A NOTE ON TWO NOTIONS OF ARBITRAGE

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A note on two notions of arbitrage

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Abstract

Since Hart's [5] and Werner's [10] seminal papers, several conditions have been proposed to show the existence of equilibrium in an asset exchange economy with short-selling. In this note, we discuss the relationship between two no-arbitrage conditions. The first condition is the assumption that the *individually rational utility set* $\mathcal{U}$ *is compact*, as considered by Dana, Le Van and Magnien [1]. The second is *inconsequential arbitrage*, introduced by Page, Wooders and Monteiro [9]. The main result of this comparison is to show that the inconsequential arbitrage condition is stronger than the assumption that $\mathcal{U}$ is compact.

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1 Introduction

In a seminal paper, Hart [5] establishes the existence of equilibrium in an asset exchange economy with general portfolio feasible sets that are not necessarily bounded below. Unlike the Arrow-Debreu model, there is no exogenous lower bound for agents' feasible portfolio assets due to the fact that unbounded short sales are allowed. Unbounded-from-below consumption sets also appear in temporary equilibrium models, Green [3], Grandmont [2], another important class of economic models. More recently, Werner [10] extends the analysis to a general equilibrium setting and provides an original and deep existence theorem. In response to these pioneering papers, a large literature has emerged on the existence of equilibrium when short sales are unbounded; see for example Milne [6], Hammond [4], Page [8] and Nielsen [7].

The purpose of the present note is to examine the relationship between the *compactness of the individually rational utility set* $\mathcal{U}$, as considered by Dana, Le Van and Magnien [1] and *inconsequential arbitrage*, introduced by Page, Wooders and Monteiro[9]. Our main result is to show that inconsequential arbitrage is stronger than the assumption that $\mathcal{U}$ is compact. The importance of these two conditions, $\mathcal{U}$ compactness and inconsequential arbitrage, is that they encompass many no-arbitrage conditions developed in the earlier literature. When the individually rational utility set $\mathcal{U}$ is compact, it follows that unbounded arbitrage opportunities are exhausted in the utility space. This is sufficient for existence because we are only concerned with the utility level of any attainable portfolio. Inconsequential arbitrage, by contrast, deals with the arbitrage direction generated by any sequence of unbounded net trades. Therefore, the existence proof using this condition is based on logic similar to the so called "Back-up" argument of Hart(1974).

The note is organised as follows. In Section 2, we introduce the basic model and the two conditions: compactness of the individually rational utility set $\mathcal{U}$ and inconsequential arbitrage. Section 3 is devoted to our main result. We will show that inconsequential arbitrage implies that the utility set is compact. The idea of the proof is to set up a sequence of n-bounded economies and then show that the utility set of the original economy coincides with the utility set of this sequence of bounded economies when n is sufficiently large provided that inconsequential arbitrage is satisfied. In the last section, we provide an example where $\mathcal{U}$ is compact and inconsequential arbitrage does not hold.

2 The basic model

We consider an asset exchange economy $\mathcal{E} = ((X_i, u_i, e_i)_{i=1\dots,m})$, with ℓ assets and m investors. For every $i = 1\dots, m$, $X_i \subset R^\ell$ is the choice set of the i-th investor, $e_i \in X_i$ her/his initial endowment vector and $u_i : X_i \to R$ her/his utility function.

We denote by $\mathcal{A}$ the set of *attainable* and *individually rational allocations* of the economy $\mathcal{E}$, that is:

$$\mathcal{A} = \{(x_i) \in \prod_{i=1}^{m} X_i \mid \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} e_i \text{ and } u_i(x_i) \geq u_i(e_i), \forall i\}.$$

Let $\mathcal{A}_i$ be the projection of $\mathcal{A}$ on X_i, that is :

$$\mathcal{A}_i = \{x_i \in X_i \mid (x_1, \dots, x_m) \in \mathcal{A}\}.$$

We also denote by $\mathcal{U}$ the *individually rational utility set* of the economy $\mathcal{E}$, that is:

$$\mathcal{U} = \{(v_i) \in R^m_+ \mid \exists x \in \mathcal{A}, \textit{ s.t. } u_i(e_i) \leq v_i \leq u_i(x_i), \forall i\}.$$

We make the following assumptions. *For all $i = 1, \dots, m$,*

[A.1] *X_i is a closed, convex subset of R^ℓ;*

[A.2] *u_i is strictly quasi-concave;*

[A.3] *u_i is upper semi-continuous.*

We say that $y \in R^{\ell m}$ is an *arbitrage* of the economy $\mathcal{E}$ if y is the limit of some sequence $\lambda^n x^n$ with $\lambda^n \downarrow 0$ and $(x^n) \in \mathcal{A}$. We shall denote by:

$$\text{arb}(\mathcal{E}) = \{y \in R^{\ell m} \mid \exists (x^n) \in \mathcal{A}, \lambda^n \downarrow 0, y = \lim_{n \to +\infty} \lambda^n x^n\}.$$

the set of all arbitrage of $\mathcal{E}$. We shall denote also by

$$\text{arbseq}(y) = \{(x^n) \in \mathcal{A} \mid \exists \lambda^n \downarrow 0, y = \lim_{n \to +\infty} \lambda^n x^n\},$$

the set of all *arbitrage sequences* corresponding to $y \in \text{arb}(\mathcal{E})$.

Definition 2.1 *The economy $\mathcal{E}$ satisfies the Inconsequential arbitrage condition if for all $y \in \text{arb}(\mathcal{E})$ and $(x^n) \in \text{arbseq}(y)$, there exists an $\epsilon > 0$ such that for all n sufficiently large*

$$x_i^n - \epsilon y_i \in X_i \textit{ and } u_i(x_i^n - \epsilon y_i) \geq u_i(x_i^n), \forall i.$$

3 Inconsequential Arbitrage implies $\mathcal{U}$ is compact

In order to prove that the inconsequential arbitrage condition implies that the individually rational utility set $\mathcal{U}$ is compact, we need a notion of bounded economies. Given a positive integer n, an *n-bounded economy* is denoted by $\mathcal{E}^n = ((X_i^n, u_i, e_i)_{i=1,\dots,m})$ where $X_i^n = X_i \cap \overline{B}(0,n)$. We choose n large enough so that $e_i \in B(0,n)$, for all $i = 1.\dots, m$. For each positive integer n, the set of individually rational attainable allocations $\mathcal{A}^n$ and the set of individually rational utility allocations $\mathcal{U}^n$ for the bounded economy $\mathcal{E}^n$ are defined in a similar way to the definition of $\mathcal{A}$ and $\mathcal{U}$. That is:

$$\mathcal{A}^n = \{(x_i) \in \prod_{i=1}^m X_i^n \mid \sum_{i=1}^m x_i = \sum_{i=1}^m e_i \text{ and } u_i(x_i) \geq u_i(e_i), \forall i\},$$

$$\mathcal{U}^n = \{(v_i) \in R_+^m \mid \exists x \in \mathcal{A}^n, \ s.t. \ u_i(e_i) \leq v_i \leq u_i(x_i), \ \forall i\}.$$

It is obvious that under [A.1]-[A.3], $\mathcal{U}^n$ is compact, since $\mathcal{A}^n$ is bounded.

We now state the main result of this section.

Proposition 3.1 *Under* [A.1]-[A.3]*, if the economy $\mathcal{E}$ satisfies the inconsequential arbitrage condition, then there exists an integer n_0 such that for all $n \geq n_0$, $\mathcal{U}^n = \mathcal{U}$ and therefore $\mathcal{U}$ is compact.*

Proof of Proposition 3.1 Suppose the contrary. Since $\mathcal{U}^n \subset \mathcal{U}^{n+1} \subset \mathcal{U}$, it follows that for all n, $\mathcal{U} \not\subset \mathcal{U}^n$. Then, we can take a sequence of attainable allocations $(x^n) \in \mathcal{A}$ such that

$$\forall x \in \mathcal{A}^n, \ \exists i \text{ such that } u_i(x_i) < u_i(x_i^n).$$

We define the set

$$\mathcal{B}^n = \{x \in \mathcal{A} \mid u_i(x_i) \geq u_i(x_i^n), \forall i\}.$$

Let us consider the optimisation problem

$$\mathcal{P}^n = \begin{cases} \inf \sum_{i=1}^m \|x_i\| \\ x \in \mathcal{B}^n \end{cases}$$

Claim 3.1 *$\mathcal{P}^n$ has a solution $z^n \in \mathcal{B}^n$.*

Proof of Claim 3.1 It is clear that $\mathcal{B}^n$ is a nonempty closed subset of $R^{\ell m}$. Moreover, the function $f^n : x \longmapsto \sum_{i=1}^m ||x_i||$ defined on $\mathcal{B}^n$ is continuous and coercive. Then the problem $\mathcal{P}^n$ has a solution $z^n \in \mathcal{B}^n$.□

Claim 3.2 $\lim_{n\to+\infty}\sum_{i=1}^m \|z_i^n\| = +\infty$.

Proof of Claim 3.2 Since $z^n \in \mathcal{B}^n$, it follows that $z^n \notin \mathcal{A}^n$. But z^n is an attainable allocation, and therefore we must have $z^n \notin \prod_{i=1}^m (X_i \cap \overline{B}(0,n))$. Hence $\sum_{i=1}^m \|z_i^n\| > n$.□

Let y denote any cluster point of the sequence $\frac{z^n}{\sum_{i=1}^m \|z_i^n\|}$.

Claim 3.3 *For n sufficiently large and for all* $0 < \epsilon \leq 1$

$$\sum_{i=1}^m \|z_i^n - \epsilon y_i\| < \sum_{i=1}^m \|z_i^n\|.$$

Proof of Claim 3.3 We first remark that, if $y_i = 0$, then $\|z_i^n - \epsilon y_i\| = \|z_i^n\|$. Moreover, $I_0 := \{i \mid y_i \neq 0\} \neq \emptyset$, since $\sum_{i=1}^m \|y_i\| = 1$. Hence for all $i \in I_0$, we have

$$\begin{aligned}\|z_i^n - \epsilon y_i\| &\leq \|z_i^n - \frac{\epsilon z_i^n}{\sum_{i=1}^m \|z_i^n\|}\| + \|\frac{\epsilon z_i^n}{\sum_{i=1}^m \|z_i^n\|} - \epsilon y_i\| \\ &= \|z_i^n\| + \epsilon(\|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|} - y_i\| - \|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|}\|).\end{aligned}$$

Since

$$\lim_{n\to+\infty}(\|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|} - y_i\| - \|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|}\|) = -\|y_i\| < 0,$$

we obtain for n sufficiently large

$$(\|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|} - y_i\| - \|\frac{z_i^n}{\sum_{i=1}^m \|z_i^n\|}\|) < 0.$$

We can conclude $\|z_i^n - \epsilon y_i\| < \|z_i^n\|$.□

To end the proof we notice that from the inconsequential arbitrage condition, it follows that for some $\epsilon > 0$, and for n sufficiently large $z^n - \epsilon y \in \mathcal{B}^n$, which contradicts Claim 3.3 since z^n is a solution of $\mathcal{P}^n$.□

4 Example

In this section we provide an example in which we have compact $\mathcal{U}$ while inconsequential arbitrage is not satisfied. Consider the economy with two consumers and three commodities.
Consumer 1 has the following characteristics:
$X_1 = R^3_+$

$u_1(x,y,z) = \begin{cases} \frac{x+y}{y+1} & \text{if } x \in [0,1] \\ x & \text{if } x \geq 1 \end{cases}$
$e_1 = (0,0,0)$.

Consumer 2 has the following characteristics:
$X_2 = [-1,+\infty[\times R_- \times R_+$
$u_2(x,y,z) = z$
$e_2 = (0,0,0)$.

The set of individually rational attainable allocations is:

$$\mathcal{A} = \{(x,y,0),(-x,-y,0) \in R^6 \mid x \in [0,1], y \in R_+\}.$$

Hence

$$\begin{aligned} u(\mathcal{A}) &= \{(\frac{x+y}{y+1},0) \mid x \in [0,1], y \in R_+\} \\ &= [0,1] \times \{0\}. \end{aligned}$$

Since $\mathcal{U} = (u(\mathcal{A}) + R^2_-) \cap R^2_+$, we have also $\mathcal{U} = [0,1] \times \{0\}$, and therefore $\mathcal{U}$ is compact.

In order to prove that inconsequential arbitrage is not satisfied, we define the sequence $(x^n) \in \mathcal{A}$, where

$$x_1^n = (0,n,0) \text{ and } x_2^n = (0,-n,0).$$

Then

$$u_1(x_1^n) = 1 - \frac{1}{n+1} \text{ and } u_2(x_2^n) = 0.$$

Let

$$y_1 = (0,1,0) \text{ and } y_2 = (0,-1,0).$$

It is clear that $y \in \mathrm{arb}(\mathcal{E})$ and $(x^n) \in \mathrm{arbseq}(y)$. But for all $\epsilon > 0$ and for all n, we have

$$u_1(x_1^n - \epsilon y_1) = 1 - \frac{1}{n-\epsilon+1} < u_1(x_1^n) = 1 - \frac{1}{n+1}.\square$$

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