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The Principle of Linearized Stability for Size-Structured Population Models

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Abstract

The principle of linearized stability for size-structured population dynamics models is proved giving validity to previous stability results reported in, for example, El-Doma (2008-1). In particular, we show that if all the roots of the characteristic equation lie to the left of the imaginary axis then the steady state is locally exponentially stable, and on the other hand, if there is at least one root that lies to the right of the imaginary axis then the steady state is unstable. We also point out cases when there is resonance.

Keywords: The principle of linearized stability; Population; Stability; Steady state; Size-structure; Inflow of newborns; Resonance; Characteristic equation.

MSC 2010: 45K05; 45M10; 35B35; 35L60; 92D25

1. Introduction

In this paper, we study the following size-structured population dynamics model:

$$\begin{cases} \dfrac{\partial p(a,t)}{\partial t} + \dfrac{\partial}{\partial a}(V(a,P(t))p(a,t)) + \mu(a,P(t))p(a,t) = 0, \ a > 0, \ t > 0, \\ V(0,P(t))p(0,t) = C + \displaystyle\int_0^\infty \beta(a,P(t))p(a,t)da, \ t \geq 0, \\ p(a,0) = p_0(a), \ a \geq 0, \\ P(t) = \displaystyle\int_0^\infty p(a,t)da, \ t \geq 0, \end{cases} \tag{1}$$

where $p(a,t)$ is the density of the population with respect to size $a \in \mathbb{R}^+ = [0,\infty)$ at time $t \geq 0$; $P(t) = \int_0^\infty p(a,t)da$ is the total population size at time t; $\beta(a,P(t)), \mu(a,P(t))$ are, respectively, the birth modulus i.e. the average number of offspring, per unit time, produced by an individual of size a when the population size is $P(t)$, and the death modulus i.e. the death rate at size a, per unit population, when the population size is $P(t)$; $0 < V(a,P)$ is the individual growth rate at the population size P; $p(0,t) = \int_0^\infty \beta(a,P(t))p(a,t)da$ is the number of births, per unit time, when the population size is $P(t)$; and, $C \geq 0$, is a constant that represents the inflow of newborns from an external source, for example, seeds, when carried by winds in plants or, eggs of fish, when carried by water.

We shall assume the following compatibility conditions:

$$V(0,P(0))p_0(0) = C + \int_0^\infty \beta(a,P(0))p_0(a)da, \tag{2}$$

$$P'(0) = V(0,P(0))p_0(0) - \int_0^\infty \mu(a,P(0))p_0(a)da, \tag{3}$$

$$\begin{aligned} V_P(0,P(0))p_0(0)P'(0) &= V(0,P(0))\Big\{\Big(V(0,P(0))p_0(0)\Big)_a + p_0(0)\Big[\mu(0,P(0)) + \beta(0,P(0))\Big]\Big\} \\ &+ \int_0^\infty \Big[\beta_P(a,P(0))P'(0) + \beta_a(a,P(0))V(a,P(0)) - \beta(a,P(0))\mu(a,P(0))\Big]p_0(a)da, \end{aligned} \tag{4}$$

$$\Big(V(a,P(0))p_0(a)\Big)_a \in L(\mathbb{R}^+); V(a,P(0))p_0(a) \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+). \tag{5}$$

As in Haimovici (1979), Gyllenberg (1982), (1983), we also assume that the total population is finite for each t i.e.,

$$P(t) = \int_0^\infty p(a,t)da < +\infty. \tag{6}$$

We note that problem (1) is studied in Calsina, et al. (1995) where C is assumed to depend on time t; they proved the existence and uniqueness of solution, the existence of a global attractor when the inflow of newborns C is a constant. El-Doma (2008-1) studied problem (1) when $C = 0$, El-Doma (2008-2) studied problem (1), determined the steady states and examined their stability. El-Doma (2009-1)-El-Doma (2009-3) studied problem (1) when $C = 0$, with the additional assumption that the population is divided into adults and juveniles; steady states are determined and stability results are obtained. Generalization of this work to the case when $C > 0$ is obtained in El-Doma (Preprint-1), El-Doma (To appear) and El-Doma (2011).

Size-structured population models are studied by many other authors, for example, see Metz, et al. (1986), and the references in the above mentioned papers.

Our motivation for the present study is to extend the stability results for age-structured population dynamics model, given in the classical work of Gurtin, el al. (1974), to the case of size-structured population dynamics models with inflow of newborns from an external source, and to prove the principle of linearized stability.

We note that in our previous papers El-Doma (2008-1), El-Doma (2008-2), and El-Doma (2009-1), we assumed linearized stability for size-structured models, a principle that has received considerable attention in recent years. This principle consists of two parts, namely, stability part and instability part, for example, see Diekmann, et al. (2007 b). The stability part says that a nontrivial steady state is locally exponentially stable if all the roots of the corresponding characteristic equation, which results from the linearization of the model equations at a steady state, lie to the left of the imaginary axis. The instability part says that a nontrivial steady state is unstable if the corresponding characteristic equation has at least one root that lies to the right of the imaginary axis. For example, Tucker, et al. (1988), proved the stability part for a general size-structured model. De Roos, et al. (1990), concluded that their numerical results are in agreement with the stability results obtained via linearization for a size-structured model of Daphnia. Calsina, et al. (1995), proved the existence of a global attractor for problem (1). Diekmann, et al. (2007 a), conjectured the principle and outlined preliminary steps for a proof. Diekmann, et al. (2007 b), Diekmann, et al. (2007 c) and Diekmann, et al. (2008), reformulated delay differential equations and delay equations as abstract integral equations and improved them progressively to prove the principle of linearized stability for physiologically structured population dynamics models. Their methods have the potential to address the principle of linearized stability for problem (1).

In this paper, we study problem (1) and consider the steady states, which are given in our previous papers, for example, see El-Doma(2008-1), El-Doma (2009-1) for the case $C \equiv 0$, and El-Doma (2008-2), El-Doma (Preprint-1) for the case $C > 0$. We also introduce a transformation that transforms problem (1) to an age-structured type model, and then use it to prove the principle of the linearized stability.

The organization of this paper as follows: in section 2 we determine the steady states; in section 3 we prove the principle of linearized stability for size-structured population models; in section 4 we conclude our results.

2. The Steady States

In this section, we determine the steady states of problem (1). A steady state of problem (1) satisfies the following:

$$\begin{cases} \frac{d}{da}[V(a,P_\infty)p_\infty(a)] + \mu(a,P_\infty)p_\infty(a) = 0, \quad a > 0, \\ \\ V(0,P_\infty)p_\infty(0) = C + \int_0^\infty \beta(a,P_\infty)p_\infty(a)da, \\ \\ P_\infty = \int_0^\infty p_\infty(a)da. \end{cases} \tag{7}$$

From (7), by solving the differential equation, we obtain that

$$p_\infty(a) = p_\infty(0)V(0,P_\infty)\frac{\pi(a,P_\infty)}{V(a,P_\infty)}, \tag{8}$$

where $\pi(a, P_\infty)$ is defined as

$$\pi(a,P_\infty) = e^{-\int_0^a \frac{\mu(\tau,P_\infty)}{V(\tau,P_\infty)}d\tau}. \tag{9}$$

Also, from (7) and (8), we obtain that $p_\infty(0)$ satisfies the following:

$$V(0,P_\infty)p_\infty(0) = C + V(0,P_\infty)p_\infty(0)\int_0^\infty \frac{\beta(a,P_\infty)}{V(a,P_\infty)}\pi(a,P_\infty)da. \tag{10}$$

Accordingly, from (10), we conclude that either $p_\infty(0) = 0 = C$ or P_∞ satisfies the following:

$$1 = \int_0^\infty \frac{\beta(a,P_\infty)}{V(a,P_\infty)}\pi(a,P_\infty)da + \frac{C}{V(0,P_\infty)p_\infty(0)}. \tag{11}$$

In order to facilitate our writing, we define a threshold parameter $R(P)$ by

$$R(P) = \int_0^\infty \frac{\beta(a,P)}{V(a,P)}\pi(a,P)da + \frac{C}{P}\int_0^\infty \frac{\pi(a,P)}{V(a,P)}da, \tag{12}$$

which when $V \equiv 1, C \equiv 0$, and a is age (the age-structured case) is interpreted as the number of children expected to be born to an individual, in a life time, when the population size is P.

We note that from equation (8), $p_\infty(0) = \dfrac{P_\infty}{V(0,P_\infty)\int_0^\infty \frac{\pi(a,P_\infty)}{V(a,P_\infty)}da}$, and accordingly, either $p_\infty(a) \equiv 0$ or $p_\infty(a)$ is completely determined by a solution $P_\infty > 0$ of equation (11).

In the following theorem, we describe the steady states of problem (1), the proof can be found in El-Doma (2008-1), for the case $C = 0$, and El-Doma(2008-2), for the case $C > 0$.

Theorem 2.1

(1) If $C = 0$, then problem (1) has the trivial steady state, $P_\infty = 0$.

(2) If $C > 0$, then problem (1) has no trivial steady state.
(3) All positive solutions of, $R(P_\infty) = 1$, are nontrivial steady states of problem (1).

3. Stability of the Steady States

In this section, we study the stability of the steady states for problem (1) as given by Theorem 2.1.

To study the stability of a steady state $p_\infty(a)$, which is a solution of (7) and is given by equation (8), we consider a perturbation $\omega(a,t)$ defined by $\omega(a,t) = p(a,t) - p_\infty(a)$, where $p(a,t)$ is a solution of problem (1). Accordingly, we obtain that $\omega(a,t)$ satisfies the following:

$$\begin{cases} \dfrac{\partial \omega(a,t)}{\partial t} + \dfrac{\partial}{\partial a}\Big(V(a,P_\infty)\omega(a,t)\Big) + D(a,P_\infty)W(t) + \mu(a,P_\infty)\omega(a,t) = \phi(a,t),\ a>0,\ t>0, \\ \omega(0,t)V(0,P_\infty) = \displaystyle\int_0^\infty \beta(a,P_\infty)\omega(a,t)da + \theta W(t) + \psi(t),\ t \geq 0, \\ \omega(a,0) = \omega_0(a) = p_0(a) - p_\infty(a), \quad a \geq 0, \end{cases} \tag{13}$$

where $W(t), \phi(a,t), \psi(t), D(a), \theta$ are given, respectively, by

$$W(t) = \int_0^\infty \omega(a,t)da, \quad t \geq 0, \tag{14}$$

$$\begin{aligned} \phi(a,t) &= -\frac{\partial}{\partial a}[\Omega(a,W(t);V)(\omega(a,t)+p_\infty(a))] - \Omega(a,W(t);\mu)[\omega(a,t)+p_\infty(a)] \\ &\quad -\Big[\mu_P(a,P_\infty)\omega(a,t) + \frac{\partial}{\partial a}\Big(V_P(a,P_\infty)\omega(a,t)\Big)\Big]W(t), \end{aligned} \tag{15}$$

$$\begin{aligned} \psi(t) &= \int_0^\infty \Omega(a,W(t);\beta)[\omega(a,t)+p_\infty(a)]da - \Omega(0,W(t);V)[\omega(0,t)+p_\infty(0)] \\ &\quad +\Big[\int_0^\infty \beta_P(a,P_\infty)\omega(a,t)da - V_P(0,P_\infty)\omega(0,t)\Big]W(t), \end{aligned} \tag{16}$$

$$D(a,P_\infty) = p_\infty(a)\mu_P(a,P_\infty) + \frac{\partial}{\partial a}\Big(V_P(a,P_\infty)p_\infty(a)\Big), \tag{17}$$

$$\theta = \int_0^\infty \beta_P(a,P_\infty)p_\infty(a)da - p_\infty(0)V_P(0,P_\infty), \tag{18}$$

where $\Omega(a, W(t); \beta)$ is given by

$$\Omega(a, W(t); \beta) = \beta(a, P_\infty + W(t)) - \beta(a, P_\infty) - \beta_P(a, P_\infty)W(t), \tag{19}$$

we also note that $\Omega(a, W(t); .)$, where " . " stands for any function, is defined similarly.

We now make the following change of variable.

$$\tau = \int_0^a \frac{d\sigma}{V(\sigma, P_\infty)} = g(a), \tag{20}$$

$$\frac{d\tau}{da} = \frac{1}{V(a, P_\infty)}, \tag{21}$$

$$\alpha = \int_0^\infty \frac{d\sigma}{V(\sigma, P_\infty)}. \tag{22}$$

Now, putting $Z(a, t, P_\infty) = \omega(a, t)V(a, P_\infty)$, and using (13), we obtain

$$\begin{cases} \dfrac{\partial \hat{Z}(\tau, t, P_\infty)}{\partial t} + \dfrac{\partial \hat{Z}(\tau, t, P_\infty)}{\partial \tau} + \hat{\mu}(\tau, P_\infty)\hat{Z}(\tau, t, P_\infty) + D^*(\tau, P_\infty)W(t) \\ = \hat{\phi}(\tau, t)\hat{V}(\tau, P_\infty), \ \tau > 0, \ t > 0, \\ Z(0, t, P_\infty) = \displaystyle\int_0^\alpha \hat{\beta}(\tau, P_\infty)\hat{Z}(\tau, t, P_\infty)d\tau + \theta W(t) + \psi(t), \ t > 0, \\ \hat{Z}(\tau, 0, P_\infty) = \hat{Z}_0(\tau, P_\infty) = \hat{\omega}_0(\tau)\hat{V}(\tau, P_\infty), \ \tau \geq 0, \end{cases} \tag{23}$$

where $\hat{Z}(\tau, t, P_\infty), D^*(\tau, P_\infty)$ are defined as

$$\hat{Z}(\tau, t, P_\infty) = Z(g^{-1}(\tau), t, P_\infty), \tag{24}$$

$$D^*(\tau, P_\infty) = \hat{\mu}_P(\tau, P_\infty)\hat{p}_\infty(\tau)\hat{V}(\tau, P_\infty) + \frac{\partial}{\partial \tau}[\hat{V}_P(\tau, P_\infty)\hat{p}_\infty(\tau)], \tag{25}$$

where g is given by equation (20), we also note that all the other functions with $\hat{}$ are defined in a manner similar to the definition of $\hat{Z}(\tau, t, P_\infty)$.

From (23), by integrating along characteristic lines, we obtain the following implicit solution:

$$\hat{Z}(\tau,t,P_\infty)=\begin{cases}\hat{Z}_0(\tau-t,P_\infty)\dfrac{\pi^*(\tau,P_\infty)}{\pi^*(\tau-t,P_\infty)}+\displaystyle\int_0^t\Big[\hat{\phi}(\tau+\sigma-t,\sigma)\hat{V}(\tau+\sigma-t,P_\infty)\\ -D^*(\tau+\sigma-t,P_\infty)W(\sigma)\Big]\dfrac{\pi^*(\tau,P_\infty)}{\pi^*(\tau+\sigma-t,P_\infty)}d\sigma, \qquad \tau>t,\\ Z(0,t-\tau,P_\infty)\pi^*(\tau,P_\infty)+\displaystyle\int_{t-\tau}^t\Big[\hat{\phi}(\tau+\sigma-t,\sigma)\hat{V}(\tau+\sigma-t,P_\infty)\\ -D^*(\tau+\sigma-t,P_\infty)W(\sigma)\Big]\dfrac{\pi^*(\tau,P_\infty)}{\pi^*(\tau+\sigma-t,P_\infty)}d\sigma, \qquad \tau<t,\end{cases} \tag{26}$$

where $\pi^*(\tau,P_\infty)$ is defined as

$$\pi^*(\tau,P_\infty)=e^{-\int_0^\tau\hat{\mu}(\sigma,P_\infty)d\sigma}. \tag{27}$$

Suppose that $B(t,P_\infty)$ is defined as

$$B(t,P_\infty)=Z(0,t,P_\infty). \tag{28}$$

Then from (23), we obtain that $B(t,P_\infty)$ satisfies the following:

$$B(t,P_\infty) \;=\; \int_0^\alpha\hat{\beta}(\tau,P_\infty)\hat{Z}(\tau,t,P_\infty)d\tau+\theta W(t)+\psi(t). \tag{29}$$

Accordingly, using (26), we obtain the following:

$$\begin{aligned}&B(t,P_\infty)-\theta W(t)-\int_0^t\hat{\beta}(\tau,P_\infty)B(t-\tau,P_\infty)\pi^*(\tau,P_\infty)d\tau+\\ &\int_0^t\int_{t-\sigma}^\alpha D^*(\tau+\sigma-t,P_\infty)\hat{\beta}(\tau,P_\infty)\frac{\pi^*(\tau,P_\infty)}{\pi^*(\tau+\sigma-t,P_\infty)}W(\sigma)d\tau d\sigma=\\ &\int_0^t\int_{t-\sigma}^\alpha\hat{\phi}(\tau+\sigma-t,\sigma)\hat{V}(\tau+\sigma-t,P_\infty)\hat{\beta}(\tau,P_\infty)\frac{\pi^*(\tau,P_\infty)}{\pi^*(\tau+\sigma-t,P_\infty)}d\tau d\sigma\\ &+\int_t^\alpha\hat{Z}_0(\tau-t,P_\infty)\frac{\pi^*(\tau,P_\infty)}{\pi(\tau-t,P_\infty)}\hat{\beta}(\tau,P_\infty)d\tau+\psi(t).\end{aligned} \tag{30}$$

Also, using equation (14), we obtain

$$W(t)=\int_0^\infty\omega(a,t)da=\int_0^\alpha\hat{Z}(\tau,t,P_\infty)d\tau. \tag{31}$$

Accordingly, using (26), we obtain

$$W(t)-\int_0^t B(t-\tau,P_\infty)\pi^*(\tau,P_\infty)d\tau+\int_0^t\int_{t-\sigma}^{\alpha} D^*(\tau+\sigma-t,P_\infty)\times$$

$$\frac{\pi^*(\tau,P_\infty)}{\pi^*(\tau+\sigma-t,P_\infty)}W(\sigma)d\tau d\sigma=\int_t^{\alpha}\hat{Z}_0(\tau-t,P_\infty)\frac{\pi^*(\tau,P_\infty)}{\pi^*(\tau-t,P_\infty)}d\tau$$

$$+\int_0^t\int_{t-\sigma}^{\alpha}\hat{\phi}(\tau+\sigma-t,\sigma)\hat{V}(\tau+\sigma-t,P_\infty)\frac{\pi^*(\tau,P_\infty)}{\pi^*(\tau+\sigma-t,P_\infty)}d\tau d\sigma. \tag{32}$$

Now, from (30), (3,20), we obtain the following integral equation:

$$Ax(t)+\int_0^t K(t-\sigma)x(\sigma)d\sigma=f(t), \tag{33}$$

where $A, x(t), K(t), f(t)$ are given, respectively, as follows:

$$A = \begin{pmatrix} 1 & 0 \\ -\theta & 1 \end{pmatrix}, \tag{34}$$

$$x(t) = \begin{pmatrix} W(t) \\ B(t,P_\infty) \end{pmatrix}, \tag{35}$$

$$K(t) = \begin{pmatrix} \int_0^{\alpha-t} D^*(\tau,P_\infty)\frac{\pi^*(\tau+t,P_\infty)}{\pi^*(\tau,P_\infty)}d\tau & -\pi^*(t,P_\infty) \\ \int_0^{\alpha-t} D^*(\tau,P_\infty)\frac{\pi^*(\tau+t,P_\infty)}{\pi^*(\tau,P_\infty)}\hat{\beta}(\tau+t,P_\infty)d\tau & -\pi^*(t,P_\infty)\hat{\beta}(t,P_\infty) \end{pmatrix}, \tag{36}$$

$$f(t) = \int_0^t\int_{t-\sigma}^{\alpha}\hat{\phi}(\tau+\sigma-t,\sigma)\hat{V}(\tau+\sigma-t,P_\infty)\frac{\pi^*(\tau,P_\infty)}{\pi^*(\tau+\sigma-t,P_\infty)}\begin{pmatrix} 1 \\ \hat{\beta}(\tau,P_\infty) \end{pmatrix}d\tau d\sigma$$

$$+\begin{pmatrix} 0 \\ \psi(t) \end{pmatrix}+\int_0^{\alpha-t}\hat{Z}_0(\tau,P_\infty)\frac{\pi^*(\tau+t,P_\infty)}{\pi^*(\tau,P_\infty)}\begin{pmatrix} 1 \\ \hat{\beta}(\tau+t,P_\infty) \end{pmatrix}d\tau. \tag{37}$$

We note that a similar integral equation to (33) is also obtained in Gurtin, et al. (1974), which has the solution

$$x(t)=A^{-1}f(t)+\int_0^t J(t-s)f(s)ds, \tag{38}$$

where $\mathcal{L}\{J\}(\xi)$ is the inverse of $\mathcal{L}\{K\}(\xi)$. We note that in Gurtin, et al. (1974), it has been shown $J(t)$ satisfies

$$J(t)\le C_0e^{-\overline{\mu}t}, \tag{39}$$

where C_0 is a constant, and $\overline{\mu}$ is a constant that satisfies

$$\overline{\mu} < \mu_* = \inf_{a \geq 0} \mu(a, P_\infty) > 0.$$

We also note that if we set $\det[\mathcal{L}\{K\}(\xi)+A] = 0$, we obtain the following characteristic equation:

$$\begin{aligned} 1 &= \Big[1 + \int_0^\alpha \int_0^\tau e^{-(\tau-\sigma)\xi} D^*(\sigma, P_\infty) \frac{\pi^*(\tau, P_\infty)}{\pi^*(\sigma, P_\infty)} d\sigma d\tau\Big] \int_0^\alpha e^{-\xi\tau} \hat{\beta}(\tau, P_\infty)\pi^*(\tau, P_\infty) d\tau \\ &+ \Big[\theta - \int_0^\alpha \int_0^\tau e^{-(\tau-\sigma)\xi} \hat{\beta}(\tau, P_\infty) D^*(\sigma, P_\infty) \frac{\pi^*(\tau, P_\infty)}{\pi^*(\sigma, P_\infty)} d\sigma d\tau\Big] \int_0^\alpha e^{-\xi\tau} \pi^*(\tau, P_\infty) d\tau \\ &- \int_0^\alpha \int_0^\tau e^{-(\tau-\sigma)\xi} D^*(\sigma, P_\infty) \frac{\pi^*(\tau, P_\infty)}{\pi^*(\sigma, P_\infty)} d\sigma d\tau. \end{aligned} \tag{40}$$

We also note that the linear version of system (23) is given by the following:

$$\begin{cases} \dfrac{\partial \hat{Z}(\tau, t, P_\infty)}{\partial t} + \dfrac{\partial \hat{Z}(\tau, t, P_\infty)}{\partial \tau} + \hat{\mu}(\tau, P_\infty)\hat{Z}(\tau, t, P_\infty) + D^*(\tau, P_\infty)W(t) = 0, \ \tau > 0, \ t > 0, \\ Z(0, t, P_\infty) = \displaystyle\int_0^\alpha \hat{\beta}(\tau, P_\infty)\hat{Z}(\tau, t, P_\infty) d\tau + \theta W(t), \ t > 0, \\ \hat{Z}(\tau, 0, P_\infty) = \hat{Z}_0(\tau, P_\infty) = \hat{\omega}_0(\tau)\hat{V}(\tau, P_\infty), \ \tau \geq 0. \end{cases} \tag{41}$$

By substituting $\hat{Z}(\tau, t, P_\infty) = m(\tau, P_\infty)e^{\xi t}$ in (41), where ξ is a complex number, and straightforward calculations, we obtain the characteristic equation (40) again.

We note that it is easy to see that the characteristic equation (40) can be rewritten in the following form:

$$\begin{aligned} 1 &= \frac{1}{V(0, P_\infty)} \int_0^\infty e^{-\int_0^a E(\tau)d\tau} \beta(a, P_\infty) da \Big(1 + \int_0^\infty \int_0^a e^{-\int_\sigma^a E(\tau)d\tau} \frac{D(\sigma, P_\infty)}{V(\sigma, P_\infty)} d\sigma da\Big) \\ &- \int_0^\infty \int_0^a e^{-\int_\sigma^a E(\tau)d\tau} \frac{D(\sigma, P_\infty)}{V(\sigma, P_\infty)} d\sigma da + \\ &\frac{1}{V(0, P_\infty)} \int_0^\infty e^{-\int_0^a E(\tau)d\tau} da \Big[\theta - \int_0^\infty \int_0^a e^{-\int_\sigma^a E(\tau)d\tau} \beta(a, P_\infty) \frac{D(\sigma, P_\infty)}{V(\sigma, P_\infty)} d\sigma da\Big], \end{aligned} \tag{42}$$

where $E(\sigma)$ is given by

$$E(\sigma) = \frac{\xi + V_\sigma(\sigma, P_\infty) + \mu(\sigma, P_\infty)}{V(\sigma, P_\infty)}. \tag{43}$$

In order to facilitate our results, we assume the following:

(A_1) $0 \leq p_0(a), p_\infty(a) \in L^1(\mathbb{R}^+) \cap C^1(\mathbb{R}^+), p_\infty^{'}(.) \in L^1(\mathbb{R}^+), \mathbb{R}^+ = [0, \infty)$.

(A_2) $V(a, P(t)), \beta(a, P(t)), \mu(a, P(t)) \geq 0 \,\&\, \in C^0(\mathbb{R}^+ \times \mathbb{R}^+)$.

(A_3) $V_a(a, P), V_P(a, P), V_{Pa}(a, P), \beta_P(a, P), \mu_P(a, P)$ exist $\forall a \geq 0, P \geq 0$.

(A_4) $V(., P), V_a(., P), V_P(., P), V_{Pa}(., P), \beta(., P), \beta_P(., P), \mu(., P), \mu_P(., P)$, as functions of $P \in C^0(\mathbb{R}^+ : L_\infty(\mathbb{R}^+))$.

(A_5) $\dfrac{\partial^k}{\partial a^k}\beta(a, P_\infty)$ exists for $k = 1, 2$, and, as functions of a belong to $L_\infty(\mathbb{R}^+)$; $\dfrac{\partial}{\partial a}\mu(a, P_\infty)$ exists, and, as a function of a belongs to $L_\infty(\mathbb{R}^+)$.

(A_6) $\dfrac{\Omega(a, W; \beta)}{W}, \dfrac{\Omega(a, W; V)}{W}, \dfrac{\Omega(a, W; \mu)}{W}, \dfrac{\Omega(a, W; V_a)}{W}$, tend to zero as $W \longrightarrow 0$ uniformly for $a \geq 0$.

(A_7) $p_a(., t) \in L^1(\mathbb{R}^+)$.

(A_8) $\mu_* = \inf\limits_{a \geq 0} \mu(a, P_\infty) > 0$.

(A_9) If $\alpha < +\infty$, then we also assume that $D^*(\tau, P_\infty) \in C^1([0, \alpha])$.

(A_{10}) $p(., t), p_t(., t) \in L^1(\mathbb{R}^+)$, and the integrals converge uniformly for $t \geq 0$.

(A_{11}) $||p_a(., t)||_{L^1(\mathbb{R}^+)}$ is bounded by a constant for all t.

We note that in our upcoming results, we shall use the order of magnitude symbols "O" and "0" as well as the behaves like symbol " $\sim$ " with the following standard meanings:

- $h_1(t) = O(h_2(t))$ as $t \longrightarrow t_0$ if $|h_1(t)| \leq constant.|h_2(t)|$ for t sufficiently close to t_0.
- $h_1(t) = 0(h_2(t))$ as $t \longrightarrow t_0$ if $\lim\limits_{t \longrightarrow t_0} \dfrac{h_1(t)}{h_2(t)} = 0$.
- $h_1(t) \sim h_2(t)$ as $t \longrightarrow +\infty$ if $\lim\limits_{t \longrightarrow \infty} \dfrac{h_1(t)}{h_2(t)} = 1$.

In the following result, we give a proof for the stability part of the linearized stability principle. We note that this result is a generalization of the result given in Gurtin, et al. (1974).

Theorem 3.1 Assume $(A_1)-(A_9), (2)-(6)$, also assume that the characteristic equation (40) has no solution ξ with $\Re(\xi) \geq 0$. Then $\exists\delta > 0$, and, $\overline{\mu} > 0$ such that for any initial size-distribution p_0 which satisfies $||p_0(.) - p_\infty(.)||_{L^1(\mathbb{R}^+)} < \delta$ the corresponding solution of problem (1) satisfies

$$P(t) - P_\infty = O(e^{-\overline{\mu}t})\ as\ t \longrightarrow +\infty$$

$$p(a,t) - p_\infty(a) = O(e^{-\overline{\mu}t})\ for\, each\, a, as\ t \longrightarrow +\infty.$$

Proof. We note that the integral equation (33) is similar to that given in Gurtin, et al. (1974), and therefore, in order to prove the theorem it suffices to check that the functions f, K, x, W, ψ satisfy the analogous conditions as in Theorem 7 of Gurtin, et al. (1974).

Indeed, using equations (37), (15), (16) and conditions $(A_1) - (A_4), A_8$, we obtain

$$|f(t)| \leq C_1\Big\{\int_0^t e^{-\mu_*(t-\sigma)}||\phi(.,\sigma)||_{L^1(\mathbb{R}^+)}d\sigma + e^{-\mu_* t}||\omega_0(.)||_{L^1(\mathbb{R}^+)} + |\psi(t)|\Big\}, \tag{44}$$

where C_1 is a constant.

We also note that from assumptions $(A_1) - (A_5), (A_8), (A_9)$, we obtain

$$\Big|\frac{d^k}{dt^k}K(t)\Big| \leq C_2 e^{-\mu_* t}, \ \ k = 0, 1, 2, \tag{45}$$

where C_2 is a constant. Accordingly, the Laplace transform of $K(t)$ denoted by $\mathcal{L}\{K\}(\xi)$ exists for $\Re(\xi) > -\mu_*$.

Also, using equations (38)-(39), (44), it is easy to see that $|x(t)|$ satisfies the following inequality:

$$\begin{aligned} |x(t)| &= (W(t)^2 + B(t, P_\infty)^2)^{\frac{1}{2}} \\ &\leq C_3\Big\{||\omega_0(.)||_{L^1}e^{-\overline{\mu}t} + |\psi(t)| + \int_0^t e^{-\overline{\mu}(t-\sigma)}[||\phi(.,\sigma)||_{L^1(\mathbb{R}^+)} + |\psi(\sigma)|]d\sigma\Big\}, \end{aligned} \tag{46}$$

where C_3 is a constant.

From the definition of $\hat{Z}(\tau, t, P_\infty)$, and the change of variable (20), we obtain

$$\int_0^\alpha |\hat{Z}(\tau, t, P_\infty)|d\tau = \int_0^\infty |\omega(a,t)|da = ||\omega(.,t)||_{L^1(\mathbb{R}^+)}. \tag{47}$$

Now, using (26), $(A_1), (A_8)$, we obtain

$$\begin{aligned} ||\omega(.,t)||_{L^1(\mathbb{R}^+)} \leq \int_0^t |B(t-\tau, P_\infty)|e^{-\mu_* t}d\tau + ||\omega_0(.)||_{L^1(\mathbb{R}^+)}e^{-\mu_* t} + \\ \int_0^t e^{-\mu_*(t-\sigma)}\Big\{||\phi(.,\sigma)||_{L^1(\mathbb{R}^+)} + ||D(.,P_\infty)||_{L^1(\mathbb{R}^+)}|W(\sigma)|\Big\}d\sigma. \end{aligned} \tag{48}$$

Also, using (46), we obtain

$$||\omega(.,t)||_{L^1(\mathbb{R}^+)} \leq C_4\Big\{||\omega_0(.)||_{L^1(\mathbb{R}^+)}e^{-\overline{\mu}t} + \int_0^t e^{-\overline{\mu}(t-\sigma)}[||\phi(.,\sigma)||_{L^1(\mathbb{R}^+)} + |\psi(\sigma)|]d\sigma\Big\}, \tag{49}$$

where C_4 is a constant.

By assumption (A_6) and for $\epsilon > 0$ fixed $\exists\overline{\delta} = \overline{\delta}(\epsilon)$ such that for $|W| < \overline{\delta}(\epsilon)$, we have $|\Omega(a, W, .)| < \epsilon|W|$, and we assume that $\overline{\delta} < \epsilon$.

Now, using $(A_1), (A_4), (A_7), (A_{11}), (2) - (6)$, we obtain

$$||\phi(., t)||_{L^1(\mathbb{R}^+)} \leq C_5\{|W|||\omega(., t)||_{L^1(\mathbb{R}^+)} + \epsilon|W|\}, \tag{50}$$

where C_5 is a constant.

From equation (16), assumptions $(A_4), (A_6)$, and equation $(13)_2$, we obtain

$$|\psi(t)|\Big[1 - |W|\Big(\frac{\epsilon + |V_P(0, P_\infty)|}{V(0, P_\infty)}\Big)\Big] \leq \epsilon|W|\Big[||\omega(., t)||_{L^1(\mathbb{R}^+)} + P_\infty + p_\infty(0)\Big] +$$

$$|W|||\beta_P(., P_\infty)||_{L_\infty(\mathbb{R}^+)}||\omega(., t)||_{L^1(\mathbb{R}^+)} + |W|\Big(\frac{\epsilon + |V_P(0, P_\infty)|}{V(0, P_\infty)}\Big)\Big[||\beta(., P_\infty)||_{L_\infty(\mathbb{R}^+)} \times$$

$$||\omega(., t)||_{L^1(\mathbb{R}^+)} + |\theta||W|\Big]. \tag{51}$$

Then if we choose $\epsilon > 0$ such that $|W|[\epsilon + |V_P(0, P_\infty)|] < V(0, P_\infty)$, for example, when $\epsilon < \min\Big(\frac{\sqrt{V(0, P_\infty)}}{\sqrt{2}}, \frac{V(0, P_\infty)}{2|V_P(0, P_\infty)|}\Big)$, we obtain

$$||\psi(., t)||_{L^1(\mathbb{R}^+)} \leq C_6\{|W|||\omega(., t)||_{L^1(\mathbb{R}^+)} + \epsilon|W|\}, \tag{52}$$

where C_6 is a constant.

Now, we can use Theorem 7 in Gurtin, et al. (1974) to obtain the theorem. This completes the proof of the theorem.

Remark 3.1: We note that in Theorem 3.1 if $V = V(a)$, then we do not need conditions $(A_7), (A_9), (A_{10})$, and in $(A_6), \Omega(a, W; V) = \Omega(a, W; V_a) = 0$, accordingly, the result follows from the age-structured case given in Gurtin, et al. (1974).

We also note that from equations (36), (34), we obtain that

$$A + \mathcal{L}\{K\}(\xi) =$$

$$\begin{pmatrix} 1 + \int_0^\infty \int_0^a e^{-\xi \int_\sigma^a \frac{d\tau}{V(\tau, P_\infty)}} D(\sigma, P_\infty) \frac{\pi(a, P_\infty)}{V(a, P_\infty)\pi(\sigma, P_\infty)} d\sigma da & -\int_0^\infty e^{-\xi \int_0^a \frac{d\tau}{V(\tau, P_\infty)}} \frac{\pi(a, P_\infty)}{V(a, P_\infty)} da \\ -\theta + \int_0^\infty \int_0^a e^{-\xi \int_\sigma^a \frac{d\tau}{V(\tau, P_\infty)}} D(\sigma, P_\infty) \frac{\beta(a, P_\infty)}{V(a, P_\infty)} \frac{\pi(a, P_\infty)}{\pi(\sigma, P_\infty)} d\sigma da & 1 - \int_0^\infty e^{-\xi \int_0^a \frac{d\tau}{V(\tau, P_\infty)}} \frac{\beta(a, P_\infty)}{V(a, P_\infty)} \pi(a, P_\infty) da \end{pmatrix}, \tag{53}$$

where $\pi(a, P_\infty)$ is given by equation (9).

In the next result, we prove the instability part of the principle of the linearized stability for size-structured models. We note that this part is not proved for many known models, for example, Gurtin, et al. (1974), Tucker, et al.(1988).

In order to facilitate our result, we define $F(\xi, f)$ to be $F(\xi, f) = \mathcal{L}\{f(t)\}(\xi)$, then

$$F^n(\xi, f) = \int_0^\infty (-a)^n e^{-\xi a} f(a)da, \tag{54}$$

where n=0,1,...

Theorem 3.2 Suppose that the characteristic equation (40) or equivalently the equation $\det[A + \mathcal{L}\{K\}(\xi)] = 0$, has m roots $\xi_j, j = 1, 2, ..., m, m = 1, 2, ...$ Let m_j be the multiplicity of the root ξ_j. Moreover, suppose that $F^0(\xi_j, f), \mathcal{L}\{K\}(\xi_j)$ exist for $1 \leq j \leq m$, and that assumptions $(A_1) - (A_5), (A_7) - (A_{10}), (2) - (5)$ hold. Then $x(t)$ is given by the following two formulas:

$$x(t) = \sum_{j=1}^{m} \sum_{k=0}^{m_j-1} \frac{p_{jk}}{k!} \int_0^\infty (t-a)^k e^{\xi_j(t-a)} f(a)da - \sum_{j=1}^{m} \sum_{k=0}^{m_j-1} \frac{p_{jk}}{k!} \int_t^\infty (t-a)^k e^{\xi_j(t-a)} f(a)da, \tag{55}$$

$$x(t) = \sum_{j=1}^{m} e^{\xi_j t} \sum_{k=0}^{m_j-1} \frac{t^k}{k!} \sum_{n=k}^{m_j-1} p_{jn} \frac{F^{n-k}(\xi_j, f)}{(n-k)!} - \sum_{j=1}^{m} e^{\xi_j t} \sum_{k=0}^{m_j-1} \frac{p_{jk}}{k!} \int_t^\infty (t-a)^k e^{-\xi_j a} f(a)da. \tag{56}$$

Moreover, the last term in the formula for $x(t)$ contains the integral $\int_t^\infty (t-a)^k e^{\xi_j(t-a)} f(a)da$, and we have the following three cases for its behavior as $t \longrightarrow +\infty$:

$$Case\,1: \ \Re(\xi_j) > 0, \quad 0(t^k e^{\xi_j t}) \ as\ t \longrightarrow +\infty, \ \ k = 0, 1, 2, ...m_j - 1.$$

$$Case\,2: \ \Re(\xi_j) = 0, \quad 0(t^k) \ as\ t \longrightarrow +\infty, \ \ k = 0, 1, 2, ...m_j - 1.$$

$$Case\,3: \ \Re(\xi_j) < 0, \quad 0(t^k e^{\xi_j t}) \ as\ t \longrightarrow +\infty, \ \ k = 0, 1, 2, ...m_j - 1.$$

Proof. We note that from (45), we obtain that

$$\mathcal{L}\{K\}(\xi) = \frac{K(0)}{\xi} + \frac{\dot{K}(0)}{\xi^2} + 0(\xi^2) \ as\ \xi \longrightarrow \infty. \tag{57}$$

From equation (57), we obtain that $\det[A + \mathcal{L}\{K\}(\xi)] \longrightarrow 1$ as $\xi \longrightarrow \infty$. Therefore, it is easy to see that $\exists\, \gamma = \sup_{1\leq j\leq m} \Re(\xi_j) < +\infty$. Accordingly, $\det[A + \mathcal{L}\{K\}(\xi)] \neq 0$ for $\Re(\xi_j) > \gamma$, and hence, we can use the Laplace inversion formula for the left-half plane to determine $x(t)$ via a Laurent series expansion, as in Miller (1974). The Laurent expansion for $[A+\mathcal{L}\{K\}(\xi)]^{-1}$ near $\xi = \xi_j, \ j = 1, 2, ..., m$, has the following form:

$$[A + \mathcal{L}\{K\}(\xi)]^{-1} = \sum_{k=0}^{m_j-1} p_{jk}(\xi - \xi_j)^{-k-1} + \sum_{k=0}^{\infty} L_{jk}(\xi - \xi_j)^k, \tag{58}$$

where p_{jk}, L_{jk}, are constant matrices.

We note that by assumptions (2) - (5), we obtain that problem (1) has a $C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ solution and therefore $f(t)$ is continuous by assumptions $(A_1) - (A_4), (A_7) - (A_{10})$; and hence, $|f(t)e^{-\xi_j t}|$ is bounded for all $t \in \mathbb{R}^+$ and as $t \longrightarrow +\infty$ as well, because $F^0(\xi_j, f)$ exists, and accordingly, f is of exponential order. Also, $x(t)$ has a continuous derivative by assumptions $(A_1) - (A_5), (A_7), (A_{10}), (2) - (5)$. We note that this fact will validate our representation for $x(t)$ by a series.

We also note that $x(t)$ has a Laplace transform because since $\gamma < +\infty$, we can choose $\gamma^* > \gamma$ and obtain the following estimate:

$$\sup_{s\in[0,t]} |e^{-\gamma^* s}x(s)| \leq \Big(\int_0^\infty |A^{-1}K(\sigma)|e^{-\gamma^*(\sigma)}d\sigma \Big) \sup_{s\in[0,t]} |e^{-\gamma^* s}x(s)| + \sup_{t\in\mathbb{R}^+} e^{-\gamma^* t}|A^{-1}f(t)|, \quad (59)$$

We note that in (59) we have used $|.|$ for two different norms one time for a vector and another time for a matrix.

We also note that γ^* can be chosen such that $\displaystyle\int_0^\infty |A^{-1}K(\sigma)|e^{-\gamma^*(\sigma)}d\sigma < 1$, since $K(t)$ satisfies (45). Accordingly, from (59), we obtain

$$\sup_{s\in[0,t]} |e^{-\gamma^* s}x(s)| \Big[1 - \int_0^\infty |A^{-1}K(\sigma)|e^{-\gamma^*(\sigma)}d\sigma \Big] \leq C_7, \quad (60)$$

where C_7 is a constant.

From (60), we obtain that $x(t)$ is of exponential order, and hence has a Laplace transform.

Accordingly, we can use Laplace inversion formula to obtain

$$x(t) \;=\; \sum_{j=1}^{m} \sum_{k=0}^{m_j-1} \frac{p_{jk}}{k!} \int_0^t (t-a)^k e^{\xi_j(t-a)} f(a)da. \quad (61)$$

From equation (61), it is easy to see that equation (55) holds. Now, using equation (54), it is easy to see that equation (56) holds.

We note that by induction, equation (61) holds for $m = 1, 2, ...$

For $\Re(\xi_j) > 0$, we note that

$$\sum_{j=1}^{m} e^{\xi_j t} \sum_{k=0}^{m_j-1} \frac{p_{jk}}{k!} \int_t^\infty (t-a)^k e^{-\xi_j a} f(a)da = \sum_{j=1}^{m} e^{\xi_j t} \sum_{k=0}^{m_j-1} \sum_{n=k}^{m_j-1} \frac{t^k}{k!} p_{jn} \int_t^\infty \frac{(-a)^{n-k} e^{-\xi_j a} f(a)}{(n-k)!} da. \quad (62)$$

But $\displaystyle\int_t^\infty (-a)^{n-k} e^{-\xi_j a} f(a)da \longrightarrow 0 \; as\, t \longrightarrow +\infty$, since by assumption $F^0(\xi_j, f)$ exists, and accordingly, $(-a)^{n-k}e^{-\xi_j a}f(a)$ is integrable, therefore, this term is $0(t^k e^{\xi_j t})$ as $t \longrightarrow +\infty$. This proves case 1.

For $\Re(\xi_j) = 0$, we can see that

$$\sum_{j=1}^{m}\sum_{k=0}^{m_j-1}\int_t^\infty \frac{p_{jk}}{k!}(t-a)^k f(a)da = \sum_{j=1}^{m}\sum_{k=0}^{m_j-1}\sum_{n=k}^{m_j-1}\frac{t^k}{k!}p_{jn}\int_t^\infty \frac{(-a)^{n-k}f(a)}{(n-k)!}da. \tag{63}$$

But $\int_t^\infty (-a)^{n-k}f(a)da \longrightarrow 0 \; as\, t \longrightarrow +\infty$, since by assumption $F^0(0,f)$ exists, and accordingly, $(-a)^{n-k}f(a)$ is integrable. Hence this term is $0(t^k)$ as $t \longrightarrow +\infty$. This proves case 2.

For $\Re(\xi_j) < 0$, we can see that

$$\sum_{j=1}^{m}e^{\xi_j t}\sum_{k=0}^{m_j-1}\frac{p_{jk}}{k!}\int_t^\infty (t-a)^k e^{-\xi_j a}f(a)da = \sum_{j=1}^{m}e^{\xi_j t}\sum_{k=0}^{m_j-1}\sum_{n=k}^{m_j-1}\frac{t^k}{k!}p_{jn}\int_t^\infty \frac{(-a)^{n-k}e^{-\xi_j a}f(a)}{(n-k)!}da. \tag{64}$$

Now, similar arguments as in Case 1 lead to the fact that this term is $0(t^k e^{\xi_j t})$ as $t \longrightarrow +\infty$. This proves case 3, and completes the proof of the theorem.

Remark 3.2: We note that if $\Re(\xi_j) > 0 \forall j$, then from equation (56), we obtain the following condition for $x(t)$ to be bounded:

$$\sum_{n=k}^{m_j-1} p_{jn}\frac{F^{n-k}(\xi_j, f)}{(n-k)!} = 0, \;\; j = 1, 2, ..., m; \;\; k = 0, 1, ..., m_j - 1.$$

Also, note that since F depends on the initial size-distribution p_0, we can alter it to obtain instability in case there is at least one root that lies to the right of the imaginary axis.

In the next result, we deduce the second half of the principle of the linearized stability, and further, we show that in the case of roots with multiplicity bigger than one and real parts zero we have resonance, we also retain Theorem 3.1.

Corollary 3.3 Suppose that the assumptions of Theorem 3.2 hold. Then the following hold:

1) If there is at least one root of the characteristic equation (42) with real part greater than zero, then the steady state is unstable.
2) If there is at least one root of the characteristic equation (42) with real part equal to zero and multiplicity greater than one, and there are no roots with real parts greater than zero, then there is resonance if the imaginary part is not equal to zero, and if the imaginary part is equal to zero, then $x(t)$ behaves as a polynomial; and in both cases the steady state is unstable.
3) If the characteristic equation (42) has no root ξ_j with $\Re(\xi_j) \geq 0$, then the steady state is locally exponentially stable.

Proof. To prove 1, we note that Case 1 of Theorem 3.2 shows that $x(t)$ grows exponentially, and therefore, the steady state is unstable. This proves 1.

To prove 2, we note that Case 2 of Theorem 3.2 shows that $x(t)$ grows as $t^k, \; k = 0, 1, ..., m_J - 1$, where m_j is at least 2, and accordingly, we obtain resonance if the imaginary part is not equal to zero, otherwise, we obtain a polynomial in t. This proves 2.

To prove 3, we note that case 3 of Theorem 3.2 shows that $x(t)$ decays exponentially, and therefore, the steady state is locally exponentially stable. This proves 3 and completes the proof of the corollary.

We note that 3. of Corollary 3.3 is exactly Theorem 3.1, and hence we retained that result.

In the following corollary, we study the stability of the special case of problem (1) when $\beta(a,P) = \beta(a)$ has compact support and not identically zero, $\mu(a,P) = \mu(a), V(a,P)$
$\equiv 1$, and other conditions hold.

Corollary 3.4 Suppose that $(A_1),(A_7),(A_9),(A_{10}),$ *and* $(2)-(5)$, hold, $\beta(a,P) = \beta(a) \not\equiv 0$, has compact support; $\mu(a,P) = \mu(a), V(a,P) \equiv 1; F^0(\xi_j, f), \mathcal{L}\{K\}(\xi_j)$ exist for $1 \leq j \leq m$, where f in the definition of F is given by equation (69) below; and ξ_j is a solution of the following equation:

$$1 = \int_0^\infty e^{-\xi_j a}\beta(a)\pi(a)da. \tag{65}$$

We also assume the following:

(A_2^*) $\beta(a), \mu(a) \geq 0 \ \& \in C^0(\mathbb{R}^+)$.

(A_5^*) $\beta^k(a)$ exists for $k = 1,2,$ *and belong to* $L_\infty(\mathbb{R}^+), \mu'(a)$ exists and belongs to $L_\infty(\mathbb{R}^+)$.

(A_8^*) $\mu_* = \inf_{a\geq 0} \mu(a) > 0$.

Then $x(t) \sim x_0 e^{pt}$ as $t \to +\infty$, where x_0 is given by

$$x_0 = \frac{\displaystyle\int_0^\infty \int_0^a e^{-p(a-\sigma)}[p_0(\sigma) - p_\infty(\sigma)]\beta(a)\frac{\pi(a)}{\pi(\sigma)}d\sigma da}{\displaystyle\int_0^\infty a e^{-pa}\beta(a)\pi(a)da} \begin{pmatrix} \displaystyle\int_0^\infty e^{-pa}\pi(a)da \\ 1 \end{pmatrix}, \tag{66}$$

p is the unique real solution of equation (65), and $\pi(a)$ is given by

$$\pi(a) = e^{-\int_0^a \mu(\tau)d\tau}. \tag{67}$$

Accordingly, if $p < 0$, the steady state is locally exponentially stable, and if $p > 0$, the steady state is unstable.

Proof. We note that, in this case, from the characteristic equation (42), we obtain

$$1 = \int_0^\infty e^{-\xi a}\beta(a)\pi(a)da, \tag{68}$$

which is exactly Lotka's characteristic equation.

We also note that it is well-known that equation (68), under our assumptions, has a unique real solution $\xi = p$, and any other root ξ satisfies $\Re(\xi_j) < p$.

Moreover, in this case, we note that from equations (15), (16), we obtain that $\phi(a,t) = \psi(t) = 0$, and accordingly, since in this case, g, given by equation (20), is the identity map, $f(t)$ satisfies

$$f(t) = \int_0^\infty Z_0(\tau, P_\infty)\frac{\pi(\tau+t, P_\infty)}{\pi(\tau, P_\infty)}\begin{pmatrix} 1 \\ \beta(\tau+t, P_\infty) \end{pmatrix} d\tau, \tag{69}$$

accordingly, from $(A_1), (A_5^*), (A_8^*)$, it is easy to see that $f''(t)$ exists, and the Laplace transform of f exists because it is bounded, and therefore, it has a representation as in equation (57).

Using Laplace transform in equation (33), we obtain that

$$[A + \mathcal{L}\{K\}(\xi)]\mathcal{L}\{x\}(\xi) = \mathcal{L}\{f\}(\xi). \tag{70}$$

We note that in the proof of Theorem 3.2 we showed that $x(t)$ has Laplace transform, and therefore, we obtain

$$\mathcal{L}\{x\}(\xi) = \frac{adj[A + \mathcal{L}\{K\}(\xi)]}{\det[A + \mathcal{L}\{K\}(\xi)]}\mathcal{L}\{f\}(\xi), \Re(\xi_j) > p. \tag{71}$$

Accordingly, the pole of $\mathcal{L}\{x\}(\xi)$ at p in the Laplace transform inversion formula is simple because

$$\lim_{\xi\longrightarrow p}(\xi - p)\frac{adj[A + \mathcal{L}\{K\}(\xi)]}{\det[A + \mathcal{L}\{K\}(\xi)]}\mathcal{L}\{f\}(\xi) = \frac{adj[A + \mathcal{L}\{K\}(p)]}{\left(\det[A + \mathcal{L}\{K\}(p)]\right)'}F^0(p, f)$$

$$= \frac{\int_0^\infty\int_0^a e^{-p(a-\sigma)}[p_0(\sigma) - p_\infty(\sigma)]\beta(a)\frac{\pi(a)}{\pi(\sigma)}d\sigma da}{\int_0^\infty ae^{-pa}\beta(a)\pi(a)da}\begin{pmatrix} \int_0^\infty e^{-pa}\pi(a)da \\ 1 \end{pmatrix}. \tag{72}$$

Using equations (56), (72), and Theorem 3.2, we obtain that $x(t) \sim x_0e^{pt}$ as $t \to +\infty$, where x_0 is given by equation (66). This completes the proof of the corollary.

We note that in Gurtin (1982), a similar result as in Corollary 3.4 is proved, except that in this case we have matrices and the integrals are improper, also see Hoppensteadt (1975) for a related result.

We also note that p, the unique real solution of equation (68), is known as the natural growth rate, and the quantity $\int_0^\infty ae^{-pa}\beta(a)\pi(a)da$, is known as the mean age of childbirth for persistent age distributions. Also, the quantity $R = \int_0^\infty \beta(a)\pi(a)da$, is known as the net reproduction rate, and is interpreted as the number of children expected to be born to an individual during its lifetime.

In the next result, we show that if the net reproduction rate R is less than one then the steady state is locally exponentially stable, whereas if R is bigger than one then the steady state is unstable.

Corollary 3.5 Suppose that the conditions of Corollary 3.4 hold. Then if $R < 1$, the steady state is locally exponentially stable, whereas if $R > 1$, the steady state is unstable.

Proof. We note that from the characteristic equation (65) if $R = \int_0^\infty \beta(a)\pi(a)da < 1$, then it is easy to see that p is negative. Also, if $R = \int_0^\infty \beta(a)\pi(a)da > 1$, then p is positive. Therefore, the result is obtained by using Corollary 3.4. This completes the proof of the corollary.

We note that Corollary 3.5 confirmed our stability results given in El-Doma (2008-1), for example, see Theorem 3.1 therein.

We also note that in El-Doma (Preprint -2) it is shown that if $C > 0$, then the unique steady state is actually globally stable. Also, note that in this case if $C > 0$, then from equation (11) it is easy to see that fertility is below replacement, and therefore, this result seems to agree with results given in Iannelli, et al. (2007) and the references therein, though the models are different.

In the next result, we prove that if $\alpha < +\infty$, then the number of the roots of the characteristic equation is finite. This fact implies that $[A + \mathcal{L}\{K(\xi)\}]^{-1}$ is a meromorphic function.

Theorem 3.6 Suppose that the following hold:

1) $\alpha < +\infty$.
2) Conditions $(A_1) - (A_5), (A_8)$, hold.

Then the number of the roots of the characteristic equation $\det[A + \mathcal{L}\{K(\xi)\}] = 0$ is finite.

Proof. We note that by assumptions 1 - 2, $K(t)$, given by equation (36), has compact support. Accordingly, $\mathcal{L}\{K(\xi)\}$ is an entire function. Also, by assumptions 1 - 2, equation (57) holds via inequality (45), and hence $\det[A + \mathcal{L}\{K(\xi)\}] \longrightarrow 1$ as $\xi \longrightarrow \infty$. Therefore, the roots of the characteristic equation are contained in a compact subset of $\mathbb{R}^{+2}$, and hence they must be finite, because the zeros of an analytic function are isolated. This completes the proof of the theorem.

In the next result, we prove that if the individuals in the population have a finite maximum size span $a_\dagger$, then the number of the roots of the characteristic equation is finite.

We note that as in Gurtin (1982), we define $\beta(a, P), \pi^*(a, P), V(a, P)$, to be zero for $a \geq a_\dagger$.

Theorem 3.7 Suppose that the following hold:

1) The individuals in the population have a finite maximum size span $a_\dagger$.
2) Conditions $(A_1) - (A_5), (A_8)$, hold.

Then the number of the roots of the characteristic equation $\det[A + \mathcal{L}\{K(\xi)\}] = 0$ is finite.

Proof. We note that from the characteristic equation (42), it is easy to see that $\mathcal{L}\{K(\xi)\}$ is an entire function since $a_\dagger < +\infty$. Now, the proof follows the same arguments as in the proof of Theorem 3.6. This completes the proof of the theorem.

In the next result, we give a corollary to Theorem 3.2, it deals with the special case when either $\alpha < +\infty$, or when the individuals in the population have a finite maximum size span $a_\dagger$. We note that in both cases we have shown that the number of the roots of the characteristic equation is finite. We also note that in the latter case, we have two choices, either $\alpha = \int_0^{a_\dagger} \frac{d\sigma}{V(\sigma, P_\infty)}$ is finite or infinite, but we found that $\alpha = +\infty$ in this case does not affect our results, for example see Corollary 3.8 and equation (73) below.

Corollary 3.8 Suppose that $\alpha < +\infty$, or the individuals in the population have a finite maximum size span $a_\dagger$. Moreover, suppose that $F^0(\xi_j, \begin{pmatrix} 0 \\ \psi(t) \end{pmatrix})$ exists for $1 \leq j \leq m$, where ξ_j is as in Theorem 3.2, and that assumptions $(A_1) - (A_5), (A_7) - (A_{10}), (2) - (5),$ hold. Then the result of Theorem 3.2 holds with m finite.

Proof. We note that from Theorem 3.6 and Theorem 3.7, we have shown that in both cases $\mathcal{L}\{K(\xi)\}$ is an entire function. Therefore, the assumption on $\mathcal{L}\{K(\xi)\}$ in theorem 3.2 can be removed in this case. Now we are only left with the assumption on $F^0(\xi_j, f)$ in Theorem 3.2 which is replaced by the assumption on $F^0(\xi_j, \begin{pmatrix} 0 \\ \psi(t) \end{pmatrix})$, and that is because from equation (37) f can be rewritten as

$$\begin{aligned} f(t) &= \int_t^\alpha \int_0^t \hat{\phi}(\tau + \sigma - t, \sigma)\hat{V}(\tau + \sigma - t, P_\infty)\frac{\pi^*(\tau, P_\infty)}{\pi^*(\tau + \sigma - t, P_\infty)} \begin{pmatrix} 1 \\ \hat{\beta}(\tau, P_\infty) \end{pmatrix} d\sigma d\tau \\ &+ \int_0^t \int_{t-\tau}^t \hat{\phi}(\tau + \sigma - t, \sigma)\hat{V}(\tau + \sigma - t, P_\infty)\frac{\pi^*(\tau, P_\infty)}{\pi^*(\tau + \sigma - t, P_\infty)} \begin{pmatrix} 1 \\ \hat{\beta}(\tau, P_\infty) \end{pmatrix} d\sigma d\tau \\ &+ \int_0^{\alpha - t} \hat{Z}_0(\tau, P_\infty)\frac{\pi^*(\tau + t, P_\infty)}{\pi^*(\tau, P_\infty)} \begin{pmatrix} 1 \\ \hat{\beta}(\tau + t, P_\infty) \end{pmatrix} d\tau + \begin{pmatrix} 0 \\ \psi(t) \end{pmatrix}. \end{aligned} \tag{73}$$

We note that the first three terms in the right-hand side of equation (73) are of compact support (in both cases), and therefore, their Laplace transform is an entire function. This completes the proof of the corollary.

In the next result, we use Corollary 3.8 to obtain a result that is similar to that in Corollary 3.3. The result follows immediately from the proof of Corollary 3.3, therefore, we omit the proof.

Corollary 3.9 Suppose that the assumptions of Corollary 3.8 hold. Then the result of Corollary 3.3 hold with the number of the roots $\xi_j, 1 \leq j \leq m,$ finite.

In the next result, we use Corollary 3.9 to obtain a result that is similar to that in Corollary 3.4, for the special case when the individuals in the population have a finite maximum size span $a_\dagger$.

Corollary 3.10 Suppose that $(A_1), (A_2^*), (A_5^*), (A_8^*),$ *and* $(2) - (5),$ hold, $\beta(a, P) = \beta(a) \not\equiv 0, V(a, P) \equiv 1, \mu(a, P) = \mu(a),$ and, the individuals in the population have a finite maximum size span $a_\dagger$. Then the result of Corollary 3.4 holds with ∞ replaced by $a_\dagger$, and the number of the roots $\xi_j, 1 \leq j \leq m,$ is finite.

Proof. We note that as before, $\mathcal{L}\{K(\xi)\}$ is an entire function. Also, note that f, given by equation

(69) has compact support, accordingly, $F^0(\xi, f)$ is an entire function. Furthermore, we do not need assumption (A_{10}) because of our assumption that the individuals in the population have a finite maximum size span $a_\dagger$. This completes the proof of the corollary.

In the next result, we use Corollary 3.10 to obtain a result that is similar to that in Corollary 3.5. The result follows immediately from Corollary 3.5, therefore, we omit the proof.

Corollary 3.11 Suppose that the assumptions of Corollary 3.10 hold. Then the result of Corollary 3.5 holds with ∞ replaced by $a_\dagger$, and the number of the roots $\xi_j, 1 \leq j \leq m$, is finite.

4. Conclusion

In this paper, we have studied a size-structured population dynamics model with vital rates i.e., the birth rate, the death rate, and the growth rate, which depend on size as well as the population size, and we also assumed that there is an inflow of newborns from an external source. Our aim here is to establish the principle of linearized stability, which consists of two parts. The first part of the principle says that if all the roots of the characteristic equation lie to the left of the imaginary axis, then the steady state is locally exponentially stable, this part is proved by extending the methods established in Gurtin, et al. (1974), for their classical model. The second part of the principle says that if at least one root of the characteristic equation lies to the right of the imaginary axis, then the steady state is unstable. We note that this part is not proved in, for example, Gurtin, et al. (1974), for their classical model, Tucker, et al. (1988), for their general size-structured model, Haimovici (1979), Gyllenberg (1982), (1983), and many other papers as well. We proved the second part of the principle by utilizing a Laplace transform inversion formula and an expansion of Laurent series, and we also identified cases where there can be resonance as well, and this is stronger statement than the principle, because this is the case where the roots have zero real parts. This method also provided another proof for the first part of the principle of linearized stability.

In the attack of problem (1), we used a transformation that revealed that there are exactly two cases, viz. $\alpha < +\infty$ and $\alpha = +\infty$, where α is given by equation (22). We also proved that if $\alpha < +\infty$, then the number of the roots of the characteristic equation is finite. For age-structured models $\alpha = +\infty$, raising the question whether some size structured models, as far as the number of the roots of the characteristic equation is concerned, are structurally simpler than some age-structured models? We also note that finite maximum size span for individuals in the population implies that the number of the roots of the characteristic equation is finite. For example, see Diekmann, et al. (2007 b) where this is also assumed in addition to the assumption that the death modulus is affine.

We note that these results justified our previous results which are reported in El-Doma (2008-1), El-Doma(2008-2). We also note that the inflow of newborns from an external source affects the steady states via equation (11), and, in fact, if $C > 0$, then problem (1) has no trivial steady state, and also it is known that it has stabilizing effects, for example, see El-Doma (Preprint -2), for a global stability result, and El-Doma(2008-2), El-Doma (Preprint 1), El-Doma (To appear), and El-Doma (2011), for local stability results.

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