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Geometric Programming Subject to System of Fuzzy Relation Inequalities

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Abstract

In this paper, an optimization model with geometric objective function is presented. Geometric programming is widely used; many objective functions in optimization problems can be analyzed by geometric programming. We often encounter these in resource allocation and structure optimization and technology management, etc. On the other hand, fuzzy relation equalities and inequalities are also used in many areas. We here present a geometric programming model with a monomial objective function subject to the fuzzy relation inequality constraints with max-product composition. Simplification operations have been given to accelerate the resolution of the problem by removing the components having no effect on the solution process. Also, an algorithm and two practical examples are presented to abbreviate and illustrate the steps of the problem resolution.

Keywords: Geometric programming; Fuzzy relation equalities and inequalities; Max- product composition

MSC 2010: 90C70, 94D05, 47S40

1. Introduction

Fuzzy relation equations (FRE), fuzzy relation inequalities (FRI) and their connected problems have been investigated by many researchers in both theoretical and applied areas Han (2006), Hassanzadeh (2011), Di Nola (1984), Zener (1971), Fang and Puthenpura (1993), Higashi and Klir (1984), Guo et al. (1988), Shivanian and Khorram (2007), Shivanian and Khorram (2010), Khorram (2008), Abbasi-Molai (2010), Perfilieva and Novák (2007), Abbasi-Molai (2010), Shieh (2007), Ghodousian and Khorram (2008). Sanchez (1977), started a development of the theory and applications of FRE treated as a formalized model for non-precise concepts. Generally, FRE and FRI have a number of properties that make them suitable for formulating the uncertain information upon which many applied concepts are usually based. The application of (FRE) and (FRI) can be seen in many areas, for instance, fuzzy control, fuzzy decision making, system analysis, fuzzy modeling, fuzzy arithmetic, fuzzy symptom diagnosis, and especially fuzzy medical diagnosis, and so on (see Alayón et al. (2007), Berrached et al. (2002), Di Nola and Russo (2007), Zener (1971), Dubois and Prade (1980), Jian-Xin (2004), Loia (2005), Nobuhara et al. (2006), Pappis and Karacapilidis (1995), Pedrycz (1981), Perfilieva and Novák (2007), Vasantha et al. (2004), Homayouni et al. (2009).

An interesting extensively investigated kind of such problems is the optimization of the objective functions on the region whose set of feasible solutions have been defined as FRE or FRI constraints Brouke et al. (1998), Fang and Li (1999), Guo and Xia (2006), Jian-Xin (2008), Shivanian (2007), Loetamonphong (2001), Wu (2008). Fang and Li (1999) solved the linear optimization problem with respect to the FRE constraints by considering the max-min composition Fang and Li (1999). The max-min composition is commonly used when a system requires conservative solutions in the sense that the goodness of one value cannot compensate the badness of another value Loetamonphong (2001). Recent results in the literature, however, show that the min operator is not always the best choice for the intersection operation. Instead, the max-product composition provided results better or equivalent to the max-min composition in some applications Alayón et al. (2007).

The fundamental result for fuzzy relation equations with max-product composition goes back to Pedrycz (1981). A recent study in this regard can be found in Bourk and Fisher (1998). They extended the study of an inverse solution of a system of fuzzy relation equations with max-product composition. They provided theoretical results for determining the complete sets of solutions as well as the conditions for the existence of resolutions. Their results showed that such complete sets of solutions can be characterized by one maximum solution and a number of minimal solutions. A problem of optimization was studied by Loetamonfong and Fang with max-product composition Loetamonphong (2001), which was improved by Jian-Xin by shrinking the search region Jian-Xin (2008). The linear objective optimization problem with FRI was investigated by Zhang et al. (2003), where the fuzzy operator is considered as max-min composition. Also, Guo and Xia presented an algorithm to accelerate the resolution of this problem Guo and Xia (2006). Zener, Duffin and Peterson proposed the geometric programming theory in 1961 Duffin et al. (1967), Peterson (1967). A large number of applications can be found in business administration, economic analysis, resource allocation, and environmental engineering Zener (1971). In 1987, Cao proposed the fuzzy geometric programming problem Cao (2001). He solved several problems of power systems Cao (1999). Liu applied it to economic management Liu (2004). Verma and Biswal have applied the theory Biswal (1992), Verma (1990). In view of the importance of geometric programming and the fuzzy relation equation in theory and applications, Yang and Cao have proposed a fuzzy relation geometric programming, discussed optimal solutions with two kinds of objective functions based on fuzzy max product operator Yang and Cao (2005a), Yang and Cao (2005b).

In this paper, we generalize the geometric programming of the FRE with the max-product operator Yang and Cao (2005b), by considering the fuzzy relation inequalities instead of the equations in the constraints. This problem can be formulated as follows:

$$\begin{aligned} \min \quad & Z = \max_{j=1,2,3,\dots,n} \{c_j . x_j^{\alpha_j}\} \\ \textit{subject to} \quad & A \bullet x \geq d^1 \\ & B \bullet x \leq d^2 \\ & x \in [0,1]^n , \end{aligned} \tag{1}$$

where $c_j, \alpha_j \in R, c_j \geq 0$ and $A = (a_{ij})_{m\times n}$, $a_{ij} \in [0,1]$, $B = (b_{ij})_{l\times n}$, $b_{ij} \in [0,1]$, are fuzzy matrices, $d^1 = (d_i^1)_{m\times 1} \in [0,1]^m, d^2 = (d_i^2)_{l\times 1} \in [0,1]^l$ are fuzzy vectors, $c = (c_j)_{n\times 1} \in R^n$ is the vector of cost coefficients, and $x = (x_j)_{n\times 1} \in [0,1]^n$ is an unknown vector, and "$\bullet$" denotes the fuzzy max-product operator as defined below. Problem (1) can be rewritten as the following problem in detail:

$$\begin{aligned} \min \quad & Z = \max_{j\in J} \{c_j . x_j^{\alpha_j}\} \\ \textit{subject to} \quad & a_i \bullet x \geq d_i^1 \qquad i \in I^1 = \{1,2,\dots,m\} \\ & b_i \bullet x \leq d_i^2 \qquad i \in I^2 = \{1,2,\dots,l\} \\ & 0 \leq x_j \leq 1 \qquad j \in J = \{1,2,\dots,n\}, \end{aligned} \tag{2}$$

where a_i and b_i are the ith row of the matrices A and B, respectively, and the constraints are expressed by the max-product operator definition as:

$$\begin{aligned} a_i \bullet x &= \max_{j\in J} \{a_{ij} \cdot x_j\} \geq d_i^1 \qquad \forall i \in I^1 \\ b_i \bullet x &= \max_{j\in J} \{b_{ij} \cdot x_j\} \leq d_i^2 \qquad \forall i \in I^2 \end{aligned} \tag{3}$$

In section 2, the set of the feasible solutions of problem 2 and its properties are studied. A necessary condition and a sufficient condition are given to realize the feasibility of problem 2. In section 3, some simplification operations are presented to accelerate the resolution process. Also, in section 4 an algorithm is introduced to solve the problem using the results of the previous sections, and two practical examples are given to illustrate the algorithm in this section. Finally, a conclusion is stated in section 5.

2. The Characteristics of the Set of Feasible Solution

Notations:

We shall use, during the paper, these notations as follows:

$$S(A,d^1)_i = \{x \in [0,1]^n : a_i \bullet x \geq d_i^1\} \text{ for each } i \in I^1$$

$$S(B,d^2)_i = \{x \in [0,1]^n : b_i \bullet x \leq d_i^2\} \text{ for each } i \in I^2$$

$$S(A,d^1) = \bigcap_{i \in I^1} S(A,d^1)_i = \{x \in [0,1]^n : A \bullet x \geq d^1\}$$

$$S(B,d^2) = \bigcap_{i \in I^2} S(B,d^2)_i = \{x \in [0,1]^n : B \bullet x \leq d^2\}$$

$$S(A,B,d^1,d^2) = S(A,d^1) \bigcap S(B,d^2) = \{x \in [0,1]^n : A \bullet x \geq d^1, B \bullet x \leq d^2\}.$$

Corollary 1:

$x \in S(A,d^1)_i$ for each $i \in I^1$ if and only if there exists some $j_i \in J$ such that $x_{j_i} \geq \dfrac{d_i^1}{a_{ij_i}}$, similarly, $x \in S(B,d^2)_i$ for each $i \in I^2$ if and only if $x_j \leq \dfrac{d_i^2}{b_{ij}}, \forall j \in J$.

Proof:

This clearly results from relations (3).

Lemma 1:

(a) $S(A,d^1) \neq \phi$ if and only if for each $i \in I^1$ there exists some $j_i \in J$ such that $a_{ij_i} \geq d_i^1$.

(b) If $S(A,d^1) \neq \phi$ then $\bar{1} = [1,1,...,1]_{1\times n}^t$ is the greatest element in set $S(A,d^1)$.

Proof:

(a) Suppose $S(A,d^1)\neq\phi$ and $x\in S(A,d^1)$. Thus, $x\in S(A,d^1)_i$, $\forall i\in I^1$ and then for each $i\in I^1$ we have $x_{j_i}\geq\frac{d_i^1}{a_{ij_i}}$ for some $j_i\in J$ from Corollary 1. Therefore, since $x\in S(A,d^1)$ then $x\in[0,1]^n$ and then $\frac{d_i^1}{a_{ij_i}}\leq 1$, $\forall i\in I^1$ which implies that there is a $j_i\in J$ such that $a_{ij_i}\geq d_i^1,\forall i\in I^1$. Conversely, suppose that there exists some $j_i\in J$ such that $a_{ij_i}\geq d_i^1,\forall i\in I^1$. Set $x=\bar{1}=[1,1,...,1]^t_{1\times n}$, since $x\in[0,1]^n$ and $x_{j_i}=1\geq\frac{d_i^1}{a_{ij_i}}$, $\forall i\in I^1$ then $x\in S(A,d^1)_i$, $\forall i\in I^1$ from Corollary 1, and then $x\in S(A,d^1)$.

(b) Proof is attained from the part (a) and Corollary 1.

Lemma 2:

(a) $S(B,d^2)\neq\phi$.

(b) The smallest element in set $S(B,d^2)$ is $\bar{0}=[0,0,...,0]^t_{1\times n}$.

Proof:

Set $x=\bar{0}=[0,0,...,0]^t_{1\times n}$. Since $d_i^2\geq 0$ and $b_{ij}\geq 0$ (in case $b_i=0$ the problem is always well defined and it is clear), then $\frac{d_i^2}{b_{ij_i}}\geq 0$. Therefore, $x_j\leq\frac{d_i^2}{b_{ij_i}},\forall i\in I^2, j\in J$, then Corollary 1 implies that $x\in S(B,d^2)$ and then part (a) and (b) are proved.

Theorem 1: Necessary Condition

If $S(A,B,d^1,d^2)\neq\phi$, then for each $i\in I^1$ there exist $j\in J$ such that $a_{ij}\geq d_i^1$.

Proof:

Suppose that $S(A,B,d^1,d^2)\neq\varphi$, then, since $S(A,B,d^1,d^2)=S(A,d^1)\bigcap S(B,d^2)$, then $S(A,d^1)\neq\phi$, at this time the theorem is proved by using part (a) of Lemma 1.

Definition 1:

Set $\bar{x} = (\bar{x}_j)_{n\times 1}$ where

$$\bar{x}_j = \begin{cases} 1, & \forall i: \quad b_{ij} \le d_i^2 \\ \min\limits_{i=1,\dots l}\left\{\dfrac{d_i^2}{b_{ij}}: \quad b_{ij} > d_i^2\right\}, & \text{otherwise.} \end{cases}$$

Lemma 3:

If $S(B,d^2) \neq \phi$ then $\bar{x}$ is the greatest element in set $S(B,d^2)$.

Proof:

See Shivanian (2010).

Corollary 2:

$S(B,d^2) = \{x \in [0,1]^n : B \bullet x \le d^2\} = [\bar{0}, \bar{x}]$, in which $\bar{x}$ and $\bar{0}$ are as defined in Definition 1 and Lemma 2, respectively.

Proof:

Since $S(B,d^2) \neq \phi$ then $\bar{0}$ and $\bar{x}$ are the single smallest element and greatest element, respectively, from Lemmas 2 and 3. Let $x \in [\bar{0}, \bar{x}]$, then $x \in [0,1]^n$ and $x \le \bar{x}$, Thus, $b_i \bullet x \le b_i \bullet \bar{x} \le d_i^2, \forall i \in I^2$ that implies $x \in S(B,d^2)$. Conversely, let $x \in S(B,d^2)$ from part (b) of Lemma 2, $\bar{0} \le x$ and also $x \in S(B,d^2)_i$, $\forall i \in I^2$. Then, Corollary 1 requires $x_j \le \dfrac{d_i^2}{b_{ij}}$, $\forall i \in I^2$ and $\forall j \in J$. Hence, $x_j \le \bar{x}_j, \forall j \in J$ that means $x \le \bar{x}$. Therefore, $x \in [\bar{0}, \bar{x}]$.

Definition 2:

Let $J_i = \{j \in J : a_{ij} \ge d_i^1\}, \forall i \in I^1$. For each $j \in J_i$, we define $i_{x(j)} = (i_{x(j)_k})_{n\times 1}$ such that

$$i_{x(j)_k} = \begin{cases} \dfrac{d_i^1}{a_{ij}}, & k = j \\ 0, & k \neq j. \end{cases}$$

Lemma 4:

Consider a fixed $i \in I^1$.

(a) If $d_i^1 \neq 0$, then the vectors $i_{x(j)}$ are the only minimal elements of $S(A,d^1)_i$ for each $j \in J_i$.

(b) If $d_i^1 = 0$ then $\bar{0}$ is the smallest element in $S(A,d^1)_i$.

Proof:

(a) Suppose $j \in J_i$ and $i \in I^1$. Since $i_{x(j)_j} = \dfrac{d_i^1}{a_{ij}}$, then $i_{x(j)} \in S(A,d^1)_i$, from Corollary 1. By contradiction, suppose $x \in S(A,d^1)_i$ and $x < i_{x(j)}$. Hence we must have $x_j < \dfrac{d_i^1}{a_{ij}}$ and $x_k = 0$ for $k \in J$ and $k \neq j$. Then $x_j < \dfrac{d_i^1}{a_{ij}}$, $\forall j \in J$ and then $x \notin S(A,d^1)_i$ from Corollary 1, which is a contradiction.

(b) It is clear from Corollary 1 and the fact that $x_j \geq 0$, $\forall j \in J$.

Corollary 3:

If $S(A,d^1)_i \neq \phi$, then $S(A,d^1)_i = \{x \in [0,1]^n : a_i \bullet x \geq d_i^1\} = \bigcup_{j \in J_i} [i_{x(j)}, \bar{1}]$, where $i \in I^1$ and $i_{x(j)}$ is as defined in Definition 2.

Proof:

If $S(A,d^1)_i \neq \phi$ then from Lemmas 1 and 4, the vector $\bar{1}$ is the maximum solution and the vectors $i_{x(j)}$, $\forall j \in J_i$ are the minimal solutions in $S(A,d^1)_i$. Let $x \in \bigcup_{j \in J_i} [i_{x(j)}, \bar{1}]$.Then $x \in [i_{x(j)}, \bar{1}]$ for some $j \in J_i$ and, then, $x \in [0,1]^n$ and $x_j \geq i_{x(j)_j} = \dfrac{d_i^1}{a_{ij}}$ from Definition 2, hence, $x \in S(A,d^1)_i$ from Corollary 1. Conversely, let $x \in S(A,d^1)_i$. Then there exits some $j' \in J$ such that $x_{j'} \geq \dfrac{d_i^1}{a_{ij'}}$ from Corollary 1. Since $x \in [0,1]^n$, then $\dfrac{d_i^1}{a_{ij'}} \leq 1$, and then, $j' \in J_i$. Therefore, $i_{x(j')} \leq x \leq \bar{1}$ that implies $x \in \bigcup_{j \in J_i} [i_{x(j)}, \bar{1}]$.

Definition 3:

Let $e=(e(1),e(2),...e(m))\in J_1\times J_2\times...\times J_m$ such that $e(i)=j\in J_i$. We define $x(e)=(x(e)_j)_{n\times 1}$, in which $x(e)_j=\max\limits_{i\in I_j^e}\{i_{x(e(i))\,j}\}=\max\limits_{i\in I_j^e}\left\{\dfrac{d_i^1}{a_{ij}}\right\}$ if $I_j^e\neq\phi$ and $x(e)_j=0$ if $I_j^e=\varphi$, where $I_j^e=\{i\in I^1:e(i)=j\}$.

Corollary 4:

(a) If $d_i^1=0$ for some $i\in I^1$, then we can remove the ith row of matrix A with no effect on the calculation of the vectors $x(e)$ for each $e\in J_I=J_1\times J_2\times...\times J_m$.

(b) If $j\notin J_i$, $\forall i\in I^1$, then we can remove the jth column of the matrix A before calculating the vectors $x(e)$, $\forall e\in J_I$ and set $x(e)_j=0$ for each $e\in J_I$

Proof:

(a) It is proved from Definition 3 and part (b) of Lemma 4, because we will get the minimal elements of $S(A,d^1)$.

(b) It is proved by only using Definition 3.

Lemma 5:

Suppose $S(A,d^1)\neq\phi$ then $S(A,d^1)=\bigcup\limits_{X(e)}[x(e),\bar{1}]$ where $X(e)=\{x(e):e\in J_I\}$.

Proof:

If $S(A,d^1)\neq\phi$, then $S(A,d^1)_i\neq\phi$, $\forall i\in I^1$. It is clear that $x(e)$ is minimal it would be the solution i.e. $x(e)\in S(A,d^1)$, so at first step, we prove it is solution. Suppose that i is fixed and $e(i)=j\in J_i$ then $x(e)_j=\max\limits_{i\in I_j^e}\{i_{x(e(i))j}\}=\max\limits_{i\in I_j^e}\left\{\dfrac{d_i^1}{a_{ij}}\right\}\geq\dfrac{d_i^1}{a_{ij}}$ from Definition 3 and so $a_{ij}x(e)_j\geq d_i^1$, hence $x(e)_j\in S(A,d^1)_i$ then $x(e)\in S(A,d^1)$. Therefore, we have

$$\begin{aligned}S(A,d^1)&=\bigcap_{i\in I^1}S(A,d^1)_i=\bigcap_{i\in I^1}[\bigcup_{j\in J_i}[i_{x(j)},\bar{1}]]=\bigcap_{i\in I^1}[\bigcup_{e(i)\in J_i}[i_{x(e(i))},\bar{1}]]\\&=\bigcup_{e\in J_I}[\bigcap_{i\in I^1}[i_{x(e(i))},\bar{1}]]=\bigcup_{e\in J_I}[\max_{i\in I^1}\{i_{x(e(i))}\},\bar{1}]=\bigcup_{e\in J_I}[x(e),\bar{1}]=\bigcup_{X(e)}[x(e),\bar{1}],\end{aligned}$$

from Corollary 3 and Definition 3.

From Lemma 5, it is obvious that $S(A,d^1)=\bigcup_{X_0(e)}[x(e),\bar{1}]$ and $X_0(e)=S_0(A,d^1)$, where $X_0(e)$ and $S_0(A,d^1)$ are the set of minimal solutions in $X(e)$ and $S(A,d^1)$, respectively.

Theorem 2:

If $S(A,B,d^1,d^2)\neq\phi$, then $S(A,B,d^1,d^2)=\bigcup_{X_0(e)}[x(e),\bar{x}]$.

Proof:

By using Corollary 2 and the result of Lemma 5, we have

$$S(A,B,d^1,d^2)=S(A,d^1)\bigcap S(B,d^2)=\{\bigcup_{X_0(e)}[x(e),\bar{1}]\}\bigcap[\bar{0},\bar{x}]=\bigcup_{X_0(e)}[x(e),\bar{x}]$$

and the proof is complete.

Corollary 5: Necessary and Sufficient Conditions

$S(A,B,d^1,d^2)\neq\phi$ if and only if $\bar{x}\in S(A,d^1)$. Equivalently, $S(A,B,d^1,d^2)\neq\phi$ if and only if there exists some $e\in J_I$ such that $x(e)\leq\bar{x}$.

Proof:

Suppose that $S(A,B,d^1,d^2)\neq\phi$, then $S(A,B,d^1,d^2)=\bigcup_{X_0(e)}[x(e),\bar{x}]$ by Theorem 2, then $\bar{x}\in S(A,B,d^1,d^2)$, and hence $\bar{x}\in S(A,d^1)$. Conversely let $\bar{x}\in S(A,d^1)$. Meanwhile we know $\bar{x}\in S(B,d^2)$, therefore $\bar{x}\in S(A,d^1)\bigcap S(B,d^2)=S(A,B,d^1,d^2)$.

3. Simplification Operations and the Resolution Algorithm

In order to solve problem (1), we first convert it into the two sub-problems below:

$$\begin{aligned} \min \quad & Z=\max_{j\in R^+}\{c_j.x_j^{\alpha_j}\} \\ s.t \quad & A\bullet x\geq d^1 \\ & B\bullet x\leq d^2 \\ & x\in[0,1]^n, \end{aligned} \qquad (4a)$$

$$\begin{aligned} \min \quad & Z=\max_{j\in R^-}\{c_j.x_j^{\alpha_j}\} \\ s.t \quad & A\bullet x\geq d^1 \\ & B\bullet x\leq d^2 \\ & x\in[0,1]^n, \end{aligned} \qquad (4b)$$

where $R^+=\{j:\alpha_j\geq 0, j\in J\}$ and $R^-=\{j:\alpha_j<0, j\in J\}$.

Lemma 6:

The optimal solution of problem (4b) is $\bar{x}$ in Definition 1.

Proof:

In objective function (4b) $\alpha_j < 0$ therefore, ${x_j}^{\alpha_j}$ is a monotone decreasing function of x_j in interval $0 \le x_j \le 1$ for each $j \in R^-$, so is $\max\limits_{j \in R^-}\{c_j.{x_j}^{\alpha_j}\}$ of x_j too. Hence $\bar{x}$ is optimal solution because $\bar{x}$ is the greatest element in the set $S(A, B, d^1, d^2)$.

Lemma 7:

The optimal solution of problem (4a) belongs to $X_0(e)$.

Proof:

In objective function (4a), $\alpha_j \ge 0$ therefore, ${x_j}^{\alpha_j}$ is a monotone increasing function of x_j in interval $0 \le x_j \le 1$ for each $j \in R^+$, so is $\max\limits_{j \in R^+}\{c_j.{x_j}^{\alpha_j}\}$ of x_j too. Now, suppose that $y \in S(A, B, d^1, d^2)$ has selected arbitrary then, there exists $x(e_0) \in X_0(e)$ such that $y \ge x(e_0)$. Since $\max\limits_{j \in R^+}\{c_j.{x_j}^{\alpha_j}\}$ is a monotone increasing function of x_j then, $\max\limits_{j \in R^+}\{c_j.{y_j}^{\alpha_j}\} \ge \max\limits_{j \in R^+}\{c_j.{x(e_0)_j}^{\alpha_j}\}$ therefore, one of the elements of $X_0(e)$ is the optimal solution of problem (4a).

Theorem 3:

Assume that $x(e_0)$ be an optimal solution of problem (4a) (it is possible not to be unique) then, the optimal solution of problem (1) is x^*, defined as follow:

$$x_j^* = \begin{cases} \bar{x}_j, & j \in R^- \\ x(e_0)_j, & j \in R^+ \end{cases}$$

Proof:

Suppose that $S(A, B, d^1, d^2)$ then by Lemmas 6 and 7, we have

$$\max_{j\in J}\{c_j.x_j^{\alpha_j}\} = \max_{j\in R^+}\{c_j.x_j^{\alpha_j}\}\max_{j\in R^-}\{c_j.x_j^{\alpha_j}\}$$
$$\geq \max_{j\in R^+}\{c_j.x(e_0)_j^{\alpha_j}\}\max_{j\in R^-}\{c_j.\overline{x}_j^{\alpha_j}\}$$
$$= \max_{j\in J}\{c_j.x^*{}_j^{\alpha_j}\}.$$

Therefore, x^* is optimal solution of problem (1) and the proof is completed.

For calculating x^* it is sufficient to find $\overline{x}$ and $x(e_0)$ from Theorem 3. While $\overline{x}$ is easily attained by Definition 1, $x(e_0)$ is usually hard to find. Since $X_0(e)$ is attained by pair wise comparison between the members of set $X(e)$, then the finding process of set $X_0(e)$ is time-consuming if $X(e)$ has many members. Therefore, a simplification operation can accelerate the resolution of problem (4a) by removing the vectors $e \in J_I$ such that $x(e)$ is not optimal in (4a). One of such operations is given by Corollary 4. Other operations are attained by the theorems below.

Theorem 4:

The set of feasible solutions for problem (1), namely $S(A,B,d^1,d^2)$, is nonempty if and only if for each $i \in I^1$ set $\overline{J}_i = \left\{ j \in J_i : \dfrac{d_i^1}{a_{ij}} \leq \overline{x}_j \right\}$ is nonempty, where $\overline{x}$ is defined by Definition 1.

Proof:

Suppose $S(A,B,d^1,d^2) \neq \phi$. From Corollary 5, $\overline{x} \in S(A,B,d^1,d^2)$ and then we have $\overline{x} \in S(A,d^1)_i$, $\forall i \in I^1$. Thus, for each $i \in I^1$ there exists some $j \in J$ such that $\overline{x}_j \geq \dfrac{d_i^1}{a_{ij}}$ from Corollary 1 which means $\overline{J}_i \neq \phi$, $\forall i \in I^1$. Conversely, suppose $\overline{J}_i \neq \phi$, $\forall i \in I^1$. Then there exists some $j \in J$ such that $\overline{x}_j \geq \dfrac{d_i^1}{a_{ij}}$, $\forall i \in I^1$. Hence, $\overline{x} \in S(A,d^1)_i$, $\forall i \in I^1$ from Corollary 1 that implies $\overline{x} \in S(A,d^1)$. These facts together with Lemma 3 imply $\overline{x} \in S(A,B,d^1,d^2)$, and therefore $S(A,B,d^1,d^2) \neq \phi$.

Theorem 5:

If $S(A,B,d^1,d^2) \neq \phi$, then

$$S(A,B,d^1,d^2) = \bigcup_{\overline{X}(e)} [x(e),\overline{x}] \text{ where } \overline{X}(e) = \{x(e) : e \in \overline{J}_I = \overline{J}_1 \times \overline{J}_2 \times \ldots \times \overline{J}_m\}.$$

Proof:

By Theorem 2, it is sufficient to show $x(e) \notin S(A,B,d^1,d^2)$ if $e \notin \overline{J}_I$. Suppose $e \notin \overline{J}_I$. Thus, there exist $i' \in I^1$ and $j' \in J_{i'}$ such that $e(i') = j'$ and $\overline{x}_{j'} < \frac{d_{i'}^1}{a_{i'j'}}$. Then $i' \in I_{j'}^e$ and by Definition 3 we have $x(e)_{j'} = \max_{i \in I_{j'}^e} \left\{ \frac{d_i^1}{a_{ij'}} \right\} \geq \frac{d_{i'}^1}{a_{i'j'}} > \overline{x}_{j'}$. Therefore, $x(e) \leq \overline{x}$ is not correct, which implies $x(e) \notin S(A,B,d^1,d^2)$ by Theorem 2.

From defined notation of theorem 4, $\overline{J}_i \subseteq J_i$, $\forall i \in I^1$,which requires $\overline{X}(e) \subseteq X(e)$, also, $S_0(A,B,d^1,d^2) \subseteq \overline{X}(e)$ by Theorem 4 in which $S_0(A,B,d^1,d^2)$ is the minimal elements of $S(A,B,d^1,d^2)$, thus Theorem 5 reduces the region of search to find set $S_0(A,B,d^1,d^2)$.

Definition 4:

We define $J_i^* = \{j: \quad j \in R^- \quad and \quad j \in \overline{J}_i\}$ for $i \in I^1$.

Theorem 6:

Suppose $x(e_0)$ is an optimal solution in (4a) and $J_{i'}^* \neq \phi$ for some $i' \in I^1$, then there exist $x(e')$ such that $e'(i') \in J_{i'}^*$, and also $x(e')$ is the optimal solution in (4a).

Proof:

Suppose $J_{i'}^* \neq \phi$ for some $i' \in I^1$ and $e_0(i') = j'$. Define $e' \in \overline{J}_I$ such that $e'(i') = k \in J_{i'}^*$ and $e'(i) = e_0(i)$ for each $i \in I^1$ and $i \neq i'$. From Definition 3 we have:

$$x(e_0)_{j'} = \max_{i \in I_{j'}^{e0}} \left\{ \frac{d_i^1}{a_{ij'}} \right\} \geq \max_{\substack{i \in I_{j'}^{e0} \\ i \neq i'}} \left\{ \frac{d_i^1}{a_{ij'}} \right\} = x(e')_{j'}$$

Also, $x(e_0)_j = x(e')_j$ for each $j \in J$ and $j \neq j', k$. Therefore, by noting that $k \notin R^+$ we have

$$\begin{aligned} \max_{j \in R^+} \{c_j . x(e_0)_j^{\alpha_j}\} &= \max\{c_{j'} . x(e_0)_{j'}^{\alpha_{j'}}, \max_{\substack{j \in R^+ \\ j \neq j'}} \{c_j . x(e_0)_j^{\alpha_j}\}\} \\ &\geq \max\{c_{j'} . x(e')_{j'}^{\alpha_{j'}}, \max_{\substack{j \in R^+ \\ j \neq j'}} \{c_j . x(e')_j^{\alpha_j}\}\} = \max_{j \in R^+} \{c_j . x(e')_j^{\alpha_j}\}. \end{aligned}$$

Therefore $x(e')$ is the optimal solution in (4a), and then the proof is complete.

Corollary 6:

If $J_i^* \neq \phi$ for some $i \in I^1$ then, we can remove the ith row of matrix A without any effect on finding an optimal solution of problem (4a).

Definition 5:

Let $j_1, j_2 \in J$, $\alpha_{j_1} > 0$ and $\alpha_{j_2} > 0$. We say j_2 dominates j_1 if and only if

(a) $j_1 \in \overline{J}_i$ implies $j_2 \in \overline{J}_i$, $\forall i \in I^1$.

(b) For each $i \in I^1$ such that $j_1 \in \overline{J}_i$ we have $c_{j_1}.(\frac{d_i^1}{a_{ij_1}})^{\alpha_{j_1}} \geq c_{j_2}.(\frac{d_i^1}{a_{ij_2}})^{\alpha_{j_2}}$.

Theorem 7:

Suppose $x(e_0)$ is the optimal solution in (4a) and j_2 dominates j_1 for $j_1, j_2 \in R^+$, then there exists $x(e')$ such that $I_{j_1}^{e'} = \phi$, and also $x(e')$ is the optimal solution in (4a). (Notification: $\alpha_{j_1} > 0$ and $\alpha_{j_2} > 0$).

Proof:

Define $e' = (e'(i))_{m \times 1}$ such that

$$e'(i) = \begin{cases} e_0(i), & i \notin I_{j_1}^{e_0}, \\ j_2, & i \in I_{j_1}^{e_0}. \end{cases}$$

It is obvious that $I_{j_1}^{e'} = \phi$ and, then, $x(e')_{j_1} = 0$. Also, $x(e_0)_j = x(e')_j$ for each $j \in J$ and $j \neq j_1, j_2$. From Definition 3, $x(e')_{j_2} = \frac{d_{i_0}^1}{a_{i_0 j_2}}$. Now, if $i_0 \notin I_{j_1}^{e_0}$, then

$$x(e_0)_{j_2} = x(e')_{j_2} = \frac{d_{i_0}^1}{a_{i_0 j_2}}.$$

So, we have

$$\max_{j\in R^+}\{c_j.x(e_0)_j^{\alpha_j}\} = \max\{c_{j_1}.x(e_0)_{j_1}^{\alpha_{j_1}}, \max_{\substack{j\in R^+\\ j\neq j_1}}\{c_j.x(e_0)_j^{\alpha_j}\}\}$$

$$\geq \max_{\substack{j\in R^+\\ j\neq j_1}}\{c_j.x(e_0)_j^{\alpha_j}\} = \max_{j\in R^+}\{c_j.x(e')_j^{\alpha_j}\}.$$

The proof is complete in this case. Otherwise, suppose $i_0 \in I_{j_1}^{e_0}$. We show $\max_{j\in R^+}\{c_j.x(e_0)_j^{\alpha_j}\} \geq \max_{j\in R^+}\{c_j.x(e')_j^{\alpha_j}\}$. By Definition 3, let $x(e_0)_{j_2} = \frac{d_i^1}{a_{ij_2}}$. Then, we have $c_{j_2}.x(e_0)_{j_2} \geq 0$ from part (a) of Corollary 4 and Definition 5. Therefore, since

$$\max_{j\in R^+}\{c_j.x(e_0)_j^{\alpha_j}\} = \max\{c_{j_1}.x(e_0)_{j_1}^{\alpha_{j_1}}, c_{j_2}.x(e_0)_{j_2}^{\alpha_{j2}}, \max_{\substack{j\in R^+\\ j\neq j_1,j_2}}\{c_j.x(e_0)_j^{\alpha_j}\}\}$$

and

$$\max_{j\in R^+}\{c_j.x(e')_j^{\alpha_j}\} = \max\{c_{j_2}.x(e')_{j_2}^{\alpha_{j2}}, \max_{\substack{j\in R^+\\ j\neq j_1,j_2}}\{c_j.x(e')_j^{\alpha_j}\}\},$$

it is sufficient to show $c_{j_1}.x(e_0)_{j_1} \geq c_{j_2}.x(e')_{j_2}$. Let $x(e_0)_{j_1} = \frac{d_{i'}^1}{a_{i'j_1}}$ from Definition 3. Since j_2 dominates j_1, then we have

$$c_{j_1}.(\frac{d_{i'}^1}{a_{i'j_1}})^{\alpha_{j1}} \geq c_{j_2}.(\frac{d_{i_0}^1}{a_{i_0j_2}})^{\alpha_{j2}},$$

which means $c_{j_1}.x(e_0)_{j_1} \geq c_{j_2}.x(e')_{j_2}$ if $i_0 = i'$. Otherwise, suppose $i_0 \neq i'$. Since $i_0 \in I_{j_1}^{e_0}$ and j_2 dominates j_1, then

$$c_{j_1}.(\frac{d_{i_0}^1}{a_{i_0j_1}})^{\alpha_{j1}} \geq c_{j_2}.(\frac{d_{i_0}^1}{a_{i_0j_2}})^{\alpha_{j2}}.$$

Also, by Definition 3, we have

$$x(e_0)_{j_1} = \max_{i\in I_{j_1}^{e_0}}\{\frac{d_i^1}{a_{ij_1}}\} = \frac{d_{i'}^1}{a_{i'j_1}}.$$

This implies

$$(\frac{d_{i'}^1}{a_{i'j_1}})^{\alpha_{j_1}} \geq (\frac{d_i^1}{a_{ij_1}})^{\alpha_{j_1}} \;, \forall i \in I_{j_1}^{e_0} .$$

Therefore,

$$c_{j_1}.(\frac{d_{i'}^1}{a_{i'j_1}})^{\alpha_{j_1}} \geq c_{j_1}.(\frac{d_{i_0}^1}{a_{i_0 j_1}})^{\alpha_{j_1}} \geq c_{j_2}.(\frac{d_{i_0}^1}{a_{i_0 j_2}})^{\alpha_{j_2}} ,$$

which requires $c_{j_1}.x(e_0)_{j_1} \geq c_{j_2}.x(e')_{j_2}$. Hence, $\max_{j\in R^+}\{c_j.x(e_0)_j{}^{\alpha_j}\} \geq \max_{j\in R^+}\{c_j.x(e')_j{}^{\alpha_j}\}$ and the proof is completed.

Corollary 7:

If j_2 dominates j_1 for some $j_1, j_2 \in R^+$, then we can remove the j_1th column of the matrix A without any effect on finding the optimal solution $x(e_0)$ in (4a).

4. Algorithm for Finding an Optimal Solution and Examples

Definition 6:

Consider problem (1). We call $\overline{A} = (\overline{a_{ij}})_{m\times n}$ and $\overline{B} = (\overline{b}_{ij})_{l\times n}$ the characteristic matrices of matrix A and matrix B, respectively, where $\overline{a}_{ij} = \frac{d_i^1}{a_{ij}}$ for each $i \in I^1$ and $j \in J$, also

$$\overline{b}_{ij} = \frac{d_i^2}{b_{ij}} \text{ for each } i \in I^2 \text{ and } j \in J . \text{ (set } \frac{0}{0} = 1 \text{ and } \frac{k}{0} = \infty \text{)}$$

Algorithm:

Given problem (2),

1. Find matrices $\overline{A}$ and $\overline{B}$ by Definition 6.

2. If there exists $i \in I^1$ such that $\overline{a}_{ij} > 1$, $\forall j \in J$, then stop. Problem 2 is infeasible (see Theorem 1).
3. Calculate $\overline{x}$ from $\overline{B}$ by Definition 1.
4. If there exists $i \in I^1$ such that $d_i^1 = 0$, then remove the i'th row of matrix $\overline{A}$ (see part (a) of Corollary 4).
5. If $\overline{a}_{ij} > \overline{x}_j$, then set $\overline{a}_{ij} = 0$, $\forall i \in I^1$ and $\forall j \in J$.

6. If there exists $i \in I^1$ such that $\overline{a}_{ij} = 0, \forall j \in J$, then stop. Problem (2) is infeasible (see Theorems 4 and 5)
7. If there exists $j' \in J$ such that $\overline{a}_{ij'} = 0, \forall i \in I^1$, then remove the j' th column of the matrix $\overline{A}$ (see part (b) of Corollary 4) and set $x(e_0)_{j'} = 0$.
8. For each $i \in I^1$, if $J_i^* \neq \phi$ then remove the ith row of the matrix $\overline{A}$ (see Corollary 6)
9. Remove each column $j \in J$ from $\overline{A}$ such that $j \in R^-$ and set $x(e_0)_j = 0$.
10. If j_2 dominates j_1, ($j_1, j_2 \in R^+$) then remove column j_1 from $\overline{A}$, $\forall j_1, j_2 \in J$ (see Corollary 7) and set $x(e_0)_{j_1} = 0$.
11. Let $J_i^{new} = \{j \in \overline{J}_i : \overline{a}_{ij} \neq 0\}$ and $J_I^{new} = J_1^{new} \times J_2^{new} \times \ldots \times J_m^{new}$. Find the vectors $x(e)$, $\forall e \in J_I^{new}$, by Definition 3 from $\overline{A}$, and $x(e_0)$ by pair wise comparison between the vectors $x(e)$.
12. Find x^* from Theorem 3.

Example 1:

We consider the problem below which was given by J. H. Yang & B. Y. Cao, Yang and Cao (2005b), and solve by above algorithm. We will see the results are the similar.

$$\min Z = \max\left\{0.4 \cdot (x_1)^{-\frac{1}{2}}, 0.7 \cdot (x_2)^{\frac{3}{2}}, 0.6 \cdot (x_3)^{\frac{1}{2}}, 0.1 \cdot (x_4)^{-2}\right\}$$

$$\begin{bmatrix} 0.5 & 0 & 0.6 & 0.8 \\ 0.5 & 0.2 & 0 & 0.4 \\ 0.2 & 0.1 & 0.3 & 0.2 \end{bmatrix} \bullet \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \\ 0.2 \end{bmatrix}, \quad 0 \le x_j \le 1, \quad j = 1,2,3,4.$$

This problem is a simple case from problem (1). Matrices A and B are equal in this problem, which means the constraints are $A \bullet x \ge b$ and $A \bullet x \le b$.

Step 1:

$$\overline{A} = \begin{bmatrix} 0.8 & \infty & \frac{2}{3} & 0.5 \\ 0.4 & 1 & \infty & 0.5 \\ 1 & 2 & \frac{2}{3} & 1 \end{bmatrix}.$$

Step 2:

Step 3:

$$\bar{x} = [0.4, 1, \frac{2}{3}, 0.5]$$

Step 4:
Step 5:

$$\bar{A} = \begin{bmatrix} 0 & 0 & \frac{2}{3} & 0.5 \\ 0.4 & 1 & 0 & 0.5 \\ 0 & 0 & \frac{2}{3} & 0 \end{bmatrix}$$

Step 6:
Step 7:
Step 8:

$J_1^* = \{4\}$ and $J_2^* = \{1,4\}$, therefore first and second rows are removed. Then

$$\bar{A} = \begin{bmatrix} 0 & 0 & \frac{2}{3} & 0 \end{bmatrix}$$

Step 9:

By this step, first and fourth columns are removed, and then the new matrix $\bar{A}$ is as follow:

$$\bar{A} = \begin{bmatrix} 0 & \frac{2}{3} \end{bmatrix}$$

Step 10:

It is clear that j_3 dominates j_2 by Definition 5, so we remove the second column from

$$\bar{A} = \begin{bmatrix} 0 & \frac{2}{3} \end{bmatrix}$$

and obtain

$$\bar{A} = \begin{bmatrix} \frac{2}{3} \end{bmatrix}$$

Step 11:

$$x(e_0) = (0,0,\frac{2}{3},0)$$

Step 12:

$$x^* = (0.4,0,\frac{2}{3},0.5)$$
$$Z^* = \max\left\{0.4\cdot(0.4)^{-\frac{1}{2}}, 0.7\cdot(0)^{\frac{3}{2}}, 0.6\cdot(\frac{2}{3})^{\frac{1}{2}}, 0.1\cdot(0.5)^{-2}\right\} = \max\{0.63, 0, 0.48, 0.4\} = 0.63.$$

Example 2: Consider the problem below:

$$\min Z = \max\left\{2\cdot(x_1)^2, 3\cdot(x_2)^{-1}, (x_3)^{\frac{1}{2}}, 3\cdot(x_4)^{-\frac{1}{2}}\right\}$$
$$\begin{bmatrix} 0.5 & 0.8 & 0.35 & 0.25 \\ 0.9 & 0.92 & 1 & 0.86 \\ 0.2 & 1 & 0.45 & 0.8 \\ 0.55 & 0.6 & 0.8 & 0.64 \end{bmatrix} \bullet \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \geq \begin{bmatrix} 0.4 \\ 0.9 \\ 0.8 \\ 0.65 \end{bmatrix}$$
$$\begin{bmatrix} 0.6 & 0.5 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.6 & 0.5 \\ 0.5 & 0.9 & 0.8 & 0.4 \end{bmatrix} \bullet \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \leq \begin{bmatrix} 0.48 \\ 0.56 \\ 0.72 \end{bmatrix}, \qquad 0 \leq x_j \leq 1, \quad j = 1,2,3,4.$$

Step 1:

Matrices, $\overline{A}$ and $\overline{B}$ are as follows:

$$\overline{A} = \begin{bmatrix} 0.8 & 0.5 & 1.14 & 1.6 \\ 1 & 0.97 & 0.9 & 1.04 \\ 4 & 0.8 & 1.77 & 1 \\ 1.18 & 1.08 & 0.81 & 1.01 \end{bmatrix}$$

$$\overline{B} = \begin{bmatrix} 0.8 & 0.96 & 4.8 & 4.8 \\ 2.8 & 0.93 & 0.93 & 1.12 \\ 1.44 & 0.8 & 0.9 & 1.8 \end{bmatrix}$$

Step 2:
Step 3:

$$\bar{x} = [0.8, 0.8, 0.9, 1]$$

Step 4:
Step 5:

By considering this step, matrix $\bar{A}$ is converted to the following:

$$\bar{A} = \begin{bmatrix} 0.8 & 0.5 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 0 & 0.8 & 0 & 1 \\ 0 & 0 & 0.81 & 0 \end{bmatrix}$$

Step 6:
Step 7:
Step 8:

According to this step, since $J_1^* = \{2\}$ and $J_3^* = \{2,4\}$, therefore we can remove the first and third rows, then we have:

$$\bar{A} = \begin{bmatrix} 0 & 0 & 0.9 & 0 \\ 0 & 0 & 0.81 & 0 \end{bmatrix}$$

Step 9:

By this step, we remove the second and fourth columns, and we set $x(e_0)_2 = x(e_0)_4 = 0$, then:

$$\bar{A} = \begin{bmatrix} 0 & 0.9 \\ 0 & 0.81 \end{bmatrix}$$

Step 10:

The first column is removed by this step.

$$\bar{A} = \begin{bmatrix} 0.9 \\ 0.81 \end{bmatrix}$$

Step 11:

$$x(e_0) = (0, 0, 0.9, 0)$$

Step 12:

By this x^* and Z^* are calculated as follows:

$$x^* = (0,0.8,0.9,1)$$

$$Z^* = \max\left\{2.(0)^2, 3.(0.8)^{-1}, (0.9)^{\frac{1}{2}}, 3.(1)^{-\frac{1}{2}}\right\} = 3.75 \ .$$

5. Conclusion

In this paper, we have studied the geometric programming with fuzzy relational inequality constraints defined by the max-product operator. Since the difficulty of this problem is finding the minimal solutions optimizing the same problem with the objective function $\min Z = \max_{j \in R^+}\{c_j .x_j^{\alpha_j}\}$, has been presented in an algorithm together with some simplifying operations to accelerate the problem resolution. Also, we have been given two examples to illustrate the proposed algorithm.

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