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DECAY OF QUASI-STATIC POROUS-THERMO-ELASTIC WAVES

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Abstract: We study the behavior in time of the solutions to several systems of equations for porous-thermo-elastic problems when one of the variables is considered to be quasi-static or, in other words, whose second time derivative can be neglected. We analyze three different situations using the classical Fourier law and also the type II or type III Green-Naghdi heat conduction models.

Keywords: types II/III thermoelasticity, quasi-static, exponential decay.

1. Introduction

Since the Cosserat brothers [3] proposed the micropolar elastic materials theory, many authors have tried to generalize it to other kind of elastic materials. In the second half of the past century new proposals were made following and according to the axioms of thermomechanics. Goodman and Cowin [13] set the foundations of a continuum theory for granular materials with interstitial voids. The basic idea lies in writing the bulk density as the product of the density matrix by the volume fraction. Later, Cowin and Nunziato [4, 5, 32] developed the theory of elastic solids with voids aiming to model the behavior of materials with pores or small voids distributed within them. Their theory has been extended to situations where the heat affects also the materials [18, 19, 20].

On the other hand, there are several theories in the literature describing the heat conduction. Green and Naghdi [14, 15, 16], for example, proposed three different models for thermoelastic materials. They just called these theories as type I, II and III. The type I coincides with the classical Fourier theory. In types II and III a new variable, the thermal displacement, is contemplated. These two theories are being deeply studied nowadays. We are convinced that a lot of interesting new results can be found using the type II and type III Green-Naghdi models because in them there are some couplings among the field equations that were not present in the classical Fourier formulation. Determine the consequences of these couplings is one of our aims.

In our opinion, it is quite important to know how the solutions to the porous-thermo-elasticity problems behave with respect to the time variable depending on the damping mechanisms present in the system and on the considered theory used to describe the situation that is being analyzed. In fact, in the last twenty years a great effort has been made to know this behavior [1, 2, 9, 10, 11, 12, 23, 33, 35, 36, 37, 38]. As a matter of example, let us recall that for the Fourier heat conduction, generically, a dissipative mechanism influencing the pores microstructure is needed to obtain the exponential decay. However, it has been recently proved that the same behavior is obtained for the type II and type III theories without any alternative damping [21, 29, 30]. Even more, if the type III with microtemperatures is considered, the exponential decay can be proved

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also in the three dimensional case [26, 27]. This fact is relevant because for the classical Fourier theory the decay is, generically, slow in the two and in the three dimensional cases.

Mosconi [31] proposed the concept of *quasi-static* for the situation in which the deformations of the microvoids are so small that their second time derivative can be neglected. One can consider this idea for the other variables present in the problem and, hence, we can speak of quasi-static displacement or quasi-static temperature, for example. In all these situations, the second time derivative of the quasi-static variable is supposed to be approximately zero.

Magaña and Quintanilla [25] studied the porous-elasticity situation with quasi-static microvoids. Recently, Magaña *et al.* [28] analyzed the type II porous-thermo-elaticity supposing again that the quasi-static variable was the voids' deformation. In this work we want to go further: we impose the quasi-static hypothesis in the displacement, in the volume fraction again but supposing a new damping mechanism in the system, and also in the temperature.

The plan of the paper is as follows. In Section 2 we set the problem and write the general system of equations we want to work with. Then we analyze different cases of the problem considering that one of the variables is quasi-static: the displacement in Section 3, the microvoids in Section 4, and the temperature in Section 5. For each case we study two different systems of equations and compare the behavior of the solutions. We summarize our results in Section 6.

2. Basic equations

First of all, we describe the problem. We will use the standard notation: a subscript of a function means its derivative with respect to the variable indicated in the subscript and a superposed dot (or dots) means time derivative (of first or second order). We try to state the more general situation and later we restrict our attention to the specific cases we want to analyze.

We consider the theory of thermo-porous-elastic materials when the heat conduction is described by the Green-Naghdi type III model that is the more general. In this model, the evolution equations for the one-dimensional situation are given by

$$\begin{array}{l} \rho\ddot{u} = t_x \\ J\ddot{\Phi} = h_x + g \\ \rho\dot{\eta} = q_x \end{array} \tag{2.1}$$

Variable u represents the displacement, Φ denotes the volume fraction, η the entropy, t the stress, h the equilibrated stress, g the equilibrated body force and q the heat flux. As usual, parameters ρ and J stand for the mass density and for the product of the mass density by the equilibrated inertia, respectively, and they are supposed to be positive.

The constitutive equations are the following (see [6]):

$$\begin{array}{l} t = \mu u_x + b\Phi - \beta\theta + \mu^*\dot{u}_x \\ h = \delta\Phi_x + l\alpha_x + \delta^*\dot{\Phi}_x + l_1^*\dot{\alpha}_x \\ g = -bu_x - \xi\Phi + m\theta \\ \rho\eta = \beta u_x + c\theta + m\Phi \\ q = k\alpha_x + l\Phi_x + k^*\dot{\alpha}_x + l_2^*\dot{\Phi}_x \end{array} \tag{2.2}$$

We recall that in the type II/III models, α denotes the thermal displacement and θ the relative temperature. Both variables are related in the following way

$$\alpha(x,t) = \alpha_0(x) + \int_0^t \theta(x,s)\,ds. \tag{2.3}$$

Substituting the constitutive equations into the evolution equations we obtain the following system of field equations:

$$\text{(2.4)} \qquad \begin{cases} \rho\ddot{u} = \mu u_{xx} + b\Phi_x - \beta\theta_x + \mu^*\dot{u}_{xx} \\ J\ddot{\Phi} = \delta\Phi_{xx} + l\alpha_{xx} - bu_x - \xi\Phi + m\theta + \delta^*\dot{\Phi}_{xx} + l_1^*\dot{\alpha}_{xx} \\ c\dot{\theta} = l\Phi_{xx} + k\alpha_{xx} - \beta\dot{u}_x - m\dot{\Phi} + l_2^*\dot{\Phi}_{xx} + k^*\dot{\alpha}_{xx} \end{cases}$$

We distinguish between the basic conservative coefficients J, ρ, μ, b, β, δ, l, ξ, m, c and k and the dissipation coefficients: μ^*, δ^*, k^*, l_1^* and l_2^*. As we said, we assume that J, ρ and c are positive. For the other basic constitutive coefficients we also suppose that

$$\text{(2.5)} \qquad \mu > 0,\ \delta > 0,\ \mu\xi > b^2,\ \delta k > l^2.$$

And for the dissipation coefficients it is commonly assumed that

$$\text{(2.6)} \qquad \mu^* \geq 0,\ 4k^*\delta^* \geq (l_1^* + l_2^*)^2.$$

The above assumptions are quite standard in the thermomechanical context. Conditions (2.5) are related with the elastic stability of the system. On the other hand, notice that if dissipation coefficients δ^* and k^* satisfy $\delta^*k^* = 0$, therefore it must be $l_1^* = l_2^* = 0$ to guarantee the dissipative structure. In fact, it would be possible to consider a dissipative case even when $\delta^*k^* = 0$ but $l_1^* = -l_2^*$. However, we restrict our analysis to the situations where Onsager's postulate applies (see, for example, [8], page 55). Following Onsager, it must be $l_1^* = l_2^*$ and, therefore, $l_i^* = 0$ whenever $\delta^*k^* = 0$.

It is quite obvious that this system, with all the dissipation mechanisms introduced, is exponentially stable. But a natural question arises: will this behavior remain if only one damping mechanism is taken into account and one of the variables is considered to be quasi-static?

In this work, we will study some different cases of system (2.4) depending on the assumptions over the dissipation terms, on the variable which is assumed to be quasi-static and on the model used to describe how the temperature behaves. Actually, we are mainly interested in the behavior of the solutions for three possible scenarios, the ones obtained by assuming the following hypotheses over the quasi-static variable and the damping coefficients:

(1) $\ddot{u} \approx 0$, $\mu^* > 0$.
(2) $\ddot{\Phi} \approx 0$, $\delta^* > 0$.
(3) $\dot{\theta} \approx 0$, $k^* > 0$.

We study system (2.4) in $B \times T$, where $B = [0, \pi]$ and $T = [0, \infty)$. To have a well posed problem we need to impose boundary and initial conditions. Nevertheless, we will use different boundary conditions depending on the case we want to study in order to simplify the analysis. Hence, we will set them for each case.

3. Case 1: Quasi-static displacement

We start our analysis supposing that the displacement is small enough to be neglected ($\ddot{u} \approx 0$). In fact, to emphasize the important results we study two different systems: first we consider the type I heat conduction theory and later we analyze the type II model.

3.1. **The classical porous-thermo-elasticity.** Let us take the classical system of displacement, porosity and temperature when this last variable is modeled using the Fourier law. We assume that there are two damping mechanisms in the system: viscoelasticity and thermal dissipation. Therefore, the system of equations is given by

$$\left\{\begin{array}{l} 0 = \mu u_{xx} + b\Phi_x - \beta\theta_x + \mu^* \dot{u}_{xx} \\ J\ddot{\Phi} = \delta\Phi_{xx} - bu_x - \xi\Phi + m\theta \\ c\dot{\theta} = -\beta\dot{u}_x - m\dot{\Phi} + k^*\theta_{xx} \end{array}\right. \tag{3.1}$$

The above system can be obtained from (2.4) by imposing $k = l = \delta^* = 0$ (and $\ddot{u} \approx 0$). As we said before, in this case it is also clear that $l_1^* = l_2^* = 0$. We suppose that μ^* and k^* are positive.

We will see that the decay of the solutions is slow. This behavior coincides with the one observed for the non quasi-static case (see [24]). Nevertheless, the results we obtain here are by no means trivially derived from the previous ones. In fact, the system of equations changes significantly because in the normal case it is composed of two second order in time partial differential equations plus one of first order while now we have two of first order and one of second order.

To make the analysis easier, we propose appropriate boundary and initial conditions. In this case, as to the boundary conditions we take

$$u_x(0,t) = u_x(\pi,t) = \Phi(0,t) = \Phi(\pi,t) = \theta(0,t) = \theta(\pi,t) = 0. \tag{3.2}$$

As for the initial conditions we consider

$$u(x,0) = u_0(x),\ \Phi(x,0) = \Phi_0(x),\ \dot{\Phi}(x,0) = \varphi_0(x),\ \theta(x,0) = \theta_0(x). \tag{3.3}$$

To avoid the possibility of having solutions uniform in the variable x which do not damp in time we assume that

$$\int_0^\pi u_0(x)\,dx = 0.$$

Let us suppose that there exists a solution to system (3.1) with the above boundary and initial conditions of the form

$$u = Ae^{\omega t}\cos(nx), \quad \Phi = Be^{\omega t}\sin(nx), \quad \theta = Ce^{\omega t}\sin(nx), \tag{3.4}$$

such that $\mathrm{Re}(\omega) > -\epsilon$ for all positive ϵ small enough. This fact implies that a solution ω as near as desired to the imaginary axis can be found, and, hence, it is impossible to have uniform exponential decay on the solutions to problem (3.1)–(3.3).

Imposing that u, Φ and θ in system (3.1) are as above, we obtain the following homogeneous system on the unknowns A, B and C:

$$\begin{pmatrix} n\mu + n\omega\mu^* & -b & \beta \\ bn & -\delta n^2 - J\omega^2 - \xi & m \\ -n\beta\omega & m\omega & k^*n^2 + c\omega \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.5}$$

This linear system has nontrivial solutions if and only if the determinant of the coefficients matrix is zero. In that case, ω would be a root of the following fourth degree polynomial:

$$p(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 \tag{3.6}$$

where

$$
\begin{aligned}
a_0 &= cJ\mu^* n \\
a_1 &= Jn\left(\beta^2 + c\mu + k^*\mu^* n^2\right) \\
a_2 &= \mu^* n\left(c\left(\delta n^2 + \xi\right) + m^2\right) + Jk^*\mu n^3 \\
a_3 &= -b^2 cn - 2b\beta mn + c\delta\mu n^3 + c\mu\xi n + \delta k^*\mu^* n^5 + k^*\mu^*\xi n^3 + \mu m^2 n + \beta^2\delta n^3 + \beta^2\xi n \\
a_4 &= k^* n^3\left(\mu\delta n^2 + \mu\xi - b^2\right)
\end{aligned}
$$

We want to prove that there are roots of $p(z)$ as near to the complex axis as desired. Equivalently, we would prove that for any $\epsilon > 0$ there are roots located on the right side of the vertical line $\mathrm{Re}(z) = -\epsilon$. So, it will be sufficient to show that there exists a root with positive real part for polynomial $p(z-\epsilon)$. We use the Routh-Hurwitz theorem to show it (see, for example, Dieudonné [7]). It says that, if $a_0 > 0$, then all the roots of polynomial $p(z)$ have negative real part if and only if a_4 and all the leading diagonal minors of matrix

$$
\begin{pmatrix}
a_1 & a_0 & 0 & 0 \\
a_3 & a_2 & a_1 & a_0 \\
0 & a_4 & a_3 & a_2 \\
0 & 0 & 0 & a_4
\end{pmatrix} \tag{3.7}
$$

are positive. Let Λ_i for $i = 1, 2, 3, 4$ be the leading diagonal minors of matrix (3.7) corresponding to polynomial $p(z-\epsilon)$.

Direct calculations prove that there exists n large enough that makes $\Lambda_3 < 0$. In fact, this minor is an eleventh degree polynomial with respect to n of the form

$$
\Lambda_3 = -2\delta J^2 (k^*)^3 (\mu^*)^3 \epsilon\, n^{11} + q_9(n),
$$

where $q_9(n)$ is a ninth degree polynomial. Notice that the main coefficient of Λ_3 is negative.

This argument proves that a uniform rate of decay of exponential type cannot be obtained for all the solutions and, hence, the decay of the solutions is slow.

3.2. **Type II with quasi-static displacement.** We still suppose that $\ddot{u} \approx 0$. However, we use now the Green-Naghdi type II model to describe the thermal behavior. In this model, the thermal displacement α plays an essential role. As to the dissipation, we take $\delta^* = k^* = 0$ and $\mu^* > 0$ in (2.4). That means that we are considering the usual viscoelastic dissipation. Hence, we obtain the following system of equations:

$$
\left\{
\begin{aligned}
0 &= \mu u_{xx} + b\Phi_x - \beta\theta_x + \mu^*\dot{u}_{xx} \\
J\ddot{\Phi} &= \delta\Phi_{xx} + l\alpha_{xx} - bu_x - \xi\Phi + m\theta \\
c\dot{\theta} &= k\alpha_{xx} + l\Phi_{xx} - \beta\dot{u}_x - m\dot{\Phi}
\end{aligned}
\right. \tag{3.8}
$$

We consider Neumann boundary conditions for the displacement u and Dirichlet conditions for the volume fraction Φ and the thermal displacement α:

$$
u_x(0,t) = u_x(\pi,t) = \Phi(0,t) = \Phi(\pi,t) = \alpha(0,t) = \alpha(\pi,t) = 0. \tag{3.9}
$$

As for the initial conditions we consider

$$
u(x,0) = u_0(x),\ \Phi(x,0) = \Phi_0(x),\ \dot{\Phi}(x,0) = \varphi_0(x),\ \alpha(x,0) = \alpha_0(x),\ \dot{\alpha}(x,0) = \theta_0(x). \tag{3.10}
$$

Assumptions (3.9) implies that the stress is null at the boundary. This is a quite reasonable condition from a mechanical point of view.

We want to prove that the problem given by (3.8)–(3.10) has a unique solution which is exponentially stable with respect to the time variable. To do so, we use the semigroup arguments.

Integrating the first equation with respect to x, and taking into account (3.9), we obtain

$$-\mu^*\dot{u}_x = \mu u_x + b\Phi - \beta\theta.$$

We introduce the symbol Δ^{-1} to denote the inverse of the Laplacian operator with Neumann boundary conditions. As we are considering boundary conditions (3.9), Δ^{-1} is a homeomorfism from L^2_* onto $H^2 \cap L^2_* \cap \{f : f_x(0,t) = f_x(\pi,t) = 0\}$, where, as usual in the literature, L^2_* is defined by

$$L^2_* = \left\{ f \in L^2 : \int_0^\pi f(x)\,dx = 0 \right\}.$$

Therefore, we can write system (3.8) in the following way:

$$\left\{ \begin{array}{l} \mu^*\dot{u} = -\Delta^{-1}\left[\mu u_{xx} + b\Phi_x - \beta\theta_x\right] \\ J\ddot{\Phi} = \delta\Phi_{xx} + l\alpha_{xx} - bu_x - \xi\Phi + m\theta \\ c\dot{\theta} = k\alpha_{xx} + l\Phi_{xx} - m\dot{\Phi} + \frac{\beta}{\mu^*}\left(\mu u_x + b\Phi - \beta\theta\right) \end{array} \right. \tag{3.11}$$

To study this system, we introduce suitable notation and consider an appropriate Hilbert space. Let us write $\varphi = \dot{\Phi}$. With the aforementioned notation, system (3.11) can be written as:

$$\left\{ \begin{array}{l} \dot{u} = -\frac{1}{\mu^*}\Delta^{-1}\left[\mu D^2 u + bD\Phi - \beta D\theta\right] \\ \dot{\Phi} = \varphi \\ \dot{\varphi} = \frac{1}{J}\left(\delta D^2\Phi + lD^2\alpha - bDu - \xi\Phi + m\theta\right) \\ \dot{\alpha} = \theta \\ \dot{\theta} = \frac{1}{c}\left[lD^2\Phi + kD^2\alpha - m\varphi + \frac{\beta}{\mu^*}\left(\mu Du + b\Phi - \beta\theta\right)\right] \end{array} \right. \tag{3.12}$$

We will analyze this system in the Hilbert space $\mathcal{H} = H^1_* \times H^1_0 \times L^2 \times H^1_0 \times L^2$, where $H^1_* = H^1 \cap L^2_*$. Let us define an inner product in $\mathcal{H}$ which is equivalent to the usual one. Given two elements of $\mathcal{H}$, $U = (u, \Phi, \varphi, \alpha, \theta)$ and $U^* = (u^*, \Phi^*, \varphi^*, \alpha^*, \theta^*)$, its inner product is defined by

$$\langle U, U^*\rangle = \int_0^\pi \left(\mu u_x\overline{u}^*_x + J\varphi\overline{\varphi}^* + c\theta\overline{\theta}^* + \delta\Phi_x\overline{\Phi}^*_x + \xi\Phi\overline{\Phi}^* + b(u_x\overline{\Phi}^* + \overline{u}^*_x\Phi) + l(\Phi_x\overline{\alpha}^*_x + \overline{\Phi}^*_x\alpha_x) + k\alpha_x\overline{\alpha}^*_x\right)dx, \tag{3.13}$$

where a bar over a variable denotes its complex conjugate.

The following step consists in analyze system (3.12), with its corresponding boundary and initial conditions, in the Hilbert space $\mathcal{H}$ using the matrix operator

$$\mathcal{A} = \begin{pmatrix} -\frac{\mu}{\mu^*}\Delta^{-1}D^2 & -\frac{b}{\mu^*}\Delta^{-1}D & 0 & 0 & \frac{\beta}{\mu^*}\Delta^{-1}D \\ 0 & 0 & \mathcal{I} & 0 & 0 \\ -\frac{b}{J}D & \frac{\delta}{J}D^2 - \frac{\xi}{J}\mathcal{I} & 0 & \frac{l}{J}D^2 & \frac{m}{J}\mathcal{I} \\ 0 & 0 & 0 & 0 & \mathcal{I} \\ \frac{\mu\beta}{c\mu^*}D & \frac{l}{c}D^2 + \frac{\beta b}{c\mu^*}\mathcal{I} & -\frac{m}{c}\mathcal{I} & \frac{k}{c}D^2 & -\frac{\beta^2}{c\mu^*}\mathcal{I} \end{pmatrix}. \tag{3.14}$$

In matrix $\mathcal{A}$ the symbol $\mathcal{I}$ denotes the identity.

Using matrix $\mathcal{A}$, the problem that we want to study can be stated as

$$\frac{dU}{dt} = \mathcal{A}U, \quad \text{with } U_0 = (u_0, \Phi_0, \varphi_0, \alpha_0, \theta_0). \tag{3.15}$$

The domain of $\mathcal{A}$, that we will denote by $\mathcal{D}(\mathcal{A})$, is the set

$$\mathcal{D}(\mathcal{A}) = \{U \in \mathcal{H} : \Phi \in H^2, \varphi \in H^1_0, \alpha \in H^2, \theta \in H^1_0\}.$$

It is dense in the Hilbert space $\mathcal{H}$.

Lemma 3.1. *The operator $\mathcal{A}$ is dissipative. This means that for any $U \in \mathcal{D}(\mathcal{A})$, $\Re\langle \mathcal{A}U, U\rangle \leq 0$.*

Proof. It is not difficult to see that

$$\Re\langle \mathcal{A}U, U\rangle = -\frac{1}{\mu^*}\int_0^\pi |\mu u_x + b\Phi - \beta\theta|^2 dx.$$

□

Lemma 3.2. *The resolvent of $\mathcal{A}$ contains the origin of the complex plane (sometimes, to shorten, this is written as $0 \in \varrho(\mathcal{A})$).*

Proof. For any $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$, we have to find $U \in \mathcal{H}$ such that $\mathcal{A}U = \mathcal{F}$. In other words, we have to prove that the following system of equations has a solution:

$$\left.\begin{aligned} -\tfrac{1}{\mu^*}\Delta^{-1}\left[\mu D^2 u + bD\Phi - \beta D\theta\right] &= f_1 \\ \varphi &= f_2 \\ \tfrac{1}{J}\left(\delta D^2\Phi + lD^2\alpha - bDu - \xi\Phi + m\theta\right) &= f_3 \\ \theta &= f_4 \\ \tfrac{1}{c}\left[lD^2\Phi + kD^2\alpha - m\varphi + \tfrac{\beta}{\mu^*}\left(\mu Du + b\Phi - \beta\theta\right)\right] &= f_5 \end{aligned}\right\} \tag{3.16}$$

To obtain the solution we will consider the development of f_i for $i = 1, ..., 5$ in Fourier series. Taking into account the Hilbert space $\mathcal{H}$, we know that

$$f_1 = \sum f_n^1 \cos nx, \text{ and } f_j = \sum f_n^j \sin nx, \text{ for } j = 2, 3, 4, 5,$$

with

$$\sum n^2 (f_n^i)^2 < \infty, \text{ for } i = 1, 2, 4 \text{ and } \sum (f_n^i)^2 < \infty, \text{ for } i = 3, 5. \tag{3.17}$$

We want to find the solution written also as Fourier series:

$$u = \sum u_n \cos nx,\ \Phi = \sum \phi_n \sin nx,\ \varphi = \sum \varphi_n \sin nx,\ \alpha = \sum \alpha_n \sin nx,\ \theta = \sum \theta_n \sin nx.$$

We want to find coefficients u_n, ϕ_n, φ_n, α_n and θ_n in terms of the f_n^i. Notice that from the second and fourth equations of system (3.16) it follows that $\varphi_n = f_n^2$ and $\theta_n = f_n^4$ for all n. Hence, system (3.16) becomes

$$\left.\begin{aligned} \mu D^2 u + bD\Phi &= -\mu^* D^2 f_1 + \beta D f_4 \\ \delta D^2\Phi + lD^2\alpha - bDu - \xi\Phi &= Jf_3 - mf_4 \\ lD^2\Phi + kD^2\alpha + \frac{\beta}{\mu^*}\left(\mu Du + b\Phi\right) &= mf_2 + \frac{\beta^2}{\mu^*}f_4 + cf_5 \end{aligned}\right\} \tag{3.18}$$

Replacing each variable by its Fourier series and simplifying, we obtain the following system for each n:

$$\left.\begin{aligned} -\mu n^2 u_n + bn\phi_n &= \mu^* n^2 f_n^1 + \beta n f_n^4 \\ bnu_n - \left(\delta n^2 + \xi\right)\phi_n - ln^2\alpha_n &= Jf_n^3 - mf_n^4 \\ -\beta\mu n u_n + \left(\beta b - l\mu^* n^2\right)\phi_n - k\mu^* n^2\alpha_n &= \mu^* m f_n^2 + \beta^2 f_n^4 + c\mu^* f_n^5 \end{aligned}\right\} \tag{3.19}$$

The solution to that system is given by

$$u_n = \frac{n^3\mu^*(\delta k - l^2) f_n^1 + r_2(n)}{n\big(n^2\mu(\delta k - l^2) + (\mu\xi - b^2)k\big)}$$
$$\phi_n = \frac{n\,(bk\mu^* + l\beta\mu)\, f_n^1 + s_0(n)}{n^2\mu(\delta k - l^2) + (\mu\xi - b^2)k}$$
$$\alpha_n = \frac{-n^3\,(bl\mu^* + \beta\delta\mu)\, f_n^1 + t_2(n)}{n^2\big(n^2\mu(\delta k - l^2) + (\mu\xi - b^2)k\big)}$$

where $r_2(n)$ and $t_2(n)$ are polynomials of degree two whose coefficients involve the system coefficients and also f_n^i for $i = 1, ..., 5$ and s_0 is a constant involving the same parameters. The denominators in the above fractions are strictly positive for all n from the hypotheses over the constitutive coefficients. Moreover, from the obtained values and from assumptions (3.17), it is not difficult to see that $\sum n^2(u_n)^2 < \infty$, $\sum n^4(\phi_n)^2 < \infty$ and $\sum n^4(\alpha_n)^2 < \infty$, and, hence, they are in the domain of the operator. Regularity conditions can also be checked.

Finally, in view of the obtained results, it is also clear that there exists a real number K such that $\|U\| \leq K\|\mathcal{F}\|$ and this implies that the inverse of $\mathcal{A}$ is continuous. □

The existence and uniqueness of the solutions is a consequence of the previous lemmas together with the Lumer–Phillips theorem. We summarize this fact in the following theorem.

Theorem 3.3. *The operator $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ in $\mathcal{H}$. Therefore, for each $U_0 \in \mathcal{D}(\mathcal{A})$, there exists a unique solution $U(t) \in \mathcal{C}^1([0,\infty), \mathcal{H}) \cap \mathcal{C}^0([0,\infty), \mathcal{D}(\mathcal{A}))$ to problem (3.15).*

We focus now on the stability of the solutions. We have to add another hypothesis to the constitutive coefficients: $l\beta \neq 0$.

We will use the characterization given by Huang [17] or Prüss [34]. In order to make this paper self-contained, we recall it bellow.

Theorem 3.4. *Let $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if $i\mathbb{R} \subset \varrho(\mathcal{A})$ and $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.*

Two conditions should to be shown. Let us prove them in two different lemmas.

Lemma 3.5. *Let $\mathcal{A}$ be the matrix operator defined before. The resolvent of $\mathcal{A}$ contains the imaginary axis of the complex plane (in short: $i\mathbb{R} \subset \varrho(\mathcal{A})$).*

Proof. The first part of the proof is related with general properties of the operator $\mathcal{A}$, it is quite standard and we prefer to omit it here in order to not enlarge the paper too much. The second part (see [22], page 25) begins by supposing that the intersection of the imaginary axis and the spectrum is non-empty. This part is specific for each particular system we analyze and we believe that is the part that deserves to be proved in detail. We suppose then that there exist a sequence of real numbers λ_n with $\lambda_n \to \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of vectors $U_n = (u_n, \Phi_n, \varphi_n, \alpha_n, \theta_n)$ in $D(\mathcal{A})$ and with unit norm such that $\|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\| \to 0$.

If we write the above expression term by term, we obtain the following conditions:

$$i\mu^*\lambda_n u_n + \Delta^{-1}\left(\mu D^2 u_n + bD\Phi_n - \beta D\theta_n\right) \to 0, \text{ in } H^1 \tag{3.20}$$

$$i\lambda_n \Phi_n - \varphi_n \to 0, \text{ in } H^1 \tag{3.21}$$

$$iJ\lambda_n\varphi_n - \delta D^2\Phi_n - lD^2\alpha_n + bDu_n + \xi\Phi_n - m\theta_n \to 0, \text{ in } L^2 \tag{3.22}$$

$$i\lambda_n\alpha_n - \theta_n \to 0, \text{ in } H^1 \tag{3.23}$$

$$ic\lambda_n\theta_n - lD^2\Phi_n - kD^2\alpha_n + m\varphi_n - \frac{\beta}{\mu^*}\left(\mu Du_n + b\Phi_n - \beta\theta_n\right) \to 0, \text{ in } L^2 \tag{3.24}$$

Selecting the real part of the product $\langle (i\lambda_n\mathcal{I} - \mathcal{A})U_n, U_n\rangle$ and taking into account Lemma 3.1, it is clear that

$$\mu Du_n + b\Phi_n - \beta\theta_n \to 0, \text{ in } L^2. \tag{3.25}$$

Hence, from (3.20), it follows that

$$\lambda_n u_n \to 0 \text{ in } H^1, \tag{3.26}$$

and, in particular, $u_n \to 0$ in H^1, which yields $Du_n \to 0$.

Notice that (3.25) reduces to $b\Phi_n - \beta\theta_n \to 0$, in L^2, or, analogously, $\theta_n \approx \frac{b}{\beta}\Phi_n$ in L^2.

Now we remove from (3.24) the term that tends to zero, substitute θ_n by $\frac{b}{\beta}\Phi_n$ and φ_n by $i\lambda_n\Phi_n$, divide all by λ_n and multiply the resulting expression by Φ_n:

$$\langle ic\lambda_n \frac{b}{\beta}\Phi_n, \Phi_n\rangle + l\langle D\Phi_n, D\Phi_n\rangle - k\langle D^2\alpha_n, \Phi_n\rangle + m\langle i\lambda_n\Phi_n, \Phi_n\rangle \to 0 \text{ in } L^2. \tag{3.27}$$

It is clear that the first and fourth terms of the above expression have null real part. Let us concentrate in the third one:

$$\langle D^2\alpha_n, \Phi_n\rangle = \langle \alpha_n, D^2\Phi_n\rangle = \langle \frac{\theta_n}{i}, \frac{D^2\Phi_n}{\lambda_n}\rangle = \langle \frac{b\Phi_n}{i\beta}, \frac{D^2\Phi_n}{\lambda_n}\rangle = -\langle \frac{bD\Phi_n}{i\beta}, \frac{D\Phi_n}{\lambda_n}\rangle. \tag{3.28}$$

We have used integration by parts, expression (3.23) and, again, $\theta_n \approx \frac{b}{\beta}\Phi_n$. Moreover, if we divide expressions (3.22) and (3.24) by λ_n and, after that, multiply both by Φ_n it is clear that $\frac{D^2\Phi_n}{\lambda_n}$ is bounded because being $\delta k > l^2$ it can be obtained as a linear combination of other bounded terms.

Hence, the only real part in (3.27) is $l\langle D\Phi_n, D\Phi_n\rangle$ and, in consequence, $D\Phi_n$ tends to 0. Therefore, $\theta_n \to 0$ in L^2.

Removing in (3.22) the terms that go to zero amb multiplying then by α_n we get

$$\langle iJ\varphi_n, \lambda_n\alpha_n\rangle - \delta\langle D^2\Phi_n, \alpha_n\rangle - l\langle D^2\alpha_n, \alpha_n\rangle - m\langle \theta_n, \alpha_n\rangle \to 0. \tag{3.29}$$

Using integration by parts, taking into account (3.23) and the fact that $\langle iJ\varphi_n, \lambda_n\alpha_n\rangle$ tends to zero, we can rewrite the above expression as

$$\delta\langle D\Phi_n, D\alpha_n\rangle + l\langle D\alpha_n, D\alpha_n\rangle - m\langle i\lambda_n\alpha_n, \alpha_n\rangle \to 0, \tag{3.30}$$

which yields $D\alpha_n \to 0$ because the first and the third terms tend to zero.

Finally, to prove that φ_n tends to 0 it is enough to multiply (3.22) by Φ_n.

This argument proves that vector U_n cannot be of unit norm and, hence, we arrive a contradiction. □

Lemma 3.6. *The operator $\mathcal{A}$ satisfies that* $\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Proof. If the statement of the lemma was not satisfied, therefore a sequence of real numbers λ_n exists such that $|\lambda_n| \to \infty$ and expressions (3.20)–(3.24) hold. With this assumption, following the same arguments used to prove Lemma 3.5, we arrive also a contradiction. Notice that the arguments we have used work when λ_n does not tend to zero and, hence, will work also when λ_n tends to any other real number or even if it tends to infinity. Notice also that from expression (3.23) $\lambda_n \alpha_n$ is bounded even though λ_n is not (moreover, $\lambda_n \alpha_n \to 0$ in L^2). □

Theorem 3.7. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ is exponentially stable. That is, there exist two positive constants M and p such that $\|S(t)\| \leq M\|S(0)\|e^{-pt}$.*

Proof. The proof is a direct consequence of Lemma 3.5, Lemma 3.6 and Theorem 3.4. □

It is worth noting that we have used $l\beta \neq 0$. Therefore, an interesting question remains to be answered: how is the behavior of the solutions when one of these coefficients is zero? Following an argument analogous to the one developed for the first system of this section, it can be shown that the decay of the solutions is, generically, slow in both cases. A fifth degree polynomial appears in each case. To be precise:

- When $\beta = 0$, the second leading diagonal minor of the corresponding matrix is negative for some n. In fact, if we denote this minor by Λ_2, direct calculations give
$$\Lambda_2 = -2cJ\epsilon\,(\mu^*)^2\,(c\delta + Jk)n^4 + q_2(n),$$
where $q_2(n)$ is a second degree polynomial in n involving the coefficients of the system. Notice that $\Lambda_2 < 0$ independently of l.
- When $l = 0$ (and $\beta \neq 0$) the fourth diagonal minor of the corresponding matrix is negative for some n. Following the same notation, it can be seen that
$$\Lambda_4 = -2Jk\delta\epsilon\,(\mu^*)^3\,(Jk - c\delta)^2\left(\beta^2 - 2c\epsilon\mu^*\right)n^{12} + q_{10}(n).$$
In this case, $\Lambda_4 < 0$ provided that $Jk \neq c\delta$. This is the reason why we said that the solutions decay *generically* in a slow way. For the specific case $Jk = c\delta$ several calculations seem to suggest the exponential decay, but a proof is needed (we prefer to focus on other results rather than enlarge the paper studying this particular case).

Finally, let us point out that replacing the type II Green-Naghdi heat conduction theory by the type III, the decay of the solutions will be also exponential because, therefore, we still have the same coupling but a stronger dissipation.

4. Case 2: Quasi-static microvoids

We now consider that the microvoids are quasi-static or, mathematically, that $\ddot{\Phi} \approx 0$. As before, we study two different systems: the classical isothermal system and the type II model.

4.1. The classical isothermal case with quasi-static microvoids.

We will see that the isothermal waves for porous-elasticity when the fraction volume is quasi-static decay in a slow way or, in other words, that a uniform exponential decay cannot be found. To this end we also use the Routh-Hurtwitz arguments.

The system that we have to study is given by

$$\left\{\begin{array}{l}\rho\ddot{u} = \mu u_{xx} + b\Phi_x\\ 0 = \delta\Phi_{xx} - bu_x - \xi\Phi + \delta^*\dot{\Phi}_{xx}\end{array}\right. \tag{4.1}$$

In this case, we set

$$u(0,t) = u(\pi,t) = \Phi_x(0,t) = \Phi_x(\pi,t) = 0. \tag{4.2}$$

As for the initial conditions we consider

$$u(x,0) = u_0(x),\ \dot{u}(x,0) = v_0(x),\ \Phi(x,0) = \Phi_0(x). \tag{4.3}$$

The solutions to problem (4.1)–(4.3) decay in a slow way. We repeat the arguments (and even the notation) we have used in Section 3 but writing only the main results. Imposing that u and Φ are of the form

$$u = Ae^{\omega t}\sin(nx),\ \Phi = Be^{\omega t}\cos(nx)$$

we obtain a 2 by 2 linear homogeneous system of equations:

$$\begin{pmatrix} \mu n^2 + \rho\omega^2 & -bn \\ -bn & \delta n^2 + \omega\delta^* n^2 + \xi \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.4}$$

Direct calculations show that exists n large enough such that $\Lambda_2 < 0$ because

$$\Lambda_2 = -2\,(\delta^*)^2\,\epsilon\mu\rho n^6 + q_4(n).$$

4.2. **Type II with quasi-static microvoids.** Let us introduce in the above situation the description of the heat transmission given by the Green-Naghdi type II model. Notice that the equation of the temperature is conservative. Nevertheless, we will see that, without any any other dissipation mechanism beyond the one that is already present in the classical porous-elasticity we have just analyzed, the solutions decay exponentially. This fact is quite remarkable.

The system of equations that we have to study can be obtained from system (2.4) when $\ddot{\Phi} \approx 0$, $\delta^* > 0$ and $\mu^* = k^* = 0$. It reduces to

$$\left\{ \begin{array}{l} \rho\ddot{u} = \mu u_{xx} + b\Phi_x - \beta\theta_x \\ 0 = \delta\Phi_{xx} + l\alpha_{xx} - bu_x - \xi\Phi + m\theta + \delta^*\dot{\Phi}_{xx} \\ c\dot{\theta} = k\alpha_{xx} + l\Phi_{xx} - \beta\dot{u}_x - m\dot{\Phi} \end{array} \right. \tag{4.5}$$

In this case, we set

$$u(0,t) = u(\pi,t) = \Phi_x(0,t) = \Phi_x(\pi,t) = \alpha_x(0,t) = \alpha_x(\pi,t) = 0. \tag{4.6}$$

As for the initial conditions we consider

$$u(x,0) = u_0(x),\ \dot{u}(x,0) = v_0(x),\ \Phi(x,0) = \Phi_0(x), \alpha(x,0) = \alpha_0(x),\ \dot{\alpha}(x,0) = \theta_0(x). \tag{4.7}$$

Let us emphasize that this system is different from the one studied in [28]. The difference lies in the considered kind of dissipation. Here we suppose $\delta^* > 0$, which means that there is a strong dissipation in the stress tensor while in the previous work the dissipation was weaker and appeared in the equilibrated body force.

As usual, we look for solutions that satisfy

$$\int_0^\pi \Phi_0(x)dx = \int_0^\pi \alpha_0(x)dx = \int_0^\pi \theta_0(x)dx = 0.$$

We denote by $v = \dot{u}$. Therefore, with the same notation used in Section 3, the system can be written as

$$
\left\{
\begin{array}{l}
\dot{u} = v \\
\dot{v} = \frac{1}{\rho}\left(\mu D^2 u + bD\Phi - \beta D\theta\right) \\
\dot{\Phi} = -\frac{1}{\delta^*}\Psi \\
\dot{\alpha} = \theta \\
\dot{\theta} = \frac{1}{c}\left(lD^2\Phi + kD^2\alpha - \beta Dv + \frac{m}{\delta^*}\Psi\right)
\end{array}
\right. \tag{4.8}
$$

where

$$\Psi = \Delta^{-1}\left[\delta D^2\Phi + lD^2\alpha - bDu - \xi\Phi + m\theta\right].$$

It is noteworthy that

$$\Psi_x = \delta\Phi_x + l\alpha_x - b\,u - \xi\int_0^x \Phi ds + m\int_0^x \theta ds.$$

We abuse a little bit the notation and denote again by $\mathcal{H}$ the Hilbert space where we are going to work in and by U its elements. In this case, we have $\mathcal{H} = H_0^1 \times L^2 \times H_*^1 \times H_*^1 \times L_*^2$.

Let $U = (u, v, \Phi, \alpha, \theta)$ and $U^* = (u^*, v^*, \Phi^*, \alpha^*, \theta^*)$ be two elements of $\mathcal{H}$. We define the following inner product in $\mathcal{H}$:

$$
\begin{aligned}
\langle U, U^*\rangle = \int_0^\pi \Big(&\rho v\overline{v}^* + c\theta\overline{\theta}^* + \mu u_x\overline{u}_x^* + \delta\Phi_x\overline{\Phi}_x^* + \xi\Phi\overline{\Phi}^* + b(u_x\overline{\Phi}^* + \overline{u}_x^*\Phi) \\
&+ l(\Phi_x\overline{\alpha}_x^* + \overline{\Phi}_x^*\alpha_x) + k\alpha_x\overline{\alpha}_x^*\Big)dx,
\end{aligned} \tag{4.9}
$$

From this system we get the following matrix operator
(4.10)

$$
\mathcal{B} = \begin{pmatrix}
0 & \mathcal{I} & 0 & 0 & 0 \\
\frac{\mu}{\rho}D^2 & 0 & \frac{b}{\rho}D & 0 & -\frac{\beta}{\rho}D \\
\frac{b}{\delta^*}\Delta^{-1}D & 0 & -\frac{\delta}{\delta^*}\Delta^{-1}D^2 + \frac{\xi}{\delta^*}\Delta^{-1} & -\frac{l}{\delta^*}\Delta^{-1}D^2 & -\frac{m}{\delta^*}\Delta^{-1} \\
0 & 0 & 0 & 0 & \mathcal{I} \\
-\frac{mb}{c\delta^*}\Delta^{-1}D & -\frac{\beta}{c}D & \frac{l}{c}D^2 + \frac{m\delta}{c\delta^*}\Delta^{-1}D^2 - \frac{m\xi}{c\delta^*}\Delta^{-1} & \frac{k}{c}D^2 + \frac{ml}{c\delta^*}\Delta^{-1}D^2 & \frac{m^2}{c\delta^*}\Delta^{-1}
\end{pmatrix}.
$$

The domain of this operator is

$$\mathcal{D}(\mathcal{B}) = \{U \in \mathcal{H} : v \in H_0^1,\ \theta \in H_*^1,\ \mu D^2u - \beta D\theta \in L^2,\ lD^2\Phi + kD^2\alpha \in L^2\}.$$

Lemma 4.1. *The operator $\mathcal{B}$ is dissipative: for any $U \in \mathcal{D}(\mathcal{B})$, $\Re\langle \mathcal{B}U, U\rangle \leq 0$.*

Proof. It is not difficult to see that

$$\Re\langle \mathcal{B}U, U\rangle = -\frac{1}{\delta^*}\int_0^\pi |\Psi_x|^2 dx.$$

□

Lemma 4.2. $0 \in \varrho(\mathcal{B})$.

Proof. If $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$ we have to prove that the following system has a solution

$$\left.\begin{aligned} v &= f_1 \\ \tfrac{1}{\rho}\left(\mu D^2 u + bD\Phi - \beta D\theta\right) &= f_2 \\ -\tfrac{1}{\delta^*}\Psi &= f_3 \\ \theta &= f_4 \\ \tfrac{1}{c}\left(lD^2\Phi + kD^2\alpha - \beta Dv + \tfrac{m}{\delta^*}\Psi\right) &= f_5 \end{aligned}\right\} \tag{4.11}$$

We consider the development of f_i for $i = 1, ..., 5$ in Fourier series. Taking into account the Hilbert space $\mathcal{H}$, we know that

$$f_i = \sum f_n^1 \sin nx \text{ for } i = 1, 2 \text{ and } f_j = \sum f_n^j \cos nx, \text{ for } j = 3, 4, 5,$$

with

$$\sum n^2 (f_n^i)^2 < \infty, \text{ for } i = 1, 3, 4 \text{ and } \sum (f_n^i)^2 < \infty, \text{ for } i = 2, 5. \tag{4.12}$$

We want to find

$$u = \sum u_n \sin nx, \; v = \sum v_n \sin nx, \; \Phi = \sum \phi_n \cos nx, \; \alpha = \sum \alpha_n \cos nx, \; \theta = \sum \theta_n \cos nx$$

in terms of the f_n^i.

Notice that from the first and fourth equations of system (4.11) if follows that $v_n = f_n^1$ and $\theta_n = f_n^4$ for all n. Hence, system (4.11) becomes

$$\left.\begin{aligned} \mu D^2 u + bD\Phi &= \rho f_2 + \beta D f_4 \\ \Psi &= -\delta^* f_3 \\ lD^2\Phi + kD^2\alpha &= cf_5 + \beta D f_1 + m f_3 \end{aligned}\right\} \tag{4.13}$$

Taking into account what Ψ is and substituting each term by its Fourier series we obtain a system of equations for each n. The solution of which is

$$\begin{aligned} u_n &= \frac{n^3\left((\delta k - l^2)\beta f_n^4 + bk f_n^3 \delta^*\right) + r_2(n)}{n^4\mu(\delta k - l^2) + n^2(\mu\xi - b^2)k} \\ \phi_n &= \frac{-n^3 f_n^3 k\delta^*\mu + s_2(n)}{n^3\mu(\delta k - l^2) + n(\mu\xi - b^2)k} \\ \alpha_n &= \frac{n^4 f_n^3 l\delta^*\mu + t_3(n)}{n^4\mu(\delta k - l^2) + n^2(\mu\xi - b^2)k} \end{aligned}$$

where $r_2(n)$, $s_2(n)$ and $t_3(n)$ are polynomials of degree two and three (the subindex indicates the degree) whose coefficients involve the system coefficients and also f_n^i for $i = 1, ..., 5$.

It is worth noting that $\mu D^2 u - \beta D\theta \in L^2$ because for each n we have

$$\mu D^2 u_n - \beta D\theta_n = -\mu n^2 \frac{n^3\left((\delta k - l^2)\beta f_n^4 + bk f_n^3\delta^*\right) + r_2(n)}{n^4\mu(\delta k - l^2) + n^2(\mu\xi - b^2)k} + \beta n f_n^4 = \frac{-\mu n^3 bk f_n^3 \delta^* + r_2'(n)}{n^2\mu(\delta k - l^2) + (\mu\xi - b^2)k},$$

where $r_2'(n)$ is a second degree polynomial on n. On the other hand, taking into account the expressions of $s_2(n)$ and $t_3(n)$ (which can be found by direct computation), it can be seen that

$$lD^2\phi_n + kD^2\alpha_n = \frac{(l^2 - \delta k)\beta\mu f_n^1 n^3 + s_2'(n)}{n^2\mu(\delta k - l^2) + (\mu\xi - b^2)k},$$

and, therefore, it is clear that $lD^2\Phi + kD^2\alpha \in L^2$. □

The above argument proves that system (4.5) has a unique solution. Let us prove now that it is exponentially stable.

Lemma 4.3. *$i\mathbb{R} \subset \varrho(\mathcal{B})$.*

Proof. We follow the same reasoning we have used in Lemma 3.5, but we focus in the essential part. Let $U_n = (u_n, v_n, \Phi_n, \alpha_n, \theta_n)$ be a sequence of vectors in $\mathcal{D}(\mathcal{B})$ with unit norm such that $\|(i\lambda_n \mathcal{I} - \mathcal{B})U_n\| \to 0$. We want to arrive a contradiction. We write the stated condition term by term:

$$i\lambda_n u_n - v_n \to 0, \text{ in } H^1 \tag{4.14}$$

$$i\rho\lambda_n v_n - \mu D^2 u_n - bD\Phi_n + \beta D\theta_n \to 0, \text{ in } L^2 \tag{4.15}$$

$$i\delta^* \lambda_n \Phi_n + \Psi_n \to 0, \text{ in } H^1 \tag{4.16}$$

$$i\lambda_n \alpha_n - \theta_n \to 0, \text{ in } H^1 \tag{4.17}$$

$$ic\lambda_n \theta_n - lD^2\Phi_n - kD^2\alpha_n + \beta D v_n - \frac{m}{\delta^*}\Psi_n \to 0, \text{ in } L^2 \tag{4.18}$$

From Lemma 4.1 it is clear that

$$D\Psi_n = \delta D\Phi_n + lD\alpha_n - bu_n - \xi \int_0^x \Phi_n ds + m \int_0^x \theta_n ds \to 0 \text{ in } L^2.$$

Therefore, from (4.16) we obtain $\lambda_n D\Phi_n \to 0$.

Let us multiply $D\Psi_n$ by $D\alpha_n$, which is bounded. Taking into account that $D\Phi_n \to 0$, performing an integration by parts and substituting θ_n by $i\lambda_n\alpha_n$ we obtain

$$l\|D\alpha_n\|^2 - b\langle u_n, D\alpha_n\rangle - i\lambda_n m\|\alpha_n\|^2 \to 0. \tag{4.19}$$

On the other hand, multiplying (4.18) by α_n and replacing v_n by $i\lambda_n u_n$ we get

$$\langle ic\lambda_n\theta_n, \alpha_n\rangle + k\|D\alpha_n\|^2 - i\beta\lambda_n\langle u_n, D\alpha_n\rangle \to 0,$$

or, equivalently, using (4.17)

$$-c\|\theta_n\|^2 + k\|D\alpha_n\|^2 - i\beta\lambda_n\langle u_n, D\alpha_n\rangle \to 0.$$

From this relation we see that the real part of $\langle u_n, D\alpha_n\rangle \to 0$ and, hence, from (4.19) $D\alpha_n \to 0$.

Multiplying again (4.18) by α_n, applying the previous results and integrating by parts, we see that

$$-c\|\theta_n\|^2 - \beta\langle v_n, D\alpha_n\rangle \to 0,$$

which means $\theta_n \to 0$ because v_n is bounded.

Finally, let us multiply (4.18) by $\frac{Du_n}{\lambda_n}$:

$$\langle lD\Phi_n + kD\alpha_n, \frac{D^2u_n}{\lambda_n}\rangle + i\beta\|Du_n\|^2 \to 0.$$

As $\frac{D^2u_n}{\lambda_n}$ is bounded (it can be seen dividing (4.15) by λ_n), it must be $Du_n \to 0$ and, in consequence, $v_n \to 0$, which finishes the proof. □

Lemma 4.4. $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathcal{I} - \mathcal{B})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

The arguments used to prove of Lemma 4.3 can be adapted easily to prove also this case, but we omit the details to shorten the paper.

The above results imply that the solutions to system (4.5) decay exponentially. We state this result in a formal way.

Theorem 4.5. *The C_0-semigroup $S(t) = \{e^{\mathcal{B}t}\}_{t\geq 0}$ is exponentially stable.*

As in the end of the previous section, we notice that the cases $\beta = 0$ or $l = 0$ deserve special attention. In both cases the decay of the solutions is, generically, slow. Direct calculation leads to the following results.

- When $l = 0$, the second leading diagonal minor is
$$\Lambda_2 = -2c\,(\delta^*)^2\,\rho\left(c\mu + k\rho + \beta^2\right)\epsilon n^6 + q_4(n).$$
- When $\beta = 0$, the fourth minor is
$$\Lambda_4 = -2c\,(\delta^*)^3\,\epsilon\mu\rho(c\mu - k\rho)^2\left(l^2 - 2k\delta^*\epsilon\right)n^{16} + q_{14}(n).$$

5. Case 3: type III with quasi-static thermal displacement

We also study two different systems of equations. The first one is the quasi-static situation for the classical thermo-porous-elasticity. The second corresponds to the same situation but using the type III Green-Naghdi theory to model the behavior of the temperature. We will see that the solutions for the first system decay slowly while for the second are exponentially stable.

5.1. **Classical porous-thermo-elasticity.** The first system is given by

$$\left\{\begin{array}{l} \rho\ddot{u} = \mu u_{xx} + b\Phi_x - \beta\theta_x \\ J\ddot{\Phi} = \delta\Phi_{xx} - bu_x - \xi\Phi + m\theta \\ 0 = -\beta\dot{u}_x - m\dot{\Phi} + k^*\dot{\alpha}_{xx} \end{array}\right. \tag{5.1}$$

with the following boundary and initial conditions:

$$u(0,t) = u(\pi,t) = \Phi_x(0,t) = \Phi_x(\pi,t) = \theta_x(0,t) = \theta_x(\pi,t) = 0, \tag{5.2}$$

and

$$u(x,0) = u_0(x),\ \dot{u}(x,0) = v_0(x),\ \Phi(x,0) = \Phi_0(x), \dot{\Phi}(x,0) = \varphi_0. \tag{5.3}$$

To prove the slow decay we repeat the method used before. Let us suppose that

$$u = Ae^{\omega t}\sin(nx), \quad \Phi = Be^{\omega t}\cos(nx), \quad \theta = Ce^{\omega t}\cos(nx), \tag{5.4}$$

therefore we obtain a 3 by 3 homogeneous system of linear equations:

$$\left(\begin{array}{ccc} \mu n^2 + \rho\omega^2 & bn & -n\beta\omega \\ bn & \delta n^2 + J\omega^2 + \xi & -m\omega \\ n\beta & m & n^2 k^* \end{array}\right)\left(\begin{array}{c} A \\ B \\ C \end{array}\right) = \left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right).$$

Direct calculations prove that there exists n large enough that, generically, makes $\Lambda_3 < 0$. In fact, this minor is a tenth degree polynomial with respect to n of the form

$$\Lambda_3 = -2J\,(k^*)^2\,\epsilon(J\mu - \delta\rho)^2\left(\beta^2 - 2k^*\epsilon\rho\right)n^{10} + q_8(n),$$

where $q_8(n)$ is an eighth degree polynomial. The main coefficient of Λ_3 is negative whenever $J\mu \neq \delta\rho$.

5.2. **Type III with quasi-static temperature.** The last case that we study comes from (2.4) when $\dot{\theta} \approx 0$, $k^* > 0$ and $\mu^* = \delta^* = 0$. The new system is given by:

$$\left\{\begin{array}{l} \rho\ddot{u} = \mu u_{xx} + b\Phi_x - \beta\theta_x \\ J\ddot{\Phi} = \delta\Phi_{xx} + l\alpha_{xx} - bu_x - \xi\Phi + m\theta \\ 0 = k\alpha_{xx} + l\Phi_{xx} - \beta\dot{u}_x - m\dot{\Phi} + k^*\dot{\alpha}_{xx} \end{array}\right. \tag{5.5}$$

In this case, we set

$$u(0,t) = u(\pi,t) = \Phi_x(0,t) = \Phi_x(\pi,t) = \alpha_x(0,t) = \alpha_x(\pi,t) = 0. \tag{5.6}$$

As for the initial conditions we consider

$$u(x,0) = u_0(x),\ \dot{u}(x,0) = v_0(x),\ \Phi(x,0) = \Phi_0(x), \dot{\Phi}(x,0) = \varphi_0, \alpha(x,0) = \alpha_0(x). \tag{5.7}$$

As usual, we look for solutions that satisfy

$$\int_0^\pi \Phi_0(x)dx = \int_0^\pi \varphi_0(x)dx = \int_0^\pi \alpha_0(x)dx = 0.$$

We denote by $v = \dot{u}$. From the third equation we get $\dot{\alpha} = -\dfrac{1}{k^*}\Omega$, where

$$\Omega = \Delta^{-1}\left[l\Phi_{xx} + k\alpha_{xx} - \beta\dot{u}_x - m\dot{\Phi}\right].$$

Notice that

$$\Omega_x = l\Phi_x + k\alpha_x - \beta\dot{u} - m\int_0^x \dot{\Phi}ds.$$

We set $\dot{\Phi} = \varphi$. Therefore, with the same notation used before, the system can be written as

$$\left\{\begin{array}{l} \dot{u} = v \\ \dot{v} = \frac{1}{\rho}\left(\mu D^2u + bD\Phi + \frac{\beta}{k^*}D\Omega\right) \\ \dot{\Phi} = \varphi \\ \dot{\varphi} = \frac{1}{J}\left(\delta D^2\Phi + lD^2\alpha - bDu - \xi\Phi - \frac{m}{k^*}\Omega\right) \\ \dot{\alpha} = -\frac{1}{k^*}\Omega \end{array}\right. \tag{5.8}$$

We abuse the notation again (for the last time) and denote by $\mathcal{H}$ the corresponding Hilbert space and by U its elements. In this case, we have $\mathcal{H} = H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times H_*^1$.

Let $U = (u, v, \Phi, \varphi, \alpha)$ and $U^* = (u^*, v^*, \Phi^*, \varphi^*, \alpha^*)$ be two elements of $\mathcal{H}$. We define the following inner product in $\mathcal{H}$:

$$\begin{aligned} \langle U, U^*\rangle = \int_0^\pi \Big(&\rho v\overline{v}^* + \mu u_x\overline{u}_x^* + J\varphi\overline{\varphi}^* + \delta\Phi_x\overline{\Phi}_x^* + \xi\Phi\overline{\Phi}^* + b(u_x\overline{\Phi}^* + \overline{u}_x^*\Phi) \\ &+ l(\Phi_x\overline{\alpha}_x^* + \overline{\Phi}_x^*\alpha_x) + k\alpha_x\overline{\alpha}_x^*\Big)dx, \end{aligned} \tag{5.9}$$

The corresponding matrix operator is

$$\mathcal{C} = \begin{pmatrix} 0 & \mathcal{I} & 0 & 0 & 0 \\ \frac{\mu}{\rho}D^2 & -\frac{\beta^2}{\rho k^*}\mathcal{I} & \frac{b+l}{\rho}D & -\frac{\beta m}{\rho k^*}D\Delta^{-1} & \frac{\beta k}{\rho k^*}D \\ 0 & 0 & 0 & \mathcal{I} & 0 \\ -\frac{b}{J}D & \frac{\beta m}{Jk^*}\Delta^{-1}D & \frac{\delta}{J}D^2 - \frac{ml}{Jk^*}\Delta^{-1}D^2 - \frac{\xi}{J}\mathcal{I} & \frac{m^2}{Jk^*}\Delta^{-1} & \frac{l}{J}D^2 - \frac{mk}{Jk^*}\Delta^{-1}D^2 \\ 0 & \frac{\beta}{k^*}\Delta^{-1}D & -\frac{l}{k^*}\Delta^{-1}D^2 & \frac{m}{k^*}\Delta^{-1} & -\frac{k}{k^*}\Delta^{-1}D^2 \end{pmatrix}. \tag{5.10}$$

The domain of this operator is

$$\mathcal{D}(\mathcal{C}) = \{U \in \mathcal{H} : u \in H^2,\ v \in H_0^1,\ \delta D^2\Phi + lD^2\alpha \in L^2_*,\ \varphi \in H^1_*\}.$$

Lemma 5.1. *The operator $\mathcal{C}$ is dissipative: for any $U \in \mathcal{D}(\mathcal{C})$, $\Re\langle \mathcal{C}U, U\rangle \leq 0$.*

Proof. It is not difficult to see that

$$\Re\langle \mathcal{C}U, U\rangle = -\frac{1}{k^*}\int_0^\pi |\Omega_x|^2 dx.$$

□

Lemma 5.2. $0 \in \varrho(\mathcal{C})$.

Proof. As in the proof of Lemma 4.2 we have to prove that the following system has a solution:

$$\left.\begin{aligned} v &= f_1\\ \mu D^2u + bD\Phi + \tfrac{\beta}{k^*}D\Omega &= \rho f_2\\ \varphi &= f_3\\ \delta D^2\Phi + lD^2\alpha - bDu - \xi\Phi - \tfrac{m}{k^*}\Omega &= f_4\\ \Omega &= -k^* f_5 \end{aligned}\right\} \tag{5.11}$$

The above system reduces to

$$\left.\begin{aligned} \mu D^2u + bD\Phi &= \rho f_2 + \beta D f_5\\ \delta D^2\Phi + lD^2\alpha - bDu - \xi\Phi &= f_4 - m f_5\\ \Omega &= -k^* f_5 \end{aligned}\right\} \tag{5.12}$$

We recall that this is a system of equations for each n. In this case the solution is given by

$$\begin{aligned} u_n &= \frac{n^3\big((\delta k - l^2)\beta + blk^*\big)f_n^5 + r_2(n)}{n^4\mu(\delta k - l^2) + n^2(\mu\xi - b^2)k}\\ \phi_n &= \frac{n^3 f_n^5 k^* l\mu + s_2(n)}{n^3\mu(\delta k - l^2) + n(\mu\xi - b^2)k}\\ \alpha_n &= \frac{n^4 f_n^5 k^*\delta\mu + t_3(n)}{n^4\mu(\delta k - l^2) + n^2(\mu\xi - b^2)k} \end{aligned}$$

Notice that $u \in H^2$ and $\varphi \in H^1_*$. □

Lemma 5.3. $i\mathbb{R} \subset \varrho(\mathcal{C})$.

Proof. Let $U_n = (u_n, v_n, \Phi_n, \varphi_n, \alpha_n)$ be a sequence of vectors in $\mathcal{D}(\mathcal{C})$ with unit norm such that $\|(i\lambda_n\mathcal{I} - \mathcal{C})U_n\| \to 0$. We write the stated condition term by term:

$$i\lambda_n u_n - v_n \to 0, \text{ in } H^1 \tag{5.13}$$

$$i\rho\lambda_n v_n - \mu D^2 u_n - bD\Phi_n - \frac{\beta}{k^*}D\Omega_n \to 0, \text{ in } L^2 \tag{5.14}$$

$$i\lambda_n\Phi_n - \varphi_n \to 0, \text{ in } H^1 \tag{5.15}$$

$$iJ\lambda_n\varphi_n - \delta D^2\Phi_n - lD^2\alpha_n + bDu_n + \xi\Phi_n + \frac{m}{k^*}\Omega \to 0, \text{ in } L^2 \tag{5.16}$$

$$ik^*\lambda_n\alpha_n + \Omega_n \to 0, \text{ in } H^1 \tag{5.17}$$

From Lemma 5.1 it is clear that

$$D\Omega_n = lD\Phi_n + kD\alpha_n - \beta v_n - m\int_0^x \varphi_n ds \to 0 \text{ in } L^2.$$

Therefore, from (5.17) we obtain $\lambda_n\alpha_n \to 0$ in H^1, which means $D\alpha_n \to 0$.

Let us multiply $D\Omega_n$ by $D\Phi_n$, which is bounded. We obtain

$$l\|D\Phi_n\|^2 + k\langle D\alpha_n, D\Phi_n\rangle - \beta\langle i\lambda_n u_n, D\Phi_n\rangle - m\langle \int_0^x \varphi_n ds, D\Phi_n\rangle \to 0. \tag{5.18}$$

Notice that we have used (5.13) and have changed v_n by $i\lambda_n u_n$. As $D\alpha_n$ tends to zero, we can remove the second term. Moreover, using integration by parts and (5.15) we can rewrite the above expression in the following way:

$$l\|D\Phi_n\|^2 - i\lambda_n\beta\langle u_n, D\Phi_n\rangle + i\lambda_n m\|\Phi_n\|^2 \to 0. \tag{5.19}$$

We multiply now (5.16) by Φ_n and perform and integration by parts:

$$iJ\lambda_n\langle\varphi_n, \Phi_n\rangle + \delta\|D\Phi_n\|^2 + l\langle D\alpha_n, D\Phi_n\rangle + b\langle Du_n, \Phi_n\rangle + \xi\|\Phi_n\|^2 \to 0.$$

Removing what tends to 0 and using (5.15) we get

$$-J\|\varphi_n\|^2 + \delta\|D\Phi_n\|^2 - b\langle u_n, D\Phi_n\rangle + \xi\|\Phi_n\|^2 \to 0.$$

From this expression we deduce that $\langle u_n, D\Phi_n\rangle$ should be real and, therefore, looking back at (5.19), it must be $D\Phi_n \to 0$. This implies also that $\varphi_n \to 0$ in L^2. And, hence, from the fact that $D\Omega \to 0$ we obtain that $v_n \to 0$. Finally, to prove that Du_n tends to zero, we multiply what remains in (5.14) after removing the terms that tend to zero by u_n:

$$i\rho\lambda_n\langle v_n, u_n\rangle - \mu\langle D^2u_n, u_n\rangle \to 0$$

or, equivalently, $-\rho\|v_n\|^2 + \mu\|Du_n\|^2 \to 0$, which gives $Du_n \to 0$. □

Lemma 5.4. $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathcal{I} - \mathcal{C})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

The arguments used to prove Lemma 5.3 can be adapted to prove also this case.

The above results imply that the solutions to system (5.5) decay exponentially.

Theorem 5.5. *The C_0-semigroup $S(t) = \{e^{\mathcal{C}t}\}_{t\geq 0}$ is exponentially stable.*

For $\beta = 0$ or $l = 0$ the decay of the solutions is again, generically, slow.

- When $l = 0$ the fourth leading diagonal minor is
$$\Lambda_4 = -2J(k^*)^3\delta\epsilon\mu(J\mu - \delta\rho)^2\left(\beta^2 - 2k^*\epsilon\rho\right)n^{16} + q_{14}(n).$$
- When $\beta = 0$, the fourth leading diagonal minor is
$$\Lambda_4 = -2J(k^*)^3\epsilon\mu\rho(J\mu - \delta\rho)^2\left(l^2 - 2k^*\delta\epsilon\right)n^{16} + q_{14}(n).$$
- When $\beta = l = 0$ the second leading diagonal minor is
$$\Lambda_2 = -2J(k^*)^2\epsilon\rho(J\mu + \delta\rho)n^6 + q_4(n).$$

6. Conclusions

We think that it is interesting to sort out the properties of the solutions to the systems of equations determined by porous-thermo-elastic materials depending on the heat conduction theory used to model the behavior of the temperature. It is quite surprising that while for the Fourier model the waves decay slowly and two well-combined dissipation mechanisms are needed to obtain exponential decay, for the type II and III Green-Naghdi theories only one damping mechanism suffices to get it. In this work we have gone deeper into these questions and we have shown that a similar solutions' behavior appears when the movement of one of the variables is supposed to be quasi-static. Specifically, we have proved the following facts:

(1) If the displacement is quasi-static and we impose viscoelasticity and thermal dissipation for the Fourier model, then the solutions decay slowly. However, if only viscoelasticity is imposed for the type II model (which is conservative) the solutions decay exponentially.
(2) In the isothermal case, when the deformation of the voids is quasi-static and we impose strong viscoporosity the decay of the solutions is slow. However, if we add the conservative mechanism established by the type II heat conduction model the decay is exponential.
(3) Finally, the same contrast is obtained for the Fourier and the type III Green-Naghdi models when the temperature is supposed to be quasi-static.

We want to highlight that, from a mathematical point of view, the kinds of systems we have studied in this work are very different from the ones needed to study the non quasi-static cases.

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