

The elementary construction of formal anafunctors

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Abstract. This article gives an elementary and formal 2-categorical construction of a bicategory of right fractions analogous to anafunctors, starting from a 2-category equipped with a family of covering maps that are fully faithful and co-fully faithful.

1 Introduction

Anafunctors were introduced by Makkai [7] as new 1-arrows in the 2-category \mathbf{Cat} to talk about category theory in the absence of the axiom of choice. The aim was to make functorial those constructions that are only defined by some universal property, rather than by some specified operation. One also recovers the characterisation of equivalences of categories as essentially surjective, fully faithful 1-arrows. The construction by Bartels [1] of the analogous bicategory $\mathbf{Cat}_{ana}(S, J)$, whose 1-arrows are anafunctors, starting from the 2-category $\mathbf{Cat}(S)$ of internal categories was extended in [10] to variable full sub-2-categories $\mathbf{Cat}'(S) \hookrightarrow \mathbf{Cat}(S)$. The canonical inclusion

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2-functor $\mathbf{Cat}'(S) \hookrightarrow \mathbf{Cat}'_{ana}(S, J)$ was there shown to be a 2-categorical localisation in the sense of Pronk [8] at the fully faithful functors which are locally weakly split in the given pretopology J on S .

In these notes I show that given a 2-category K equipped with a (strict) singleton pretopology J whose elements are fully faithful and co-fully faithful arrows, one can construct an analogue K_J of the bicategory $\mathbf{Cat}'_{ana}(S, J)$. The 1-arrows of K_J are formal 2-categorical versions of anafunctors, here dubbed *J-fractions*. The construction of K_J is elementary in the sense of only needing the first-order theory of 2-categories, and the construction is Choice-free. The original 2-category K is a wide and locally full sub-bicategory of K_J and the inclusion 2-functor $A_J: K \hookrightarrow K_J$ is a bicategorical localisation; this result uses Pronk's comparison theorem from [8], but it should be possible to prove directly using the construction given here.

The following quote from [13] should be kept in mind when reading the elementary calculations in these notes, as no such details have fully appeared in the literature, let alone at the level of generality here:

Nonetheless, it is interesting to note the prevalence of formulations leaving “to the reader” parts of the proofs of details of the localization constructions. . . . Another interesting reference is Pronk's paper on localization of 2-categories [21]¹, pointed out to me by I. Moerdijk. This paper constructs the localization of a 2-category by a subset of 1-morphisms satisfying a generalization of the right fraction condition. . . . the full set of details for the coherence relations on the level of 2-cells is still too much, so the paper ends with:

[21 , p. 302:] “It is left to the reader to verify that the above defined isomorphisms a , l and r are natural in their arguments and satisfy the identity coherence axioms.”

One pleasant feature of the current approach, at least for the author, is that one could take the opposite 2-category everywhere in the current notes and everything will still work fine, only exchanging pullbacks for pushouts everywhere. In this way, one could also localise suitable 2-categories using *cospans*, rather than spans, for instance 2-categories whose objects are more

¹ [8] in the References below.

algebraic in nature, rather than geometric, like Hopf algebroids. It is not clear that for alternative presentations of the localisation, say one using of left-principal bibundles, such an approach would still work, or what should play the rôle of the 1-arrows when one is working with a general 2-category and not a 2-category of structured or internal groupoids.

Remark 1.1. This article has had a long and tortuous history. It started out as the second half of what was published as [12], in an attempt to give an elementary and completely self-contained proof omitting no details of the results of [8] dealing with 2-categories of stacks as bicategorical localisations. In this the author was partially influenced by the late Vladimir Voevodsky's insistence on details and constructions in what might otherwise seem merely bureaucratic proofs. The appearance of [9] gave a much more satisfactory (and conceptual) proof of these results, after which the author lost hope that there was any merit in the current naive approach. However, work in preparation using cospans to localise 2-categories of topological groupoids required the machinery of the present paper, in its dual incarnation, so it is hoped there is some merit in the elementary approach in it.

2 Preliminaries

We refer to [3, Chapter 2] for background on bicategories.

Definition 2.1. If \mathcal{P} is a property of functors, then we say that a 1-arrow $f: x \rightarrow y$ in a 2-category K is *representably* \mathcal{P} (or just \mathcal{P}) if for all objects z of K we have that $f_*: K(z, x) \rightarrow K(z, y)$ has property \mathcal{P} .

The most important case for the present paper is the property ‘fully faithful’, and we will abbreviate ‘representably fully faithful’ to ff , and will denote by ff the class of ff 1-arrows in a 2-category. Note however that Definition 2.1 can be rewritten as a first-order property of a 1-arrow in a 2-category (and even in a bicategory).

Definition 2.2. A 1-arrow $f: x \rightarrow y$ in a bicategory is ff if for all $g, h: w \rightarrow x$ and $\tilde{a}: f \circ g \Rightarrow f \circ h$ there is a unique $a: g \Rightarrow h$ such that $\tilde{a} = 1_f \circ a$.

Example 2.3. Any equivalence in a 2-category is ff ; it is a neat exercise to directly construct the required 2-arrow a . It is less easy to do the analogous

construction for an equivalence in a bicategory, showing that such a 1-arrow is ff in an elementary fashion, but still possible.²

Lemma 2.4. *If $f: y \rightarrow z \in \text{ff}$ and $g: x \rightarrow y$ is any other arrow then $g \in \text{ff}$ if and only if $f \circ g \in \text{ff}$. If $h: y \rightarrow z$ is another arrow that is isomorphic to f in $K(y, z)$ then $h \in \text{ff}$. Moreover, if f is isomorphic to $f': y' \rightarrow z'$ (in the arrow 2-category) then $f' \in \text{ff}$.*

The following lemma will be a major workhorse in the construction below.

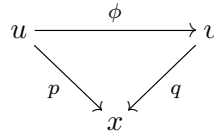
Lemma 2.5. *Let $p: u \rightarrow a$ be an ff 1-arrow in a 2-category K . Then for any 1-arrow $f: b \rightarrow a$ and any two lifts $k, l: b \rightarrow u$ of f through p , there is a unique 2-arrow $k \Rightarrow l$ lifting the identity 2-arrow on f through p .*

The proof of this lemma follows almost immediately from the definition of ff arrows.

Example 2.6. Given an ff 1-arrow $p: u \rightarrow x$ in K such that the strict pullback $u \times_x u$ exists, then there is a unique 2-arrow $\ell_p: \text{pr}_1 \Rightarrow \text{pr}_2: u \times_x u \rightarrow u$ lifting $\text{id}: p \Rightarrow p$.

In this paper, all pullbacks are likewise **strict**.

Example 2.7. Consider a commutative triangle



with $q: v \rightarrow x$ ff. Then assuming the relevant strict pullbacks exist, there is an equality

$$u \times_x u \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \Downarrow \ell_p \\ \xrightarrow{\text{pr}_2} \end{array} u \xrightarrow{\phi} v \quad = \quad u \times_x u \xrightarrow{\phi \times \phi} v \times_x v \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \Downarrow \ell_q \\ \xrightarrow{\text{pr}_2} \end{array} v$$

between the pasted 2-cells, as the source and target 1-arrows all lift $u \times_x u \rightarrow x$ through q .

²In fact the computation shows that a 1-arrow with a representably faithful pseudo-retract is ff, but we don't need this level of generality.

Example 2.8. A more complicated example is the equality of pasted 2-cells in

$$\begin{array}{c}
 \begin{array}{ccc}
 & & u \times_x u \\
 & \text{pr}_{12} \curvearrowright & \\
 u \times_x u & \xrightarrow{\Delta \times \text{id}_u} & u \times_x u \times_x u \\
 & \text{pr}_{23} \curvearrowleft & \\
 & & u \times_x x \\
 & & \text{pr}_2
 \end{array}
 \end{array}
 =
 \begin{array}{c}
 \begin{array}{ccc}
 & & u \\
 & \text{pr}_1 \curvearrowright & \\
 u \times_x u & \xrightarrow{\quad} & u \\
 & \text{pr}_2 \curvearrowleft & \\
 & & u
 \end{array}
 \end{array}$$

The structure of a site on a 2-category is not a common notion so we need to specify what we mean. There are at least two different ways to describe this in the 1-categorical case, namely using sieves and using pretopologies, and it is not clear a priori that they generalise to the same thing for 2-categories. Our definition will be as follows, as this paper only deals with unary sites.

Definition 2.9. A *singleton strict pretopology* on a 2-category K is a class J of 1-arrows which contains all identity arrows, is closed under composition and the strict pullback an element of J exists and is again in J . We will assume that *specified* strict pullbacks are given—rather than merely assuming they exist—and that the pullback of an identity 1-arrow is again an identity 1-arrow.

Since this is the same thing as a singleton pretopology on the 1-category underlying the 2-category, we refrain from placing the prefix ‘2-’ in the name. If one merely asks for existence of pullbacks, then one may use a global axiom of choice to make the pullback of a cover an operation.

Example 2.10. Let K be a 2-category which admits specified strict pullbacks. Then ff is a singleton strict pretopology.

This is in some sense a degenerate example. The following is more of interest.

Example 2.11. Let S be a finitely complete category with specified limits and J_0 a singleton pretopology on S . Then we have the 2-categories $\mathbf{Cat}(S)$ and $\mathbf{Gpd}(S)$ of internal categories and groupoids. Let J denote the class of internal functors in either of those 2-categories whose object component is

an arrow in J_0 . Then J is a singleton strict pretopology on both $\mathbf{Cat}(S)$ and $\mathbf{Gpd}(S)$.

In addition, we need to consider a 2-categorical version somewhat analogous to subcanonicity, and here we cannot avoid involving the 2-arrows. This makes the notion essentially 2-categorical, and not just a structure on the underlying 1-category as is the case for Definition 2.9.³

Definition 2.12. Given a singleton strict pretopology J , we will call it *bi-ff* if every $j: u \rightarrow x$ in J is ff and also co-fully faithful (co-ff): for all $g, h: x \rightarrow y$ and $\tilde{a}: g \circ j \Rightarrow h \circ j$ there is a unique $a: g \Rightarrow h$ such that $\tilde{a} = 1_j \circ a$.

In this paper we do not need to descend 1-arrows down a cover in the pretopology (which is a consequence of representable presheaves being sheaves), but only 2-arrows, so the weaker notion of ff + co-ff is sufficient. We will not here dwell on how this relates to 2-dimensional sheaf theory à la Street [14].

Example 2.13. Continuing Example 2.11, if we additionally assume the pretopology J_0 consists of regular epimorphisms (hence is a subcanonical pretopology on S), then J is a bi-ff singleton strict pretopology on $\mathbf{Cat}(S)$ and on $\mathbf{Gpd}(S)$. Indeed, the arrows in this pretopology are also regular epimorphisms, though we do not need this here.

This example partly recovers the examples that were used in [10, Section.8]; variants on this definition will give all examples from *loc. cit.*

The main object of study of this paper are 2-categories K with a choice of bi-ff singleton strict pretopology J . We shall just refer to these as **2-sites** for brevity, though properly speaking it is a very special case of this notion.

As a consequence of our definition of 2-site, we don't just get descent of 2-arrows along covers, but along maps between covers.

Lemma 2.14. *Given a 2-site (K, J) , and a diagram*

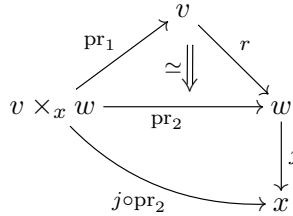
$$\begin{array}{ccc} v & \xrightarrow{r} & w \\ & \searrow k & \swarrow j \\ & & x \end{array}$$

where $k, j \in J$, then r is also co-ff.

³For the sense in which this deserves to be considered a type of subcanonicity, cf [2, Lemma 52], which states that an ff regular epimorphism is also co-ff.

The following proof, simplifying the author’s, is due to the anonymous referee.

Proof. Recall that r being co-ff means that given a 2-arrow $\bar{a}: f \circ r \Rightarrow g \circ r$, there is a unique 2-arrow $a: f \Rightarrow g$ such that $a \circ \text{id}_r = \bar{a}$. From our definition of 2-site, j and k are both ff and co-ff. This also implies that the two projection maps pr_1, pr_2 in the next diagram are in \mathbf{J} , hence are both co-ff. Since $j \circ \text{pr}_2 = j \circ r \circ \text{pr}_2$ and j is ff, there is a unique lift of $\text{id}_{j \circ \text{pr}_2}$ to an invertible 2-arrow $r \circ \text{pr}_1 \xrightarrow{\cong} \text{pr}_2$:



But pr_2 is co-ff, implying $r \circ \text{pr}_1$ is co-ff, and since pr_1 is co-ff, then so is r (applying two of the cases of Lemma 2.4 in K^{op}). □

3 The bicategory of J-fractions

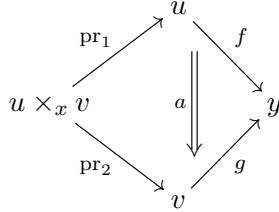
We are aiming to localise a 2-category, and in time-honoured tradition we shall call the arrows in the localised 2-category fractions. Fractions are defined relative to a strict pretopology.

Definition 3.1. Let (K, \mathbf{J}) be a 2-site. A *J-fraction* is a span $x \xleftarrow{j} u \xrightarrow{f} y$ in K where $j \in \mathbf{J}$, to be denoted (j, f) .

For example, given any 1-arrow $f: x \rightarrow y$ in K , we have the fraction (id_x, f) . In particular, we have for any object a the *identity fraction*, which is $(\text{id}_x, \text{id}_x)$

Definition 3.2. Let $x \xleftarrow{j} u \xrightarrow{f} y$ and $x \xleftarrow{k} v \xrightarrow{g} y$ be a pair of J-fractions in

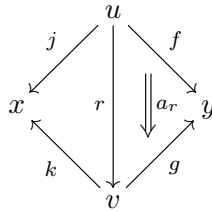
K . A map of J-fractions $(j, f) \Rightarrow (k, g)$ consists of a 2-arrow



Sometimes we will also write the 1-arrow $x \leftarrow u \times_x v$ in such a diagram for emphasis, so that the 0-source object is clear.⁴

There are certain maps of fractions which are easier to describe and to compose, and the coherence maps of the bicategory we are going to define all turn out to be examples, so we shall spend some time detailing these.

Definition 3.3. A renaming map⁵ r from the fraction (j, f) to the fraction (k, g) is a map of spans in a 2-category of the form:



We can compose renaming maps and so get a category $K_J^R(x, y)$ with objects the fractions from x to y and arrows the renaming maps.

As we shall see, we will also have a category $K_J(x, y)$ with objects the J-fractions and arrows the maps of fractions, and a functor including $K_J^R(x, y)$ into this latter category. For now we will be content with giving the definition of the arrow component of this functor, without proving functoriality;

⁴Note that, as presented, this 2-arrow alone does not allow us to reconstruct its (1-)source and (1-)target; we take the source and target as implicitly part of the data (cf the definition of arrow in the category of ZFC-sets).

⁵The ‘renaming transformations’ of [7, Section.1] in the case when $K = \mathbf{Cat}$ are a special case of the notion here. Makkai requires that r is invertible and a_r is the identity 2-arrow.

namely, a renaming map r from (j, f) to (k, g) as above is sent to the map $\iota(r)$ of fractions specified by the 2-arrow

$$\begin{array}{ccc}
 & & u \\
 & \nearrow \text{pr}_1 & \searrow f \\
 u \times_x v & \Downarrow & y \\
 & \searrow \text{pr}_2 & \nearrow g \\
 & & v
 \end{array}
 \quad , \quad (3.1)$$

where the 2-arrow on the left is the canonical lift of the identity 2-arrow $k \circ r \circ \text{pr}_1 = k \circ \text{pr}_2$ through the ff arrow k , using Lemma 2.5.

Definition 3.4. The identity map $1: (j, f) \Rightarrow (j, f)$ on a J -fraction $x \xleftarrow{j} u \xrightarrow{f} y$ is given by $\iota(\text{id}_u)$.

The (vertical) composition of maps of J -fractions proceeds as follows. Given

$$\begin{aligned}
 t_1 &: (j_1, f_1) \Rightarrow (j_2, f_2) \\
 t_2 &: (j_2, f_2) \Rightarrow (j_3, f_3)
 \end{aligned}$$

where $x \xleftarrow{j_i} u_i \xrightarrow{f_i} b$, consider the 2-arrow $t_1 \oplus t_2$ filling the diagram⁶

$$\begin{array}{ccccc}
 & & u_1 \times_x u_2 & \longrightarrow & u_1 \\
 & \nearrow & \searrow & \Downarrow t_1 & \searrow f_1 \\
 & & & & u_2 \xrightarrow{f_2} y \\
 u_1 \times_x u_2 \times_x u_3 & & & \Downarrow t_2 & \nearrow f_3 \\
 & \searrow & \nearrow & & \\
 & & u_2 \times_x u_3 & \longrightarrow & u_3
 \end{array}
 \quad , \quad (3.2)$$

which we shall call the *precomposition* of t_1 and t_2 . We need to show that this 2-arrow descends along the arrow $u_1 \times_x u_2 \times_x u_3 \xrightarrow{\text{pr}_{13}} u_1 \times_x u_3 \in \mathbf{J}$.

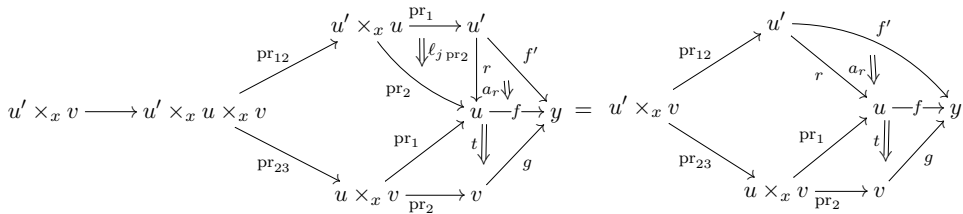
⁶One might be concerned with the bracketing of the triple pullback here; for concreteness we can take $(u_1 \times_x u_2) \times_x u_3$, it would not change the final result if we used the other option.

But pr_{13} is co-ff, and the source and target of $t_1 \oplus t_2$ factor as $u_1 \times_x u_2 \times_x u_3 \xrightarrow{\text{pr}_{13}} u_1 \times_x u_3 \rightarrow u_i \xrightarrow{f_i} y$ for $i = 1$ and $i = 3$ respectively. Thus $t_1 \oplus t_2$ descends uniquely, and we call this descended 2-arrow $t_1 + t_2$ (note that $+$ is *not* a commutative operation!), and it gives a map of J-fractions $(u_1, f_1) \Rightarrow (u_3, f_3)$.

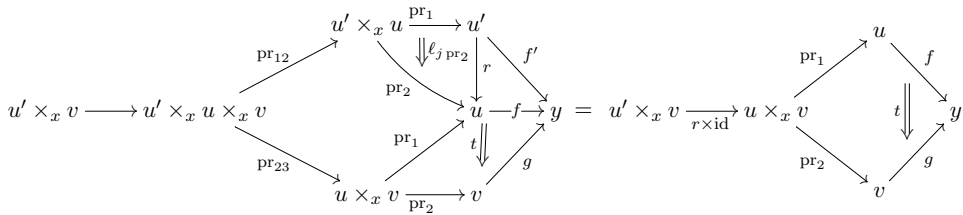
Remark 3.5. If $u_1 \times_x u_2 \times_x u_3 \rightarrow u_1 \times_x u_3$ has a section, then the vertical composition $t_1 + t_2$ is the whiskering of $t_1 \oplus t_2$ on the left with this section.

An example of such a section arises when composing two maps of J-fractions, where one of the maps arises from an invertible renaming map. Here we say a renaming map with data (r, a_r) as in Definition 3.3 is *invertible* if r is an invertible 1-arrow and a_r is an invertible 2-arrow. We record a special case of this as a lemma for future reference.

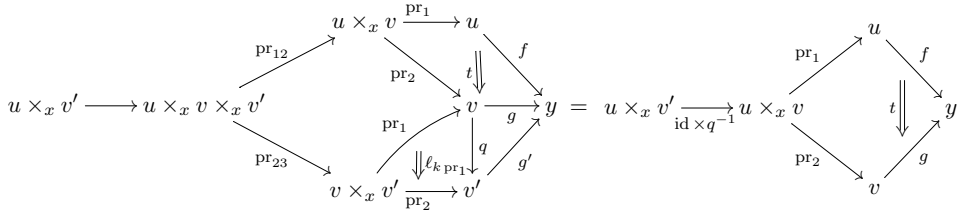
Lemma 3.6. Let $x \xleftarrow{j} u \xrightarrow{f} y$ and $x \xleftarrow{k} v \xrightarrow{g} y$ be J-fractions, $t: (j, f) \Rightarrow (k, g)$ be a map of fractions, and let $r: u' \rightarrow u$ be a renaming map from $x \xleftarrow{j'} u' \xrightarrow{f'} y$ to (j, f) , with $j' = jr$ and $a_r: f' \Rightarrow fr$. Then $\iota(r) + t$ is given by the 2-arrow



Hence if $a_r = \text{id}$, $\iota(r) + t$ is given by the 2-arrow



Further, given an invertible renaming map q from (k, g) to $x \xleftarrow{k'} v' \xrightarrow{g'} y$, with $k = k'q$ and $g = g'q$ (hence $a_q = \text{id}$), the composite $t + \iota(q)$ is given by the 2-arrow



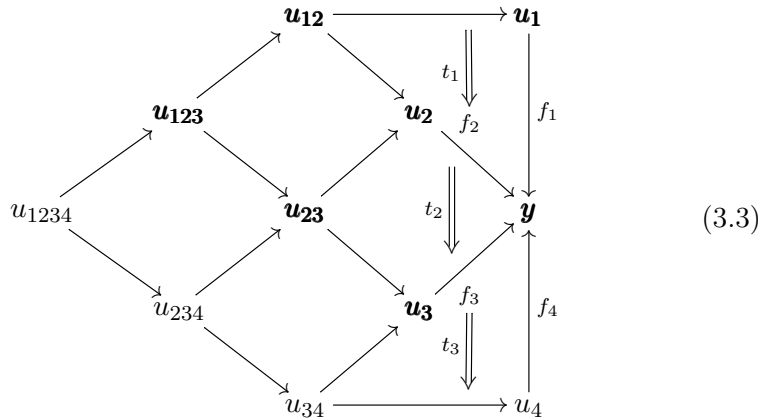
Corollary 3.7. *The assignment ι is a functor $K_J^R(x, y) \rightarrow K_J(x, y)$.*

Proof. Apply the first case in Lemma 3.6 to when $t = \iota(r')$ for an arbitrary renaming transformation q and use Lemma 2.5. \square

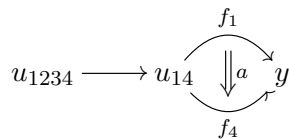
Proposition 3.8. *We have a category $K_J(x, y)$ with objects the J -fractions from x to y and arrows the maps of J -fractions.*

Proof. That $1_{(j,f)}$ is the identity arrow for $x \xleftarrow{j} u \xrightarrow{f} y$ follows from the second and third cases of Lemma 3.6, taking $r = \text{id}_u$ and $q = \text{id}_v$ respectively.

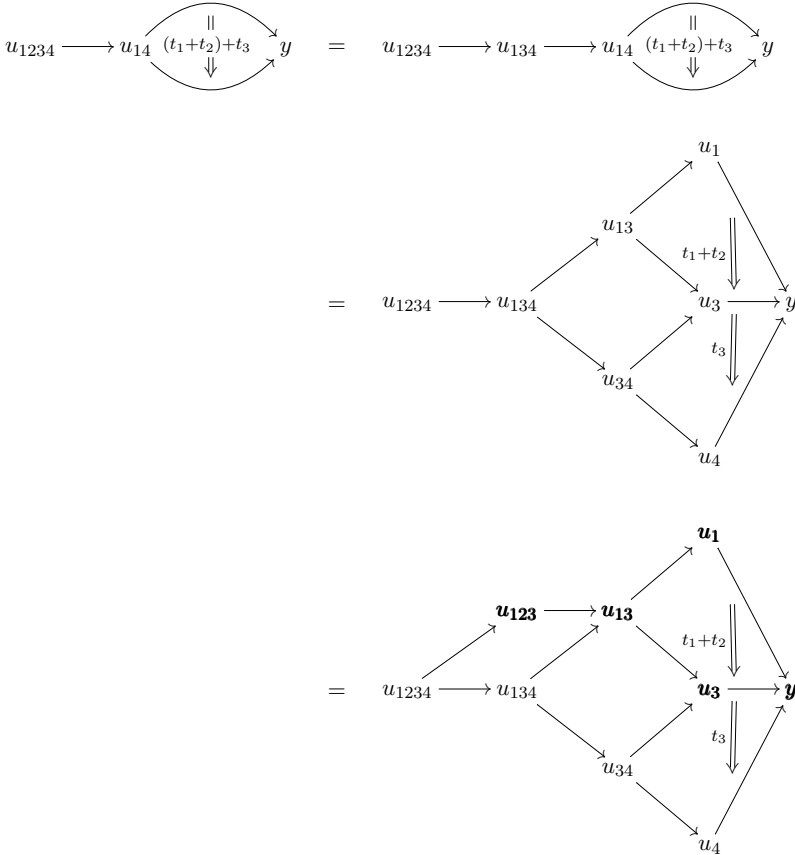
We thus need only to show composition is associative. Consider the diagram



where $u_{i\dots k} := u_i \times_x \dots \times_x u_k$. The bold objects form a sub-diagram we will refer to below. We will show that the composites



are equal to (3.3) for $a = (t_1 + t_2) + t_3$ and $a = t_1 + (t_2 + t_3)$. First consider $(t_1 + t_2) + t_3$:

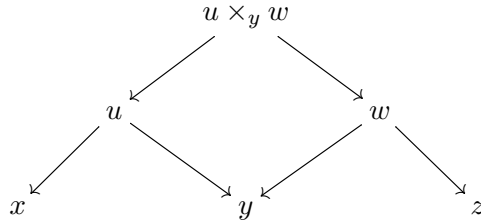


Omitting some of the labels on the 1-arrows for clarity. Now the whiskered 2-arrow in the subdiagram on the bold symbols above is equal to the composite 2-arrow in the subdiagram of (3.3) on the bold symbols, hence the whole diagram equals (3.3). A symmetric argument shows that $t_1 + (t_2 + t_3) \circ 1_{u_{1234} \rightarrow u_{14}}$ is also equal to (3.3). By uniqueness of descent, composition of maps of J-fractions is associative, and $K_J(x, y)$ is a category. \square

3.1 Defining the bicategory K_J Now we want to show that $K_J(x, y)$ is the hom-category of a bicategory, so we need a composition functor. Com-

posing 1-arrows is easy:

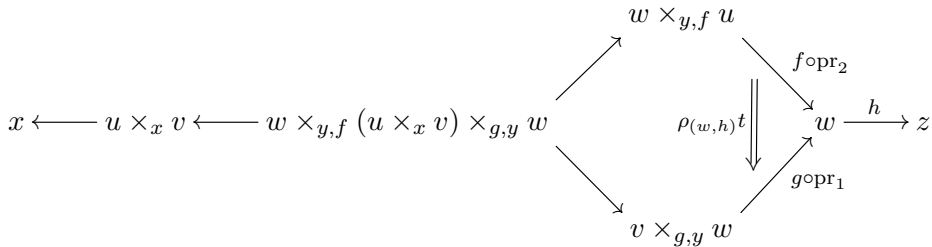
Definition 3.9. The composition of J-fractions is the composite span



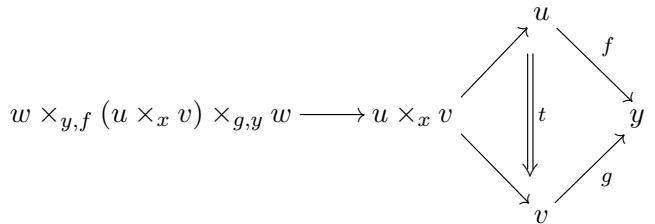
where recall we are assuming we have specified pullbacks of 1-arrows in \mathbf{J} , so this is well-defined.

We shall define the composition in the bicategory $K_{\mathbf{J}}$ by defining left and right whiskering functors and proving the interchange law as outlined in [7, pp 126-127]⁷ for the case where $K = \mathbf{Cat}$ and \mathbf{J} is the class of fully faithful, surjective-on-objects functors.

Definition 3.10. Let t be a map of fractions from $x \leftarrow u \xrightarrow{f} y$ to $x \leftarrow v \xrightarrow{g} y$. The *right whiskering* of t by the J-fraction $y \leftarrow w \xrightarrow{h} z$ is given by



where the 2-arrow $\rho_{(w,h)} t: \text{pr}_1 \Rightarrow \text{pr}_4$ is the unique lift through $l: w \rightarrow y$ of



⁷Makkai says, helpfully, “Next, we need to verify that thus we have defined functors ... we leave the task to the reader.” [*ibid.* page 127]

Proposition 3.11. *Right whiskering with $y \xleftarrow{l} w \xrightarrow{h} z$ is a functor $K_J(x, y) \rightarrow K_J(x, z)$.*

Proof. First, let us show right whiskering preserves identity 2-arrows. That is, the horizontal composition of a pair of identity 2-arrows is the identity 2-arrow of the composition of the 1-arrows. Let $x \xleftarrow{j} u \xrightarrow{f} y$ be a fraction and consider the right whiskering of the map $\text{id}_{(j,f)}$ by (l, h) . This is the map of fractions given by

$$x \longleftarrow w \times_y u^{[2]} \times_y w \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \Downarrow \\ \xrightarrow{\text{pr}_4} \end{array} w \xrightarrow{h} z \tag{3.4}$$

where the 2-arrow is the unique lift of

$$w \times_y u^{[2]} \times_y w \longrightarrow u^{[2]} \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \Downarrow \\ \xrightarrow{\text{pr}_2} \end{array} u \xrightarrow{f} y ,$$

the unlabeled maps being the obvious projections. But we have the equality

$$w \times_y u^{[2]} \times_y w \begin{array}{c} \xrightarrow{\text{pr}_1} \\ \Downarrow \\ \xrightarrow{\text{pr}_4} \end{array} w = w \times_y u^{[2]} \times_y w \begin{array}{c} \xrightarrow{\text{pr}_{12}} \\ \Downarrow \\ \xrightarrow{\text{pr}_{34}} \end{array} u \times_x y \xrightarrow{f \circ \text{pr}_1} w$$

hence (3.4) is

$$x \longleftarrow w \times_y u^{[2]} \times_y w \begin{array}{c} \xrightarrow{\text{pr}_{12}} \\ \Downarrow \\ \xrightarrow{\text{pr}_{34}} \end{array} u \times_y w \xrightarrow{h \circ f \circ \text{pr}_1} z = \text{id}_{(l,h) \circ (j,f)} .$$

Thus whiskering is unital.

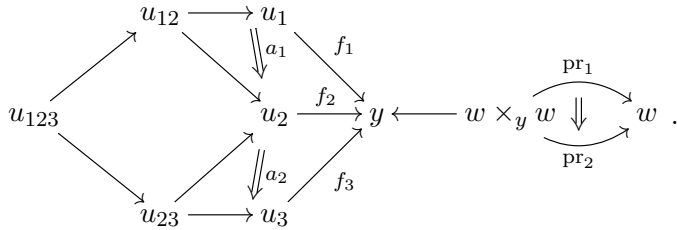
Now to prove that right whiskering preserves composition we will again use uniqueness of descent, and prove equal a pair of 2-arrows with 0-source a cover of the 0-source of the 2-arrows we are interested in. Without loss of generality, we can right whisker by the fraction $(l, \text{id}) = y \xleftarrow{l} w \xrightarrow{\text{id}} w$,

as the component of the fraction pointing in the forward direction plays no substantial rôle in what is to follow.

Consider the composable pair of maps of fractions given by the data

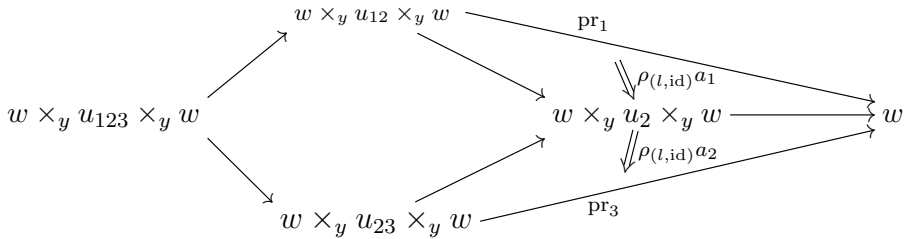
$$u_1 \times_x u_2 \begin{array}{c} \xrightarrow{f_1 \circ \text{pr}_1} \\ \Downarrow a_1 \\ \xrightarrow{f_2 \circ \text{pr}_2} \end{array} y \quad \text{and} \quad u_2 \times_x u_3 \begin{array}{c} \xrightarrow{f_2 \circ \text{pr}_1} \\ \Downarrow a_2 \\ \xrightarrow{f_3 \circ \text{pr}_2} \end{array} y .$$

Let $u_{123} := u_1 \times_x u_2 \times_x u_3$ and similarly for u_{12} , u_{23} , and consider the diagram



We need to prove equal the pair of 2-arrows $(\rho_{(l,\text{id})}a_1) + (\rho_{(l,\text{id})}a_2)$ and $\rho_{(l,\text{id})}(a_1 + a_2)$ between the two 1-arrows $(w \times_y u_1) \times_x (u_3 \times_y w) \simeq w \times_y u_{13} \times_y w \xrightarrow{\text{pr}_i} w$, for $i = 1, 3$.

In Figure 1 the sub-diagram consisting of just the solid arrows together with the 2-arrows between them 2-commutes, so the precomposition $(\rho_{(l,\text{id})}a_1) \oplus (\rho_{(l,\text{id})}a_2)$ is given by the top layer of the diagram, namely



and $(\rho_{(l,\text{id})}a_1) + (\rho_{(l,\text{id})}a_2)$ is given by the unique descent of this 2-arrow along p . The 2-arrow marked $(*)$ is the whiskering $\rho_{(l,\text{id})}(a_1 + a_2)$, and forms a 2-commuting diagram with $a_1 + a_2$ and the 1-arrows $w \times_y u_{13} \times_y w \rightarrow u_{13}$

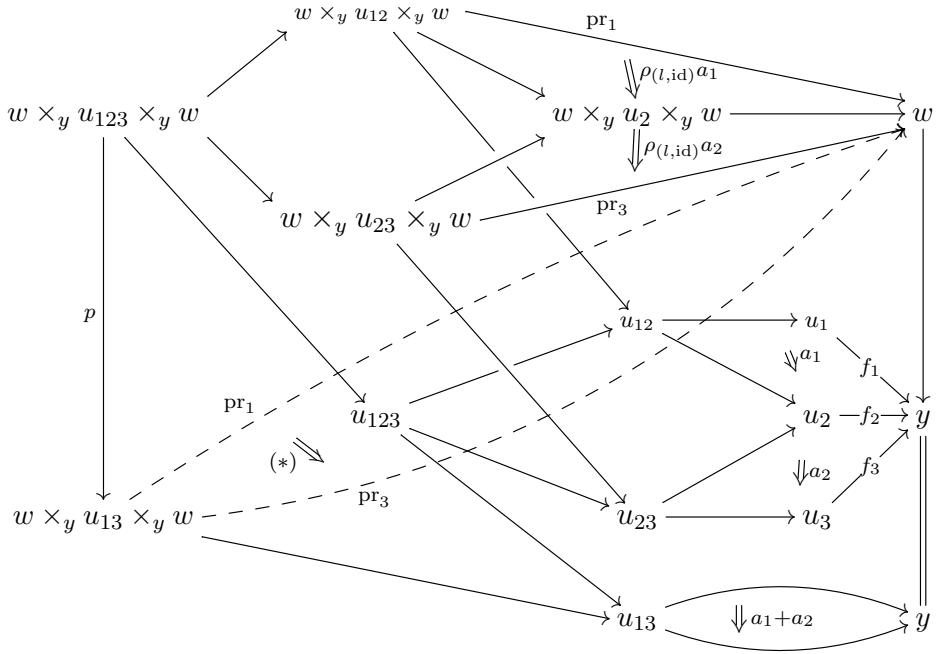


Figure 1: Right whiskering is functorial

and $w \rightarrow y$. The 2-cell

$$w \times_y u_{123} \times_y w \longrightarrow w \times_y u_{13} \times_y w \begin{array}{c} \text{pr}_1 \\ \Downarrow (*) \\ \text{pr}_3 \end{array} w$$

in Figure 1 is, by uniqueness of lifts through p and $w \rightarrow y$ (both in \mathcal{J}) equal to $(\rho_{(l,\text{id})}a_1) \oplus (\rho_{(l,\text{id})}a_2)$. Thus the descent of $(\rho_{(l,\text{id})}a_1) \oplus (\rho_{(l,\text{id})}a_2)$ along p is just $\rho_{(l,\text{id})}(a_1 + a_2)$, which is what we needed to prove. \square

The definition of the left whiskering is slightly more complicated, as it is such that it doesn't permit us to nearly ignore half of the span as we can for right whiskering. What we shall do is define left whiskering

by a general J-fraction $x \xleftarrow{j} u \xrightarrow{f} y$ in two cases, using the factorisation $(j, f) = (\text{id}_u, f) \circ (j, \text{id}_u)$.

3.1.1 Case I: left whiskering by (id_u, f)

Let a be a map of fractions from $y \leftarrow v_1 \xrightarrow{g} z$ to $y \leftarrow v_2 \xrightarrow{h} z$, and $f: u \rightarrow y$ an arrow in K . The whiskered 2-arrow will be a map of fractions from $u \xleftarrow{\text{pr}_1} u \times_y v_1 \xrightarrow{g \circ \text{pr}_2} z$ to $u \xleftarrow{\text{pr}_1} u \times_y v_2 \xrightarrow{h \circ \text{pr}_2} z$, and so the desired 2-arrow in K will be of the form

$$u \times_y v_{12} \begin{array}{c} \xrightarrow{g \circ \text{pr}_1} \\ \Downarrow a \\ \xrightarrow{h \circ \text{pr}_2} \end{array} z$$

where $v_{12} := v_1 \times_y v_2$.

Definition 3.12. The left whiskering of the map a of J-fractions by the fraction $u \xleftarrow{\text{id}_u} u \xrightarrow{f} y$ is given by the 2-arrow $\lambda_{(\text{id}_u, f)}^I a$, defined as

$$u \times_y v_{12} \longrightarrow v_{12} \begin{array}{c} \xrightarrow{\quad} \\ \Downarrow a \\ \xrightarrow{\quad} \end{array} z .$$

3.1.2 Case II: left whiskering by (j, id_u)

We have $v_i \rightarrow y = u$ J-covers for $i = 1, 2$, and now here $v_{12} := v_1 \times_u v_2$. We also let $V_{12} := v_1 \times_x v_2$, and there is a canonical map $v_{12} \rightarrow V_{12}$ fitting into a commutative diagram

$$\begin{array}{ccc} v_{12} & \longrightarrow & V_{12} \\ \downarrow & & \downarrow \\ u & \xrightarrow{j} & x \end{array}$$

where the left, bottom and right arrows are all in J. Thus from Lemma 2.14 we have that the top arrow is co-ff. Notice also that there is a trivial factorisation of $\text{pr}_i: v_{12} \rightarrow v_i$ as $v_{12} \rightarrow V_{12} \xrightarrow{\text{pr}_i} v_i$.

Definition 3.13. The left whiskering of the map a by $x \xleftarrow{j} u \xrightarrow{\text{id}} u$ is given by the 2-arrow $\lambda_{(j, \text{id})}^{II} a$ in K defined via unique descent along the co-ff arrow $v_{12} \times_x V_{12} \rightarrow V_{12}$ by the equation

$$\begin{array}{c}
 v_{12} \times_x V_{12} \xrightarrow{\text{pr}_2} V_{12} \xrightarrow{\text{pr}_2} z \\
 \begin{array}{c}
 \text{gopr}_1 \\
 \curvearrowright \\
 \lambda_{(j,\text{id})}^{II} \Downarrow a \\
 \curvearrowleft \\
 \text{hopr}_2
 \end{array}
 \end{array}
 =
 \begin{array}{c}
 v_{12} \times_x V_{12} \xrightarrow{\text{pr}_2} V_{12} \xrightarrow{\text{pr}_2} z \\
 \begin{array}{c}
 \text{pr}_2 \searrow \quad \nearrow \text{pr}_2 \\
 \Downarrow \quad \Downarrow \\
 v_{12} \xrightarrow{\text{pr}_2} z \\
 \Downarrow \quad \Downarrow \\
 V_{12} \xrightarrow{\text{pr}_2} z \\
 \text{gopr}_1 \searrow \quad \nearrow \text{hopr}_2 \\
 \Downarrow a
 \end{array}
 \end{array}
 .$$

Left whiskering by an arbitrary fraction $x \xleftarrow{j} u \xrightarrow{f} y$ will then be the composite of the two (putative) functors given by cases I and II.

Proposition 3.14. *Left whiskering with $x \xleftarrow{j} u \xrightarrow{f} y$ is a functor $K_J(y, z) \rightarrow K_J(x, z)$.*

The proof that left whiskering preserves (vertical) composition will be deferred to appendix A, as it is a sizable calculation.

Proof. (Left whiskering is unital) We want to do the whiskering

$$x \xleftarrow{j} u \xrightarrow{f} y \longleftarrow v \times_y v \begin{array}{c} \curvearrowright \\ \Downarrow a \\ \curvearrowleft \end{array} v \xrightarrow{g} z .$$

Note that without loss of generality we can assume $g = \text{id}_v$, the general case follows exactly the same argument merely with g right whiskered onto all the 2-cells involved. We treat case I and case II of the definition of left whiskering separately.

Case I. Note that v_{12} in this case is $v \times_y v$. The left whiskering of the identity map on $y \leftarrow v \xrightarrow{\text{id}} v$ has 2-cell component

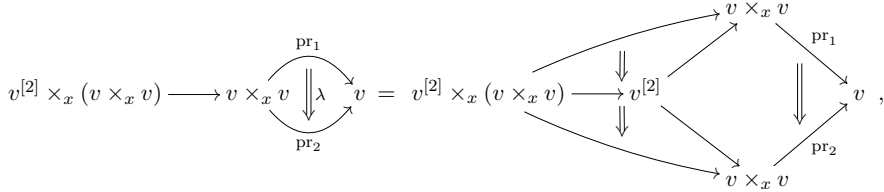
$$u \times_y (v \times_y v) \longrightarrow v \times_y v \begin{array}{c} \curvearrowright \\ \Downarrow a \\ \curvearrowleft \end{array} v$$

but $u \times_y (v \times_y v) \simeq (u \times_y v) \times_u (u \times_y v)$, and by Lemma 2.5 this is equal to

$$(u \times_y v) \times_u (u \times_y v) \begin{array}{c} \curvearrowright \\ \Downarrow a \\ \curvearrowleft \end{array} u \times_y v \longrightarrow v$$

and this is the identity map on the composite $u \leftarrow u \times_y v \rightarrow v$, as required.

Case II. Again, in this case, $v_{12} = v \times_u v$, which for now will be denoted $v^{[2]}$ and $V_{12} = v \times_x v$. Recall that the 2-cell component of the whiskered identity map will be the unique 2-cell $\lambda := \lambda_{(j,\text{id})}^{II} a$ in the diagram



which exists by descent along the \mathbf{J} -cover $v^{[2]} \times_x (v \times_x v) \rightarrow v \times_x v$. However, by Lemma 2.5 the canonical 2-arrow $\text{pr}_1 \Rightarrow \text{pr}_2 : v \times_x v \rightarrow v$ fits into such an equation of 2-arrows, and this is none other than the 2-cell component of the identity 2-arrow on the composite fraction $x \xleftarrow{j} v \xrightarrow{\text{id}} v$.

Putting case I and case II together, we have that left whiskering

$$\lambda_{(j,f)} : K_{\mathbf{J}}(y, z) \rightarrow K_{\mathbf{J}}(x, z)$$

preserves identity maps. □

With Propositions 3.14 and 3.11 we can define a composition functor.

Lemma 3.15. *Left and right whiskering fit together to give a functor*

$$K_{\mathbf{J}}(x, y) \times K_{\mathbf{J}}(y, z) \rightarrow K_{\mathbf{J}}(x, z).$$

Proof. The only thing that remains to check is that middle-four interchange holds, as per the hypothesis of [6, Proposition II.3.1]. This proof is deferred to Appendix C. □

In order for this to be the composition functor for a bicategory we just need to now show that it is coherently associative and unital. In fact, by virtue of Definition 2.9, this composition is *strictly* unital, since the composition of any fraction with the identity fraction of its source or target is unchanged. We define the left and right *unitors* to be the appropriate identity 2-arrows.

Lemma 3.16. *The unitors are natural.*

Proof. Recall that the unitors are meant to be natural transformations as in the diagram (of categories)

$$\begin{array}{ccc}
 K_J(x, y) \hookrightarrow K_J(x, y) \times K_J(y, y) & & K_J(x, y) \hookrightarrow K_J(x, x) \times K_J(x, y) \\
 \searrow \scriptstyle = & \swarrow \scriptstyle l_y & \searrow \scriptstyle = \\
 & K_J(x, y) & \\
 \end{array}
 \qquad
 \begin{array}{ccc}
 K_J(x, y) \hookrightarrow K_J(x, x) \times K_J(x, y) & & K_J(x, y) \hookrightarrow K_J(x, x) \times K_J(x, y) \\
 \searrow \scriptstyle = & \swarrow \scriptstyle r_x & \searrow \scriptstyle = \\
 & K_J(x, y) & \\
 \end{array}$$

Our unitors consist of identity arrows, so we need to prove that these diagrams commute *on the nose*. This is true at the level of objects, so we just need to check that the appropriate whiskerings of an arbitrary map of J-fractions (i.e. an arrow in $K_J(x, y)$) by identity fractions result in the original map of fractions.

In the case of r_x , we can apply Definition 3.12, whiskering by the fraction $(\text{id}_x, \text{id}_x)$. But then we just get the original map of J-spans. Thus the left triangle above commutes and r_x is natural.

In the case of l_y , we use Definition 3.10, with $y \xleftarrow{l} w \xrightarrow{h} z$ being $(\text{id}_y, \text{id}_y)$. But then $w \times_{y,f} (u \times_x v) \times_{g,y} w = u \times_x v$, and we are lifting through an identity arrow, and then whiskering (in K) with id_y . The result is then the original map of J-spans, making the right triangle above commute and so l_y is natural. \square

Definition 3.17. The *associator* for the 3-tuple of composable fractions

$$\begin{array}{ccccccc}
 & & u & & v & & w \\
 & j \swarrow & & f \searrow & k \swarrow & & g \searrow & l \swarrow & & h \searrow \\
 x_1 & & & & x_2 & & & & x_3 & & & & x_4
 \end{array}$$

is the invertible map $\iota(a_{uvw}) : [(l, h) \circ (k, g)] \circ (j, f) \Rightarrow (l, h) \circ [(k, g) \circ (j, f)]$ of J-fractions associated to the renaming map arising from the canonical isomorphism

$$a_{uvw} : u \times_{x_2} (v \times_{x_3} w) \xrightarrow{\cong} (u \times_{x_2} v) \times_{x_3} w$$

over x_1 , together the appropriate identity 2-arrow.

Lemma 3.18. *The associator 2-arrow in Definition 3.17 is natural.*

Proof. This is proved in Appendix B. □

We can check that the associator satisfies the necessary coherence diagrams in the bicategory of fractions and renaming maps, since it will then hold in the bicategory of fractions and maps of fractions. In fact, since the renaming map in question is the associator for products in the strict slice K/x_1 (i.e. strict pullbacks in K), it satisfies coherence by the universal property of pullbacks.

Remark 3.19. If we do not assume that pullbacks of identity arrows are again identity arrows, then we do get nontrivial unitors, but they are, like the associator, renaming maps, and one can check they are coherent.

We have thus proved:⁸

Proposition 3.20. *There is a bicategory K_J with the same objects as K , fractions as 1-arrows and maps of fractions as 2-arrows.*

We now define an identity-on-objects strict 2-functor $A_J: K \rightarrow K_J$ as follows. For a 1-arrow $f: x \rightarrow y$ of K , let $A_J(f)$ be the fraction $x \xleftarrow{\text{id}_x} x \xrightarrow{f} y$. Given a 2-arrow $a: f \Rightarrow g: x \rightarrow y$ in K , let $A_J(a)$ be the map of fractions $(\text{id}_x, f) \Rightarrow (\text{id}_x, g)$ determined by

$$x \times_x x = x \begin{array}{c} \xrightarrow{f} \\ \Downarrow a \\ \xrightarrow{g} \end{array} y ,$$

To check that A_J is a strict 2-functor, we need to check first that it is functorial for vertical composition of 2-arrows. In the definition of vertical composition of 2-cells, the diagram (3.2) in the case of maps of fractions in the image of A_J collapses as all objects u_i and their fibre products reduce to x , with all arrows between them identity arrows. The descended 2-arrow is then just the vertical composite in K , and so A_J preserves vertical composition. It is also simple to show that A_J preserves identity 2-arrows.

⁸cf Bartels, who says “The various coherence conditions in a (weak) 2-category are now tedious but straightforward to check.” [1]

Secondly, we need to show that A_J is functorial for horizontal composition. Identity 1-arrows are preserved strictly, as is composition of 1-arrows, so it is just a matter of checking that horizontal composition of 2-cells is preserved. Since horizontal composition is defined via left and right whiskering, we need to check that whiskering a map of fractions in the image of A_J by a fraction in the image of A_J is of the same form. The right whiskering of $A_J(a: f_1 \Rightarrow f_2)$ where $f_1, f_2: x \rightarrow y$ by $y \xleftarrow{\text{id}_y} y \xrightarrow{g} z$ involves a 2-cell $\rho_{(\text{id},g)}a$ (see Definition 3.10). Since our fractions are in the image of A_J , the diagram again collapses so that all appearances of $u \times_x v$ are equal to x , and $w = y$, so that $\rho_{(\text{id},g)}a = a$, and the final result has the 2-cell component the right whiskering of a by g . The left whiskering we need is case I, so we consider Definition 3.12. Consider the map of fractions $A_J(a: g_1 \Rightarrow g_2)$ where $g_1, g_2: y \rightarrow z$ and whisker it by $x \xleftarrow{\text{id}_x} x \xrightarrow{f} y$. Now in the definition of the 2-cell $\lambda_{(\text{id},f)}^I a$, we have $v_{12} = v_1 = v_2 = y$, the maps between them are identity maps, $u = x$, and $u \times_y v_{12} \rightarrow v_{12}$ is just f . Thus the whiskered map of fractions is again in the image of A_J , and we have proved that A_J is a strict 2-functor.

Lemma 3.21. *The 2-functor A_J is locally fully faithful, that is, $K(x, y) \rightarrow K_J(x, y)$ is fully faithful for all objects x and y of K .*

Proof. A map of J-fractions $(\text{id}_x, f) \Rightarrow (\text{id}_x, g)$ is precisely the same data as a 2-arrow $f \Rightarrow g$ in K . □

Definition 3.22. Given J , a 1-arrow in $q: x \rightarrow y$ in K is *J-locally split* if there is an arrow $j: u \rightarrow y$ in J and a diagram of the form

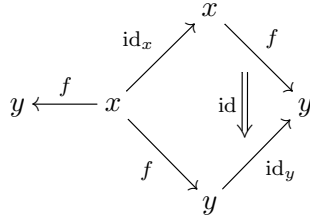
$$\begin{array}{ccc}
 & & x \\
 & \nearrow s & \\
 u & & \\
 & \searrow j & \\
 & & y
 \end{array}
 \begin{array}{c}
 \\
 \\
 \Downarrow \\
 \\
 q
 \end{array}$$

with the 2-arrow invertible. A 1-arrow in K is a *weak equivalence* if it is ff and J-locally split. Denote the class of weak equivalences by W_J .

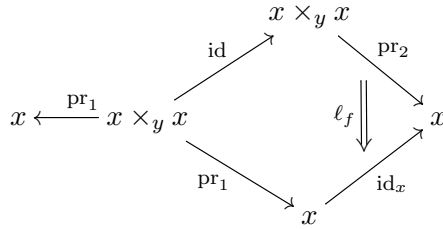
Clearly $J \subset W_J$ as we are assuming all arrows in J are ff, and every arrow in J is trivially J-locally split.

Proposition 3.23. *Let f be a 1-arrow of K . Then $A_J(f)$ an equivalence if and only if $f \in W_J$.*

Proof. First assume $f: x \rightarrow y$ is in \mathbf{J} ; we will show (id_x, f) is an equivalence, with quasi-inverse (f, id_x) . This is because $(\text{id}_x, f) \circ (f, \text{id}_x) = (f, f)$, which is isomorphic to $(\text{id}_y, \text{id}_y)$ by the invertible map of fractions



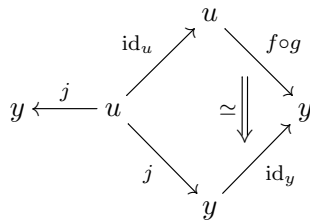
In the other direction, $(f, \text{id}_y) \circ (\text{id}_y, f) = (\text{pr}_1, \text{pr}_2)$, where $\text{pr}_i: x \times_y x \rightarrow x$, $i = 1, 2$, are the projections (both of which are in \mathbf{J}). There is the canonical invertible 2-cell $\ell_f: \text{pr}_1 \Rightarrow \text{pr}_2$, which gives an isomorphism of \mathbf{J} -fractions



In the other direction, let $f: x \rightarrow y$ be a 1-arrow in K such that (id_x, f) is an equivalence in $K_{\mathbf{J}}$, i.e. there is a \mathbf{J} -fraction $y \xleftarrow{j} u \xrightarrow{g} x$ such that

1. $(\text{id}_x, f) \circ (j, g) \xrightarrow{\sim} (\text{id}_y, \text{id}_y)$
2. $(\text{id}_x, \text{id}_x) \xrightarrow{\sim} (j, g) \circ (\text{id}_x, f)$

Point 1 implies that we have an isomorphism of \mathbf{J} -fractions



The right hand half of this diagram means that f is J -locally split. Since (id_x, f) is an equivalence it is ff in the bicategory K_J . Then as A_J is locally fully faithful it reflects ff 1-arrows, hence f is ff in K . Thus f is both J -locally split and ff, hence is in W_J . \square

For a number of diverse examples of weak equivalences in various categories of internal categories and groupoids, see [10, Section.8].

3.2 K_J as a localisation Given a 2-category (or bicategory) B with a class W of 1-arrows, we say that a 2-functor $Q: B \rightarrow \tilde{B}$ is a *localisation of B at W* if it sends the 1-arrows in W to equivalences in \tilde{B} and is universal with this property. This latter means that for any bicategory A precomposition with Q ,

$$Q^*: \mathbf{Bicat}(\tilde{B}, A) \rightarrow \mathbf{Bicat}_W(B, A),$$

is an equivalence of hom-bicategories, with \mathbf{Bicat}_W meaning the full sub-bicategory on those 2-functors sending arrows in W to equivalences.

Theorem 3.24. *A 2-site (K, J) admits a bicategory of fractions for W_J , and the inclusion 2-functor $A_J: K \rightarrow K_J$ is a localisation at the class W_J of weak equivalences.*

Proof. That (K, J) admits a bicategory of fractions for W_J is [12, Theorem 6] (the weaker hypotheses there on 2-sites are implied by the ones here). The proof that A_J is a localisation proceeds via Pronk's comparison theorem [8, Proposition 24], the conditions of which imply that the canonical 2-functor $K[W_J^{-1}] \rightarrow K_J$ is an equivalence of bicategories. Here $K[W_J^{-1}]$ is the bicategory of fractions constructed by Pronk, and we recall the conditions of the comparison theorem for ease of reference, using the current notation:

EF1. A_J is essentially surjective,

EF2. For every 1-arrow f of K_J there are 1-arrows $w \in W_J$ and g of K such that $A_J(g) \xrightarrow{\sim} f \circ A_J(w)$,

EF3. A_J is locally fully faithful.

We now show these conditions hold. To begin with, the 2-functor A_J sends weak equivalences to equivalences by Proposition 3.23.

EF1. A_J is the identity on objects, and hence surjective on objects.

EF2. This is equivalent to showing that for any J -fraction $x \xleftarrow{j} u \xrightarrow{f} y$ there are 1-arrows w, g in K such that w is in W_J and

$$(j, f) \xrightarrow{\sim} A_J(g) \circ \overline{A_J(w)}$$

where $\overline{A_J(w)}$ is some pseudoinverse for $A_J(w)$. We can take $w = j$ and $g = f$, since by the proof of Proposition 3.23, (j, id_u) is a pseudoinverse for (id_u, j) , and the composite fraction of (j, id_u) and (id_u, f) is just (j, f) .

EF3. This holds by Lemma 3.21.

Thus A_J is a localisation of K at W_J . □

As a last remark, one would like to know if the localisation of K at the weak equivalences is locally essentially small. This can be assured by the following result, where we have used the condition WISC from [10], which states that every object x of K has a set of covers that are weakly initial in the subcategory of K/x on the J -covers.

Proposition 3.25. *If the locally essentially small 2-site (K, J) satisfies WISC, then K_J is locally essentially small, and hence so is any localisation of K at W_J .*

Notice that local essential smallness is *not* automatic, as there are well-pointed toposes with a natural numbers object, otherwise very nice categories, for which the 2-category of internal categories fails the hypothesis of Proposition 3.25. For example the toposes of material sets in models of ZF as given by Gitik (see [15]) and Karagila [4], or the well-pointed topos of structural sets arising from [11]. Karagila has also described an explicit model of ZF in which the category of anafunctors from the discrete groupoid \mathbb{N} to the one-object groupoid $\mathbf{B}(\mathbb{Z}/2)$ is not essentially small.⁹

Finally, note that nothing in this paper relies on K being a $(2,1)$ -category, namely a 2-category with only invertible 2-arrows. This is usually assumed

⁹See the answers to the MathOverflow question <https://mathoverflow.net/q/264585/>. User ‘aws’ also gave a model of ZFA—choiceless Zermelo–Fraenkel set theory with atoms—with the same property.

for results subsumed by Theorem 3.24, but is unnecessary in the framework presented here. The following example will be treated in a forthcoming paper.

Example 3.26. Take a pretopos E with stable reflexive coequalisers, and define K to be the wide, locally full sub-2-category of $\mathbf{Cat}(E)^{op}$ taking only those internal functors whose object component is a coproduct inclusion, which we shall call a *complemented cofibration*. Let \mathbf{J} consist of the class W of complemented cofibrations $f: X \rightarrow Y$ that are ff and essentially surjective (i.e. $X_0 \times_{f_0, Y_0, s} Y_1 \xrightarrow{t \text{pr}_2} Y_0$ is a regular epimorphism). Then W consists of ff and co-ff arrows, contains all identity arrows, and is closed under composition and—most crucially—pushout. This makes $(\mathbf{Cat}(E)^{op}, W)$ a 2-site as defined in this paper, and the constructions here involving W -fractions in $\mathbf{Cat}(E)^{op}$ correspond to analogous dual constructions involving *left* W -fractions (certain cospans) in $\mathbf{Cat}(E)$. The analogous result holds for $\mathbf{Gpd}(E)$ in place of $\mathbf{Cat}(E)$, and in this case E can be an arbitrary pretopos.

This gives a bicategorical perspective on a generalisation of the case of small *groupoids*, studied in [5] using cofibration categories as a presentation of $(\infty, 1)$ -categories.

Appendices

A Proof that left whiskering in K_J preserves vertical composition

The definition of left whiskering in K_J is slightly more complicated, as it is such that it doesn't permit us to ignore half of the span as we can for right whiskering. Recall that we define left whiskering by a general J-fraction $x \xleftarrow{j} u \xrightarrow{f} y$ in two cases, using the factorisation $(j, f) = (\text{id}_u, f) \circ (j, \text{id}_u)$.

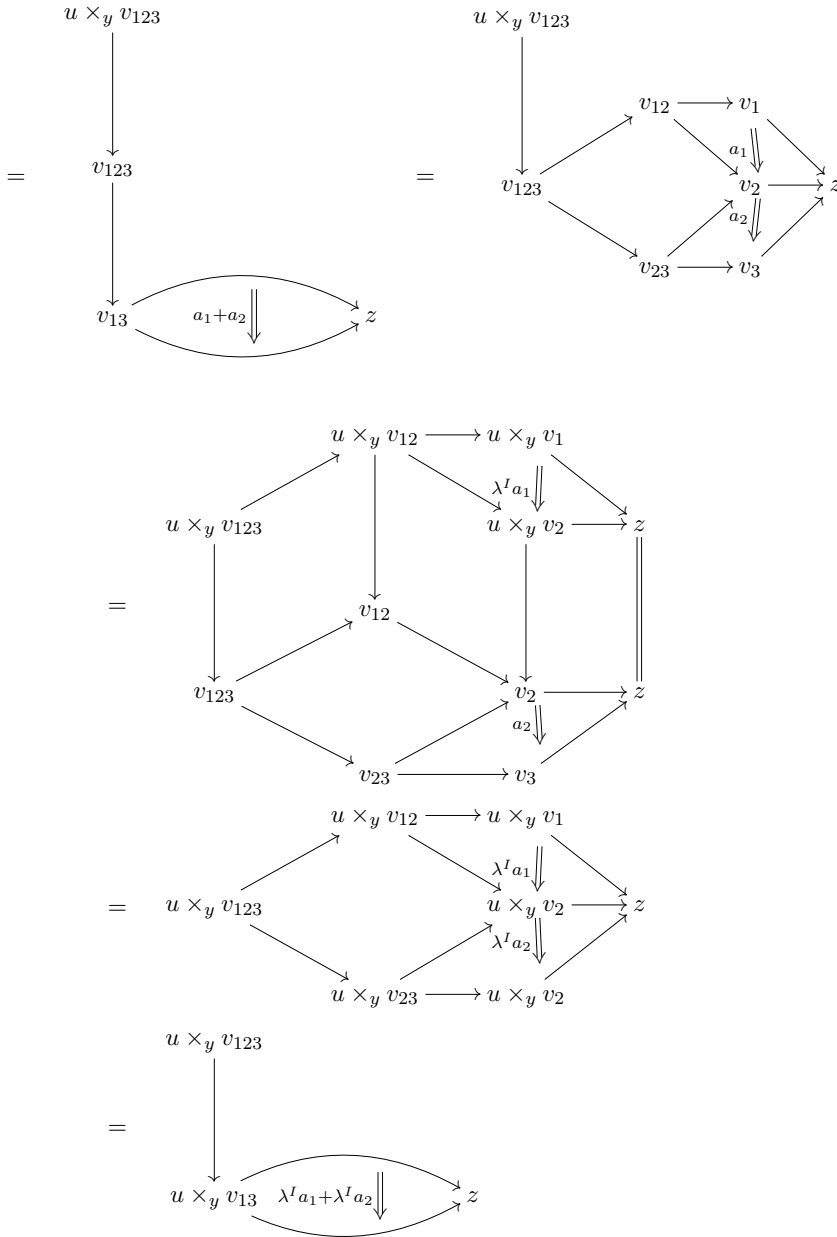
A.1 Case I: left whiskering by (id_u, f) Recall the definition of case I of left whiskering (Definition 3.12).

Definition A.1. The left whiskering of $a: (j, g) \Rightarrow (k, h)$ by (id_u, f) is given by the 2-arrow $\lambda^I_{(\text{id}, f)} a$, defined as

$$u \times_y v_{12} \longrightarrow v_{12} \begin{array}{c} \curvearrowright \\ a \Downarrow \\ \curvearrowleft \end{array} z .$$

We now show left whiskering by (id, f) preserves composition. In the following, let $\lambda^I(-) = \lambda^I_{(\text{id}, f)}(-)$

$$\begin{array}{ccc} u \times_y v_{123} & & u \times_y v_{123} \\ \downarrow & & \downarrow \\ u \times_y v_{13} & \xrightarrow{\lambda^I(a_1+a_2)} & z \\ \downarrow & & \downarrow \\ u \times_y v_{13} & & v_{13} \\ \downarrow & & \downarrow \\ v_{13} & \xrightarrow{a_1+a_2} & z \end{array} =$$



By uniqueness of descent, $\lambda^I(a_1 + a_2) = \lambda^I a_1 + \lambda^I a_2$.

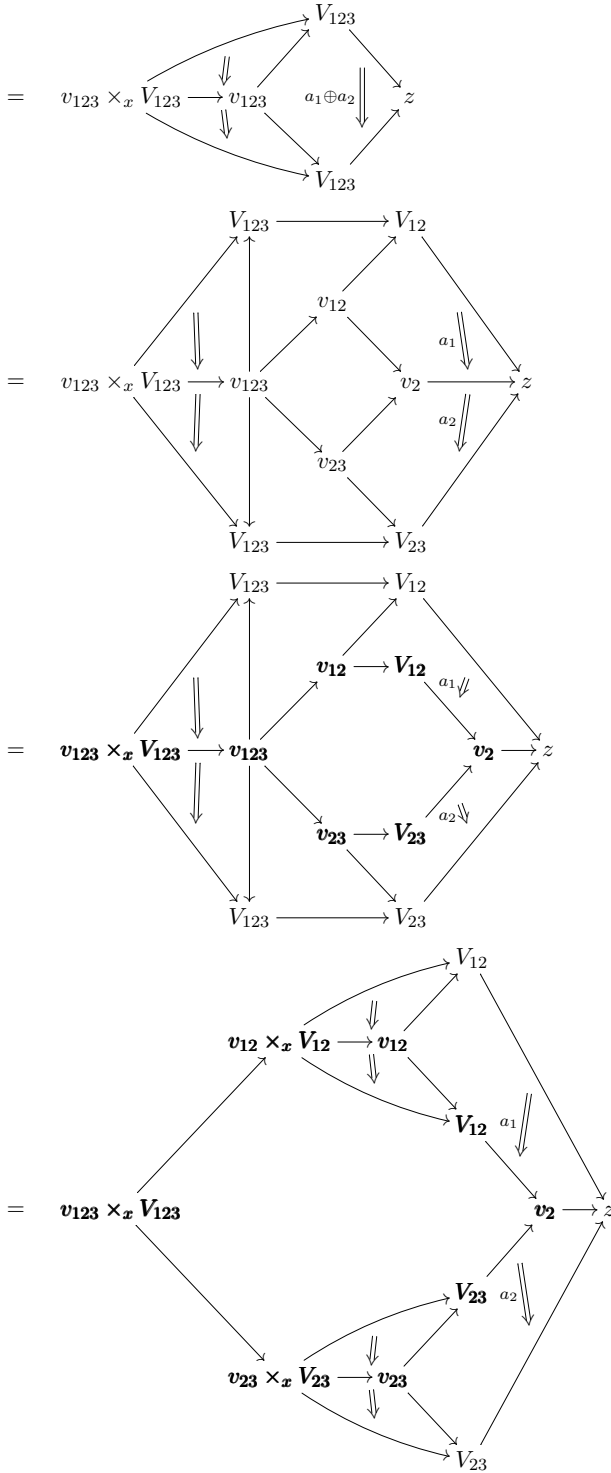
A.2 Case II: left whiskering by (j, id_u) Recall the notations $V_{12} := v_1 \times_x v_2$, $V_{ij\dots} := v_i \times_x v_j \times_x \dots$ and the canonical maps $v_{ij\dots} \rightarrow V_{ij\dots}$. Recall the definition of case II of left whiskering (Definition 3.13):

Definition A.2. The left whiskering of $a: (j, g) \Rightarrow (k, h)$ by (j, id_u) is given by the 2-arrow $\lambda_{(j, \text{id})}^{II} a$ in K defined via unique descent by the equation

$$v_{12} \times_x V_{12} \xrightarrow{\text{pr}_2} V_{12} \xrightarrow{\lambda_{(j, \text{id})}^{II} a} z = v_{12} \times_x \begin{array}{ccc} & & V_{12} \\ & \text{pr}_2 \nearrow & \nearrow \\ & v_{12} & \\ & \searrow & \searrow \\ & & V_{12} \end{array} \xrightarrow{a} z .$$

We now prove left whiskering by (j, id_u) preserves composition. In the following, let $\lambda^{II}(-) := \lambda_{(j, \text{id})}^{II}(-)$

$$\begin{array}{ccc} v_{123} \times_x V_{123} & v_{123} \times_x V_{123} & \\ \downarrow & \downarrow & \\ v_{13} \times_x V_{13} & = v_{13} \times_x \begin{array}{ccc} & & V_{13} \\ & \text{pr}_2 \nearrow & \nearrow \\ & v_{13} & \\ & \searrow & \searrow \\ & & V_{13} \end{array} \xrightarrow{a_1+a_2} z & = \\ \downarrow & & \\ V_{13} & \xrightarrow{\lambda^{II}(a_1+a_2)} z & \end{array} = \begin{array}{ccc} & & V_{123} \\ & \text{pr}_2 \nearrow & \nearrow \\ v_{123} \times_x V_{123} & \xrightarrow{\text{pr}_2} v_{123} & \\ & \searrow & \searrow \\ & & V_{123} \\ \downarrow & & \downarrow \\ v_{13} & \xrightarrow{a_1+a_2} V_{13} & \\ & \searrow & \searrow \\ & & V_{13} \end{array} \xrightarrow{a} z$$



(where the subdiagrams on the bold symbols are equal)

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & v_{12} \times_x V_{12} & \longrightarrow & V_{12} & \longrightarrow & v_1 & & \\
 & & \searrow & & \searrow & & \searrow & & \\
 & & & & & & \lambda^{II} a_1 & \Downarrow & \\
 = & v_{123} \times_x V_{123} & & & & & & & v_2 & \longrightarrow & z \\
 & & \searrow & & \searrow & & \searrow & & \lambda^{II} a_2 & \Downarrow & \\
 & & & & v_{23} \times_x V_{23} & \longrightarrow & V_{23} & \longrightarrow & v_3 & &
 \end{array} \\
 \\
 \begin{array}{ccccc}
 v_{123} \times_x V_{123} & & V_{12} & \longrightarrow & v_1 & & \\
 \downarrow & \nearrow & \searrow & & \searrow & & \\
 V_{123} & & & & \lambda^{II} a_1 & \Downarrow & \\
 & & & & & & v_2 & \longrightarrow & z \\
 & \searrow & \nearrow & & \searrow & & \lambda^{II} a_2 & \Downarrow & \\
 & & V_{23} & \longrightarrow & v_3 & &
 \end{array} \\
 \\
 \begin{array}{c}
 v_{123} \times_x V_{123} \\
 \downarrow \\
 V_{123} \\
 \downarrow \\
 V_{13} \xrightarrow{\lambda^{II} a_1 + \lambda^{II} a_2} z
 \end{array}
 \end{array}$$

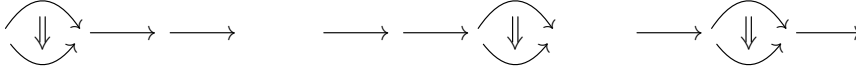
By uniqueness of descent, we have $\lambda^{II}(a_1 + a_2) = \lambda^{II} a_1 + \lambda^{II} a_2$.

Putting the two results this appendix together, arbitrary left whiskering preserves vertical composition.

B Proof that the associator is natural

We will use [6, Proposition II.3.2], which says the naturality condition for a transformation between functors $A \times B \times C \rightarrow D$ can be checked in each

component separately. Thus we only need to show the putative associator for K_J is natural for 2-arrows arising from double right whiskering, double left whiskering and left+right whiskering:



where these 1-arrows are J-fractions, and the 2-arrows are maps of J-fractions.

The major tool here is Lemma 3.6, in particular the second and third cases, since the associators arise from invertible renaming transformations.

For the rest of this appendix, fix composable triples of J-fractions:

$$\begin{array}{ccccc}
 x_1 \xleftarrow{j_1} u_1 \xrightarrow{f_1} x_2 & x_2 \xleftarrow{j_2} u_2 \xrightarrow{f_2} x_3 & x_3 \xleftarrow{j_3} u_3 \xrightarrow{f_3} x_4 \\
 x_1 \xleftarrow{k_1} v_1 \xrightarrow{g_1} x_2 & x_2 \xleftarrow{k_2} v_2 \xrightarrow{g_2} x_3 & x_3 \xleftarrow{k_3} v_3 \xrightarrow{g_3} x_4
 \end{array}$$

and maps of fractions $a_i: (j_i, f_i) \Rightarrow (k_i, g_i)$ for $i = 1, 2, 3$. Each step will only use one of a_1, a_2 or a_3 at a time.

We also need some notation for pulled back arrows, else there will be a confusing proliferation of projection maps. So, given f, g in the pullback square below (in our given 2-category K), the projection maps will be denoted \tilde{f} and \tilde{g} as shown:

$$\begin{array}{ccc}
 x \times_y z & \xrightarrow{\tilde{f}} & z \\
 \tilde{g} \downarrow & & \downarrow g \\
 x & \xrightarrow{j} & y
 \end{array}$$

If we need to pull an arrow g back along two different maps $x_1 \xrightarrow{j_1} y$ and $x_2 \xrightarrow{j_2} y$, the results will be denoted \tilde{g}^{j_1} and \tilde{g}^{j_2} respectively. To save space, horizontal composition will be denoted by juxtaposition (in function composition order), and the identity 2-arrow in K on a 1-arrow f will be denoted 1_f .

B.1 Double right whiskered See Definition 3.10 for how right whiskering is defined. We need to both right whisker a_1 twice in succession (by (j_2, f_2) and (j_3, f_3)), and also whisker it by $(j_3, f_3) \circ (j_2, f_2) = (j_2 j_3, f_3 f_2)$. The results then need to be vertically composed with associators in the

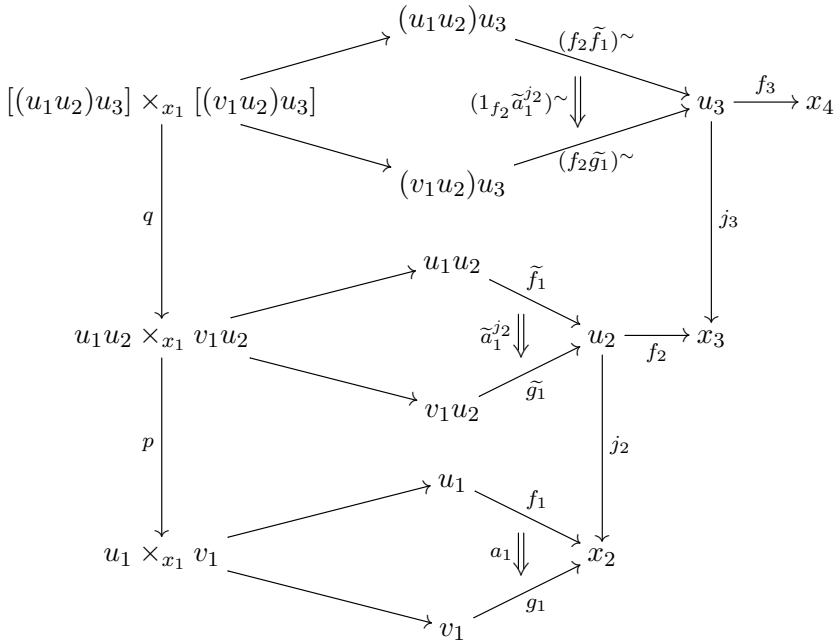


Figure 2: Constructing the double right whiskering

appropriate order and compared. Namely, we need to show the following 2-arrows are equal:

$$\rho_{(j_2\tilde{j}_3, f_3\tilde{f}_2)} a_1 + \iota(a_{v_1, u_2, u_3}) \stackrel{?}{=} \iota(a_{u_1, u_2, u_3}) + \rho_{(j_3, f_3)}(\rho_{(j_2, f_2)} a_1).$$

Or rather, we will show

$$(\iota(a_{u_1, u_2, u_3}))^{-1} + \rho_{(j_2\tilde{j}_3, f_3\tilde{f}_2)} a_1 + \iota(a_{v_1, u_2, u_3}) \stackrel{?}{=} \rho_{(j_3, f_3)}(\rho_{(j_2, f_2)} a_1).$$

Denote the 2-arrow data of $\rho_{(j_2\tilde{j}_3, f_3\tilde{f}_2)} a_1$ by $\rho_{f_3\tilde{f}_2} a$ and the 2-arrow data of $\rho_{(j_3, f_3)}(\rho_{(j_2, f_2)} a_1)$ by $\rho_{f_3}(\rho_{f_2} a)$. Then $\rho_{f_3}(\rho_{f_2} a) := 1_{f_3}(1_{f_2}\tilde{a}_1^{j_2})^\sim$, as shown the diagram in Figure 2, where $(u_1u_2)u_3 := (u_1 \times_{x_2} u_2) \times_{x_3} u_3$, $(v_1u_2)u_3 := (v_1 \times_{x_2} u_2) \times_{x_3} u_3$ and similarly for u_1u_2 and v_1u_2 . The 2-arrows $\tilde{a}_1^{j_2}$ and

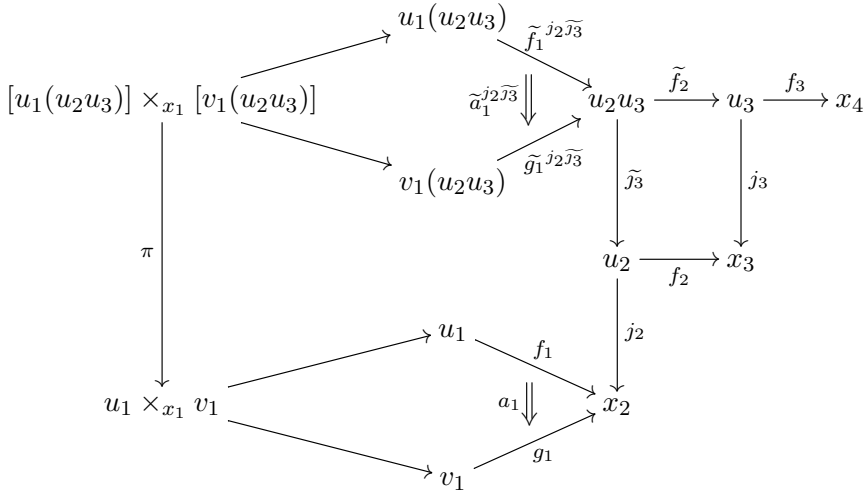


Figure 3: Constructing the right whiskering by the composite

$(1_{f_2} \tilde{a}_1^{j_2})^\sim$ are the unique 2-arrows satisfying

$$1_{j_3}(1_{f_2} \tilde{a}_1^{j_2})^\sim = 1_{f_2} \tilde{a}_1^{j_2} 1_q \tag{B.1}$$

$$1_{j_2} \tilde{a}_1^{j_2} = a_1 1_p \tag{B.2}$$

respectively, using the fact j_2 and j_3 are ff. We also have $\rho_{f_3\tilde{f}_2} a := 1_{f_3} 1_{f_2} \tilde{a}_1^{j_2\tilde{j}_3}$, as shown in the diagram in Figure 3. Again, using the fact $j_2\tilde{j}_3$ is ff, $\tilde{a}_1^{j_2\tilde{j}_3}$ is the unique 2-arrow satisfying

$$1_{j_2} 1_{\tilde{j}_3} \tilde{a}_1^{j_2\tilde{j}_3} = a_1 1_\pi \tag{B.3}$$

Finally, we note that we have

$$\pi\alpha = pq \tag{B.4}$$

where $\alpha := (a_{u_1, u_2, u_3} \times a_{v_1, u_2, u_3})^{-1}$. Using Lemma 3.6, we see that $(\iota(a_{u_1, u_2, u_3}))^{-1} + \rho_{(j_2\tilde{j}_3, f_3\tilde{f}_2)} a_1 + \iota(a_{v_1, u_2, u_3})$ is given by

$$[(u_1 u_2) u_3] \times_{x_1} [(v_1 u_2) u_3] \xrightarrow{\alpha} [u_1 (u_2 u_3)] \times_{x_1} [v_1 (u_2 u_3)] \begin{array}{c} \xrightarrow{\quad} \\ \tilde{a}_1^{j_2 \tilde{j}_3} \downarrow \\ \xrightarrow{\quad} \end{array} u_3 \xrightarrow{f_3 \tilde{f}_2} x_4$$

We can thus calculate

$$\begin{aligned} 1_{j_2} 1_{\tilde{j}_3} \tilde{a}_1^{j_2 \tilde{j}_3} 1_\alpha &= a_1 1_\pi 1_\alpha && \text{using (B.3)} \\ &= a_1 1_p 1_q && \text{using (B.4)} \\ &= 1_{j_2} \tilde{a}_1^{j_2} 1_q && \text{using (B.2)} \end{aligned}$$

and hence, since j_2 is ff, we have $1_{\tilde{j}_3} \tilde{a}_1^{j_2 \tilde{j}_3} 1_\alpha = \tilde{a}_1^{j_2} 1_q$. Continuing,

$$\begin{aligned} 1_{j_3} 1_{\tilde{f}_2} \tilde{a}_1^{j_2 \tilde{j}_3} 1_\alpha &= 1_{f_2} 1_{\tilde{j}_3} \tilde{a}_1^{j_2 \tilde{j}_3} 1_\alpha && \text{by definition of } \tilde{f}_2 \text{ and } \tilde{j}_3 \\ &= 1_{j_3} \tilde{a}_1^{j_2} 1_q \\ &= 1_{j_3} (1_{f_2} \tilde{a}_1^{j_2})^\sim && \text{using (B.1),} \end{aligned}$$

but now since j_3 is ff, we have $1_{\tilde{f}_2} \tilde{a}_1^{j_2 \tilde{j}_3} 1_\alpha = (1_{f_2} \tilde{a}_1^{j_2})^\sim$, and hence

$$\begin{aligned} (\iota(a_{u_1, u_2, u_3}))^{-1} + \rho_{(j_2 \tilde{j}_3, f_3 \tilde{f}_2)} a_1 + \iota(a_{v_1, u_2, u_3}) &= 1_{f_3} 1_{\tilde{f}_2} \tilde{a}_1^{j_2 \tilde{j}_3} 1_\alpha \\ &= 1_{f_3} (1_{f_2} \tilde{a}_1^{j_2})^\sim \\ &= \rho_{(j_3, f_3)} (\rho_{(j_2, f_2)} a_1), \end{aligned}$$

which is what we needed to show.

B.2 Double left whiskered See Definitions 3.12 and 3.13 for how left whiskering is defined. We need to both left whisker a_3 twice in succession (by (j_2, f_2) and (j_1, f_1)), and also whisker it by $(j_2, f_2) \circ (j_1, f_1)$. The results then need to be vertically composed with associators in the appropriate order and compared. Namely, we need to show the following 2-arrows are equal:

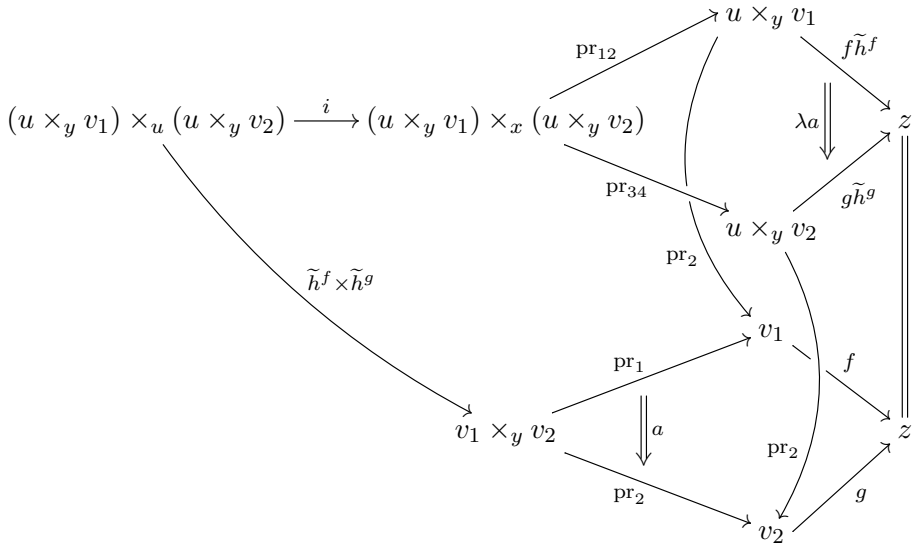
$$\lambda_{(j_1, f_1)} (\lambda_{(j_2, f_2)} a_3) + \iota(a_{u_1, u_2, v_3}) \stackrel{?}{=} \iota(a_{u_1, u_2, u_3}) + \lambda_{(j_1 \tilde{j}_2, f_2 \tilde{f}_1)} a_3$$

What we will show is the following

$$\lambda_{(j_1, f_1)} (\lambda_{(j_2, f_2)} a_3) \stackrel{?}{=} \iota(a_{u_1, u_2, u_3}) + \lambda_{(j_1 \tilde{j}_2, f_2 \tilde{f}_1)} a_3 + \iota(a_{u_1, u_2, v_3})^{-1}.$$

We can give a direct construction of the left whiskering, combining the two cases given in Definitions 3.12 and 3.13, as follows.¹⁰

Lemma B.1. *Let $y \xleftarrow{j} v_1 \xrightarrow{f} z$ and $y \xleftarrow{k} v_2 \xrightarrow{g} z$ be J-fractions. The left whiskering of the map $a: (j, f) \Rightarrow (k, g)$ by $x \xleftarrow{l} u \xrightarrow{h} y$ is given by the 2-arrow λa that is the unique descent of $a \circ 1_{\tilde{h}^f \times \tilde{h}^g}$ along the co-ff arrow i in the following diagram*



Proof. We first need to verify that i is indeed co-ff. But this follows from the fact it fits into a commutative square

$$\begin{array}{ccc}
 (u \times_y v_1) \times_u (u \times_y v_2) & \xrightarrow{i} & (u \times_y v_1) \times_x (u \times_y v_2) \\
 \downarrow & & \downarrow \\
 u & \xrightarrow{l} & x
 \end{array}$$

where the left and right vertical arrows are in \mathbf{J} (and $l \in \mathbf{J}$ by assumption), hence i is a map between \mathbf{J} -covers, and we can apply Lemma 2.14.

¹⁰This construction was found during revision of the article, after referee comments. I thank Fosco Loregian for the whimsical suggestion to call it the “teapot lemma”.

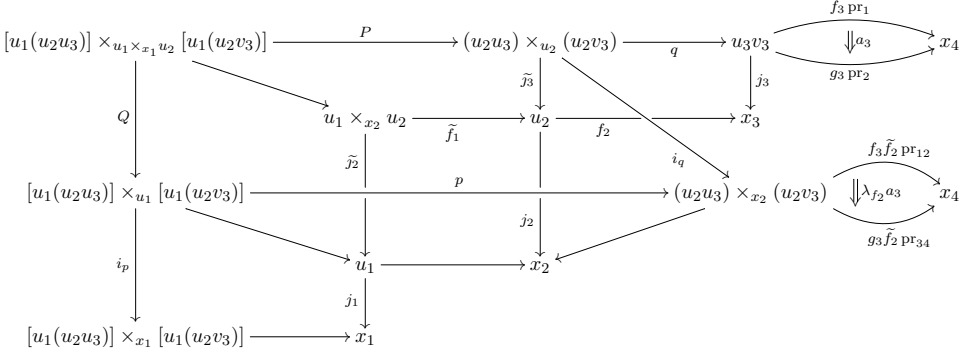


Figure 4: Constructing the double left whiskering

We thus need to verify that the left whiskerings (in K) of $\lambda_{(l,h)}a$ and λa with i are equal, since then i being co-ff means $\lambda_{(l,h)}a = \lambda a$. But this follows from the definition of the left whiskering of $\lambda_{(\text{id}_u, h)}a$ with (l, id_u) . \square

We first construct the 2-arrow $\lambda_{f_1}(\lambda_{f_2}a_3)$ specifying $\lambda_{(j_1, f_1)}(\lambda_{(j_2, f_2)}a_3)$, using the diagram in Figure 4.

Where $\lambda_{f_2}a_3$ is the unique 2-arrow satisfying

$$(\lambda_{f_2}a_3)1_{i_q} = a_31_q. \quad (\text{B.5})$$

The last step is that $\lambda_{f_1}(\lambda_{f_2}a_3)$ is the unique 2-arrow satisfying

$$(\lambda_{f_1}(\lambda_{f_2}a_3))1_{i_p} = (\lambda_{f_2}a_3)1_p. \quad (\text{B.6})$$

Note that from the diagram we also have

$$pQ = i_qP. \quad (\text{B.7})$$

We next construct the 2-arrow $\lambda_{f_2 \tilde{f}_1}a_3$ specifying $\lambda_{(j_1 \tilde{j}_2, f_2 \tilde{f}_1)}a_3$, using the diagram in Figure 5. Here $\lambda_{f_2 \tilde{f}_1}a_3$ is the unique 2-arrow satisfying

$$(\lambda_{f_2 \tilde{f}_1}a_3)1_{i_\pi} = a_31_\pi, \quad (\text{B.8})$$

and we note the identities

$$i_\pi \tilde{\alpha} = \alpha i_p Q \quad (\text{B.9})$$

$$qP = \pi \tilde{\alpha} \quad (\text{B.10})$$

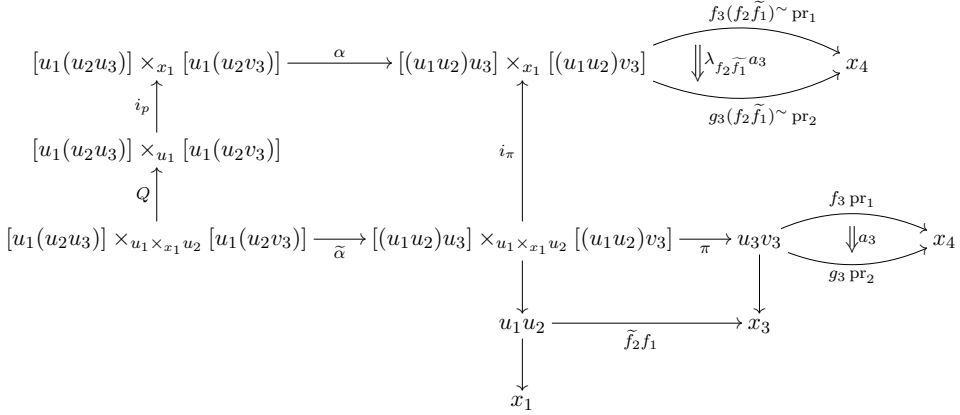


Figure 5: Constructing the left whisking by the composite

Further, we have that the map of J-fractions

$$\iota(a_{u_1, u_2, u_3})^{-1} + \lambda_{(j_1 \tilde{j}_2, f_2 \tilde{f}_1)} a_3 + \iota(a_{u_1, u_2, v_3})$$

is specified by the 2-arrow $(\lambda_{f_2 \tilde{f}_1} a_3) 1_\alpha$. Hence we can make the following calculation

$$\begin{aligned} (\lambda_{f_2 \tilde{f}_1} a_3) 1_\alpha 1_{i_p Q} &= (\lambda_{f_2 \tilde{f}_1} a_3) 1_{i_\pi} 1_{\tilde{\alpha}} && \text{using (B.9)} \\ &= a_3 1_\pi 1_{\tilde{\alpha}} && \text{using (B.8)} \\ &= a_3 1_q 1_P && \text{using (B.10)} \\ &= (\lambda_{f_2} a_3) 1_{i_q} 1_P && \text{using (B.5)} \\ &= (\lambda_{f_2} a_3) 1_p 1_Q && \text{using (B.7)} \\ &= (\lambda_{f_1} (\lambda_{f_2} a_3)) 1_{i_p} 1_Q && \text{using (B.6)} \end{aligned}$$

Notice that Q is co-ff as \tilde{j}_2 is a J-cover, as well as the other two maps in the square defining Q , and we can apply Lemma 2.14. Since i_p is also co-ff (by the proof of Lemma B.1), and co-ff 1-arrows are closed under composition, we have $(\lambda_{f_1} (\lambda_{f_2} a_3)) = \lambda_{f_2 \tilde{f}_1} a_3 1_\alpha$, as we needed to show.

B.3 Left+right whiskered We need to both whisker a_2 by (j_1, f_1) and (j_3, f_3) , in that order, and also whisker it by (j_3, f_3) and (j_1, f_1) , in that

order. The results then need to be vertically composed with associators in the appropriate order and compared. Namely, we need to show the following 2-arrows are equal:

$$\lambda_{(j_1, f_1)}(\rho_{(j_3, f_3)}a_2) + \iota(a_{u_1, v_2, u_3}) \stackrel{?}{=} \iota(a_{u_1, u_2, u_3}) + \rho_{(j_3, f_3)}(\lambda_{(j_1, f_1)}a_3).$$

What we will show, though, is the following:

$$\rho_{(j_3, f_3)}(\lambda_{(j_1, f_1)}a_3) \stackrel{?}{=} \iota(a_{u_1, u_2, u_3})^{-1} + \lambda_{(j_1, f_1)}(\rho_{(j_3, f_3)}a_2) + \iota(a_{u_1, v_2, u_3}),$$

where $\iota(a_{u_1, u_2, u_3})^{-1} + \lambda_{(j_1, f_1)}(\rho_{(j_3, f_3)}a_2) + \iota(a_{u_1, v_2, u_3})$ is specified by the 2-arrow

$$[(u_1u_2)u_3] \times_{x_1} [(u_1v_2)u_3] \xrightarrow{\alpha} [u_1(u_2u_3)] \times_{x_1} [v_1(u_2u_3)] \begin{array}{c} \xrightarrow{\lambda_{f_1}(\rho_{f_3}a_2)} \\ \Downarrow \\ \xrightarrow{\quad} \end{array} x_4$$

where $\lambda_{f_1}(\rho_{f_3}a_2)$ specifies the map $\lambda_{(j_1, f_1)}(\rho_{(j_3, f_3)}a_2)$ of J-fractions.

We first construct the 2-arrow data of $\lambda_{(j_1, f_1)}(\rho_{(j_3, f_3)}a_2)$, using the diagram in Figure 6. We have $\rho_{f_3}a_2 := 1_{f_3}\tilde{a}_2$ where \tilde{a}_2 is the unique 2-arrow satisfying

$$1_{j_3}\tilde{a}_2 = a_21_{\tilde{j}_3^2}. \tag{B.11}$$

Similarly $\lambda_{f_1}(\rho_{f_3}a_2) := 1_{f_3}\lambda_{f_1}(\tilde{a}_2)$, where $\lambda_{f_1}(\tilde{a}_2)$ is the unique 2-arrow satisfying

$$\lambda_{f_1}(\tilde{a}_2)1_{i_p} = \tilde{a}_21_p. \tag{B.12}$$

We next construct the 2-arrow data of $\rho_{(j_3, f_3)}(\lambda_{(j_1, f_1)}a_3)$, using the diagram in Figure 7. We have $\rho_{f_3}(\lambda_{f_1}a_2) := 1_{f_3}(\lambda_{f_1}a_2)^\sim$, where $(\lambda_{f_1}a_2)^\sim$ is the unique 2-arrow satisfying

$$1_{j_3}(\lambda_{f_1}a_2)^\sim = (\lambda_{f_1}a_2)1_{\tilde{j}_3^{12}} \tag{B.13}$$

Similarly, $\lambda_{f_1}a_2$ is the unique 2-arrow satisfying

$$(\lambda_{f_1}a_2)1_{i_q} = a_21_q. \tag{B.14}$$

Finally, we note that we have commutative squares

$$\begin{array}{ccccc} u_1u_2 \times_{u_1} u_1v_2 & \xleftarrow{J} & (u_1u_2)u_3 \times_{u_1} (u_1v_2)u_3 & \xrightarrow{\tilde{\alpha}} & u_1(u_2u_3) \times_{u_1} u_1(v_2u_3) \\ \downarrow i_q & & \downarrow \pi & & \downarrow i_p \\ u_1u_2 \times_{x_1} u_1v_2 & \xleftarrow{\tilde{j}_3^{12}} & (u_1u_2)u_3 \times_{x_1} (u_1v_2)u_3 & \xrightarrow{\alpha} & u_1(u_2u_3) \times_{x_1} u_1(v_2u_3) \end{array} \tag{B.15}$$

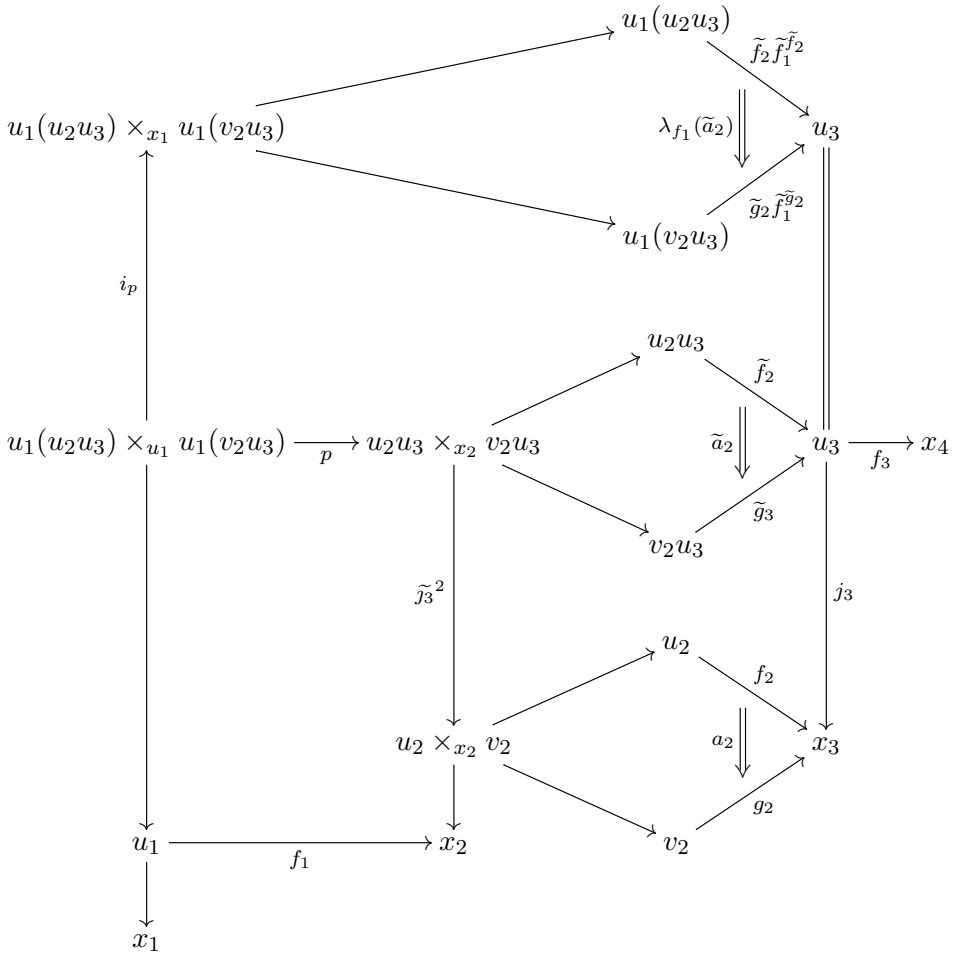


Figure 6: Constructing the right then left whiskering

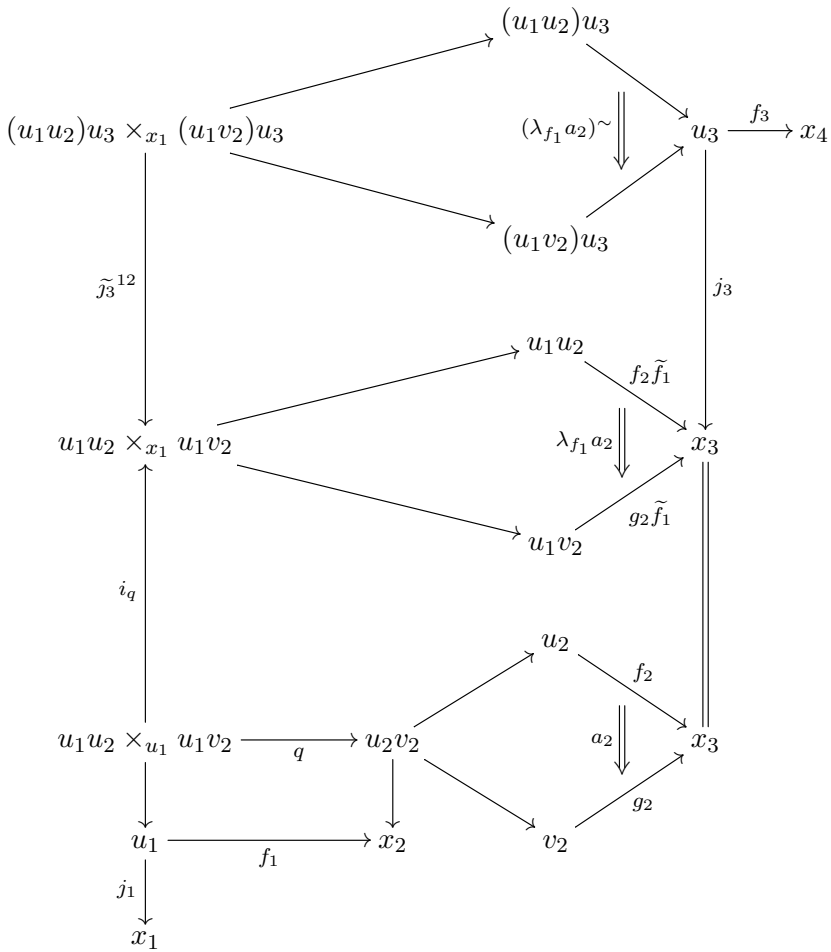


Figure 7: Constructing the left then right whiskering

and we need to show that

$$\lambda_{f_1}(\rho_{f_3} a_2) 1_\alpha = 1_{f_3}(\lambda_{f_1} \tilde{a}_2) 1_\alpha = 1_{f_3}(\lambda_{f_1} \tilde{a}_2) 1_\alpha \stackrel{?}{=} 1_{f_3}(\lambda_{f_1} a_2)^\sim = \rho_{f_3}(\lambda_{f_1} a_2). \tag{B.16}$$

We can now calculate:

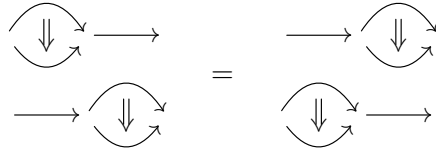
$$\begin{aligned} 1_{j_3}(\lambda_{f_1} \tilde{a}_2) 1_{i_p} &= 1_{j_3} \tilde{a}_2 1_p && \text{using (B.12)} \\ &= a_2 1_{\tilde{j}_3^2} 1_p && \text{using (B.11)} \\ &= a_2 1_q 1_J 1_{\tilde{\alpha}^{-1}} \\ &= (\lambda_{f_1} a_2) 1_{i_q} 1_J 1_{\tilde{\alpha}^{-1}} && \text{using (B.14)} \\ &= (\lambda_{f_1} a_2) 1_{\tilde{j}_3^{12}} 1_\pi 1_{\tilde{\alpha}^{-1}} && \text{using (B.15)} \\ &= 1_{j_3}(\lambda_{f_1} a_2)^\sim 1_\pi 1_{\tilde{\alpha}^{-1}} && \text{using (B.13)} \\ &= 1_{j_3}(\lambda_{f_1} a_2)^\sim 1_{\alpha^{-1}} 1_{i_p} && \text{using (B.15)}. \end{aligned}$$

Since j_3 is ff and i_p is co-ff, we have $(\lambda_{f_1}(\tilde{a}_2)) 1_\alpha = (\lambda_{f_1} a_2)^\sim$, from which (B.16) follows.

This completes the proof of Lemma 3.18.

C Proof that the middle-four interchange holds

We need to show the following equality of vertical compositions in K_J :



(Here the 1-arrows are J-fractions, and the 2-arrows are maps of J-fractions.)
 For the rest of this appendix, fix composable pairs of J-fractions

$$\begin{aligned} x_1 \xleftarrow{j_1} u_1 \xrightarrow{f_1} x_2 & \quad x_2 \xleftarrow{j_2} u_2 \xrightarrow{f_2} x_3 \\ x_1 \xleftarrow{k_1} v_1 \xrightarrow{g_1} x_2 & \quad x_2 \xleftarrow{k_2} v_2 \xrightarrow{g_2} x_3 \end{aligned}$$

and maps of fractions $a_i : (j_i, f_i) \Rightarrow (k_i, g_i)$ for $i = 1, 2$. In everything that follows, unlabeled 1-arrows are canonical projection maps.

We first calculate $\rho_{(j_2, f_2)} a_1 + \lambda_{(g_1, k_1)} a_2$. Define the arrows $p: v_1 u_2 \times_{x_1} v_1 v_2 \rightarrow u_2 v_2$, $i_p: v_1 u_2 \times_{x_1} v_1 v_2 \rightarrow v_1 u_2 \times_{x_1} v_1 v_2$, and $\tilde{j}_2 = \text{pr}_{13}: \cdot$, and for $a, b \in \{u, v\}$, the notation $a_1 b_2 := a_1 \times_{x_2} b_2$. The arrow i_p is, by Lemma 2.14, co-ff. Using the definition of right whiskering (Definition 3.10), and left whiskering as in Lemma B.1, we get that the 2-cell representing $\rho_{(j_2, f_2)} a_1 + \lambda_{(g_1, k_1)} a_2$ is the unique descent along the co-ff arrow $u_1 u_2 \times_{x_1} (v_1 u_2 \times_{x_1} v_1 v_2) \rightarrow u_1 u_2 \times_{x_1} v_1 v_2$ (using Lemma 2.14) of

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & u_1 u_2 \times_{x_1} v_1 u_2 & \longrightarrow & u_1 u_2 \\
 & \nearrow & & \searrow & \\
 & & & & \tilde{f}_1^{j_2} \\
 u_1 u_2 \times_{x_1} (v_1 u_2 \times_{x_1} v_1 v_2) & & & & v_1 u_2 \xrightarrow{\tilde{g}_1^{j_2}} u_2 \xrightarrow{f_2} x_3 \\
 & \searrow & & \nearrow & \\
 & & v_1 u_2 \times_{x_1} v_1 v_2 & \longrightarrow & v_1 v_2 \\
 & & & & \downarrow \lambda_{g_1} a_2 \\
 & & & & v_2 \xrightarrow{\tilde{g}_1^{k_2}} v_2 \xrightarrow{g_2} x_3
 \end{array} \\
 \text{(C.1)}
 \end{array}$$

where $\tilde{a}_1^{j_2}$ is the unique lift through j_2 of $a_1 1_{\tilde{j}_2}$ (using j_2 is ff), and $\lambda_{g_1} a_2$ is the unique descent of $a_1 1_p$ along the co-ff arrow i_p .

We next calculate $\lambda_{(j_1, f_1)} a_2 + \rho_{(k_2, g_2)} a_1$. Define the arrows $q: u_1 u_2 \times_{u_1} u_1 v_2 \rightarrow u_2 v_2$, $i_q: u_1 u_2 \times_{u_1} u_1 v_2 \rightarrow u_1 u_2 \times_{x_1} u_1 v_2$, and $\tilde{k}_2 = \text{pr}_{13}: u_1 v_2 \times_{x_1} v_1 v_2 \rightarrow u_1 v_1$. The arrow i_q is, by Lemma 2.14, co-ff. Then the 2-arrow representing $\lambda_{(j_1, f_1)} a_2 + \rho_{(k_2, g_2)} a_1$ is the unique descent along the co-ff arrow $u_1 u_2 \times_{x_1} (u_1 v_2 \times_{x_1} v_1 v_2) \rightarrow u_1 u_2 \times_{x_1} v_1 v_2$ (using Lemma 2.14) of

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & u_1 u_2 \times_{x_1} u_1 v_2 & \longrightarrow & u_1 u_2 \xrightarrow{\tilde{f}_1^{j_2}} u_2 \\
 & \nearrow & & \searrow & \\
 & & & & \downarrow \lambda_{f_1} a_2 \\
 u_1 u_2 \times_{x_1} (u_1 v_2 \times_{x_1} v_1 v_2) & & & & u_1 v_2 \xrightarrow{\tilde{f}_1^{k_2}} v_2 \xrightarrow{f_2} x_3 \\
 & \searrow & & \nearrow & \\
 & & u_1 v_2 \times_{x_1} v_1 v_2 & \longrightarrow & v_1 v_2 \\
 & & & & \downarrow \tilde{a}_1^{k_2} \\
 & & & & v_2 \xrightarrow{\tilde{g}_1^{k_1}} v_2 \xrightarrow{g_2} x_3
 \end{array} \\
 \text{(C.2)}
 \end{array}$$

where $\tilde{a}_1^{k_2}$ is the unique lift through k_2 of $a_1 1_{\tilde{k}_2}$ (using k_2 is ff), and $\lambda_{f_1} a_2$ is the unique descent of $a_1 1_q$ along the co-ff arrow i_q .

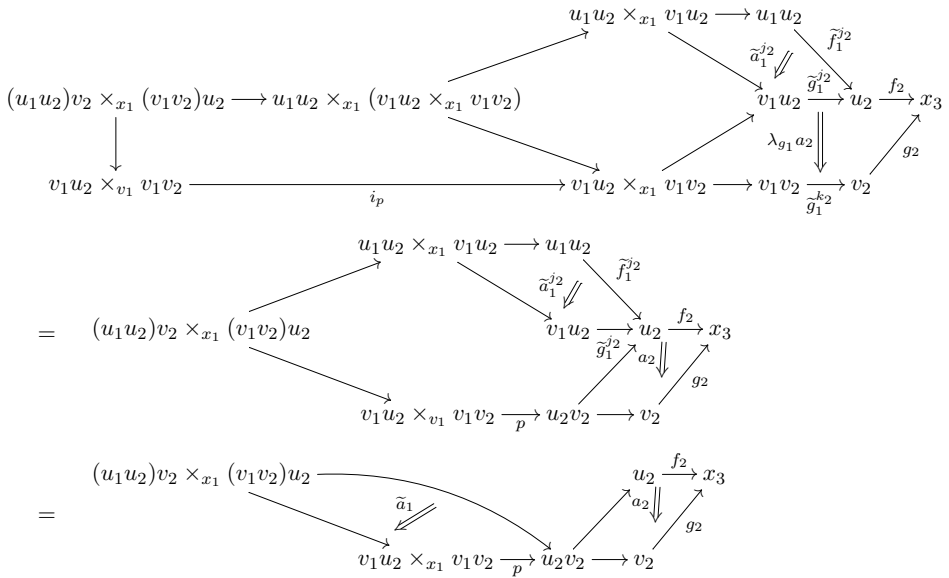
From here, we will left whisker (in K) both of $\rho_{(j_2, f_2)} a_1 + \lambda_{(g_1, k_1)} a_2$ and

$\lambda_{(j_1, f_1)} a_2 + \rho_{(k_2, g_2)} a_1$ by the co-ff 1-arrow

$$\pi : (u_1 u_2) v_2 \times_{x_1} (v_1 v_2) u_2 \rightarrow u_1 u_2 \times_{x_1} v_1 v_2$$

(again using Lemma 2.14) and show the resulting 2-arrows are equal. Here $(u_1 u_2) v_2 = (u_1 \times_{x_2}) \times_{x_2} v_2$ and similarly for $(v_1 v_2) u_2$. From this it will follow that $\rho_{(j_2, f_2)} a_1 + \lambda_{(g_1, k_1)} a_2 = \lambda_{(j_1, f_1)} a_2 + \rho_{(k_2, g_2)} a_1$, and hence that middle-four interchange holds.

Note first that using (C.1), the 2-arrow $(\rho_{(j_2, f_2)} a_1 + \lambda_{(g_1, k_1)} a_2) 1_\pi$ is equal to



where the last equality holds as $u_2 v_2 \rightarrow u_2$ is ff, so we can lift $\tilde{a}_1^{j_2}$ to \tilde{a}_1 . We note that \tilde{a}_1 is a lift of $a_1 1_Q$ through $u_2 v_2 \rightarrow x_2$, for $Q : (u_1 u_2) v_2 \times_{x_1} (v_1 v_2) u_2 \rightarrow u_1 v_1$.

Now using (C.2), the 2-arrow $(\lambda_{(j_1, f_1)} a_2 + \rho_{(k_2, g_2)} a_1) 1_\pi$ is equal to

$$\begin{array}{c}
 \begin{array}{c}
 u_1 u_2 \times_{u_1} u_1 v_2 \xrightarrow{i_q} u_1 u_2 \times_{x_1} u_1 v_2 \xrightarrow{\tilde{f}_1^{j_2}} u_2 \xrightarrow{f_2} x_3 \\
 \uparrow \\
 (u_1 u_2) v_2 \times_{x_1} (v_1 v_2) u_2 \longrightarrow u_1 u_2 \times_{x_1} (u_1 v_2 \times_{x_1} v_1 v_2) \longrightarrow u_1 v_2 \times_{x_1} v_1 v_2 \longrightarrow v_1 v_2 \xrightarrow{\tilde{f}_1^{k_2}} v_2 \xrightarrow{g_2} x_3 \\
 \downarrow \lambda_{f_1 a_2} \\
 u_1 v_2 \times_{x_1} v_1 v_2 \longrightarrow v_1 v_2 \xrightarrow{\tilde{f}_1^{k_2}} v_2 \xrightarrow{g_2} x_3 \\
 \downarrow \tilde{a}_1^{k_2} \quad \downarrow \tilde{g}_1^{k_1} \\
 u_1 v_2 \times_{x_1} v_1 v_2 \longrightarrow v_1 v_2 \longrightarrow v_2 \xrightarrow{g_2} x_3
 \end{array} \\
 = \\
 \begin{array}{c}
 u_1 u_2 \times_{u_1} u_1 v_2 \xrightarrow{q} u_2 v_2 \longrightarrow u_2 \xrightarrow{f_2} x_3 \\
 \uparrow \\
 (u_1 u_2) v_2 \times_{x_1} (v_1 v_2) u_2 \longrightarrow u_1 v_2 \times_{x_1} v_1 v_2 \longrightarrow v_1 v_2 \xrightarrow{\tilde{f}_1^{k_2}} v_2 \xrightarrow{g_2} x_3 \\
 \downarrow \tilde{a}_1^{k_2} \quad \downarrow \tilde{g}_1^{k_1} \\
 u_1 v_2 \times_{x_1} v_1 v_2 \longrightarrow v_1 v_2 \longrightarrow v_2 \xrightarrow{g_2} x_3
 \end{array} \\
 = \\
 \begin{array}{c}
 u_1 u_2 \times_{u_1} u_1 v_2 \xrightarrow{q} u_2 v_2 \longrightarrow u_2 \xrightarrow{f_2} x_3 \\
 \uparrow \\
 (u_1 u_2) v_2 \times_{x_1} (v_1 v_2) u_2 \xrightarrow{\tilde{a}'_1} u_2 v_2 \longrightarrow u_2 \xrightarrow{f_2} x_3 \\
 \downarrow a_2 \\
 v_2 \xrightarrow{g_2} x_3
 \end{array}
 \end{array}$$

where the last equality holds as $u_2 v_2 \rightarrow v_2$ is ff, so we can lift $\tilde{a}_1^{k_2}$ to \tilde{a}'_1 . But now \tilde{a}'_1 is also a lift of $a_1 1_Q$ through $u_2 v_2 \rightarrow x_2$, hence $\tilde{a}_1 = \tilde{a}'_1$. This give us $(\rho_{(j_2, f_2)} a_1 + \lambda_{(g_1, k_1)} a_2) 1_\pi = (\lambda_{(j_1, f_1)} a_2 + \rho_{(k_2, g_2)} a_1) 1_\pi$, and hence the desired result, using the fact π is co-ff.

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