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Harmonic C^*-Categories of Longitudinal Pseudo Differential Operators over Flag Variety

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Abstract

Let $K = SU(3)$ be the special unitary group and maximal compact subgroup of the special linear group $SL(3;\mathbb{C})$.by depending on order n, The main aim of this paper is to use Gelfand- Tsetlin bases to show that the set of longitudinal pseudodifferential operators $\psi^n_{\mathcal{F}_i}$ on $K-$ homogeneous vector bundles E, E^* is the subset of simultaneous multiplier category $\mathcal{A} = \bigcap_{s\subseteq\sum} \mathcal{A}_s$, for C^*-categories $\mathcal{A}_s$ and $\mathcal{K}_s$ operators between $K-$ spaces, with simple roots $\propto_1, \propto_2$ of Lie group $SL(3;\mathbb{C})$ by using the Lie algebra $sl(3;\mathbb{C})$ and weight $s \subseteq \sum = \{\propto_1, \propto_2\}$.

Keywords: Gelfand-Tsetlin pattern; harmonic analysis on flag variety; longitudinal psudodifferential operators; Lie algebras and Lie group.

1. Introduction

In this paper we show that the set of longitudinal psudodifferential operators of n, (n is positive integer) on $K-$ homogeneous vector bundles over flag variety $\mathbb{X}$ is the subset of simultaneous multiplier category. This problem requires some lengthy computations in noncommutative harmonic analysis. The approach is connected to the idea of the harmonic C^*-categories and with Bernstein- Gelfand- complex and Kasparov theory for the action of the group $SL(3;\mathbb{C})$ [1]. As in [22] the key computation will be made using Gelfand-Tsetline (GT) bases It is possible to relate each (GT) pattern of integers array with a vector in the irreducible representation with highest weight.

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These vectors form an orthogonal basis for this representation and our work depends on the expository paper [17] together with some remarks in [22]. We specialize the case of $sl(3;\mathbb{C})$. Furthermore, in [22] another construction of regarding harmonic analysis on flag manifolds for $SL(n;\mathbb{C})$ has been done. We are particularly interested in the case where $n = 3$. For the definition and basic properties of longitudinal psudodifferential operators, we refer the reader to [18]. For $SL(3;\mathbb{C}) = G$, the simple roots $\propto_i$ $(i = 1,2)$ are G-equivariant fibrations $\mathbb{X} \to \mathbb{X}_i$ where $\mathbb{X}_i$ is the Grassmannians of lines and planes in the complex Bernstein, Gelfand and Gelfand made a homological complex by assembling interwiners between Verma modules. Refer to [6] for details. The C^*- algebra $\mathbb{K}_{\propto_i}$ of operators on the L^2-section space of any homogenous line bundle over flag variety $\mathbb{X}$ associated to each of these fibration. The–fibration is tangent to the longitudinal pseudodifferential operators. The intersection of C^*- algebras $\mathbb{K}_{\propto_i}$ of compact operators is the important property. For more information, see [1].

2. Notations and Preliminaries

First, we introduce some notation. Let $\boldsymbol{K = SU(3)}$ be the maximal compact subgroup of Lie group $\boldsymbol{SL(3;\mathbb{C})}$. We denote the set of longitudinal pseudodifferential operators of order at most $\boldsymbol{P}$ by $\boldsymbol{\psi^{p}_{\mathcal{F}_i}}$ and $\boldsymbol{\mathcal{A}}$ is the simultaneous multiplier category of $\boldsymbol{C^*}$-algebra $\boldsymbol{\mathcal{K}_{\propto_i}}$ $(\boldsymbol{i = 1,2})$ of operators on the $\boldsymbol{L^2}$- section space of any homogenous vector bundles over $\boldsymbol{\mathcal{X}}$ [1]. We are only interested in $\boldsymbol{p = 0}$ and for $\boldsymbol{i \neq j}$ we will answer the question that: $\boldsymbol{\psi^{0}_{\mathcal{F}_i}(E, E^*) \subset \mathcal{A}_j}$.

Let $\boldsymbol{n}: \begin{bmatrix} \boldsymbol{t_1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{t_2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{t_3} \end{bmatrix} \to \sum_i \boldsymbol{n_i\, t_i}$ be a map and the weights for $\boldsymbol{gl(3,\mathbb{C})}$ match the $\boldsymbol{n = (n_1, n_2, n_3)}$ via the-map. The triples $\boldsymbol{n_1 \geq n_2 \geq n_3}$ correspond to the dominant weights. We have a triangular array of integers with conditions:

$$\boldsymbol{\mu_{(k+1,j)}} \geq \boldsymbol{\mu_{(k,j)}} \text{ and } \boldsymbol{\mu_{(k,j)}} \geq \boldsymbol{\mu_{(k+1,j+1)}} \quad (1)$$

which is the Gelfand-Tsetline pattern (or GT-pattern)

There are vectors $\boldsymbol{\xi_\Lambda}$ form an orthogonal basis for the irreducible representation $\boldsymbol{\pi_n}$ with highest weight n= $(\boldsymbol{\mu_{31}, \mu_{32}, \mu_{33}})$, such that to each GT-pattern there is associated a vector $\boldsymbol{\xi_\Lambda}$ in $\boldsymbol{\pi_n}$. $\boldsymbol{\xi_\Lambda}$ is a weight vector, with weight $(\boldsymbol{s_1 - s_0, s_1 - s_2, s_3 - s_2})$ and the sum of the entries of the **k**th row is $\boldsymbol{s_k = \sum_{j=1}^{k} \mu_{k,j}}$; we obtain the GT-pattern $\boldsymbol{l_{k,j} = \mu_{k,j} - j + 1}$; and $\boldsymbol{\Lambda \mp \delta_{k,j}}$ from $\boldsymbol{\Lambda}$ by adding $\boldsymbol{\mp 1}$ to the $(\boldsymbol{k, j})$-entry.

$$\boldsymbol{\pi(X_1)\xi_\Lambda = -(l_{11} - l_{21})(l_{11} - l_{22})\, \xi_{\Lambda + \delta_{11}}}.$$

We switch the longest element $\boldsymbol{\omega_\rho \in W}$ to the element

$$\boldsymbol{\omega_\rho} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-1} \\ \boldsymbol{0} & \boldsymbol{-1} & \boldsymbol{0} \\ \boldsymbol{-1} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \in \boldsymbol{K} \quad (2)$$

Conjugation by $\boldsymbol{\omega_\rho}$ interchanges the subgroups $\boldsymbol{K_1}$ and $\boldsymbol{K_2}$. We define vectors form as an alternative orthogonal basis for $\boldsymbol{\pi_m}$ with related properties $\boldsymbol{\eta_\Lambda} = \boldsymbol{\pi_m(\omega_\rho)\xi_\Lambda}$. Equations and formulae should be typed and numbered consecutively with Arabic numerals in parentheses on the right hand side of the page (if referred to explicitly in the text).

3. Harmonic analysis for longitudinal-pseudodifferential operators

Let $\boldsymbol{m} = (\boldsymbol{m}, \boldsymbol{0}, -\boldsymbol{m})$ be the $\boldsymbol{\pi_m}$ representation with highest weight such that $\boldsymbol{m} \in \mathbb{N}$.

The Gelfand-Tsetlin vectors span the $\boldsymbol{0}$-weight space of $\boldsymbol{V}^{(\boldsymbol{m},\boldsymbol{0},-\boldsymbol{m})}$

$\boldsymbol{\xi_{m,j}} = \boldsymbol{\xi_\Lambda}$, with $\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{m} & \boldsymbol{0} & -\boldsymbol{m} \\ \boldsymbol{j} & & -\boldsymbol{j} \\ & \boldsymbol{0} & \end{bmatrix}$, for $\boldsymbol{j} = \boldsymbol{0}, \dots, \boldsymbol{m}$. The vectors span the $(\boldsymbol{0}, -\boldsymbol{1}, \boldsymbol{1})$-weight space

$\boldsymbol{\xi'_{m,j}} = \boldsymbol{\xi_\Lambda}$, for $\boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{m} & \boldsymbol{0} & -\boldsymbol{m} \\ (\boldsymbol{j}-\boldsymbol{1}) & & -\boldsymbol{j} \\ & \boldsymbol{0} & \end{bmatrix}$, $\boldsymbol{j} = \boldsymbol{1}, \dots, \boldsymbol{m}$. Via the Gelfand-Tsetlin formulas-mentioned above

$$\boldsymbol{\pi_m}(\boldsymbol{X_2^*})\boldsymbol{\xi_{m,j}} = \frac{\boldsymbol{j}}{\boldsymbol{2j+1}}\boldsymbol{\xi'_{m,j}} + \frac{\boldsymbol{j+1}}{\boldsymbol{2j+1}}\boldsymbol{\xi'_{m,j+1}} \qquad (3)$$

$$\boldsymbol{\pi_m} = (\boldsymbol{X_2})\boldsymbol{\xi'_{m,j}} = \frac{\boldsymbol{1}}{\boldsymbol{2}}((\boldsymbol{m}+\boldsymbol{1})^2 - \boldsymbol{j}^2)\boldsymbol{\xi_{m,j-1}} + \frac{\boldsymbol{1}}{\boldsymbol{2}}((\boldsymbol{m}+\boldsymbol{1})^2 - \boldsymbol{j}^2)\boldsymbol{\xi_{m,j}} \qquad (4)$$

$$\boldsymbol{\pi_m} = (\boldsymbol{X_2})\boldsymbol{\pi_m}(\boldsymbol{X_2^*})\boldsymbol{\xi_{m,j}} = \frac{\boldsymbol{j}}{\boldsymbol{2(2j+1)}}((\boldsymbol{m}+\boldsymbol{1})^2 - \boldsymbol{j}^2)\boldsymbol{\xi_{m,j-1}} + \frac{\boldsymbol{1}}{\boldsymbol{2}}\left((\boldsymbol{m}+\boldsymbol{1})^2 - (\boldsymbol{j}^2+\boldsymbol{j}+\boldsymbol{1})\right)\boldsymbol{\xi_{m,j}} + \frac{\boldsymbol{j+1}}{\boldsymbol{2(2j+1)}}((\boldsymbol{m}+\boldsymbol{1})^2 - (\boldsymbol{j}+\boldsymbol{1})^2\boldsymbol{\xi_{m,j+1}} \qquad (5)$$

$$\boldsymbol{\pi_m}(\boldsymbol{X_1^*})\boldsymbol{\pi_m}(\boldsymbol{X_1})\boldsymbol{\xi_{m,j}} = \boldsymbol{j}(\boldsymbol{j}+\boldsymbol{1})\boldsymbol{\xi_{m,j}} \qquad (6)$$

The vectors norms are

$$\left\|\boldsymbol{\xi_{m,j}}\right\|^2 = \frac{\boldsymbol{1}}{\boldsymbol{2j+1}}\boldsymbol{m}!^{\boldsymbol{2}}\,(\boldsymbol{2m}+\boldsymbol{1})! \qquad (7)$$

$$\left\|\boldsymbol{\xi'_{m,j}}\right\|^2 = \frac{\boldsymbol{1}}{\boldsymbol{2j}}((\boldsymbol{m}+\boldsymbol{1})^2 - \boldsymbol{j}^2)\boldsymbol{m}!^{\boldsymbol{2}}\,(\boldsymbol{2m}+\boldsymbol{1})! \qquad (8)$$

We define vectors

$\boldsymbol{\eta_{m,j}} = \boldsymbol{\pi_m}(\boldsymbol{\omega_\rho})\,\boldsymbol{\xi_{m,j}} \quad (\boldsymbol{0} \le \boldsymbol{j} \le \boldsymbol{m}), \qquad \boldsymbol{\eta'_{m,j}} = \boldsymbol{\pi_m}(\boldsymbol{\omega_\rho})\,\boldsymbol{\xi'_{m,j}} \quad (\boldsymbol{1} \le \boldsymbol{j} \le \boldsymbol{m}).$

with weights

$\boldsymbol{\omega_\rho}.\boldsymbol{0} = \boldsymbol{0}$ has norm $\left\|\boldsymbol{\eta_{m,j}}\right\| = \left\|\boldsymbol{\xi_{m,j}}\right\|$ and

$\boldsymbol{\omega_{\rho}}.(\mathbf{0},-\mathbf{1},\mathbf{1}) = (\mathbf{1},-\mathbf{1},\mathbf{0}) = \boldsymbol{\alpha_1}$ has norm $\left\|\boldsymbol{\eta'}_{m,j}\right\| = \left\|\boldsymbol{\xi'}_{m,j}\right\|$.

"Eq. (3)" and "Eq. (6)" yield

$$\pi_m(X_1^*)\eta'_{m,j} = \frac{1}{2}((m+1)^2 - j^2)\eta_{m,j-1} + \frac{1}{2}((m+1)^2 - j^2)\eta_j \qquad (9)$$

$$\pi_m(X_1^*)\pi_m(X_1)\eta_{m,j}$$
$$= \frac{1}{1(2j+1)}((m+1)^2 - j^2)_{\eta_{m,j-1}} + \frac{1}{2}((m+1)^2 - (j^2+j+1))_{\eta_{m,j}}$$
$$+ \frac{j+1}{2(2j+1)}((m+1)^2 - (j+1)^2)_{\eta_{m,j+1}} \qquad (\mathbf{10})$$

We define

$$a_{m,j,k} = \frac{(-1)^j\,\overline{\omega_m}}{m!^2(2m+1)!}\langle \xi_{m,j}, \eta_{m,k}\rangle. \qquad (11)$$

If $\mathbf{0} \le j, k \le m$ does not hold we write $a_{m,j,k} = \mathbf{0}$.

Lemma 3.1 [1] For any $m \in \mathbb{N}$, $\eta_{m,0} = \omega_m \sum_{j=0}^{m} (-1)^j \frac{2j+1}{m+1} \xi_{m,j}$, $|\omega_m| = \mathbf{1}$, where $\omega_m \in \mathbb{C}$ is some phase factor.

Lemma 3.2 For $\mathbf{0} \le j, k \le m$, we have the recurrence relation in k with initial condition $a_{m,j,0} = \frac{1}{(m+1)}$.

$$k((m+1)^2 - k^2)_{a_{m,j,k-1}} + (2k+1)\Big((m+1)^2 - (k^2+k+1) - 2j(j+1)\Big)_{a_{m,j,k}} + (k+1)((m+1)^2 - (k+1)^2)_{a_{m,j,k+1}} = \mathbf{0} \qquad (12)$$

Proof Apply "Eq. (6)" and "Eq. (10)"

to the equality

$$\langle \pi_m(X_1^*)\pi_m(X_1)\xi_{m,j}, \eta_{m,k} \rangle = \langle \xi_{m,j}, \pi_m(X_1^*)\pi_m(X_1)\eta_{m,k} \rangle$$

to obtain

$$j(j+1)\langle \xi_{m,j}, \eta_{m,k} \rangle = \frac{k}{2(2k+1)}((m+1)^2 - k^2)\langle \xi_{m,j}, \eta_{m,k-1} \rangle +$$

$$\frac{1}{2}\Big((m+1)^2 - (k^2+k+1)\Big)\langle \xi_{m,j}, \eta_{m,k} \rangle +$$

$$\frac{k+1}{2(2k+1)}((m+1)^2-(k+1)^2)\langle \xi_{m,j}, \eta_{m,k+1} \rangle,$$

which reduces to (12). Lemma (3.1) gives

$$a_{m,j,0} = (-1)^j \frac{2j+1}{m+1} \frac{1}{m!^2(2m+1)!} \left\| \xi_{m,j} \right\|^2 = \frac{(-1)^j}{(m+1)}.$$

Corollary 3.3 For $0 \leq m-2\in, m-\in \leq m, (m-2\in) < (m-\in)$, we have the recurrence relation in $m-\in$ with respect to the initial condition $a_{m,m-2\epsilon,0} = \frac{1}{(m+1)}$,

$$(m-\in)((m+1)^2-(m-\in)^2)a_{m,m-2\epsilon,m-\epsilon-1} +$$

$$(2(m-\in)+1)\Big((m+1)^2-((m-\in)^2+m-\in+1)-2(m-2\epsilon)(m-2\epsilon+1)\Big)_{a_{m,m-2\epsilon,m-\epsilon}} + (m-\in +1)((m+1)^2-(m-\in+1)^2{}_{a_{m,m-2\epsilon,m-\epsilon+1}} = 0 \qquad (13)$$

Proof From "Eq. (6)" and "Eq. (10)" we obtain $\langle \pi_m(X_1^*)\pi_m(X_1)\xi_{m,m-2\in,}\eta_{m,m-\in} \rangle = \langle \xi_{m,m-2\in,\pi_m} \ (X_1^*)\pi_m(X_1)\eta_{m,m-\in} \rangle$

$$(m-2\in)(m-2\in+1)\langle \xi_{m,m-2\in}, \eta_{m,m-\in} \rangle = \frac{m-\in}{2(2(m-\in)+1)}((m+1)^2-(m-\in)^2)\langle \xi_{m,m-2\in}, \eta_{m,m-\in-1} \rangle +$$

$$\frac{1}{2}\Big((m+1)^2-((m-\in)^2+m-\in+1)\Big)\langle \xi_{m,m-2\in}, \eta_{m,m-\in} \rangle +$$

$$\frac{m-\in+1}{2(2(m-\in)+1)}((m+1)^2-(m-\in+1)^2)\langle \xi_{m,m-2\in}, \eta_{m,m-\in+1} \rangle,$$

which reduces to "Eq. (12)". Lemma (3.1) gives

$$a_{m,m-2\in,0} = (-1)^{(m-2\in)} \frac{2(m-2\in)+1}{m+1} \frac{1}{m!^2(2m+1)!} \left\| \xi_{m,m-2\in} \right\|^2 = \frac{(-1)^{(m-2\in)}}{(m+1)}.$$

For $j > 0, and\ (phX_1)\xi_{m,j} = 0$, we obtain the next equation from "Eq. (6)"

$$(X_1)\xi_{m,j} = X_1.(X_1{}^*X_1)^{-\frac{1}{2}}\xi_{m,j} = \frac{1}{\sqrt{j(j+1)}}X_1\xi_{m,j} \qquad (14)$$

We suppose that $y_{m,j}$ and $y'_{m,j}$ are the corresponding orthonormal bases,

$$y_{m,j} = \eta_{m,j}/\|\eta_{m,j}\| = \frac{1}{m!(2m+1)!^{\frac{1}{2}}}\eta_{m,j} \qquad (15)$$

and

$$y'_{m,j} = \eta'_{m,j} / \left\| \eta'_{m,j} \right\| = \frac{1}{m!(2m+1)!^{\frac{1}{2}}} \left(\frac{2j}{(m+1)^2 - j^2}\right)^{1/2} \eta_{m,j} \qquad (16)$$

Lemma 3.4

$\lim_{m\to\infty} \left\| \left\langle (X_1) y_{m,0}, y'_{m,k} \right\rangle \right\| = \sqrt{2k}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$, for each fixed $k \epsilon \mathbb{N}$.

Proof We use "Eq. (14)" and "Eq. (15)", lemma (3.1), "Eq. (13)", "Eq. (10)", and "Eq. (11)", to compute

$$\left\langle (phX_1) y_{m,0}, y'_{m,k} \right\rangle = \frac{1}{m!^2\,(2m+1)!} \sqrt{\frac{2k}{(m+1)^2 - k^2}} \left\langle (phX_1)\eta_{m,0}, \eta'_{m,k} \right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!} \sqrt{\frac{2k}{(m+1)^2 - k^2}} \sum_{j=0}^{m} (-1)^j \frac{2j+1}{m+1} \left\langle (phX_1)\xi_{m,j}\,, \eta'_{m,k} \right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!} \sqrt{\frac{2k}{(m+1)^2 - k^2}} \sum_{j=0}^{m} (-1)^j \frac{2j+1}{m+1} \left\langle \frac{1}{\sqrt{j(j+1)}} X_1 \xi_{m,j}\,, \eta'_{m,k} \right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!} \sqrt{\frac{2k}{(m+1)^2 - k^2}} \sum_{j=0}^{m} (-1)^j \frac{2j+1}{m+1} \frac{1}{\sqrt{j(j+1)}} \left\langle X_1 \xi_{m,j}\,, \eta'_{m,k} \right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!} \frac{1}{m+1} \sqrt{\frac{2k}{(m+1)^2 - k^2}} \sum_{j=0}^{m} (-1)^j \frac{2j+1}{\sqrt{j(j+1)}} \left\langle \xi_{m,j}\,, X_1{}^{*} \eta'_{m,k} \right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!} \frac{\sqrt{2k(m+1)^2 - k^2}}{m+1} \sum_{j=0}^{m} (-1)^j \frac{2j+1}{2\sqrt{j(j+1)}} \left\langle \xi_{m,j}\,, \eta_{m,k-1} + \eta_{m,k} \right\rangle$$

$$= \omega_m \sqrt{2k\left(1 - \frac{k^2}{(m+1)^2}\right.} \sum_{j=0}^{m} \frac{j+\frac{1}{2}}{\sqrt{j(j+1)}} \left(a_{m,j,k-1} + a_{m,j,k}\right) \qquad (17)$$

$$\sum_{j=0}^{m} \frac{j+\frac{1}{2}}{\sqrt{j(j+1)}} \left(a_{m,j,k-1} + a_{m,j,k}\right) \qquad (18)$$

$$= \sum_{j=0}^{m} \left(\frac{j+\frac{1}{2}}{\sqrt{j(j+1)}} - 1\right)\left(a_{m,j,k-1} + a_{m,j,k}\right) \qquad (19)$$

$$= \sum_{j=0}^{m} \left(a_{m,j,k-1} b_{m,j,k-1}\right) + \left(a_{m,j,k} - b_{m,j,k}\right) \qquad (20)$$

$$= \sum_{j=0}^{m} \left(b_{m,j,k-1} + b_{m,j,k}\right). \qquad (21)$$

In "Eq. (18)"

$$\left(\frac{j+\frac{1}{2}}{\sqrt{j(j+1)}}-1\right)=\sqrt{\frac{j^2+j+\frac{1}{4}}{j^2+j}}-1\leq\frac{1}{8j^2},$$

Fix k, and by remark[1,(A.6], $a_{m,j,k-1}$ and $a_{m,j,k}$ are both $O(m^{-1})$, so (18) goes to 0 as $m\to\infty$. From [1,(A.5)], $(m+1)[C(k-1)+C(k)](m+1)^{-2}$ bounds the equation (19), as $m\to\infty$ also this equation approaches- 0.

The Riemann sum of the integral $(-1^j)\int_{t=0}^{1}P_k\,(2t^2-1)dt$ is

$$\sum_{j=0}^{m}b_{m,j,k}=\frac{1}{m+1}P_k(2\left(\frac{j}{m+1}\right)^2-1) \qquad (22)$$

We substitute $(2t^2-1)=v$, "Eq. (21)"converges to

$$\left(2^{-\frac{3}{2}}\right)\int_{-1}^{1}(1-v)^{-\frac{1}{2}}\,P_k(-v)dv=\frac{(-1)^k}{2k+1}$$

As $m\to\infty$, "Eq. (18)"converges to $(-1)^{k-1}(\frac{1}{2k-1}-\frac{1}{2k+1})$. to complete our proof We put this into "Eq. (17)" to obtain

$$\left\langle(phX_1)y_{m,0},{y'}_{m,k}\right\rangle=\omega_m\sqrt{2k(1-\frac{k^2}{(m+1)^2}}\left((-1)^{k-1}(\frac{1}{2k-1}-\frac{1}{2k+1})\right).$$

Corollary 3.5 ,

$$\lim_{m\to\infty}\left\|\left\langle(phX_i)y_{m,0},{y'}_{m,(1+\epsilon)}\right\rangle\right\|=\sqrt{2(1+\epsilon)}(\frac{1}{2(1+\epsilon)-1}-\frac{1}{2(1+\epsilon)+1}),\text{ for each fixed }(1+\epsilon)\epsilon\mathbb{N},\ \epsilon\geq 0.$$

Proof Use "Eq. (15)"and "Eq. (16)", lemma (3.1), "Eq. (14)", "Eq. (11)", and "Eq. (12)", to compute

$$\left\langle(phX_1)y_{m,0},{y'}_{m,(1+\epsilon)}\right\rangle$$

$$=\frac{1}{m!^2\,(2m+1)!}\sqrt{\frac{2(1+\epsilon)}{(m+1)^2-(1+\epsilon)^2}}\left\langle(phX_1)\eta_{m,0},{\eta'}_{m,(1+\epsilon)}\right\rangle$$

$$=\frac{\omega_m}{m!^2\,(2m+1)!}\sqrt{\frac{2(1+\epsilon)}{(m+1)^2-(1+\epsilon)^2}}\sum_{(1+\epsilon)=0}^{m}(-1)^{(1+\epsilon)}\frac{(3+2\,\epsilon)}{m+1}\left\langle(phX_1)\xi_{m,(1+\epsilon)}\,,{\eta'}_{m,(1+\epsilon)}\right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\sqrt{\frac{2(1+\in)}{(m+1)^2-(1+\in)^2}}\sum_{(1+\in)=0}^{m}(-1)^{(1+\in)}\frac{(3+2\in)}{m+1}\left\langle\frac{1}{\sqrt{(1+\in)(2+\in)}}X_1\xi_{m,(1+\in)}\,,\eta'_{m,(1+\in)}\right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\sqrt{\frac{2(1+\in)}{(m+1)^2-(1+\in)^2}}\sum_{(1+\in)=0}^{m}(-1)^{(1+\in)}\frac{(3+2\in)}{m+1}\frac{1}{\sqrt{(1+\in)(2+\in)}}\left\langle X_1\xi_{m,(1+\in)}\,,\eta'_{m,(1+\in)}\right\rangle$$

$$= \frac{\omega_m}{m!^2\,(2m+1)!}\frac{1}{m+1}\sqrt{\frac{2(1+\in)}{(m+1)^2-(1+\in)^2}}\sum_{(1+\in)=0}^{m}(-1)^{(1+\in)}\frac{(3+2\in)}{\sqrt{(1+\in)(2+\in)}}\left\langle \xi_{m,(1+\in)}\,,X_1{}^{*}\eta'_{m,(1+\in)}\right\rangle$$

$$= \frac{\omega_m}{m!^2(2m+1)!}\frac{\sqrt{2(1+\in)(m+1)^2-(1+\in)^2}}{m+1}\sum_{(1+\in)=0}^{m}(-1)^{(1+\in)}\frac{(3+2\in)}{2\sqrt{(1+\in)(2+\in)}}\left\langle\xi_{m,(1+\in)}\,,\eta_{m,\in}+\eta_{m,(1+\in)}\right\rangle =$$
$$\omega_m\sqrt{2(1+\in)(1-\frac{(1+\in)^2}{(m+1)^2}}\sum_{(1+\in)=0}^{m}\frac{(\in+\frac{3}{2})}{\sqrt{(1+\in)(2+\in)}}\left(a_{m,(1+\in),\in}+a_{m,(1+\in),(1+\in)}\right)$$
(23)

$$\sum_{(1+\in)=0}^{m}\frac{(\in+\frac{3}{2})}{\sqrt{(1+\in)(2+\in)}}\left(a_{m,(1+\in),\in}+a_{m,(1+\in),(1+\in)}\right) \qquad (24)$$

$$= \sum_{(1+\in)=0}^{m}\left(\frac{(\in+\frac{3}{2})}{\sqrt{(1+\in)(2+\in)}}-1\right)\left(a_{m,(1+\in),\in}+a_{m,(1+\in),(1+\in)}\right) \qquad (25)$$

$$= \sum_{(1+\in)=0}^{m}\left(a_{m,(1+\in),\in}\,b_{m,(1+\in),\in}\right)+\left(a_{m,(1+\in),(1+\in)}-b_{m,(1+\in),(1+\in)}\right) \qquad (26)$$

$$= \sum_{(1+\in)=0}^{m}\left(b_{m,(1+\in),\in}+b_{m,(1+\in),(1+\in)}\right). \qquad (27)$$

In "Eq. (24)"

$$\left(\frac{(\in+\frac{3}{2})}{\sqrt{(1+\in)((2+\in)}}-1\right)=\sqrt{\frac{(1+\in)^2+\in+\frac{5}{4}}{(1+\in)^2+(1+\in)}}-1\le\frac{1}{8(1+\in)^2},$$

Fix $(1+\in)$, and by remark[1,(A.6], $a_{m,(1+\in),\in}$ and $a_{m,(1+\in),(1+\in)}$ are both $O(m^{-1})$, so "Eq. (25)"goes to $\mathbf{0}$ as $m\to\infty$. From [1,(A.5)], $(m+1)[C(\in)+C((1+\in))](m+1)^{-2}$ bounds "Eq. (26)", as $m\to\infty$ also this equation approaches to $\mathbf{0}$.

The Riemann sum of the integral $(-1^{(1+\in)})\int_{t=0}^{1}P_{(1+\in)}(2t^2-1)dt$ is

$$\sum_{(1+\in)=0}^{m}b_{m,(1+\in),(1+\in)}=\frac{1}{m+1}P_{(1+\in)}\left(2\left(\frac{(1+\in)}{m+1}\right)^2-1\right) \qquad (28)$$

We substitute $(2t^2-1)=v$, "Eq. (27)"converges to

$$\left(2^{-\frac{3}{2}}\right)\int_{-1}^{1}(1-v)^{-\frac{1}{2}}\ P_{(1+\in)}(-v)dv = \frac{(-1)^{(1+\in)}}{2(1+\in)+1} = \frac{(-1)^{(1+\in)}}{3+2\in}$$

As $m \to \infty$, "Eq. (23)"converges to $(-1)^{\in}(\frac{1}{1+2\in} - \frac{1}{3+2\in})$. to complete our proof. We put this into "Eq. (22)":

$$\left\langle (phX_1)y_{m,0}, y'_{m,(1+\in)}\right\rangle = = \omega_m \sqrt{2(1+\in)(1-\frac{(1+\in)^2}{(m+1)^2}}\left((-1)^{\in}(\frac{1}{1+2\in} - \frac{1}{3+2\in})\right).$$

Lemma 3.6 on any unitary K- representation $\mathcal{H}$ the operators $(phX_1{}^{*})p_{\sigma_0}$, and therefore $p_{\sigma_0}(phX_1)$, are in $\mathcal{K}_{\beta_2}(\mathcal{H})$. Proof. Let U be a unitary representation of K on $\mathcal{H}$. The antilinear map $J: \mathcal{H} \to \mathcal{H}^{\dagger}; \xi \to \langle \xi, . \rangle$ Intertwines the representations U and $U^{\dagger}$. for any X in the complexification $\iota_{\mathbb{C}}$, $J^{-1}\ U^{\dagger}(X)J = -U(X)^{*}$. Since J is anti-unitary, $J^{-1}\ ph\left(U^{\dagger}(X)\right)J = -ph\big(U(X^{*})\big)$. If $\xi \in \mathcal{H}$ has K_2 −type σ, then $J\xi$ has K_2 −type $\sigma^{\dagger}$, so $p\sigma = J^{-1}p_{\sigma}^{\dagger}J$. By conjugating by J, the estimate $\|p_F^{\perp}(PhX_1)p_{\sigma_0}\| < \epsilon$ implies $\|p_{F^{\dagger}}^{\perp}(PhX_1{}^{*})p_{\sigma_0}\| < \epsilon$, where $F^{\dagger} = \{\sigma^{\dagger} | \sigma \in F\}$. Corollary 3.7 on any unitary K- representation $\mathcal{H}$, the operators $(phX_1)p_{\sigma_0}$, and therefore $p_{\sigma_0}(phX_1{}^{*})$, are in $\mathcal{K}_{\beta_2}(\mathcal{H})$. Proof. Let U be a unitary representation of K on $\mathcal{H}$. The antilinear map $J: \mathcal{H} \to \mathcal{H}^{\dagger}$; $\xi \to \langle \xi, . \rangle$ intertwines the representations U for any X^{*} in the complexification $\iota_{\mathbb{C}}$, J^{-1} $U^{\dagger}(X)^{*}J = -U(X)$. Since J is anti-unitary, $J^{-1}\ ph\left(U^{\dagger}(X^{*})\right)J = -ph\big(U(X)\big)$. If $\xi \in \mathcal{H}$ has K_2 −type σ, then $J\xi$ has K_2 −type $\sigma^{\dagger}$, so $p\sigma = J^{-1}p_{\sigma}^{\dagger}J$. By conjugating by J, the estimate $\|p_F^{\perp}(PhX_1)p_{\sigma_0}\| < \epsilon$ implies $\|p_{F^{\dagger}}^{\perp}(PhX_1)p_{\sigma_0}\| < \epsilon$, where $F^{\dagger} = \{\sigma^{\dagger} | \sigma \in F\}$. Lemma 3.8 let v be a weight of K. For any $f \in C(K)$, $[PhX_1, M_f]pv$ and $[PhY_1, M_f]pv$ are in $\mathcal{K}_{\propto_1}(L^2(K))$. Proof. Assume that f is a weight vector for the right regular representation, i.e, $f \in C(\mathcal{X}; E_{-\mu})$ for some μ. Through Lemma[3.19,1] we have

$$[PhX_1, M_f]: pvL^2(K) \to p_{v+}\mu + \propto_1 L^2(K)$$

Is in $\mathcal{K}_{\propto_1}$, which implies the result. The subspace spanned by these weight vectors contains all matrix units, so is uniformly dense in $C(K)$. A density argument completes the proof. Similarly, $[PhY_1, M_f]pv \in \mathcal{K}_{\propto_1}$. corollary 3.9 let v be a weight of K. For any $f_j \in C(K)$, $\left[PhX_1, M_{\sum f_j}\right]pv$ and $\left[PhY_1, M_{\sum f_j}\right]pv$ are in $\mathcal{K}_{\propto_1}(L^2(K))$. Proof. Assume that f_j is a weight vector for the right regular representation, i.e, $f_j \in C(\mathcal{X}; E_{-\mu})$ for some μ. Through Lemma[3.19,1] we have

$$\left[PhX_1, M_{\sum f_j}\right]: pvL^2(K) \to p_{v+}\mu + \propto_1 L^2(K)$$

is in $\mathcal{K}_{\propto_1}$, which implies the result. The subspace spanned by these weight vectors contains all matrix units, so is uniformly dense in $C(K)$. A density argument completes the proof. Similarly, $\left[PhY_1, M_{\sum f_j}\right]pv \in \mathcal{K}_{\propto_1}$. Theorem 3.10 On any unitary K representation on $\mathcal{H}$, PhX_i and PhY_i are in $\mathcal{A}(\mathcal{H})$ for $i = 1, 2$. *Proof.* We first have $\mathcal{H} = L^2(K)$ with right regular representation, and consider PhX_1. With respect to lemma[3.7,1], the finite multiplicity of K-types in $L^2(K)$ implies that $\mathcal{A}_{\sum}\left(L^2(K)\right) = \mathcal{L}\left(L^2(K)\right)$, so $PhX_1 \in \mathcal{A}_{\sum}$ trivially. Since

$\boldsymbol{PhX_1}$ maps the $\boldsymbol{\mu}$-weight space into the $(\boldsymbol{\mu}+\propto_1)$-weight space for each weight $\boldsymbol{\mu}$, it is $\boldsymbol{M}$-harmonically proper, so in $\mathcal{A}_\emptyset$. Since X_1in $(\iota_1)_\mathbb{C}$, PhX_1preserves K_1-types, so $PhX_1 \in \mathcal{A}_{\propto_1}$. And eventually we are going to show that $PhX_1 \in \mathcal{A}_{\propto_2}$. Let $\sigma \in \widehat{K}_2$ and let $\psi_{1,\dots,}\psi_n \in C(K)$ be as in lemma [A.10,1]. Then

$$(PhX_1)p\sigma = \sum_{j=1}^{n}(PhX_1)M_{\psi j}\, P\sigma_0\, M_{\overline{\psi j}} = \sum_{j=1}^{n} M_{\psi j}(PhX_1)\, P\sigma_0\, M_{\overline{\psi j}} + \sum_{j=1}^{n}[(PhX_1), M_{\psi j}]\, P\sigma_0\, M_{\overline{\psi j}}\,.$$

since $P\sigma_0$ projects into the $\mathbf{0}$-weight space, lemmas [A.8, 1], (1.6) and [3.11,1], show that $(PhX_1)\, p\sigma \in \mathcal{K}_{\propto_2}$. By analogous computation lemma (1.4) illustrate that $(PhY_1)p\sigma = (PhX_1{}^*)p\sigma \in \mathcal{K}_{\propto_2}$, so $p\sigma(PhX_1) \in \mathcal{K}_{\propto_2}$. By proposition 3.6, $PhX_1 \in \mathcal{A}_{\alpha_2}$

then $PhX_1 \in \mathcal{A}$. By taking adjoints $PhY_1 \in \mathcal{A}$.

$\boldsymbol{conjugationby\ the\ longest\ weyl\ group\ element\ interchanges}$ $\mathcal{A}_{\alpha_1}$ and $\mathcal{A}_{\alpha_2}$ and fixes $\mathcal{A}_\emptyset$ and $\mathcal{A}_\Sigma$, so fixes $\mathcal{A}$. It also sends $\mathbf{X_1}$ and Y_1 to Y_2 and $\mathbf{X_2}$, respectively. We obtain PhY_2 , $PhX_2 \epsilon \mathcal{A}$. The theorem remains true if $\mathcal{H}$ is a direct sum of arbitrarily many copies of the regular representation. Since every unitary K-representation can be equivariantly embedded into such a direct sum, we are done. Corollary 3.11 on any unitary K_{r-1} representation on $\mathcal{H}$, PhX_i and PhY_i are in $\mathcal{A}(\mathcal{H})$ for $i = 1, 2.$ *Proof.* We first have $\mathcal{H} = L^2(K_{r-1})$ with right regular representation, and consider PhX_1. With respect to lemma[3.7,1], the finite multiplicity of K_{r-1} -types in $L^2(K_{r-1})$ implies that $\mathcal{A}_\Sigma\left(L^2(K_{r-1})\right) = \mathcal{L}\left(L^2(K_{r-1})\right)$, so $PhX_1 \in \mathcal{A}_\Sigma$ trivially. Since PhX_1 maps the $\boldsymbol{\mu}$-weight space into the $(\boldsymbol{\mu}+\propto_1)$-weight space for each weight $\boldsymbol{\mu}$, it is $\boldsymbol{M}$-harmonically proper, so in $\mathcal{A}_\emptyset$. Since X_1in $(\iota_1)_\mathbb{C}$, PhX_1preserves K_r-types, so $PhX_1 \in \mathcal{A}_{\propto_1}$. And eventually we are going to show that $PhX_1 \in \mathcal{A}_{\propto_2}$. Let $\sigma \in \widehat{K}_{r+1}$ and let $\psi_{1,\dots,}\psi_n \in C(K_{r-1})$ be as in lemma [A.10,1]. Then

$$(PhX_1)p\sigma = \sum_{j=1}^{n}(PhX_1)M_{\psi j}\, P\sigma_0\, M_{\overline{\psi j}} = \sum_{j=1}^{n} M_{\psi j}(PhX_1)\, P\sigma_0\, M_{\overline{\psi j}} + \sum_{j=1}^{n}[(PhX_1), M_{\psi j}]\, P\sigma_0\, M_{\overline{\psi j}}\,.$$

since $P\sigma_0$ projects into the $\mathbf{0}$-weight space, lemmas [A.8, 1], (1.6) and [3.11,1], show that $(PhX_1)\, p\sigma \in \mathcal{K}_{\propto_2}$. By analogous computation lemma (1.4) illustrate that $(PhY_1)p\sigma = (PhX_1{}^*)p\sigma \in \mathcal{K}_{\propto_2}$, so $p\sigma(PhX_1) \in \mathcal{K}_{\propto_2}$. By proposition 3.6, $PhX_1 \in \mathcal{A}_{\alpha_2}$

then $PhX_1 \in \mathcal{A}$. By taking adjoints $PhY_1 \in \mathcal{A}$.

$\boldsymbol{conjugationby\ the\ longest\ weyl\ group\ element\ interchanges}$ $\mathcal{A}_{\alpha_1}$ and $\mathcal{A}_{\alpha_2}$ and fixes $\mathcal{A}_\emptyset$ and $\mathcal{A}_\Sigma$, so fixes $\mathcal{A}$. It also sends $\mathbf{X_1}$ and Y_1 to Y_2 and $\mathbf{X_2}$, respectively. We obtain PhY_2 , $PhX_2 \epsilon \mathcal{A}$. The theorem remains true if $\mathcal{H}$ is a direct sum of arbitrarily many copies of the regular representation. Since every unitary $\mathbf{K_{r-1}}$ -representation can be equivariantly embedded into such a direct sum, we are done. Theorem 3.12 Let E, E^* be $K-$ vector bundles over flag variety $\mathbb{X}$, then $\psi^n_{\mathcal{F}_i}(E, E^*) \subseteq \mathcal{A}$. Proof. It is clear that $\overline{\psi_{\mathcal{F}_\iota}^{-1}}(E_n) \subseteq \mathcal{A}$. **Let** $E_n = E = E^* = E_\rho = E_\sigma$, the trivial line bundle over flag variety $\mathbb{X}$.

$$n \to \overline{\psi_{\mathcal{F}_\iota}^{-1}}(E_n) \to \overline{\psi_{\mathcal{F}_\iota}^{n}}(E_n) \to C(S^*\mathcal{F}_i) \to n$$

According to the Stone-Weierstrass theorem and since the points in different fibres of $S^*\mathcal{F}_i$ are separated by the function algebrac we prove that $C(S^*\mathcal{F}_i) = C$ by showing that it separates the points of $S^*\mathcal{F}_i$. The multiplication operator $M_f \in \psi_{\mathcal{F}_i}^{n}(E_n) \cap \mathcal{A}$ for any $f \epsilon C(\mathbb{X})$. Let $\emptyset \in C(\mathbb{X}, E_\sigma)$ be any non-zero smooth section of E_σ at the identity coset. $M_\emptyset phx_i \in \psi_{\mathcal{F}_i}^{n}(E_n) \cap \mathcal{A}$ and its principal symbol separates points of the fiber at the identity coset. By Lemma 2.1 $\emptyset_1, \dots, \emptyset_n \in C(\mathbb{X}, E_\mu)$, $\emptyset_1{}', \dots, \emptyset_m{}' \in C(\mathbb{X}, E_v)$. For each j, μ and $k \in K$, $\overline{M_{\emptyset_j'}}\ A\, M_{\emptyset k} \in \overline{\psi_{\mathcal{F}_\iota}^{n}}(E_n) \subseteq \mathcal{A}$, for $A \in \overline{\psi_{\mathcal{F}_\iota}^{n}}$.

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