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Kink, singular soliton and periodic solutions to class of nonlinear equations

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Abstract

In this paper, we extend the ordinary differential Duffing equation into a partial differential equation. We study the traveling wave solutions to this model by using the G'/G expansion method. Then, based on the obtained results given for the Duffing equation, we generate kink, singular soliton and periodic solutions for a coupled integrable dispersionless nonlinear system. All the solutions given in this work are verified.

Keywords: Duffing equation; Coupled dispersionless system; G'/G expansion method

MSC(2010): 74J35, 35G20

1. Introduction

Many physical phenomena are modeled by nonlinear systems of partial differential equations (PDEs). An important problem in the study of nonlinear systems is to find exact solutions and explicitly describe traveling wave behaviors. Motivated by potential applications in physics, engineering, biology and communication theory, the damped Duffing equation

$$x''(t)+\alpha x'(t)+\beta x(t)+\gamma x^3(t)=0, \tag{1.1}$$

has received much interest. In the above, α is the coefficient of viscous damping and the term $\beta x(t)+\gamma x^3(t)$ represents the nonlinear restoring force, acting like a hard spring, and the prime denotes differentiation with respect to time. The Duffing equation is a typical model arising in many areas of physics and engineering such as the study of oscillations of a rigid pendulum undergoing with moderately large amplitude motion [Jordan and Smith (1977)], vibrations of a buckled beam, and so on [Thompson and Stewart (1986), Pezeshki and Dowell (1987) and Moon (1987)]. Exact solutions of (1.1) were discussed by [Chen (2002)] using the target function method, but no explicit solutions were shown. In [Lawden (1989)], exact solutions were presented by using the elliptic function method for various special cases. Senthil and Lakshmanan (1995) dealt with equation (1.1) by using the Lie symmetry method and derived an exact solution from the properties of the symmetry vector fields. Finally, approximate solutions of (1.1) were investigated by Alquran and Al-khaled (2012) using the poincare method and differential transform method.

Many nonlinear PDEs can be converted into nonlinear ordinary differential equations (ODEs) after making traveling wave transformations. Seeking traveling wave solutions for those nonlinear systems is equivalent to finding exact solutions of their corresponding ODEs. Now, we extend the ODE given in (1.1) into the following $(1+1)-$dimensional PDE

$$u_{tt}+\alpha u_t+\beta u+\gamma u^3=0, \tag{1.2}$$

where α, β, γ are real physical constants and $u=u(x,t)$. The aim of this current work is to study the solution of the PDE given in (1.2) with $\alpha=0$ [Qawasmeh (2013)] by implementing the G'/G-expansion method [Alquran and Qawasmeh (2014) and Qawasmeh and Alquran (2014 a,b)]. Then, we will use the obtained results to retrieve solutions to another interesting model called the coupled integrable disperssionless system.

2. Construction and analysis of G'/G method

Consider the following nonlinear partial differential equation PDE:

$$P(u,u_t,u_x,u_{tt},u_{xt},...)=0, \tag{2.1}$$

where $u=u(x,t)$ is an unknown function, P is a polynomial in $u=u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. By the wave variable $\zeta=x-ct$ the PDE (2.1) is then transformed to the ODE

$$P(u,-cu',u',c^2u'',-cu'',u'',...) = 0, \tag{2.2}$$

where $u = u(\zeta)$. Suppose that the solution of (2.2) can be expressed by a polynomial in G'/G as follows

$$u(\zeta) = a_m\left(\frac{G'}{G}\right)^m + \cdots + a_1\left(\frac{G'}{G}\right) + a_0, \tag{2.3}$$

where $G = G(\zeta)$ satisfies the second order differential equation in the form

$$G'' + \lambda G' + \mu G = 0, \tag{2.4}$$

$a_0, a_1, ..., a_m, \lambda$ and μ are constants to be determined later, provided that $a_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in the ODE (2.2).

Now, if we let

$$Y = Y(\zeta) = \frac{G'}{G}, \tag{2.5}$$

then by the help of (2.4) we get

$$Y' = \frac{GG'' - G'^2}{G^2} = \frac{G(-\lambda G' - \mu G) - G'^2}{G^2} = -\lambda Y - \mu - Y^2, \tag{2.6}$$

or, equivalently

$$Y' = -Y^2 - \lambda Y - \mu. \tag{2.7}$$

By result (2.7) and by implicit differentiation, one can derive the following two formulas

$$Y'' = 2Y^3 + 3\lambda Y^2 + (2\mu + \lambda^2)Y + \lambda\,\mu, \tag{2.8}$$

$$Y''' = -6Y^4 - 12\lambda Y^3 - (7\lambda^2 + 8\mu)Y^2 - (\lambda^3 + 8\lambda\mu)Y - (\lambda^2\mu + 2\mu^2). \tag{2.9}$$

Combining equations (2.3), (2.5) and (2.7)-(2.9), yieldspolynomial of powers of Y. Then, collecting all terms of the same order of Y and equating to zero, yields a set of algebraic equations for $a_0, a_1, ..., a_m, \lambda$, and μ.

It is known that the solution of equation (2.4) is a linear combination of sinh and cosh or of sine and cosine, respectively, if $\Delta = \lambda^2 - 4\mu > 0$ or $\Delta < 0$. Without lost of generality, we consider the first case and therefore

$$G(\zeta) = e^{-\frac{\lambda\zeta}{2}}\left(A\sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}\,\zeta}{2}\right) + B\cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}\,\zeta}{2}\right)\right), \tag{2.10}$$

where A and B are any real constants.

3. The extended Duffing equation

In this section we derive kink, singular soliton and periodic solutions of the following PDE

$$u_{tt} + \beta u + \gamma u^3 = 0, \tag{3.1}$$

where $u = u(x,t)$. By the wave variable $\zeta = x - ct$, the above PDE is transformed into the ODE

$$c^2 u'' + \beta u + \gamma u^3 = 0, \tag{3.2}$$

where $u = u(\zeta)$. Consider

$$u(\zeta) = a_m \left(\frac{G'}{G}\right)^m + ... + a_1 \left(\frac{G'}{G}\right) + a_0 \tag{3.3}$$

with $G = G(\zeta)$. Using the assumption given in (2.4) and the result obtained in (2.6) we have

$$u^3(\zeta) = a_m^3 \left(\frac{G'}{G}\right)^{3m} + ..., \tag{3.4}$$

and

$$u''(\zeta) = m(m+1)\, a_m \left(\frac{G'}{G}\right)^{m+2} + \cdots. \tag{3.5}$$

Balancing the nonlinear term u^3 in (3.4) with the linear term u'' in (3.5), requires that $3m = m + 2$. Thus, $m = 1$, and t (3.3) can be rewritten as

$$u(\zeta) = a_1 \left(\frac{G'}{G}\right) + a_0 = a_1 Y + a_0. \tag{3.6}$$

Differentiating the above function u twice yields

$$u''(\zeta) = a_1 Y''. \tag{3.7}$$

Now, we substitute equations (3.6), (3.7) and (2.8) in (3.2) to get the following algebraic system:

$$
\begin{aligned}
0 &= a_0\beta + a_0^3\gamma + a_1 c^2\lambda\mu, \\
0 &= a_1\beta + 3a_0^2 a_1\gamma + a_1 c^2\lambda^2 + 2a_1 c^2\mu, \\
0 &= 3a_0 a_1^2\gamma + 3a_1 c^2\lambda, \\
0 &= 2a_1 c^2 + a_1^3\gamma.
\end{aligned} \tag{3.8}
$$

Solving the above system produces two different solution sets involving the parameters λ, μ, a_0, a_1 and c. The first set is

$$\lambda = 0, \quad \mu = -\frac{\beta}{2c^2}, \quad a_1 = \pm\frac{i\sqrt{2}c}{\sqrt{\gamma}}, \quad a_0 = 0, \tag{3.9}$$

and the second set is

$$\lambda = \pm\frac{i\, a_0\sqrt{2\gamma}}{c}, \quad \mu = \frac{-\beta - a_0^2\gamma}{2c^2}, \quad a_1 = \mp\frac{i\sqrt{2}c}{\sqrt{\gamma}}. \tag{3.10}$$

Considering the first obtained set, the solution of (3.1) is

$$u(x,t) = \frac{i\sqrt{\beta}}{\sqrt{\gamma}}\,\frac{A + B\tanh\left(\sqrt{\frac{\beta}{2c^2}}(x-ct)\right)}{B + A\tanh\left(\sqrt{\frac{\beta}{2c^2}}(x-ct)\right)}, \tag{3.11}$$

where the parameters β, γ, c, A and B are free real constants. For example:

Case I:

If we choose $\beta = 1, \gamma = -1, c = 1, A = 0, B = 1$, then $u(x,t) = \tanh\left(\frac{x-t}{\sqrt{2}}\right)$ is a solution of equation (3.1) which is of thie kink type.

Case II:

If $\beta = 1, \gamma = -1, c = 1, A = 1, B = 0$, then

$$u(x,t) = \coth\left(\frac{x-t}{\sqrt{2}}\right),$$

which is singular soliton.

Case III:

If $\beta = -1,\ \gamma = -1,\ c = 1,\ A = 0, B = 1$, then

$$u(x,t) = -\tan\left(\frac{x-t}{\sqrt{2}}\right)$$

is a periodic solution that (3.1) possess and by swapping the values of A and B in this case another periodic solution

$$u(x,t) = \cot\left(\frac{x-t}{\sqrt{2}}\right)$$

does the (3.1) have. See Figures 1 and 2.

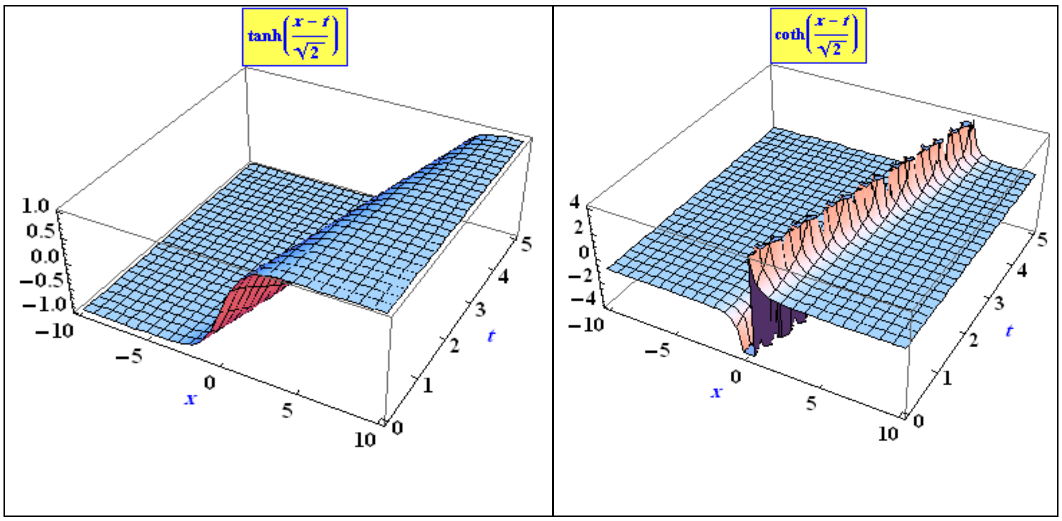

Figure 1. Plots of solutions for (3.1) obtained in Case I and II respectively

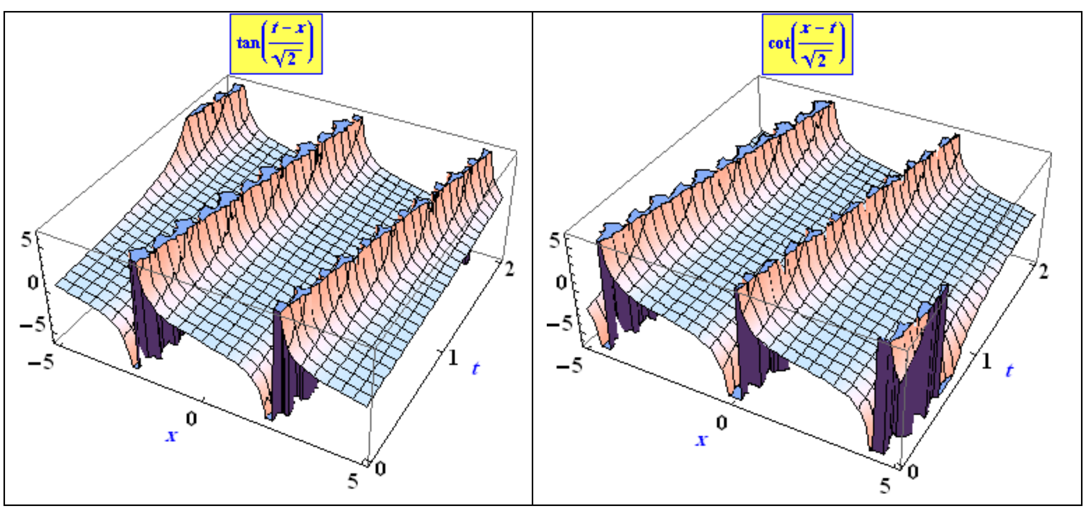

Figure 2. Plots of solutions for (3.1) obtained in Case III

It is noteworthy here that solitons are the solutions in the form $\sec h$ and $\sec h^2$; the graph of the soliton is a wave that goes up only. It is not like the periodic solutions sine, cosine, etc, as in trigonometric function, that goes above and below the horizontal. Kink is also called a soliton; it is in the form $\tanh$ not $\tanh^2$. In kink the limit as $x \to \infty$, gives the answer as a constant, not like solitons where the limit goes to 0 [Alquran and Al-Khaled (2011 a,b), Alquran (2012) and Alquran el at (2012)].

Now, by using the second set, another solution for (3.1) is given by:

$$u_2(x,t) = a_0 - \frac{i}{\sqrt{\gamma}} \frac{(i\sqrt{\beta}A + a_0 B\sqrt{\gamma}) + (i\sqrt{\beta}B + a_0 A\sqrt{\gamma})\tanh\left(\sqrt{\frac{\beta}{2c^2}}(x-ct)\right)}{B + A\tanh\left(\sqrt{\frac{\beta}{2c^2}}(x-ct)\right)}, \quad (3.12)$$

provided that $A \neq \pm B$ to avoid obtaining the constant solution. The general solution given in (3.12) produces the same types of solutions obtained by (3.11).

4. Coupled integrable dispersionless equations

The coupled integrable dispersionless equations [Kono and Onon (1994) and Bekir and Unsal (2013)] are:

$$u_{xt} + (vw)_x = 0, \quad (4.1)$$

$$v_{xt} - 2vu_x = 0, \quad (4.2)$$

$$w_{xt} - 2wu_x = 0. \quad (4.3)$$

Physically, the above system describes a current-fed string interacting with an external magnetic field in a three-dimensional Euclidean space. It also appears geometrically as the parallel transport of each point of the curve along the direction of time where the connection is magnetic-valued. The wave variable $\zeta = x - ct$ transform the above PDEs to the ODEs:

$$-cu'' + (vw)' = 0, \quad (4.4)$$

$$-cv'' - 2vu' = 0, \quad (4.5)$$

$$-cw'' - 2wu' = 0. \quad (4.6)$$

From equation (4.4), we deduce the following relation:

$$cu' = vw + R, \quad (4.7)$$

where R is the constant of integration. Accordingly, both equation (4.5) and (4.6) are symmetric in the functions $v(\zeta)$ and $w(\zeta)$. Therefore, w is proportional to v, i.e.,

$$w(\zeta) = k\, v(\zeta), \tag{4.8}$$

where k is the proportionality constant. Based on the above analysis we finally get the following ODE in terms of v only

$$c^2 v'' + 2Rv + 2k v^3 = 0. \tag{4.9}$$

Now, recalling equation (3.2), the function $v(x,t)$ admits the same obtained solutions for the extended Duffing equation (3.1) by replacing β by $2R$ and γ by $2k$. Thus, using the relations (4.7) and (4.8) the solutions to the dispersionless system are:

$$v_1(x,t) = \frac{i\sqrt{R}}{\sqrt{k}} \frac{A + B\tanh\left(\sqrt{\frac{R}{c^2}}(x-ct)\right)}{B + A\tanh\left(\sqrt{\frac{R}{c^2}}(x-ct)\right)},$$

$$w_1(x,t) = \frac{i\,k\sqrt{R}}{\sqrt{k}} \frac{A + B\tanh\left(\sqrt{\frac{R}{c^2}}(x-ct)\right)}{B + A\tanh\left(\sqrt{\frac{R}{c^2}}(x-ct)\right)},$$

$$u_1(x,t) = \frac{-c\sqrt{R}}{B} \frac{(A^2 - B^2)}{A + B\coth\left(\sqrt{\frac{R}{c^2}}(x-ct)\right)}. \tag{4.10}$$

And

$$v_2(x,t) = a_0 - \frac{1}{\sqrt{k}} \frac{(i\sqrt{R}A + a_0 B\sqrt{k}) + (i\sqrt{R}B + a_0 A\sqrt{k})\tanh\left(\sqrt{\frac{R}{c^2}}(x-ct)\right)}{B + A\tanh\left(\sqrt{\frac{R}{c^2}}(x-ct)\right)},$$

$$w_2(x,t) = a_0\, k - \sqrt{k}\, \frac{(i\sqrt{R}A + a_0 B\sqrt{k}) + (i\sqrt{R}B + a_0 A\sqrt{k})\tanh\left(\sqrt{\frac{R}{c^2}}(x-ct)\right)}{B + A\tanh\left(\sqrt{\frac{R}{c^2}}(x-ct)\right)},$$

$$u_2(x,t) = \frac{-c\sqrt{R}}{B}\frac{(A^2-B^2)}{A+B\coth\left(\sqrt{\frac{R}{c^2}}(x-ct)\right)}. \tag{4.11}$$

5. Discussion and conclusion

It is worth of mention in this work that there are other physical models that possess the same solutions obtained for the extended Duffing equation as well as the dispersionless system. For example, the Klein-Gordon equation

$$u_{tt} - u_{xx} + u - \frac{1}{6}u^3 = 0. \tag{5.1}$$

This equation appears in many scientific fields such as solid state physics, nonlinear optics, and dislocations in metals [Biswas et al. (2012)]. The wave variable $\zeta = x - ct$ transforms (5.1) into the ODE

$$(c^2-1)u'' + u - \frac{1}{6}u^3 = 0. \tag{5.2}$$

Comparying (5.2) with (3.2), it is clear that $\beta = 1$, $\gamma = -1/6$ and c^2 is replaced by $c^2 - 1$.

Another example is the Landau-Ginzburg-Higgs equation.

$$u_{tt} - u_{xx} - m^2u + n^2u^3 = 0, \tag{5.1}$$

where m and n are real constants [Hu et al. (2009)]. It possesses the same solution by considering $\beta = -m^2$, $\gamma = n^2$ and c^2 to be replaced by $c^2 - 1$.

In summary we have succeeded in recovering solutions for the coupled integrable dispersionless system when it was connected with the extended Duffing equation, so roughly speaking, there are many nonlinear physical models that possess the same class of solutions.

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