A hesitant fuzzy perceptual-based approach to model linguistic assessments

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UNIVERSITAT POLITÈCNICA DE CATALUNYA

DOCTORAL THESIS

A hesitant fuzzy perceptual-based approach to model linguistic assessments

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A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy in Applied Mathematics

in the

Applied Mathematics Doctoral Program
Department of Mathematics

February 21, 2021

Declaration of Authorship

I, Olga Porro Martorell, declare that this thesis titled, “A hesitant fuzzy perceptual-based approach to model linguistic assessments” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.
- I have worked under the supervision of Dr.Nuria Agell and Dr.Monica Sanchez

Signed: Olga Porro Martorell

Date: January 2021

“The brain thinks, but the heart knows”

Joe Dispenza

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Abstract

School of Mathematics and Statistics (FME)
Department of Mathematics

Doctor of Philosophy in Applied Mathematics

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by Olga PORRO MARTORELL

Multiple-criteria or multiple-attribute group decision-making is a sub-field of operations research that seek to find a common and representative solution given the preferences elicited by a pre-defined group, over a set of alternatives and with respect to a set of coherent criteria (or attributes). Recently, the modelling of natural language in these processes has captured the attention of many researchers. Most of the evaluations in a group-decision making context are inherently imprecise, incomplete or vague, and therefore, experts feel more comfortable using their language rather than numerical values. The use of hesitant fuzzy linguistic term sets is one of the recent tools that enables the modelling of linguistic assessments in multiple-criteria decision-making. Nonetheless, advances in hesitant linguistic multi-attribute group decision making require the development of structures flexible enough to deal with unbalanced and multi-granular linguistic information. More tools are needed in order to really grasp the differences in the qualitative reasoning processes of each individual. This thesis, firstly, introduces a perceptual-based distance able to capture differences between unbalanced linguistic assessments, which is based on a lattice structure of hesitant fuzzy linguistic terms. Secondly, this distance is used to define a perceptual-based centroid or central opinion which, in turn, is used to define a consensus measure or degree of agreement within the group. Thirdly, with the aim to deal with multi-perceptual group decision-making contexts, where each decision maker has its own qualitative reasoning approach, a perceptual-based transformation function and a projected algebraic structure are defined. The developed tools can deal with different multi-granularity linguistic environments. Two applications are presented to demonstrate the utility, relevancy and feasibility of the methods. On the one hand, a specific perceptual-based classification and ranking method is introduced and applied to a real group decision making problem in an educational setting. This framework is used to classify and rank a set of secondary students according to their degree of entrepreneurial competency, which is based on real data provided by the Andorra Government. On the other hand, an extended fuzzy multi-perceptual linguistic TOPSIS is designed and applied to a real group decision making problem in the context of smart city governance. This perceptual extension is used to assess the criteria governing the strategic decision making process of energy multinational companies when deciding where to expand its sustainable services and products.

Acknowledgements

L'any 2020, el recordarem tots per les mascaretes, la distància social, la constant neteja de mans i, evidentment, pels confinaments totals, municipals, comarcals i perimetrals, de tot tipus. L'any 2020 o l'any de la Covid, va ser un any molt difícil per molta gent. Jo també el vaig patir molt, però si una cosa positiva puc treure del confinament, és que em va permetre focalitzar i disposar de més temps per redactar aquesta tesis. Escriure una tesis ha estat per mi, com córrer una marató. És un camí que he recorregut en solitari i he estat capaç de ser constant i persistent per arribar amb prou forces a la meta final. Però, a diferència d'una marató, en el camí de la tesis m'he trobat el recolzament i l'acompanyament de persones extraordinàries que m'han ajudat a no defallir. No voldria deixar de mencionar a ningú.

Sense cap mena de dubte, els primers agraïments que vull donar són per a la Mónica i la Núria. D'aquesta tesis, no només m'emporto coneixements en l'àmbit de la Intel.ligència Artificial en el camp del MCDM, sinó que m'enduc infinitat de moments viscuts amb elles dues. Vam viure situacions molt complicades al principi però les tres som dones resilients i persistents. Quin gran equip que hem format! Us estaré sempre tremendament agraïda. A tu Núria, per tot el que has arribat a fer per mi i la confiança que sempre m'has dipositat. Sense tu, segur que no hagués acabat aquest doctorat. I a tu Mónica, per tot el teu infinit i constant recolzament, la teva paciència, i per preocupar-te sempre tant per mi. Sort n'he tingut de tenir-te.

Al llarg d'aquest camí també he pogut tenir la sort de rebre l'ajuda d'antics doctorants (ara ja doctors!), com en Jordi Montserrat i la Jenn Nguyen. Els vull donar les gràcies per compartir amb mi el seu coneixement i per també, donar-me suport moral sempre que ho he necessitat. Com bé diu un proverbi xinès: 'Para conocer el camino que aún te falta por recorrer, pregúntale a los que ya vienen de vuelta'.

Molt especialment, vull donar les gràcies al Francesc Pardo. Quina gran descoberta!. Espero emportar-me la seva amistat per sempre. Gràcies a ell, el meu primer article va poder sortir a la llum. Sense ell, no hagués estat possible, no en tinc cap dubte. Gràcies Francesc per la confiança que em vas fer i sobretot, per ser una persona tan excepcional.

També, vull agrair al Francisco el seu suport en la elaboració del meu segon paper. Recordo molt bé aquella reunió en la què vam passar de les matemàtiques a la filosofia amb menys de tres segons. Les teves recomanacions sempre han sigut fonts d'inspiració i insights claus per avançar en la meva investigació.

També vull expressar paraules d'agraïment d'aquesta tesi a tots aquells doctors i doctorands amb qui he anat conversant en les diverses conferències, workshops, viatges, etc. que he assistit com a doctoranda (com les EWG-MCDA meetings, European Conference on Operational Research o la International Conference of the Catalan Association for Artificial Intelligence).

A més a més, ha estat un plaer formar part de l'equip d'investigadors de l'Institut d'Emprenedoria d'ESADE (EEI). En concret, vull donar les gràcies al Marcel Planelles per deixar-me ser part important del projecte d'assessorament i avaluació del sistema educatiu andorrà en relació a la competència emprenedora dels seus estudiants de secundària. També, molt especialment, vull agrair al Josep Alias. Sempre ha trobat les paraules per animar-me a seguir amb el doctorat i m'ha donat tot el recolzament que podia, i més!. Gràcies a aquest projecte, he pogut interactuar amb grans professionals del sector educatiu andorrà, des de la Ministra d'Educació al professor de matemàtiques de l'escola d'Ordino. També, a la Montse i l'Anna, dos grans investigadores de la universitat Blanquerna.

Mirant enrere, tampoc em puc oblidar de donar les gràcies a qui va fer possible que jo un dia anés a estudiar matemàtiques al Courant Institute of Mathematical Science de New York University. Sense cap mena de dubte, no seria on sóc ara si no hagués conegut al Valentí Giró. Un amic, un mentor, un coach qui entremig de les muntanyes de l'Himalaya, em va animar a seguir endavant amb els meus propòsits sense pors. Després, arribaria l'Elsa i aviat farà més de 10 anys que ens coneixem i que compartim aventures increïbles. Elsa moltes gràcies per ajudar-me a ser valenta. A l'Alfred Vernis per obrir-me al món de l'emprenedoria i a la innovació social. Des d'aconseguir la beca Fulbright fins a obrir-me nous projectes, l'Alfred sempre m'ha dedicat infinitat de temps. Ell ha estat una font d'inspiració per a mi.

En l'àmbit personal i no tan acadèmic, m'agradaria en primer lloc donar les gràcies als meus pares, en Fonso i la Farners. Jo no hauria arribat fins aquí ni seria qui sóc, sense ells. Han estat sempre el pal de paller que m'ha permès ser una mica més valenta: afrontar els meus dos anys a NY, la meva vivència al Nepal, les etapes de tristesa, les victòries, etc. .... Com diuen ells, són el millor equip de logística del món. La veritat és que no hi ha paraules que puguin descriure l'agraïment que sento cap a ells. Tampoc seria la mateixa persona, sense els meus dos germans, l'Alfons i en Joan. L'Alfons, el meu germà petit, significa suport incondicional sense demanar res. En Joan, el meu germà gran, és un escalf de proximitat, amor i generositat plena tot i la distància. A ells, m'he pogut obrir per explicar-los les meves emocions i patiments en moments difícils. Són el millor tàndem!

També paraules d'agraïment per les emocions que em dona la meva neboda, la Lea. I pel seu germanet, en Soan, recent nascut mentre jo escrivia les últimes línies d'aquesta tesis. A ells dos, els desitjo de tot cor un camí professional alineat amb el seu propòsit vital i espero que la societat els deixem un món millor.

A les meves amigues vull agrair-los el seu afecte i amor durant aquests anys. M'han ajudat a confiar una mica més amb mi mateixa. A la Maria i la Carlota, perquè ens coneixem des de que vam néixer i sembla que tot segueix igual. El nostre grup Carpe Diem – Hakuna Matata em recorda sempre que hem de viure intensament el moment (amb tesis o sense tesis!). A la Rebeca i a la Carla, quin plaer tan gran haver-vos conegut aquí a Girona. Conèixer-vos ha estat una gran descoberta. Al grup Hachiko d'ESADE, elles m'han acceptat tal com sóc; sense fer màrqueting, ni finances ni seguir el camí establert d'una noia llicenciada per ESADE. I si miro més enrere en el temps, no vull perdre ocasió d'agrair a tota la petita família de la residència Sant Jordi els anys viscuts a Barcelona. Ens hem fet grans....!

A la Elisa i la Javiera també vull dedicar-los-hi unes paraules. Elles van estar amb mi tot el viatge del MSc al Courant Institute. Si no hagués superat aquella etapa, aquesta tesis no seria possible.

També a la Cintia, qui, al llarg dels últims quatre mesos d'aquesta tesis, amb tot el seu amor, m'ha ajudat a no perdre el rumb del meu propòsit vital.

Ara bé, si hi ha alguna persona que es mereix totes les gràcies que tinc per donar és en Miquel Sunyer, la meva parella i el meu company d'aventures, alegries, tristeses i converses dels últims 6 anys. Ell és qui ha viscut aquesta tesis doctoral dia a dia, en tots els moments bons i no tant bons. No hagués estat capaç de sobreposar-me als moments difícils sense tenir-lo al meu costat. Em sento molt afortunada de tenir una persona com ell al meu costat. Miquel, que per molts anys puguem fer semblances obres amb *més* manobres!.

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List of Abbreviations

AHP	**A**nalytic **H**ierarchy **P**rocess
AI	**A**rtificial **I**ntelligence
CWW	**C**omputing **w**ith **W**ords
DM	**D**ecision **M**aker
EC	**E**ntrepreneurial **C**ompetency
EHFLTS	**E**xtended **H**estitant **F**uzzy **L**inguistic **T**erm **S**et
GDM	**G**roup **D**ecision **M**aking
HFLD	**H**estitant **F**uzzy **L**inguistic **D**escription
HFLTS	**H**estitant **F**uzzy **L**inguistic **T**erm **S**et
LTS	**L**inguistic **T**erm **S**et
MAGDM	**M**ultiple-**A**ttribute **G**roup **D**ecision-**M**aking
MAGDA	**M**ultiple-**A**ttribute **G**roup **D**ecision-**A**iding
MCDM	**M**ultiple-**C**riteria **D**ecision-**M**aking
MCDA	**M**ultiple-**C**riteria **D**ecision-**A**iding
MADM	**M**ultiple-**A**ttribute **D**ecision-**M**aking
MADA	**M**ultiple-**A**ttribute **D**ecision-**A**iding
OR	**O**perations **R**esearch
PHFLTS	**P**roportional **H**estitant **F**uzzy **L**inguistic **T**erm **S**et
TOPSIS	**T**echnique for **O**rder of **P**reference by **S**imilarity to **I**deal **S**olution
ULTS	**U**nbalanced **L**inguistic **T**erm **S**et

Dedicated to: la Mama, el Papa, l'Alfons, en Joan i en Miquel

Chapter 1

Introduction

1.1 Motivation

One of my first motivations for writing a PhD Thesis in Applied Math was to make a theoretical contribution to the field of artificial intelligence (AI). Turing's paper 'Computing Machinery and Intelligence' (1950), and it's subsequent Turing Test, established the fundamental goal and vision of artificial intelligence: "Can machines think?" Now, in the 21st century, we are still working to incorporate learning, reasoning and perception to machines so they can mimic and execute tasks the way we do. For instance, we, humans, are used to make quick decisions based on imprecise and vague linguistic information. The modelling of natural human reasoning with imprecise, incomplete or vague linguistic information is currently one the challenges of Artificial Intelligent Systems. This area interacts with other fields such as Qualitative Reasoning (QR), Computing with Words (CWW) and MCDM (multi-criteria decision-making).

MCDM (or MADM) is a sub-discipline of Operations Research (OR) which has been very active for the last 50 years. The origins of decision analysis go back to many centuries ago According to [100], the first known recorded work on MCDM (although not using that name) was carried out by Benjamin Franklin. Even before Franklin's times, Aristotle (384–322 bc), a famous Greek philosopher and polymath, defined 'preferences' as 'rational desires'. This might have been the first time where someone made the connection between rational decision making and human desires (preferences). However, formally speaking, the economist Vilfredo Pareto (1848–1923) was probably the first researcher whose work might be classified as MCDM.

On the one hand, in the context of MCDM, due to the rapid growth of information and communication exchange, the interaction between experts is more common than a unique individual governing the decision making process. These contexts are included in the field of Group Decision Making (GDM). On the other hand, the contributions to develop linguistic tools have been increasing since the last 20 years. Many MCDM problems are better evaluated by means of qualitative descriptions, rather than quantitative values. For example, suppose a professor has to evaluate the creativity skills of a student. The use of numerical values is not aligned with his way of thinking. However, if he can express his opinions using a linguistic term set such as *exceptional, very good, average, poor, very poor* his evaluation becomes more realistic (as compared, to assign a crisp number to each student).

This leads to the research area of Linguistic Multi-Attribute or Multi-Criteria Group Decision-Making or Aiding (MAGDM, MCGDM, MCGDA, MAGDA) which is used when a group of experts (or Decision Makers) express their linguistic assessments or qualitative preferences on a set of attributes (or criteria) for a set of alternatives and an optimal representative or common solution is needed to solve the problem [63, 64].

Many practical applications have used hesitant fuzzy linguistic term sets (HFLTSs) to deal with the linguistic information involved in linguistic MAGDM problems [108, 173]. HFLTSs were developed by Rodriguez et al. in 2012 [142] to deal with situations in which experts hesitate between several values to assess an indicator, alternative, variable, etc. HFLTSs were introduced to handle the uncertainty and imprecision inherent in many multicriteria linguistic decision-making models. Initially, they were based on the use of a balanced linguistic term set. The use of HFLTSs provides now a linguistic and computational frame to model the opinions provided by the group of experts, based on the fuzzy linguistic approach and the use of context-free grammars [24, 67, 142].

Nonetheless, as the MAGDM problems become more complex and interact with AI, the help of human intuition becomes essential as significance gain importance over precision. With respect to linguistic MAGDM, there is a need to develop more flexible tools. In the case of HFLTSs, these are limited tools when it comes to model linguistic expressions which present two or several sources of fuzziness simultaneously.

In many real-life MAGDM contexts, unbalanced linguistic term sets (ULTSs) [81, 89, 115] are usually needed to model DMs or experts opinions. The sources of unbalance might be different. It may be that the weight or 'length' of each linguistic label of the ULTS is different due to the personal characteristics of each DM. Given a ULTSs of a given granularity, the type of symmetry or uniformity might differ based on each individual profile. For instance, suppose two different patients, A and B, from a hospital, have to weekly express their pain verbally to their doctor. Unfortunately, both patients are in the same stage of a lung cancer. However, Patient A is a strong young man with no relevant medical history while Patient B is a weak elderly man whose clinical history includes past serious illnesses, three kidney surgeons and lots of medications. Suppose each week, the doctor asks them about their pain intensity. He uses a universal pain assessment tool consisting of six linguistic labels. His question is: *'How do you feel about your pain today? No pain, mild, moderate, severe, very severe or worst possible?'*. Suppose both patients, give the same answer; 'Severe'. My concern is: Do these two exact same answers weight the same? Can we attribute the same significance or importance to them? Do both patents place the expression 'severe' in the same position in the given linguistic unbalanced scale of six labels? My intuition tells me that we would respond no to these questions.

Extensions of HFLTSs based on ULTSs are found in the literature [28, 30, 196] to address this issue. However, what happens now if the doctor decide to pose different questions to the patients? Suppose that, for clarity purposes, he decides to ask a different question to patient A based on just three labels of intensity pain. The new question for Patient A is now: *How do you feel about your pain today? No pain, moderate, or very severe?*. In these circumstances, patient A gives an answer on a set of three linguistic labels while Patient B keeps answering on a set of six linguistic labels. Most of the extensions of HFLTSs based on ULTs which are found in MAGDM problems, are restricted to use the same ULTS, i.e., with the same granularity. Inspired by this example, it is clear to me that we can find many situations in which experts, decisions makers or individuals might need to give an answer over different unbalanced LTS.

These two problematics can be found simultaneously in a MAGDM problem. As far as I know, very few HFLTS-based linguistic representation methods for MAGDM problems can simultaneously deals with hesitant unbalanced and multi-granular linguistic information. Specifically, in this thesis, we focus on developing new methods to manage multigranular- ULTS in hesitant fuzzy linguistic MAGDM contexts.

The specific motivation of this thesis is to make a contribution to the field of linguistic MAGDM/MCGDM with the aim to study tools and frameworks to model the different qualitative reasoning processes of individuals. This is done through the formulation of the perceptual-map concept over unbalanced HFLTSs. The developed tools include the definition of new distance and consensus measures to handle situations where individuals who express their opinions have different knowledge and backgrounds. Since one of the main goals of artificial intelligence is to include and model human reasoning and perception, my motivation is also to provide insights and put my small sand grain in the advancement of Artificial intelligence.

1.2 Objectives of the thesis

The main theoretical objective of this Ph.D. thesis is to develop a mathematical framework to handle, at the same time, uncertainty, multi-granularity and multi-qualitative reasoning processes in multi-attribute group decision-making situations with linguistic information. To this end, a new algebraic structure, a distance and specific aggregation methods have been developed. These have been incorporated into existing MAGDM methodologies, in particular, TOPSIS method, to enrich the elicitation of linguistic judgements by means of unbalanced HFLTSs. Moreover, new ranking and classification methods based on these theoretical developed tools have been introduced as a response of two real multi-attribute group decision making problems, one requested by the Andorra government and the other as part of an EU Project for smart cities. More precisely, this manuscript seeks to respond to the following objectives:

O1. To define a new algebraic structure and a distance over unbalanced and multigranular HFLTSs. In practical terms, these theoretical tools are useful to model the different qualitative reasoning processes of experts, evaluators or DMs. The presented methodology allows them to express their opinions and judgements with different linguistic assessments' semantics. To this end, different lengths of the linguistic terms and different weights for each basic linguistic label are considered.

O2. To define a perceptual map as a new normalized measure over this structure of unbalanced and multigranular HFLTSs and study its properties.

O3. To extend existing definitions of unbalanced linguistic term set or possibility distributions over HFLTSs based on the new introduced normalized measure.

O4. To define a new distance between HFLTSs in the distributive lattice of the extended set of HFLTSs in order to measure differences between linguistic assessments modelled by unbalanced HFLTSs, incorporating the information provided by the perceptual maps.

O5. To define a consensus measure for linguistic MAGDM contexts to quantify the level of agreement within a group that uses different perceptual maps.

O6. To develop a transformation function for multi-perceptual MAGDM environments to simultaneously model multi-granularity and operate with different perceptual maps, through a new projected linguistic structure.

O7. To introduce new ranking and classification framework for solving multi-attribute group decision making problems involving heterogenous experts or DMs who provide their linguistic assessments in an unbalanced and multi-granular manner while dealing with heterogeneity and uncertain data. This means that we allow them to use the linguistic expressions that they feel more confident with and that attributes can be of very different type.

O8. To adapt existing multiple-attribute group decision making methods such as TOPSIS to simultaneously model the hesitancy, multi-perceptual and multi-granularity of linguistic assessments.

O9. To apply and test the developed methodologies to different social and sustainable projects in real business environments.

1.3 Contributions of the thesis

The contributions of this Ph.D. thesis can be summarized as follows:

C1. The first contribution of this thesis is the introduction of the concept of a perceptual-map over unbalanced HFLTSs. The properties and characteristics of the perceptual map defined over the set of positive HFLTSs are presented in detail. This contribution corresponds to the objectives O1 and O2.

C2. The second contribution of this thesis is the introduction of a new distance over the extended lattice of HFLTSs based on a context of unbalanced linguistic assessments. The novelty of this distance is that it can work with unbalanced HFLTSs, hence it can model situations where the elements of the linguistic term sets are not uniform or symmetrically distributed. The basics of the new distance were introduced and explained in the conference:

- CCIA 2018, the 21st International Conference of the Catalan Association for Artificial Intelligence. *October 8-10th, 2018 – Roses, Catalunya*. The main goal of this international conference is to foster discussion around the last advances in Artificial Intelligence. CCIA is the usual meeting point of the Catalan AI scientific community.
(https://ccia2018.upc.edu/en/conference-programme/conference-programme)

This contribution responds to the objectives O1, O2, O3 and O4.

C3. The third contribution is the study of a consensus measure in a multi-granular and unbalanced GDM linguistic context and use it in a real GDM application. The values of the consensus measure are used as a weighting factors in the GDM framework developed for the Andorra Government to assess students on their entrepreneurial competency development. This contribution matches the objectives O3, O4, O5, O6 and O9. Results of this contribution have been submitted to the following journal and are currently under review:

- Porro Olga, Agell Núria, Sánchez Mónica, Ruiz Francisco-Javier. A multi-attribute group decision model based on unbalanced and multi-granular linguistic information: an application to assess entrepreneurial competencies in secondary schools. *Applied Soft Computing Journal*. Currently under review. Impact Factor: 5.472 (2019). JCR Categories: Computer Sciences & Artificial Intelligence and Computer Sciences, Interdisciplinary applications.

Preliminary works on this topic were initially presented in the following conference:

- 25th International Conference on Multiple Criteria Decision Making. *June 16-21st, 2019 – Istanbul Technical University, Istanbul, Turkey*. International society on MCDM: (https://mcdm2019.org/) A presentation done on 'Entrepreneurship Education: Towards the validity of an impactful and effective framework for business schools'.

C4. The forth contribution is the extension and improvement of existing MAGDM methods to incorporate the possibility of different experts or DMs to have different perceptual maps. The aim of this extension is to model and operate with the linguistic assessments into a projected space where the differences in the qualitative reasoning processes are captured. Specifically, this perceptual-based transformation framework is used to present a new version of the traditional TOPSIS (Technique for Order of Preference by Similarity to Ideal Solution) method which is suitable for modeling the different qualitative reasoning approaches of experts prior to make a ranking of alternatives. This has been used and applied in a real group decision making process to assess relevant sub-criteria location factors for multinational energies, under the umbrella of a smart city research project. The results of this study have been published in the following article of the Journal Energies, with current Impact Factor 2.702 (2019) and JCR Category 'Energy & Fuels'.

- Porro, O., Pardo-Bosch, F., Agell, N., & Sánchez, M. (2020). Understanding Location Decisions of Energy Multinational Enterprises within the European Smart Cities' Context: An Integrated AHP and Extended Fuzzy Linguistic TOPSIS Method. *Energies*, 13(10), 2415.

Preliminary works on this topic have been presented in the following international conference:

- 89th meeting of the EURO Working Group in Multi Criteria Decision Aiding (EWG-MCDA). *April 11-13th, 2019 - Department of Industrial Engineering of the University of Trento, Italy*. (https://event.unitn.it/ewg-mcda2019/) 'A location decision problem based on AHP: Strategic priorities of European energy companies' was presented.

C5. Another practical contribution is the training and testing of the different developed tools along with existing MCDM/A methods in a social business enterprise, Vies Braves. The enterprise served as a 'simulation' to validate hypothesis. Vies Braves (Sea Swimming Lanes) is a pioneer company in designing, promoting, and revitalizing open water and marine itineraries, protected by buoys, and dedicated to health, leisure, educational and environmental protection activities. The management team's strategic priority is the implementation of new Vies Braves in new coastal locations. The purpose of this contribution was to test and investigate the feasibility of different sorting or/and classification methods. Works on this topic have been presented in the following international conferences:

- 87th Meeting of the European Working Group on Multicriteria Decision Aiding (EWG/MCDA). *April 5th-7th, 2018 - Delft University of Technology*. EURO Working Group Multicriteria Decision Aiding: (www.cs.put.poznan.pl/) 'Towards a definition of the social entrepreneurship concept in traditional small and medium enterprises' was published in the Book of Abstracts.

- EURO 2019, the 30th European Conference on Operational Research. *June 23-26th, 2019 – Dublin, Ireland*. (www.euro2019dublin.com/) The work on 'Categorizing coastal municipalities for sustainable development: A multi-criteria decision aiding sorting tool based on ELECTRE-Tri-C' was presented.

- 90th meeting of the EURO Working Group in Multi Criteria Decision Aiding (EWG-MCDA). 'Decision aiding for a sustainable development of the ocean'. *September 26-28th, 2019 – IMT Atlantique, Brest Campus*. (http://conferences.imt-atlantique.fr/mcda90). The abstract and presentation 'A multi-criteria decision-aiding approach to designing sustainable marine itineraries' was shared and explained to all attendants.

This contribution covers the objectives O7, O8 and O9.

C6. Another practical contribution is the training and testing of the developed tools with a medium size business. The chosen enterprise (Pellets Farners) was a 'green enterprise' generating an environmental positive impact. Preliminary results were presented in:

- EURO 2018, the 29th European Conference on Operational Research. *July 8-11th, 2018 – Valencia*. EURO 2018 is the largest and most important conference for Operational Research and Management Science (OR/MS) in Europe organized by EURO – the European Association of Operational Research Society. (http://euro2018valencia.com/). 'Towards the generation of social entrepreneurial impact in the traditional wood and biomass sector: A multi-criteria decision aid perspective' was presented in a workshop-presentation session during the conference. Feedback was positively received.

This contribution seek to meet the objectives O7, O8 and O9.

C7. Participation in non-competitive and competitive research projects:

- The INVITE Research Project TIN2016-80049-C2-1-R and TIN2016-80049-C2-2-R AEI/FEDER, UE.

- Horizon 2020 Research Innovation Programme under the grant agreements No 731297).

- Private-funded projects (Esade Entrepreneurship Institute and Andorra Ministery of Education).

- Non-for-profit projects (Vies Braves)

From a collaborative point of view, this thesis has also contributed to continue exchanging knowledge between UPC-BarcelonaTech, ESADE Business School and the Andorra Government. This collaboration has aroused due to the consultancy project developed by the Esade Entrepreneurship Institute for the Ministry of Education of Andorra with the aim to develop and auditing and evaluation tool for measuring entrepreneurial competencies among secondary students. I personally worked on this project and interacted with many school professors as well as public agents. The theoretical framework developed in this thesis is the basis for the materials and tools developed for the Andorra Government.

Furthermore, the development of this thesis research have contributed to a H2020 EU project related to Smart Cities and Sustainability under the auspices of ESADEgov, led by Dr Francisco Pardo Bosch. The possibility to capture the different reasoning approaches of experts, by means of perceptual maps, is key to get the most accurate and adjusted-to-reality results and conclusions, when questionnaires are used to elicit opinions from experts.

Last but not least, during summer 2018, I participated in the Euro PhD Summer School on MCDM/A celebrated in Chania, Greece. For two weeks I successfully completed all the courses of the Summer School, which included theoretical lectures, case project work, and preparation of project report, earning 5 ECTS credits. Besides, this allowed me to share my findings with other students in the field. This Ph.D. development and results have been partially supported by the INVITE Research Project (TIN2016-80049-C2-1-R and TIN2016-80049-C2-2-R (AEI/FEDER, UE)), funded by the Spanish Ministry of Science and Information Technology, the Andorra Government and the European Union 'Horizon 2020 Research and Innovation Programme', under the grant agreement No 731297.

1.4 Outline of the Thesis

The present manuscript is structured in several chapters which are summarized as follows:

- Chapter 2 is an overview of the State of the Art and theoretical framework needed for the development of this thesis.
- Chapter 3 includes all the new mathematical tools that I have developed within the context of the HFLTS lattice structure embedded with a perceptual-map. Two specific frameworks which allow for multi-granularity and multi-perceptual maps are developed.
- Chapter 4 applies the developed methodology to the first real case example based on smart city governance. The framework is used for assessing the potential of smart cities to attract new multinational companies of the energy sector.
- Chapter 5 applies the developed methodology to the second real case example in the context of secondary schools. The Andorra Government requested to develop a new evaluation tool for assessing the entrepreneurial profile of its secondary students.
- Chapter 6 presents some conclusions of the thesis and introduces some future work directions.

Chapter 2

Preliminaries

2.1 Introduction

Multiple attribute group decision analysis (MAGDA) or multi-criteria group decision analysis (MCGDA) is used when a group of experts or Decision Makers (DM) express their assessments or preferences on a set of attributes (or criteria) for a set of alternatives and an optimal representative or common solution is needed to solve the problem [63, 64]. It is very common that, in these group decision making (GDM) environments, experts or decision makers (DMs) do not feel at ease in using numerical values to express their preferences or judgements, but rather feel more comfortable using linguistic terms, i.e., words. Actually, the natural language is what governs uncertain human's cognitive process and it is more appropriate for conveying uncertain assessments whose nature is vague, imprecise or incomplete [109, 142].

The modeling of linguistic information has been done with several tools (type-2 fuzzy sets, 2-tuple model, proportional 2-tuple model, etc). Recently, the introduction of HFLTSs [142], which effectively captures the hesitancy in linguistic assessments, has attracted significant attention from researchers and many practical applications have used HFLTSs to deal with linguistic MCGDA problems [108, 173]. The use of HFLTSs provides a linguistic and computational frame to model the linguistic expressions and opinions provided by the expert or group of experts, based on the fuzzy linguistic approach and the use of context-free grammars [23, 24, 67, 142]. Some contributions on the definition, properties, computational functions, distances, etc. in relation to HFLTSs can be found in the literature [104, 106, 109, 121, 123, 142, 182]. In addition, different extensions of HFLTSs have also been studied and applied to linguistic MCGDA applications, such as hesitant intuitionistic fuzzy linguistic term sets [16, 135] which allow experts to use some membership and non-membership values, interval-valued hesitant fuzzy linguistic term sets [175], which allow DM's to use interval values, or the extended hesitant fuzzy linguistic term sets [172] which enables any non-consecutive linguistic term appear in the expression.

In this chapter 2, the state of the theory with respect to HFLTSs, its associated distances and some techniques of MAGDA that will become useful for the development of the new methodology proposed in chapter 3, are illustrated in detail. The rest of this chapter is organized into two main sections, Section 2.2 and Section 2.3. On the one hand, Section 2.2 provides all the theoretical framework related to the concept and properties of HFLTSs. This section includes the explanation of the extended lattice of HFLTSs and its operations of extended connected union and intersection of HFLTSs. It also introduces the needed knowledge of some extensions of HFLTSs which will be then used in chapter 3 and 4. It also provides an specific subsection devoted to provide an overview of existing distances and similarity measures between HFLTSs. Many examples are provided in order to clarify concepts. In addition, an specific subsection is also developed to explain consensus measures in the context of HFLTSs. Finally, the issues of unbalanced HFLTSs and multigranularity are presented. On the other hand, Section 2.3 is dedicated to provide the theoretical background needed to develop the perceptual-based extension of existing MAGDM methods which will be used to solve the application in Chapter 4. The methods of AHP and TOPSIS are introduced in more detail.

2.2 Hesitant Fuzzy Linguistic Term Sets for decision making

This section is devoted to provide the preliminary needed theoretical framework on the specific fuzzy linguistic approach used in this thesis to model experts' assessments, i.e., the hesitant fuzzy linguistic term sets. A brief review about the distances and consensus measures specifically developed for this fuzzy linguistic modeling is provided. Furthermore, emphasis is given to the algebraic structure of the extended set of HFLTSs and also, some newer extensions of HFLTSs, such as the extended hesitant fuzzy linguistic term sets (EHFLTSs) and the proportional hesitant fuzzy linguistic term sets (PHFLTSs) are given special attention.

2.2.1 The concept of Hesitant Fuzzy Linguistic Term Sets

The concept of hesitant fuzzy linguistic term sets (HFLTSs) was introduced by Rodriguez et al. in [142], based on the notions of fuzzy linguistic approach [197] and hesitant fuzzy sets [166], to provide a linguistic and computational basis to increase the richness of linguistic elicitation. The use of HFLTSs allow experts to hesitate among several linguistic terms and use richer and more complex linguistic expressions to asses an indicator, alternative or variable. Experts or DMs instead of being forced to provide a precise and crisp linguistic term set, this new linguistic modelling is flexible enough to deal with expressions such as for example "more than moderate", "less than appropriate", "between good and extremely good". A state of the art survey on HFLTSs and its applications in decision-making can be found in [108].

The notion of HFLTS is based on balanced linguistic term sets as follows:

Definition 2.1. ([142]) Let S be a totally ordered and balanced set of linguistic terms, S=$\{s_1, \ldots, s_n\}$, with $s_1 < \cdots < s_n$. *A hesitant fuzzy linguistic term set (HFLTS)* over S is a subset of consecutive linguistic terms of S, i.e., $\{x \in S \,|\, s_i \leq x \leq s_j\}$, for some $i, j \in \{1, \ldots, n\}$ with $i \leq j$.

The set of all possible HFLTSs over S is denoted by $\mathcal{H}_S$ and it includes the empty HFLTS, denoted as, $\emptyset$. HFLTSs are usually identified by their envelope.

Definition 2.2. ([142]) Let S be a linguistic term set and H_S be a HFLTS as introduced in Definition 2.1. The *upper bound*, H_{S+}, and the *lower bound*, H_{S-}, of H_S are defined as:

- $H_{S+} = \max(s_i) = s_j, s_i \in H_S$ and $s_i \leq s_j, \forall i$.
- $H_{S-} = \min(s_i) = s_j, s_i \in H_S$ and $s_i \geq s_j, \forall i$.

Definition 2.3. ([142]) The *envelope* of the HFLTS, $env(H_S)$, is a linguistic interval whose limits are obtained by means of the upper bound and lower bound, as defined in Definition 2.2. Hence, $env(H_S) = [H_{S-}, H_{S+}]$.

HFLTSs were developed to represent complex linguistic expressions based on a context-free grammar G_H [142]. A new linguistic group decision model that facilitates the elicitation of more complex and richer linguistic expressions was developed based on context-free grammars and the use of HFLTSs [141].

Definition 2.4. ([141]) Let G_H be a context-free grammar and $S = \{s_0, \ldots, s_g\}$ a linguistic term set. The elements of $G_H = (V_N, V_T, I, P)$ are defined as follows:

$V_N = \{\langle \textsf{primary term} \rangle, \langle \textsf{composite term} \rangle, \langle \textsf{unary relation} \rangle,$

$\langle \textsf{binary relation} \rangle, \langle \textsf{conjunction} \rangle\}$

$V_T = \{\textsf{lower than}, \textsf{greater than}, \textsf{at least}, \textsf{at most},$

$\textsf{between}, \textsf{and}, s_0, s_1, \ldots, s_g\}$

$I \in V_N$

The production rules are defined in an extended Backus Naur Form so that the brackets enclose optional elements and the symbol | indicates alternative elements [24]. For the context-free grammar, G_H, the production rules are as follows:

$P = \{I ::= \textsf{primary term}\rangle \mid \langle \textsf{composite term} \rangle$
$\langle \textsf{composite term} \rangle ::== \textsf{unary relation}\rangle \langle \textsf{primary relation} \rangle \mid \langle \textsf{binary relation} \rangle$
$\langle \textsf{primary term} \rangle \langle \textsf{conjuntion} \rangle \langle \textsf{primary term} \rangle$
$\langle \textsf{primary term} \rangle ::= s_0 \mid s_1 \mid \cdots \mid s_g$
$\langle \textsf{unary relation} \rangle ::= \textsf{lower than} \mid \textsf{greater than} \mid \textsf{at least} \mid \textsf{at most}$
$\langle \textsf{binary relation} \rangle ::= \textsf{between}$
$\langle \textsf{conjunction} \rangle ::= \textsf{and}\}$

Depending on the specific problem, the context-free grammar can generate different linguistic expressions and it can accommodate comparative linguistic expressions similar to the expressions commonly used by experts in GDM problems [141]. This idea is illustrated in example 2.1.

For computational purposes, besides the envelope, it is also necessary to introduce the concept of the possibility distribution for HFLTS developed in [187]. Zhang et al. [200] initially proposed the concept of distribution assessment in a linguistic term set. Then, in [187], Wu and Xu introduced the concept of the possibility distribution in the framework of HFLTSs and developed some aggregation operators to manage the fusion over HFLTS. Wu and Xu framework can be seen as a special case of Zhang et al.'s approach, since they assume that the experts have an equal possibility to employ the linguistic terms in S. The notion is also based on balanced linguistic term sets as follows:

Definition 2.5. ([187]) Let S=$\{s_1, s_2, \ldots, s_n\}$ be a linguistic term set. Let $H_S = [s_L, s_U]$ be a HFLTS given by an expert. The *possibility distribution* for H_S on S is represented by $P = \{p_1, p_2, \ldots, p_n\}$, where p_l is given by the following:

$$p_l = \begin{cases} 0, & \textit{if } l = 1, 2, \ldots L\text{-}1; \\ \dfrac{1}{(U-L+1)}, & \textit{if } l = L, L+1, \ldots, U; \\ 0, & \textit{if } l = U+1, \ldots, n. \end{cases}$$

p_l denotes the possibility degree under which the alternative has an assessment value s_l provided by the expert, such that $\sum_{l=1}^{n} p_l = 1$ and $0 \leq p_l \leq 1, l = 1, 2, \ldots, n$.

Moreover, Wu and Xu [187] developed an HFLWA operator for the aggregation of HFLTSs, which is based on their possibility distributions.

Definition 2.6. ([187]) Let S be as before. Let $H_S^j = \{s_{L_j}, s_{L_{j+1}}, \ldots, s_{U_j}\}$ $(j = 1, 2, \ldots, k)$ a collection of k HFLTSs and $w = (w_1, w_2, \ldots, w_k)^T$ be the associated weight vector, thus satisfying $0 \leq w_j \leq 1$ and $\sum_{j=1}^{k} w_j = 1$. Each HFLTSs can be transformed into a possibility distribution $P_j = (p_{j1}, \ldots, p_{jl}, \ldots, p_{jn})$. The HFLWA operator is also defined as a possibility distribution $P = (p_1, \ldots, p_l, \ldots, p_n)$

$$HFLWA(H_S^1, H_S^2, \ldots, H_S^k) = HFLWA(P_1, P_2, \ldots, P_k) = (p_1, \ldots, p_l, \ldots, p_n) \tag{2.1}$$

where $p_l = \sum_{j=1}^{k} w_j p_{jl}$.

For comprehensiveness purposes, let us introduce the following example.

Example 2.1. Let S be a set of five linguistic term sets, $S = \{s_1, s_2, s_3, s_4, s_5\}$, with meaning s_1: not important, s_2: low importance, s_3: somewhat important, s_4: very important, s_5: extremely important. Four experts are asked to give their opinion about a given attribute, based on the set S, and the resulting linguistic assessments are: "Low importance", "At least very important", "Not extremely important" and "Between low importance and very important". Their corresponding HFLTSs, H_i, to model each assessment along with their corresponding envelope and possibility distribution are provided in the second, third and fourth columns, respectively, of Table 2.1. Note that when the envelop is $[s_i, s_i]$, then I denote it as $\{s_i\}$.

Linguistic assessments	H_i	$env(H_i)$	Possibility Distribution
"Low importance"	$H_1 = \{s_2\}$	$\{s_2\}$	$P_1 = (0, 1, 0, 0, 0)$
"At least very important"	$H_2 = \{s_4, s_5\}$	$[s_4, s_5]$	$P_2 = (0, 0, 0, 0.5, 0.5)$
"Not extremely important"	$H_3 = \{s_1, s_2, s_3, s_4\}$	$[s_1, s_4]$	$P_3 = (0.25, 0.25, 0.25, 0.25, 0)$
"Between low importance and very important"	$H_4 = \{s_2, s_3, s_4\}$	$[s_2, s_4]$	$P_4 = (0, 0.33, 0.33, 0.33, 0)$

TABLE 2.1: HFLTSs, envelopes and possibility distributions corresponding to the linguistic assessments provided by experts of example 2.1

Now, suppose a situation where all four experts are equally important in the decision-making problem, i.e., $w_A = (0.25, 0.25, 0.25, 0.25)$ and represents the associated weighting vector presented in Definition 2.6. In contrast, also consider a different situation where the first two decision makers have a greater voting power, i.e., $w_B = (0.40, 0.40, 0.10, 0.10)$. Given the possibility distributions and applying the HFLWA operator, then the following aggregated possibility distributions P_A and P_B are computed, respectively:

P_A=$(0.062, 0.396, 0.146, 0.271, 0.125)$

P_B=$(0.025, 0.458, 0.058, 0.258, 0.2)$

2.2.2 The extended lattice of HFLTSs

An algebraic extension of the set of HFLTSs, denoted as $\overline{\mathcal{H}_S}$, was introduced by Montserrat-Adell et al. [121]. $\overline{\mathcal{H}_S}$ is presented as the union of the *positive HFLTSs*, $\mathcal{H}_S^*$, being $\mathcal{H}_S^* = \mathcal{H}_S - \{\emptyset\}$, the *negative HFLTSs*,$-\mathcal{H}_S^* = \{-H | H \in \mathcal{H}_S^*\}$ and the *zero HFLTSs*, $\mathcal{A} = \{\alpha_0, \ldots, \alpha_n\}$. The introduction of the *negative HFLTSs* proved very useful for distinguishing, based on their gap, the very separated non consecutive linguistic terms from those non consecutive terms which are nearer. The use of the *zero HFLTSs* allows for the identification of consecutive terms. Given any two HFLTSs, H_S^A and H_S^B, when $H_S^A \sqcap H_S^B = \alpha_i$, $\alpha_i \in \mathcal{A}$, then H_S^A and H_S^B are consecutive (see figure 2.1).

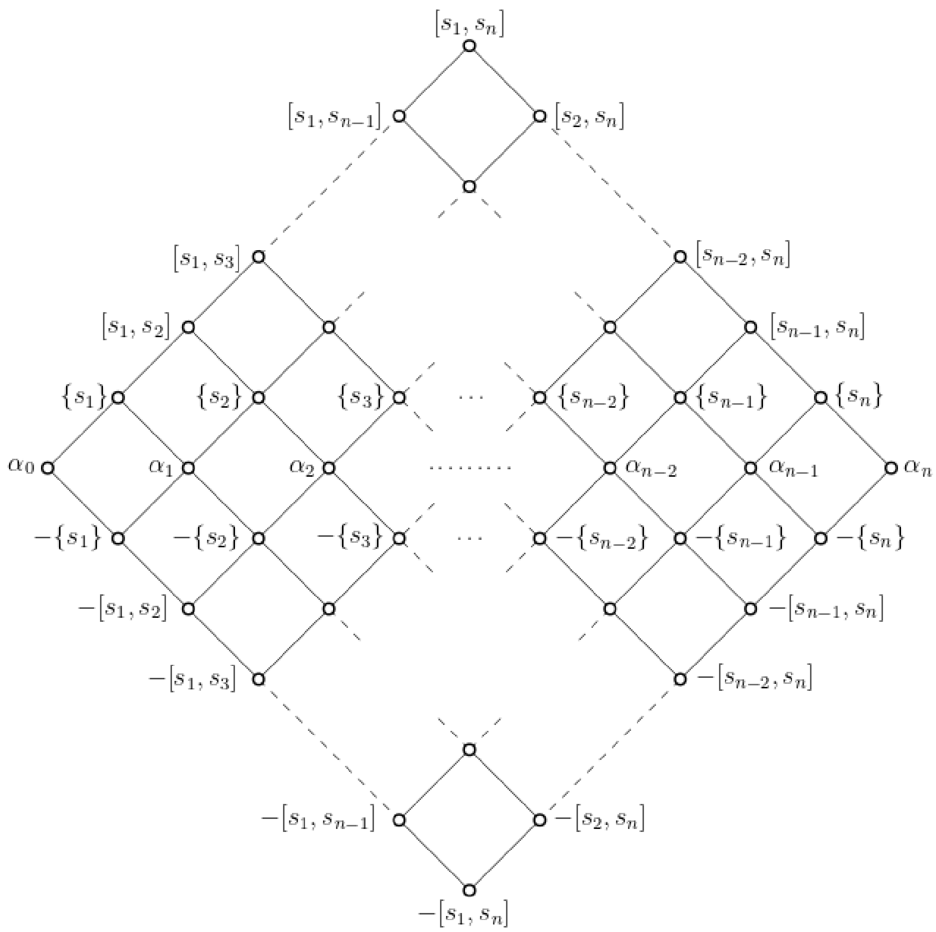

FIGURE 2.1: Graph of the extended set of HFLTSs, $\overline{\mathcal{H}_S}$, over a uniformly distributed set of LTS, $S = \{s_1, s_2, s_3, \ldots, s_n\}$.

Definition 2.7. ([121]) The *extended inclusion relation* in $\overline{\mathcal{H}_S}$, $\preceq$, is defined as:

$$\forall H_1, H_2 \in \overline{\mathcal{H}_S}, H_1 \preceq H_2 \iff H_1 \in cov(H_2). \tag{2.2}$$

where $cov(H_2)$ is the number of basic labels contained in H_2 as defined in [121].

In the context of this algebraic structure and based on the extended inclusion relation, the extended connected union and the extended intersection as closed operations within the set $\overline{\mathcal{H}_S}$ are presented.

Definition 2.8. ([121]) Given $H_S^1, H_S^2 \in \overline{\mathcal{H}_S}$, then:

1. The *extended connected union of* H_S^1 *and* H_S^2, $H_S^1 \sqcup H_S^2$, is defined as the least element that contains H_S^1 and H_S^2, according to the extended inclusion relation.
2. The *extended intersection of* H_S^1 *and* H_S^2, $H_S^1 \sqcap H_S^2$, is defined as the largest element being contained in H_S^1 and H_S^2, according to the extended inclusion relation.

The authors proved that $(\overline{\mathcal{H}_S}, \sqcup, \sqcap)$ is a distributive lattice. The graphical representation of the proposed extended lattice of HFLTSs over a uniform set of linguistic terms S based on [121] is shown in Figure 2.1.

Example 2.2. In Figures 2.2 and 2.3, two examples are given to graphically illustrate the extended connected union and the extended intersection of two HFLTSs of a uniformly distributed LTS of 5 elements. Let $H_S^1 = \{s_2\}$, $H_S^2 = [s_3, s_5]$, $H_S^3 = [s_1, s_2]$ and $H_S^4 = [s_4, s_5]$. Then $H_S^1 \sqcup H_S^2 = [s_2, s_5]$, $H_S^1 \sqcap H_S^2 = \{\alpha_2\}$ and $H_S^3 \sqcup H_S^4 = [s_1, s_5]$, $H_S^3 \sqcap H_S^4 = -\{s_2\}$

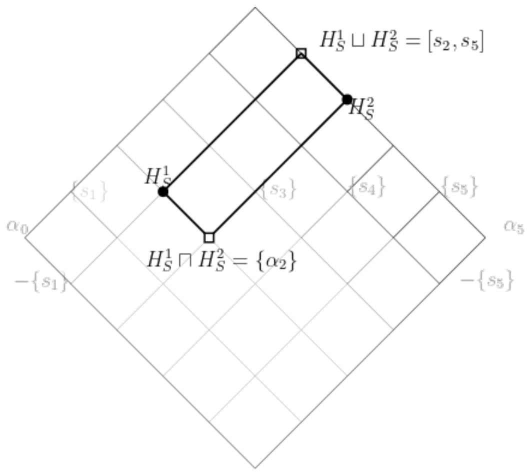

FIGURE 2.2: The extended connected union and the extended intersection of H_S^1 and H_S^2.

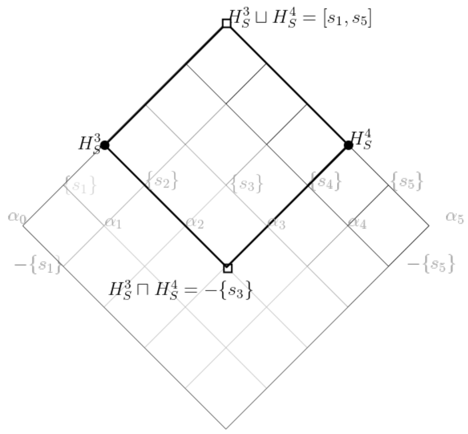

FIGURE 2.3: The extended connected union and the extended intersection of H_S^3 and H_S^4.

Given any two HFLTSs, H_S^A and H_S^B, note that any time $H_S^A \sqcap H_S^B = \alpha_i$, $\alpha_i \in \mathcal{A}$, then H_S^A and H_S^B are consecutive.

The concept of an *L-fuzzy set* on a non-empty set, which was introduced by Goguen in [69], is applied to the lattice $(\overline{\mathcal{H}_S}, \sqcup, \sqcap)$ to obtain the concept of $\mathcal{H}_S$-fuzzy sets on a set of alternatives Λ and the concept of Hesitant Fuzzy Linguistic Description (HFLD).

Definition 2.9. ([123]) The set $\mathcal{F}_\mathcal{H}$ of $\mathcal{H}_S$-fuzzy sets on Λ is:

$$\mathcal{F}_\mathcal{H} = (\mathcal{H}_S)^\Lambda = \{F_H \mid F_H : \Lambda \to \mathcal{H}_S\} \tag{2.3}$$

Definition 2.10. ([123]) A *hesitant fuzzy linguistic description (HFLD)* of the set Λ by $\mathcal{H}_S$ is a function F_H: $\Lambda \to \mathcal{H}_S^*$ such that for all $\lambda \in \Lambda$, $F_H(\lambda) \in \mathcal{H}_S^*$.

Example 2.3. Let $G = \{d_1, d_2, d_3\}$ be a group of 3 DM assessing a set of 2 alternatives, i.e., $\Lambda = \{\lambda_1, \lambda_2\}$, over one criterion, by means of HFLTSs over a set $S =$

$\{s_1, s_2, s_3, s_4, s_5\}$, as in example 2.2. With respect to λ_1, the opinion of d_1 results in $\{s_1\}$ while the opinions of d_2 and d_3 are $[s_1, s_3]$ and $\{s_3\}$, respectively. With respect to λ_2, the opinion of d_1 results in $\{s_4\}$ while the opinions of d_2 and d_3 are $[s_4, s_5]$ and $[s_3, s_5]$, respectively. The HFLDs describing their corresponding assessments, denoted as F_H^1, F_H^2 and F_H^3, are the following vectors.

$$F_H^1 = (\{s_1\}, \{s_4\})$$

$$F_H^2 = ([s_1, s_3], [s_4, s_5])$$

$$F_H^3 = (\{s_3\}, [s_3, s_5])$$

Hence, if $G = \{d_1, \ldots, d_j, \ldots, d_k\}$ denote a group of k DMs, then each HFLD, F_H^j, of the set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ by $\mathcal{H}_S^*$, can be identified with an r-dimensional vector $(H_1, H_2, ..., H_r) \in (\mathcal{H}_S^*)^r = (\mathcal{H}_S^* \times \cdots \times \mathcal{H}_S^*)$ whose components are $H_{ij} = F_H^j(\lambda_i)$, for all $i \in \{1, \ldots, r\}$.

2.2.3 Extended Hesitant Fuzzy Linguistic Term Sets and Proportional Hesitant Fuzzy Linguistic Term Sets

The use of only HFLTSs, i.e., a set of consecutive linguistic terms, might not appropriate in some group decision-making contexts. When evaluating an indicator, variable or alternative, the resulting aggregated linguistic labels from a group of experts might not always result in consecutive terms [172]. Then, the concept of extended hesitant fuzzy linguistic term sets (EHFLTSs) and proportional hesitant fuzzy linguistic term sets (PHFLTSs) are needed to develop a more complex decision framework. As compared to HFLTSs, in an EHFLTS or a PHFLTS the linguistic terms do not need to be consecutive. EHFLTS where introduced by Wang [172] and can be constructed by the union of HFLTSs given by individual experts, representing evaluations with uncertainties. Extended hesitant fuzzy linguistic term sets (EHFLTSs) are a powerful tool for modeling uncertain linguistic information in group decision-making. Inspired by [172], I propose a re-definition of EHFTLS as follows:

Definition 2.11. Let S be a linguistic term set, then any ordered subset $S\prime \subseteq S$, that is:

$$H_{S\prime} = \{s_i \mid s_i \in S\}, \tag{2.4}$$

is called an extended hesitant fuzzy linguistic term set (EHFLTS).

Example 2.4. Recall the assessments of expert one and expert two from example 2.1. Suppose they are part of a team. Notice that the set of different linguistic terms that emerge from this team are not consecutive subsets of S. The evaluation of this team could be represented by an EHFLTS and not a HFLTS, i.e., $H_{S\prime} = \{s_2, s_4, s_5\}$. Hence, when modelling the aggregation of group linguistic assessments by means of EHFLTSs, I take into account all possible linguistic terms provided by experts without a pre-aggregation process and less information is lost. In comparison, if I had chosen other aggregation procedures, such as the connected union of two HFLTSs defined in [123], without considering the existence of EHFLTSs, the aggregation would have resulted in $H_{S\prime} = \{s_2, s_3, s_4, s_5\} = [s_2, s_4]$.

On the other hand, with the aim to include proportional information to deal with EHFLTSs, the idea of Wu and Xu, is extended by Chen et al. [38] to develop the concept of proportional hesitant fuzzy linguistic term sets (PHFLTSs).

Definition 2.12. ([38]) Let $S = \{s_1, s_2, \dots, s_n\}$ be a LTS. Let $H_\theta = (H_1, H_2, \dots, H_k)$ a set of k HFLTSs given by a group of k experts. A PHFLTS for a linguistic variable ν, formed by the union of H_θ, namely P_{H_θ}, is a set of ordered finite proportional linguistic pairs:

$$P_{H_\theta}(\nu) = \{(s_i, p_i) \mid s_i \in S, i \in \{1, 2, \dots, n\}\} \tag{2.5}$$

where $\boldsymbol{P} = (p_1, p_2, \dots, p_n)^T$ is a proportional vector and p_i denotes the degree of possibility that the alternative carries an assessment value s_i provided by a group of experts with the condition that $\sum_{i=1}^{n} p_i = 1$ and $0 \leq p_i \leq 1$, for all $i \in \{1, 2, \dots, n\}$.

Example 2.5. Let $H_A = (H_1, H_2, H_3, H_4)$ be the set of four HFLTS, assessing a variable ν, corresponding to the assessments presented in example 2.1. Assume that the four experts have equal voting power. The PHFLTS formed by the union of the elements in H_A is:

$$P_{H_A}(\nu) = \{(s_1, 0.062), (s_2, 0.396), (s_3, 0.146), (s_4, 0.271), (s_5, 0.125)\} \tag{2.6}$$

with $\boldsymbol{P_A} = (0.062, 0.396, 0.146, 0.271, 0.125)^T$.
Notice that this resulting vector is the same as if I had applied the HFLWA operator over H_A, as defined in Definition 2.6. Similarly, let $H_B = (H_1, H_2)$, which includes only the assessments of the team presented in example 2.4. Then the resulting PHFLTS formed by the union of the elements in H_B is:

$$P_{H_B}(\nu) = \{(s_1, 0), (s_2, 0.5), (s_3, 0), (s_4, 0.25), (s_5, 0.25)\} \tag{2.7}$$

with $\boldsymbol{P_B} = (0, 0.5, 0, 0.25, 0.25)^T$.

2.2.4 Distances and similarity measures between HFLTSs

In order to accurately use HFLTSs in GDM situations, it is relevant to put more efforts on understanding the basic characteristics of HFLTSs, and in particular, the distance, similarity measures and comparison methods which I believe also constitute a relevant issue for other fields such as machine learning or text-mining.

The preference degree between two HFLTSs has been widely analyzed by many researchers [104, 106, 108, 138, 142, 182, 188] and it has been used to derive distances and pseudo-distances between HFLTSs. In [142], the authors proposed to use the concept of envelope, a linguistic interval based on the upper and lower bounds of HFLTS, and applied the comparison theory of interval values [180] to compare HFLTSs. Then, other authors [104, 129, 182] provided other comparison methods of HFLTSs based on the probability theory. Furthermore, two other approaches have also motivated researchers to develop new distance measures between two HFLTSs. On the one hand, traditional distances, such as the Hamming distance, the Euclidean distance and the Hausdorff metric are extended to operate with HFLTSs [17, 106]. On the other hand, other authors have adapted some weighted distance operators to get linguistic distance operators, such as the hesitant fuzzy linguistic (ordered) weighted distance operator in [188, 189] or the generalized hesitant fuzzy linguistic weighted distance measure and the generalized hesitant fuzzy linguistic 2-additive Shapley weighted distance in [118].

In the following paragraphs, I review some existing distances that operate with HFLTSs.

Definition 2.13. ([175]) Let $\mathcal{S} = \{s_0, s_1, s_2, ..., s_g\}$ be a linguistic term set, $H^1_{\mathcal{S}}$ and $H^2_{\mathcal{S}}$ be two arbitrary HFLTSs on $\mathcal{H}_S$, and then the distance introduced by Wang, between

H^1_S and H^2_S can be defined as follows:

$$d(H^1_S, H^2_S) = \sqrt{\left(I(H^1_{S^+}) - I(H^2_{S^+})\right)^2 + \left(I(H^1_{S^-}) - I(H^2_{S^-})\right)^2} \tag{2.8}$$

where I represents the function of the position index i for the element s_i in $\mathcal{S}$ and $H^1_{S^+}$ and $H^2_{S^+}$ are the upper bounds respectively, and $H^1_{S^-}$ and $H^2_{S^-}$ are the lower bounds respectively.

In [142], the authors used the comparison theory of interval values to compare HFLTSs while other authors [104, 182] provided other comparison methods of HFLTSs based on the probability theory.

Definition 2.14. ([104, 138]) Let H^1_S and H^2_S be two HFLTSs on $\mathcal{S} = \{s_0, s_1, s_2, ..., s_g\}$, $env(H^1_S) = [s_p, s_q]$, $env(H^2_S) = [s_{p'}, s_{q'}]$ Then the preference degree $p(H^1_S \geq H^2_S)$ between H^1_S and H^2_S is as follows:

$$p(H^1_S \geq H^2_S) = max\{1 - max\{\frac{q' - p}{(q-p) + (q' - p')}, 0\}, 0\} \tag{2.9}$$

Definition 2.14 in [104] is called the likelihood-based comparison relation between two hesitant fuzzy linguistic term sets. Based on it, the authors defined the HFLWA, HFLWG, HFLOWA and HFLOWG operators of a collection of hesitant fuzzy linguistic term sets. These are useful tools for fuzzy linguistic GDM situations. The likelihood-based comparison relation depends on the subscripts used and assumes that the set $\mathcal{S}$ is balanced.

Motivated by the definition of distance measure between any two linguistic terms given by Xu [191] and using the subscript-symmetric linguistic term set in ascending order for the set $\mathcal{S}$, in Ref. [106], Liao et al. developed a family of distance and similarity measures between HFLTSs as well as a variety of ordered weighted distance measures between two collections of HFLTSs. This is relevant as in real life group decision problems, the evaluation of alternatives is usually done with respect to several criteria.

Definition 2.15. ([106]) Let $\mathcal{S}$=$\{s_t \mid t = -\tau, ..., -1, 0, 1, ..., \tau\}$ be a linguistic term set, $H^1_S(x_i) = \cup_{s_{\delta^1_l} \in H^1_S} \{s_{\delta^1_l} \mid l = 1, ..., \#H^1_S\}$ ($\#H^1_S$ be the number of linguistic terms in H^1_S) and $H^2_S(x_i) = \cup_{s_{\delta^2_l} \in H^2_S} \{s_{\delta^2_l} \mid l = 1, ..., \#H^2_S\}$ ($\#H^2_S$ be the number of linguistic terms in H^2_S). A generalised distance of $H^1_S(x_i) \in \mathcal{H}_S$ and $H^2_S(x_i) \in \mathcal{H}_S$ can be defined as:

$$d_{g_d}(H^1_S(x_i), H^2_S(x_i)) = \left(\frac{1}{L}\sum_{l=1}^{L}\left(\frac{|\delta^1_l - \delta^2_l|}{2\tau + 1}\right)^\lambda\right)^{1/\lambda} \tag{2.10}$$

where $\lambda > 0$. In particular if $\lambda = 1$, then the measure becomes the Hamming distance; if $\lambda = 2$, then the generalized distance becomes the Euclidian distance.

Liao et al. also introduced the generalized Hausdorff distance measure.

Definition 2.16. ([106]) The generalized Hausdorff distance measure of $H^1_S(x_i) \in \mathcal{H}_S$ and $H^2_S(x_i) \in \mathcal{H}_S$ can be defined as:

$$d_{ghaud}(H^1_S(x_i), H^2_S(x_i)) = \left(\max_{l=1,2,...,L}\left(\frac{|\delta^1_l - \delta^2_l|}{2\tau + 1}\right)^\lambda\right)^{1/\lambda} \tag{2.11}$$

where $\lambda > 0$. In particular if $\lambda = 1$, then the Hausdorff distance becomes the Hamming-Hausdorff distance; if $\lambda = 2$, then the generalized distance becomes the Euclidian-Hausdorff distance.

Combining the first generalized distance measure in Definition 2.15 with the generalized Hausdorff measure in Definition 2.16, the authors developed another hybrid distance measure which is the result of the two [106].

Inspired by some of the limitations of the HLWA operator introduced in [182], Wang et al. [179] proposed the directional Hausdorff distance which is then used to define the dominance relations between hesitant fuzzy linguistic term sets.

Definition 2.17. ([179]) Let $H_{\mathcal{S}}^1$ and $H_{\mathcal{S}}^2$ be two arbitrary HFLTSs on $\mathcal{S}$. A hesitant directional Hausdorff distance D_{hdh} from $H_{\mathcal{S}}^1$ to $H_{\mathcal{S}}^2$ can be defined as follows.

$$D_{hdh}(H_{\mathcal{S}}^1, H_{\mathcal{S}}^2) = \begin{cases} \frac{1}{|H_{\mathcal{S}}^1|} \sum\limits_{s_i \in H_{\mathcal{S}}^1} \min\limits_{s_j \in H_{\mathcal{S}}^2} \{\max\{0, f(s_i) - f(s_j)\}\}, & \text{if} \quad H_{\mathcal{S}^+}^1 \neq H_{\mathcal{S}^+}^2; \\ \frac{1}{|H_{\mathcal{S}}^2|} \sum\limits_{s_j \in H_{\mathcal{S}}^2} \min\limits_{s_i \in H_{\mathcal{S}}^1} \{\max\{0, f(s_i) - f(s_j)\}\}, & \text{otherwise.} \end{cases}$$

Here, $\mid H_{\mathcal{S}}^i \mid$ denotes the number of the linguistic terms in $H_{\mathcal{S}}^i$. This distance [179] also represents the degree to which $H_{\mathcal{S}}^1$ outranks $H_{\mathcal{S}}^2$.Nevertheless, this distance D_{hdh} does not take into account a multi-granular term set $\mathcal{S}$ and depends on the subscripts labels.

An extended lattice of HFLTSs is introduced by Montserrat-Adell et al [121] and the binary operations of extended connected union and the extended intersetion of two HFLTSs are defined. Given any $H \in \overline{\mathcal{H}_S}$, a distance between HFLTSs that solves the subscripts problem is proposed in [121]. This distance is based on the operator of the *width* of H, $\mathcal{W}(H)$. The authors define the *width* of H as the cardinal of H, $card(H)$ if $H \in \mathcal{H}_{\mathcal{S}}^*$, the negative of the cardinal $-card(-H)$ if $H \in -\mathcal{H}_{\mathcal{S}}^*$ or, 0 if $H \in \mathcal{A}$. Based on this definition of *width*, a distance between two HFLTSs is considered:

Definition 2.18. ([121]) Let $H_1, H_2 \in \overline{\mathcal{H}_S}$, then $D(H_1, H_2) := \mathcal{W}(H_1 \sqcup H_2) - \mathcal{W}(H_1 \sqcap H_2)$ provides a distance in $\overline{\mathcal{H}_S}$.

On the contrary of the standard definition of distance (connected union minus intersection), distance proposed in Definition 2.18 takes into account the gap between two non-overlapping HFLTSs. Besides, as compared to other distances proposed for HFLTSs [106, 205], any previous transformations are needed when two HFLTSs with different levels of precision are compared. In [106], there is a need to add virtual linguistic terms to compare HFLTSs of different length. However, the distance introduced in Definition 2.18 works under the assumption of a balanced LTS since its $\mathcal{W}$ operator is based on the concept of the cardinal. To overcome this limitation, in the following chapter 3, a measure μ on $\mathcal{H}_{\mathcal{S}}^*$ is defined such that $\mu(H)$, for all $H \in \mathcal{H}_{\mathcal{S}}^*$, represents the "size" or "length" of the semantic content of H. Note that a similar measure μ was considered in absolute order-of-magnitude spaces in [144].

Comparison methods have also been studied for other types of fuzzy linguistic approaches derived from the use of HFLTSs. In [176], the authors introduced the concept of interval-valued hesitant fuzzy linguistic set (IVHFLS) and proposed a comparison method for interval-valued hesitant fuzzy numbers (IVHFLNs) based on a defined score and accuracy function. Also in this case the functions depend on the subscripts and the linguistic scale function used. In [172], the concept of extended hesitant fuzzy linguistic set (EHFLTS) is proposed and a comparison method based on the expected linguistic term, i.e., the averaging linguistic term of an EHFLT, and the degree of hesitancy is developed to compare two EHFLTs. Again, this method depends on the subscripts used and assumes similar distances between individual original linguistic term sets. Beg and Rashid (2014) introduced the concept of an hesitant intuitionistic fuzzy linguistic term set (HIFLTS) to provide a linguistic and computational basis to manage the situations in which experts assess an alternative in possible linguistic interval and impossible linguistic interval. Their proposed distance measure between any two elements of hesitant intuitionistic fuzzy linguistic term set [16] assumes that distances between consecutive linguistic terms are the same, regardless its positioning within the ordered set S. The authors use this method in combination with TOPSIS to solve a multi-criteria GDM situation.

In the following paragraphs an example is provided to illustrate the differences in the aforementioned distances between HFLTSs. The example is based on a real linguistic MAGDM situation in the business setting. The aforementioned distances will be used to compute differences in linguistic opinions. The details are explained in Example 2.6.

Example 2.6. In many multinationals, human resources department uses a 360- employee evaluation as a system or process in which employees receive confidential, anonymous feedback from the people who work around them. It typically includes the employee's manager, peers, and direct subordinates. Suppose that employee x is being evaluated by his manager y his direct co-subordinate z on criterion i. Criterion i refers to the creativity and innovation capabilities of employee x. y and z are asked to express with linguistic expressions the degree to which they disagree or agree to a given statement in relation to criterion i, based on a finite linguistic term set, $\mathcal{S}$, of 5 ordered and uniformly distributed basic linguistic labels, which are: *strongly disagree*, *disagree*, *neither agree or disagree*,*agree* and *strongly agree*.

Following the proposed distance measures in Definition 2.13 and setting $\mathcal{S}$ as $\mathcal{S} = \{s_1, s_2, s_3, s_4, s_5\}$ different distance values can be obtained, when combining all possible pairwise comparisons of $\mathcal{H}_S$. Table 2.2 represents all the possible distances between any judgement given by y and any linguistic assessment expressed by z, according to the distance measure proposed in [175].

	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_4\}$	$\{s_5\}$	$[s_1,s_2]$	$[s_2,s_3]$	$[s_3,s_4]$	$[s_4,s_5]$	$[s_1,s_3]$	$[s_2,s_4]$	$[s_3,s_5]$	$[s_1,s_4]$	$[s_2,s_5]$	$[s_1,s_5]$
$\{s_1\}$	0	1.414	2.828	4.243	5.657	1	2.236	3.606	5	2	3.162	4.472	3	4.123	4
$\{s_2\}$	1.414	0	1.414	2.828	4.243	1	1	2.236	3.606	1.414	2	3.162	2.236	3	3.162
$\{s_3\}$	2.828	1.414	0	1.414	2.828	2.236	1	1	2.236	2	1.414	2	2.236	2.236	2.828
$\{s_4\}$	4.243	2.828	1.414	0	1.414	3.606	2.236	1	1	3.162	2	1.414	3	2.236	3.162
$\{s_5\}$	5.657	4.243	2.828	1.414	0	5	3.606	2.236	1	4.472	3.162	2	4.123	3	4
$[s_1,s_2]$	1	1	2.236	3.606	5	0	1.414	2.828	4.243	1	2.236	3.606	2	3.162	3
$[s_2,s_3]$	2.236	1	1	2.236	3.606	1.414	0	1.414	2.828	1	1	2.236	1.414	2	2.236
$[s_3,s_4]$	3.606	2.236	1	1	2.236	2.828	1.414	0	1.414	2.236	1	1	2	1.414	2.236
$[s_4,s_5]$	5	3.606	2.236	1	1	4.243	2.828	1.414	0	3.606	2.236	1	3.162	2	3
$[s_1,s_3]$	2	1.414	2	3.162	4.472	1	1	2.236	3.606	0	1.414	2.828	1	2.236	2
$[s_2,s_4]$	3.162	2	1.414	2	3.162	2.236	1	1	2.236	1.414	0	1.414	1	1	1.414
$[s_3,s_5]$	4.472	3.162	2	1.414	2	3.606	2.236	1	1	2.828	1.414	0	2.236	1	2
$[s_1,s_4]$	3	2.236	2.236	3	4.123	2	1.414	2	3.162	1	1	2.236	0	1.414	1
$[s_2,s_5]$	4.123	3	2.236	2.236	3	3.162	2	1.414	2	2.236	1	1	1.414	0	1
$[s_1,s_5]$	4	3.162	2.828	3.162	4	3	2.236	2.236	3	2	1.414	2	1	1	0

TABLE 2.2: Pairwise distance measurements of example 2.6 according to distance defined in [175]

Similarly, the likelihood-based comparison method is also computed for all possible combinations of linguistic assessments provided by y and z in Table 2.3

	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_4\}$	$\{s_5\}$	$[s_1,s_2]$	$[s_2,s_3]$	$[s_3,s_4]$	$[s_4,s_5]$	$[s_1,s_3]$	$[s_2,s_4]$	$[s_3,s_5]$	$[s_1,s_4]$	$[s_2,s_5]$	$[s_1,s_5]$
$\{s_1\}$	0.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\{s_2\}$	1	0.5	0	0	0	1	0	0	0	0.5	0	0	0.333	0	0.25
$\{s_3\}$	1	1	0.5	0	0	1	1	0	0	1	0.5	0	0.667	0.333	0.5
$\{s_4\}$	1	1	1	0.5	0	1	1	1	0	1	1	0.5	1	0.667	0.75
$\{s_5\}$	1	1	1	1	0.5	1	1	1	1	1	1	1	1	1	1
$[s_1,s_2]$	1	0	0	0	0	0.5	0	0	0	0.333	0	0	0.25	0	0.2
$[s_2,s_3]$	1	1	0	0	0	1	0.5	0	0	0.667	0.333	0	0.5	0.25	0.4
$[s_3,s_4]$	1	1	1	0	0	1	1	0.5	0	1	0.667	0.333	0.75	0.5	0.6
$[s_4,s_5]$	1	1	1	1	0	1	1	1	0.5	1	1	0.667	1	0.75	0.8
$[s_1,s_3]$	1	0.5	0	0	0	0.667	0.333	0	0	0.5	0.25	0	0.4	0.2	0.333
$[s_2,s_4]$	1	1	0.5	0	0	1	0.667	0.333	0	0.75	0.5	0.25	0.6	0.4	0.5
$[s_3,s_5]$	1	1	1	0.5	0	1	1	0.667	0.333	1	0.75	0.5	0.8	0.6	0.667
$[s_1,s_4]$	1	0.667	0.333	0	0	0.75	0.5	0.25	0	0.6	0.4	0.2	0.5	0.333	0.429
$[s_2,s_5]$	1	1	0.667	0.333	0	1	0.75	0.5	0.25	0.8	0.6	0.4	0.667	0.5	0.571
$[s_1,s_5]$	1	0.75	0.5	0.25	0	0.8	0.6	0.4	0.2	0.667	0.5	0.333	0.571	0.429	0.5

TABLE 2.3: Likelihood-based comparison relation of Definition 2.14

New pairwise distance measures can be obtained if distances are computed with Definitions 2.15 and 2.16, respectively, with with $\lambda = 2$ and using the subscripts in $\mathcal{S} = \{s_{-2} =$ strongly disagree,$s_{-1} =$ disagree,$s_0 =$ neither agree or disagree,$s_1 =$ agree,$s_2 =$ strongly agree$\}$. The results are shown in table 2.4 and 2.5.

	$\{s_{-2}\}$	$\{s_{-1}\}$	$\{s_0\}$	$\{s_1\}$	$\{s_2\}$	$[s_{-2},s_{-1}]$	$[s_{-1},s_0]$	$[s_0,s_1]$	$[s_1,s_2]$	$[s_{-2},s_0]$	$[s_{-1},s_1]$	$[s_0,s_2]$	$[s_{-2},s_1]$	$[s_{-1},s_2]$	$[s_{-2},s_2]$
$\{s_{-2}\}$	0	0.2	0.4	0.6	0.8	0.141	0.316	0.51	0.707	0.258	0.432	0.622	0.374	0.548	0.49
$\{s_{-1}\}$	0.2	0	0.2	0.4	0.6	0.141	0.141	0.316	0.51	0.163	0.258	0.432	0.245	0.374	0.346
$\{s_0\}$	0.4	0.2	0	0.2	0.4	0.316	0.141	0.141	0.316	0.258	0.163	0.258	0.245	0.245	0.283
$\{s_1\}$	0.6	0.4	0.2	0	0.2	0.51	0.316	0.141	0.141	0.432	0.258	0.163	0.374	0.245	0.346
$\{s_2\}$	0.8	0.6	0.4	0.2	0	0.707	0.51	0.316	0.141	0.622	0.432	0.258	0.548	0.374	0.49
$[s_{-2},s_{-1}]$	0.141	0.141	0.316	0.51	0.707	0	0.2	0.4	0.6	0.129	0.311	0.507	0.249	0.427	0.367
$[s_{-1},s_0]$	0.316	0.141	0.141	0.316	0.51	0.2	0	0.2	0.4	0.129	0.129	0.311	0.149	0.249	0.235
$[s_0,s_1]$	0.51	0.316	0.141	0.141	0.316	0.4	0.2	0	0.2	0.311	0.129	0.129	0.249	0.149	0.235
$[s_1,s_2]$	0.707	0.51	0.316	0.141	0.141	0.6	0.4	0.2	0	0.507	0.311	0.129	0.427	0.249	0.367
$[s_{-2},s_0]$	0.258	0.163	0.258	0.432	0.622	0.129	0.129	0.311	0.507	0	0.2	0.4	0.125	0.309	0.245
$[s_{-1},s_1]$	0.432	0.258	0.163	0.258	0.432	0.311	0.129	0.129	0.311	0.2	0	0.2	0.125	0.125	0.141
$[s_0,s_2]$	0.622	0.432	0.258	0.163	0.258	0.507	0.311	0.129	0.129	0.4	0.2	0	0.309	0.125	0.245
$[s_{-2},s_1]$	0.374	0.245	0.245	0.374	0.548	0.249	0.149	0.249	0.427	0.125	0.125	0.309	0	0.2	0.122
$[s_{-1},s_2]$	0.548	0.374	0.245	0.245	0.374	0.427	0.249	0.149	0.249	0.309	0.125	0.125	0.2	0	0.122
$[s_{-2},s_2]$	0.49	0.346	0.283	0.346	0.49	0.367	0.235	0.235	0.367	0.245	0.141	0.245	0.122	0.122	0

TABLE 2.4: Pairwise distance measurements of example using the generalized distance measure of Definition 2.15

	$\{s_{-2}\}$	$\{s_{-1}\}$	$\{s_0\}$	$\{s_1\}$	$\{s_2\}$	$[s_{-2},s_{-1}]$	$[s_{-1},s_0]$	$[s_0,s_1]$	$[s_1,s_2]$	$[s_{-2},s_0]$	$[s_{-1},s_1]$	$[s_0,s_2]$	$[s_{-2},s_1]$	$[s_{-1},s_2]$	$[s_{-2},s_2]$
$\{s_{-2}\}$	0	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8	0.4	0.6	0.8	0.6	0.8	0.8
$\{s_{-1}\}$	0.2	0	0.2	0.4	0.6	0.2	0.2	0.4	0.6	0.2	0.4	0.6	0.4	0.6	0.6
$\{s_0\}$	0.4	0.2	0	0.2	0.4	0.4	0.2	0.2	0.4	0.4	0.2	0.4	0.4	0.4	0.4
$\{s_1\}$	0.6	0.4	0.2	0	0.2	0.6	0.4	0.2	0.2	0.6	0.4	0.2	0.6	0.4	0.6
$\{s_2\}$	0.8	0.6	0.4	0.2	0	0.8	0.6	0.4	0.2	0.8	0.6	0.4	0.8	0.6	0.8
$[s_{-2},s_{-1}]$	0.2	0.2	0.4	0.6	0.8	0	0.2	0.4	0.6	0.2	0.4	0.6	0.4	0.6	0.6
$[s_{-1},s_0]$	0.4	0.2	0.2	0.4	0.6	0.2	0	0.2	0.4	0.2	0.2	0.4	0.2	0.4	0.4
$[s_0,s_1]$	0.4	0.2	0.2	0.4	0.6	0.2	0	0.2	0.4	0.2	0.2	0.4	0.2	0.4	0.4
$[s_1,s_2]$	0.8	0.6	0.4	0.2	0.2	0.6	0.4	0.2	0	0.6	0.4	0.2	0.6	0.4	0.6
$[s_{-2},s_0]$	0.4	0.2	0.4	0.6	0.8	0.2	0.2	0.4	0.6	0	0.2	0.4	0.2	0.4	0.4
$[s_{-1},s_1]$	0.6	0.4	0.2	0.4	0.6	0.4	0.2	0.2	0.4	0.2	0	0.2	0.2	0.2	0.2
$[s_0,s_2]$	0.8	0.6	0.4	0.2	0.4	0.6	0.4	0.2	0.2	0.4	0.2	0	0.4	0.2	0.4
$[s_{-2},s_1]$	0.6	0.4	0.4	0.6	0.8	0.4	0.2	0.4	0.6	0.2	0.2	0.4	0	0.2	0.2
$[s_{-1},s_2]$	0.8	0.6	0.4	0.4	0.6	0.6	0.4	0.2	0.4	0.4	0.2	0.2	0.2	0	0.2
$[s_{-2},s_2]$	0.8	0.6	0.4	0.6	0.8	0.6	0.4	0.4	0.6	0.4	0.2	0.4	0.2	0.2	0

TABLE 2.5: Pairwise distance measurements using the generalized Hausdorff distance measure in Definition 2.16 with $\lambda = 2$.

New pairwise distance measures can be obtained if distances are computed with Definition 2.18,regardless of the subscripts used in $\mathcal{S}$, as shows Table 2.6

	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_4\}$	$\{s_5\}$	$[s_1,s_2]$	$[s_2,s_3]$	$[s_3,s_4]$	$[s_4,s_5]$	$[s_1,s_3]$	$[s_2,s_4]$	$[s_3,s_5]$	$[s_1,s_4]$	$[s_2,s_5]$	$[s_1,s_5]$
$\{s_1\}$	0	2	4	6	8	0	3	5	7	2	4	6	3	5	4
$\{s_2\}$	2	0	2	4	6	1	1	3	5	2	2	4	2	3	4
$\{s_3\}$	4	2	0	2	4	3	1	1	3	2	2	2	3	3	4
$\{s_4\}$	6	4	2	0	2	5	3	1	1	4	2	2	3	3	4
$\{s_5\}$	8	6	4	2	0	7	5	3	1	6	4	2	5	3	4
$[s_1,s_2]$	0	1	3	5	7	0	2	4	6	1	3	5	2	4	3
$[s_2,s_3]$	3	1	1	3	5	2	0	2	4	1	1	3	2	2	3
$[s_3,s_4]$	5	3	1	1	3	4	2	0	2	3	1	1	2	2	3
$[s_4,s_5]$	7	5	3	1	1	6	4	2	0	5	3	1	4	2	3
$[s_1,s_3]$	2	2	2	4	6	1	1	3	5	0	2	4	1	3	2
$[s_2,s_4]$	4	2	2	2	4	3	1	1	3	2	0	2	1	1	2
$[s_3,s_5]$	6	4	2	2	2	5	3	1	1	4	2	0	3	1	2
$[s_1,s_4]$	3	2	3	3	5	2	2	2	4	1	1	3	0	2	1
$[s_2,s_5]$	5	3	3	3	3	4	2	2	2	3	1	1	2	0	1
$[s_1,s_5]$	4	4	4	4	4	3	3	3	3	2	2	2	1	1	0

TABLE 2.6: Pairwise distance measurements of example using Definition 2.18

2.2.5 HFLTSs in an unbalanced context

In the majority of the GDM problems found in the literature where the assessments provided by DMs are represented by means of HFLTSs, these are assumed to be built over an uniform and symmetrically distributed linguistic term set (LTS) [106, 108, 123, 168, 172, 175]. This seems to be appropriate for cases where the semantics of each term have a proportional uncertainty and are usually equally placed around a central label. However, there exist many group decision situations in which attributes relate to qualitative aspects that need to be assessed in linguistic terms represented by unsymmetrical and not uniformly distributed linguistic term sets. [28, 30, 53, 55, 81, 110, 145, 164, 165, 196, 204]. The unbalanced linguistic information may arise from the nature and characteristics of some linguistic variables such as the ones involved in a grading system. In other circumstances, the use of unbalanced linguistic information is appropriate to capture the different psychological aspects of decision makers. The decision aiding framework proposed in this paper focus on modelling expert's linguistic expressions by means of unbalanced HFLTSs.

In the literature, several methods have been proposed to deal with unbalanced linguistic term sets (ULTSs), i.e., LTS which are neither uniformly distributed nor symmetric. Some models are based on the evidential reasoning approach [165], others are built upon the linguistic hierarchy and the use of a 2-tuple fuzzy linguistic computational model [41, 53, 55, 56, 81, 110] while some other approaches were developed in the structure of the generalized absolute orders of magnitude qualitative spaces [144, 145] or the asymmetric sigmoid semantics [204]. Also, different aggregation operators for unbalanced linguistic labels have been suggested and proposed [89, 115].

As with the initial 2-tuple linguistic models, the origin of HFLTSs [142] worked under the assumption of linguistic information represented by linguistic variables with equidistant labels. Recently, in the context of HFLTSs, several linguistic computational models have been developed to handle ULTSs [28, 30, 53, 164, 196]. For instance, in [196] an unbalanced HFLTS method is developed and applied to a personnel selection process, an investment alternative selection process and a telecommunication service provider selection process. In [53] a novel CW methodology where the hesitant fuzzy linguistic term sets are constructed based on ULTSs using a numerical scale is proposed. The authors defined several possibility degree formulas for comparing HFLTSs and introduced some operators to aggregate the hesitant fuzzy unbalanced linguistic information. In [196], Yu et al. formulated a gain and loss formula for an unbalanced HFLTSs over another and provide a practical example to demonstrate the application of unbalanced HFLTSs in MCGDM.

The linguistic 2-tuple representation model [83], which was initially appropriate for dealing with linguistic term sets that were uniformly and symmetrically distributed, is the basis of the most extensively studied approaches to deal with ULTs. Two model extensions were developed to address ULTSs. On the one hand, Herrera et al. [81] presented an unbalanced linguistic methodology using the 2-tuple computational model that used the concept of linguistic hierarchy [34, 44, 82, 84, 92] to deal with unbalanced linguistic term sets.

On the other hand, the second approach was based on transformations of linguistic terms into interval numbers and was initiated by Wang and Hao [177, 178] with the development of the proportional 2-tuple linguistic representation model, which allows experts to express their opinions using two adjacent ordinals, and it was further extended by Dong et al. [54, 56]. Dong et al [56] generalized the numerical scale approach to set the interval numerical scale, by considering the context where semantics of linguistic terms are defined by interval type-2 fuzzy sets (IT2 FSs). Then, in [53] Dong et al. analytically proved the equivalence of the linguistic computational models by equating the model based on a linguistic hierarchy and the numerical scale model to address ULTSs.

The concept of balanced and unbalanced linguistic term set was formally proposed based on the numerical scale introduced in [54].

Definition 2.19. ([54]) Let $S=\{s_1, s_2, \ldots, s_n\}$ be a linguistic term set and R be a real number set. I define the function $NS : S \rightarrow R$ as a numerical scale of S and call $NS(s_i)$ the numerical index of i

Definition 2.20. ([55, 196]) Let $S=\{s_1, s_2, \ldots, s_n\}$ be a linguistic term set and $NS(s_i)$ be the numerical scale of s_i, $i = 1, 2, \ldots, n$. S is a linguistic term set which is uniformly and symmetrically distributed, if the following conditions are satisfied:

1. There exists a unique constant $\lambda > 0$ such that $NS(s_i) - NS(s_j)=\lambda(i-j)$ for all $i, j = 1, 2, \ldots, n$.

2. Let $S^R = \{s \mid s \in S, s > s^*\}$ and $S^L = \{s \mid s \in S, s < s^*\}$, where s^* is the midterm of S. Let $\#(S^R)$ and $\#(S^L)$ be the cardinality of S^R and S^L, respectively, then $\#(S^R) = \#(S^L)$.

If S is uniformly and symmetrically distributed linguistic term set, then S is called a balanced linguistic term set with respect to the numerical scale NS. Otherwise, S is called an unbalanced linguistic term set with respecto to the numerical scale NS.

Afterwards, nonetheless, researchers realized that in many GDM problems, decision makers or experts feel more comfortable if they can provide several terms at the same time to express their preferences, instead of a single linguistic term. This was a main drawback of the 2 tuple linguistic model which was, in particular, solved by the proposed concept of HFLTS introduced by Rodríguez et al. [142]. However, the HFLTS model developed by Rodríguez et al. [141, 142], as the initial 2-tuple linguistic model, works under the assumption of linguistic information represented by linguistic variables with equidistant labels, i.e., balanced linguistic term sets.

In [53, 196], based on numerical scale and inspired by the works of Herrera et al. [81] and Wang and Hao [177, 178], a definition of HFLTS was proposed for the unbalanced case.

Definition 2.21. ([53]) Let S=$\{s_1, s_2, \ldots, s_n\}$ be a an unbalanced linguistic term set. An unbalanced HFLTS H^S on S is an ordered finite subset of consecutive linguistic terms in S. Moreover, the *score function* of H^S is defined by:

$$E(H^S) = \frac{1}{N_{H^S}} \sum_{s_i \in H^S} NS(s_i), \tag{2.12}$$

where N_{H^S} is the number of elements in H^S.

In Chapter 3, where the new methodology to deal with unbalanced HFLTSs is developed, the concept of numerical scale to define HFLTS is reviewed again. In Remark 3.2, a comparison between the proposed perceptual map and the previous definitions of unbalanced HFLTSs based on the numerical scale is given.

2.2.6 Consensus measures in the context of HFLTSs

With respect to consensus measures in GDM, different linguistic modelling have been adapted to deal with unbalanced fuzzy linguistic information [28, 30, 76, 130, 145, 164, 186, 201]. Nonetheless, consensus frameworks dealing with unbalanced linguistic information modelled specifically by means of HFLTSs are limited. Some references can be found in [28, 76, 164]. In [28], the authors study different approaches to obtain soft consensus degrees using a strict concept of coincidence, a soft concept of coincidences and the approaches based on solutions. Recently, an attempt to model experts attitudes in group-decision making problems has been developed in [76]. Hao and Chiclana define the concept of an attitude linguistic quantifiers and associate it to the attitude and subjective preference of an expert. The authors develop an attitude quantifier deriving method as the basis to generate possibility distribution in the HFLTSs framework [76] that extends the previous works of Wu and Xu [187] and Chen et al.[38]. In addition, fewer approaches have been developed to simultaneously deal with multi-granular ULTSs. In particular, in [164], the authors introduce a signed distance measure between HFLTSs based on the ordinal semantics of linguistic terms and the possibility distribution method introduced in

[187]. A family of consensus measures are developed on the proposed signed distance.

With respect to consensus measures for MAGDM in a fuzzy linguistic context, several frameworks have been proposed in the literature with different linguistic modelling: such as the 2-tuple fuzzy linguistic model [28, 30], the order-of-magnitude qualitative metric spaces [144, 145], by means of absolute assessments with hesitant fuzzy linguistic term sets [28, 122, 164], hesitant fuzzy linguistic preference relations (HFLPR) [186, 202, 203], intuitionistic fuzzy preference relations (IFPRs) [107], interval-valued intuitionistic fuzzy sets (IVIFSs) [40] and preference relations (IVIFPRs) [171] or probabilistic linguistic preference relation (PLPR) [201].

Some of these existing consensus measures in GDM are adapted to deal with unbalanced fuzzy linguistic information [28, 30, 76, 130, 145, 164, 186, 201]. For instance, in [30], the authors propose a computational model based on symbolic models and design a consensus model to reach an acceptable consensus level based on this representation model for unbalanced linguistic term sets.

2.3 Multiple-attribute group decision making methods

In this section, the notions and background needed for the illustrative cases, specifically the one presented in Chapter 4, with respect to methods of MAGDM are provided.
On the one hand, a review on multi-criteria decision making (MCDM) and group decision making (GDM) is provided. Following this contextual explanation, the basic notions and characteristics of two widely used MCDM methods, i.e., AHP and TOPSIS, are given.

2.3.1 Multi-Criteria Decision Making (MCDM) and Group Decision Making (GDM)

The field of operations research (OR) develops models and optimization procedures to help the business sector analyze and solve complex problems in the presence of multiple and conflicting criteria or objectives. Depending on perspective, MCDM techniques are considered to be both, past and modern part of OR [99]. Foundations of modern MCDM were developed in 1950s and 1960s and since then, several authors have attempted to review the multi-criteria techniques and give an overview of the existing situation of MCDM methods [19, 198, 199]. The scope of MCDM methods is enormous, being applied to a wide range of different sectors such as: economics, health care, logistics, industrial engineering, environmental sciences, bio economy, urban studies or public policy [9, 114, 116, 139].

Some contexts of the MCDM field are sometimes referred as multi-criteria decision aiding (MCDA) situations [146]. This is a constructivist or, also known as, "European" approach of a multi-criteria decision situation. Two main actors are involved in an MCDM process: the analyst, who is responsible for designing the method and the decision maker (DM), for whom this aiding method is offered. This decision aiding process better reflects the co-construction process followed in Chapters 4 and 5 of this thesis. MCDA tools have to be seen as keys to doors giving access to elements of knowledge contributing to acceptance of a final recommendation [158].

There are various mathematical tools for developing MCDM models. Possibly the most well-recognized are Elimination and Choice Expressing the Reality (ELECTRE) [20], Analytical Hierarchy Process (AHP) [153], Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [87], PROMETHEE [25] or VIKOR [127]. In our methodology, an AHP and an extended version of TOPSIS to deal with linguistic information in a group decision situation are combined. The basics of AHP and TOPSIS are explained in the following sections, 2.3.2 and 2.3.3.

Despite the increasingly number of recently new developed methods, the structure of any decision-aiding context or process is founded on the following pillars:

- $A = \{a_1, a_2, \ldots, a_i, \ldots, a_m\}$, where each a_i is a distinct alternative (action, object, etc.) to be evaluated. MCDM/MCDA methods can solve problems of three type: choice, classification or ranking of these alternatives.
- $F = \{g_1, g_2, \ldots, g_j, \ldots, g_n\}$. It is the coherent set of n criteria.
- w_j for all $g_j \in F$. This is the relative importance coefficient of each criterion.
- $g_j(a_i)$ for all $a_i \in A$ and $g_j \in F$. It refers to the performances (consequences, characteristics or attributes) of each alternative with respect to each criterion which allow to compare one with another.
- There is a decision maker (DM) or a group of DM who provides preference information and a process that models this preference system.

When more than one decision maker is involved, the situation is commonly referred as to group decision-making (GDM) or group decision-aiding (GDA) environment. The building of the family of criteria and the relative importance assigned to it, which is elicited from of DM preferences are important steps in a GDM context. This study will focus on analyzing these aspects of the decision aiding process as a fundamental phase prior to evaluating alternatives in a second stage research project.

Group decision making based on linguistic assessments or preference relations provided by the DMs is a research topic that has been widely studied among researchers [80, 193] and received a great deal of interest over the last years [2, 31]. In some studies, I see that experts feel more comfortable providing linguistic information rather than exact numerical values and this allows to better capture the ambiguity and impreciseness inherent in human's reasoning [4]. A detailed explanation of the use of a fuzzy approach to deal with linguistic information is provided further below.

2.3.2 Analytic hierarchy process

Analytic hierarchy process (AHP) is one of the common methods used in multi-criteria decision-making tools developed by Saaty in [153]. AHP is a theory of measurement through pairwise comparisons and relies on the judgements of experts to derive priority scales. It is therefore a theory of relative measurement [27]. AHP has been extensively adopted in many practical decision-making applications [169]. For example, in the business or corporate sector, the more traditional AHP procedure has been used to select a logistics or software provider [47, 57] to solve a variety of marketing problems [49] to deal with corporate social responsibility programs [95], planning renewable energy projects [157] or rating sovereign debt [93]. The AHP represents a commonly used mathematical method in MCDM [51] and it is usually decomposed in the same steps, explained in [151].

In the AHP method, the comparisons are made using a scale of absolute judgements that represents, how much more, one element dominates another with respect to a given attribute [151]. In the classical AHP, the pairwise comparisons are done by using the crisp numbers within the 1–9 scale and from simple judgments on two elements, priority vectors are computed [152]. In the application of 4, the eigenvalue method, which is the one proposed by Saaty himself and the most popular method to estimate the priority vector [27], is used to derive the priorities from the comparison matrix.

The judgments are usually inconsistent, and there is a mathematical way to measure inconsistency. However, a perfectly consistent matrix is obtained following the method proposed for constructing consistent fuzzy preference relations from a set of $n-1$ preference data [85]. This method allows us to ask less questions to the experts and obtain the linguistic judgements for the main criteria assessments. To construct a consistent multiplicative preference relation A' on $X = \{x_1, \cdots, x_n\}$, with $n \leq 2$, from $n-1$ preference values; for instance $\{a_{12}, a_{23}, \cdots, a_{n-1n}\}$, the authors propose these steps:

1. Compute the following preference values: $B = \{a_{ij} = a_{ii+1} \times a_{i+1i+2} ... \times a_{j-1j}\}$ such that $i < j \; \forall a_{ij} \notin \{a_{12}, a_{23}, \cdots, a_{n-1n}\}$
2. Set $a = \max B$
3. $A = \{a_{12}, a_{23}, \cdots, a_{n-1n}\} \cup B \cup \{a_{12}, a_{23}, \cdots, a_{n-1n}\}^{-1} \cup B^{-1}$
4. The consistent multiplicative preference relation A' is obtained as $A' = f(A)$ such that:
   $f : [\frac{1}{a}, a] \rightarrow [\frac{1}{9}, 9]$
   $f(x) = x^{\frac{1}{log_9 a}}$

In the following lines, Example 2.7 is provided to show how to apply this procedure. I provide a practical example to illustrate how to obtain a consistent multiplicative matrix, as explained in the AHP method, from the minimum number of preference relations given by an expert, in a scale 1–9, as the input.

Example 2.7. Suppose that one expert has provided his judgements on a set of six criteria $C = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ by answering only five questions. He has certain knowledge to assure that criterion six c_6 has demonstrated importance over criteria c_1, c_4, c_5 and extremely more important than criteria c_2. Besides, he says that criterion c_3 is moderately more important than criterion c_6. From these judgements, the pairwise comparison matrix could be filled in, as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 7 & 9 & 0 & 7 & 7 & 1 \end{pmatrix}$$

Following the proposed process [101], I derive the rest of the values and build a consistent multiplicative preference relation which does not preserve the Saaty's ratio. Each entry i, j denotes the comparison of importance between row C_i with column C_j:

$$M' = \begin{pmatrix} 1 & 1.29 & 0.05 & 1 & 1 & 0.14 \\ 0.78 & 1 & 0.04 & 0.78 & 0.78 & 0.11 \\ 21 & 27 & 1 & 21 & 21 & 3 \\ 1 & 1.29 & 0.05 & 1 & 1 & 0.14 \\ 1 & 1.29 & 0.05 & 1 & 1 & 0.14 \\ 7 & 9 & 0.33 & 7 & 7 & 1 \end{pmatrix}$$

Then, fixing $a = 27$, the transformation function is applied to obtain the consistent multiplicative preference relation with the Saaty's ratio. Note that in row 3 and column 2, there is now a value of 9:

$$M'^{*} = \begin{pmatrix} 1 & 1.18 & 0.13 & 1 & 1 & 0.27 \\ 0.84 & 1 & 0.11 & 0.84 & 0.84 & 0.23 \\ 7.61 & 9 & 1 & 7.61 & 7.61 & 2.08 \\ 1 & 1.18 & 0.13 & 1 & 1 & 0.27 \\ 1 & 1.18 & 0.13 & 1 & 1 & 0.27 \\ 3.65 & 4.32 & 0.48 & 3.65 & 3.65 & 1 \end{pmatrix}$$

A part from this explanation, it is also relevant to mention that many new versions and extensions of the traditional AHP have been developed. For example, the fuzzy AHP (FAHP) is a popular methodology to account for uncertainty and is extracted from the theory of fuzzy sets. A state-of-the-art of FAHP can be found in [101]. In our proposed methodology in the application presented in Chapter 4, I will apply the traditional AHP from a set of minimum preference relations and fuzziness will be incorporated throughout the TOPSIS phase.

2.3.3 Technique for Order Preference by Similarity to Ideal Solution

TOPSIS stands for Technique for Order Preference by Similarity to Ideal Solution. It was initially proposed and developed by Hwang and Yoon [87], Lai et al. [103] and Yoon and Hwang [195]. The fundamental idea behind this method is to simultaneously compute distances, for each alternative, to both the positive-ideal solution (PIS), which presents the extreme performance on each criterion, and the negative-ideal solution (NIS), which represents the reverse extreme performance on each criterion [126]. The ranking of alternatives of the method is based on the relative closeness coefficient (CC_i) which is based on "the shortest distance from the positive ideal solution and the farthest form the negative ideal solution" [138].

As AHP, TOPSIS is also a very well-known MCDM technique and it has been applied in a wide range of real-world applications. An identification and analysis of the current level of development of issues related to TOPSIS methodology is performed in [18]. More recently, for instance, TOPSIS method has been recently used to evaluate the multidimensional concept of sustainable development in European countries [13], to assess the food and nutrition security in Iran [8], as a non-parametric classifier method to predict bankruptcy [128] as well as to assess the consequences of Great Britain leaving the European Union in its electricity market from different stakeholders' perspectives [117].

TOPSIS was also extended to the fuzzy environment and, in the business sector, has been proposed for selecting top management positions [96], selecting suppliers [174] or solving group decision making [42]. Besides, the integration of (fuzzy) AHP with (fuzzy) TOPSIS to solve multi-criteria problems have been widely used in the literature [91, 94, 102, 162, 163]. In a business setting, TOPSIS and AHP are combined to determine a cost-benefit decision-making tool applicable for the shipping operators [94] as well as used conjointly to select the best supplier providing the highest satisfaction for the criteria determined [91, 94, 162].

Specifically, the use of TOPSIS in MAGDM problems where the opinion of the experts is represented by hesitant fuzzy linguistic term sets was first proposed by Beg and Rashid [17]. New approaches of fuzzy linguistic TOPSIS method for group multi-criteria linguistic decision-making were latter developed [138, 181, 185]. For instance, Ren et al. [138] used their new concept of pseudo-distance between two HFLTSs to compute distances between the individual HFLTS and the corresponding ideal solutions and Wu et al. [185] developed a new linguistic operator (HFLWA) to aggregate individual preferences.

In the application of a TOPSIS technique for group decision-making, it should be noted that the ranking of alternatives depends on mainly three aspects, which have to be decided by the experts and DMs participating in the decision process: (a) the aggregation operator for individual assessments, (b) the choice of the positive and negative ideal solutions and (c) the choice of the distance measure used to compute the relative closeness coefficient. In the proposed TOPSIS of Chapter 4, the aggregated linguistic assessments are modeled by proportional hesitant fuzzy linguistic term sets (PHFLTSs). Secondly, albeit taking into account the issue of rank reversal, the relative ideal solutions are identified as the most adequate and appropriate by the experts who participated in the decision process of the smart city problem. As in [185], results of different combinations, using the absolute ideal solutions were also computed and shown to the experts. Thirdly, with respect to the distance measure, the proposed framework, introduced in subsection 3.3.2 of Chapter 3, is based on the cosine distance function [46, 105]. I will apply this distance to PHFLTSs, which are vectors of dimensionality equal to the cardinality of the linguistic term set.

In the following paragraphs, a numerical example, Example 2.8 is shown with the purpose to understand the computation of distances between PHFLTSs in the proposed TOPSIS version, used in Chapter 4 of this thesis.

Example 2.8. Let $A = \{a_1, a_2, a_3\}$ be a set of three alternatives which are evaluated, over one criterion, by a group of two decision makers, $M = \{d_1, d_2\}$, each one representing the same weight in the decision. DMs express their opinion on the three alternatives using a set S of 5 linguistic term sets; $S = \{VU : \text{very unsatisfied}, U : \text{unsatisfied}, N : \text{neutral}, S : \text{satisfied}, VS : \text{very satisfied}\}$. The individual and aggregated linguistic assessments, by means of PHFLTSs are shown in figure 2.4.

Alternatives	Assessment by d_1	Assessment by d_2	PHFLTSs
a_1	$H_1^1 = [VU, U]$	$H_1^2 = \{U\}$	$P_1 = (0.25, 0.75, 0, 0, 0)$
a_2	$H_2^1 = [S, VS]$	$H_2^2 = [N, VS]$	$P_2 = (0, 0, 0.25, 0.5, 0.25)$
a_3	$H_3^1 = \{N\}$	$H_3^2 = [U, N]$	$P_3 = (0, 0.25, 0.75, 0, 0)$

FIGURE 2.4: The hesitant fuzzy linguistic assessment of alternatives provided by DMs, their aggregated PHFLTSs.

To determine which relative PHFLTSs are the negative and positive ideal solutions, the DMs decide to assign a weight of 1,2,3,4,5 respectively to each of the basic labels of S, and for each PHFLTSs I compute:

$P_1 = (0.25, 0.75, 0, 0, 0) = 0.25(1) + 0.75(2) + 0(3) + 0(4) + 0(5) = 1.75$

$P_2 = (0, 0, 0.25, 0.5, 0.25) = 0(1) + 0(2) + 0.25(3) + 0.5(4) + 0.25(5) = 4$
$P_3 = (0, 0.25, 0.75, 0, 0) = 0(1) + 0.25(2) + 0.75(3) + 0(4) + 0(5) = 2.75$

In this way, $P^- = (0.25, 0.75, 0, 0, 0)$ and $P^+ = (0, 0, 0.25, 0.5, 0.25)$ are considered as the PHFLTS negative and positive-ideal solutions respectively. Now, I compute the cosine similarity function between all vectors and these ideal solutions, using the following formula $similarity(P_i, P_j) = \cos(\theta) = \frac{P_i \cdot P_j}{\|P_i\| \|P_j\|}$. Then: $distance(P_i, P_j) = 1 - similarity(P_i, P_j)$, where $P_i \cdot P_j$ denotes the dot product and $\|P_i\|$ is the norm of the vector. Then, the distances of each alternative to the positive, D^+, and negative, D^-, ideal solutions are shown in figure 2.5

Alternatives	**Similarity to P^-**	**Similarity to P^+**	D^-	D^+	CC_i
a_1	1	0	0	1	0
a_2	0	1	1	0	1
a_3	0.3	0.3873	0.7	0.6127	0.5332

FIGURE 2.5: Similarity and distances of PHFLTS to the positive and negative ideal solutions.

The relative closeness coefficient is computed as: $CC_i = \frac{D^-}{D^- + D^+}$. Alternatives are then ranked according to their CC_i, from the highest to the lowest. In this case, as expected, since it was identified as the positive ideal PHFLTS, alternative number two is the one that have satisfied the most to all DMs, followed by alternative three.

Chapter 3

A perceptual-based approach for MAGDM

3.1 Introduction

Multiple-attribute group decision analysis (MAGDA) or multi-criteria group decision analysis (MCGDA) is used when a group of experts or decision makers (DMs) express their assessments or preferences on a set of attributes (or criteria) for a set of alternatives and an optimal representative or solution is needed to solve the problem [63, 64]. It is very common that, in these group decision making (GDM) environments, experts or DMs do not feel at ease in using numerical values to express their judgements, but rather feel more comfortable using linguistic terms, i.e., words. Actually, the natural language is what governs uncertain human's cognitive process and it is more appropriate for expressing uncertain assessments whose nature is vague, imprecise or incomplete [109, 142]. Recently, the introduction of hesitant fuzzy linguistic term sets (HFLTSs) has attracted significant attention from researchers, as introduced in Chapter 2.

Many practical applications have used HFLTSs to deal with the linguistic information involved in MAGDA problems. An state of the art and list of applications can be found in [108, 173]. Additionally, very recent publications have also operated with this hesitant fuzzy linguistic approach to solve complex MAGDA problems [37, 39]. The use of HFLTSs provides a linguistic and computational frame to model the opinions provided by the group of experts, based on the fuzzy linguistic approach and the use of context-free grammars [24, 67, 142]. Since its introduction, many contributions on HFLTSs properties and theory can be found in the literature, for example, in relation to aggregation operators [108, 182], comparison methods [53, 104, 108, 182], correlation coefficients of HFLTSs [109], similarities and distance measures between HFLTSs [76, 106, 123] or consensus degrees [164, 186].

Most of the GDM applications found in the literature, which are framed as MCDM problems with linguistic assessments modelled by means of HFLTSs, are assumed to be built over a uniform and symmetrically distributed linguistic term set (LTS), as for example, in some AHP [168], ELECTRE [135, 175, 179] or TOPSIS [138, 185] applications. Moreover, specific decision making supporting tools have also been designed based on balanced HFLTSs, such as the multiple-expert multi-criteria decision making (MEMCDM) model developed in [120] for a real estate web site in the housing market or the model based on balanced extended hesitant fuzzy linguistic term sets (EHFLTSs) for the evaluation of university faculty members for tenure and promotion in [172]. This seems to be appropriate for cases where the semantics of each term have a proportional uncertainty and are usually equally placed around a central label. However, there exist many GDM situations where attributes relate to qualitative characteristics that need to be assessed by linguistic terms represented by unsymmetrical or not uniformly distributed LTSs, i.e., unbalanced LTS, such as for example, the evaluation of creditworthiness and credit risk quality of bonds [41] or factors affecting the comfort of passengers [37]. Similarly, it is also very common to find GDM situations with DMs having different backgrounds or knowledge and this also needs to be modelled by unbalanced LTS. For instance, the classical TODIM method is extended with unbalanced HFLTSs to model the psychological behaviours of DMs [196]. Some consensus measures and consensus reaching process have been adapted to flexibly handle MCGDM problems with unbalanced HFLTSs [28, 30, 76, 164]. Note that the unbalanced linguistic information may arise from the nature and characteristics of some linguistic variables such as the ones involved in a grading system. In other circumstances, the use of unbalanced linguistic information is appropriate to capture the different physiological aspects of DMs.

In the literature, several methods were proposed to deal with unbalanced linguistic term sets (ULTSs). Some models are built upon the linguistic hierarchy and the use of a 2-tuple fuzzy linguistic model [41, 53, 55, 56, 81, 110] while other approaches used the generalized absolute orders of magnitude qualitative spaces [144, 145] or the asymmetric sigmoid semantics [204]. As with the initial 2-tuple linguistic models, the origin of HFLTSs worked under the assumption of linguistic terms with equidistant labels [142]. Recently, with respect to HFLTSs modeling, several linguistic computational models have been developed to handle ULTSs [28, 30, 53, 164, 196]. For instance, in [53], the authors introduce a methodology to build HFLTSs based on ULTSs using a numerical scale and propose a mixed 0-1 linear programming model to aggregate unbalanced linguistic information. In [196] another unbalanced HFLTS method is developed to consider the psychological behaviour of DMs which is then applied to a personnel selection process, an investment alternative selection process and a telecommunication service provider selection process. Furthermore, in [36], a framework containing several algorithms for implementing attitudinal HFLTS possibility distribution generation is developed which is based on the similarity measure of linguistic terms. This method is used, in combination with several aggregation algorithms, for solving real business MAGDM problem, such as the selection of professional third-party reverse logistics providers in in [39] or the prioritization of factors affecting in-cabin passenger comfort on high-speed rail in China [37]. Nonetheless, even if all these approaches can effectively deal with unbalanced HFLTSs, very few can simultaneously deal with multi-granularity and unbalanced hesitant linguistic information.

With respect to the issue of multi-granularity, managing information assessed in different linguistic term sets (multi-granularity LTSs) has always represented another challenge for collective performance evaluations [82]. In a multicriteria group decision situation, not all DM might feel comfortable using the same linguistic term set when expressing their judgements. It may happen that some attributes or criteria are better evaluated using different linguistic term set (for instance, some may be more appropriately evaluated with a LTS of a higher granularity). Some methodologies were introduced to deal with multigranular linguistic term sets in a MCDM problem based on the concept of linguistic hierarchy and the use of fuzzy sets with membership functions or fuzzy preference relations [34, 44, 82, 84, 130]. An extensive range of methods are proposed for uniform and aggregation of multigranular linguistic information, without loss of information. [2, 34, 92, 130, 144, 145, 174]. Recently, with respect to HFLTSs modeling, several approaches have been developed to handle multi-granularity [118, 203]. But, the majority of the existing methods that focus on multi-granularity lack the treatment of ULTSs.

An example of a decision aiding framework that focus simultaneously on modeling unbalanced and multi-granular DMs linguistic information by means of HFLTSs can be found in [164]. In this paper, the authors introduce a signed distance measure between HFLTSs based on the ordinal semantics of linguistic terms and the possibility distribution method [187].

With respect to consensus measures in GDM, different linguistic models have been adapted to deal with unbalanced linguistic information [28, 30, 76, 130, 164, 186, 201]. Nonetheless, consensus measures modeled by means of unbalanced HFLTSs are limited. Some references can be found in [28, 76, 164]. For instance, Hao and Chiclana defined the concept of attitude linguistic quantifiers and associated it to the subjective preference of an expert [76]. The authors developed an attitude quantifier deriving method as the basis to generate possibility distribution in the HFLTSs framework that extends the previous works of Wu and Xu [187] and Chen et al.[38]. But again, in these previous studies, DMs are limited to using the same ULTSs and hence, the proposed measures fail to capture the complete heterogeneity of DMs. Therefore, the development of consensus measures that deal with multi-granular ULTSs, by means of HFLTSs, are necessary.

In this chapter, a new linguistic representation methodology for group decision-making problems that simultaneously deals with hesitant unbalanced and multi-granular linguistic information is developed. The modelling is based on the algebraic structure of the extended lattice of HFLTSs [121] and the measures developed on it. Compared to previous linguistic frameworks modelled by HFLTSs, there are some different aspects: (1) Subscript independence; (2) Basic labels can be freely distributed, no need for uniformity neither symmetry; (3) flexibility for different degrees of uncertainty and granularity over the experts.

This chapter responds mainly to objectives O1-O6 stated in the Objectives section 1.2 and it reflects the contributions C1-C4 explained in the Contributions section 1.3. The chapter is divided into two main sections. In the first section, section 3.2, all the new tools and operations needed for the method are developed. Within this first section, the perceptual map is initially defined over ULTSs in subsection 3.2.1. Then, in subsections 3.2.2 and 3.2.3, a perceptual-based distance and consensus measure for GDM are developed on the context of multi-granularity and unbalanced LTSs. Finally, in subsection 3.2.4, a transformation function for multi-perceptual GDM contexts dealing with multi-granular LTSs is developed. In the second section, section 3.3, two specific methodologies are designed and presented, step by step, to deal with several MAGDM objectives. First, a classification and ranking perceptual-based method is presented in 3.3.1 and then, an extended fuzzy multi-perceptual linguistic TOPSIS is designed to make TOPSIS a more human-oriented method in 3.3.2

3.2 A perceptual-based approach for hesitant unbalanced linguistic information

In this section, I introduce the needed tools to develop a new computing with words (CWW) methodology which uses ULTSs to build HFLTSs in a linguistic GDM context. The concept of perceptual map is introduced in the next subsection 3.2.1. This concept allows us to redefine the previous distance between HFLTS and consider a new one, in subsection 3.2.2. In addition, a consensus degree based on this new distance is also introduced in subsection 3.2.3. Objectives O1, O2, O3, O4, O5 and O6 are fulfilled throughout these subsections Finally, everything is used in subsection 3.2.4

to develop a transformation method for dealing with multi-perceptual and multi-granular GDM contexts. This responds to the objectives O7, O8 and O9, which are related to develop new methods to solve MAGDM problems based on the defined perceptual-based tools.

3.2.1 The perceptual map on the structure of the lattice with ULTSs

In this subsection, I define the concept of a normalized measure over a linguistic term set S, which may not be balanced: the perceptual map.

Definition 3.1. Let S be a totally ordered finite LTS, $S = \{s_1, s_2, ..., s_n\}$. Let μ' denote a measure over S such that $\mu'(s_i) > 0, \forall i \in \{1, 2, \ldots, n\}$. Then, the *perceptual map*, μ, induced by μ', is a function $\mathcal{H}_S^* \to [0, 1]$ defined as:

$$\mu(H_S) = \frac{\sum\limits_{s_i \in H_S} \mu'(s_i)}{\sum\limits_{i=1}^{n} \mu'(s_i)} \tag{3.1}$$

for any $H_S \in \mathcal{H}_S^*$.

Proposition 3.1. The perceptual map μ provides a normalized measure on $\mathcal{H}_S^*$.

Proof. From the definition and properties of a measure [74], the defined normalized measure, μ on $\mathcal{H}_S^*$, satisfies:

1. Non negativity. For all $H_S \in \mathcal{H}_S^*$, $\mu(H_S) \geq 0$. Since μ' is a measure defined over S, the numerator and the denominator in equation 3.1 are non negative terms, and, therefore $\mu(H_S) \geq 0$.

2. Null empty set. Since μ' is a measure, $\mu'(\emptyset) = 0$, then it is straight forward to see that $\mu(\emptyset) = 0$. However, by definition, $\mathcal{H}_S^*$ does not include $\emptyset$, since $\mathcal{H}_S^* = \mathcal{H}_S - \{\emptyset\}$ and hence, the domain of the perceptual map does not include an emtpy HFLTSs.

3. Additivity. For any finite collection of $\{H_S^k\}$, $k \in \{1, 2, \ldots, m\}$, of pairwise disjoint sets in $\mathcal{H}_S^*$, i.e., they do not overlap, then: $\mu(\cup H_S^k) = \sum \mu(H_S^k)$. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a LTS and μ' a given measure on S such that $\sum\limits_{i=1}^{n} \mu'(s_i) = b$. Let μ denote the normalized measured induced by μ' and let $\mathbb{H}_S$=$\{H_S^1, \ldots, H_S^k, \ldots, H_S^m\}$, $k \in \{1, 2, \ldots, m\}$, be a finite collection of pairwise disjoint HFLTSs. Then:

$$\sum_{k=1}^{m} \mu(H_S^k) = \mu(H_S^1) + \mu(H_S^2) + \cdots + \mu(H_S^m) = \sum_{k=1}^{m} \sum_{s_i \in H_S^k} \frac{\mu'(s_i)}{b}$$

Alternatively, given that $\mathcal{H}_S^*$ is a finite set, the unions and sums are finite and the union of disjoint elements of $\mathbb{H}_S$ is as follows:

$$\cup H_S^k = \{s_i \mid s_i \in H_S^k, k \in \{1, 2, \ldots, m\}\}$$

Since the elements of $\mathbb{H}_S$ do not overlap and μ' is a measure on S, then:

$$\mu(\cup H_S^k) = \frac{\sum\limits_{s_i \in \cup H_S^k} \mu'(s_i)}{b} = \sum_{k=1}^{m} \sum_{s_i \in H_S^k} \frac{\mu'(s_i)}{b}$$

4. The measure μ is a normalized measure because it satisfies

$$\sum_{s_i \in S} \mu(s_i) = 1$$

It is straight forward to see from equation 3.1, that $\mu([s_1, s_n]) = 1$, since numerator and denominator become the same term b.

□

Proposition 3.2. Let $[0, b]$ be any closed interval in $\mathbb{R}$. There exists a one-to-one correspondence (bijective function) between the set of measures μ' on S with $\mu'(s_1) + \cdots + \mu'(s_n) = b$, denoted as $\mathbb{F}_{\mathcal{B}}$, and the set of partitions of the set $[0, b]$ with n non-empty subsets, denoted as $\mathbb{P}_{\mathcal{B}}$.

Proof. Let $P_{\mathcal{B}}$ be any partition of $\mathbb{P}_{\mathcal{B}}$ defined as $P_{\mathcal{B}}$=$\{[\alpha_0, \alpha_1), [\alpha_1, \alpha_2), \ldots, [\alpha_{n-1}, \alpha_n] \mid 0 = \alpha_0 < \cdots < \alpha_n = b\}$. Given the collection of α_i, $i \in \{0, 1, \ldots, n\}$ which denotes the set of points defining the partition $P_{\mathcal{B}}$, then: $\mu'(s_i) = \alpha_i - \alpha_{i-1}$.
Reciprocally, let μ' be a measure belonging to $\mathbb{F}_{\mathcal{B}}$. Then, there exists one partition $P_{\mathcal{B}} \in \mathbb{P}_{\mathcal{B}}$, defined as $P_{\mathcal{B}}$=$\{[\alpha_0, \alpha_1), [\alpha_1, \alpha_2), \ldots, [\alpha_{n-1}, \alpha_n] \mid 0 = \alpha_0 < \cdots < \alpha_n = b\}$ such that $\alpha_i = \sum_{k=1}^{i} \mu'(s_k)$ □

An immediate consequence of proposition 3.2 is the following:

Corollary 3.1. There exists a bijective function between the set of perceptual maps over a set S of granularity n, denoted as $\mathbb{F}_n$ and the set of partitions of $[0, 1]$ with n non-empty subsets, denoted as $\mathbb{P}_n$.

Remark 3.1. In their attempt to find a way to make transformation between linguistic 2-tuples and numerical values, Dong et al. [54], proposed the concept of *numerical scale*. The *NS* is set to 0 for the first linguistic term set. In comparison with the proposed method, the partition elements of $P_{\mathcal{A}}$ of $[0, b]$ associated to the measure μ' over a LTS, S, would be equivalent to the values of the $NS(s_i)$, $i \in \{1, 2, \ldots, n\}$, with $NS(s_n) = b$.

Example 3.1. Let S be an ULTS of granularity 6. Let P=$\{0, 1, 3, 7, 10, 11, 12\}$ be a partition of the closed interval $[0, 12]$. Then, its corresponding measure μ' over S is defined as: $\mu'(s_1) = \mu'(s_5) = \mu'(s_6) = 1$, $\mu'(s_2) = 2$, $\mu'(s_3) = 4$ and $\mu'(s_4) = 3$ (See figure 3.1).

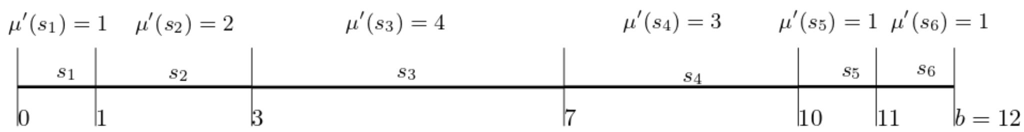

FIGURE 3.1: The equivalence relation between partition $P_{\mathcal{B}}$ of example 3.1 and its corresponding measure μ'.

By extension, applying Definition 3.1 to the given measure μ' of example 3.1, its corresponding perceptual map over $\mathcal{H}_S^*$, would be defined in $\{s_i\}$ as follows: $\mu(\{s_1\}) = \mu(\{s_5\}) = \mu(\{s_6\}) = \frac{1}{12}$, $\mu(\{s_2\}) = \frac{2}{12}$, $\mu(\{s_3\}) = \frac{4}{12}$ and $\mu(\{s_4\}) = \frac{3}{12}$. An extended lattice based on this perceptual map is graphically illustrated in figure 3.2.

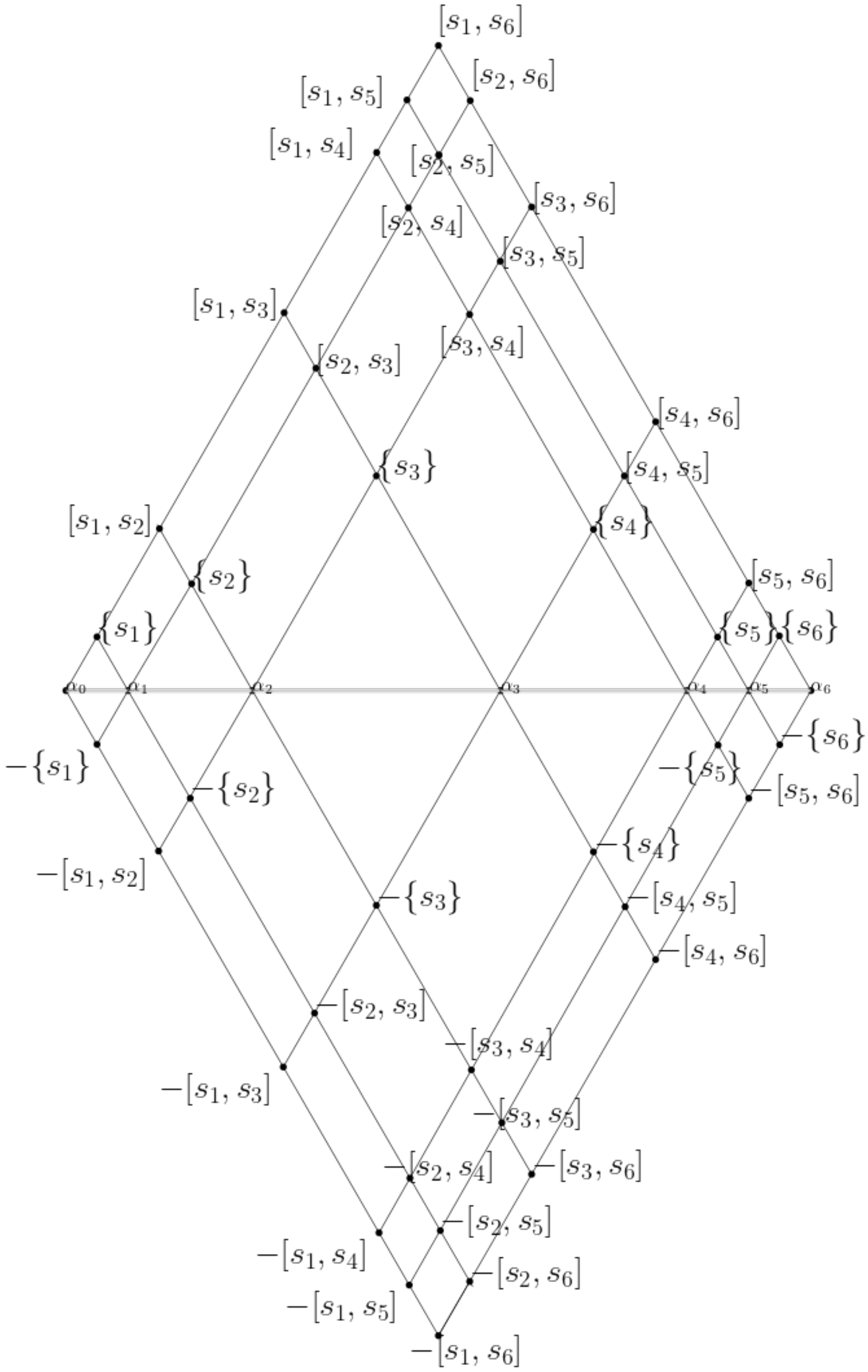

FIGURE 3.2: The extended distributive lattice of example 3.1

Based on the perceptual map defined in 3.1, I propose a more straight-forward definition of balanced LTS.

Definition 3.2. Let $S=\{s_1, s_2, \ldots, s_n\}$ be a linguistic term set and μ its perceptual map. S is a *balanced linguistic term set* with respect to μ, i.e. it is uniformly and symmetrically distributed, if there exists a unique constant $\xi > 0$ such that $\mu(s_i) = \xi$ $\forall\ i = 1, 2, \ldots, n$.

If the condition is unmet, S is called an *unbalanced linguistic term set* with respect to the perceptual map μ.

I can also define a not uniform but symmetrically distributed S, when the previous condition is unmet but the followings hold:

1. Let $S^R = \{s_i \mid s_i \in S, i > \frac{n}{2}\}$ and $S^L = \{s_i \mid s_i \in S, i \leq \frac{n}{2}\}$, where S is $S=\{s_1, s_2, \ldots, s_n\}$. Then, $\sum_{i \in S^R} \mu(s_i) = \sum_{i \in S^L} \mu(s_i)$.

2. There exists a set of constants $\xi_i > 0$ such that $\mu(s_i) = \mu(s_{n-i+1}) = \xi_i \; \forall \, i \leq \frac{n}{2}$.

Remark 3.2. Following the comparisons with the numerical scale used to define unbalanced LTS in [54, 55] introduced in Remark 3.1, the concept of (normalized) numerical scale can be compared to the perceptual map as follows:

$$NS(s_l) = \sum_{i=1}^{i=l} \mu(s_i) \qquad \forall i \in \{1, 2, \ldots, n\} \tag{3.2}$$

Inspired by the linguistic distribution assessments [200] and several possibility-distribution based approaches in the context of HFLTSs [164, 187], I propose a new definition of the possibility distribution for a given HFLTS, H_S, over S, based on the concept of the perceptual map.

Definition 3.3. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a LTS and μ its associated perceptual map. Let $H_S = \{s_L, s_{L+1}, \ldots, s_R\}$ be a HFLTS on S representing the opinion or judgement given by an expert or decision maker. The perceptual-based possibility distribution for H_S on S is represented by $P = \{p_1, p_2, \ldots, p_n\}$, where p_l is given by the following:

$$p_l = \begin{cases} 0, & \text{if } l = 1, 2, \ldots L-1; \\ \dfrac{\mu(s_l)}{\mu(H_S)}, & \text{if } l = L, L+1, \ldots, R; \\ 0, & \text{if } l = R+1, \ldots, n. \end{cases}$$

p_l denotes the perceptual distribution degree under which the alternative has an assessment of value s_l, such that $\sum_{l=1}^{n} p_l = 1$ and $0 \leq p_l \leq 1, l = 1, 2, \ldots, n$.

This definition can be flexibly applied to HFLTSs when the LTS are balanced as well as unbalanced. Notice that when S is balanced, its corresponding perceptual map, μ, is defined as $\mu(s_i) = \frac{1}{n}$, with n being the cardinality of S. Therefore, the proposed Definition 3.3 is the same as the possibility distribution in [187].

Example 3.2. Let H_S^1, H_S^2, H_S^3 be three HFLTSs defined over the unbalanced set S as defined in example 3.1. Let $H_S^1 = \{s_1\}$, $H_S^2 = \{s_2, s_3, s_4\}$, $H_S^3 = \{s_5, s_6\}$. Then, their corresponding perceptual-based possibility distributions are represented by $H_S^1 = (1, 0, 0, 0, 0, 0)$, $H_S^2 = (0, \frac{2}{9}, \frac{4}{9}, \frac{3}{9}, 0, 0)$, $H_S^3 = (0, 0, 0, 0, \frac{1}{2}, \frac{1}{2})$, respectively.

3.2.2 A perceptual-based distance for unbalanced HFLTSs

In this subsection, a new distance for unbalanced HFLTSs, in the context of the extended lattice, $(\overline{\mathcal{H}_S}, \sqcup, \sqcap)$, is introduced based on the perceptual map defined on S.

A new distance measure between HFLTSs which is not subscript dependent and takes into account the unbalanced structure of the extended set, $\overline{\mathcal{H}_S}$, is proposed.

Hence, this measure in the set $\mathcal{H}_S$ can represent situations where linguistic information is not uniformly ordered (i.e. distances between basic labels s_i are not proportional) and therefore, in the context the $\overline{\mathcal{H}_S}$ over a set S, the "steps" of the path between two HFLTSs, represented in the lattice illustration with the axis, are not equal. As compared to figure 2.1, the graphical representation of the proposed extended lattice of HFLTSs over a not uniformly distributed set of linguistic terms is represented in figure 3.3.

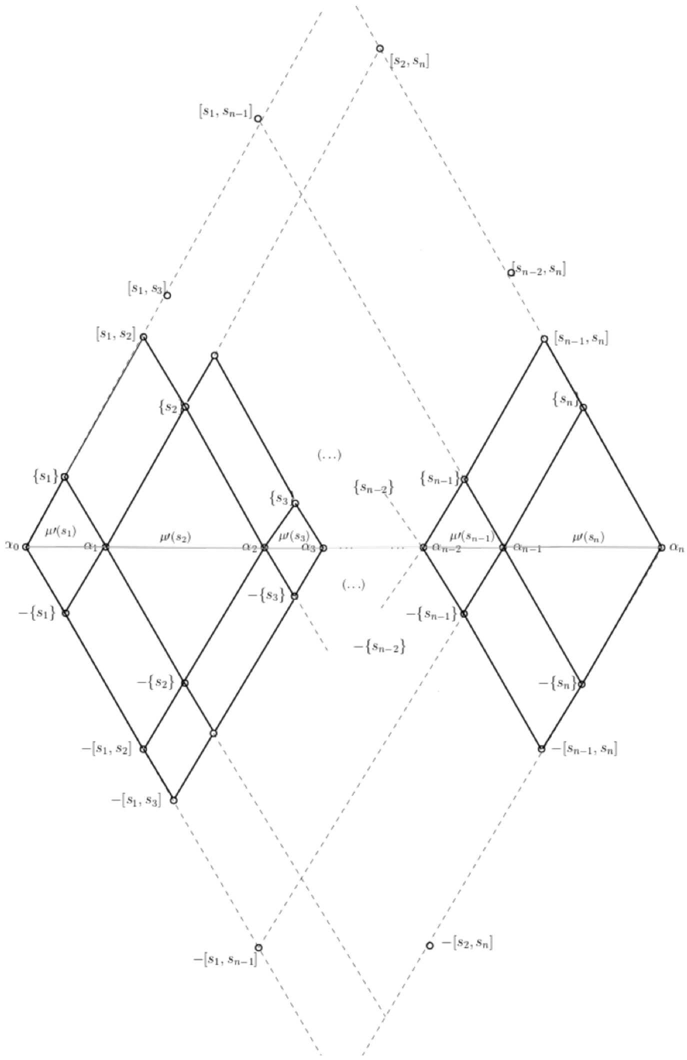

FIGURE 3.3: The extended distributive lattice of an unbalanced HFLTSs

Given any perceptual map, μ, the definition of width of a HFLTS $H_S \in \overline{\mathcal{H}_S}$ used in [121] is extended as follows:

Definition 3.4. Given a perceptual map μ and $H_S \in \overline{\mathcal{H}_S}$, the *width* of H_S with respect to μ is defined as:

$$W_\mu(H_S) = \begin{cases} \mu(H_S), & H_S \in \mathcal{H}_S^*; \\ 0, & H_S \in \mathcal{A}; \\ -\mu(-H_S), & H_S \in (-\mathcal{H}_S^*). \end{cases}$$

with $\mathcal{H}_S^*$, $\mathcal{A}$ and $-\mathcal{H}_S^*$ considered as defined in subsubsection 2.2.2

Proposition 3.3. Let $H_S^1, H_S^2 \in \mathcal{H}_S^*$, then:

$$D_\mu(H_S^1, H_S^2) := \mathcal{W}_\mu(H_S^1 \sqcup H_S^2) - \mathcal{W}_\mu(H_S^1 \sqcap H_S^2) \tag{3.3}$$

provides a distance in the lattice ($\overline{\mathcal{H}_S}$,$\sqcup$,$\sqcap$).

Proof. $D_\mu(H_S^1, H_S^2)$ defines a distance because it is equivalent to the geodesic distance in the distributive lattice $\overline{\mathcal{H}_S}$. The geodesic distance between H_S^1 and H_S^2 is defined by the length of the shortest path to go from H_S^1 to H_S^2. According to the definition of the extended inclusion relation [121] in $\overline{\mathcal{H}_S}$, I have that $H_S^1 \sqcap H_S^2 \preceq H_S^1 \sqcup H_S^2$. Then $\mathcal{W}_\mu(H_S^1 \sqcup H_S^2)$ - $\mathcal{W}_\mu(H_S^1 \sqcap H_S^2)$ is the length of the minimum path between $H_S^1 \sqcup H_S^2$ and $H_S^1 \sqcap H_S^2$. Thus, I should check that the length of the shortest path between $H_S^1 \sqcup H_S^2$ and $H_S^1 \sqcap H_S^2$ is equivalent to the length of the shortest path between H_S^1 and H_S^2. On the one hand, if $H_S^1 \sqcap H_S^2$, then $H_S^1 \sqcup H_S^2 = H_S^2$ and $H_S^1 \sqcap H_S^2 = H_S^1$ and hence, the proof becomes straightforward. On the other hand, when none of them belongs to the coverage of the other one, then the points H_S^1,H_S^2,$H_S^1 \sqcup H_S^2$,$H_S^1 \sqcap H_S^2$ are the vertices that form a parallelogram with two pair of parallel sides. Facing sides are of equal length and, particularly, if the set S is balanced, then all four sides are equal. As can be checked in the graph 2.1, the sum of the length of two consecutive sides of this formed parallelogram define the shortest path between H_S^1 and H_S^2, which is the same path defined by the other two consecutive sides. Thus, this is the shortest path between $H_S^1 \sqcup H_S^2$ and $H_S^1 \sqcap H_S^2$. □

Definition 3.5. The distance introduced in Proposition 3.3 and defined by equation 3.3 is the *perceptual-based distance* for HFLTSs.

Example 3.3. In an IT service company, the human resources manager is asked to evaluate the programming skills of three candidates, (A, B, C), using a common grading system which is an unbalanced linguistic term set $S = \{s_1 =$ "newcomer", $s_2 =$ "novice", $s_3 =$ "apprentice", $s_4 =$ "talented", $s_5 =$ "expert"$\}$. The perceptual map of the manager is set to be as: $\mu(s_1) = \frac{1}{20}$, $\mu(s_2) = \frac{3}{20}$, $\mu(s_3) = \frac{4}{20}$, and $\mu(s_4) = \mu(s_5) = \frac{6}{20}$. His assessments for candidates A, B and C are "between a newcomer and an apprentice", "apprentice" and "at most talented", respectively. The corresponding HFLTSs are $H_S^A = [s_1, s_3]$, $H_S^B = \{s_3\}$ and $H_S^C = [s_1, s_4]$. According to Definition 3.5 the perceptual-based distances between candidates are the following:

$$D_\mu(H_S^A, H_S^B) = \mathcal{W}_\mu([s_1, s_3]) - \mathcal{W}_\mu(\{s_3\}) = 0.40 - 0.20 = 0.20$$

$$D_\mu(H_S^B, H_S^C) = \mathcal{W}_\mu([s_1, s_4]) - \mathcal{W}_\mu(\{s_3\}) = 0.70 - 0.20 = 0.50$$

$$D_\mu(H_S^A, H_S^C) = \mathcal{W}_\mu([s_1, s_4]) - \mathcal{W}_\mu([s_1, s_3]) = 0.70 - 0.40 = 0.30$$

Given these results, I can derive that, with respect to the programming skills and based on the manager reasoning process, candidate A is more similar to B than to C and that candidate B and C are the most distant ones. For comparison purposes, these same distances between pairwise candidates have been computed using other existing distances measures in the literature. These assume the linguistic term set S to be balanced. See Table 3.1.

	D_μ defined in Proposition 3.3	Montserrat-Adell et al. [121]	Wang et al. [175]	Liao et al. (Euclidean) [106]	Liao et al. (Euclidian-Hausdorff) [106]
H_S^A and H_S^B	0.20	2	2	0.258	0.4
H_S^B and H_S^C	0.50	3	2.236	0.245	0.4
H_S^A and H_S^C	0.30	1	1	0.125	0.2

TABLE 3.1: Pairwise distances of candidates of example 3.3 using different distance measures

Considering the presence of an unbalanced μ associated to the set S, the less "separated" candidates are candidates A and B. However, if the assumption of S being a balanced and symmetric LTS is made, then the closest candidates are A and C, according to the different proposed distances in the literature [106, 121, 175]. Notice that if I assume S to be a balanced set, and I use the concept of the perceptual map μ to model the opinions of the manager, then $\mu(s_i) = 1/5$ for $i \in \{1, 2, 3, 4, 5\}$. It follows that, in this case, the pairwise distances between candidates would be $D_\mu(H_S^A, H_S^B)$ = 0.40, $D_\mu(H_S^B, H_S^C)$ = 0.60 and $D_\mu(H_S^A, H_S^C)$ = 0.20. This is in line with the rest of the balanced distances measures of Table 3.1

Example 3.4. Following example 2.6 of preliminaries Chapter 2, let consider now an unbalanced LTS with a measure on it such as $\mu'(s_1) = 1, \mu'(s_2) = 3, \mu'(s_3) = 8, \mu'(s_4) = 4, \mu'(s_5) = 2$. Then, according to the perceptual-based distance, all the pairwise distances for this unbalanced set are expressed in the following Table 3.2.

	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_4\}$	$\{s_5\}$	$[s_1,s_2]$	$[s_2,s_3]$	$[s_3,s_4]$	$[s_4,s_5]$	$[s_1,s_3]$	$[s_2,s_4]$	$[s_3,s_5]$	$[s_1,s_4]$	$[s_2,s_5]$	$[s_1,s_5]$
$\{s_1\}$	0	0.222	0.833	1.5	1.833	0.167	0.667	1.056	1.611	0.611	0.889	1.167	0.833	1	0.944
$\{s_2\}$	0.222	0	0.611	1.278	1.611	0.056	0.444	0.833	1.389	0.5	0.667	0.944	0.667	0.778	0.833
$\{s_3\}$	0.833	0.611	0	0.667	1	0.667	0.167	0.222	0.778	0.222	0.389	0.333	0.444	0.5	0.556
$\{s_4\}$	1.5	1.278	0.667	0	0.333	1.333	0.833	0.444	0.111	0.889	0.611	0.556	0.667	0.722	0.778
$\{s_5\}$	1.833	1.611	1	0.333	0	1.667	1.167	0.778	0.222	1.222	0.944	0.667	1	0.833	0.889
$[s_1,s_2]$	0	0.056	0.667	1.333	1.667	0	0.5	0.889	1.444	0.444	0.722	1	0.667	0.833	0.778
$[s_2,s_3]$	0.667	0.444	0.167	0.833	1.167	0.5	0	0.389	0.944	0.056	0.222	0.5	0.278	0.333	0.389
$[s_3,s_4]$	1.056	0.833	0.222	0.444	0.778	0.889	0.389	0	0.556	0.444	0.167	0.111	0.222	0.278	0.333
$[s_4,s_5]$	1.611	1.389	0.778	0.111	0.222	1.444	0.944	0.556	0	1	0.722	0.444	0.778	0.611	0.667
$[s_1,s_3]$	0.611	0.5	0.222	0.889	1.222	0.444	0.056	0.444	1	0	0.278	0.556	0.222	0.389	0.333
$[s_2,s_4]$	0.889	0.667	0.389	0.611	0.944	0.722	0.222	0.167	0.722	0.278	0	0.278	0.056	0.111	0.167
$[s_3,s_5]$	1.167	0.944	0.333	0.556	0.667	1	0.5	0.111	0.444	0.556	0.278	0	0.333	0.167	0.222
$[s_1,s_4]$	0.833	0.667	0.444	0.667	1	0.667	0.278	0.222	0.778	0.222	0.056	0.333	0	0.167	0.111
$[s_2,s_5]$	1	0.778	0.5	0.722	0.833	0.833	0.333	0.278	0.611	0.389	0.111	0.167	0.167	0	0.056
$[s_1,s_5]$	0.944	0.833	0.556	0.778	0.889	0.778	0.389	0.333	0.667	0.333	0.167	0.222	0.111	0.056	0

TABLE 3.2: Pairwise distance measurements of example 2.6 of preliminaries Chapter 2, using the developed perceptual-based distance

The following Lemma provides a useful tool to compute the perceptual-based distance, D_μ, defined in 3.5.

Lemma 3.1. The perceptual-based distance, D_μ, can be equivalently expressed as:

$$D_\mu(H_S^1, H_S^2) = 2 \cdot \mathcal{W}_\mu(H_S^1 \sqcup H_S^2) - \mathcal{W}_\mu(H_S^1) - \mathcal{W}_\mu(H_S^2) \tag{3.4}$$

Proof. Let H_S^1=$[s_i, s_j]$ and let H_S^2=$[s_k, s_l]$ be two HFLTSs over S. I consider the four cases:

- Consecutive HFLTS, i.e., $i \leq j < k \leq l$ and $k = j+1$. Then $H_S^1 \sqcap H_S^2 \in \mathcal{A}$ and hence, $\mathcal{W}_\mu(H_S^1 \sqcap H_S^2) = 0$. Then, the perceptual-based distance of Definition 3.5 becomes $D_\mu(H_S^1, H_S^2) = \mathcal{W}_\mu(H_S^1 \sqcup H_S^2) - 0$. Since $\mathcal{W}_\mu(H_S^1 \sqcup H_S^2) = \mathcal{W}_\mu(H_S^1) + \mathcal{W}_\mu(H_S^2) - \mathcal{W}_\mu(H_S^1 \sqcap H_S^2)$, I have that $\mathcal{W}_\mu(H_S^1 \sqcup H_S^2) - \mathcal{W}_\mu(H_S^1) - \mathcal{W}_\mu(H_S^2) = 0$. Therefore, adding this term to the perceptual-based expression of equation 3.3, I obtain: $D_\mu(H_S^1, H_S^2) = 2 \cdot \mathcal{W}_\mu(H_S^1 \sqcup H_S^2) - \mathcal{W}_\mu(H_S^1) - \mathcal{W}_\mu(H_S^2)$.

- Nested HFLTS, i.e., $i \leq k \leq l \leq j$. Then $H_S^1 \sqcap H_S^2 = H_S^2$ and hence, $\mathcal{W}_\mu(H_S^2) = \mathcal{W}_\mu(H_S^1 \sqcap H_S^2)$. Then, the perceptual-based distance becomes $D_\mu(H_S^1, H_S^2) = \mathcal{W}_\mu(H_S^1 \sqcup H_S^2) - \mathcal{W}_\mu(H_S^2)$. Since $\mathcal{W}_\mu(H_S^1 \sqcup H_S^2) = \mathcal{W}_\mu(H_S^1) + \mathcal{W}_\mu(H_S^2) - \mathcal{W}_\mu(H_S^2)$, I get $\mathcal{W}_\mu(H_S^1 \sqcup H_S^2) - \mathcal{W}_\mu(H_S^1) = 0$. Adding this term to the perceptual-based distance of equation 3.3, I obtain: $D_\mu(H_S^1, H_S^2) = 2 \cdot \mathcal{W}_\mu(H_S^1 \sqcup H_S^2) - \mathcal{W}_\mu(H_S^1) - \mathcal{W}_\mu(H_S^2)$.

- Overlapped HFLTS, i.e., $i \leq k \leq j \leq l$. Then $H_S^1 \sqcap H_S^2 = [s_k, s_j]$, $H_S^1 \sqcup H_S^2 = [s_i, s_l]$. In this case, using $\mathcal{W}_\mu(H_S^1 \sqcup H_S^2) = \mathcal{W}_\mu(H_S^1) + \mathcal{W}_\mu(H_S^2) - \mathcal{W}_\mu(H_S^1 \sqcap H_S^2)$ and isolating the term $\mathcal{W}_\mu(H_S^1 \sqcap H_S^2)$, is is then substituted into the perceptual-based distance of equation 3.3 to obtain: $D_\mu(H_S^1, H_S^2) = 2 \cdot \mathcal{W}_\mu(H_S^1 \sqcup H_S^2) - \mathcal{W}_\mu(H_S^1) - \mathcal{W}_\mu(H_S^2)$.

- Disjoint HFLTS (with gap), i.e., $i \leq j < k \leq l$ and $k > j+1$. Then $H_S^1 \sqcap H_S^2 \in -\mathcal{H}_S^*$, $H_S^1 \sqcap H_S^2 = -[s_{j+1}, s_{k-1}]$. In this case, $\mathcal{W}_\mu(H_S^1 \sqcap H_S^2) = -\mathcal{W}_\mu([s_{j+1}, s_{k-1}])$. Then, $\mathcal{W}_\mu(H_S^1 \sqcup H_S^2) = \mathcal{W}_\mu(H_S^1) + \mathcal{W}_\mu(H_S^2) - \mathcal{W}_\mu(H_S^1 \sqcap H_S^2)$, being this last term a negative term. If I isolate this term $\mathcal{W}_\mu(H_S^1 \sqcap H_S^2)$ and substitute it into the perceptual-based distance of equation 3.3, I get, as the previous case: $D_\mu(H_S^1, H_S^2) = 2 \cdot \mathcal{W}_\mu(H_S^1 \sqcup H_S^2) - \mathcal{W}_\mu(H_S^1) - \mathcal{W}_\mu(H_S^2)$.

□

Proposition 3.4. Given two HFLTSs, $H_S^1, H_S^2 \in \mathcal{H}_S^*$, then $D_\mu(H_S^1, H_S^2) \leq 2 - \big(\mu(\{s_1\}) + \mu(\{s_n\})\big)$.

Proof. If H_S^1, $H_S^2 \in \mathcal{H}_S^*$, the most distant pairs are $\{s_1\}$ and $\{s_n\}$. Then, based on Lemma 3.1, their distance is computed as follows:

$$
\begin{aligned}
D_\mu(\{s_1\}, \{s_n\}) &= 2 \cdot \mathcal{W}_\mu(\{s_1\} \sqcup \{s_n\}) - \mathcal{W}_\mu(\{s_1\}) - \mathcal{W}_\mu(\{s_n\}) = \\
&2 \cdot \mathcal{W}_\mu([s_1, s_n]) - \mu(\{s_1\}) - \mu(\{s_n\}) = 2 - \big(\mu(\{s_1\}) + \mu(\{s_n\})\big)
\end{aligned}
$$

□

3.2.3 A perceptual-based collective consensus measure for hesitant fuzzy unbalanced linguistic group decision-making

In a GDM context, let us consider a set of alternatives $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ and a group of DMs, $G = \{d_1, d_2, \ldots, d_k\}$. The assessments provided by each d_j over the set Λ are denoted as F_H^j, $j \in \{1, \ldots, k\}$. F_H^j is an r-dimensional vector $F_H^j(\Lambda) = (H_{1j}, H_{2j}, \ldots, H_{rj}) \in (\mathcal{H}_S^*)^r$. Each component of the vector is the HFLTS provided by d_j over the alternative λ_i, i.e, $F_H^j(\lambda_i) = H_{ij}, H_{ij} \in \mathcal{H}_S^*$. Based on this context, the perceptual-based distance D_μ between two HFLTSs is extended to a distance between two HFLDs, denoted as $D_\mu^{\mathcal{F}}$, as the weighted aggregation of the distances between the corresponding HFLTSs of each λ_i, $i \in \{1, \ldots, r\}$, as follows.

Definition 3.6. Let F_H^1 and F_H^2 be two HFLDs, $F_H^1, F_H^2 \in (\mathcal{H}_S^*)^r$, $F_H^1(\lambda_i) = H_{i1}$ and $F_H^2(\lambda_i) = H_{i2}$, $i \in \{1, \dots, r\}$. Let $(w_1, w_2, \dots, w_r)$ be a vector of weights such that $\sum_{i=1}^r w_i = 1$. Then, the *weighted perceptual-based distance* $D_\mu^\mathcal{F}$ between F_H^1 and F_H^2 is defined as:

$$D_\mu^\mathcal{F}(F_H^1, F_H^2) = \sum_{i=1}^r w_i D_\mu(H_{i1}, H_{i2}) \tag{3.5}$$

The distance $D_\mu^\mathcal{F}$ can be used to calculate a central opinion (or centroid) of a group G of DMs about a set of alternatives Λ as the HFLD that minimizes the summation of distances to the HFLD of each expert.

Definition 3.7. Given a set of alternatives, $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$ and a group of DMs, $G = \{d_1, d_2, \dots, d_k\}$, let $F_H^j \in (\mathcal{H}_S^*)^r$ denote the HFLD provided by d_j over Λ with $j \in \{1, \dots, k\}$. Then, the *centroid of the group*, denoted as $F_H^{C_\mu}$, is defined as:

$$F_H^{C_\mu} = arg \min_{F_H^x \in (\mathcal{H}_S^*)^r} \sum_{j=1}^k D_\mu^\mathcal{F}(F_H^x, F_H^j) \tag{3.6}$$

Example 3.5. Let G be a group of three external consultants evaluating the programming skills of the three candidates, (A, B, C), of example 3.3, by means of S. Considering their reasoning process and background, I know that the consultants share the same perceptual map. However, to model their opinions, I consider the possibility of two perceptual maps: a perceptual map, μ_x, as defined for the manager in example 3.3 and a different one, μ_y, defined as: $\mu_y(s_1) = \frac{1}{21}$, $\mu_y(s_2) = \frac{2}{21}$, $\mu_y(s_3) = \frac{8}{21}$, $\mu_y(s_4) = \frac{8}{21}$, $\mu_y(s_5) = \frac{2}{21}$. The corresponding HFLDs of each consultant are: $F_H^1 = ([s_1, s_3], [s_3, s_4], \{s_4\})$, $F_H^2 = (\{s_3\}, \{s_2\}, [s_2, s_4])$ and $F_H^3 = ([s_1, s_4], [s_2, s_4], \{s_2\})$.
Using equation 3.6, the corresponding centroids computed for each possible perceptual map are:
$F_H^{C_{\mu x}} = ([s_1, s_3], [s_2, s_4], [s_2, s_4])$
$F_H^{C_{\mu y}} = ([s_1, s_3], [s_2, s_4], [s_2, s_4])$

The following Proposition 3.5 proves that when all DMs express their linguistic assessments over the same lattice structure of S, $(\overline{\mathcal{H}_S}, \sqcup, \sqcap)$, i.e. considering the same perceptual map, the obtained centroid is independent of the considered shared perceptual map.

Proposition 3.5. Let S be a LTS and let μ be its associated perceptual map, which is shared by a group of DMs, $G = \{d_1, d_2, \dots, d_k\}$. Let $\Lambda = \{\lambda_1, \dots, \lambda_r\}$ be a set of alternatives. Let $H_{ij} = [s_{L_{ij}}, s_{R_{ij}}]$, $j \in \{1, \cdots, k\}$ and $i \in \{1, \cdots, r\}$, denote a HFLTS modeling the linguistic assessment provided by d_j over a given λ_i. Then, for a given λ_i, the centroid of the set $\{H_{ij}\}$ is:

$$H^c = \{[s_L, s_R] \in \mathcal{H}_S^* \mid L \in \mathbb{M}(L_1, L_2, \cdots, L_k), R \in \mathbb{M}(R_1, R_2, \cdots, R_k)\} \tag{3.7}$$

where $\mathbb{M}$ is the set that contains just the median if k is an odd number or the central values and any integer number between them if k is even.

Proof. The demonstration of Proposition 3.5 is based on *reductio ad absurdum*, considering four cases. For a given λ_i, let $H^c = [s_L, s_R]$ be the centroid of the set $\{H_{ij}\}$, with $H_{ij} \in \mathcal{H}_S^*$.

- If $L < a \; \forall \; a \in \mathbb{M}(L_1, L_2, \cdots, L_k)$ then $\sum_{j=1}^k D_\mu([s_{L+1}, s_R], H_{ij})$= $\sum_{j=1}^k D_\mu([s_L, s_R], H_{ij}) - p \cdot \mu(s_R)$ with $p > 0$ and $p \in \mathbb{N}$, therefore H^c does not satisfy equation 3.7.

- If $L > a \; \forall \, a \in \mathbb{M}(L_1, L_2, \cdots, L_k)$ then $\sum_{j=1}^{k} D_\mu([s_{L-1}, s_R], H_{ij})=$ $\sum_{j=1}^{k} D_\mu([s_L, s_R], H_{ij}) - p \cdot \mu(s_{R-1})$ with $p > 0$ and $p \in \mathbb{N}$, therefore H^c does not satisfy equation 3.7.

- If $R < b \; \forall \, b \in \mathbb{M}(R_1, R_2, \cdots, R_k)$ then $\sum_{j=1}^{k} D_\mu([s_L, s_{R+1}], H_{ij})=$ $\sum_{j=1}^{k} D_\mu([s_L, s_R], H_{ij}) - p \cdot \mu(s_{R+1})$ with $p > 0$ and $p \in \mathbb{N}$, therefore H^c does not satisfy equation 3.7.

- If $R > b \; \forall \, b \in \mathbb{M}(R_1, R_2, \cdots, R_k)$ then $\sum_{j=1}^{k} D_\mu([s_L, s_{R-1}], H_{ij})=$ $\sum_{j=1}^{k} D_\mu([s_L, s_R], H_{ij}) - p \cdot \mu(s_R)$ with $p > 0$ and $p \in \mathbb{N}$, therefore H^c does not satisfy equation 3.7.

Therefore

$$H^c = \{[s_L, s_R] \in \mathcal{H}_S^* \mid L \in \mathbb{M}(L_1, L_2, \cdots, L_k), R \in \mathbb{M}(R_1, R_2, \cdots, R_k)\}$$

□

Corollary 3.2. By extension to HFLDs, the centroid $F_H^{C_\mu}$ of equation 3.6 is also independent of the perceptual map.

Remark 3.3. When the number of DMs is odd, the centroid is unique. However, when the number of DM is even, the centroid is not unique. From now on, in this paper, whenever I refer to the centroid of a set $\{H_{ij}\}, j \in \{1, 2, \cdots, k\}, i \in \{1, 2, \cdots, r\}$ I choose the centroid of highest hesitancy, i.e, highest value of $\mu(H^c)$.

Based on the proposed $D_\mu^{\mathcal{F}}$ of Definition 3.6, a new collective degree of consensus is proposed with the aim to quantify the level of agreement of a group G over a set $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$. This measure is also appropriate to measure the harmony or accordance in opinion in a context where a unique expert is evaluating a set of alternatives over various criteria or attributes. Similarly, it can also be applied to both situations, multi-attribute and GDM. This new measure is independent of the cardinality of the set S but is dependent on the perceptual map μ and the number of DMs.

To define this measure, I previously need to find the upper bound of the total addition of distances between the centroid of a group and each individual HFLDs. First, a Lemma is introduced.

Lemma 3.2. Let F_H^1, F_H^2 be two HFLDs over a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$, modeled by means of $S = \{s_1, s_2, \ldots, s_n\}$ and the same perceptual map μ. Then,

$$D_\mu^{\mathcal{F}}(F_H^1, F_H^2) \leq r \cdot (2 - \mu(\{s_1\}) - \mu(\{s_n\})) \tag{3.8}$$

Proof. According to Proposition 3.4, the most distant HFLTSs in $\mathcal{H}_S^*$ are $\{s_1\}$ and $\{s_n\}$ and $D_\mu(H_S^1, H_S^2) \leq 2 - \mu(\{s_1\}) - \mu(\{s_n\})$. It follows that the most distant HFLDs correspond to those where each corresponding pair of HFLTSs are the most distant ones. In this case, it follows:

$$D_\mu^{\mathcal{F}}(F_H^1, F_H^2) = \sum_{i=1}^{r} (2 - \mu(\{s_1\}) - \mu(\{s_n\})) = r \cdot (2 - \mu(\{s_1\}) - \mu(\{s_n\})) \tag{3.9}$$

□

Proposition 3.6. Let $\{F_H^j\}$ be the set of HFLDs provided by a group of k DMs over a set of alternatives $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ by means of $S = \{s_1, s_2, \ldots, s_n\}$ and the same perceptual map μ. Let F_H^C denote the centroid of the group. Then,

$$\sum_{j=1}^{k} D_\mu^{\mathcal{F}}(F_H^C, F_H^j) \leq k \cdot r \cdot \left(1 - \frac{(\mu(\{s_1\}) + \mu(\{s_n\}))}{2}\right) \tag{3.10}$$

Proof. When k is an even number, the scenario of maximum disagreement is reached when, for each λ_i, half of the DMs provide an assessment, without hesitancy, with the worst linguistic label and the other half give the best linguistic term. This means that, for each alternative, $k/2$ DMs provide an evaluation with $\{s_1\}$ and the other $k/2$ give an assessment of $\{s_n\}$. In this case, notice that regardless of the μ and the value of k, any $F_H^C \in (\mathcal{H}_S^*)^r$ give the same addition of distances to $\{F_H^j\}$. Without lost of generality, let $F_H^C(\lambda_i) = [s_1, s_n]$ $\forall\, i = 1, 2, \ldots, r$. Then,

$$\begin{aligned}\sum_{j=1}^{k} D_\mu^{\mathcal{F}}(F_H^C, F_H^j) = \frac{k}{2} \cdot r \cdot D_\mu([s_1, s_n], \{s_1\}) + \frac{k}{2} \cdot r \cdot D_\mu([s_1, s_n], \{s_1\}) =\\ \frac{k}{2} \cdot r \cdot (1 - \mu(\{s_1\})) + \frac{k}{2} \cdot r \cdot (1 - \mu(\{s_n\})) =\\ k \cdot r - \frac{k \cdot r}{2} \cdot \mu(\{s_1\}) - \frac{k \cdot r}{2} \cdot \mu(\{s_n\}) =\\ \frac{k \cdot r}{2} \cdot (2 - \mu(\{s_1\}) - \mu(\{s_n\}))\end{aligned}$$

When k is an odd number, the maximum disagreement is obtained when, for each alternative, $\frac{k+1}{2}$ DMs assess it with $\{s_1\}$ and the rest $\frac{k-1}{2}$ DMs provide an assessment of $\{s_n\}$ (or, conversely). By Proposition 3.5, for a given λ_i, the centroid of $F_H^j(\lambda_i)$, $j \in \{1, \ldots, k\}$ is $\{s_1\}$ (or $\{s_n\}$ if its the other case). Then,

$$\begin{aligned}\sum_{j=1}^{k} D_\mu^{\mathcal{F}}(F_H^C, F_H^j) = \frac{k+1}{2} \cdot r \cdot D_\mu(\{s_1\}, \{s_1\}) + \frac{k-1}{2} \cdot r \cdot D_\mu(\{s_1\}, \{s_n\}) =\\ \frac{k+1}{2} \cdot r \cdot 0 + \frac{k-1}{2} \cdot r \cdot (1 + 1 - \mu(\{s_1\}) - \mu(\{s_n\})) =\\ \frac{(k-1) \cdot r}{2} \cdot (2 - \mu(\{s_1\}) - \mu(\{s_n\}))\end{aligned}$$

□

The upper bound provided in Proposition 3.6 is used for normalization and leads to the definition of our proposed measure of agreement or consensus between HFLDs.

Definition 3.8. Let $\{F_H^j\}$ be the set of HFLDs provided by a group G of k DMs over a set $\Lambda = \{\lambda_1, \lambda_i, \ldots, \lambda_r\}$ by means of $S = \{s_1, s_2, \ldots, s_n\}$ and the same perceptual map μ. Let F_H^C denote the centroid of the group and $F_H^j(\lambda_i)$=H_{ij}, for $j = \{1, 2, \ldots, k, C\}$. The *degree of agreement* of G on λ_i is defined as:

$$\delta_{\lambda_i}(G) = 1 - \frac{\sum_{j=1}^{k} D_\mu(H_{ij}, H_{iC})}{\zeta} \tag{3.11}$$

with $\zeta = \frac{k}{2} \cdot (2 - \mu(\{s_1\}) - \mu(\{s_n\}))$ if k is even and $\zeta = \frac{k-1}{2} \cdot (2 - \mu(\{s_1\}) - \mu(\{s_n\}))$ if k is odd. Similarly, the *degree of agreement* of G on Λ is defined as:

$$\delta_\Lambda(G) = 1 - \frac{\sum_{j=1}^{k} D_\mu^{\mathcal{F}}(F_H^j, F_H^C)}{\zeta \cdot r} \tag{3.12}$$

By Proposition 3.6, the degree of agreement takes values between 0 and 1, i.e., $0 \leq \delta_\Lambda(G) \leq 1$. The closer the numerator of the fraction is to the maximum value of the addition of distances, the closer $\delta_\Lambda(G)$ goes to zero, meaning a high degree of disagreement. Notice that this maximum is adapted to both, a situation when k is odd and when k is even so that a degree of consensus with zero value can be reached in all GDM contexts, regardless of an even or odd number of DMs.

Proposition 3.7. Let G be a group of k DMs evaluating a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ by means of $S = \{s_1, s_2, \ldots, s_n\}$ and the same perceptual map μ. Let $\delta_{\lambda_i}(G)$ and $\delta_\Lambda(G)$ denote their *degree of agreement* on a specific λ_i and on Λ, respectively. Then:

$$\delta_\Lambda(G) = \frac{\sum_{i=1}^{r} \delta_{\lambda_i}(G)}{r} \tag{3.13}$$

Proof. Let $F_H^1, \ldots, F_H^j, \ldots, F_H^k$ denote the HFLDs of the group G and $F_H^j(\lambda_i)=H_{ij}$, $j = \{1, 2, \ldots, k\}$. Let F_H^C denote the centroid of the group.

$$\frac{\sum_{i=1}^{r} \delta_{\lambda_i}(G)}{r} = \frac{\sum_{i=1}^{r} 1 - \frac{\sum_{j=1}^{k} D_\mu(H_{ij}, H_{iC})}{\zeta}}{r} =$$

$$= \frac{r - \frac{\sum_{j=1}^{k} \sum_{i=1}^{r} D_\mu(H_{iC}, H_{ij})}{\zeta}}{r} = 1 - \frac{\sum_{j=1}^{k} D_\mu^{\mathcal{F}}(F_H^j, F_H^C)}{\zeta \cdot r} = \delta_\Lambda(G)$$

□

3.2.4 A transformation function for multi-perceptual GDM

In this subsection, I seek to establish the basis for modelling multiple perceptual maps with multi-granularity. In order to compare and operate with unbalanced HFLTSs based on different perceptual maps, I develop a perceptual-based transformation function to project linguistic assessments built over different perceptual maps onto a projected linguistic structure. Inspired by some of the ideas developed for linguistic hierarchies [44, 84], multi-granular contexts [34, 82, 86] and the extension of a discrete linguistic term set [190, 192], I firstly present and develop the needed definitions and notions. In the following paragraphs, G is assumed to be a set of DMs, $G=\{d_j \mid j \in 1, \ldots, k\}$. Each d_j express his or her opinions based on his or her own perceptual map, μ_j, over his or her appropriate (unbalanced) linguistic term set, $S_j = \{\{s_1^j, s_2^j, \ldots, s_{n_j}^j\} \mid j \in \{1, 2, \ldots, k\}\}$.

Note that, based on Proposition 3.2, from a given perceptual map μ_j defined on a LTS S_j with cardinality n_j, I can obtain its partition of the unit interval with n_j non-empty subsets. If P_{μ_j} denotes this partition, then:

$P_{\mu_j} = \{[0, \mu_j(\{s_1\})), [\mu_j(\{s_1\}), \mu_j(\{s_1\}) + \mu_j(\{s_2\})), \ldots, [\sum_{i=0}^{n_j-1} \mu_j(s_i), 1]\}$

First, I need to define a refinement of the set of partitions $\{P_{\mu_j}, j \in 1, \ldots, k\}$.

Definition 3.9. Let $\{P_{\mu_j} \mid j \in 1, \dots, k\}$ be the set of partitions associated to the set of perceptual maps $\{\mu_j \mid j \in 1, \dots, k\}$ and the set of LTS $\{S_{\mu_j} \mid j \in 1, \dots, k\}$. Each P_{μ_j} is a partition of the unit interval defined by $\{\lambda_0^j, \lambda_1^j, \lambda_2^j, \cdots, \lambda_{n_j}^j\}$, with $\lambda_0^j = 0$, $\lambda_{n_j}^j = 1$ and n_j denotes the cardinality of each S_j. The *projected partition* associated to $\{P_{\mu_j} \mid j \in 1, 2, \dots, k\}$ is P_p, defined by $\bigcup_{j=1}^{k} \bigcup_{l=0}^{n_j} \{\lambda_l^j\}$.

Definition 3.10. Let P_p be the projected partition of the set $\{P_{\mu_j} \mid j \in 1, 2, \dots, k\}$ defined by $\{\lambda_0, \lambda_1, \dots, \lambda_{n^*}\}$. The *projected LTS*, S^*, is the set that contains the *projected basic labels*, s_α^*, i.e., $S^* = \{s_\alpha^* \mid \alpha \in 1, 2 \dots, n^*\}$, where n^* is the cardinality of the set $\bigcup_{j=1}^{k} \bigcup_{l=0}^{n_j} \{\lambda_l^j\}$.

Note that the projected basic labels are only considered for computational purposes and the semantics that apply to each S_j do not apply for S^*.

Definition 3.11. Let P_p be the projected partition of the set $\{P_{\mu_j} \mid j \in 1, 2, \dots, k\}$ defined by $\{\lambda_0, \lambda_1, \dots, \lambda_{n^*}\}$ and let S^* be the projected LTS. Then, the *projected normalized measure* over S^*, μ'_* induced by this partition is defined as:

$$\mu'_*(s_\alpha^*) = \lambda_\alpha - \lambda_{\alpha-1}, \alpha \in 1, 2, \dots, n^* \tag{3.14}$$

where $s_\alpha^* \in S^*$.

Definition 3.12. Let S^* be a projected LTS, $S^* = \{s_1^*, s_2^*, ..., s_{n^*}^*\}$ and μ'_* its projected normalized measure. Then, the *projected perceptual map* μ_* is the perceptual map induced by μ'_* in $\mathcal{H}_{S^*}^*$ (as defined in equation 3.1).

Note that the previous Definitions 3.9, 3.10, 3.11 and 3.12 not only deal with unbalanced LTS but are also adapted to contexts of multi-granularity when the LTS used by each DM, S_j, are of different cardinality.

Example 3.6. Let S_1 be an unbalanced LTS with granularity 9 and let μ_1 be a perceptual map defined as $\mu_1(s_1) = \mu_1(s_2) = \mu_1(s_3) = \mu_1(s_4) = \mu_1(s_7) = \frac{3}{26}$, $\mu_1(s_5) = \mu_1(s_6) = \frac{4}{26}$, $\mu_1(s_8) = \frac{2}{26}$ and $\mu_1(s_9) = \frac{1}{26}$. Let μ_2 be a different perceptual map defined over S_2, with granularity 6 and with $\mu_2(s_1) = \mu_2(s_6) = \frac{3}{26}$, $\mu_2(s_2) = \frac{4}{26}$, $\mu_2(s_3) = \mu_2(s_4)\frac{5}{26}$, $\mu_2(s_5) = \frac{6}{26}$. Let S_3 be another unbalanced LTS with granularity 3 with μ_3 such that $\mu_3(s_1) = \mu_3(s_3) = \frac{8}{26}$ and $\mu_3(s_2) = \frac{10}{26}$. Following Definition 3.9, their projected partition P_{μ_*} is formed by the ordered set of points $\{0, \frac{3}{26}, \frac{3}{13}, \frac{7}{26}, \frac{8}{26}, \frac{9}{26}, \frac{6}{13}, \frac{8}{13}, \frac{17}{26}, \frac{18}{26}, \frac{10}{13}, \frac{23}{26}, \frac{25}{26}, 1\}$ and is illustrated in figure 3.4. Based on Definition 3.10, P_{μ_*} is associated to a projected LTS defined as $S^* = \{s_1^*, s_2^*, \dots, s_{13}^*\}$.

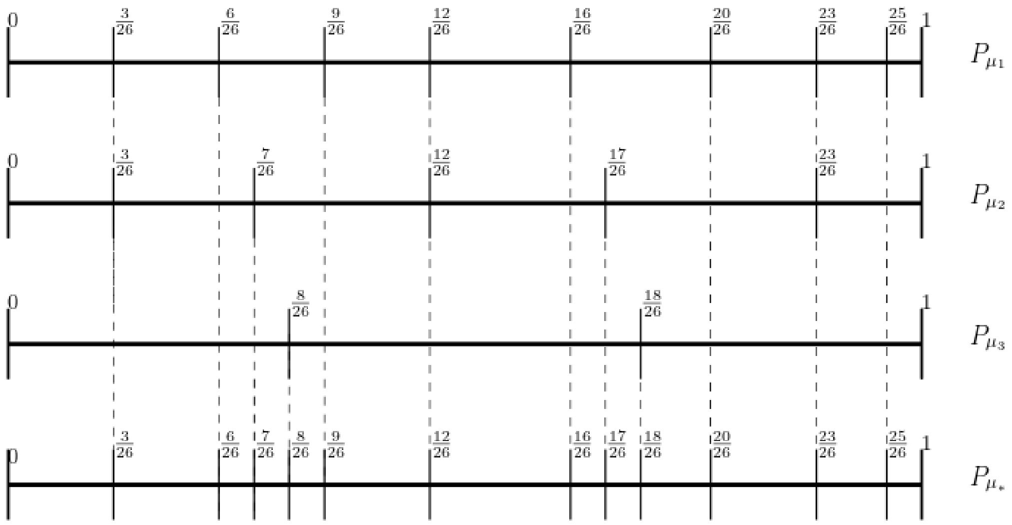

FIGURE 3.4: The *projected partition*,P_{μ_*}, obtained from the three partitions of example 3.6

Following Definition 3.11, the μ'_* is obtained:

$$\mu'_*(s^*_\alpha) = \begin{cases} \dfrac{1}{26}, & \alpha \in \{3,4,5,8,9,13\}; \\ \dfrac{1}{13}, & \alpha \in \{10,12\}; \\ \dfrac{3}{26}, & \alpha \in \{1,2,6,11\}; \\ \dfrac{4}{26}, & \alpha = 7. \end{cases}$$

From now on let $\mathcal{H}^*_{\mu_*}$ denote the set of positive projected HFLTSs, based on the obtained S^* and μ_* the projected perceptual map defined on it. I can develop a *perceptual-based transformation function* that allow us to transform any $H_{\mu_j} \in \mathcal{H}^*_{\mu_j}$ into the projected space, i.e., $H_{\mu_*} \in \mathcal{H}^*_{\mu_*}$.

Definition 3.13. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a LTS, μ its corresponding perceptual map and $H = [s_L, s_R]$ any HFLTS in $\mathcal{H}^*_\mu$. Then, the functions $T^L : \mathcal{H}^*_\mu \to [0,1]$ and $T^R : \mathcal{H}^*_\mu \to [0,1]$ are defined as:

- $T^L(H) = \sum_{i=1}^{L-1} \mu(s_i)$
- $T^R(H) = \sum_{i=R+1}^{n} \mu(s_i)$

Definition 3.14. Let $\{P_{\mu_j} \mid j \in 1,2,\ldots,k\}$ be a set of different partitions of the unit interval and let $\{\mu_j \mid j \in 1,2,\ldots,k\}$ and $\{S_j = \{s^j_1, s^j_2, \ldots, s^j_{n_j}\} \mid j \in \{1,2,\ldots,k\}\}$ be its corresponding set of perceptual maps and LTS, respectively. Let $S^* = \{s^*_1, s^*_2, \ldots, s^*_{n_*}\}$ denote the projected LTS and μ_*, the projected perceptual map. The *perceptual-based transformation function* is a map from $\mathcal{H}^*_{\mu_j} \to \mathcal{H}^*_{\mu_*}$, such that to each $H_{\mu_j} = [s^j_L, s^j_R] \in \mathcal{H}^*_{\mu_j}$, it assigns the image, $H_{\mu_*} = [s^*_L, s^*_R] \in \mathcal{H}^*_{\mu_*}$, holding $T^L(H_{\mu_j}) = T^L(H_{\mu_*})$ and $T^R(H_{\mu_j}) = T^R(H_{\mu_*})$, i.e. $\sum_{i=1}^{L-1} \mu_j(s^j_i) = \sum_{\alpha=1}^{L-1} \mu_*(s^*_\alpha)$ and $\sum_{i=R+1}^{n_j} \mu_j(s^j_i) = \sum_{\alpha=R+1}^{n*} \mu_*(s^*_\alpha)$.

Example 3.7. Let $G = \{d_1, d_2, d_3\}$ be a group of 3 different evaluators, each one coming from a different background and expertise. The three evaluators are asked to give their opinion with respect to a set of new candidates, $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. The linguistic judgements they provide are modelled via different algebraic structures. Suppose that, based on example 3.6, linguistic assessments of d_1 are modelled via HFLTSs over $(\overline{\mathcal{H}_{S_{\mu_1}}}, \sqcup, \sqcap)$ and opinions of d_2 and d_3 are represented by the structure $(\overline{\mathcal{H}_{S_{\mu_2}}}, \sqcup, \sqcap)$ and $(\overline{\mathcal{H}_{S_{\mu_3}}}, \sqcup, \sqcap)$, respectively. Let $F^1_{\mu_1}$, $F^2_{\mu_2}$ and $F^3_{\mu_3}$ be the HFLDs modeling their corresponding assessments, which are shown in Table 3.3. Using Definition 3.14 about the perceptual-based transformation function, I are able to map the individual linguistic assessments onto the projected LTS, found in example 3.6, to obtain $F^1_{\mu_*}$, $F^2_{\mu_*}$ and $F^3_{\mu_*}$. Results are illustrated in Table 3.3.

	$F^1_{\mu_1}$	$F^2_{\mu_2}$	$F^3_{\mu_3}$	$F^1_{\mu_*}$	$F^2_{\mu_*}$	$F^3_{\mu_*}$
λ_1	$[s_3, s_5]$	$\{s_3\}$	$\{s_2\}$	$[s^*_3, s^*_7]$	$[s^*_4, s^*_6]$	$[s^*_5, s^*_9]$
λ_2	$[s_8, s_9]$	$[s_4, s_5]$	$\{s_3\}$	$[s^*_{12}, s^*_{13}]$	$[s^*_7, s^*_{11}]$	$[s^*_{10}, s^*_{13}]$
λ_3	$[s_1, s_2]$	$\{s_2\}$	$\{s_1\}$	$[s^*_1, s^*_2]$	$[s^*_2, s^*_3]$	$[s^*_1, s^*_4]$
λ_4	$[s_7, s_9]$	$[s_5, s_6]$	$\{s_3\}$	$[s^*_{11}, s^*_{13}]$	$[s^*_9, s^*_{13}]$	$[s^*_{10}, s^*_{13}]$

TABLE 3.3: Projected HFLDs, $F^1_{\mu_1}$, $F^2_{\mu_2}$ and $F^3_{\mu_3}$, of the group G of evaluators of example 3.7.

If the projected perceptual map and projected LTS are identified and all HFLDs are mapped onto the same structure by using the perceptual-based transformation function, then all propositions and definitions with respect to distance and collective agreement measures presented in subsection 3.2.3 apply and can be used to operate. Results involving linguistic expressions in the projected LTS can latter be transformed back to the individual algebraic structure of each individual, by means of the perceptual-based transformation function.

3.3 A multi-granular and multi-perceptual method for hesitant fuzzy linguistic MAGDM

It is already known that, when assessing alternatives with respect to specific criteria, using linguistic variables is more in line with human intuition than using specific discrete numerical values. Linguistic variables are more alienated with uncertain information and more user-friendly in real world applications. When using linguistic variables, the decision-making process become more realistic [112]. As previously illustrated and analyzed in Chapter 2, there exist several techniques to model the hesitancy or lack of knowledge involved in these linguistic judgements. The use of HFLTSs is an example.

Nonetheless, in the context of group decision making, it is sometimes as relevant to take into account the different reasoning processes, rationality and logic behind each DM' linguistic opinion. This is particularly important to be considered when performing aggregation of fuzzy linguistic information. Each DM, d_j, involved in a GDM situation may express his or her preferences according to his or her own perceptual map, μ_j, and his or her preferred linguistic term set, S_j. The reasoning processes can be embodied as the form of perceptual maps in ULTSs and it's obvious to observe that the information contained in linguistic variables provided by each DM can be different based on these individual characteristics and hence, it can directly influence the evaluation results.

Based on the previous developed tools, in this section, I seek to build a framework for GDM problems involving multiple types of perceptual maps which, at the same time, can simultaneously deal with multi-granularity. The developed framework can suitably be used to model a MCDM problem involving just one DM but with a highly heterogeneous family of criteria G, which are evaluated by means of different perceptual maps. This might happen when the evaluation of alternatives is characterized by attributes with very different nature. Hence, even when considering a decision-making situation of only one DM, it is also relevant and useful to develop a transformation system between linguistic information built over different perceptual maps.

This section responds to objectives O7 and O8 presented in section Objectives 1.2. In addition, it defines the methodologies needed to respond to objective O7. As a consequence, contributions C3 and C4 can be achieved in chapters 4 and 5. Moreover, the developed methods in the following subsections have been tested in other real case situations as explained in contributions C5 and C6.

First, a novel MAGDM ranking and classification method is developed in the following paragraphs of subsection 3.3.1. This method is different from any other existing MCGDM/MAGDM and it is build over the three main tools developed in the previous subsections: the perceptual-based distance (subsection 3.2.2), the perceptual-based consensus measure (subsection 3.2.3) and the transformation function for muulti-perceptual contexts (subsection 3.2.4) Secondly, the TOPSIS method is adapted to incorporate the use of perceptual-maps. In subsection 3.3.2, the steps to adapt the classical TOPSIS method are illustrated so the qualitative reasoning processes of each DM can be taken into account, via the transformation function for multi-perceptual contexts (in 3.2.4).

3.3.1 A classification or ranking perceptual-based method

According to the aforementioned distances and consensus analysis based on the perceptual map structure, I define a new MAGDM method using multi-granular and unbalanced HFLTSs. The main steps of the method are outlined below:

Step 1: Settings. Establish a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_i, \ldots, \lambda_r\}$ and a group $G = \{d_1, \ldots, d_j, \ldots, d_k\}$ of k DMs or experts to evaluate the set Λ over a coherent and relevant set A of m attributes, i.e., $A = \{a_1, \ldots, a_l, \ldots, a_m\}$. For each d_j, define his or her appropriate LTS and perceptual map upon which his or her linguistic assessments will be modeled. This implies the identification of $S_j = \{s^j_1, s^j_2, \ldots, s^j_{n_j}\}$, with $n_j \in \mathbb{N}\ \forall\, j \in \{1, 2, \ldots, k\}$ and the perceptual maps $\mu_j : S_j \to [0, 1]$, such that $\sum_{i=1}^{n_j} \mu(s^j_i) = 1\ \forall\, j \in \{1, 2, \ldots, k\}$.

Step 2: Eliciting assessments. Get $H^{jl}_i = F^j(\lambda^l_i), \forall\, i \in \{1, \ldots, r\}, j \in \{1, \ldots, k\}$ and $l \in \{1, \ldots, m\}$, which is the HFLTS modelling the opinion of DM or expert d_j with respect to alternative λ_i over criterion a_l.

Step 3: Mapping onto the projected space. Employ Definitions 3.10 and 3.12 to find the projected LTS, S^*, and the projected perceptual map, μ_*, associated to the perceptual characteristics of group G defined in step 1 of settings. Then, using the perceptual-based transformation function of Definition 3.14 find the projected HFLTSs, $H^{jl^*}_i$, for each H^{jl}_i, $\forall\, i \in \{1, \ldots, r\}, j \in \{1, \ldots, k\}$ and $l \in \{1, \ldots, m\}$.

Step 4: Computing projected centroid and consensus. For each a_l and λ_i, obtain the centroid, $H_i^{Cl^*}$ and the collective degree of consensus of G, $\delta_{(\lambda_i^l)}$, from the k projected HFLTs, using equations 3.7 and 3.11, respectively. As a result, each λ_i can be represented by a HFLD, F_i^*, of dimensionality m whose components are $F_i^* = (H_i^{Cl^*}, H_i^{Cl^*}, \ldots, H_i^{Cl^*}, \ldots, H_i^{Cl^*})$. Besides, each $H_i^{Cl^*}$ is paired with a number, $\delta_{(\lambda_i^l)}$, corresponding to the degree of agreement which will be used as the weighting factor for computing the weighted distances of the following steps.

Now, depending on the MAGDM objective, the last steps can be redifined accordingly. Next, I give the details fo the proposed method for two specific purposes: classification and ranking. The steps are as follows:

Step 5a: Classification. Considering a classification problem, identify the category set, $X = \{x_1, \ldots, x_q, \ldots, x_t\}$ and establish the F_q^* with $q \in \{1, 2, \ldots, t\}$, which are the prototype m dimensional HFLDs describing each category. Using the degree of consensus, $\delta_{(\lambda_i^l)}$, as the weighting factor, employ equation 3.5 to calculate, $\hat{D_{\mu_*}}(F_i^*, F_q^*)$ $\forall$ $q \in \{1, 2, \ldots, t\}$ and $\forall$ $i \in \{1, 2, \ldots, r\}$. A total of $t \times r$ distances are obtained. Classify the alternatives according to these distances. Alternative λ_i is classified to category x_q if the distance of F_i^* to F_q^* is the smallest $\forall$ $q \in \{1, 2, \ldots, t\}$.

Step 5b: Ranking. Considering a ranking problem, establish the best dimensional HFLD, F_m^*. Using the degree of consensus, $\delta_{(\lambda_i^l)}$, as the weighting factor, employ equation 3.5 to calculate, $\hat{D_{\mu_*}}(F_i^*, F_m^*)$, $\forall$ $i \in \{1, 2, \ldots, r\}$. A total of r distances are obtained. Rank the alternatives according to these distances, in ascending order.

3.3.1.1 An illustrative example on the Documentary Film Industry I

Let $G = \{d_1, d_2, d_3\}$ be a group of 3 experts. Each expert, d_j, comes from a different country. These three experts are members of the final jury participating in the "Awards for the Best Documentary Films". This Festival is celebrated every two years in Europe. In particular, the three members of the jury are responsible for awarding the *New Talent Award*, which is awarded based on three attributes: Film Editing, Topic Originality and Acting. There are four documentary films that have reached the final phase of the selection process. The members of the jury are asked to give their linguistic opinions with respect to each documentary film and each attribute, based on their expertise, knowledge and professional background. The nature of this decision implies the possibility of hesitancy, ambiguity and impreciseness in the evaluation process. Since these experts come from different cultural regions as well as their qualitative reasoning and attitudinal process are different, this particular MAGDM situation is a good candidate to be modeled by means of multi-granular and unbalanced HFLTSs, following steps proposed in 3.3.1.

Step 1:Establish a set of alternatives. The set of alternatives is a group of 4 documentary films, which is, $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. The group of experts $G = \{d_1, d_2, , d_3\}$ is choosen to evaluate the set Λ over a coherent and relevant set A of 3 attributes, i.e., $A = \{a_1, a_2, a_3\}$, where a_1 refers to Film Editing, a_2 is Topic Originality and a_3 means Acting. For each, $d_j, j \in \{1, 2, 3\}$, his appropiate LTS upon which his linguistic opinions are modeled, have been identified as:

$S_1 = \{s_1^1 : \text{bad}, s_2^1 : \text{good}, s_3^1 : \text{excelent}\}$

$S_2 = \{s^2_1 : \text{bad}, s^2_2 : \text{medium}, s^2_3 : \text{good}, s^2_4 : \text{very good}, s^2_5 : \text{excellent}\}$
$S_3 = \{s^3_1 : \text{bad}, s^3_2 : \text{medium}, s^3_3 : \text{good}, s^3_4 : \text{excelent}\}$

Hence, $n_1 = 3$, $n_2 = 5$ and $n_3 = 4$. Similarly, their perceptual maps have also been identified as follows:

$\mu_1 : S_1 \to [0,1]$,
where $S_1 = \{s^1_1, s^1_2, s^1_3\}$, such that:

$$\mu_1(s^1_i) = \begin{cases} 0.4, & i \in \{1,2\}; \\ 0.2, & i = 3. \end{cases}$$

$\mu_2 : S_2 \to [0,1]$,
where $S_2 = \{s^2_1, s^2_2, s^2_3, s^2_4, s^2_5\}$, such that:

$$\mu_2(s^2_i) = \{0.2, \quad i \in \{1,2,3,4,5\}.$$

$\mu_3 : S_3 \to [0,1]$,
where $S_3 = \{s^3_1, s^3_2, s^3_3, s^3_4\}$, such that:

$$\mu_3(s^3_i) = \begin{cases} 0.2, & i \in \{1,4\}; \\ 0.3, & i \in \{2,3\}. \end{cases}$$

The projected partitions associated to these perceptual maps are illustrated in figure 3.5.

Step 2: Eliciting assessments. Get $H^{jl}_i = F^j(\lambda^l_i), \forall\, i \in \{1,\ldots,4\}, j \in \{1,2,3\}$ and $l \in \{1,2,3\}$. Recall that each H^{jl}_i denote the linguistic opinion, modeled by HFLTSs provided by expert j with respect to the attribute l of the documentary i. ? Denotes total hesitancy.

The evaluations are provided in Tables 3.4, 3.5 and 3.6.

Film editing	$F^1(\lambda^1_i)$	$F^2(\lambda^1_i)$	$F^3(\lambda^1_i)$
λ_1	$\{s^1_1\}$	$[s^2_1, s^2_2]$	$\{s^3_1\}$
λ_2	$\{s^1_3\}$	$\{s^2_4\}$	$[s^3_2, s^3_4]$
λ_3	$[s^1_1, s^1_2]$	$\{s^2_5\}$	$\{s^3_3\}$
λ_4	?	$\{s^2_2\}$	$[s^3_2, s^3_3]$

TABLE 3.4: HFLTSs modelling the linguistic evaluations provided by the group of experts of example 3.3.1 when evaluating the four documentary films with respect to Film Editing.

Topic Originality	$F^1(\lambda_i^2)$	$F^2(\lambda_i^2)$	$F^3(\lambda_i^2)$
λ_1	$\{s_2^1\}$	$[s_3^2, s_4^2]$	$\{s_4^3\}$
λ_2	$\{s_3^1\}$	?	$\{s_4^3\}$
λ_3	$[s_1^1, s_2^1]$	$\{s_1^2\}$	$\{s_1^3\}$
λ_4	$\{s_2^1\}$	?	$[s_1^3, s_3^3]$

TABLE 3.5: HFLTSs modelling the linguistic evaluations provided by the group of experts of example 3.3.1 when evaluating the four documentary films with respect to Topic Originality.

Acting	$F^1(\lambda_i^3)$	$F^2(\lambda_i^3)$	$F^3(\lambda_i^3)$
λ_1	$\{s_3^1\}$	$\{s_4^2\}$	$[s_3^3, s_4^3]$
λ_2	$\{s_2^1\}$	$[s_2^2, s_4^2]$	$[s_2^3, s_3^3]$
λ_3	$\{s_2^1\}$	$\{s_5^2\}$	$\{s_4^3\}$
λ_4	$\{s_3^1\}$	$[s_3^2, s_5^2]$	$\{s_4^3\}$

TABLE 3.6: HFLTSs modelling the linguistic evaluations provided by the group of experts of example 3.3.1 when evaluating the four documentary films with respect to Acting.

Step 3: Mapping onto the projected space. The projected LTS, S^* and the projected perceptual map, μ_* are computed. The resulting S^* has granularity 6 as shown in figure 3.5 and the resulting μ_* is:

$$\mu_*(s_\alpha^*) = \begin{cases} 0.2, & \alpha \in \{1,2,5,6\}; \\ 0.1, & \alpha \in \{3,4\}. \end{cases}$$

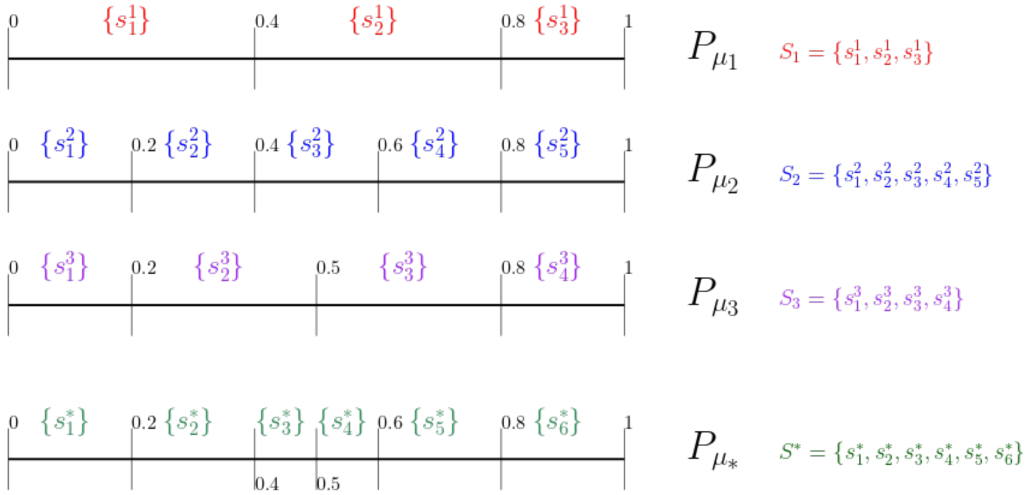

FIGURE 3.5: LTS and partitions modeling the linguistic opinions of the group of expert of example 3.3.1 and its projected LTS and projected partition

Following the instructions of step 3, the projected HFLTSs, $H_i^{jl^*}$, are computed for each H_i^{jl}, $\forall\, i \in \{1,2,3,4\}$, $j \in \{1,2,3\}$ and $l \in \{1,2,3\}$. The projected evaluations are shown in Tables 3.7, 3.8 and 3.9.

Film editing	$F^1 * (\lambda_i^1)$	$F^2 * (\lambda_i^1)$	$F^3 * (\lambda_i^1)$
λ_1	$[s_1^*, s_2^*]$	$[s_1^*, s_2^*]$	$\{s_1^*\}$
λ_2	$\{s_6^*\}$	$\{s_5^*\}$	$[s_2^*, s_6^*]$
λ_3	$[s_1^*, s_5^*]$	$\{s_6^*\}$	$[s_4^*, s_5^*]$
λ_4	$[s_1^*, s_6^*]$	$\{s_2^*\}$	$[s_2^*, s_5^*]$

TABLE 3.7: Projected HFLTSs corresponding to evaluations of Table 3.4

Topic Originality	$F^1(\lambda_i^2)$	$F^2(\lambda_i^2)$	$F^3(\lambda_i^2)$
λ_1	$[s_3^*, s_5^*]$	$[s_3^*, s_5^*]$	$\{s_6^*\}$
λ_2	$\{s_6^*\}$	$[s_1^*, s_6^*]$	$\{s_6^*\}$
λ_3	$[s_1^*, s_5^*]$	$\{s_1^*\}$	$\{s_1^*\}$
λ_4	$[s_3^*, s_5^*]$	$[s_1^*, s_6^*]$	$[s_1^*, s_5^*]$

TABLE 3.8: Projected HFLTSs corresponding to evaluations of Table 3.5

Acting	$F^1(\lambda_i^3)$	$F^2(\lambda_i^3)$	$F^3(\lambda_i^3)$
λ_1	$\{s_6^*\}$	$\{s_5^*\}$	$[s_4^*, s_6^*]$
λ_2	$[s_3^*, s_5^*]$	$[s_2^*, s_5^*]$	$[s_2^*, s_5^*]$
λ_3	$[s_3^*, s_5^*]$	$\{s_6^*\}$	$\{s_6^*\}$
λ_4	$\{s_6^*\}$	$[s_3^*, s_6^*]$	$\{s_6^*\}$

TABLE 3.9: Projected HFLTSs corresponding to evaluations of Table 3.6

Step 4: Computing projected centroid and consensus. For each attribute and documentary film (i.e., each row of Tables 3.4, 3.5 and 3.5, the centroid, $H_i^{Cl^*}$, and the collective degree of consensus, $\delta_{(\lambda_i^l)}$, using the projected HFLTs are computed. Note that here an odd number of decision makers should be considered.

Each documentary film, λ_i, is now represented by a HFLD, F_i^*, of dimensionality 3 whose components are $F_i^* = (H_i^{C1^*}, H_i^{C2^*}, H_i^{C3^*})$. Besides, each $H_i^{Cl^*}$, $l \in \{1,2,3\}$ is paired with a number, $\delta_{(\lambda_i^l)}$, corresponding to the degree of agreement.

Documentary film	Film Editing	Topic Originality	Acting
λ_1	$[s_1^*, s_2^*]$, 0.875	$[s_3^*, s_4^*]$, 0.625	$[s_5^*, s_6^*]$, 0.6875
λ_2	$[s_5^*, s_6^*]$, 0.5	$\{s_6^*\}$, 0.5	$[s_2^*, s_5^*]$, 0.875
λ_3	$[s_4^*, s_5^*]$, 0.375	$\{s_1^*\}$, 0.625	$\{s_6^*\}$, 0.625
λ_4	$[s_2^*, s_5^*]$, 0.5	$[s_1^*, s_4^*]$, 0.625	$\{s_6^*\}$, 0.75

TABLE 3.10: Centroids, $H_i^{Cl^*}$, $\forall i \in \{1,2,3,4\}$ and $l \in \{1,2,3\}$ and its corresponding degree of agreement, based on evaluations of example 3.3.1

Step 5b: Ranking. The objective of the *New Talent Award* Committee is to obtain a ranking of the four documentaries based on the opinions of the experts group. Following the instructions of step 5, we set F_m^* to be $(\{s_6^*\}, \{s_6^*\}, \{s_6^*\})$ and we use the degrees of consensus, shown in Table 3.10, as weighting factors to compute the distance of each centroide to F_m^*, which is shown in last column of Table 3.11. Finally, the documentary films are ranked in ascending order in Table 3.12.

Doc. film	$d(H_i^{C1^*}, \{s_6^*\})$	w_i^1	$d(H_i^{C2^*}, \{s_6^*\})$	w_i^2	$d(H_i^{C3^*}, \{s_6^*\})$	w_i^3	$d(F_i^*, F_m^*)$
λ_1	1.4	0.4	0.6	0.286	0.2	0.314	0. 7943
λ_2	0.2	0.267	0	0.267	0.8	0.467	0.4267
λ_3	0.5	0.231	1.6	0.385	0	0.385	0.7308
λ_4	0.8	0.267	1	0.333	0	0.4	0.5467

TABLE 3.11: Distances of each $H_i^{C1^*}$, $\forall i \in \{1,2,3,4\} \forall l \in \{1,2,3\}$ to $\{s_6^*\}$ and the resulting four final distances $d(F_i^*, F_m^*)$ of each documentary film.

Documentary film	Ranking
λ_1	4
λ_2	1
λ_3	3
λ_4	2

TABLE 3.12: Final ranking of documentary films of example 3.3.1, based on experts opinions and following methodology presented in 3.3

3.3.2 An extended fuzzy multi-perceptual linguistic TOPSIS

The classical TOPSIS method has been widely adopted in many practical applications [18]. Our purpose here is to adapt this method so it can incorporate the information contained in the perceptual-map of each DM. Here, the main steps that should be followed if a TOPSIS is used in combination with multi-granular and unbalanced HFLTSs are presented:

Step 1: Settings. Establish a set of alternatives $\Lambda = \{\lambda_1, \ldots, \lambda_i, \ldots, \lambda_r\}$ and a group $G = \{d_1, \ldots, d_j, \ldots, d_k\}$ of k choosen experts to evaluate the set Λ over a coherent and relevant set A of m attributes, i.e., $A = \{a_1, \ldots, a_l, \ldots, a_m\}$. For each d_j, define his or her appropriate LTS and perceptual map upon which his or her linguistic assessments will be modeled. This implies the identification of $S_j = \{s_1^j, s_2^j, \ldots, s_{n_j}^j\}$, with $n_j \in \mathbb{N}\ \forall\, j \in \{1,2,\ldots,k\}$ and the perceptual maps $\mu_j : S_j \to [0,1]$, such that $\sum_{i=1}^{n_j} \mu(s_i^j) = 1, \forall\, j \in \{1,2,\ldots,k\}$.

Step 2: Eliciting assessments. Get $H_i^{jl} = F^j(\lambda_i^l), \forall\, i \in \{1,\ldots,r\}, j \in \{1,\ldots,k\}$ and $l \in \{1,\ldots,m\}$, which is the HFLTS modelling the opinion of DM or expert d_j with respect to alternative λ_i over criterion a_l.

Step 3: Mapping onto the projected space. Employ Definitions 3.10 and 3.12 to find the projected LTS, S^*, and the projected perceptual map, μ_*, associated to the perceptual characteristics of group G defined in step 1 of settings. Then, using the perceptual-based transformation function of Definition 3.14, find the projected HFLTSs, $H_i^{jl^*}$, for each $H_i^{jl}, \forall\, i \in \{1,\ldots,r\}, j \in \{1,\ldots,k\}$ and $l \in \{1,\ldots,m\}$.

Step 4: Compute the PHFLTS formed by the union of elements in the projected space. For each alternative i, and each attribute l, compute the PHFLTS as introduced in Definition 2.12. This means to find $P_{H_i^{l*}} = \{(s_i^*, p_i) \mid s_i^* \in S^*, i \in \{1,2,\ldots,n^*\}\}$, with $\sum_{i=1}^{n^*} p_i = 1$ and $0 \le p_i \le 1, \forall\, i \in \{1,2,\ldots,n^*\}$. Then, take each $P_{H_i^{l*}}, \forall l \in \{1,2,\ldots,m\}$ and form a description of PHFLTS, denoted as, F_{P_i}. This will result in vectors of dimension $n^* \times l$ which have to be normalized.

Step 5: Obtain the PHFLTS positive description (PHFLTS-PIS) and the PHFLTS negative description (PHFLTS-NIS) ideal solutions and calculate the cosine distances for each alternative to both. Let P^+ and P^- denote the PHFLTS-PIS and PHFLTS-NIS, respectively, both have to be normalized and with dimension $n^* \times l$. Then, for each F_{P_i} two distances are computed.

Given, P^+, for each F_{P_i}, compute $D_{F_i}^+ = distance(F_{P_i}, P^+) = 1 - \frac{F_{P_i} \cdot P^+}{\|F_{P_i}\| \|P^+\|}$.

Also, given P^-, for each F_{P_i}, compute $D_{F_i}^- = distance(F_{P_i}, P^-) = 1 - \frac{F_{P_i} \cdot P^-}{\|F_{P_i}\| \|P^-\|}$

Step 6: Calculate the relative closeness coefficient to the ideal solution. The original TOPSIS method is based on computing the shortest distance to the positive-ideal solution and the farthest distance from the negative-ideal solution. This is captured by the relative closeness coefficient. This is obtained with, for each F_{P_i}, $CC_i = \frac{D_{F_i}^-}{D_{F_i}^- + D_{F_i}^+}$

Step 7: Ranking. Rank alternatives i according to the closeness coefficient, CC_i.

3.3.2.1 An illustrative example on the Documentary Film Industry II

Let's consider the exact same MAGDM situation explained in 3.3.1. In this case, we will solve it using a classical TOPSIS method with a prior adaptation to handle multi-granular and unbalanced HFLTSs, based on the steps defined in subsection

3.3.2.

Step 1: Settings, Step 2: Eliciting assessments and Step 3: Mapping onto the projected space. As we follow the same context and settings explained in the illustrative example 3.3.1, the first three steps of the proposed method are the same. Therefore, the linguistic evaluations of the four documentary films, $\lambda_i, i \in \{1,2,3,4\}$, can be found in Tables 3.4, 3.5 and 3.6. Based on the identified perceptual maps of each expert as shown in figure 3.5, the mapping onto the projected space is performed and results are shown in tables 3.7, 3.8 and 3.9 for each attribute.

Step 4: Compute the PHFLTS formed by the union of elements in the projected space. For each documentary i, and each attribute l, we have compute the PHFLTS which shows the aggregation of the projected HFLTSs expressed by the experts. We have form the description of PHFLTS with 18 dimensions. The resulting (normalized) vectors, F_{P_i}, for each documentary are listed below:

F_{P_1}=(0.222, 0.111, 0, 0, 0, 0, 0, 0, 0.074, 0.074, 0.074, 0.111, 0, 0, 0, 0.037, 0.148, 0.148)
F_{P_2}=(0, 0.022, 0.022, 0.022, 0.133, 0.133, 0.019, 0.019, 0.019, 0.019, 0.019, 0.241, 0, 0.056, 0.093, 0.093, 0.093, 0)
F_{P_3}=(0.022, 0.022, 0.022, 0.078, 0.078, 0.111, 0.244, 0.022, 0.022, 0.022, 0.022, 0, 0, 0, 0.037, 0.037, 0.037, 0.222)
F_{P_4}=(0.019, 0.157, 0.046, 0.046, 0.046, 0.019, 0.041, 0.041, 0.078, 0.078, 0.078, 0.019, 0, 0, 0.028, 0.028, 0.028, 0, 25)

Step 5: Obtain the PHFLTS positive description (PHFLTS-PIS) and the PHFLTS negative description (PHFLTS-NIS) ideal solutions and calculate the cosine distances for each alternative to both.
Let
P^-=(0.333, 0, 0, 0, 0, 0, 0.333, 0, 0, 0, 0, 0, 0.333, 0, 0, 0, 0, 0) and
P^+ =(0, 0, 0, 0, 0, 0.333, 0, 0, 0, 0, 0, 0.333, 0, 0, 0, 0, 0, 0.333)
denote the normalized PHFLTS-NIS and PHFLTS-PIS, respectively. Then, for each F_{P_i} two distances are computed. Results are shown in Table 3.13

F_{P_i}	$D^-_{F_i}$	$D^+_{F_i}$	CC_i
F_{P_1}	0.6528	0.5938	0.4767
F_{P_2}	0.9698	0.3904	0.2870
F_{P_3}	0.5902	0.4878	0.4525
F_{P_4}	0.9006	0.5186	0.3654

TABLE 3.13: Distances of F_{P_i} to PHFLTS-NIS and PHFLTS-PIS and closeness coefficient

Step 6: Calculate the relative closeness coefficient to the ideal solution.. Each CC_i is shown in the last colum of Table 3.13

Step 7: Ranking. Rank alternatives i according to the closeness coefficient, CC_i. The results are presented in Table 3.14

Documentary film	Ranking
λ_1	4
λ_2	1
λ_3	3
λ_4	2

TABLE 3.14: Final ranking of documentary films of example 3.3.2, based on experts opinions and following methodology presented in 3.3.2

Chapter 4

Understanding city location decisions of energy MNEs

This chapter corresponds mainly to the contribution forth explained in section 1.3. The main goal of this application is to show the feasibility and practicability of the perceptual-based transformation framework, developed in chapter 3, in a real world multi attribute group decision making situation involving several experts who elicitate their opinions with linguistic assessments. The application is framed in the scheme of location decisions made by multinationals enterprises (MNEs) of the energy sector within the European Smart city context. As already mentioned, the development of this application has been supported by the INVITE Research Project (TIN2016-80049-C2-1-R and TIN2016-80049-C2-2-R (AEI/FEDER, UE)), funded by the Spanish Ministry of Science and Information Technology and the European Union 'Horizon 2020 Research and Innovation Programme', under the grant agreement No 731297. The initial results derived from this chapter have already been published in the Journal Energies. [1]

Becoming a smart city is one of the top priorities in the urban agenda of many European cities. Among the various strategies in this transition path, local governments seek to bring innovation to their cities by encouraging MNEs to deploy their green energy services and products in their municipalities. Knowing how to attract these enterprises implies that political leaders understand the multi-criteria decision problem that the energy enterprises face when deciding whether to expand to one city or another. To this end, the purpose of this first application is to design a novel manageable and controllable framework oriented to European cities' public managers, based on the assessment of criteria and sub-criteria governing the strategic location decision made by these enterprises. In this chapter, the challenge of helping political leaders to obtain insights from energy MNEs location decisions is developed step by step using an integration of an AHP technique with an extended fuzzy multi-perceptual linguistic TOPSIS method based on the steps presented in section 3.3.2.

The framework is build based on extracting the relative importance of sub-criteria by asking the opinion to ten experts of the field. Three different initial assumptions with respect to the type of perceptual-map owned by each of the ten experts involved in the group decision-aiding situation are considered. For each scenario, the evaluation results are obtained. As will be shown in the following sections, even if the resulting set of the five most important sub-criteria are the same for each hypothesis situation, significant differences are found among the whole list of sub-criteria. The sub-criterion 'City's potential customers' is considered to be the most important

[1] Porro, O., Pardo-Bosch, F., Agell, N., & Sanchez, M. (2020). Understanding Location Decisions of Energy Multinational Enterprises within the European Smart Cities' Context: An Integrated AHP and Extended Fuzzy Linguistic TOPSIS Method. *Energies*, 13(10), 2415.

factor in this particular location multi-criteria problem. And this holds, regardless of the hypothesis considered. Also, the sub-criterion 'Host country GDP per capita' is the least important factor, regardless of the perceptual-map' hypothesis considered. Nonetheless, the second, third, fourth and fifth positions are different depending on the perceptual-map parameters.

Considering the comparison analysis, these results can be great assets to current European leaders. Moreover, this application shows the feasibility of the method and open up the possibility to replicate the proposed framework to other sectors or geographical areas. For instance, the same methodology could be replicated to understand the city location decision of IT corporations or could be replicated for energy multinational enterprises in the United States. With respect to the Thesis objectives, this chapter clearly illustrates the achievement of Contribution C4.

The chapter is organized as follows. Section 4.1 introduces the topic and the phenomenon of location decisions within the multinational energy firms of Europe. Section 4.2 presents the research method of an integrated AHP with a fuzzy multi-perceptual based TOPSIS, which has been designed specifically for this application. In this section, the method is applied step by step. Section 4.3 provides a discussion of the results and comparative analysis. Finally, section 4.4 concludes this chapter.

4.1 Introduction

Urban systems are dynamic spaces of cohabitation and development of human and industrial activities that have experienced a great evolution during the last decades. The consolidation of human well-being and the generation of opportunities for its inhabitants in many different fields have encouraged cities' growth, so much so that since 2007, according to United Nations [124], more than half the world's population is living in urban areas, consuming over 60% of total resource and generating around 70% of global carbon emissions. It means that cities are one of the most significant contributors to climate change [45], but it also makes cities one of the key actors having an influence and the ability to fight for the sustainable development [3, 71], by implementing low carbon development plans [72, 119]. In Europe, where more than 70% of the population live in cities, the role of these human settlements is particularly important. In this sense, some experts [26, 140] consider that it is essential to evoke the interests of the business sector, since they can be a key actor alongside public authorities and citizens [98]. World Commission on Environment and Development [88] stated that multinational companies, within the private sector, have the power to contribute to sustainable development and to bring far-reaching changes and improvements needed in the face of climate change and unsustainable practices.

Aware of that, city mayors are highly interested in attracting multinational enterprises, especially those working on the green energy field. This is because the multitude of positive impact of the inherent innovations and the social benefits of the given services provided by the new companies have already been verified [43, 77, 90]. Local governments need to identify what key variables companies consider in their strategic decision-making processes, when entering new markets. If, and only if public authorities know what MNE's companies are looking for, will they be capable of being attraction poles to these organizations. The decision-making process is far from being homogenous. Each sector demands different features, although it is possible to find common requirements and behavioral patterns. Considering how important the energy sector is for the sustainable development, this study focuses on understanding which variables are most significant for the green energy European MNEs, when making the choice for new locations to offer their services.

This complex phenomenon of location decision, involving many interrelated and conflicting criteria that can vary over time and over industry type, has been widely studied in specific industries: the business service industry [148], retail industry and stores [32, 59, 137], industrial plants and facilities for supply chain management [5, 6, 33, 78, 97, 160, 206], hospitals and medical facilities [161, 184], agro-industrial firms [133], logistics companies [50, 170], bank industry and financial service providers [134], entrepreneurship [60, 62], and even, the aerospace industry [183]. However, to the best of my knowledge, there is no study, and despite the growing interest in the issue, that incorporates a comprehensive and complete set of variables specific for the green energy services. Precisely, there is no model gathering all the variables that might be significant for the energy industry offering services, such as district heating or retrofitting, to cities. Besides, most of the location frameworks used in other fields do not deal with fuzzy linguistic terms that can model the hesitancy which is always intrinsic to human reasoning.

Considering this important gap in the literature, this application aims to contribute on location theories by providing a novel and original linguistic framework based on the use of tools from the multi-criteria decision aiding field [146] and the application of the developed structure of perceptual maps, presented in previous chapter 3. It is important to highlight the fact that the presented methodology is part of a broader project that aims to help city political leaders, in a comprehensive manner, to prioritize investments based on a defined problem or need. Figure 4.1 is a schematic illustration of the proposed cyclical procedure to move from a specific set of ordered criteria to policy development. This is a tool intended for city political leaders use. This framework can be used for different needs or multi-criteria problems faced by the cities. As can be seen in Figure 4.1, multi-criteria decision-making tools as well as linguistic modelling are of great use thorough the entire procedure that a city should follow in order to go from the theoretical framework to impactful actions.

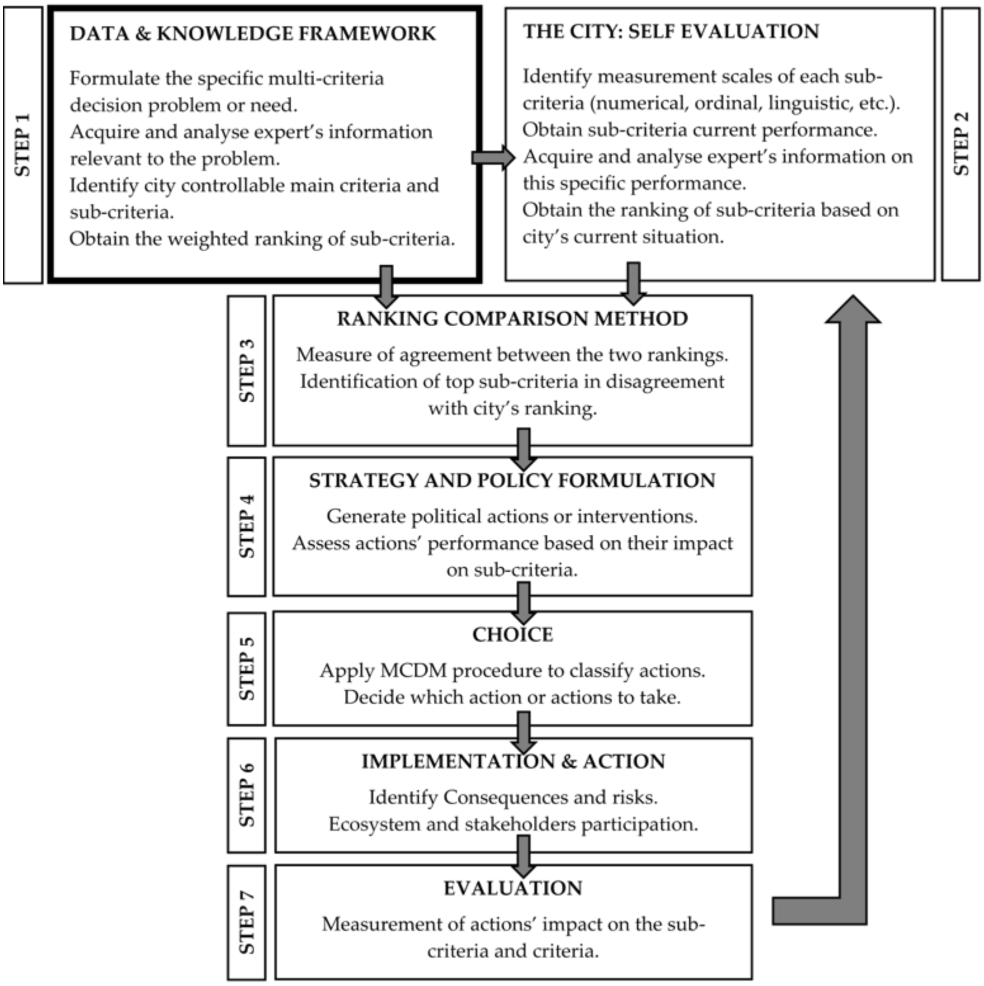

FIGURE 4.1: Holistic plan to help city leaders move from the theoretical framework to actions.

In the following subsections, a specific methodology to solve the step one of this holistic framework (4.1 is developed. As already presented, in this case, the specific problem is to help European smart cities better attract energy MNEs and hence, the need is to assess the importance of main criteria and sub-criteria governing the strategic location decision made by these enterprises. Therefore, the acquisition of data and the proposed ranking framework is obviously generated from opinions and judgements given by experts of this sector. The proposed procedure could be applied to different multi-criteria problems, such as how cities can attract tech start-ups, resulting in a different set of controllable criteria.

4.2 An integrated AHP and multi-perceptual linguistic TOPSIS application

The main goal is to develop a decision support framework combining an AHP with an extended version of a fuzzy linguistic TOPSIS with unbalanced EHFLTSs in order to assess the influential factors governing the strategic location decision made by European MNEs from the energy sector. In the proposed method, firstly, a classic AHP is performed to obtain the relative importance of the identified first-level criteria and secondly, the extended version of TOPSIS with fuzzy linguistic information as presented in 3.3.2 is used to obtain a rank of the sub-criteria. Hence, in this context, sub-criteria are treated as the alternatives of the MCDM context.

Let $G = \{E_1, E_2, \ldots, E_l, \ldots, E_K\}$ be a group of K experts, $F = \{g_1, \ldots, g_j, \ldots, g_m\}$ be a coherent set of m criteria and $C = \{c_1, c_2, \ldots, c_t, \ldots, c_h\}$ represent the set of all sub-criteria. Also, let w_j ,with $j \in \{1, 2, \ldots, m\}$ be the aggregated relative importance (or, weight) of each criterion g_j and z_t ,with $t \in \{1, 2, \ldots, h\}$, denote the relative importance (or, weight) of each sub-criterion c_t with respect to its corresponding main criteria group. Then, the proposed method can be divided in the following five steps:

Step 1: Analyzing the criteria and sub-criteria and forming a hierarchical structure. The first step is to clearly define the group decision-aiding situation, the main goal and gather the criteria and sub-criteria needed for the problem throughout a systematic literature review process. Then, following the AHP steps, the goal is situated at the top and the main criteria on the subsequent level. The lowest level is composed by the set of all sub-criteria.

Step 2: Getting the criteria weights by means of AHP method. From the judgements group G , K pairwise comparison matrices are built. Each expert l makes comparisons between two criteria to determine the dominance of one over another, using the fundamental scale of absolute numbers 1–9. Experts are asked $m - 1$ questions. Individual pairwise comparisons are used to obtain the individual relative importance of the main criteria, using the proposed eigenvalue method. Individual results are aggregated to obtain the corresponding weights, w_j.

Step 3: Elicitation of the individual and aggregated sub-criteria opinion. The group of experts is asked to assess the set of sub-criteria, C, identified in step 1 with respect to the relative influence or power the sub-criteria has in the decision problem. To obtain the aggregated sub-criteria opinion be means of proportional information, the steps 1,2,3 and 4 defined in 3.3.2 are used here:

Step 3a: Settings. For each E_l, define his or her appropriate LTS and perceptual map upon which his or her linguistic assessments will be modeled. This implies the identification of $S_l = \{s^l_1, s^l_2, \ldots, s^l_{n_l}\}$, with $n_l \in \mathbb{N}$, $\forall\, l \in \{1, 2, \ldots, K\}$ and the perceptual maps $\mu_l : S_l \to [0, 1]$, such that $\sum_{i=1}^{n_l} \mu(s^l_i) = 1, \forall\, l \in \{1, 2, \ldots, K\}$.

Step 3b: Eliciting assessments. Get H^l_t, $\forall\, t \in \{1, \ldots, h\}$ and $l \in \{1, \ldots, K\}$, which is the HFLTS modelling the opinion of expert E_l with respect to the importance of the sub-criterion c_t.

Step 3c: Mapping onto the projected space. Employ Definitions 3.10 and 3.12 to find the projected LTS, S^*, and the projected perceptual map, μ_*, associated to the perceptual characteristics of group G defined in the settings. Then, using the perceptual-based transformation function of Definition 3.14 find the projected HFLTSs, $H^{l^*}_t$, for each H^l_t, $\forall\, t \in \{1, \ldots, h\}$ and $l \in \{1, \ldots, K\}$.

Step 3d: Compute the PHFLTS formed by the union of elements in the projected space. For each sub-criterion c_t, compute the PHFLTS as introduced in Definition 2.12. This means to find $P_{H_t} = \{(s^*_i, p_i) \mid s^*_i \in S^*, i \in \{1, 2, \ldots, n^*\}\}$, with $\sum_{i=1}^{n^*} p_i = 1$ and $0 \le p_i \le 1, \forall\, i \in \{1, 2, \ldots, n^*\}$.

Step 4: Computing the relative weight of each sub-criteria by means of TOPSIS: first, along with the experts, the relative positive and negative ideal solutions need to be identified and then, the distances of each sub criteria normalized vector to the positive and negative ideal solutions are calculated, respectively. Following TOPSIS traditional procedure, based on these distances the closeness coefficient is obtained for each c_t and the resulting z_t is computed. The c_t are ranked within each criteria group, g_j, according to z_t.

Step 5: Integration of AHP and TOPSIS results: Combining the criteria weight, w_j, with the relative importance of each sub-criteria, z_t, a final ranking is obtained.

The proposed steps are illustrated in Figure 4.2

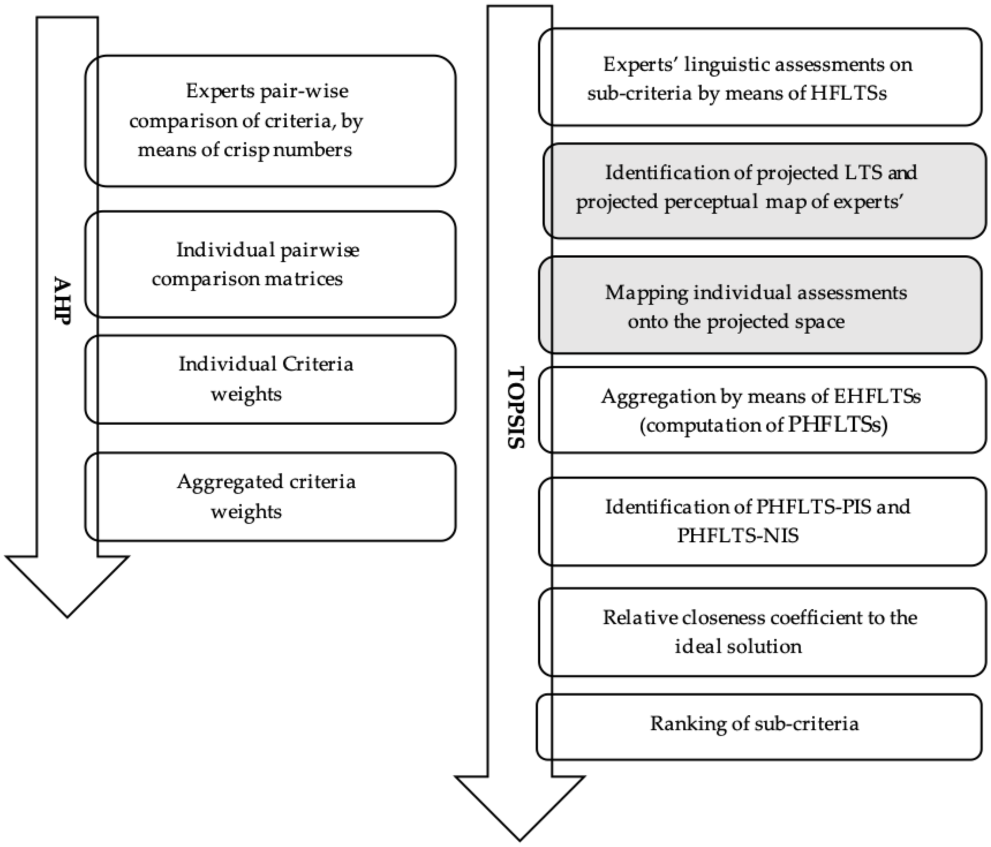

FIGURE 4.2: A framework for GDM with multi-granular and multi-perceptual linguistic information based on a combination of AHP and fuzzy linguistic TOPSIS techniques.

4.2.1 Analyzing the criteria and sub-criteria and forming a hierarchical structure

A systematic literature review process related to site location decision problems faced by business with similar characteristics to the MNEs of the energy sector was performed. Firstly, the selected articles were published only by academic peer-reviewed journals, written solely in English, containing the keywords such as "location", "decision(s)", "factor(s)" and "business" and not older than 5 years. Filters were used in Web of Science and Scopus. Secondly, the titles and abstract of the papers were carefully reviewed to reject the ones, whose objectives and topics were not related to the purpose of this particular application. Thirdly,the resulting papers were explored in detail to identify the key explanatory factors for site location. Furthermore, a more detailed literature review on specific journals using additional keywords such as "municipalities" or "energy business locations" or "renewable energy decision-making" was performed. Finally,a workshop with practitioners was organized to share the results of the literature process and some final adjustments were made. In the following paragraphs, a brief explanation of the resulting main criteria and its associated sub-criteria is presented.

Characteristics of the City's Host Country or Region

This category refers to the main geographic, economic, social and political factors that characterize the city's host country or region. The sub-criteria corresponding to this group is described as follows:

- Home-Host Country Distance: The geographic distance between the MNE headquarters or its main area of operations and the city (new location)
- Host country GDP per capita: The country's economic output per person.
- Host country level of welfare state: The degree to which the city's host country (or region) protects and promotes the well-being of its citizens in terms of as health, equal opportunities, equitable distribution, etc.
- Host country political stability perception: The perception of a country's political order and system (e.g., safe, predictable, uncertain, with several political coups, etc.).
- Host country's corruption perception: The perceived level of public sector corruption, i.e., the misuse of public power for private benefits.

City Structural Factors

These are the predominant characteristics that distinguish one city from another in terms of long-term stablished or structural factors. The set of sub-criteria corresponding to this criterion is the following:

- The city size: The city size in terms of inhabitants living in the full municipal area or urban system.
- City's cultural and language distance perception: The perceived differences between the values, communication styles and language of the city and the MNE's own organizational culture.
- City's climate characteristics: The main features of the predominant climate of the city (temperature, rain, wind, etc.).
- City's connectivity—infrastructural features: Transport infrastructure, in terms of service quality, rail and road networks, public transport level, airport connections, etc., both within the city and with other cities.
- City's reputation, image and prestige: The business sector's long-term impression regarding the city and its "positioning" efforts in comparison with other cities.

The City's Government and its Policies

The conditions and environment offered by the city government in terms of doing business. The identified sub-criteria are:

- City government degree of transparency: Transparency of the city government in terms of holding public officials accountable, fighting corruption, opening decisions and law to discussion and government meetings with the press and public.
- City government bureaucracy level: The friendliness and ease (or the opposite) of the city's regulatory framework for setting up new businesses. For instance, are administrative procedures for starting a new enterprise in the city highly complicated?
- Access to financial support provided by city government: The financial support and aid (e.g., tax incentives) given by the city government for the creation or development of new ventures or projects.
- City government support to public-private partnerships (PPP): The extent to which the city government promotes PPPs, creating a good regulatory environment for collaborations

Socioeconomic Context of the City

This refers to the quantitative economic features and subjective aspects of the city's economic and social environment. The identified sub-criteria are:

- City GDP per capita: The city's economic output per person.
- Municipal economic budget: The capacity of the city's annual budget revenues to cover expenditures and finance all type of necessities for the city.
- City R&D expenditure: The relative importance of research and development expenditure in the city's annual budget.
- The service economy of the city: The city's provision of services such as financial services, information technology, retail services or education.
- Stakeholders' pressure in the city: The perception of the presence of stakeholders in the city and their influence on the way businesses operate in the city.

Environmental conditions of the city

It reflects the progress of the city towards a greener and more environmentally sustainable model. The sub-criteria corresponding to the environmental criteria are:

- Citizens' environmental awareness: The awareness and understanding of the city's citizens regarding the environment and environmental problems.
- City's air quality: The air quality of the city and levels of urban air pollution.
- Degree of city transition to renewables: The extent to which the city relies on renewable energy sources for electricity generation or heat supply.

Market Conditions for Energy Firms in the City

The specific market conditions and agglomeration effects related to the services and products offered by the energy MNE. The sub-criteria corresponding to this group is described as follows:

- Competition intensity in the city: The concentration of competitors in the city, who offer similar services to those of the MNE.
- Pool of skilled labor in the city: The availability of specific human resources needed by the MNE to implement its services in the city.
- Access to needed suppliers: The accessibility of the inputs and materials needed to implement or construct the services offered by the MNE.
- City's potential customers: The number of potential clients, living in the city, willing to buy the MNE green services or products.
- City's degree of know-how, innovation and technological exchanges: The innovative environment of the city in terms of know-how and technological best practices transfer between economic agents such as universities, clusters, R&D departments, etc.

Table 4.1 summarizes the resulting list of six criteria and its corresponding sub-criteria, resulting from literature review and experts and practitioners' feedback.In Figure 4.3, following the first steps of AHP, a decision framework is structured as a hierarchy from the top with the goal of the decision.

Criteria	Sub-Criteria	Literature
Characteristics of the city's host country or region	Home-Host Country Distance Host country GDP per capita Host country level of welfare state Host country political stability perception Host country's corruption perception	[1, 134, 155] [22, 70, 147] [97, 148]
City structural factors	The city size City's cultural and language distance perception City's climate characteristics City's connectivity—infrastructural features City's reputation, image and prestige	[147, 149, 160, 206] [7, 21, 48, 125]
The city's government and its policies	City government degree of transparency City government bureaucracy level Access to financial support provided by city gov. City government support to PPP	[21, 61, 150] [132, 143]
Socioeconomic context of the city	City GDP per capita Municipal economic budget City R&D expenditure The service economy of the city Stakeholders' pressure in the city	[58, 97, 131, 160]
Environmental conditions of the city	Citizens' environmental awareness City's air quality Degree of city transition to renewables	[6, 75, 150, 167]
Market conditions for energy firms in the city	Competition intensity in the city Pool of skilled labor in the city Access to needed suppliers City's potential customers City's degree of know-how, innovation...	[147, 148, 206] [21, 35, 58, 159]

TABLE 4.1: List of relevant criteria and sub-criteria for location decisions in the energy MNEs context.

In the first (or top) level of figure 4.3, the overall goal of the decision-makers (the MNE of the energy sector) is specified, i.e., choosing a new European municipality to implement its green products or services. The MNE selects the best possible city (or municipality) among a given set of alternatives considering multiple criteria. In the second level of the hierarchical structure, the six main criteria governing this complex location decision are positioned. In the third level, the sub-criteria are included. Each criterion is explained by several sub-criteria (ranging from three to five sub-criteria per criterion) and the framework is composed by a total of 27 sub-criteria. It is relevant to highlight the fact that some sub-criteria are intangible attributes (such as, city government bureaucracy level), while others are quantitative in their nature (for instance, city size).

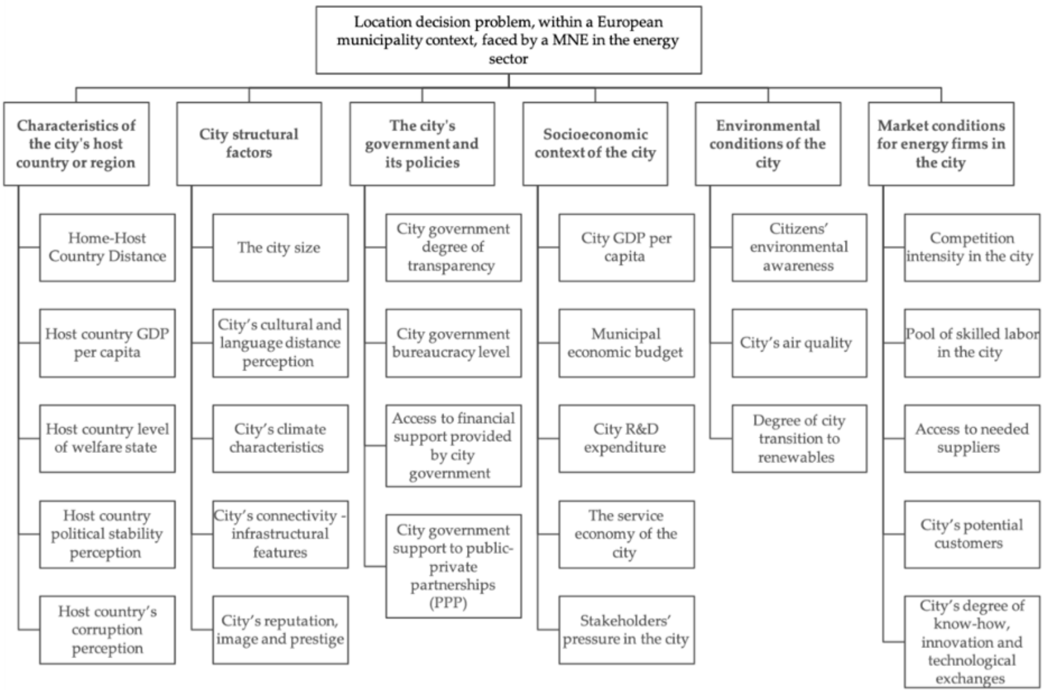

FIGURE 4.3: The AHP hierarchical framework for the European city selection problem of MNE in the energy sector.

As will be presented in the following sections, the relative importance associated to each of the identified sub-criteria will depend on the perceptual-map considered for each expert. For instance, when the assumption of an equal and balanced perceptual map is made, results indicate a higher relative importance of access to financial support provided by the city as compared to city government degree of transparency. In contrast, if half of the experts provide their linguistic opinions based on a more strict perceptual-map, this statement is not true and the city degree of transparency is, in aggregated terms, more important than the city government support.

4.2.2 Getting the criteria weights by means of AHP method

Once the research framework is constructed, ten experts with abundance professional services in the energy sector and having more than ten years' experience and company managers were chosen to participate in the AHP survey. They were contacted by telephone, and in some cases, I could personally meet the respondents in person. The purpose of the study was clearly explained to all of them. The interviews were designed and facilitated in such a way that respondents naturally used simple or complex linguistic expressions of a given linguistic term set to express their opinions. With a half a dozen or eight responses from experts gathered, the methodologies proposed in the following paragraphs are consistent and stable [47]. The number of experts considered for this study is enough as accumulated knowledge in top strategic positions in MNEs in the energy sector is concentrated in few people. The target companies have the following particular characteristics: they are all well-established companies founded before the nineties; headquarters are not necessary placed in big cities or European capitals, their growth is a result of first, organic growth and then, mergers and acquisitions, a vast majority were initially owned by the state. Some still have a public shareholder, their current revenues are usually thousands of millions of Euros, they all operate in the international market, beyond Europe.

These experts were asked five pairwise comparison questions of the type: "Which of the two criteria being compared (e.g., market conditions in the city or socioeconomic context of the city), is considered more important by your organization when looking for a new European city to expand your green and renewable services?". The sixth criteria of market conditions for the energy firms was taken as the basis to build the five questions. The Saaty's pairwise comparison scale of 9 numerical values was used. Hence, if an expert considered that market conditions in the city is very strongly more important than city host country characteristics, the intersection row "market conditions" and column "city host country characteristics", in the pairwise comparison matrix, would contain a value of 7. The reciprocal of this value (1/7) would be placed in the city host country characteristics—market conditions cell. Following the procedure in [85], the resulting ten consistent multiplicative matrices in a ratio 1/9–9 of the pairwise comparisons of the criteria, given by the each of the ten respondents, are shown in figure 4.4.

$$E1\begin{pmatrix}1.00&1.00&1.19&1.19&1.19&0.13\\1.00&1.00&1.19&1.19&1.19&0.13\\0.84&0.84&1.00&1.00&1.00&0.11\\0.84&0.84&1.00&1.00&1.00&0.11\\0.84&0.84&1.00&1.00&1.00&0.11\\7.56&7.56&9.00&9.00&9.00&1.00\end{pmatrix};\ E2\begin{pmatrix}1.00&1.18&0.13&1.00&1.00&0.27\\0.84&1.00&0.11&0.84&0.84&0.23\\7.61&9.00&1.00&7.61&7.61&2.08\\1.00&1.18&0.13&1.00&1.00&0.27\\1.00&1.18&0.13&1.00&1.00&0.27\\3.65&4.32&0.48&3.65&3.65&1.00\end{pmatrix};E3\begin{pmatrix}1.00&8.38&1.07&9.00&8.38&3.00\\0.11&1.00&0.12&1.07&1.00&0.35\\0.93&7.81&1.00&8.38&7.81&2.79\\0.11&0.93&0.11&1.00&0.93&0.33\\0.11&1.00&0.12&1.07&1.00&0.35\\0.33&2.79&0.35&3.00&2.79&1.00\end{pmatrix}$$

$$E4\begin{pmatrix}1.00&0.12&0.11&0.20&0.27&0.45\\8.05&1.00&0.89&1.64&2.21&3.64\\9.00&1.11&1.00&1.84&2.47&4.07\\4.88&0.60&0.54&1.00&1.34&2.21\\3.64&0.45&0.40&0.74&1.00&1.64\\2.21&0.27&0.24&0.45&0.60&1.00\end{pmatrix};\ E5\begin{pmatrix}1.00&1.64&0.23&0.27&0.18&0.52\\0.60&1.00&0.14&0.16&0.11&0.31\\4.30&7.09&1.00&1.18&0.78&2.25\\3.63&5.98&0.84&1.00&0.66&1.90\\5.46&9.00&1.26&1.50&1.00&2.86\\1.90&3.13&0.44&0.52&0.34&1.00\end{pmatrix};E6\begin{pmatrix}1.00&1.00&1.00&2.60&2.60&0.28\\1.00&1.00&1.00&2.60&2.60&0.28\\1.00&1.00&1.00&2.60&2.60&0.20\\0.38&0.38&0.38&1.00&1.00&0.11\\0.38&0.38&0.38&1.00&1.00&0.11\\3.45&3.45&3.45&9.00&9.00&1.00\end{pmatrix}$$

$$E7\begin{pmatrix}1.00&0.82&0.11&0.22&0.61&0.38\\1.22&1.00&0.13&0.29&0.75&0.46\\9.00&7.37&1.00&2.13&5.57&3.45\\4.21&3.45&0.46&1.00&2.60&1.61\\1.61&1.32&0.17&0.38&1.00&0.61\\2.60&2.13&0.29&0.61&1.61&1.00\end{pmatrix};\ E8\begin{pmatrix}1.00&1.19&0.13&0.79&0.57&0.32\\0.83&1.00&0.11&0.66&0.47&0.27\\7.51&9.00&1.00&5.94&4.27&2.43\\1.26&1.51&0.16&1.00&0.72&0.41\\1.75&2.10&0.23&1.39&1.00&0.57\\3.08&3.69&0.41&2.43&1.75&1.00\end{pmatrix};E9\begin{pmatrix}1.00&1.00&1.00&0.11&1.00&0.11\\1.00&1.00&1.00&0.11&1.00&0.11\\1.00&1.00&1.00&0.11&1.00&0.11\\9.00&9.00&9.00&1.00&9.00&1.00\\1.00&1.00&1.00&0.11&1.00&0.11\\9.00&9.00&9.00&1.00&9.00&1.00\end{pmatrix}$$

$$E10\begin{pmatrix}1.00&1.00&0.11&1.00&0.13&0.34\\1.00&1.00&0.11&1.00&0.13&0.34\\9.00&9.00&1.00&9.00&1.17&3.11\\1.00&1.00&0.11&1.00&0.13&0.34\\7.69&7.69&0.85&7.69&1.00&2.65\\2.89&2.89&0.32&2.89&0.37&1.00\end{pmatrix}$$

FIGURE 4.4: Pairwise comparison matrices from expert's preference relations, based on AHP

Following the AHP procedure, a priority vector is obtained for each of the 10 matrices and the resulting individual criteria weights are computed. A crisp or classical AHP is performed since it is not expect to find uncertainty or vagueness in the comparison judgements of the main criteria. This is in line with the obtained results, i.e., individual criteria weights are similarly distributed. Moreover, since all ten experts have equal voting power, the average is computed to obtain the resulting weights. In the Table 4.2, the resulting average weights for this group of experts are illustrated.

Criteria	Weight
The city's government and its policies	30%
Market Conditions for energy firms in the city	25%
Socioeconomic context of the city	14%
Environmental conditions of the city	13%
Characteristics of the city's host country or region	10%
City structural factors	8%

TABLE 4.2: Aggregated criteria weights, obtained from AHP, following step 2 described in section 4.2

4.2.3 Elicitation of the individual and aggregated sub-criteria opinions

Once the weight of main criteria is obtained, the experts are asked to assess the degree of importance of the 27 sub-criteria, using the same linguistic term set S with cardinality 5. This linguistic information is captured via HFLTSs. Figure 4.5 shows the linguistic expressions given by the 10 experts based on a given and fixed linguistic term set of five elements, $S = \{N = \text{not important}, L = \text{low importance}, S = \text{somewhat important}, V = \text{very important}, E = \text{extremely important}\}$ and ? denotes total hesitancy, i.e., $? = \{N, L, S, V, E\}$. It is relevant to note the fact that, in this case, multi-granularity has not been considered for comparison purposes so the effect of the variability in perceptual-maps can be solely analyzed. Nonetheless, incorporating different granularities for each expert is in the list of future research directions and has been used in other applications.

Sub-Criteria	E1	E2	E3	E4	E5	E6	E7	E8	E9	E10
Home-Host Country Distance	V	N, L	V	V	N	L	V	L	N	S
Host country GDP per capita	L	L, S	L	S	L	L	S	L	N	S
Host country level of welfare state	L	S, V	V	L	L	S	S	L	N	S
Host country political stability perception	S	V	E	V	S	V	V	S	N	V
Host country's corruption perception	L	V	E	V	L	E	V	S	N	V
The city size	V	S	V	L	L	L	L	L	S	V
City's cultural and language distance perception	S	N	L, S	S	L	S	S	S	L	N
City's climate characteristics	N	V, E	N, L	E	S	L	V	S	S	L
City's connectivity—infrastructural features	V	L	S, V	V	L	V	V	S	N	S
City's reputation, image and prestige	S	L	S, V	S	S	S	S	L	N	S
City government degree of transparency	L	V	V, E	E	?	E	V	V	N	V
City government bureaucracy level	L	E	S, V	V	L	V	V	V	E	V
Access to financial support provided by city government	V	V	S, V, E	S	S	S	V	V	E	V
City government support to public-private partnerships (PPP)	V	E	V, E	S	?	S	V	V	V	V
City GDP per capita	S	S	S	L	?	L	S	L	N	V
Municipal economic budget	S	S, V	V, E	L	S	L	V	L	L, S	S
City R&D expenditure	S	S	S, V	L	L	L	N	L	L, S	V
The service economy of the city	S	L	L, S, V	V	S	V	V	L	L, S	S
Stakeholders' pressure in the city	S	S, V	V, E	V	L	S	V	S	L, S	V
Citizens' environmental awareness	L	V	V, E	E	E	S	L	L	L, S	V
City's air quality	L	V	S	S	V	S	N	L	L	V
Degree of city transition to renewables	L	V, E	S	E	N	L	V	V	E	V
Competition intensity in the city	V	S, V	V, E	L	L	S	S	L	N	S
Pool of skilled labor in the city	V	S, V	S, V, E	L	V	V	E	L	N, L	V
Access to needed suppliers	V	S	S, V, E	S	S	S	V	S	L, S	V
City's potential customers	V	V	V, E	E	V	V	S	S	S	E
City's degree of know-how, innovation and technological exchanges	S	L	S, V	S	E	S	N	L	S, V	V

FIGURE 4.5: Linguistic expressions given by the ten experts in relation to the importance of each sub-criteria.

Hereinafter, to stay on the same terminology, let the set S be denoted as $S = \{s_1 :$ not important, $s_2 :$ low importance, $s_3 :$ somewhat important, $s_4 :$ very important, $s_5 :$ extremely important$\}$. In this step, for comparison purposes, three different assumptions (starting-points) are defined, which are based on the existence of different perceptual-maps within the group of experts:

- **Situation A.** The setting with respect to the perceptual maps is assumed to be $\mu_l(s_i) = 0.2$, $\forall\ i \in \{1, 2, 3, 4, 5\}$ and $\forall\ l \in \{1, 2, \ldots, K\}$. This means that the qualitative reasoning process of all experts can be modelled by means of an equally and symmetrically distributed LTS. The partition associated to this perceptual map, $\mu_{balanced}$, is shown in figure 4.7.
- **Situation B.** The setting with respect to the perceptual maps is assumed to be $\mu_l(s_1) = \mu_l(s_2) = 0.3$, $\mu_l(s_3) = 0.2$, $\mu_l(s_4) = \mu_l(s_5) = 0.1$, $\forall\ l \in \{1, 2, 3, 4, 5\}$ and $\mu_l(s_i) = 0.2\ \forall\ i \in \{1, 2, 3, 4, 5\}, \forall\ l \in \{6, 7, 8, 9, 10\}$. This represents a situation where the first five experts have a qualitative reasoning process that could be considered 'strict' or 'perfectionist', owing a perceptual map, μ_{strict}, illustrated in figure 4.7. The rest of the experts are assumed to elicit their opinions based on $\mu_{balanced}$.

- **Situation C.** The setting with respect to the perceptual maps is assumed to be $\mu_l(s_1) = \mu_l(s_2) = 0.1$, $\mu_l(s_3) = 0.2$, $\mu_l(s_4) = \mu_l(s_5) = 0.3$, $\forall\ l \in \{1,2,3,4,5\}$ and $\mu_l(s_i) = 0.2\ \forall\ i \in \{1,2,3,4,5\}$, $\forall\ l \in \{6,7,8,9,10\}$. In this situation, the first five experts are assumed to be 'generous' or 'soft' when eliciting their opinions. This is modeled with a perceptual map, μ_{soft}, whose associated partition is also illustrated in figure 4.7. The rest of the experts are assumed to elicit their opinions based on $\mu_{balanced}$.

Starting-points A, B and C are graphically illustrated in figure 4.6.

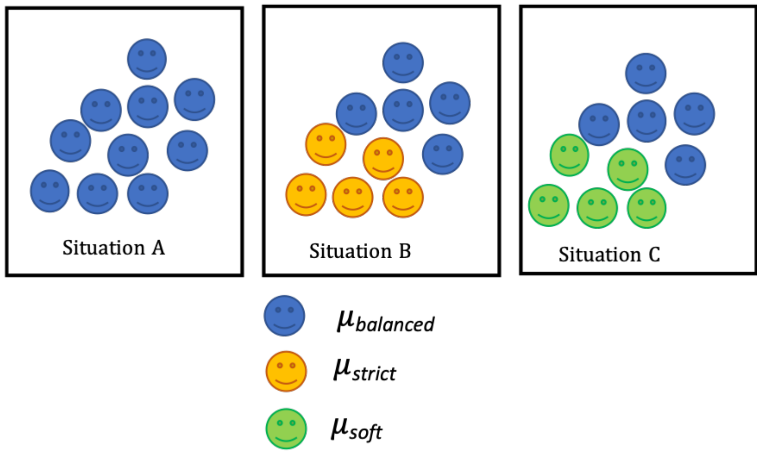

FIGURE 4.6: An illustration of the three different starting-points, based on the different perceptual-map hypothesis considered

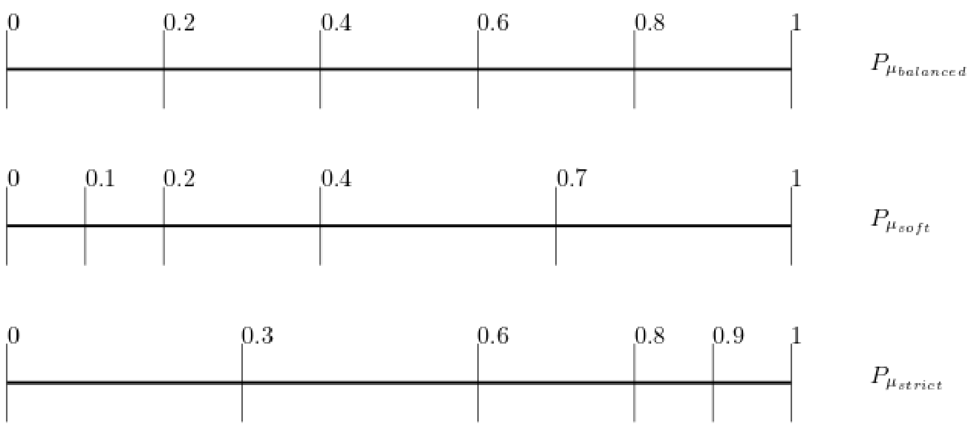

FIGURE 4.7: Partitions corresponding to the different perceptual maps assumed to model experts' opinions in situation A, situation B and situation C.

Following step 3c, Definitions 3.10 and 3.12 are used to find the projected LTS, S^* and the projected perceptual map, μ_*, for each situation:

- **Situation A.** The projected LTS is S^*_A=$\{s^*_1, s^*_2, s^*_3, s^*_4, s^*_5\}$ and μ^A_*=$\mu^A_*(s^*_i) = 0.2\ \forall\ i \in \{1,2,3,4,5\}$. The projected partition modelling situation A, $P_{\mu^A_*}$, is illustrated in figure 4.8.

- **Situation B.** The projected LTS is $S_B^*=\{s_1^*, s_2^*, s_3^*, s_4^*, s_5^*, s_6^*, s_7^*\}$ with $\mu_*^B=\mu_*^B(s_i^*) = 0.2\ \forall\, i \in \{1,4,5\}$ and $\mu_*^B=\mu_*^B(s_i^*) = 0.1\ \forall\, i \in \{2,3,6,7\}$. The projected partition modelling situation B, $P_{\mu_*^B}$, is illustrated in figure 4.8.

- **Situation C.** The projected LTS is $S_C^*=\{s_1^*, s_2^*, s_3^*, s_4^*, s_5^*, s_6^*, s_7^*\}$ with $\mu_*^C=\mu_*^C(s_i^*) = 0.2\ \forall\, i \in \{3,4,7\}$ and $\mu_*^C=\mu_*^C(s_i^*) = 0.1\ \forall\, i \in \{1,2,5,6\}$. The projected partition modelling situation C, $P_{\mu_*^C}$, is illustrated in figure 4.8.

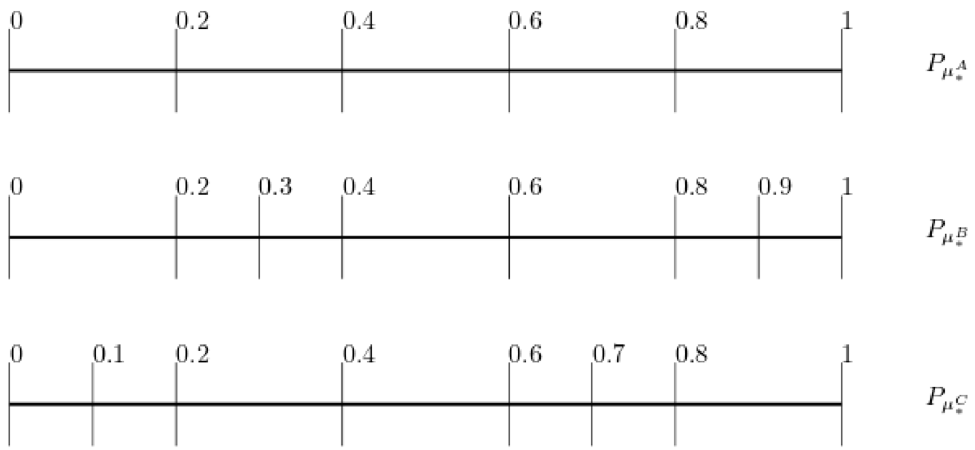

FIGURE 4.8: The projected partitions resulting of situation A, B and C

Note that the development and results published in the journal Energies, mentioned earlier in the introduction of this chapter, relate to the assumption of situation A.

Then, for each situation A,B and C, using the perceptual-based transformation function of Definition 3.14, each individual linguistic assessments is mapped onto the corresponding projected space, i.e. find $H_t^{l^*}$, for each H_t^l, $\forall\, t \in \{1, \ldots, 23\}$ and $\forall\, l \in \{1, \ldots, 10\}$.

The individual projected assessments are aggregated, for each sub-criterion, by means of projected EHFLTSs. As explained in step 3d in the proposed method in 4.2, the proportional information is simultaneously calculated, by means of PHFLTSs for each sub-criteria, as shown in Table 4.3 for situation A, Table 4.4 for situation B and Table 4.5 for situation C. The second column of each table denotes the vector corresponding to the projected PHFLTSs of each sub-criterion.

Sub-criteria	PHFLTS (Sit. A)
Home-Host Country dist.	$(0.25, 0.25, 0.10, 0.40, 0.00)$
Host country GDP per capita	$(0.10, 0.55, 0.35, 0.00, 0.00)$
Host country level of welfare state	$(0.10, 0.40, 0.35, 0.15, 0.00)$
Host country political stability per.	$(0.10, 0.00, 0.30, 0.50, 0.10)$
Host country's corruption per.	$(0.10, 0.20, 0.10, 0.40, 0.20)$
The city size	$(0.00, 0.50, 0.20, 0.30, 0.00)$
City's cultural and language distance per,	$(0.20, 0.25, 0.55, 0.00, 0.00)$
City's climate characteristics	$(0.15, 0.25, 0.30, 0.15, 0.15)$
City's connectivity-infrastructural features	$(0.10, 0.20, 0.25, 0.45, 0.00)$
City's reputation, image and prestige	$(0.10, 0.20, 0.65, 0.05, 0.00)$
City government degree of transparency	$(0.12, 0.12, 0.02, 0.47, 0.27)$
City government bureaucracy level	$(0.00, 0.20, 0.05, 0.55, 0.20)$
Access to financial support...	$(0.00, 0.00, 0.33, 0.53, 0.13)$
City government support to PPP	$(0.02, 0.02, 0.22, 0.57, 0.17)$
City GDP per capita	$(0.12, 0.32, 0.42, 0.12, 0.02)$
Municipal economic budget	$(0.00, 0.35, 0.40, 0.20, 0.05)$
City R&D expenditure	$(0.10, 0.45, 0.30, 0.15, 0.00)$
The service economy	$(0.00, 0.28, 0.38, 0.33, 0.00)$
Stakeholders' pressure	$(0.00, 0.15, 0.40, 0.40, 0.05)$
Citizens' environmental awareness	$(0.00, 0.35, 0.15, 0.25, 0.25)$
City's air quality	$(0.10, 0.30, 0.30, 0.30, 0.00)$
Degree of city transition to renew.	$(0.10, 0.20, 0.10, 0.35, 0.25)$
Competition intensity in the city	$(0.10, 0.30, 0.35, 0.20, 0.05)$
Pool of skilled labor in the city	$(0.05, 0.25, 0.08, 0.48, 0.13)$
Access to needed suppliers	$(0.00, 0.05, 0.58, 0.33, 0.03)$
City's potential customers	$(0.00, 0.00, 0.30, 0.35, 0.35)$
City's degree of know-how, innovation...	$(0.10, 0.20, 0.40, 0.20, 0.10)$

TABLE 4.3: PHFLTSs of the aggregated linguistic information for each sub-criterion for situation A

For example, with respect to sub-criterion 'City's potential customers', the HFLTSs provided by the then experts, according to figure 4.5, are: $\{s_4\}$, $\{s_5\}$, $[s_4, s_5]$, $\{s_5\}$, $\{s_4\}$, $\{s_4\}$, $\{s_3\}$, $\{s_3\}$, $\{s_3\}$, $\{s_5\}$. Based on situation A, where all experts share the same perceptual-map, the corresponding aggregated EHFLTS is $[s_3^*, s_4^*, s_5^*]$ and the PHFLTS is the vector $(0.00, 0.00, 0.30, 0.35, 0.35)$ as can be seen in Table 4.3. Similarly, if sub-criterion 'Host country political stability perception' is taken, the HFLTSs modeling the opinion of the ten experts, according to figure 4.5, are: $\{s_3\}$, $\{s_4\}$, $\{s_5\}$, $\{s_4\}$, $\{s_3\}$, $\{s_4\}$, $\{s_4\}$, $\{s_3\}$, $\{s_1\}$, $\{s_4\}$. Based on situation A, the corresponding aggregated EHFLTS is $[s_1^*, s_3^*, s_4^*, s_5^*]$ and the PHFLTS is the vector $(0.10, 0.00, 0.30, 0.50, 0.10)$ as can be seen in Table 4.3. Note here that when using PHFLTSs, the linguistic term sets might not be consecutive. As compared to the former sub-criterion, the latter causes more controversy and variability among respondents.

Sub-criteria	PHFLTS (Sit. B)
Home-Host Country dist.	$(0.175, 0.175, 0.125, 0.125, 0.1, 0.3, 0)$
Host country GDP per capita	$(0.1, 0.1, 0.283, 0.383, 0.133, 0, 0)$
Host country level of welfare...	$(0.1, 0.05, 0.2, 0.45, 0.05, 0.15, 0)$
Host country political stability per.	$(0.1, 0, 0, 0.1, 0.5, 0.2, 0.1)$
Host country's corruption per.	$(0.1, 0, 0.1, 0.2, 0.2, 0.25, 0.15)$
The city size	$(0, 0.15, 0.25, 0.2, 0.2, 0.2, 0)$
City's cultural and language distance per,	$(0.15, 0.1, 0.133, 0.383, 0.233, 0, 0)$
City's climate characteristics	$(0.075, 0.175, 0.125, 0.225, 0.2, 0.05, 0.15)$
City's connectivity - infrastructural...	$(0.1, 0, 0.1, 0.3, 0.25, 0.25, 0)$
City's reputation, image and prestige	$(0.1, 0.05, 0.1, 0.35, 0.35, 0.05, 0)$
City government degree of transparency	$(0.114, 0.014, 0.064, 0.064, 0.314, 0.214, 0.214)$
City government bureaucracy level	$(0, 0, 0.1, 0.1, 0.45, 0.2, 0.15)$
Access to financial support...	$(0, 0, 0, 0.1, 0.533, 0.283, 0.083)$
City government support to PPP	$(0.014, 0.014, 0.014, 0.114, 0.514, 0.164, 0.164)$
City GDP per capita	$(0.114, 0.114, 0.164, 0.164, 0.414, 0.014, 0.014)$
Municipal economic budget	$(0, 0.133, 0.183, 0.183, 0.35, 0.1, 0.05)$
City R&D expenditure	$(0.1, 0.133, 0.233, 0.133, 0.35, 0.05, 0)$
The service economy	$(0, 0.083, 0.158, 0.208, 0.425, 0.125, 0)$
Stakeholders' pressure	$(0, 0.033, 0.083, 0.283, 0.35, 0.2, 0.05)$
Citizens' environmental awareness	$(0, 0.133, 0.183, 0.183, 0.1, 0.15, 0.25)$
City's air quality	$(0.1, 0.1, 0.15, 0.15, 0.3, 0.2, 0)$
Degree of city transition to renewables	$(0.05, 0.1, 0.1, 0.05, 0.4, 0.1, 0.2)$
Competition intensity in the city	$(0.1, 0.05, 0.15, 0.4, 0.05, 0.2, 0.05)$
Pool of skilled labor in the city	$(0.033, 0.083, 0.133, 0.05, 0.283, 0.333, 0.083)$
Access to needed suppliers	$(0, 0.033, 0.033, 0.233, 0.533, 0.133, 0.033)$
City's potential customers	$(0, 0, 0, 0.3, 0.1, 0.3, 0.3)$
City's degree of know-how, innovation...	$(0, 0, 0, 0.3, 0.1, 0.3, 0.3)$

TABLE 4.4: PHFLTSs of the aggregated linguistic information for each sub-criterion for situation B

As shown in Tables 4.4 and 4.5, different EHFLTSs and PHFLTSs are obtained depending on the perceptual-maps considered. In fact, the computation of PHFLTS is performed in a projected linguistic term set of seven projected terms, as illustrated in 4.8. For example, with respect to sub-criterion 'City's potential customers', based on situation B, where half of the experts are assumed to hold a more strict perceptual-map, the corresponding aggregated EHFLTS is $[s_4^*, s_5^*, s_6^*, s_7^*]$ and the PHFLTS is the vector $(0, 0, 0, 0.3, 0.1, 0.3, 0.3)$. In contrast, based on situation C, where half of the experts are assumed to hold a softer perceptual-map, the corresponding EHFLTS is the same but the PHFLTS is the vector $(0, 0, 0, 0.425, 0.175, 0.175, 0.225)$.

Sub-criteria	PHFLTS (Sit. C)
Home-Host Country dist.	$(0.2, 0.1, 0.2, 0.25, 0.2, 0.05, 0)$
Host country GDP per capita	$(0.05, 0.4, 0.35, 0.2, 0, 0, 0)$
Host country level of welfare	$(0.05, 0.35, 0.133, 0.383, 0.083, 0, 0)$
Host country political stability percep.	$(0.05, 0.05, 0.2, 0.2, 0.25, 0.2, 0.05)$
Host country's corruption percep.	$(0.05, 0.25, 0, 0.2, 0.2, 0.15, 0.15)$
The city size	$(0, 0.2, 0.4, 0.2, 0.15, 0.05, 0)$
City's cultural and language dis. percep.	$(0.15, 0.2, 0.35, 0.3, 0, 0, 0)$
City's climate characteristics	$(0.15, 0.05, 0.3, 0.225, 0.075, 0.125, 0.075)$
City's connectivity - infrastructural...	$(0.05, 0.25, 0.033, 0.333, 0.233, 0.1, 0)$
City's reputation, image and prestige	$(0.05, 0.15, 0.433, 0.333, 0.033, 0, 0)$
City government degree of transparency	$(0.064, 0.164, 0.014, 0.089, 0.239, 0.239, 0.189)$
City government bureaucracy level	$(0, 0.2, 0.033, 0.083, 0.283, 0.25, 0.15)$
Access to financial support...	$(0, 0, 0.22, 0.22, 0.27, 0.17, 0.12)$
City government support to PPP	$(0.014, 0.014, 0.114, 0.189, 0.289, 0.289, 0.089)$
City GDP per capita	$(0.064, 0.164, 0.514, 0.114, 0.064, 0.064, 0.014)$
Municipal economic budget	$(0, 0.1, 0.483, 0.208, 0.108, 0.075, 0.025)$
City R&D expenditure	$(0.05, 0.25, 0.483, 0.083, 0.083, 0.05, 0)$
The service economy of the city	$(0, 0.125, 0.375, 0.225, 0.175, 0.1, 0)$
Stakeholders' pressure in the city	$(0, 0.1, 0.183, 0.358, 0.208, 0.125, 0.025)$
Citizens' environmental awareness	$(0, 0.1, 0.25, 0.225, 0.125, 0.175, 0.125)$
City's air quality	$(0.05, 0.15, 0.4, 0.2, 0.15, 0.05, 0)$
Degree of city transition to renewables	$(0.1, 0.1, 0.2, 0.025, 0.175, 0.225, 0.175)$
Competition intensity in the city	$(0.05, 0.25, 0.133, 0.408, 0.108, 0.025, 0.025)$
Pool of skilled labor in the city	$(0.033, 0.133, 0.187, 0.153, 0.253, 0.12, 0.12)$
Access to needed suppliers	$(0, 0, 0.37, 0.32, 0.17, 0.12, 0.02)$
City's potential customers	$(0, 0, 0, 0.425, 0.175, 0.175, 0.225)$
City's degree of know-how, innovation...	$(0.05, 0.15, 0.333, 0.167, 0.117, 0.133, 0.05)$

TABLE 4.5: PHFLTSs of the aggregated linguistic information for each sub-criterion for situation C

4.2.4 Computing the relative weight of each sub-criteria by mmeans of TOPSIS

Based on TOPSIS methodology, the positive ideal solution (PHFLTS-PIS) and the negative ideal solution (PHFLTS-NIS) are first identified. Due to the systematic literature review process carried out and the fact that the strategic location decision problem does not deal with a set of highly dynamic or changing city variables, the relative positive and negative ideal solutions are considered the best choice by the experts. This is relevant considering the rank reversal problem of the TOPSIS method. With respect to situation A, based on results illustrated in Table 4.3, the PHFLTS-NIS is set to be the host country GDP per capita, which is modelled by $(0.10, 0.55, 0.35, 0.00, 0.00)$ and the PHFLTS-PIS corresponds to the city's potential customers, which is modelled by $(0.00, 0.00, 0.30, 0.35, 0.35)$.

On the one hand, with respect to situation B and based on results shown in Table 4.4, the PHFLTS-NIS is set to be the host country GDP per capita, which is modelled by $(0.10, 0.10, 0.283, 0.383, 0.133, 0.00, 0.00)$ and the PHFLTS-PIS corresponds to the city's potential customers, which is modelled by $(0.00, 0.00, 0.00, 0.30, 0.10, 0.30, 0.30)$. On the other hand, with respect to situation C and based on Table 4.5, the PHFLTS-NIS is set to be the host country GDP per capita, which is modelled by

$(0.05, 0.40, 0.350, 0.20, 0.00, 0.00, 0.00)$ and the PHFLTS-PIS corresponds to the city's potential customers, which is modelled by $(0.00, 0.00, 0.00, 0.425, 0.175, 0.175, 0.225)$.

Secondly, for each sub-criteria PHFLTS, its cosine distance function to the defined PHFLTS-NIS and PHFLTS-PIS are computed, respectively. The set of distances, for situation A, are detailed in Table 4.6. Both distances are used simultaneously to compute the closeness coefficient, shown in the last column of Table 4.6. A sub-criterion is closer to the PHFLTS-PIS and farther from PHFLTS-NIS as this coefficient approaches 1. Since PHFLTS-NIS corresponds to the host country GDP per capita vector and the PHFLTS-PIS corresponds to the city's potential customer vector, in all situations, it is straight forward that sub-criteria city's potential customers has a coefficient of 1 while the value for the sub-criteria host country GDP per capita is 0. The same computations are performed considering the PHFLTSs obtained for situation B and C. Table 4.7 synthesizes the resulting closeness coefficient for each situation.

Sub-criteria	distance to PHFLTS-PIS	distance to PHFLTS-NIS	CC_i^A
Home-Host Country Dist.	0.459226	0.44867	0.4941864
Host country GDP per cap.	0.724943	0	0
Host country level of welfare	0.515155	0.047731	0.084798
Host country political stability percep.	0.136131	0.709395	0.8389980
Host country's corruption percep.	0.186791	0.53910	0.7426758
The city size	0.537544	0.151439	0.2198014
City's cultural and language distt percep.	0.552045	0.16613	0.2313273
City's climate charact.	0.281708	0.167620	0.3730467
City's connectivity - infrastructural	0.28427	0.439445	0.6072018
City's reputation, image ...	0.467291	0.235524	0.3351158
City government degree of transparency	0.19439	0.773236	0.799102
City government bureaucracy ...	0.227301	0.688444	0.7517856
Access to financial support	0.104210	0.724861	0.8743049
City government support to PPP	0.115478	0.785049	0.871765
City GDP per capita	0.455196	0.084781	0.1570092
Municipal economic budget	0.371139	0.115688	0.2376368
City R&D expenditure	0.568132	0.035900	0.059435
The service economy	0.311872	0.244071	0.4390210
Stakeholders' pressure	0.183735	0.425651	0.6984913
Citizens' environmental awareness	0.268493	0.285109	0.5150073
City' s air quality	0.363302	0.197704	0.3524100
Degree of transition to renewables	0.162266	0.525207	0.7639668
Competition intensity	0.353921	0.12376	0.2590963
Pool of skilled labor	0.268239	0.542262	0.6690453
Access to needed suppliers	0.223051	0.479269	0.682407
City's potential customers	0	0.724943	1
City's degree of know-how and...	0.237616	0.226888	0.4884523

TABLE 4.6: Distances to the PHFLTS-PIS and to the PHFLTS-NIS and its closeness coefficient (CC_i), in situation A.

Sub-criteria	CC_i^B	CC_i^C
Home-Host Country distance	0,50563543	0,46611208
Host country GDP per capita	0	0
Host country level of welfare state	0,18917988	0,26579861
Host country political stability perception	0,58340989	0,6443307
Host country's corruption perception	0,74814463	0,64335487
The city size	0,31188829	0,16697726
City's cultural and language distance perception	0,12342951	0,14963354
City's climate characteristics	0,35728449	0,43417475
City's connectivity - infrastructural features	0,49284436	0,58879031
City's reputation, image and prestige	0,27072834	0,20668762
City government degree of transparency	0,68993513	0,67573129
City government bureaucracy level	0,59772867	0,62744216
Access to financial support	0,61614863	0,72124411
City government support to PPP	0,6017607	0,76377622
City GDP per capita	0,30446857	0,16870146
Municipal economic budget	0,34975606	0,26043826
City R&D expenditure	0,26668419	0,10918848
The service economy of the city	0,37734407	0,30279098
Stakeholders' pressure in the city	0,52557742	0,67969702
Citizens' environmental awareness	0,64400214	0,55027018
City' s air quality	0,41223411	0,2115593
Degree of city transition to renewables	0,55208287	0,51429386
Competition intensity in the city	0,37579645	0,43787634
Pool of skilled labor in the city	0,63038307	0,54895753
Access to needed suppliers	0,49146639	0,5191419
City's potential customers	1	1
City's degree of know-how, innovation and...	0,41264171	0,25339861

TABLE 4.7: Closeness coefficients (CC_i) for situations B and C

Based on the relative closeness coefficients' values, the partial weight of each sub-criterion within each criteria group is distributed, i.e., the weight percentages of each criteria group sum up to 100%. The sub criteria are then ranked within each criteria group. Results are shown in Tables 4.8 to 4.13, considering all three situations. Column W means weight or relative importance and column R means ranking position. The complete name of the sub-criteria have been abbreviated and synthesized for styling purposes and to fit the content into the margins of the present manuscript.

Hypothesis	Situation A		Situation B		Situation C	
Sub-Criteria	W	R	W	R	W	R
Home-Host Country dist.	22.87%	3	24.95%	3	23.08%	3
Host country GDP...	0%	5	0%	5	0%	5
Host country welfare...	3.92%	4	9.34%	4	13.16%	4
Host country political stab...	38.83%	1	28.79%	2	31.90%	1
Host country's corruption...	34.37%	2	36.92%	1	31.86%	2

TABLE 4.8: Sub-criteria relative weight and rank of **Characteristics of the city's host country or region**, depending on the situation considered.

Hypothesis	Situation A		Situation B		Situation C	
Sub-Criteria	W	R	W	R	W	R
The city (C.) size	12.44%	5	20.04%	3	10.79%	4
C. cultural and language dist. percep.	13.10%	4	7.93%	5	9.68%	5
C. climate characteristics	21.12%	2	22.96%	2	28.08%	2
C. conn.—infrastructural feat...	34.37%	1	31.67%	1	38.08%	1
C. reputation, image...	18.97%	3	17.40%	4	13.37%	3

TABLE 4.9: Sub-criteria relative weight and rank of **City structural factors**, depending on the situation considered.

Hypothesis	Situation A		Situation B		Situation C	
Sub-Criteria	W	R	W	R	W	R
C. govern. degree of transparency	24.24%	3	27.54%	1	24.23%	3
C. govern. bureaucracy level	22.80%	4	23.86%	4	22.50%	4
Access to financial support	26.52%	1	24.59%	2	25.86%	2
C. government support to PPP	26.44%	2	24.02%	3	27.39%	1

TABLE 4.10: Sub-criteria relative weight and rank of **City's government and its policies**, depending on the situation considered.

Hypothesis	Situation A		Situation B		Situation C	
Sub-Criteria	W	R	W	R	W	R
C. GDP per capita	9.86%	4	16.69%	4	11.09%	4
Municipal economic budget	14.93%	3	19.17%	3	17.12%	3
C. R&D expenditure	3.73%	5	14.62%	5	7.18%	5
The service economy	27.58%	2	20.69%	2	19.91%	2
Stakeholders' pressure	43.89%	1	28.82%	1	44.69%	1

TABLE 4.11: Sub-criteria relative weight and rank of **Socioeconomic context of the city**, depending on the situation considered.

Hypothesis	Situation A		Situation B		Situation C	
Sub-Criteria	W	R	W	R	W	R
Citizens' environmental awa.	31.57%	2	40.04%	1	43.12%	1
C. air quality	21.60%	3	25.63%	3	16.58%	3
C. transition to renewables	46.83%	1	34.33%	2	40.30%	2

TABLE 4.12: Sub-criteria relative weight and rank of **Environmental conditions of the city**, depending on the situation considered.

Hypothesis	Situation A		Situation B		Situation C	
Sub-Criteria	W	R	W	R	W	R
Competition intensity	8.36%	5	12.91%	5	15.87%	4
Pool of skilled labor	21.59%	3	21.66%	2	19.89%	2
Access to needed supp.	22.02%	2	16.88%	3	18.81%	3
C. potential customers	32.27%	1	34.36%	1	36.24%	1
C. degree of know-how, inn...	15.76%	4	14.17%	4	9.18%	5

TABLE 4.13: Sub-criteria relative weight and rank of **Market conditions for energy firms in the city**, depending on the situation considered.

Note that in some cases, even if the percentage associated to each sub-criterion may vary depending on the perceptual-maps considered, as illustrated in Table 4.14, the rank of the sub-criterion within the group remains the same. This is the case of the 'Socioeconomic context of the city' sub-criterion, as shows Table 4.11. Focusing on the resulting rank of sub-criteria within the group, the higher differences among perceptual-maps hypothesis are found in 'City structural factors'. As seen in Table 4.9, the city size position might move between the third and the fifth.

Sub-criteria	Range of percentage difference $\mid \max z_t - \min z_t \mid$
Home-Host Country Distance	2.08%
Host country GDP per capita	0%
Host country level of welfare state	9.24%
Host country political stability perception	10.04%
Host country's corruption perception	5.06%
The city size	9.24%
City's cultural and language distance perception	5.16%
City's climate characteristics	6.96%
City's connectivity - infrastructural features	6.41%
City's reputation, image and prestige	5.60%
City government degree of transparency	3.30%
City government bureaucracy level	1.35%
Access to financial support	1.93%
City government support to PPP	3.38%
City GDP per capita	6.38%
Municipal economic budget	4.25%
City R&D expenditure	10.89%
The service economy of the city	7.67%
Stakeholders' pressure in the city	15.88%
Citizens' environmental awareness	11.55%
City' s air quality	9.05%
Degree of city transition to renewables	12.05%
Competition intensity in the city	7.51%
Pool of skilled labor in the city	1.77%
Access to needed suppliers	5.13%
City's potential customers	3.97%
City's degree of know-how, innovation and...	6.58%

TABLE 4.14: Percentage points of difference between the maximum partial weight and the minimum partial weight of each sub-criterion

4.2.5 Integration of AHP and TOPSIS results

Combining the average weights of the main criteria, obtained in 4.2, with the relative importance of each sub-criteria within each group, a final ranking is obtained. The ranking is illustrated in Table 4.15.

Sub-criteria	Sit. A	Sit. B	Sit. C
City's potential customers	8.07%	8.59%	9.06%
Access to financial support provided by city government	7.96%	7.37%	7.76%
City government support to public-private partnerships	7.93%	7.20%	8.21%
City government degree of transparency	7.27%	8.26%	7.27%
City government bureaucracy level	6.84%	7.15%	6.75%
Stakeholders' pressure in the city	6.14%	4.03%	6.25%
Degree of city transition to renewables	6.09%	4.46%	5.24%
Access to needed suppliers	5.51%	4.22%	4.70%
Pool of skilled labor in the city	5.40%	5.41%	4.97%
Citizens' environmental awareness	4.10%	5.20%	5.60%
City's degree of know-how, innovation and...	3.94%	3.54%	2.29%
Host country political stability perception	3.88%	2.87%	3.19%
The service economy of the city	3.86%	2.89%	2.78%
Host country's corruption perception	3.44%	3.69%	3.18%
City's air quality	2.81%	3.33%	2.15%
City's connectivity—infrastructural features	2.75%	2.53%	3.04%
Home-Host Country Distance	2.29%	2.49%	2.30%
Municipal economic budget	2.09%	2.68%	2.39%
Competition intensity in the city	2.09%	3.22%	3.97%
City's climate characteristics	1.69%	1.83%	2.24%
City's reputation, image and prestige	1.52%	1.39%	1.07%
City GDP per capita	1.38%	2.33%	1.55%
City's cultural and language distance perception	1.05%	0.63%	0.77%
The city size	1.00%	1.60%	0.86%
City R&D expenditure	0.52%	2.04%	1.00%
Host country level of welfare state	0.39%	0.93%	1.31%
Host country GDP per capita	0.00%	0.00%	0.00%

TABLE 4.15: Sub-criteria overall relative weight, for each situation considered

According to the results of Table 4.15, the most relevant factor is 'City's potential customers', regardless of the perceptual-map hypothesis. Besides, the relevant percentage weight of this sub-criterion is quite similar in the three situations: 8.0671 % , 8.590% and 9.06% for situations A, B and C, respectively. The rest of the sub-criteria related to market conditions for energy firms, which are access to needed suppliers, pool of skilled labor, city's degree of know-how and competition intensity in the city are placed in the 8th, 9th, 11th, 19th positions of the rank (situation A), respectively. In the case of situation B and C, these sub-criteria positions are: 9th, 6th, 12th, 14th and 10th, 9th, 18th and 11th, respectively. It is important to highlight that customers and suppliers' environments are more relevant than the factor of competition intensity in the city, regardless of the hypothesis.

As expected, the least valued sub-criterion is 'Host Country GDP per capita' in all situations.

It is also relevant to notice that the resulting TOP 5 sub-criterion are all the same, regardless of the hypothesis. Moreover, without considering the first ranked sub-criterion which belongs to market conditions, the following four sub-criteria all belong to the group of 'City's government and its policies'. Only a few differences in the sub-criterion positions are noticeable as detailed in Table 4.16

Sub-criteria position ranking	Sit. A	Sit. B	Sit. C
Access to financial support provided by the city	2	3	3
City government support to PPP	3	4	2
City government degree of transparency	4	2	4
City government bureaucracy level	5	5	5

TABLE 4.16: Ranking positions of sub-criteria included in 'City's government and its policies'

Therefore, in this case, the consideration of more or less strict perceptual maps does not have an influence on the TOP 5 sub-criteria. This fact suggests the relevant importance and influence of city's governance policies and decisions over multinational firms behavior. Policies directed to provide some type of financial incentives, offer mentorship/accelerator programs or reduce the administrative procedures could have a great impact on the location decision of an energy multinational firm. According to the ten experts, these actions have a higher influence on the location decision of these companies than the economic power of the host country or the perceived distance in terms of cultural or languages issues.

Nonetheless, as illustrated in Table 4.15, considerable differences in the sub-criterion ranking can be found in middle positions. On the one hand, the sub-criterion 'Stakeholders' pressure in the city' is placed 6th if experts all have a balanced perceptual-map. In contrast, this sub-criterion is placed 10th in the ranking if situation B is considered. On the other hand, higher variability have been found in specific sub-criteria. For instance, sub-criterion 'City's degree of know-how, innovation and technological exchanges' presents a difference of 7 positions. While in situation A this sub-criterion is ranked 11th, it is found in position 18th in situation C, when five of the experts are assumed to elicit their opinions with a softer perceptual-map. A similar difference is found in sub-criterion 'City's air quality', being placed 13th in situation B but 20th in situation C.

With respect to the group of 'Characteristics of the city's host country or region', which include aspects inherent to the municipality and holds a total weight of 10%, it is relevant to identify the fact that all sub-criteria represent a relative weight of less than 4%, regardless of the hypothesis. However, the ranking positions of some sub-criteria are slightly different. This is the case of 'Host country political stability perception', which is ranked 12th in situation C and A but ranks 16th in situation B, as well as 'Host country level of welfare state', which is situated 26th in situation A, 25th in situation B and 22nd in situation C. Within this group of criteria, 'Host country political stability perception' and 'Host country's corruption perception' are the most important sub-criteria.

In addition, Table 4.15 illustrates the fact that the three sub-criteria included in environmental conditions of the city are also placed relatively high in the ranking. Precisely, 'Degree of city transition to renewables' and 'Citizen's environmental awareness' are always in the TOP 10, regardless of the hypothesis. Decision makers have expressed their preference for cities which are in a process of transitioning to renewables. This sub-criterion has an influence of between 4.46% and 6.08% , resulting higher than municipal economic budget, city GDP per capita or infrastructural features. Experts also did not express a preference for cities with high R&D expenditure, which is placed 25th, 21st and 24th respectively. This might suggest that they do prefer a municipality which offers specific financial support for their sector or related to their products/services or agility in the bureaucratic processes rather than an innovative city with plenty of R&D hubs.

4.3 Discussion

When business managers of energy MNEs have to decide which European city is best to go and sell its green services and products, they are clearly facing a decision which involve multiple and sometimes conflicting criteria and usually do not find a unique optimal solution. The relevant factors guiding this decision-making process are usually unknown by the city leaders. Retrieving the key information from decision makers and experts of this field to consequently identify and assess the determinants of this process, would allow policy makers to better take actions, in advance, aimed at improving their attractiveness to energy enterprises.

In the existing literature, many techniques for assessing the determinants of location decisions are based on the use of quantitative variables measured in numbers [134, 137] or qualitative variables modelled and categorized by crisp numbers [62]. Statistical methods such as multiple regressions [148] are usually the predominant type of techniques found in location theories. These methods of gathering information cannot fully capture the hesitancy, uncertainty and fuzziness nature of human thoughts and opinions. In contrast, qualitative information expressed with linguistic variables, which is the basis of CWW processes, most closely resembles how human mind works. These techniques have been successfully applied to many fields, [52, 173, 196]. Moreover, the use of LTSs that are neither uniformly nor symmetrically distributed improves the capacity of these techniques to capture the variability and specificity of the different DMs' qualitative reasoning processes.

In this chapter, a novel approach to the location problem is introduced. It is based on a combination of AHP with a fuzzy multi-perceptual linguistic TOPSIS, two multiple-criteria decision-making techniques, which have been proven to work well to solve other business challenges [91, 94, 102, 162, 163]. In general, MCDA approaches for criteria ranking or selection that apply an integration of AHP [113, 194] and TOPSIS [10, 156] methods consider either crisp or fuzzy attitudes, however, the methodology presented in this chapter takes a hybrid position. This makes the presented approach more realistic and better adapted to the specific problem considered. Considering the background and expertise of energy experts, the main five criteria should not be subject of ambiguity, whereas the uncertainty and vagueness inherent in the respondents' evaluations with respect to the sub-criteria and has been incorporated through the use of PHFLTSs in the modified TOPSIS. Besides, the introduction of several perceptual-map tools in the developed methodology allows us to better capture and understand the obtained results, based on the qualitative reasoning processes which are inherent to all experts or DMs eliciting linguistic opinions.

The results indicate that MNEs in the energy sector consider 'City aspects related to government and its policies' and 'Market conditions for energy firms in the city' key when they make a location decision in the European municipalities context. These two criteria represent, according to results obtained using a crisp AHP, 55% of the weight in this decision. In contrast, only a maximum of ten percent (10%) of the decision' weight is due to the 'City's host country characteristics' and 'City structural factors', which are decision factors over which city governments have less influential power.

The perceptual-based approach based on the TOPSIS method used to capture the relevancy of sub-criteria indicates that 'City's potential customers' is the most influential factor and ranks first in any of the three situations considered. Besides, the rest of the top five sub-criteria which are considered the most valuable for location strategic decisions in energy MNEs are, regardless of the variability in perceptual-maps: 'Access to financial support provided by city government', 'City government support to public-private partnerships', 'City government degree of transparency' and 'city government bureaucracy level'. Nonetheless, the identification and use of the appropriate perceptual-maps for each DMs becomes decisive when other sub-criteria such as 'City's degree of know-how, innovation and technological exchanges' or 'City's air quality' are analyzed. If city leaders were interested in the complete ranking of sub-criterion and a generous amount of economic resources would be ready to proportionally be allocated to all type of initiatives, the use of the correct perceptual-maps would become crucial.

As compared to the conclusions reached by Rubalcaba and Gago in their study applied to the business services sector [147], the importance of traditional location factors (demand, supply and market factors) are, in general terms, less relevant factors in the energy sector. Another interesting point is that skilled labour is considered less important than the impact of any of the sub-criteria related to city's government and policies. This is a different conclusion if it is compared to other studies [33, 160], where this was a very important variable for location decisions. Moreover, I have also incorporated all factors related to the triple bottom line assumption of sustainability and these, regardless of the perceptual-map used, are gaining importance in the final rank compared to other recent decision support frameworks [6, 73, 167] as well. Actually, the 'Degree of city transition to renewables' is above access to needed suppliers in all situations considered and pool of skilled labour in the city (in situation A and situation C).

Besides, according to our results, 'Citizen's environmental awareness' is also a preferred sub-criterion for multinationals, surprisingly, much more than 'City R&D expenditure', 'Host country GDP per capita' or 'The city's reputation and prestige'. Differences appear in situation A, where a balanced linguistic term set is used for all 10 experts. In this situation, 'Citizen's environmental awareness' is ranked 10th whilst in situation B and C, this sub-criterion is ranked 7th. Therefore, depending on the appropriate perceptual-map used in the projection of linguistic variables, this sub-criterion may be more or less relevant than 'Competition intensity in the city', which is ranked 9th, 14th and 11th respectively for situation A, B and C. This contrasts with other specific research results aimed at understanding manufacturing plan location selection [33] which indicates a minor impact of environmental issues on plant location decision.

A priori, with respect to the applicability of the framework obtained, it would seem like a city's government has little to say about its potential customers as compared to the rest of sub-criteria directly related to government policies. However, municipalities are potential customers themselves and part of the demand of sustainable energy solutions for their public buildings and facilities. According to the results, PPP would seem an appropriate organizational tool to encourage these first public early-adopters. Public administrations should lead by example and show citizens and private companies the technical feasibility, economic viability and environmental impact of their interventions. Nowadays, with respect to local authorities developing specific financial supporting tools for businesses and citizens willing to become part of the demand, it seems that a major setback might cause the postponement of these initiatives to a longer term. Due to the unexpected outbreak of the Covid-19 crisis, public administrations financial priorities will probably dramatically change in the short term and their limited resources will be allocated to avoid the breakdown of SMEs and maintain the employment rate. Nonetheless, meanwhile, in the short or medium term, local governments could offer tax incentives to both individuals and businesses to stimulate the demand of green and sustainable services.

With respect to the limitations of the method, I identify some aspects related to the techniques used. From a technical point of view, the limitations of the proposed method are basically concerned with the main issues that frequently emerge from the TOPSIS method. The use of TOPSIS for sub-criteria linguistic assessments is based on a pre-defined set of sub-criteria. It is well known that one of the main limitations of the fuzzy TOPSIS method can be the rank reversal problem when the positive and negative ideal solutions are set to be the best and the worst choices considered by the experts and not the absolute ones. Nevertheless, the relative ideal solutions were identified as the most adequate and appropriate by the experts who participated in this study. As future research, different options for the positive and negative ideal solutions to analyze differences in the results will be compared.

On the other hand, the final sub-criteria ranking is dependent on the perceptual-map considered. This is now a limitation since part of the results obtained involve variability and the perceptual-maps considered in the hypothesis are somehow difficult to verify. A future research direction is the development of an algorithm to obtain the perceptual-map parameters which are more adjusting to the qualitative reasoning process of each expert. Moreover, the proposed framework is based on expert knowledge gathering. This means that it works well when the information related to the location decision processes is centralized and deployed mainly by a group of experts or managers. Hence, the techniques used for the data gathering process and the analysis should be modified and adapted if the knowledge and expertise needed for a specific multi-criteria problem was held by a large amount of people and required Big Data.

From a practical point of view, the obtained results are framed within the context of location decision-making in European municipalities. If the tool was used by political leaders from other continents, the set of criteria and sub-criteria would be first reviewed. Location determinants vary to a great extent depending on the geographical area considered and differences might arise in the results if the study was done, for example, in underdeveloped countries. Similarly, the obtained results meet the needs and priorities of the MNEs of the energy sector. If other researchers, for instance, were interested in analysing the preferences of IT companies which seek to expand their services to new European cities, they might need to adapt, modify or add the questions posed to the experts in the context of IT. Moreover, the proposed method would need to be adapted to this high dynamic sector before replication. Special attention should be given to the choice of the ideal solutions in the TOPSIS phase since the use of the absolute ideal PHFLTSs would seem more appropriate a priori. Besides, the hypothesis of the perceptual-maps used should be revised. Comparing location results of different sectors within the European context is an interesting direction for future work as well.

4.4 Conclusions

The present chapter presents a dual contribution, one from an academic perspective and one from a managerial point of view. First of all, it contributes to the existing literature, filling a theoretical gap on location theories, by providing a new MCDM framework specifically for the energy sector, combining AHP and fuzzy multi-perceptual linguistic TOPSIS; secondly, it offers local public managers the possibility of understanding what exists behind energy company location decisions, more precisely, it explains what variables are more relevant for European energy firms looking for new sites to operate. This is done under the perspective of the different qualitative reasoning process that predominate in human's opinions.

Due to the critical role that the reformulation of the policy strategy [68] related to the energy sector can play at a municipality level [11], the result of this applications provides European city leaders with a framework that could help them make more data-driven investment decisions with regard to the attraction of MNE energy firms, which could create economic, social and environmental positive effects. The results, which were obtained from the perspective of multiple companies, highlight the value of certain city government policies, such as the financial facilities, the support for public-private partnerships, the level of transparency or the degree of bureaucracy on location decisions for energy MNEs. Whereas, less controllable factors such as the economic situation of the city's host country or city climate characteristics have little weight on the decision. Consequently, this application shows the importance and the possible impacts of local government decisions and contributes to the development of more data-driven urban-policy making in Europe in the sustainable energy ecosystem.

Chapter 5

Entrepreneurial competency evaluation in secondary schools

The results derived from this chapter have already been accepted with major revisions by the Applied Soft Computing. At the time this Thesis is deposited, I am waiting for the final acceptance. This chapter corresponds mainly to the contribution three explained in section 1.3. As already mentioned, the development of this application has been supported by the INVITE Research Project (TIN2016-80049-C2-1-R and TIN2016-80049-C2-2-R (AEI/FEDER, UE)), funded by the Spanish Ministry of Science and Information Technology and the ESADE Entrepreneurship Institute.

The feasibility of the framework presented in subsection 3.3.1 is demonstrated in this second application consisting of the assessment of young students' entrepreneurial competency by evaluators who have different perceptual maps. Entrepreneurship is a highly complex process influenced by an enormous range of variables [15] and it is not limited to a business phenomenon. The entrepreneurial concept focus upon the development of personal entrepreneurial behaviors, attributes and skills [66]. Due to the complexity inherent in entrepreneurship [15] the evaluation of this competency is certainly a challenging task. In comparison, the level of mathematics or the comprehension of certain foreign languages are competencies which are, in general, more easily evaluated in a common and shared manner within educational systems.

It is an area of great interest since some literature have studied the impact of entrepreneurial educational programs on a country's development [79]. Due to its ambiguity, subjectivity and complexity,it is difficult for professors as well as for parents or students themselves to assess the attributes related to the entrepreneurial competency by precise values as compared to the grading system of languages or mathematics. It is more close to their thinking and reasoning process, if we allow them to use complex and rich linguistic expressions to determine the performance in each attribute, such as 'Very often', 'excellent', 'less than good' or 'not bad'. Furthermore, the different evaluators involved in the assessment have different backgrounds and qualitative reasoning characteristics. An evaluation of 'excellent' given by a professor might not hold the same significance as if it was provided by a parent.

This application contributes to the development of innovative evaluation systems in educational contexts and helps to better design new personalized learning and teaching programs. With respect to the Thesis Objectives, this chapter fulfills contribution C3. Besides, the method has also been tested in other MAGDM contexts, as explained in contributions C5 and C6.

5.1 Introduction

The goal of the following real multi-attribute group decision-aiding situation is to evaluate, by means of both, a classification and a ranking objective, the entrepreneurial competency of some secondary students. On the one hand, classifying students according to their entrepreneurial profile is an effective and successful way to identify the best match between student's needs and teaching techniques. A common pedagogical approach to entrepreneurship is to group students who perform similarly in some specific entrepreneurial attributes, which in turn can be taught and trained with similar teaching methods and materials. On the other hand, a ranking might be useful in situations where this competency is the criterion for choosing the best candidate student or group of students for a program or scholarship. From a secondary school, a random sample of 25 students was selected for an initial pilot test. Next, we consider the proposed method steps to classify and rank the students of the test set.

5.2 Application of the method: step by step

In the following sections, we present the results of a pilot test developed for the Andorra secondary school system, to show the feasibility of the proposed approach. The methodology presented in subsection 3.3.1 provides the theoretical framework for the application of perceptual-based distances with unbalanced HFLTSs in this field of educational competencies assessment.

5.2.1 Step 1: Settings

A set of students, Λ=$\{St_1, \ldots, St_i, \ldots, St_r\}$ with $i \in \{1, \ldots, r\}$, was chosen for an initial pilot test. Students' names were kept anonymous for confidential reasons. Since an evaluation derived from a variety of perspectives provides a more comprehensive and closer to reality result, a group of three different profiles, G=$\{d_a, d_t, d_p\}$ was set up as the set of evaluators profiles, where d_a represents the auto-evaluation (student) profile, d_t is the teacher' s profile and d_p corresponds to the parental profile (father or mother). With respect to the relevant set A of attributes, it is important to highlight the fact that the majority of research developed in terms of evaluating entrepreneurship has been done in undergraduate or graduate level courses [12, 136, 154]. Therefore, a literature review about entrepreneurial competencies in young students was performed along with a workshop with professors from secondary schools to discuss and agree on the specific attributes embedded in the entrepreneurial competency concept in this setting. This resulted in an entrepreneurial set based on six attributes, $A = \{a_l\}$ with $l \in \{1, 2, 3, 4, 5, 6\}$. These are: { a_1: Self-confidence and self-esteem, a_2: Creativity and Originality, a_3: Leadership influence and relationships, a_4: Opportunity recognition and exploitation, a_5: Planning and organizing, a_6: Achievement orientation }.

Therefore, for each student, St_i, linguistic information with respect to each attribute, a_l, was provided by the teacher, d_t, by the students themselves, d_a, and by a parent representative, d_p. Precisely, each profile was requested to express their opinion on the frequency behaviour displayed by each student with respect to each attribute, a_l, during the academic year. Responses were modelled by means of unbalanced HFLTSs, using a linguistic term set, S, with the same granularity of uncertainty for all profiles. The linguistic basic labels of S are $\{s_1 : never, s_2 : rarely, s_3 : occasionally, s_4 : frequently, s_5 : veryfrequently\}$. However, due to their differences among their qualitative reasoning, three different perceptual maps are defined over S, μ_a for the auto evaluation profile, μ_p for parents and μ_t for teachers. The perceptual maps considered are:

$$\mu_a(s_i) = 0.2\ \forall\ i \in \{1,2,3,4,5\}$$

$$\mu_p(s_i) = 0.1\ \forall\ i \in \{1,2\},\ \mu_p(s_i) = 0.2 \text{ for } i = 3 \text{ and } \mu_p(s_i) = 0.3\ \forall\ i \in \{4,5\}$$

$$\mu_t(s_i) = 0.3\ \forall\ i \in \{1,2\},\ \mu_t(s_i) = 0.2 \text{ for } i = 3 \text{ and } \mu_t(s_i) = 0.1\ \forall\ i \in \{4,5\}$$

The corresponding partitions to each perceptual map are illustrated in figure 5.1.

FIGURE 5.1: Partitions of the unit interval corresponding to the perceptual map considered for each profile, d_a, d_p, d_t.

5.2.2 Step 2: Eliciting assessments

The complete set of linguistic evaluations, H_i^{jl}, $\forall\ i \in \{1,2,3,\ldots,25\}$, $\forall\ j \in \{1,2,3\}$ and $\forall\ l \in \{1,2,3,4,5,6\}$, expressed by each profile with respect to each attribute' s frequency behaviour is shown below. A complete matrix of 150×3 is obtained.

All assessments provided by the three evaluation profiles, d_a, d_p, d_t for a class of 25 students, with respect to the complete set of attributes, are provided in the following Tables 5.1 and 5.2.

	Attribute a_1			**Attribute** a_2			**Attribute** a_3		
id.	d_a	d_p	d_t	d_a	d_p	d_t	d_a	d_p	d_t
St1	$[s_4, s_5]$	$\{s_5\}$	$\{s_5\}$	?	$[s_3, s_5]$	$\{s_4\}$	$[s_3, s_5]$	$\{s_5\}$	$[s_4, s_5]$
St2	$\{s_3\}$	$\{s_1\}$	$[s_2, s_3]$	$\{s_3\}$	$\{s_2\}$	$\{s_2\}$	$[s_2, s_3]$	$[s_1, s_4]$	$[s_2, s_4]$
St3	$\{s_4\}$	$\{s_5\}$	$\{s_5\}$	$\{s_5\}$	$\{s_4\}$	$[s_4, s_5]$	$[s_4, s_5]$	$\{s_5\}$	$\{s_5\}$
St4	$\{s_4\}$	$\{s_5\}$	$\{s_5\}$	$[s_3, s_4]$	$\{s_4\}$	$[s_4, s_5]$	$[s_3, s_4]$	$\{s_5\}$	$[s_4, s_5]$
St5	$\{s_3\}$	$[s_1, s_4]$	$\{s_2\}$	$[s_3, s_4]$	$\{s_2\}$	$\{s_1\}$	$\{s_3\}$	$\{s_2\}$	$[s_1, s_2]$
St6	$\{s_4\}$	$[s_4, s_5]$	$[s_4, s_5]$	$\{s_4\}$	$[s_4, s_5]$	$\{s_4\}$	$\{s_4\}$	$\{s_5\}$	$[s_4, s_5]$
St7	$[s_2, s_4]$	$\{s_4\}$	$[s_2, s_4]$	$[s_2, s_3]$	$[s_4, s_5]$	$\{s_3\}$	$[s_3, s_4]$	$\{s_4\}$	$[s_3, s_4]$
St8	$\{s_3\}$	$[s_3, s_4]$	$[s_4, s_5]$	?	$\{s_4\}$	$\{s_5\}$	$\{s_4\}$	$[s_4, s_5]$	$[s_4, s_5]$
St9	$[s_2, s_3]$	$[s_3, s_5]$	$[s_4, s_5]$	$\{s_4\}$	$\{s_3\}$	$\{s_4\}$	$[s_3, s_4]$	$\{s_4\}$	$\{s_5\}$
St10	$[s_2, s_5]$	$[s_4, s_5]$	$[s_4, s_5]$	$[s_3, s_4]$	$[s_3, s_4]$	$[s_3, s_4]$	$[s_4, s_5]$	$[s_4, s_5]$	$[s_4, s_5]$
St11	$[s_2, s_3]$	$\{s_5\}$	$\{s_5\}$	$\{s_3\}$	?	$[s_3, s_4]$	$[s_3, s_4]$	$\{s_5\}$	$[s_4, s_5]$
St12	$[s_2, s_3]$	$[s_3, s_5]$	$\{s_4\}$	$\{s_3\}$	$[s_3, s_4]$	$\{s_3\}$	$[s_2, s_5]$	$[s_4, s_5]$	$[s_3, s_5]$
St13	$\{s_4\}$	$[s_4, s_5]$	$[s_3, s_4]$	$[s_4, s_5]$	$[s_3, s_4]$	$\{s_4\}$	$[s_4, s_5]$	$[s_4, s_5]$	$[s_3, s_4]$
St14	$[s_2, s_3]$	$[s_4, s_5]$	$[s_3, s_5]$	$\{s_4\}$	$[s_3, s_4]$	$[s_3, s_4]$	$\{s_4\}$	$[s_4, s_5]$	$\{s_4\}$
St15	$[s_3, s_5]$	$\{s_4\}$	$\{s_4\}$	$\{s_3\}$	$[s_3, s_4]$	$\{s_4\}$	$[s_4, s_5]$	$[s_4, s_5]$	$[s_4, s_5]$
St16	$\{s_4\}$	$[s_4, s_5]$	$[s_2, s_3]$	$[s_3, s_4]$	?	$\{s_2\}$	$[s_3, s_4]$	$\{s_5\}$	$[s_2, s_4]$
St17	$[s_2, s_5]$	$[s_3, s_5]$	$\{s_2\}$	$\{s_3\}$	$[s_4, s_5]$	$[s_1, s_2]$	?	$[s_3, s_4]$	$[s_2, s_4]$
St18	$[s_4, s_5]$	$[s_4, s_5]$	$\{s_4\}$	$[s_3, s_4]$	$[s_4, s_5]$	$\{s_3\}$	$[s_4, s_5]$	$[s_4, s_5]$	$[s_4, s_5]$
St19	$[s_3, s_4]$	$\{s_5\}$	$[s_3, s_4]$	$\{s_3\}$	$[s_4, s_5]$	$\{s_3\}$	$\{s_3\}$	$[s_4, s_5]$	$[s_2, s_4]$
St20	$[s_3, s_4]$	$\{s_4\}$	$\{s_2\}$	$[s_3, s_4]$	$[s_2, s_3]$	$[s_2, s_3]$	$[s_3, s_5]$	$[s_3, s_5]$	$[s_1, s_3]$
St21	$[s_4, s_5]$	$\{s_5\}$	$\{s_4\}$	$[s_3, s_4]$	$[s_4, s_5]$	$\{s_3\}$	$[s_4, s_5]$	$\{s_5\}$	$[s_3, s_4]$
St22	$\{s_3\}$	$\{s_5\}$	$\{s_4\}$	$[s_2, s_3]$	$[s_4, s_5]$	$[s_3, s_4]$	$[s_3, s_5]$	$[s_2, s_3]$	$[s_3, s_4]$
St23	$\{s_3\}$	$\{s_5\}$	$\{s_3\}$	$[s_2, s_3]$	$[s_2, s_3]$	$\{s_4\}$	$[s_3, s_5]$	$[s_2, s_5]$	$[s_3, s_4]$
St24	$[s_2, s_3]$	$\{s_4\}$	$[s_2, s_4]$	$\{s_2\}$	$\{s_3\}$	$[s_1, s_2]$	$\{s_4\}$	$[s_4, s_5]$	$[s_3, s_4]$
St25	$[s_1, s_2]$	$[s_3, s_5]$	$[s_4, s_5]$	$[s_3, s_4]$	$[s_4, s_5]$	$\{s_4\}$	$[s_3, s_5]$	$[s_3, s_5]$	$\{s_4\}$

TABLE 5.1: Linguistic assessments provided for attribute a_1, self-confidence and self-esteem, attribute a_2, creativity and originality, and attribute a_3, leadership influence and relationships

For instance, according to Table 5.1, student with id St2 has auto-evaluated his/her creativity and originality with a frequency of occasionally, whereas the professor and the teacher, both have evaluated him/her as rarely in this attribute. Note that, the use of HFLTSs allow respondents to be hesitant. For example, student with id St17 has given a complete hesitant opinion of the frequency with respect his/her development in leadership and relationships. The method allow the student to answer in this way if he or she is not sure about the answer. The teacher profile also hesitates between rarely and frequently. The teacher is only sure that the development of student ST17 with respect to this attribute is nor never neither very frequently.

In other cases, respondents can answer with a precise linguistic label. For example, according to Table 5.2, student St2 has obtained a precise opinion of occasionally with respect opportunity recognition, which is shared by the three profiles. As will be analyzed in the following subsections, this situation will bring to the maximum level of consensus.

	Attribute a_4			**Attribute** a_5			**Attribute** a_6		
id.	d_a	d_p	d_t	d_a	d_p	d_t	d_a	d_p	d_t
St1	$[s_3,s_4]$	$\{s_4\}$	$\{s_4\}$	$\{s_4\}$	$\{s_4\}$	$\{s_5\}$	$\{s_4\}$	$[s_4,s_5]$	$\{s_5\}$
St2	$\{s_3\}$	$\{s_3\}$	$\{s_3\}$	$\{s_2\}$	$[s_1,s_3]$	$\{s_2\}$	$[s_2,s_3]$	$[s_1,s_2]$	$[s_2,s_3]$
St3	$[s_4,s_5]$	$\{s_4\}$	$\{s_5\}$	$\{s_4\}$	$\{s_5\}$	$\{s_5\}$	$\{s_5\}$	$[s_4,s_5]$	$\{s_5\}$
St4	$\{s_4\}$	$[s_4,s_5]$	$[s_4,s_5]$	$[s_4,s_5]$	$\{s_5\}$	$\{s_5\}$	$[s_4,s_5]$	$\{s_5\}$	$\{s_5\}$
St5	$\{s_3\}$	$[s_2,s_3]$	$\{s_1\}$	$\{s_3\}$	$\{s_2\}$	$\{s_2\}$	$[s_3,s_4]$	$[s_2,s_3]$	$\{s_2\}$
St6	$\{s_4\}$	$\{s_5\}$	$\{s_5\}$	$[s_4,s_5]$	$[s_4,s_5]$	$[s_4,s_5]$	$[s_4,s_5]$	$[s_4,s_5]$	$\{s_5\}$
St7	$[s_3,s_4]$	$[s_3,s_5]$	$[s_3,s_4]$	$\{s_4\}$	$[s_4,s_5]$	$\{s_3\}$	$\{s_3\}$	$[s_4,s_5]$	$[s_3,s_4]$
St8	?	$\{s_5\}$	$\{s_5\}$	?	$[s_4,s_5]$	$\{s_5\}$	$[s_3,s_4]$	$[s_4,s_5]$	$\{s_5\}$
St9	$[s_2,s_3]$	$[s_2,s_3]$	$[s_4,s_5]$	$[s_3,s_4]$	$[s_4,s_5]$	$\{s_5\}$	$[s_3,s_4]$	$[s_3,s_4]$	$[s_4,s_5]$
St10	$\{s_4\}$	$\{s_4\}$	$\{s_5\}$	$\{s_4\}$	$\{s_4\}$	$\{s_5\}$	$\{s_4\}$	$[s_4,s_5]$	$\{s_5\}$
St11	$[s_2,s_3]$	?	$[s_3,s_4]$	$[s_3,s_4]$	$\{s_5\}$	$\{s_5\}$	$[s_3,s_4]$	$\{s_5\}$	$[s_4,s_5]$
St12	$\{s_3\}$	$[s_2,s_3]$	$\{s_4\}$	$\{s_3\}$	$\{s_4\}$	$\{s_4\}$	$\{s_3\}$	$[s_4,s_5]$	$\{s_4\}$
St13	$[s_3,s_4]$	$\{s_5\}$	$[s_2,s_4]$	$\{s_3\}$	$[s_4,s_5]$	$\{s_3\}$	$\{s_3\}$	$\{s_4\}$	$[s_2,s_3]$
St14	$[s_3,s_5]$	$\{s_4\}$	$[s_2,s_3]$	$\{s_3\}$	$[s_4,s_5]$	$[s_3,s_4]$	$\{s_4\}$	$[s_4,s_5]$	$\{s_4\}$
St15	$[s_2,s_3]$	$\{s_4\}$	$\{s_4\}$	$[s_3,s_5]$	$\{s_4\}$	$\{s_5\}$	$[s_4,s_5]$	$[s_4,s_5]$	$[s_4,s_5]$
St16	$\{s_3\}$	$\{s_4\}$	$\{s_2\}$	$[s_2,s_4]$	$[s_3,s_4]$	$\{s_2\}$	$[s_3,s_4]$	$\{s_3\}$	$\{s_2\}$
St17	$\{s_4\}$	$[s_4,s_5]$	$[s_2,s_3]$	$\{s_3\}$	$\{s_5\}$	$\{s_2\}$	$\{s_3\}$	$\{s_4\}$	$[s_2,s_3]$
St18	$[s_3,s_4]$	$\{s_5\}$	$\{s_4\}$	$\{s_4\}$	$\{s_5\}$	$\{s_4\}$	$\{s_4\}$	$\{s_5\}$	$\{s_4\}$
St19	$[s_3,s_4]$	$[s_4,s_5]$	$\{s_3\}$	$[s_3,s_4]$	$\{s_4\}$	$\{s_4\}$	$\{s_3\}$	$[s_4,s_5]$	$[s_3,s_4]$
St20	$[s_3,s_4]$	$[s_3,s_4]$	$[s_3,s_4]$	$[s_4,s_5]$	$\{s_4\}$	$\{s_4\}$	$[s_4,s_5]$	$[s_4,s_5]$	$[s_3,s_4]$
St21	$\{s_4\}$	$[s_4,s_5]$	$[s_2,s_4]$	$\{s_4\}$	$\{s_4\}$	$\{s_4\}$	$[s_4,s_5]$	$[s_4,s_5]$	$\{s_4\}$
St22	$\{s_4\}$	$\{s_4\}$	$\{s_5\}$	$[s_3,s_4]$	$[s_4,s_5]$	$\{s_5\}$	$[s_3,s_5]$	$\{s_5\}$	$[s_4,s_5]$
St23	$[s_3,s_5]$	$[s_4,s_5]$	$\{s_3\}$	$\{s_4\}$	$[s_2,s_4]$	$\{s_2\}$	$\{s_4\}$	$[s_3,s_5]$	$[s_2,s_3]$
St24	$\{s_4\}$	$[s_4,s_5]$	?	$[s_3,s_4]$	$[s_4,s_5]$	$[s_2,s_3]$	$\{s_5\}$	$[s_4,s_5]$	$[s_2,s_4]$
St25	$[s_4,s_5]$	$[s_4,s_5]$	$\{s_5\}$	$\{s_4\}$	$\{s_5\}$	$\{s_5\}$	$[s_4,s_5]$	$\{s_5\}$	$\{s_5\}$

TABLE 5.2: Linguistic assessments provided for attribute a_4, opportunity recognition and exploitation, attribute a_5, planning and organizing, and attribute a_6, achievement orientation

5.2.3 Step 3: Mapping onto the projected space

With the context of step 1, the projected partition is computed and a projected LTS, S^*, is obtained with cardinality nine, resulting in a projected perceptual map, μ_*, which is defined as:
$\mu_*(s_i) = 0.1 \ \forall \ i \in \{1,2,3,4,6,7,8,9\}$
$\mu_*(s_i) = 0.2$ for $i = 5$

This forms the lattice structure where all HFLTSs of Step 2 were projected onto, using the perceptual-based transformation function in Definition 3.14. The upper part of this extended distributive projected lattice is illustrated in figure 5.2. Note that the projected partition is the result of the union of the landmarks in figure 5.1.

5.2.4 Step 4: Computing the projected centroid and consensus

For each St_i and a_l, the centroid and the degree of consensus were calculated, based on Definitions 3.7 and 3.8 and using the projected H_i^{jl*}, obtained in the previous step. Based on equation 3.11, the term ζ used for the computation of the consensus degree is 1.8. The results are shown in Tables 5.4, 5.5, 5.6, 5.7, 5.8 and 5.9. Therefore, each student, St_i, was represented by a 6-dimensional HFLD, F_i^*, where each HFLTS is the calculated centroid of each attribute, i.e., $F_i^*=(H_i^{C1*}, H_i^{C2*}, H_i^{C3*}, H_i^{C4*}, H_i^{C5*}, H_i^{C6*})$, where H_i^{Cl*} is the projected HFLTS representing the centroid of attribute l. Besides,

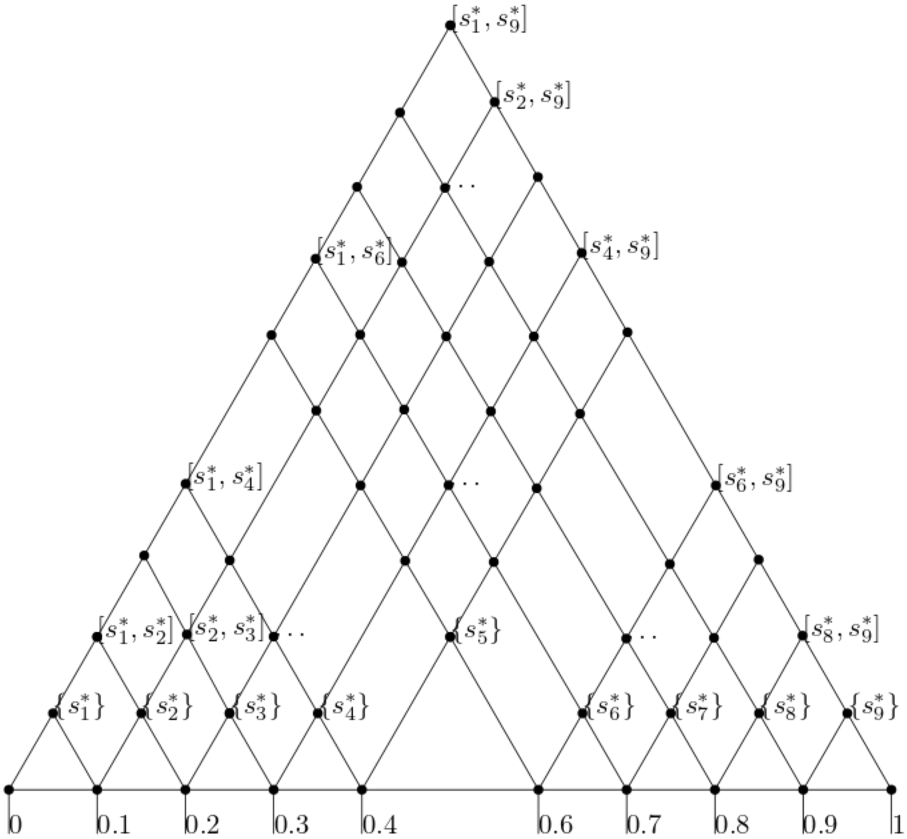

FIGURE 5.2: Upper part of the extended projected lattice, with μ_*

each H_i^{Cl*} weight is determined by its corresponding degree of consensus, $\delta_{(\lambda_i^l)}$. For instance, the 6-dimensional HFLD corresponding to $St1$ is

$$F_1^* = ([s_7^*, s_9^*], [s_3^*, s_9^*], [s_7^*, s_9^*], [s_5^*, s_7^*], [s_6^*, s_7^*], [s_6^*, s_9^*])$$

. Table 5.3 shows these results for $St1$.

Attribute	centroid, H_1^{Cl*}	Degree of agreement, $\delta_{(\lambda_1^l)}$	Percentage weight
Self esteem	$[s_7^*, s_9^*]$	0.8333	17.65
Creativity and originality	$[s_3^*, s_9^*]$	0.5	10.59
Leadership and relationships	$[s_7^*, s_9^*]$	0.7777	16.47
Opportunity recognition and exploitation	$[s_5^*, s_7^*]$	0.6666	14.12
Planning and organizing	$[s_6^*, s_7^*]$	0.5555	11.76
Achievement orientation	$[s_6^*, s_9^*]$	0.6111	12.94

TABLE 5.3: Centroids and its weighting values of student St_1

Student id	centroid a_1	Degree of agreement a_1
St1	$[s_7^*, s_9^*]$	0.8333
St2	$[s_4^*, s_5^*]$	0.3889
St3	$[s_7^*, s_9^*]$	0.7222
St4	$[s_7^*, s_9^*]$	0.7222
St5	$[s_4^*, s_5^*]$	0.7222
St6	$[s_6^*, s_9^*]$	0.6667
St7	$[s_4^*, s_7^*]$	0.7778
St8	$[s_5^*, s_6^*]$	0.4444
St9	$[s_3^*, s_9^*]$	0.4444
St10	$[s_5^*, s_9^*]$	0.6667
St11	$[s_7^*, s_9^*]$	0.3889
St12	$[s_3^*, s_8^*]$	0.4444
St13	$[s_6^*, s_8^*]$	0.7778
St14	$[s_5^*, s_9^*]$	0.5556
St15	$[s_5^*, s_8^*]$	0.6111
St16	$[s_5^*, s_7^*]$	0.7222
St17	$[s_3^*, s_9^*]$	0.7222
St18	$[s_6^*, s_9^*]$	0.7222
St19	$[s_6^*, s_8^*]$	0.7222
St20	$[s_5^*, s_6^*]$	0.8333
St21	$[s_7^*, s_9^*]$	0.8333
St22	$[s_7^*, s_8^*]$	0.5556
St23	$[s_6^*, s_7^*]$	0.6111
St24	$[s_4^*, s_6^*]$	0.7222
St25	$[s_3^*, s_9^*]$	0.2222

TABLE 5.4: Centroids and degree of agreement with respect to self-confidence and self-esteem

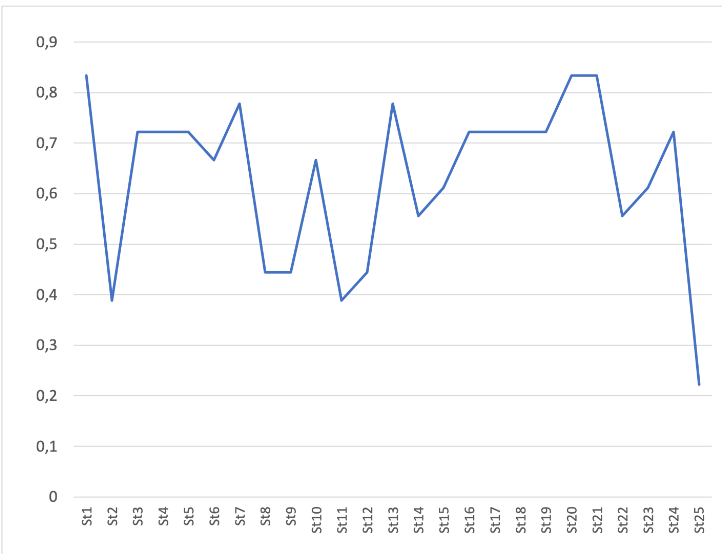

FIGURE 5.3: Plot of consensus values for self-confidence and self-esteem

Student id	centroid a_2	Degree of agreement a_2
St1	$[s_3^*, s_9^*]$	0.5
St2	$[s_4^*, s_5^*]$	0.6111
St3	$[s_8^*, s_9^*]$	0.6111
St4	$[s_5^*, s_7^*]$	0.6111
St5	$[s_2^*, s_3^*]$	0.4444
St6	$[s_6^*, s_7^*]$	0.7778
St7	$[s_5^*, s_7^*]$	0.5556
St8	$[s_5^*, s_9^*]$	0.3333
St9	$[s_6^*, s_7^*]$	0.3889
St10	$[s_5^*, s_7^*]$	0.6667
St11	$[s_5^*, s_8^*]$	0.4444
St12	$[s_5^*, s_6^*]$	0.6667
St13	$[s_6^*, s_8^*]$	0.5
St14	$[s_6^*, s_7^*]$	0.6667
St15	$[s_5^*, s_6^*]$	0.5
St16	$[s_4^*, s_7^*]$	0.5556
St17	$\{s_5^*\}$	0.5556
St18	$[s_5^*, s_7^*]$	0.7778
St19	$[s_5^*, s_7^*]$	0.6667
St20	$[s_4^*, s_7^*]$	0.6111
St21	$[s_5^*, s_7^*]$	0.7778
St22	$[s_5^*, s_8^*]$	0.5556
St23	$[s_3^*, s_5^*]$	0.3333
St24	$[s_3^*, s_4^*]$	0.7778
St25	$[s_5^*, s_8^*]$	0.6667

TABLE 5.5: Centroids and degree of agreement, for each student, with respect to creativity and originality

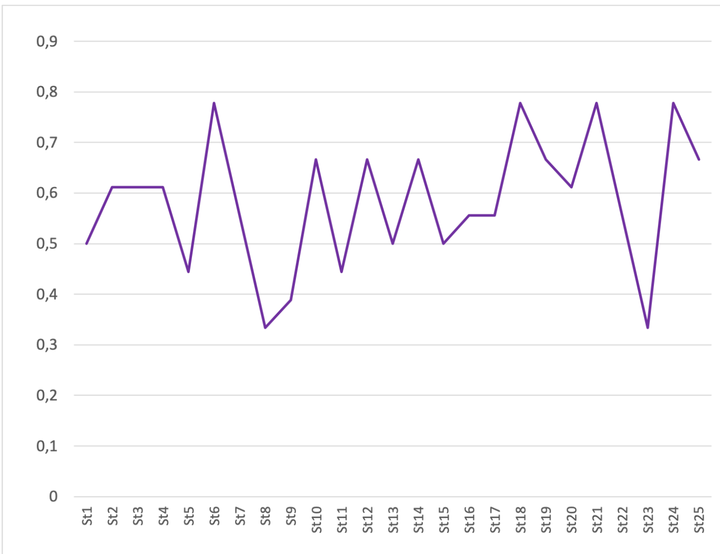

FIGURE 5.4: Plot of consensus values for creativity and originality

Student id	centroid a_3	Degree of agreement a_3
St1	$[s_7^*, s_9^*]$	0.7778
St2	$[s_3^*, s_6^*]$	0.6667
St3	$[s_7^*, s_9^*]$	0.8333
St4	$[s_7^*, s_9^*]$	0.6667
St5	$[s_2^*, s_5^*]$	0.5556
St6	$[s_7^*, s_9^*]$	0.7778
St7	$[s_5^*, s_7^*]$	0.7778
St8	$[s_6^*, s_9^*]$	0.6667
St9	$[s_5^*, s_7^*]$	0.5556
St10	$[s_6^*, s_9^*]$	0.7778
St11	$[s_7^*, s_9^*]$	0.6667
St12	$[s_5^*, s_9^*]$	0.7778
St13	$[s_6^*, s_9^*]$	0.8333
St14	$[s_6^*, s_8^*]$	0.6667
St15	$[s_6^*, s_9^*]$	0.7778
St16	$[s_5^*, s_8^*]$	0.6667
St17	$[s_3^*, s_8^*]$	0.6667
St18	$[s_6^*, s_9^*]$	0.7778
St19	$[s_5^*, s_8^*]$	0.7222
St20	$[s_3^*, s_9^*]$	0.6667
St21	$[s_6^*, s_9^*]$	0.8889
St22	$[s_5^*, s_8^*]$	0.3889
St23	$[s_5^*, s_9^*]$	0.6667
St24	$[s_6^*, s_8^*]$	0.7778
St25	$[s_5^*, s_9^*]$	0.6111

TABLE 5.6: Centroids and degree of agreement, for each student, with respect to leadership influence and relationships

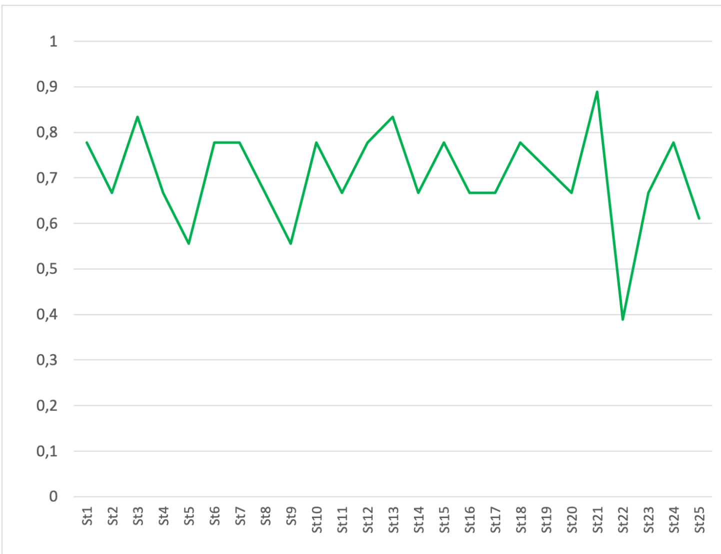

FIGURE 5.5: Plot of consensus values for leadership influence and relationships

Student id	centroid a_4	Degree of agreement a_4
St1	$[s_5^*, s_7^*]$	0.6667
St2	$\{s_5^*\}$	0.5556
St3	$[s_6^*, s_9^*]$	0.5556
St4	$[s_6^*, s_9^*]$	0.6667
St5	$[s_2^*, s_4^*]$	0.6111
St6	$[s_7^*, s_9^*]$	0.7222
St7	$[s_5^*, s_8^*]$	0.6667
St8	$[s_7^*, s_9^*]$	0.5
St9	$[s_3^*, s_5^*]$	0.2778
St10	$[s_6^*, s_7^*]$	0.5556
St11	$[s_3^*, s_8^*]$	0.4444
St12	$\{s_5^*\}$	0.3333
St13	$[s_5^*, s_8^*]$	0.6667
St14	$[s_5^*, s_7^*]$	0.7778
St15	$[s_5^*, s_6^*]$	0.5
St16	$\{s_5^*\}$	0.8889
St17	$[s_5^*, s_7^*]$	0.7222
St18	$[s_7^*, s_8^*]$	0.6667
St19	$[s_5^*, s_7^*]$	0.7778
St20	$[s_5^*, s_7^*]$	0.6667
St21	$[s_5^*, s_8^*]$	0.7222
St22	$[s_6^*, s_7^*]$	0.5556
St23	$[s_5^*, s_9^*]$	0.7778
St24	$[s_5^*, s_9^*]$	0.5556
St25	$[s_6^*, s_9^*]$	0.7222

TABLE 5.7: Centroids and degree of agreement, for each student, with respect to opportunity recognition and exploitation

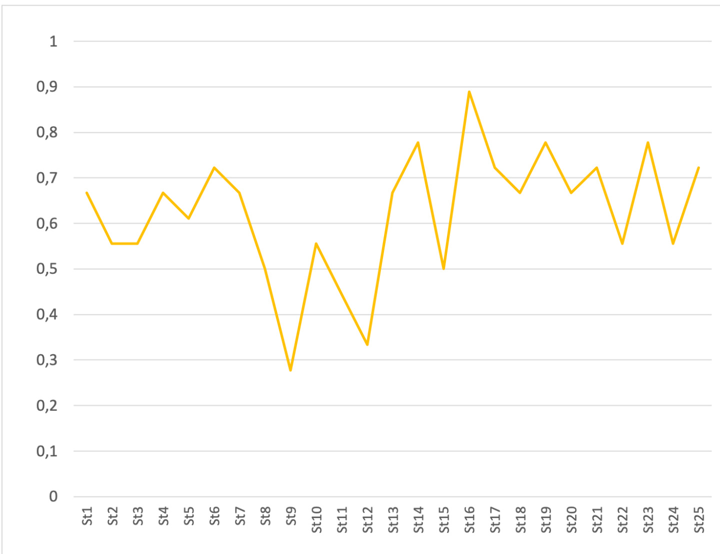

FIGURE 5.6: Plot of consensus values for opportunity recognition and exploitation

Student id	centroid a_5	Degree of agreement a_5
St1	$[s_6^*, s_7^*]$	0.5556
St2	$[s_3^*, s_4^*]$	0.7222
St3	$[s_7^*, s_9^*]$	0.7222
St4	$[s_7^*, s_9^*]$	0.8333
St5	$[s_4^*, s_5^*]$	0.6111
St6	$[s_6^*, s_9^*]$	0.7778
St7	$[s_6^*, s_7^*]$	0.7778
St8	$[s_5^*, s_9^*]$	0.5
St9	$[s_5^*, s_9^*]$	0.6111
St10	$[s_6^*, s_7^*]$	0.5556
St11	$[s_7^*, s_9^*]$	0.6111
St12	$[s_5^*, s_6^*]$	0.6111
St13	$[s_5^*, s_7^*]$	0.6667
St14	$[s_5^*, s_8^*]$	0.6667
St15	$[s_5^*, s_9^*]$	0.5556
St16	$[s_3^*, s_6^*]$	0.8333
St17	$\{s_5^*\}$	0.5556
St18	$[s_7^*, s_8^*]$	0.7778
St19	$[s_5^*, s_7^*]$	0.6667
St20	$[s_6^*, s_8^*]$	0.6111
St21	$[s_6^*, s_7^*]$	0.6667
St22	$[s_5^*, s_9^*]$	0.6111
St23	$[s_4^*, s_6^*]$	0.6111
St24	$[s_5^*, s_7^*]$	0.8333
St25	$[s_7^*, s_9^*]$	0.7222

TABLE 5.8: Centroids and degree of agreement, for each student, with respect to planning and organizing

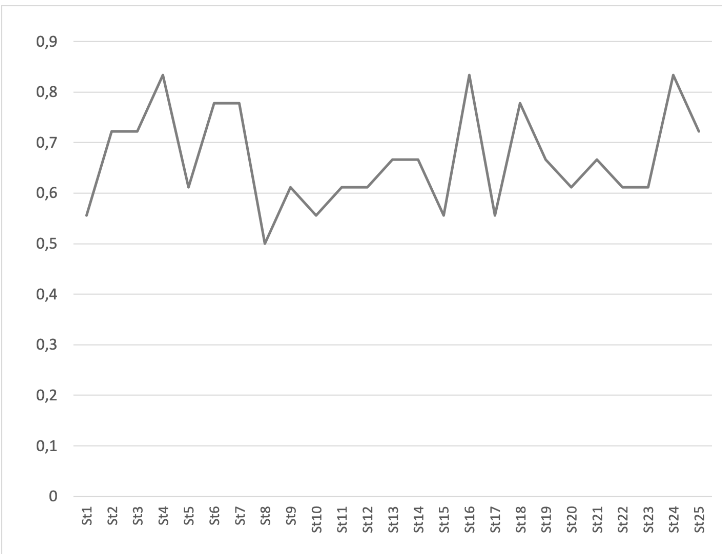

FIGURE 5.7: Plot of consensus values for planning and organizing

Student id	centroid a_6	Degree of agreement a_6
St1	$[s_6^*, s_9^*]$	0.6111
St2	$[s_3^*, s_5^*]$	0.5
St3	$[s_8^*, s_9^*]$	0.7222
St4	$[s_7^*, s_9^*]$	0.8333
St5	$[s_4^*, s_5^*]$	0.6111
St6	$[s_6^*, s_9^*]$	0.7222
St7	$[s_5^*, s_8^*]$	0.6667
St8	$[s_5^*, s_9^*]$	0.6111
St9	$[s_5^*, s_7^*]$	0.5
St10	$[s_6^*, s_9^*]$	0.6111
St11	$[s_7^*, s_9^*]$	0.6667
St12	$[s_5^*, s_8^*]$	0.5556
St13	$[s_5^*, s_6^*]$	0.8333
St14	$[s_6^*, s_8^*]$	0.6667
St15	$[s_6^*, s_9^*]$	0.7778
St16	$[s_4^*, s_5^*]$	0.6667
St17	$[s_5^*, s_6^*]$	0.8333
St18	$[s_7^*, s_8^*]$	0.7778
St19	$[s_5^*, s_8^*]$	0.6667
St20	$[s_6^*, s_9^*]$	0.8333
St21	$[s_6^*, s_9^*]$	0.7222
St22	$[s_7^*, s_9^*]$	0.7778
St23	$[s_4^*, s_7^*]$	0.6667
St24	$[s_5^*, s_9^*]$	0.6667
St25	$[s_7^*, s_9^*]$	0.8333

TABLE 5.9: Centroids and degree of agreement, for each student, with respect to Achievement orientation

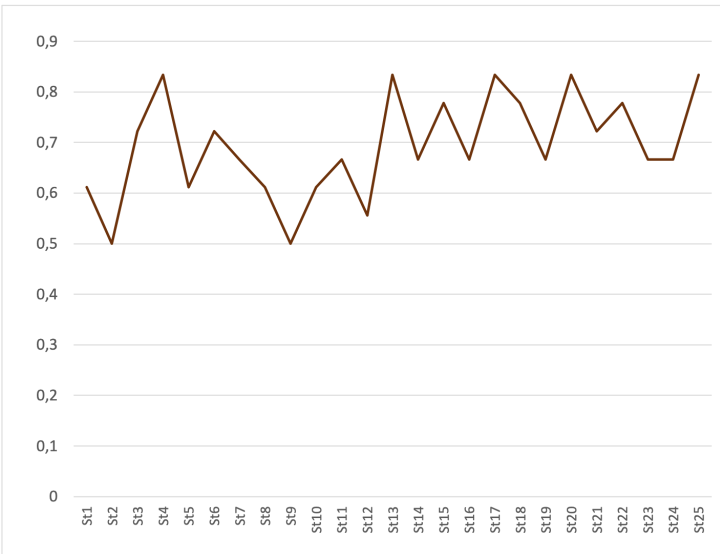

FIGURE 5.8: Plot of consensus values for achievement orientation

Then, in parallel, depending on the MAGDM purpose, we proceed with step 5a for classification and step 5b for ranking, as follows:

5.2.5 Step 5a: Classification

For a classification purpose, the representative HFLD of each category type have to be identified. The six entrepreneurial attributes were grouped in pairs according to their similarities. For instance, students can be trained to be better at recognizing opportunities and this pattern recognition perspective is linked to alertness to opportunities which in turn, is related to another aspect of cognition, the creativity [14]. This implies that these two attributes, opportunity recognition and creativity, are related and can be taught simultaneously, with similar teaching methods and materials. Similarly, self-confidence and self-esteem is paired with achievement orientation and leadership influence and relationships is trained along with planning and organizing. As a result, a set $X = \{A, B, C\}$ of three entrepreneurial categories were identified. Category A is characterized by a low level of self-esteem and achievement orientation, category B is poor in creativity and opportunity recognition skills and the third type C demonstrate few abilities of leadership and planning. Three representative 6-dimensional HFLD, F_q^*, $(q = A, B, C)$, were build as prototype vectors representing each category. These are defined as follows:

- F_A^*=$([s_1^*, s_2^*], \{s_5^*\}, \{s_5^*\}, \{s_5^*\}, \{s_5^*\}, [s_1^*, s_2^*])$
- F_B^*=$(\{s_5^*\}, [s_1^*, s_2^*], \{s_5^*\}, [s_1^*, s_2^*], \{s_5^*\}, \{s_5^*\})$
- F_C^*=$(\{s_5^*\}, \{s_5^*\}, [s_1^*, s_2^*], \{s_5^*\}, [s_1^*, s_2^*], \{s_5^*\})$

Notice that $[s_1^*, s_2^*]$ is the projected centroid of the lowest evaluations given by the auto-avaluation profile, the parent and the teacher, i.e., the centroid of $(\{s_1\}, \{s_1\}, \{s_1\})$, which, in the projected lattice, is the centroid of $([s_1^*, s_2^*], \{s_1^*\}, [s_1^*, s_3^*])$. Similarly, the $\{s_5^*\}$ is the centroid of the midterm evalutaions given by each profile, i.e., the centroid of $(\{s_3\}, \{s_3\}, \{s_3\})$, which, in the projected lattice, is the centroid of $(\{s_5^*\}, [s_3^*, s_4^*], [s_6^*, s_7^*])$.

Following with the proposed method, for each St_i, three distances, using equation 3.5 are computed, i.e., $D_{\mu_*}^{\mathcal{F}}(F_i^*, F_A^*)$, $D_{\mu_*}^{\mathcal{F}}(F_i^*, F_B^*)$ and $D_{\mu_*}^{\mathcal{F}}(F_i^*, F_C^*)$. The weighting vector is formed by the normalization of $\delta_{(\lambda_i^l)}$. The resulting distances to each entrepreneurial category are provided in Table 5.10 along with the assigned category.

5.2.6 Step 5b: Ranking

For a ranking purpose, the representative best HFLD has to be identified. In this case, the best 6-dimensional HFLD, denoted as F_m^*, is build with the projection of the best linguistic labels given to all six attributes, with no hesitancy. This results in F_m^*=$([s_8^*, s_9^*], [s_8^*, s_9^*], [s_8^*, s_9^*], [s_8^*, s_9^*], [s_8^*, s_9^*], [s_8^*, s_9^*])$. Notice that each component, $[s_8^*, s_9^*]$, is the projected centroid of the highest evaluations given by the auto-avaluation profile, the parent and the teacher, i.e., the centroid of $(\{s_5\}, \{s_5\}, \{s_5\})$, which, in the projected lattice, is the centroid of $([s_8^*, s_9^*], [s_7^*, s_9^*], \{s_9^*\})$. Following the proposed method and using equation 3.5, for each St_i, the distance $D_{\mu_*}^{\mathcal{F}}(F_i^*, F_m^*)$ is computed. The weighting vector is formed by the normalization of $\delta_{(\lambda_i^l)}$. The resulting distances are provided in Table 5.11 along with the resulting ranking.

Student id	Distance to F_A^*	Distance to F_B^*	Distance to F_C^*	Category
St1	0.838	0.731	0.8155	B
St2	0.3258	0.4355	0.2774	C
St3	0.996	0.9427	1.0173	B
St4	0.9013	0.85	0.891	B
St5	0.4875	0.1781	0.3969	B
St6	0.8488	0.8688	0.8788	A
St7	0.5237	0.5184	0.5816	B
St8	0.7255	0.6673	0.7545	B
St9	0.544	0.488	0.672	B
St10	0.7043	0.6928	0.7159	B
St11	0.8793	0.7828	0.9345	B
St12	0.4279	0.4803	0.5721	A
St13	0.6649	0.5818	0.172	B
St14	0.6222	0.6667	0.174	A
St15	0.6851	0.6015	0.6731	B
St16	0.4167	0.4321	0.391	C
St17	0.4808	0.4973	0.4205	C
St18	0.7975	0.7877	0.8074	B
St19	0.5474	0.511	0.5474	A,C
St20	0.6961	0.124	0.5592	C
St21	0.7446	0.7349	0.7446	B
St22	0.7823	0.7306	0.7048	C
St23	0.5727	0.24	0.5758	B
St24	0.5692	0.4487	0.166	B
St25	0.7559	0.85	0.8382	A

TABLE 5.10: Distance of each St_i to the three entrepreneurial styles, based on equation 3.5 and assigned category.

5.3 Results and comparative analysis

According to the results from the classification problem, shown in Table 5.10, 5 students are classified in category A, 15 students in category B and 6 in category C. With this approach, only one student (Student $St19$) is classified simultaneously in two categories as the corresponding distances to category A and C are equal. This means that this student, for teaching purposes, could be considered as both profiles, low in self-esteem and achievement orientation or low in leadership and planning skills. The rest can be assigned to specific classes. According to the results from the ranking problem, shown in Table 5.11, $St3$, $St4$ and $St6$ are classified as the first, second and third respectively. Their corresponding distances to F_m^* are the lowest among the class. In contrast, $St16$, $St2$ and $St5$ are ranked in the last three positions.

With the proposed methodology, the aforementioned results are influenced by the use of the degree con consensus as a weighting factor as well as the considered perceptual maps. For instance, results are affected by the fact that, in creativity, the average degree of consensus is 0.5822, which is the lowest value in contrast with leadership whose average degree of agreement is above 0.70. This means that, on average, the computed centroids for leadership have a higher influence on the distances calculations as compared to creativity centroids. On average, each student

Student id	Distance to F_m^*	Ranking
St1	0.3056	7
St2	0.9612	24
St3	0.08133	1
St4	0.1858	2
St5	1.0625	25
St6	0.2012	3
St7	0.55	17
St8	0.3509	10
St9	0.568	18
St10	0.3623	11
St11	0.2379	4
St12	0.6081	20
St13	0.4363	14
St14	0.4222	13
St15	0.4134	12
St16	0.7423	23
St17	0.6917	22
St18	0.2691	6
St19	0.5157	16
St20	0.5118	15
St21	0.3253	8
St22	0.3274	9
St23	0.5757	19
St24	0.6307	21
St25	0.2676	5

TABLE 5.11: Distance of each St_i to the best HFLD, based on equation 3.5 and final ranking.

obtained, a degree of agreement of 0.6504 among all six attributes, which considering the imprecise and ambiguous nature of entrepreneurship, it is a very reasonable value to get conclusions. Similarly, the use of different perceptual maps for each evaluation profile has a direct influence on the resulting projected partition and hence, the resulting centroids which configure the 6-dimensional HFLD for each student.

In order to further analyze the results obtained in the pilot test and demonstrate the necessity of introducing the perceptual maps and the consensus weights in the MAGDM proposed method, some calculations and comparisons are provided. Firstly, calculations have been done, with the unbalanced approach, but without considering the degrees of consensus as a weighing factor. Hence, each H_i^{Cl*} has the same weight for all $l \in \{1,2,3,4,5,6\}$ and $i \in \{1,2,\ldots,25\}$. Secondly, we have computed the same exact steps considering the same reasoning and background for the three profiles. This means, considering the same balanced LTS of granularity 5 and the same perceptual map, i.e., $\mu_a(s_i) = \mu_p(s_i) = \mu_t(s_i) = \mu(s_i) = 0.2$ for all $i = 1,2,\ldots,5$. With respect to the classification problem, category assignments based on these two comparative approaches are shown in Table 5.12. As can be seen, the other two approaches does not work so well in discriminating students between the three categories. With respect to the ranking problem, the different classifications are shown in Table 5.13. As compared to the algorithm with equal weighting, the ranking obtained with the proposed method clearly works better in determining a precise classification. For example, $St6$ and $St4$ are equally placed in the second position as their distances to F_m^* resulted in the same value. With this third approach, a total of 6 pairs of students are ranked in the same position.

Student id	Proposed method	Balanced μ	Algorithm with equal weighting
St1	B	B	B
St2	C	C	C
St3	B	B	A,B,C
St4	B	C	A,B,C
St5	B	C	B
St6	A	B	A,B,C
St7	B	A	A
St8	B	B	A,B,C
St9	B	B	A,B
St10	B	A	A,B,C
St11	B	B	B
St12	A	A	A
St13	B	B	A,B,C
St14	A	A,B	A,B,C
St15	B	B	A,B,C
St16	C	B	C
St17	C	C	A,C
St18	B	B	A,B,C
St19	A,C	A,B	A,B,C
St20	C	A	C
St21	B	B	A,B,C
St22	C	A,B,C	A,B,C
St23	B	B	B
St24	B	B	B
St25	A	A	A

TABLE 5.12: Assignments comparison with balanced LTS and with no weighting

5.4 Conclusions

This case study contributes to further develop MAGDM methods for ranking and classification that can simultaneously deal with hesitant unbalanced and multi-granular linguistic information. The proposed methodology improves existing approaches [28, 76, 122, 164] since it can accommodate both balanced and unbalanced LTSs as well as model multi-granularity linguistic information, allowing each DM to choose his or her preferred LTS. For instance, in the recently developed multi-perspective MADM in [37] the questionnaire used to collect pairwise comparisons of the attributes, expressed by means of generalized comparative linguistic expressions based on HFLTSs requires the use of the same LTS with granularity 9 for all respondents. Also, in [39], even if the background of the five experts considered with

Student id	Proposed method	Balanced μ	Algorithm with equal weighting
St1	7	6	5
St2	24	24	18
St3	1	1	1
St4	2	2	2
St5	25	25	19
St6	3	3	2
St7	17	20	12
St8	10	5	7
St9	18	13	13
St10	11	10	7
St11	4	7	3
St12	20	17	15
St13	14	15	9
St14	13	14	8
St15	12	8	10
St16	23	23	17
St17	22	22	16
St18	6	11	3
St19	16	19	11
St20	15	16	11
St21	8	9	5
St22	9	12	6
St23	19	21	14
St24	21	18	14
St25	5	4	4

TABLE 5.13: Ranking comparison with balanced LTS and with no weighting

respect to job, education or work experience is different, the online questionnaire is designed so it uses the same set S for all the panel.

In the existing literature, distances and consensus measures in unbalanced linguistic contexts are limited to use the same unbalanced LTS or/and the same granularity [29] and hence, are not capable of capturing the complete heterogeneity of DMs. For instance, as compared to our proposed approach, the proposed geodestic distance in in [29] to model unbalanced linguistic information assumes that the universe of every assessment space is the same and hence, only different granules of this same universe can be considered.

Besides, as compared to other distances between HFLTSs [53, 201], the proposed perceptual-based distance is subscript independent, does not depend on an parameter and takes into account the gap of non-overlapping HFLTSs. In addition, as compared to other existing aggregation operators and consensus measures [111], our proposed approach does not require the use of linguistic preference degrees over pairs of alternatives but hesitant fuzzy linguistic terms to asses each alternative and in practical terms, it requires less time from the evaluators' perspective.

An illustrative example with three different perceptual maps considered for each specific profile of DM and six entrepreneurial attributes is presented. The proposed method can also suitably be used to model MAGDM problems where DMs might have a similar background but the evaluation of alternatives is characterized by attributes with very different nature.

As a future research direction, based on the proposed framework, since the degree of agreement is used as the weighting factor to compute distances, it is planned to analyze how different levels of consensus influence the output of the classification or ranking results. On the other hand, a relevant future research work on the area of multi-attribute group decision aiding is to develop a learning algorithm to identify the most appropriate perceptual map and granularity of the ULTS used by each DM.

In practical terms and considering results from the illustrative example, the method will be applied to a large data set of students. Similarly, the method will also be used to tackle the evaluation of other specific competencies whose different attributes or participating evaluators may require the use of different perceptual maps, such as the evaluation of candidates in a business setting.

Chapter 6

Conclusions and future work directions

6.1 Conclusions

This thesis contributes to further develop mathematical structures to model MAGDM problems under HFLTSs linguistic assessments in a multi-granular and unbalanced context. It is oriented to solve situations in which the different experts or DMs can hesitate and be uncertain when providing their evaluations, are allowed to use their preferred linguistic term set to do so and besides, each of their LTS can be either be balanced or unbalanced. Studying the modelling of these characteristics is of great interest since many real life MAGDM contexts happen under these circumstances. Experts or DM's linguistic evaluations and assessments are very often influenced by their attitudes, experiences, backgrounds and knowledge.

With respect to the linguistic modelling, this thesis is framed in the use of Hesitant Fuzzy Linguistic Term Sets (HFLTSs), which were introduced by Rodríguez et al. in 2012 [142]. In this field, the main theoretical contributions of my thesis are developed in Chapter 3 whereas, Chapters 4 and 5 provides useful insights and conclusions derived from MAGDM real life applications.

Chapter 3 proposes a new theoretical framework to model linguistic MAGDM situations, under the use of unbalanced and multi-granular HFLTSs. For this purpose, the perceptual map μ is defined on the structure of the unbalanced HFLTSs lattice. The properties of a normalized measure are proven. Then, the perceptual map is the basis to develop a perceptual-based distance for unbalanced HFLTSs, a perceptual-based collective consensus measure (based on the previously defined centroid of the group) and a transformation function for multi-perceptual GDM contexts. Finally, using these developed tools, a new ranking and classification MAGDM method as well as an extension of TOPSIS are presented, step by step. These frameworks are then used in the two real life applications.

Therefore, as already presented in the first pages of this thesis, in section 1.3, the first two main contributions of this thesis are the introduction of the concept of perceptual-map over unbalanced HFLTSs and the new distance over the extended lattice of unbalanced HFLTs. This distance is inspired by previous work done by Montserrat-Adell in [121] and it is based on the operator of the *width* and, this *width* is defined as the cardinality of H. This means that this distance works under the assumption of a balanced linguistic context. It is not flexible enough to capture the different semantic content of H. This is a drawback as, according to human common sense, it is not always true that all experts share the same HFLTSs lattice structure, i.e., assign the same semantic meaning to each linguistic label. For instance, when a professor provides a 'Very good' to a students' evaluation, from a LTS of 4 labels, this might contain different semantic information compared to a 'Very good' provided by the father, over the same LTS.

In order to overcome this relevant limitation, the concept of perceptual-map is developed in this thesis as a normalised measure defined over the set of positive HFLTSs. Using the perceptual map, a new distance measure is developed taken into account the possibility of a given unbalanced context. Hence, the introduced distance provides is used to compute the distance between two HFLTSs that are built over an unbalanced lattice of HFLTSs. In addition, Chapter 3 uses this (weighted) perceptual-based distance to provide a new definition of centroid in a MAGDM situation as the HFLD that minimizes the addition of distances to the assessments to all DMs in the group G. As a result, a new degree of agreement of G is developed in the context of unbalanced HFLTSs.

Another relevant contribution of Chapter 3 is the development of a perceptual-based transformation function which is key to simultaneously model multi granularity and unbalanced HFLTS linguistic information. This tool allows to handle situations where, for instance, one expert feels more comfortable to provide his linguistic assessments over an unbalanced LTS of cardinality 4 while another might feel more familiar with an unbalanced LTS of 6 linguistic labels. Thanks to the perceptual-based transformation function defined in Definition 3.14, all linguistic assessments can be projected to the same projected linguistic space. Then, distances and consensus measures can be computed within this projected space.

At this point of the thesis, I consider relevant to offer an extensive comparative summary with respect to other existing soft consensus measures [65] for MAGDM in uncertainty and hesitancy linguistic contexts. This is provided in Table 6.1. As compared to other approaches based on HFLTSs, my perceptual-based degree of agreement entails different levels of granularity and precision among DMs and can accommodate both, balanced and unbalanced LTSs. This demonstrate again the contributions of my thesis to further develop more flexible tools within the field of HFLTSs.

Article	Parreiras et al.[130]	Roselló et al. [145]	Wu and Xu [186]	Zhang et al. [201]	Montserrat-Adell et al. [122]	Hao and Chiclana [76]	Zhang et al. [203]	Proposed approach
Year	2010	2014	2016	2017	2018	2020	2020	2021
Linguistic modeling	Trapezoidal fuzzy numbers	Absolute order-of-magnitude qualitative spaces	HFLPR and 2-tuple model	PLPRs	HFLTSs	HFLTSs	FLPRs	HFLTSs
Dealing with multi-granular LTS	Yes	Yes	No	No	No	No	No	Yes
Type of LTS	Odd, symmetry and balanced	Odd or even, balanced or unbalanced	Odd, symmetry and balanced	Odd, symmetry and balanced or unbalanced	Odd or even, balanced	Odd or even, balanced or unbalanced	Odd, symmetry and balanced	Odd or even, balanced or unbalanced
Degree of consensus measure type	Distance-based	Distance in the metric space defined from the geodesic distance	Consistency indexes and similarity matrices	Distance-based (from Hamming and Euclidean) and similarity degree	Distance (cardinal) in the extended distributive lattice	Similarity degrees	Proximity degree of FLPRs	Perceptual distance-based over the extended lattice.
Requires pairwise comparisons	Yes	No	Yes	Yes	No	No	Yes	No

TABLE 6.1: Comparison of the proposed method of this thesis with other existing frameworks in the literature

In the method provided in Parreiras et al.[130], the linguistic information provided by each expert is given in terms of multi-granular fuzzy estimates which is based on a linguistic hierarchical model. It is true that the choice of the most suitable set is prerogative of each expert but as compared to my approach, the experts are limited to choose the linguistic variables from a given linguistic hierarchy. Hence, if a linguistic hierarchy of trapezoidal linguistic estimates is chosen, then experts in the same MAGDM problem can only choose to evaluate alternatives using for instance, level 1 (containing 3 labels), level 2 (containing 5 labels) or level 3 (containing 9 labels). This means that an expert wouldn't be able to choose from a set of 4 labels if this could reflect more adequately his level uncertainty.

This drawback is improved in the mathematical framework designed in Rosello et al. [145], which allows different sets of ordinal labels to qualify features. Hence, in this method the multi-granularity of linguistic assessments is total, as is the case of my proposed approach. Nonetheless, their method is framed in the use of multi-dimensional qualitative assessments. The work presented in this paper is based on the use of the recently developed HFLTSs. I overcome this limitation with the perceptual-based approach on HFLTSs.

As compared to [122], the perceptual-based proposed collective degree of consensus takes into account the case of unbalanced linguistic term sets. Therefore, the different qualitative reasoning processes made by DMs when expressing their opinions are captured. My methodology adapts to different perceptual maps for different DMs.

As compared to [201], the perceptual-based distance between linguistic assessments provided by experts does not depend on any parameter. The authors in [201], get different distance and similarity degrees based on the different values of parameters λ. Moreover, in contrast with other consensus measures such as in [186] or [203], in my approach, I do not work with linguistic preference degrees over alternatives and in practical terms, my framework requires less time from the evaluators' perspective. The proposed approach introduced in 3.3 is designed so that experts or decision makers have to provide linguistic assessments over alternatives with respect to each attribute or criteria.

HFLTSs information is managed in other frameworks such as in [76]. Hao and Chiclana propose a novel possibility distribution generation method with linguistic quantifier, which allow for different importance values of elements in HFLTS. However, experts' HFLTSs are all based on the same set S. As can be seen in the steps of the method, it requires the use of the exact same linguistic term set in order to aggregate HFLTSs and to calculate the levels of similarity to measure the consensus levels for experts and alternatives. My approach which is also based on the use of HFLTSs, overcomes this limitation.

Last but not least, in Chapters 4 and 5, the developed perceptual-based methodologies are tested and applied in two real MAGDM problems. Preliminary works on these applications have been presented in several conferences as mentioned in section 1.3.

Moreover, part of the results of Chapter 4 have been published in the *Energies* Journal: Porro, O., Pardo-Bosch, F., Agell, N., & Sánchez, M. (2020). Understanding Location Decisions of Energy Multinational Enterprises within the European Smart Cities' Context: An Integrated AHP and Extended Fuzzy Linguistic TOPSIS Method. *Energies*, 13(10), 2415. Besides, the results obtained in Chapter 5 have been submitted (as 'A multi-attribute group decision model based on unbalanced and multi-granular linguistic information: an application to assess entrepreneurial competencies in secondary schools') to *Applied Soft Computing Journal* and are currently under second review.

In summary, the main contributions of this thesis are summarized as follows:

- Development of a new measure over unbalanced HFLTSs, named perceptual-map, and proposals for new definitions: perceptual-based distance, consensus measure and transformation function.
- Development of a multi-granular and multi-perceptual MAGDM where linguistic evaluations are modelled via unbalanced HFLTSs and consensus measures are used as weighting factors.
- Application of a developed perceptual-based TOPSIS for a ranking MAGDM problem in the relevant field of smart cities.
- Application of this new computational linguistic method for assessing students according to their entrepreneurial competency with evaluators having

different qualitative reasoning processes. This contributes to the development of innovative evaluation system, in educational settings, where ambiguity, subjectivity and impreciseness in the evaluations are predominant (as compared to the grading system of languages or mathematics).

6.2 Managerial implications

The use of perceptual-based techniques can have a great influence on any managerial decision based on results from any system, process or algorithm whose input training data is data gathered from human opinions. Here, in this subsection, some relevant managerial implications and recommendations derived from the use of perceptual maps, based on the findings of this thesis, are highlighted.

In customer services of big corporations, it might be interesting to classify or segment clients according to their degree of satisfaction with the product or service offered. In most of the cases, when data is pre-processed and treated, it is assumed that all clients share the same perceptual map. But this does not reflect reality. Clients have different backgrounds, are influenced by different cultures or have different type of personalities. All these factors have an influence on the way and intensity of their opinions. Therefore, if companies seek to correctly classify clients into groups based on linguistic opinions, first it is recommended, prior to train a model and get results, to identify clients who share the same perceptual map and find the projected-space of the whole group.

Another area where perceptual-maps can have an implication is in patient-related data for health-care or medical research studies. For instance, the patient perception of pain is susceptible to be modelled by means of perceptual-maps. Also, the quality of sleep is another interesting linguistic related-data which the incorporation of perceptual-maps would better reflect reality.

The developed theory of perceptual-maps can have a relevant influence on the way recommender systems algorithms are designed and trained. A perceptual-based parameter or set of parameters should be incorporated into the list of hyper parameters and parameters of the machine learning algorithms that are trained with huge amounts of data in order to improve the satisfaction of the recommender system' results.

For instance, let consider the case of the Netflix recommender system. Netflix estimates the likelihood that you will watch a particular movie or series based on several factors. Some of these factors are the preferences and tastes of similar users and the past ratings you have provided for your favorite movies or shows. In our Netflix homepages, this first titles that we see are the ones that results in the higher positions of the resulting ranking of the algorithm.

Nonetheless, to rate a title you are only asked to select a number of starts from one to five. You can't hesitate on your answer, neither you can't choose two or three starts simultaneously. Moreover, regardless of the profile of user, you have to always choose from a set of five starts and use a number to represent your opinion instead of a qualitative label which might be more appropriate to the way we think. Imagine you can 'say' to Netflix: "This movie is extraordinary", "This movie was more boring than the one I saw yesterday", "This series was not bad", and so on. Obviously, when a movie is rated based only on a fix number, the lost of information is high.

In addition to rating individual shows, Netflix offers you the possibility to set your taste preferences by answering "never", "sometimes" or "often" to indicate how often you watch specific programs. In this case, although you can leave some questions in blank, you again are limited to answer from a set of three linguistic term sets and you are not allowed to hesitate. For some users, the elicitation of frequency preferences would be more adequate if five terms are used such as, for instance: "never", "rarely", "sometimes", "often", "very often". Also, in other circumstances, a user might feel more comfortable responding as: "Between rarely and often". This answer, again, is not feasible in the current system of Netflix.

The incorporation of several techniques and tools used in multi-granular and multi-perceptual linguistic fuzzy systems into the Netflix recommender systems would result in a more personalized and intelligent system. If we want to move from soft artificial intelligence systems to hard artificial intelligence systems, multi-granular and multi-perceptual linguistic techniques are of great help.

6.3 Future research directions

The work presented in this thesis contributes to further develop and improve the flexibility of linguistic MAGDM methods. It can be framed in the intersection between the fields of MCDM and Artificial Intelligence. The perceptual map aims to enhance and improve the set of tools that researchers have at their disposal to model linguistic opinions. In addition, it contributes to improve the way the human qualitative reasoning processes are modelled. For reaching better results, MAGDM methods have to incorporate the ability to capture the cognitive style, attitudes or qualitative reasoning processes of each person involved in the process.

In this section, I present several directions of future work that I will follow to continue enhancing the work developed in this thesis.I can divide these directions into two main research areas: theoretical development and applied research. From a mathematical and theoretical point of view, I have identified several lines of future work. Some of them have initially been explored during the elaboration of this thesis.

In relation to the work developed in the first subsections of Chapter 3, the perceptual based distance and degree of consensus will be adapted to deal with extensions of HFLTSs. Therefore, in the short or mid-term, I plan to extend the perceptual-based tools to other forms of HFLTSs such as the extended or proportional HFLTSs. In addition, a consensus reaching process algorithm to aid and assess experts in the process to reach a final solution in an attempt to derive an acceptable group decision is also in my priorities as a future work directions.

As a long-term project, I am very enthusiastic on studying an algorithm to identify the perceptual-map of each DM based on a set of historical data collected from previous decisions. I think that the development of a learning algorithm to identify the most appropriate perceptual map and granularity of the ULTS used by each DM is a relevant future research work on the area of multi-attribute group decision aiding. Actually, I see this research direction closely related to Machine Learning techniques as this will require the use of data and supervised techniques.

From a practical point of view, I am interested in designing a new extension of the TOPSIS method using the perceptual-based distance to compute the closeness coefficient of each alternative. Then, I would use the application of Chapter 4, to test this new TOPSIS version and compare results. It is also in my short-term plans, to apply the developed method to other business settings such as technological companies.

Also, considering results from Chapter 5, the developed method will be applied to a large data set of students. In the frame of the Andorra national project, the whole population of secondary students will be used as input. Similarly, the method will also be tested to evaluate other specific competencies whose different attributes or participating evaluators may require the use of different perceptual maps, such as the evaluation of candidates in a business setting. Based on the proposed framework, since the degree of agreement is used as the weighting factor to compute distances, it is planned to analyse how different levels of consensus influence the output of the classification or ranking results.

I hope this future work directions will provide functional and practical results to all analyst who face a MAGDM problem with unbalanced and multi-granular linguistic information. Besides, I hope that my current and future work will help other researchers in other fields, from machine learning to psychology.

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