-1}^2 b_{-1}^2,$$

$$R_{10} = k^5 a_0 b_{-1}^4 + 30 k a_{-1}^3 b_0 b_{-1} - k^5 a_{-1} b_0 b_{-1}^3 + c a_0 b_{-1}^4 + 45 k a_0 a_{-1}^2 b_{-1}^2 + 4 c a_{-1} b_0 b_{-1}^3,$$
$$+ 15 k^3 a_{-1} a_0 b_{-1}^3 - 15 k^3 a_{-1}^2 b_0 b_{-1}^2$$

$$R_{11} = c a_{-1} b_{-1}^4 + 15 k a_{-1}^3 b_{-1}^2. \tag{27}$$

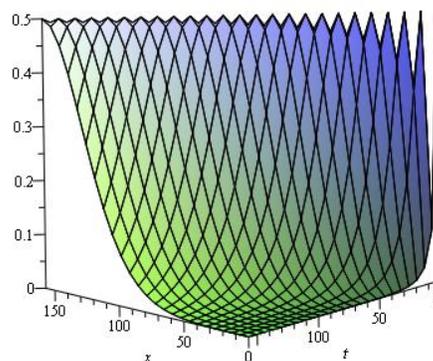Solving the above algebraic equations by using Maple for $a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, c$ and k, we have

the solution if $a_1 = 0$, $a_0 = b_0$, $a_{-1} = 0$, $b_1 = b_1$, $b_0 = b_0$ and $b_{-1} = \frac{1}{4}\frac{b_0^2}{b_1}$, $k = -1$ and $c = -1$.

We take $b_1 = \frac{1}{2}$ and $b_0 = 1$, then $a_0 = 1$ and $b_{-1} = \frac{1}{2}$ so

$$u(x,t) = \frac{1}{2}sech^2\left[\frac{1}{2}\left(\frac{-x^\alpha}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)}\right)\right]. \tag{28}$$
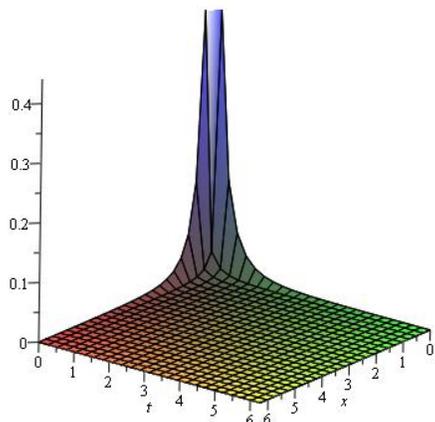
(a) $u(x,t)$ of equation(18)   (b) $u(x,t)$ of equation(28)

Figure 1. a. solution of Burger-like equation for $\alpha = \frac{1}{2}$ b. solution of Sawada-Kotera equation for $\alpha = \frac{1}{2}$

6. Conclusion

In the present paper, the exact solution of some fractional differential equations is obtained by the exp-function method. Also, by balancing the first sentence of the highest degree and first sentence the highest derivative order of nonlinear (ODE) and similarly, balancing the last sentence of the highest degree and last sentence the highest derivative order of nonlinear (ODE), leads $m_2 = n_2$ and $m_1 = n_1$. Consequently, the simplest choice $m_2 = n_2 = 1$ and $m_1 = n_1 = 1$ has to be only considered. Hence, this method can be useful to solve other nonlinear FDEs in mathematical physics, because it is an easy, direct, concise, basic and powerful method to implement, also the exp-function method generalized solitary solutions and periodic solutions of nonlinear equations. The principal merits of this method over the other methods are that it provides more general solutions with some arbitrary parameters. It should be noticed that the solutions are accurate.

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REFERENCES

Ali, M.Y., Hafez, M.G., Chowdary, M.K.H. and Akter, M.T. (2016). Analytical and traveling wave solutions to the fifth order stender Sawada-Kotera Equation via the generalized $\exp(-\phi(\xi))$-expansion method, Journal of Applied Mathematics and Physics, Vol. 4, No. 2, pp. 262-271.

Bulut, H., Tuz, M. and Aktürk, T. (2013). New multiple solution to the Boussinesq equation and the Burgers-like Equation, Journal of Applied Mathematics, Vol. 2012, Artick ID952614, pp. 1-6.

Cerdik, Y.H. and Girgin, A. (2019). (G'/G)-expansion method for the conformable space-time fractional Jimbo-Miwa and Burger-like equations, Mathematical Sciences and Applications E-Notes (MSAEN), Vol. 7, No. 1, pp. 47-53.

Ebaid, A. (2012). An improvement on the Exp-function method when balancing the highest order linear and nonlinear terms, Journal of Mathematical Analysis and Applications, Vol. 392, No. 1, pp. 1-5.

Guner, O.A. and Behir, A. (2017). Exact solution to the time-fractional differential equations via local derivatives, Waves in Random and Complex Media, Vol. 28, pp. 139-149.

Inan, I.E., Duran, S. and Ugurlu, Y. (2017). F-expansion method for traveling wave solutions of AKNS and Burgers-like equations modified method of simplest equation and its applications to the Bogoyavlenskii equation, Optik, Vol. 138, pp. 15-20.

Jumarie, G. (2006). Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, Comput Math Appl, Vol. 54, pp. 1367-1376.

Jumarie, G. (2009). Table of some basic fractional calculus formulae derived froma modified Riemann-Liouville derivative for nondifferentiable functions, Applied Mathematics Letters, Vol. 22, No. 3, pp. 378–385.

Kilbas, A., Srivastava, H. and Trujillo, J. (2006). *Theory and Applications of Fractional Differential Equations*, Elsevier, San Diego.

Mille, K. and Ross, B. (1993). *An Introduction to the Fractional Calculus and Fractional Differential Equations,* Wiley, New York.

Podlubny, I. (1999). *Fractional Differential Equations,* Academic Press, San Diego.

Thaghizadeh, N., Najand Foumani, M. and Soltani Mohmmadi, V. (2015). New exact solutions of the perturbed nonlinear fractional Schrodinger equation using two reliable methods, Applications and Applied Mathematics, Vol. 10, pp. 139-148.

Thaghizadeh, N. and Najand Foumani, M. (2016). Complex solutions of the time fractional Gross-Pitaevskii (FGP) equation with external potential by using a reliable method, Applicatiions and Applied Mathematics, Vol. 11, pp. 943-956.

Wang, X.M., Bilige, S.D. and Bai, Y.X. (2017). A general sub-equation method to the Burgers-like Equation, Thermal Science, Vol. 21, pp. 1681-1687.