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Generalized problem of thermal bending analysis in the Cartesian domain

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Abstract

This is an attempt for mathematical formulation and general analytical solution of the most generalized thermal bending problem in the Cartesian domain. The problem has been formulated in the context of non-homogeneous transient heat equation subjected to Robin's boundary conditions. The general solution of the generalized thermoelastic problem has been discussed for temperature change, displacements, thermal stresses, deflection, and deformation. The most important feature of this work is any special case of practical interest may be readily obtained by this most generalized mathematical formulation and its analytical solution. There are 729 such combinations of possible boundary conditions prescribed on parallelepiped shaped region in the Cartesian coordinate system. The key idea behind the solution of heat equation is to transform the original initial and boundary value problem into eigenvalue problem through the Strum-Liouville theory. The finite Fourier transform has been applied with respect to space variables by choosing suitable normalized kernels. The well-posedness of the problem has been discussed by the existence, uniqueness, and stability of series solutions obtained analytically. The convergence of infinite series solutions also been discussed.

Keywords: Thermal bending problem; Thermal stresses; Deflection; Eigenvalue problem; Strum-Liouville theory; Fourier transform

MSC 2010 No.: 35A01, 35A02, 35K05, 74A10, 74C05, 74F05

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1. Introduction

During recent years, the theory of elasticity has found considerable applications in the solution of engineering problems. There are many cases in which the elementary methods of a strength of materials are inadequate to furnish satisfactory information regarding stress distribution in engineering structures. In such cases, advanced theory of elasticity plays an important role in the development and extensions of interest and practical applicability.

A method of determining the thermal stresses in a flat rectangular isotropic plate of constant thickness with an arbitrary temperature distribution in the plane of the plate and with no variation in temperature through the thickness is presented by Iyengar and Chandrashekhara (1966). An analytical solution is presented for three-dimensional thermomechanical deformations of a simply supported functionally graded rectangular plate subjected to time-dependent thermal loads on its top and/or bottom surfaces by Senthil and Batra (2003). Tanigawa and Komatsubara (2007) studied a transient thermal stress problem of a rectangular plate due to a non-uniform heat supply and developed the corresponding thermal stress analysis on the basis of the two-dimensional plane stress problem using Airy's stress function method and analyzed the stress intensity factor for the bi-axial stress state. Xu et al. (2010) proposed an analytical method to find temperature, stress components of simply supported rectangular plates with variable thickness and subjected to thermomechanical loads. Deshmukh et al. (2010) derived results for thermal deflection and bending moments in a simply supported thin rectangular plate subjected to Dirichlet conditions on its boundaries. The thermal bending of a simply supported strip and a rectangular plate caused by a difference between the temperatures of the surroundings on the faces and the coordinate dependent heat exchange coefficients on them were investigated by Khapko et al. (2011). Based on the exact three-dimensional thermoelasticity theory, the elasticity solution of the simply supported laminated rectangular plates subjected to uniform temperature load was studied by Qian et al. (2014). Cheng and Fan (2015) discussed the analytical solution for deflection and bending of the thin rectangular plate of two opposite edges clamped, one edge simply supported and one edge free under temperature disparity. Transverse vibrations of nonhomogeneous rectangular Kirchhoff plates of variable thickness was studied by Lal and Saini (2015). Dhaba et al. (2015) gave a simple method to solve a static, plane boundary value problem in elasticity for an isotropic rectangular region using infinite Fourier transform. Kumar et al. (2017) derived the solutions for the coupled thermoelastic beam. A numerical technique based on the Laplace transformation is used to calculate the lateral deflection, thermal moment, and axial stress average. The effects of thermal and mechanical inhomogeneity on temperature and thermal stress distributions are examined by Manthena et al. (2017).

The mathematical formulation and analytical solution of the most generalized thermoelastic problem in the Cartesian coordinate system has been discussed in this manuscript. The most important feature of this work is any special case of practical interest may be readily obtained by this most generalized formulation and its analytical solution. There are 729 such combinations of possible boundary conditions prescribed on parallelepiped shaped region in the Cartesian coordinate system. The mathematical analysis for existence, uniqueness, and stability of series solution also has been discussed.

2. The Mathematical Model

Consider a solid polygon in the form of a rectangular plate in the Cartesian coordinate system (x, y, z) occupying space $\Omega \subset \mathbb{R}^3$ defined by

$$\Omega = \{(x, y, z) \in R^3/\ 0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c\}. \tag{1}$$

Let $T = T(x, y, z, t)$ denotes the temperature of the solid polygon, where t denote the time variable. The governing boundary value problem of heat conduction with Robin's boundary conditions on the open domain Ω for $T : \Omega \times (0, \infty) \longrightarrow \mathbb{R}$ is given by

$$\nabla^2 T + \frac{g(x, y, z, t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad (x, y, z) \in \Omega, \quad t > 0, \tag{2}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$.

The plane boundary $\partial\Omega_i$ is subjected to dissipation by convection $f_i : \partial\Omega \times [0, \infty) \longrightarrow \mathbb{R}$ which attempts to stimulate the heat transfer between solid and surrounding media. The heat is generated in the interior of Ω given by $g : \Omega \times (0, \infty) \longrightarrow \mathbb{R}$ at the rate $g(x, y, z, t)$ per unit volume. The initial temperature of solid polygon is arbitrary function $F : \Omega \times (0, \infty) \longrightarrow \mathbb{R}$. All f_i, F and g are continuous and absolutely integrable functions depends on the space variables (x, y, z) and time variable t.

The boundary conditions are given by

$$(-1)^i\ k_i\ \frac{\partial T}{\partial x} + h_i\ T = f_i(y, z, t), \quad (x, y, z) \in \partial\Omega_i,\ i = 1, 2, \tag{3}$$

$$(-1)^i\ k_i\ \frac{\partial T}{\partial y} + h_i\ T = f_i(x, z, t), \quad (x, y, z) \in \partial\Omega_i,\ i = 3, 4, \tag{4}$$

$$(-1)^i\ k_i\ \frac{\partial T}{\partial z} + h_i\ T = f_i(x, y, t), \quad (x, y, z) \in \partial\Omega_i,\ i = 5, 6. \tag{5}$$

The initial condition is given by

$$T(x, y, z, t) = F(x, y, z), \quad (x, y, z) \in \Omega, \quad t = 0. \tag{6}$$

k, α, h_i are thermal conductivity, thermal diffusivity and relative heat transfer coefficients of the material of solid polygon, respectively.

The boundaries of the solid polygon are denoted as

$$\partial\Omega_1 = \{(x, y, z) \in \Omega | x = 0\}, \quad \partial\Omega_2 = \{(x, y, z) \in \Omega | x = a\},$$

$$\partial\Omega_3 = \{(x, y, z) \in \Omega | y = 0\}, \quad \partial\Omega_4 = \{(x, y, z) \in \Omega | y = b\},$$

$$\partial\Omega_5 = \{(x, y, z) \in \Omega | z = 0\}, \quad \partial\Omega_6 = \{(x, y, z) \in \Omega | z = c\},$$

$$\partial\Omega = \bigcup_{i=1}^{6} \partial\Omega_i.$$

Following Eslami et al. (2013), in the thermal bending of a thin rectangular plate, it will be assumed that deflection, which means a deformation in the out of plane direction of the plate is small. The Kirchhoff-Love hypothesis assumes that a mid-surface plane can be used to represent a three-dimensional plate in two-dimensional form.

The fundamental equation for deflection of the plate is given as

$$\nabla^2\nabla^2 w = -\frac{1}{(1-\nu)\, D}\nabla^2 M_T, \tag{7}$$

where $D = \dfrac{Ec^3}{12(1-\nu^2)}$ is the bending rigidity of the material, ν is the Poisson's ratio and thermally induced resultant moment M_T is defined as

$$M_T = \alpha\, E \int_0^c \tau\, z\, dz. \tag{8}$$

The corresponding resultant moments are expressed in terms of deflection ω and thermally induced resultant moment M_T as

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\,\frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{1-\nu}\, M_T, \tag{9}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\,\frac{\partial^2 w}{\partial x^2}\right) - \frac{1}{1-\nu}\, M_T, \tag{10}$$

$$M_{xy} = (1-\nu)\, D\, \frac{\partial^2 w}{\partial x \partial y}. \tag{11}$$

The governing equation for thermal stress function is given by

$$\nabla^2\nabla^2 F = -\nabla^2 N_T, \tag{12}$$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

Thermally induced resultant force N_T is defined as

$$N_T = \alpha E \int_0^c \tau\, dz, \tag{13}$$

where τ represent the temperature change given by $\tau = T - T_i$, where T_i is the initial temperature.

The corresponding resultant forces in terms of thermal stress function F are expressed as

$$N_x = \frac{\partial^2 F}{\partial y^2}, \tag{14}$$

$$N_y = \frac{\partial^2 F}{\partial x^2}, \tag{15}$$

$$N_{xy} = -\frac{\partial^2 F}{\partial x \partial y}. \tag{16}$$

The thermal stress components in terms of resultant forces and resultant moments are given by

$$\sigma_{xx} = \frac{1}{c}\, N_x + \frac{12\, z}{c^3}\, M_x + \frac{1}{1-\nu}\left(\frac{1}{c}\, N_T + \frac{12\, z}{c^3}\, M_T - \alpha\, E\, \tau\right), \tag{17}$$

$$\sigma_{yy} = \frac{1}{c}\, N_y + \frac{12\, z}{c^3}\, M_y + \frac{1}{1-\nu}\left(\frac{1}{c}\, N_T + \frac{12\, z}{c^3}\, M_T - \alpha\, E\, \tau\right), \tag{18}$$

$$\sigma_{xy} = \frac{1}{c}\, N_{xy} - \frac{12\, z}{c^3}\, M_{xy}. \tag{19}$$

The thermal stress components for a traction-free surface are given by

$$\sigma_{xx} = 0 \text{ at } x = 0,\ x = a, \tag{20}$$

$$\sigma_{yy} = 0 \text{ at } y = 0,\ y = b, \tag{21}$$

$$\sigma_{xy} = 0 \text{ at } x = 0,\ x = a,\ y = 0,\ y = b. \tag{22}$$

Equations (1) to (22) constitutes the mathematical formulation of the problem.

3. The Solution

The key idea behind the existence and uniqueness of the solution of heat equation is to transform the original initial and boundary value problem into eigenvalue problem through the Strum-Liouville theory and application of triple finite Fourier transform by choosing suitable normalized kernels. Then the convergence and stability of resulting series solution have been discussed. The existence depends on the conditions on the functions f_i.

3.1. Transformation to Eigenvalue Problem

Following Özisik et al. (1968), the triple finite Fourier integral transform with respect to the space variables x, y and z of temperature function $T(x,y,z,t)$ in the finite range $0 \leq x \leq a,\ 0 \leq y \leq b,\ 0 \leq z \leq c$ is given by

$$\overline{\overline{\overline{T}}}(\xi_m, \chi_n, \eta_p, t) = \int_{x'=0}^{a} \int_{y'=0}^{b} \int_{z'=0}^{c} K(\xi_m, x')\ K(\chi_n, y')\ K(\eta_p, z')\ T(x', y', z', t)\ dx'\ dy'\ dz'. \tag{23}$$

The corresponding triple finite Fourier inversion formula with respect to the transformed space variables ξ, χ, η of temperature function $\overline{\overline{\overline{T}}}(\xi, \chi, \eta, t)$ is given by

$$T(x,y,z,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} K(\xi_m, x)\ K(\chi_n, y)\ K(\eta_p, z)\ \overline{\overline{\overline{T}}}(\xi_m, \chi_n, \eta_p, t), \tag{24}$$

where summation is taken over all eigenvalues $\xi_m,\ \chi_n,\ \eta_p$.

In order to find the Kernel $K(\xi_m, x)$, consider the eigenvalue problem

$$\frac{d^2 X(x)}{dx^2} + \xi^2\ X(x) = 0 \qquad \text{in } 0 \leq x \leq a, \tag{25}$$

$$-k_1\ \frac{dX(x)}{dx} + h_1\ X(x) = 0 \quad \text{at } x = 0, \tag{26}$$

$$k_2\ \frac{dX(x)}{dx} + h_2\ X(x) = 0 \quad \text{at } x = a. \tag{27}$$

This system is a special case of the Sturm-Liouville system.

The eigenfunctions $X(\xi_m, x)$ constitute an orthogonal set in the interval $0 \leq x \leq a$ with respect to a weighting function $w(x) = 1$.

Define kernel $K(\xi_m, x)$ to denote the normalized eigenfunctions as

$$K(\xi_m, x) = \frac{X(\xi_m, x)}{\sqrt{N}}, \tag{28}$$

where the eigenfunction $X(\xi_m, x)$ is a particular solution of eigenvalue problem $(25-27)$ given by

$$X(\xi_m, x) = \cos \xi_m x + \frac{H_1}{\xi_m}\ \sin \xi_m x. \tag{29}$$

The normality constant N is derived by using orthogonality property of eigenfunctions of the Sturm-Liouville problem and boundary conditions

$$N = \int_0^a X^2(\xi_m, x')\ dx' = \frac{1}{2}\left[\frac{\xi_m^2 + H_1^2}{\xi_m^2}\left(a + \frac{H_2}{\xi_m^2 + H_2^2}\right) + \frac{H_1}{\xi_m^2}\right], \tag{30}$$

and ξ_m are the non-zero positive roots of the transcendental equation

$$\tan \xi a = \frac{\xi\,(H_1 + H_2)}{\xi^2 - H_1 H_2}. \tag{31}$$

Hence the kernels for finite triple Fourier transform are summarized as

$$K(\xi_m, x) = \frac{\xi_m \cos(\xi_m x) + H_1 \sin(\xi_m x)}{N_1}, \tag{32}$$

$$K(\chi_n, y) = \frac{\chi_n \cos(\chi_n y) + H_3 \sin(\chi_n y)}{N_2}, \tag{33}$$

$$K(\eta_p, z) = \frac{\eta_p \cos(\eta_p z) + H_5 \sin(\eta_p z)}{N_3}, \tag{34}$$

and N_1, N_2 and N_3 are normality constants which are determined by the orthogonal properties of eigenfunctions as

$$N_1 = \frac{\sqrt{2}}{\left[(\xi_m^2 + H_1^2)\left(a + \dfrac{H_2}{\xi_m^2 + H_2^2}\right) + H_1\right]^{1/2}}, \tag{35}$$

$$N_2 = \frac{\sqrt{2}}{\left[(\chi_n^2 + H_3^2)\left(b + \dfrac{H_4}{\chi_n^2 + H_4^2}\right) + H_3\right]^{1/2}}, \tag{36}$$

$$N_3 = \frac{\sqrt{2}}{\left[(\eta_p^2 + H_5^2)\left(c + \dfrac{H_6}{\eta_p^2 + H_6^2}\right) + H_5\right]^{1/2}}. \tag{37}$$

Eigenvalues ξ_m, χ_n, η_p are non-zero positive roots of transcendental equations

$$\tan(\xi a) = \frac{\xi\,(H_1 + H_2)}{\xi^2 - H_1 H_2}, \tag{38}$$

$$\tan(\chi b) = \frac{\chi\,(H_3 + H_4)}{\chi^2 - H_3 H_4}, \tag{39}$$

$$\tan(\eta c) = \frac{\eta\,(H_5 + H_6)}{\eta^2 - H_5 H_6}, \tag{40}$$

respectively.

3.2. The Application of Finite Fourier Transform

Applying triple Fourier integral transforms and corresponding inversion formula to the initial and boundary value problem $(1-6)$, one obtain the final solution for temperature function as

$$\begin{aligned} T(x,y,z,t) = & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} e^{-\alpha(\xi_m^2+\chi_n^2+\eta_p^2)t}\ K(\xi_m,x)\ K(\chi_n,y)\ K(\eta_p,z) \\ & \times \left[\overline{\overline{\overline{F}}}(\xi_m,\chi_n,\eta_p) + \int_{t'=0}^{t} e^{\alpha(\xi_m^2+\chi_n^2+\eta_p^2)t'}\ A(\xi_m,\chi_n,\eta_p,t')\ dt'\right], \end{aligned} \tag{41}$$

$$\begin{aligned} \text{where } A(\xi_m,\chi_n,\eta_p,t) = & \frac{\alpha}{k}\ \overline{\overline{\overline{g}}}(\xi_m,\chi_n,\eta_p,t) \\ & + \frac{\alpha\ K(\xi_m,x)}{k_1}\Big|_{x=0} \int_{z=0}^{c}\int_{y=0}^{b} K(\eta_p,z)\ K(\chi_n,y)\ f_1(y,z,t)\ dy\ dz \\ & + \frac{\alpha\ \ K(\xi_m,x)}{k_2}\Big|_{x=a} \int_{z=0}^{c}\int_{y=0}^{b} K(\chi_n,y)\ K(\eta_p,z)\ f_2(y,z,t)\ dy\ dz \\ & + \frac{\alpha\ \ K(\chi_n,y)}{k_3}\Big|_{y=0} \int_{x=0}^{a}\int_{z=0}^{c} K(\xi_m,x)\ K(\eta_p,z)\ f_3(x,z,t)\ dx\ dz \\ & + \frac{\alpha\ \ K(\chi_n,y)}{k_4}\Big|_{y=b} \int_{x=0}^{a}\int_{z=0}^{c} K(\xi_m,x)\ K(\eta_p,z)\ f_4(x,z,t)\ dx\ dz \\ & + \frac{\alpha\ \ K(\eta_p,z)}{k_5}\Big|_{z=0} \int_{x=0}^{a}\int_{y=0}^{b} K(\xi_m,x)\ K(\chi_n,y)\ f_5(x,y,t)\ dx\ dy \\ & + \frac{\alpha\ \ K(\eta_p,z)}{k_6}\Big|_{z=c} \int_{x=0}^{a}\int_{y=0}^{b} K(\xi_m,x)\ K(\chi_n,y)\ f_6(x,y,t)\ dx\ dy, \end{aligned} \tag{42}$$

$$\overline{\overline{\overline{g}}}(\xi_m,\chi_n,\eta_p,t) = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} K(\xi_m,x)\ K(\chi_n,y)\ K(\eta_p,z)\ g(x,y,z,t)\ dx\ dy\ dz, \tag{43}$$

$$\overline{\overline{\overline{F}}}(\xi_m,\chi_n,\eta_p) = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} K(\xi_m,x)\ K(\chi_n,y)\ K(\eta_p,z)\ F(x,y,z)\ dx\ dy\ dz. \tag{44}$$

4. The Convergence Analysis

Following Wang et al. (2008), the well-posedness of the boundary value problem has been studied by discussing existence, uniqueness, and stability of the solution.

4.1. Existence and Convergence

Consider the transcendental equation for eigenvalues ξ_m

$$\tan(\xi a) = \frac{\xi\ (H_1+H_2)}{\xi^2 - H_1 H_2}. \tag{45}$$

Assuming $H_1 = H_2$, this equation can be written in the form

$$Z = \cot x = \frac{1}{2}\left(\frac{x}{B} - \frac{B}{x}\right), \tag{46}$$

where $B = \dfrac{ha}{k} = Ha$ (Biot number) and $x = \xi a$.

The intersection of these two curves corresponds to $x_m = \xi_m a$ where $\xi_m \approx m\pi$ for $m \in \mathbb{Z}^+$.

Consider

$$K(\xi_m, x) = \sqrt{2}\left[\frac{\xi_m \cos(\xi_m x) + H_1 \sin(\xi_m x)}{\left[(\xi_m^2 + H_1^2)\left(a + \dfrac{H_2}{\xi_m^2 + H_2^2}\right) + H_1\right]^{1/2}}\right]. \tag{47}$$

As ξ_m is a non-zero root, dividing numerator and denominator by ξ_m, then

$$K(\xi_m, x) = \sqrt{2}\left[\frac{\cos\xi_m x + \dfrac{H_1}{\xi_m}\ \sin\xi_m x}{\left[\left(1 + \dfrac{{H_1}^2}{{\xi_m}^2}\right)\left(a + \dfrac{H_2}{{\xi_m}^2 + {H_2}^2}\right) + \dfrac{H_1}{{\xi_m}^2}\right]^{1/2}}\right]. \tag{48}$$

Using $\xi_m \longrightarrow \infty$ as $m \longrightarrow \infty \Longrightarrow \|K(\xi_m, x)\| \leq C_1$, for some constant C_1.

Similarly one can find constants $C_2,\ C_3$ such that $\|K(\chi_n, y)\| \leq C_2$ and $\|K(\eta_p, z)\| \leq C_3$.

Let $T_{mnp}(x, y, z, t)$ be general term of solution (41) of boundary value problem as

$$\begin{aligned} T_{mnp}(x, y, z, t) = \Big[& e^{-\alpha(\xi_m^2+\chi_n^2+\eta_p^2)t}\ K(\xi_m, x)\ K(\chi_n, y)\ K(\eta_p, z) \\ & \times \left(\overline{\overline{\overline{F}}}(\xi_m, \chi_n, \eta_p) + \int_{t'=0}^{t} e^{\alpha(\xi_m^2+\chi_n^2+\eta_p^2)t'}\ A(\xi_m, \chi_n, \eta_p, t')\ dt'\right)\Big], \end{aligned} \tag{49}$$

$$\|T_{mnp}(x, y, z, t)\| \ \leq \frac{1}{e^{\alpha\xi_m^2 t}}\ \frac{1}{e^{\alpha\chi_n^2 t}}\ \frac{1}{e^{\alpha\eta_p^2 t}}\ C_1\ C_2\ C_3\ C_4, \tag{50}$$

where C_4 is a constant satisfying

$$\left\|\overline{\overline{\overline{F}}}(\xi_m, \chi_n, \eta_p) + \int_{t'=0}^{t} e^{\alpha(\xi_m^2+\chi_n^2+\eta_p^2)t'}\ A(\xi_m, \chi_n, \eta_p, t')\ dt'\right\| \leq C_4. \tag{51}$$

For any fixed natural number N, $\lim\limits_{k\to\infty} \dfrac{k^N}{e^{k^2}} = 0.$

Therefore for every $\epsilon > 0 \ \exists \ n_0 \in \mathbb{N}$ such that $\left\| \dfrac{k^2}{e^{k^2}} \right\| < \epsilon, \quad \forall \ k \geq n_0$, i.e. for sufficiently large k, $\dfrac{1}{e^{k^2}} < \dfrac{\epsilon}{k^2}$.

From Equation (50), one can conclude that for every $\epsilon > 0 \ \exists \ N_0 \in \mathbb{N}$ such that

$$\|T_{mnp}(x, y, z, t)\| \leq \frac{\epsilon^3 M}{\xi_m^2 \ \chi_n^2 \ \eta_p^2}, \quad \text{for all } m, n, p \geq N_0, \tag{52}$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \|T_{mnp}(x, y, z, t)\| \ \leq \ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \frac{\epsilon^3 M}{\xi_m^2 \ \chi_n^2 \ \eta_p^2} = \epsilon^3 M \ \sum_{m=1}^{\infty} \frac{1}{\xi_m^2} \ \sum_{n=1}^{\infty} \frac{1}{\chi_n^2} \ \sum_{p=1}^{\infty} \frac{1}{\eta_p^2}, \tag{53}$$

which is a convergent series.

$$\frac{\partial^2 T_{mnp}}{\partial x^2} + \frac{\partial^2 T_{mnp}}{\partial y^2} + \frac{\partial^2 T_{mnp}}{\partial z^2} = -(\xi_m^2 + \chi_n^2 + \eta_p^2) \ K(\xi_m, x) \ K(\chi_n, y) \ K(\eta_p, z)$$

$$\times \left[\overline{\overline{\overline{F}}}(\xi_m, \chi_n, \eta_p) + \int_{t'=0}^{t} e^{\alpha(\xi_m^2 + \chi_n^2 + \eta_p^2)t'} \ A(\xi_m, \chi_n, \eta_p, t') \ dt' \right] \ e^{-\alpha(\xi_m^2 + \chi_n^2 + \eta_p^2)t}, \tag{54}$$

$$\frac{\partial T_{mnp}}{\partial t} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} \ K(\xi_m, x) \ K(\chi_n, y) \ K(\eta_p, z) \ \left(-\alpha \ (\xi_m^2 + \chi_n^2 + \eta_p^2) \right) \ e^{-\alpha(\xi_m^2 + \chi_n^2 + \eta_p^2)t}$$

$$\times \ A(\xi_m, \chi_n, \eta_p, t) + \left[\overline{\overline{\overline{F}}}(\xi_m, \chi_n, \eta_p) + \int_{t'=0}^{t} e^{\alpha(\xi_m^2 + \chi_n^2 + \eta_p^2)t'} \ A(\xi_m, \chi_n, \eta_p, t') \ dt' \right]. \tag{55}$$

Then by a similar argument as above, one obtains

$$\left\| \frac{\partial^2 T_{mnp}}{\partial x^2} + \frac{\partial^2 T_{mnp}}{\partial y^2} + \frac{\partial^2 T_{mnp}}{\partial z^2} \right\| \ \leq \ \frac{\epsilon^3 \ M}{\xi_m^2 \ \chi_n^2 \ \eta_p^2}, \tag{56}$$

$$\left\| \frac{\partial T_{mnp}}{\partial t} \right\| \ \leq \ \frac{\epsilon^3 \ M}{\xi_m^2 \ \chi_n^2 \ \eta_p^2}. \tag{57}$$

Therefore the series in (41) is uniformly convergent, so derivatives of this series can be taken term by term. Thus the solution (41) satisfies the boundary value problem (1) to (6).

4.2. Uniqueness and Stability

The solution of boundary value problem in terms of a partial differential equation is in the form of Fourier series expansion. So uniqueness of the solution follows from the uniqueness of Fourier series expansion in terms of trigonometric polynomials. Also suppose that whenever there is some small variation in convective heat flux and initial temperature, the corresponding temperature changes say T_1 and T_2, one must have $\|T_1(x,y,z,t) - T_2(x,y,z,t)\| < \epsilon$. Therefore the solution is stable.

Thus one may conclude that the generalized problem of heat conduction in the Cartesian domain in the form of boundary value problem is well-posed.

5. Thermal Bending Analysis

The induced resultant moment is given by equation (8) as

$$M_T = \alpha E \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} D(x,y,\xi_m,\chi_n,\eta_p,t). \tag{58}$$

The deflection in thin rectangular plate due to convective heat flux is given by the solution of the equation (7) as

$$w = \frac{\alpha\, E}{(1-\nu)\, D} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{1}{(\xi_m^2 + \chi_n^2)}\, D(x,y,\xi_m,\chi_n,\eta_p,t). \tag{59}$$

The corresponding resultant moments follows from equations (9 - 11) as

$$M_x = \frac{\alpha\, E}{1-\nu} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \left(\frac{\xi_m^2 + \nu\, \chi_n^2}{\xi_m^2 + \chi_n^2} - 1\right) D(x,y,\xi_m,\chi_n,\eta_p,t), \tag{60}$$

$$M_y = \frac{\alpha\, E}{1-\nu} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \left(\frac{\chi_n^2 + \nu\, \xi_m^2}{\xi_m^2 + \chi_n^2} - 1\right) D(x,y,\xi_m,\chi_n,\eta_p,t), \tag{61}$$

$$\begin{aligned} M_{xy} = \alpha\, E \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{N_1\, N_2\, N_3}{\xi_m^2 + \chi_n^2}\, e^{-\alpha(\xi_m^2+\chi_n^2+\eta_p^2)t} \left[-\xi_m^2 \sin(\xi_m x) + H_1 \xi_m \cos(\xi_m x)\right] \\ \times \left[-\chi_n^2 \sin(\chi_n y) + H_3 \chi_n \cos(\chi_n y)\right] \left[\int_0^t e^{\alpha(\xi_m^2+\chi_n^2+\eta_p^2)t'}\, A(\xi_m,\chi_n,\eta_p,t')\, dt'\right] \\ \times \left[\left(c + \frac{H_5}{\eta_p^2}\right) \sin(\eta_p c) + \left(\frac{1 - H_5 c}{\eta_p}\right) \cos(\eta_p c) - \frac{1}{\eta_p}\right]. \end{aligned} \tag{62}$$

The induced resultant force is given by equation (13),

$$N_T = \alpha E \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} N_3 \left[\sin(\eta_p c) + \frac{H_5}{\eta_p}\,(1 - \cos(\eta_p c))\right] B(x,y,\xi_m,\chi_n,\eta_p,t). \tag{63}$$

On solving the governing differential equation (12) for thermal stress function, one obtains

$$F = \alpha E \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{1}{\xi_m^2+\chi_n^2}\, C(x,y,\xi_m,\chi_n,\eta_p,t), \tag{64}$$

and corresponding resultant forces follows from definitions (14 - 16) as

$$N_x = \alpha E \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{-\chi_n^2}{\xi_m^2+\chi_n^2}\, C(x,y,\xi_m,\chi_n,\eta_p,t), \tag{65}$$

$$N_y = \alpha E \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{-\xi_m^2}{\xi_m^2+\chi_n^2}\, C(x,y,\xi_m,\chi_n,\eta_p,t), \tag{66}$$

$$\begin{aligned} N_{xy} =& -\alpha E \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{1}{\xi_m^2+\chi_n^2}\, e^{-\alpha(\xi_m^2+\chi_n^2+\eta_p^2)t} \left[N_1 \left(H_1\xi_m\cos(\xi_m x)-\xi_m^2\sin(\xi_m x)\right)\right] \\ & \times \left[N_2 \left(H_3\chi_n\cos(\chi_n y)-\chi_n^2\sin(\chi_n y)\right)\right] . \left[N_3 \left(\sin(\eta_p c)+\frac{H_5}{\eta_p}\left(1-\cos(\eta_p c)\right)\right)\right] \\ & \times \int_0^t e^{\alpha(\xi_m^2+\chi_n^2+\eta_p^2)t'} A(\xi_m,\chi_n,\eta_p,t')\, dt'. \end{aligned} \tag{67}$$

Substituting these values in Equations (17 - 19), one obtain thermal stress components as

$$\begin{aligned} \sigma_{xx} =& \left[\frac{\alpha\, E}{c}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{-\chi_n^2}{\xi_m^2+\chi_n^2}\, C(x,y,\xi_m,\chi_n,\eta_p,t)\right] \\ & + \left[\frac{12\, z}{c^3}\,\frac{\alpha\, E}{1-\nu}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \left(\frac{\xi_m^2+\nu\,\chi_n^2}{\xi_m^2+\chi_n^2}-1\right) D(x,y,\xi_m,\chi_n,\eta_p,t)\right] \\ & + \left[\frac{1}{1-\nu}\frac{\alpha\, E}{c}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} C(x,y,\xi_m,\chi_n,\eta_p,t)\right] \\ & + \left[\frac{1}{1-\nu}\frac{12\, z}{c^3}\,\alpha\, E\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} D(x,y,\xi_m,\chi_n,\eta_p,t)\right] \\ & - \left[\frac{1}{1-\nu}\,\alpha\, E\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} k(\eta_p,z)\, B(x,y,\xi_m,\chi_n,\eta_p,t)\right], \end{aligned} \tag{68}$$

$$\sigma_{yy} = \left[\frac{\alpha\ E}{c}\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\ \frac{-\xi_m^2}{\xi_m^2+\chi_n^2}\ C(x,y,\xi_m,\chi_n,\eta_p,t)\right]$$

$$+\left[\frac{12\ z}{c^3}\ \frac{\alpha\ E}{1-\nu}\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\ \left(\frac{\chi_n^2+\nu\ \xi_m^2}{\xi_m^2+\chi_n^2}-1\right)\ D(x,y,\xi_m,\chi_n,\eta_p,t)\right]$$

$$+\left[\frac{1}{1-\nu}\ \frac{\alpha\ E}{c}\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\ C(x,y,\xi_m,\chi_n,\eta_p,t)\right]$$

$$+\left[\frac{1}{1-\nu}\ \frac{12\ z}{c^3}\ \alpha\ E\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\ D(x,y,\xi_m,\chi_n,\eta_p,t)\right]$$

$$-\left[\frac{1}{1-\nu}\ \alpha\ E\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\ k(\eta_p,z)\ B(x,y,\xi_m,\chi_n,\eta_p,t)\right], \tag{69}$$

$$\sigma_{xy} = \left[\frac{-\alpha\ E}{c}\ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\ \frac{1}{\xi_m^2+\chi_n^2}\ \left[N_1\ \left(H_1\ \xi_m\ \cos(\xi_m\ x)-\xi_m^2\ \sin(\xi_m\ x)\right)\right]\right.$$

$$\times\left[N_2\ \left(H_3\ \chi_n\ \cos(\chi_n\ y)-\chi_n^2\ \sin(\chi_n\ y)\right)\right]\ \left[e^{-\alpha\ (\xi_m^2+\chi_n^2+\eta_p^2)\ t}\right]$$

$$\left.\times\left[\int_0^t e^{\ \alpha\ (\xi_m^2+\chi_n^2+\eta_p^2)\ t'}\ A(\xi_m,\chi_n,\eta_p,t')\ dt'\right]\ \left[N_3\ \left(\sin(\eta_p\ c)+\frac{H_5}{\eta_p}(1-\cos(\eta_p\ c))\right)\right]\right]$$

$$-\left[\frac{12\ z\ \alpha\ E}{c^3}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\ \frac{1}{\xi_m^2+\chi_n^2}\ N_1\ N_2\ N_3\ e^{-\alpha\ (\xi_m^2+\chi_n^2+\eta_p^2)\ t}\right.$$

$$\times\left[-\xi_m^2\ \sin(\xi_m\ x)+H_1\ \xi_m\ \cos(\xi_m\ x)\right]\left[-\chi_n^2\ \sin(\chi_n\ y)+H_3\ \chi_n\ \cos(\chi_n\ y)\right]$$

$$\times\left[\left(c+\frac{H_5}{\eta_p^2}\right)\ \sin(\eta_p\ c)+\left(\frac{1-H_5\ c}{\eta_p}\right)\cos(\eta_p\ c)-\frac{1}{\eta_p}\right]$$

$$\left.\times\int_0^t e^{\ \alpha(\xi_m^2+\chi_n^2+\eta_p^2)\ t'}\ A(\xi_m,\chi_n,\eta_p,t')\ dt'\right]. \tag{70}$$

Unknown Coefficients

$$
\begin{aligned}
A(x,y,\xi_m,\chi_n,\eta_p,t) = {} & \frac{\alpha}{k}\ \overline{\overline{\overline{g}}}(\xi_m,\chi_n,\eta_p,t) \\
& + \left.\frac{\alpha\ K(\xi_m,x)}{k_1}\right|_{x=0} \int_{z=0}^{c}\int_{y=0}^{b} K(\eta_p,z)\ K(\chi_n,y)\ f_1(y,z,t)\ dy\ dz \\
& + \left.\frac{\alpha\ K(\xi_m,x)}{k_2}\right|_{x=a} \int_{z=0}^{c}\int_{y=0}^{b} K(\chi_n,y)\ K(\eta_p,z)\ f_2(y,z,t)\ dy\ dz \\
& + \left.\frac{\alpha\ K(\chi_n,y)}{k_3}\right|_{y=0} \int_{x=0}^{a}\int_{z=0}^{c} K(\xi_m,x)\ K(\eta_p,z)\ f_3(x,z,t)\ dx\ dz \\
& + \left.\frac{\alpha\ K(\chi_n,y)}{k_4}\right|_{y=b} \int_{x=0}^{a}\int_{z=0}^{c} K(\xi_m,x)\ K(\eta_p,z)\ f_4(x,z,t)\ dx\ dz \\
& + \left.\frac{\alpha\ K(\eta_p,z)}{k_5}\right|_{z=0} \int_{x=0}^{a}\int_{y=0}^{b} K(\xi_m,x)\ K(\chi_n,y)\ f_5(x,y,t)\ dx\ dy \\
& + \left.\frac{\alpha\ K(\eta_p,z)}{k_6}\right|_{z=c} \int_{x=0}^{a}\int_{y=0}^{b} K(\xi_m,x)\ K(\chi_n,y)\ f_6(x,y,t)\ dx\ dy,
\end{aligned}
$$

$$
B(x,y,\xi_m,\chi_n,\eta_p,t) = \left[\overline{\overline{\overline{F}}}(\xi_m,\chi_n,\eta_p) + \int_{t'=0}^{t} e^{\,\alpha\ (\xi_m^2+\chi_n^2+\eta_p^2)\ t'}\ A(\xi_m,\chi_n,\eta_p,t')\ dt'\right]
$$

$$
\times\ e^{-\alpha\ (\xi_m^2+\chi_n^2+\eta_p^2)\ t}\ K(\xi_m,x)\ K(\chi_n,y)\ ,
$$

$$
C(x,y,\xi_m,\chi_n,\eta_p,t) = N_3\ \left[\sin(\eta_p\ c) + \frac{H_5}{\eta_p}\ (1-\cos(\eta_p\ c)\right]\ B(x,y,\xi_m,\chi_n,\eta_p,t),
$$

$$
D(x,y,\xi_m,\chi_n,\eta_p,t)
$$

$$
= \left[\left(c+\frac{H_5}{\eta_p^2}\right)\ \sin(\eta_p\ c) + \left(\frac{1}{\eta_p}-\frac{H_5\ c}{\eta_p}\right)\cos(\eta_p\ c) - \frac{1}{\eta_p}\right]\ N_3\ B(x,y,\xi_m,\chi_n,\eta_p,t).
$$

Illustrative Example

Consider a thin rectangular iron plate initially at zero temperature ($i.e.$ $F(x,y,z)=0$). For time $t>0$, the internal heat source is an instantaneous line heat source $g(x,y,z,t)$ of constant strength

g_0. All boundary surfaces are kept insulated ($i.e.\ h_i = 0,\ k_i = 1,\ f_i = 0,$ for $i = 1, 2, \cdots 6$). The internal heat source is defined as

$$g(x, y, z, t) = g_0\ e^{\omega t} \left[\sum_{i=1}^{4} g_i(x_i, y, 0.01)\right],$$

where $g_0 = 323.15\ K$ is the strength of internal heat source and $\omega = 3.9,\ 0.03 \leq y \leq 0.09,$

$$g_i(x, y, z) = \delta(x - x_i),\ x_1 = 0.03,\ x_2 = 0.05,\ x_3 = 0.07,\ x_4 = 0.09.$$

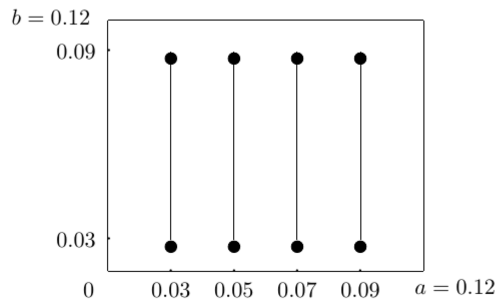

Figure 1. Partial line heat source

Dimensions

The length $a = 0.12m$, breadth $b = 0.12m$ and thickness $c = 0.02m$.

Material Properties

The material properties of cast iron are as follows

$\alpha = 20.34 \times 10^{-6}\ m^2/s,\ \ k = 72.7\ W/mK,\ \ \nu = 0.35.$

Roots of Transcendental Equations

The first ten positive roots of transcendental equation $\tan(0.12\xi) = 0$ are
$\xi_m = [26.18,\ 52.36,\ 78.54,\ 104.72,\ 130.9,\ 157.08,\ 183.26,\ 209.44,\ 235.62,\ 261.8]$.

The first ten positive roots of transcendental equation $\tan(0.12\chi) = 0$ are
$\chi_n = [26.18,\ 52.36,\ 78.54,\ 104.72,\ 130.9,\ 157.08,\ 183.26,\ 209.44,\ 235.62,\ 261.8]$.

The first ten positive roots of transcendental equation $\tan(0.02\eta) = 0$ are
$\eta_p = [157.08,\ 314.16,\ 471.24,\ 628.32,\ 785.4,\ 942.48,\ 1099.56,\ 1256.64,\ 1413.72,\ 1570.8]$.

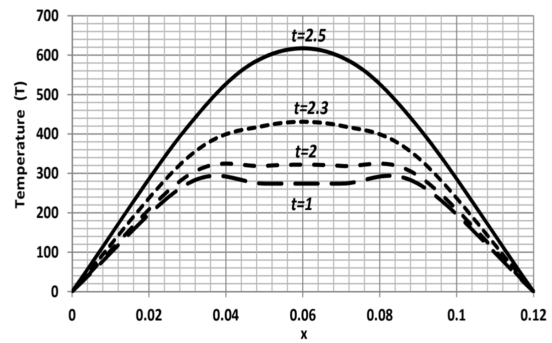

Figure [1] : Temperature Distribution (T) along x -axis

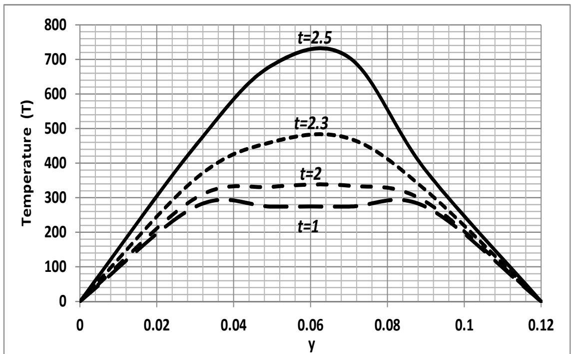

Figure [2] : Temperature Distribution (T) along y -axis

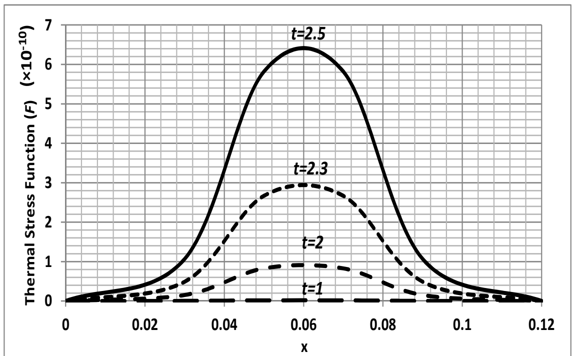

Figure [3] : Thermal Stress Function (F) along x -axis

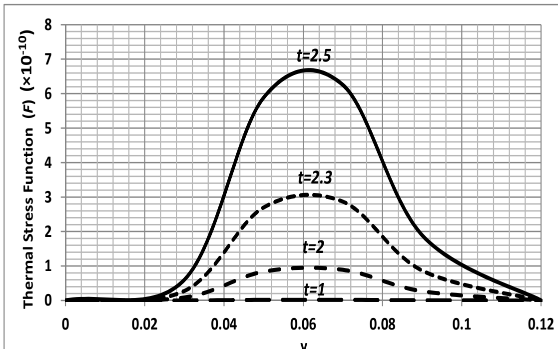

Figure [4] : Thermal Stress Function (F) along y -axis

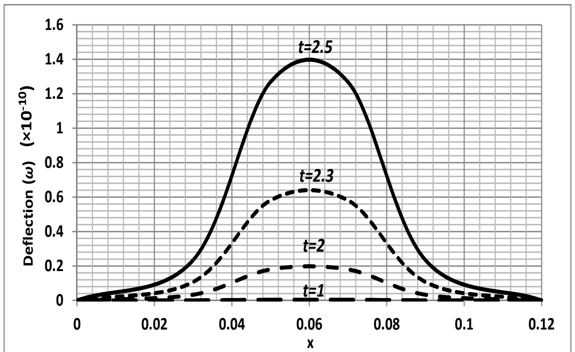

Figure [5] : Deflection (ω) along x -axis

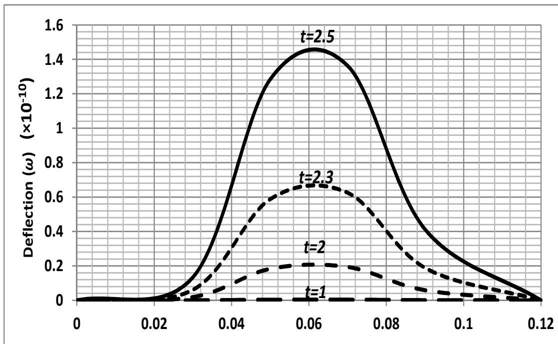

Figure [6] : Deflection (ω) along y-axis

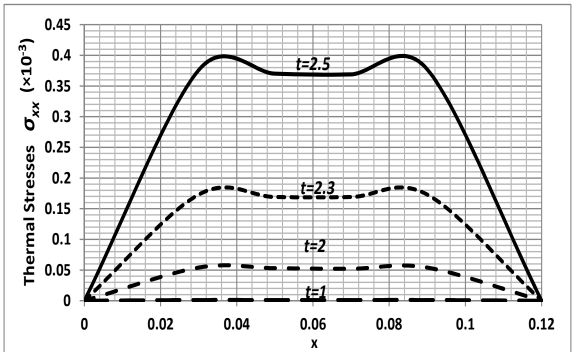

Figure [7] : Thermal Stresses σ_{xx} along x -axis

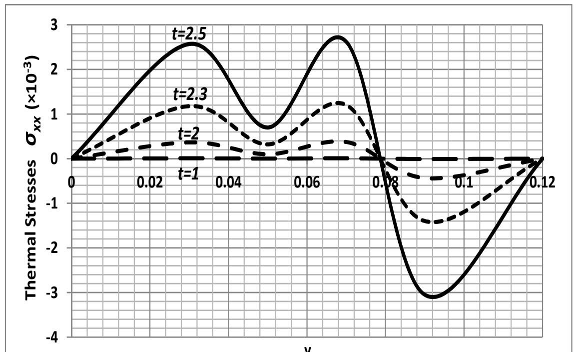

Figure [8] : Thermal Stresses σ_{xx} along y -axis

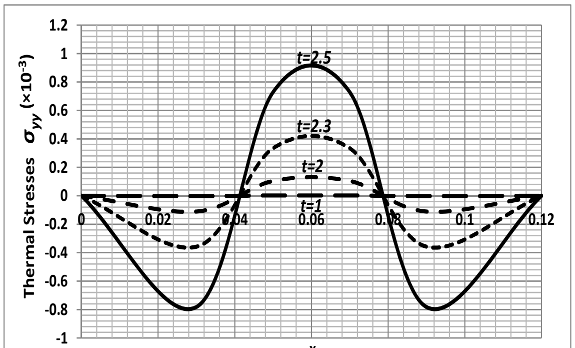

Figure [9] : Thermal Stresses σ_{yy} along x -axis

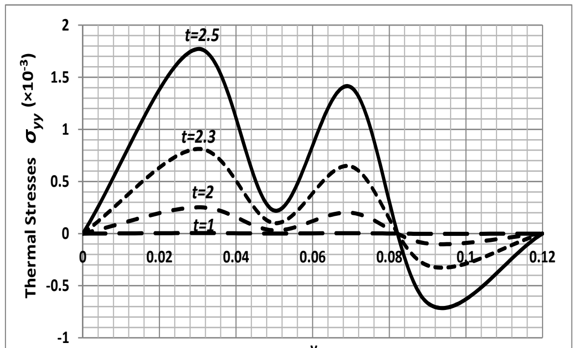

Figure [10] : Thermal Stresses σ_{yy} along y-axis

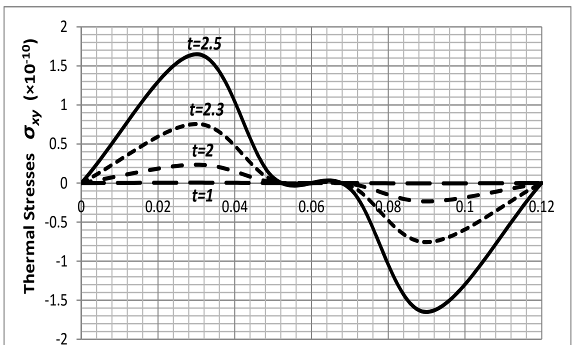

Figure [11] : Thermal Stresses σ_{xy} along x -axis

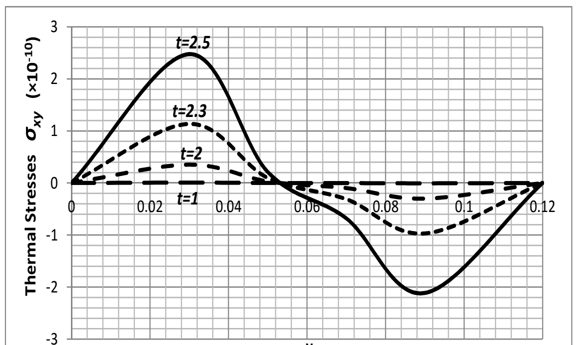

Figure [12] : Thermal Stresses σ_{xy} along y -axis

6. Conclusion

The mathematical formulation and corresponding analytical solution of most generalized thermal bending problem in the Cartesian domain have been discussed in this manuscript. The problem has been formulated in the context of the non-homogeneous transient heat equation with internal heat generation subjected to Robin's boundary conditions. The general analytical solution of heat equation has been obtained by transforming the original initial and boundary value problem into eigenvalue problem through the Strum-Liouville theory. The finite Fourier transform has been applied with respect to space variables by choosing suitable normalized kernels. The general solution of the generalized thermoelastic problem has been discussed for temperature change, displacements, thermal stresses, deflection, and deformation. In order to have a well-posed problem, the mathematical analysis for existence, uniqueness, and stability of series solution also has been carried out. The convergence of infinite series solutions also been discussed.

The most important feature of this work is any special case of practical interest may be readily obtained by this most generalized mathematical formulation and its analytical solution. There are 729 such combinations of possible boundary conditions prescribed on parallelepiped shaped region in the Cartesian coordinate system. The analytical and numerical results obtained in this manuscript for all thermal parameters are validated by the equilibrium and compatibility equations of the classical thermoelasticity theory.

As a special case, an isotropic thin rectangular iron plate subjected to instantaneous line heat source has been considered and discussed its thermal variation due to change in temperature. The internal heat generation is an instantaneous line heat source situated concentrically inside the rectangular plate. The Dirac delta function has been used to formulate the internal heat source. The sinusoidal patterns of temperature distribution can be seen along both x and y axis. The temperature attains maximum value at the middle of the plate and decreases towards the boundaries. Due to insulated boundaries, no dissipation due to convection of heat is observed. The thermal stress function and deflection are seen in the proportion and increases with respect to time. The thermal stress function and deflection both are maximum in the middle and tends to zero towards the boundaries. Due to insulated built-in boundaries, the mixed development of compressive and tensile thermal stresses can be observed. All the thermal stresses are zero at the traction free boundaries. This results in small deflection and deformation in the middle of the plate. Due to a negligible amount of deflection, there will not be any considerable creep or shrinkage in solid.

The results obtained in this manuscript are applicable for thermal analysis in the designing of material and structures. The similar most generalized thermal bending problem in cylindrical and spherical coordinate system may be attempted for mathematical formulation and corresponding analytical solution. Moreover, generalized thermal bending problem may be formulated in terms of temperature-dependent thermal parameters like thermal conductivity, thermal diffusivity, specific heat capacity, density. Also, the model presented in this manuscript can be extended to fractional thermoelasticity by using Caputo time fractional derivative of order $\alpha \in (0, 1]$ which is applicable for microstructural thermal analysis of nanomaterials.

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