POROUS-ELASTIC PLATES: FOURIER VERSUS TYPE III

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ABSTRACT. In this paper we investigate the time decay of the solutions for a thermoelastic plate with voids in the cases when the heat conduction is modeled by the Fourier law and when it is modeled by the type III theory (with and without the inertial term). In all situations we show that, in general, the strong stability holds. In particular, we show slow decay of solutions for the Fourier case, that is, the solutions do not decay exponentially to zero (in general). However, if the coefficients satisfy a new relationship involving the inertial coefficient (singular case), we characterize the exponential decay of solutions. On the other hand, for the type III theory the situation is very different and we prove that generically the solutions decay to zero exponentially. This is another striking aspect when we compare both theories. This difference is a consequence of the couplings appearing in the type III case which are not present in the case of the Fourier law.

1. INTRODUCTION

The theory of elastic materials with voids (also called porous elasticity) was introduced by Cowin and Nunziato [8, 9, 32] in the second part of the last century. For this theory the materials have two types of structures: on one side at the macrostructure level we have the displacement concerning the elastic part, and on the other side we have the porosity defined by means of the volume fraction. The basic idea is that we have a basic matrix material with holes and the kind of voids is determined by the volume fraction. This theory has deserved much attention in the recent years and the quantity of contributions studying this class of materials is huge ([1, 2, 14, 12, 13, 15, 26, 27, 37], see among others), this is because the wide applicability of these materials in engineering and biology. The two structures composing the solid are coupling in a *weak* sense and we cannot (generically) expect that the dissipation imposed to only a level of the structure is sufficient to bring all the system to an exponential decay of the perturbations. For this reason several kinds of mechanism have been introduced in the study of these materials to clarify its consequences on the time decay of the thermomechanical deformations. In particular thermal and microthermal effects have been considered [5, 6].

The most usual constitutive law to define the heat flux for the heat conduction in solids or fluids was proposed by Fourier. In this situation the heat flux vector is proportional to the gradient of temperature. A mathematical consequence of it is that the thermal waves propagate with unbounded speed and therefore the heat spreads instantaneously regardless of how far the point is from the heat source. This fact contradicts the *causality principle* and as a consequence the Fourier law has received different criticism. Several authors have tried to propose alternative laws for the heat flux that were free from this paradox. The most known alternative law is the one proposed by Cattaneo and Maxwell that introduces a relaxation parameter to the Fourier law. In the last 25 years a big interest has been developed to understand the thermoelastic theories proposed by Green and Naghdi [20, 21, 22]. In these basic contributions the authors proposed three theories that they called type I, II and III respectively. The difference between them correspond to the way as the heat conduction is determined. Type I recovers Fourier's law and the heat flux is proportional to the gradient of temperature. For the type II theory the heat flux is proportional to the gradient of the thermal displacement. This variable had deserved few attention before the works of Green and Naghdi and it can be defined as the time integral of the temperature. Type III theory is the most general because the other two theories can be obtained as limit cases. It is worth recalling that type II theory overcomes the causality paradox, but type III theory falls back into the same problem of the Fourier law. A way to overcome this fact can be the same as Cattaneo and Maxwell proposed in the case of the Fourier law to obtain the so-called Moore-Gibson-Thompson theory for the heat conduction [36].

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In this paper we center our attention in the type I and type III theories. It is worth remarking that in the recent years there has been proved a relevant difference in the time behavior of the solutions of different thermomechanical theories when these two heat conduction theories are present [28]. In particular different decay estimates has been obtained in the case that we consider the elasticity with voids when these two thermal mechanisms are present [30, 29]. Even more, several striking facts have been noted even in the case that we consider type II theory [23, 25, 31]. These new and remarkable effects are consequence of the fact that the type II and type III theories impose *new coupling mechanisms* between the independent variables in the field equations. These couplings are not present in the case of the Fourier law (type I).

In this paper we want to continue with this kind of comparisons and studies. We want to analyze the thermo-porous-elastic plate in the case of type I and type III theories in the cases that inertial term is considered or not. It is worth recalling that the time decay of the solutions of thermoleatic plates has been a topic deeply studied, see for example [3, 7, 16] and references therein. In our case we want to prove that, for a porous-elastic plates with thermal effects given by type I theory, the decay of the solutions is (generically) slow. That is, the solutions are not controlled by any exponential. However for the type III theory we will prove that generically the solutions decay in an exponential way. Therefore, we obtain again a difference in the behavior of the solutions depending of the kind of heat conduction theory we select. Being more specific, we will prove that the thermo-porous-elastic system in the case of type I thermal effects (Fourier law) will be exponentially stable if and only if a specific relationship between some constants of the system holds. This relationship involves directly the presence of the inertial rotational constant α which means that $\alpha \geq 0$ plays an important role in the characterization of exponential and non-exponential stability, see Remarks 2.3 and 2.9. On the other hand, considering the thermo-porous-elastic system in the case of type III's thermal effects, we prove that the exponential stability result does not depend on $\alpha > 0$ or $\alpha = 0$. Additionally, strong stability conditions are formulated for both cases.

The organization of the paper is given as follows. In Section 2 we study well-posedness and stability results for the thermo-porous-elastic system with type I thermal effects. Similar results are formulated for the thermo-porous-elastic system with type III thermal effects.

2. Fourier's Thermal Effects

From now on we denote by Ω a two dimensional domain with boundary smooth enough to apply the divergence theorem and compactness embeddings. We start considering thermal effects given by Fourier's law. In this case the system can be written as

$$
\begin{aligned}
\rho u_{tt} - \alpha \Delta u_{tt} + \mu \Delta^2 u - d\Delta\theta - \gamma\Delta\varphi &= 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+ \\
J\varphi_{tt} - b\Delta\varphi + \xi\varphi - n\theta - \gamma\Delta u &= 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+ \\
a^*\theta_t - k^*\Delta\theta + n\varphi_t + d\Delta u_t &= 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+,
\end{aligned}
\tag{1}
$$

with boundary conditions

$$
\begin{aligned}
u(x,t) = \Delta u(x,t) &= 0 \quad \text{in} \quad \partial\Omega \times \mathbb{R}^+ \\
\varphi(x,t) = \theta(x,t) &= 0 \quad \text{in} \quad \partial\Omega \times \mathbb{R}^+,
\end{aligned}
\tag{2}
$$

and initial conditions

$$
\begin{aligned}
u(\cdot,0) = u_0(\cdot) \quad &, \quad u_t(\cdot,0) = u_1(\cdot) \quad \text{in} \quad \Omega \\
\varphi(\cdot,0) = \varphi_0(\cdot) \quad &, \quad \varphi_t(\cdot,0) = \varphi_1(\cdot) \quad \text{in} \quad \Omega \\
\theta(\cdot,0) &= \theta_0(\cdot) \quad \text{in} \quad \Omega.
\end{aligned}
\tag{3}
$$

Here, the hypotheses on the constants are the following,

$$
\mu > 0, \quad \xi\mu - \gamma^2 > 0, \quad b > 0, \quad \rho, J, a^*, k^* > 0, \quad d \neq 0, \quad n, \gamma \in \mathbb{R}
\tag{4}
$$

with $\alpha \geq 0$. Conditions (4) are motivated to guarantee that we can define an inner product in the Hilbert space $\mathcal{M}_\alpha$ to be defined shortly by means of the functional considered later. In system (1), u describes the displacement, φ the volume fraction and θ the temperature.

Remark 2.1. *Let us mention some comments about the case $d < 0$ in conditions (4). Physically, it is usual to accept that the temperature generates dilatation of the elastic material implying that the corresponding parameters should be positive. In this sense the mathematical analysis could be worked in the same way, this is, assuming $d \geq 0$. On the other side: Is it possible that, for certain kind of materials, the consequences of*

the thermal effects would be the contraction of the material? For this reason we believe that it is suitable to considerer the case $d < 0$. Similar comments can be applied to the parameters n or γ.

In this section we will characterize the exponential stability of system (1)-(3). In fact, the main result of this section is given by the following theorem.

Theorem 2.2. *For damped solutions, system (1)-(3) is exponentially stable if and only if*

$$\mu - \frac{\alpha b}{J} = 0 \quad and \quad \gamma \neq 0. \tag{5}$$

Remark 2.3. *The first condition of (5) is relative new because it does not involve the coefficient $\rho > 0$, which is different when compared with other type of porous-elastic systems with thermal effects (see for example [33]) or even when we compare condition (5) with similar conditions used to stabilize Timoshenko systems with or without thermal effects, see for example [10, 18, 19, 38].*

Remark 2.4. *In Theorem 2.2, we define "damped solutions" as the solutions which are strongly stable, this is when the associated energy $E(t)$ goes to zero when t goes to infinity. So, in the context of linear semigroups, in order to obtain "damped solutions", it is sufficient to show that $i\mathbb{R} \subset \rho(\mathcal{A})$, where $\mathcal{A}$ is the infinitesimal generator to the semigroup associated to the system.*

In order to prove Theorem 2.2, we use semigroup techniques dividing our analysis in the next subsections, each situation associated to $\alpha > 0$ or $\alpha = 0$.

2.1. **Well-posedness.** For the well-posedness we define the Hilbert spaces

$$\mathcal{M}_0 := [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega), \quad (\alpha = 0)$$

$$\mathcal{M}_\alpha := [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega), \quad (\alpha > 0)$$

with inner products for $U_i = (u_i, v_i, \varphi_i, \phi_i, \theta_i)^T$, $i = 1, 2$ given by

$$(U_1, U_2)_{\mathcal{M}_\alpha} := \mu(\Delta u_1, \Delta u_2)_{L^2} + \rho(v_1, v_2)_{L^2} + \alpha(\nabla v_1, \nabla v_2)_{L^2} + b(\nabla \varphi_1, \nabla \varphi_2)_{L^2} + \xi(\varphi_1, \varphi_2)_{L^2}$$

$$+ J(\phi_1, \phi_2)_{L^2} + a^*(\theta_1, \theta_2)_{L^2} - \gamma(\varphi_1, \Delta u_2)_{L^2} - \gamma(\Delta u_1, \varphi_2)_{L^2},$$

which implies the norms, for $U = (u, v, \varphi, \phi, \theta)^T$

$$||U||^2_{\mathcal{M}_\alpha} = \mu||\Delta u||^2_{L^2} + \rho||v||^2_{L^2} + \alpha||\nabla v||^2_{L^2} + b||\nabla \varphi||^2_{L^2} + \xi||\varphi||^2_{L^2} + J||\phi||^2_{L^2} + a^*||\theta||^2_{L^2}$$

$$- 2\gamma \text{Re}(\varphi, \Delta u)_{L^2}.$$

Additionally, let $\mathcal{B}_\alpha$ be the associated operators

$$\mathcal{B}_\alpha \begin{pmatrix} u \\ v \\ \varphi \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{\rho}(I - \frac{\alpha}{\rho}\Delta)^{-1}\Delta(-\mu\Delta u + d\theta + \gamma\varphi) \\ \phi \\ \frac{1}{J}[b\Delta\varphi - \xi\varphi + n\theta + \gamma\Delta u] \\ \frac{1}{a^*}[k^*\Delta\theta - \phi - \Delta v] \end{pmatrix}, \tag{6}$$

with respective domains

$$D(\mathcal{B}_0) = \left\{ (u, v, \varphi, \phi, \theta)^T \in \mathcal{M}_0 \; : \; \Delta u, \, v, \, \varphi, \, \theta \in H^2(\Omega) \cap H_0^1(\Omega); \; \phi \in H_0^1(\Omega) \right\},$$

$$D(\mathcal{B}_\alpha) = \left\{ (u, v, \varphi, \phi, \theta)^T \in \mathcal{M}_\alpha \; : \; v, \, \varphi, \, \theta \in H^2(\Omega) \cap H_0^1(\Omega); \; \Delta u, \, \phi \in H_0^1(\Omega) \right\},$$

which are defined for $\alpha = 0$ and $\alpha > 0$, respectively. In this context, the operators $\mathcal{B}_\alpha$ associated to system (1)-(3) when $\alpha \geq 0$ are infinitesimal generators of C_0-semigroups on the Hilbert spaces $\mathcal{M}_\alpha$, respectively. In fact, for the proof it is sufficient to see that $\overline{D(\mathcal{B}_\alpha)} = \mathcal{M}_\alpha$, $\mathcal{B}_\alpha$ are dissipative and $0 \in \varrho(\mathcal{B}_\alpha)$, for all $\alpha \geq 0$, see [24, 34]. Note that the density results are standard. Also, by straightforward calculations we have

$$\text{Re}(\mathcal{B}_\alpha U, U)_{\mathcal{M}_\alpha} = -k^*||\nabla\theta||^2_{L^2}, \qquad \forall U = (u, v, \varphi, \phi, \theta)^T \in \mathcal{M}_\alpha,$$

implying that the operators $\mathcal{B}_\alpha$ are dissipative. Additionally, in order to show that $0 \in \varrho(\mathcal{B}_\alpha)$, let $F = (f_1, f_2, \cdots, f_5) \in \mathcal{M}_\alpha$, then the equation

$$-\mathcal{B}_\alpha U = F \quad \text{in} \quad \mathcal{M}_\alpha$$

implies $v = -f_1 \in H^2 \cap H^1_0$, $\phi = -f_3 \in H^1_0$, $\theta = \frac{1}{k^*}(-\Delta)^{-1}(a^* f_5 + n f_3 + d\Delta f_1) \in H^2 \cap H^1_0$ and (u, φ) should satisfy

$$\Delta(\mu\Delta u - \gamma\varphi) \;=\; g_1 \quad \text{in} \quad H^{-1} \qquad (\text{or } L^2, \text{ if } \alpha = 0)$$

$$-b\Delta\varphi + \xi\varphi - \gamma\Delta u \;=\; g_2 \quad \text{in} \quad L_2,$$

where

$$g_1 := \rho f_2 - \alpha\Delta f_2 - \frac{d}{k^*}(a^* f_5 + n f_3 + d\Delta f_1) \quad \text{and} \quad g_2 := J f_4 + \frac{n}{k^*}(-\Delta)^{-1}(a^* f_5 + n f_3 + d\Delta f_1).$$

So, using conditions (4) and the Lax-Milgram Theorem we obtain a unique solution $(u, \varphi) \in [H^2 \cap H^1_0] \times H^1_0$ satisfying the conditions of $D(\mathcal{B}_\alpha)$, for each $\alpha \geq 0$. Then $\mathcal{B}_\alpha$ is bijective. Also, working with the components of $\mathcal{B}_\alpha U = F$, it is not difficult to prove that $\|U\|_{\mathcal{M}_\alpha} \leq C\|F\|_{\mathcal{M}_\alpha}$, which implies that $\mathcal{B}_\alpha^{-1}$ is bounded. Therefore $0 \in \varrho(\mathcal{B}_\alpha)$ for all $\alpha \geq 0$, which completes the proof that the operators $\mathcal{B}_\alpha$ are infinitesimal generators of C_0-semigroups on the Hilbert spaces $\mathcal{M}_\alpha$, respectively. So the associated systems (1)-(3) are well-posed.

Before to prove the characterization of exponential stability we finish this section showing that, under suitable conditions, system (1)-(3) is strongly stable. For this purpose, we define ν_j as the eigenvalues of the Laplacian operator with Dirichlet boundary conditions in $L^2(\Omega)$ and e_j their corresponding eigenfunctions, for each $i \in \mathbb{N}$, this is

$$-\Delta e_j = \nu_j e_j \quad \text{with} \quad \nu_j \to +\infty \;\; (j \to +\infty), \tag{7}$$

where $e_0 := \|e_j\|_{L^2} < \infty$ is a constant, for all $j \in \mathbb{N}$. Now, let us start by showing a characterization of the set $i\mathbb{R} \cap \sigma(\mathcal{B}_a)$.

Lemma 2.5. *Let $n \neq 0$ and ν_j, e_j defined in (7). Then*

$$i\mathbb{R} \cap \sigma(\mathcal{B}_\alpha) \neq \emptyset \qquad \Longrightarrow \qquad \mu + \frac{\gamma d}{n} > 0 \quad \text{and} \quad P(\nu_j) = 0 \quad \text{for some} \quad j \in \mathbb{N},$$

where

$$P(X) := \left[J\left(\mu + \frac{\gamma d}{n}\right) - \alpha b \right] X^2 - \left[\rho b + \alpha\left(\xi + \frac{\gamma n}{d}\right) \right] X - \rho\left(\xi + \frac{\gamma n}{d}\right).$$

Proof. Assuming that $\lambda \in i\mathbb{R} \cap \sigma(\mathcal{B}_\alpha)$, then $\lambda \neq 0$ because, from the well-posedness, we have $0 \in \varrho(\mathcal{B}_\alpha)$. Then, by the definition of $D(\mathcal{B}_\alpha)$ and using appropriately the compact embedding $H^s(\Omega) \hookrightarrow H^{s-\epsilon}(\Omega)$, $\forall s \in \mathbb{R}, \forall \varepsilon > 0$, we have the compact embedding $D(\mathcal{B}_\alpha) \hookrightarrow \mathcal{M}_\alpha$. So, we have that $\mathcal{B}_\alpha$ has compact resolvent and thus $\sigma(\mathcal{B}_\alpha) = \sigma_p(\mathcal{B}_\alpha)$, see [11]. Therefore $i\lambda$ is an eigenvalue of $\mathcal{B}_a$, which implies that there exists $U \in D(\mathcal{B}_\alpha)$, $U \neq 0$, satisfying the resolvent equation

$$i\lambda U - \mathcal{B}_\alpha U = 0 \quad \text{in} \quad \mathcal{M}_\alpha.$$

So, multiplying this equation by $U \in \mathcal{M}_\alpha$ and taking the real part, we can deduce that

$$-k^* \|\nabla\theta\|^2_{L^2} = \text{Re}(\mathcal{B}_\alpha U, U)_{\mathcal{M}_\alpha} = 0 \qquad \text{where} \qquad U = (u, v, \varphi, \phi, \theta)^T,$$

which implies $\theta \equiv 0$. So substituting in the resolvent equation, results

$$i\lambda u - v \;=\; 0 \quad \text{in} \quad H^2(\Omega) \cap H^1_0(\Omega)$$

$$i\lambda\rho v + \left(I - \frac{\alpha}{\rho}\Delta\right)^{-1}\Delta(\mu\Delta u - \gamma\varphi) \;=\; 0 \quad \text{in} \quad L^2(\Omega) \qquad (\text{or } H^1_0(\Omega))$$

$$i\lambda\varphi - \phi \;=\; 0 \quad \text{in} \quad H^1_0(\Omega)$$

$$i\lambda J\phi - b\Delta\varphi + \xi\varphi - \gamma\Delta u \;=\; 0 \quad \text{in} \quad L^2(\Omega)$$

$$n\phi + d\Delta v \;=\; 0 \quad \text{in} \quad L^2(\Omega),$$

which implies

$$-\lambda^2 \rho u + \lambda^2 \alpha \Delta u + \mu \Delta^2 u - \gamma \Delta \varphi = 0 \quad \text{in} \quad L^2(\Omega) \qquad (\text{or } H^{-1}(\Omega)) \tag{8}$$

$$-\lambda^2 J \varphi - b \Delta \varphi + \xi \varphi - \gamma \Delta u = 0 \quad \text{in} \quad L^2(\Omega) \tag{9}$$

$$n\varphi + d\Delta u = 0, \quad \text{in} \quad L^2(\Omega). \tag{10}$$

Then, we reduce the existence of $i\lambda \in \sigma(\mathcal{B}_\alpha)$, $\lambda \neq 0$, to the existence of a nontrivial solution of system (8)-(10) which, in particular, can be solved by $u = A_j e_j$ and $\varphi = B_j e_j$, where e_j is defined in (7), with $A_j \neq 0$ and $B_j \neq 0$. In fact, substituting into (8)-(9) and using (10) we obtain

$$\left[-\lambda^2 \left(\rho + \alpha \nu_j \right) + \left(\mu + \frac{\gamma d}{n} \right) \nu_j^2 \right] A_j = 0 \quad \text{and} \quad \left[-\lambda^2 J + b\nu_j + \xi + \frac{\gamma n}{d} \right] B_j = 0,$$

which implies

$$\lambda^2 = \frac{\left(\mu + \dfrac{\gamma d}{n} \right)}{\rho + \alpha \nu_j} \nu_j^2 = \frac{1}{J} \left[b\nu_j + \xi + \frac{\gamma n}{d} \right] > 0.$$

Therefore, the first equality implies $\mu + \dfrac{\gamma d}{n} > 0$ and the second equality implies $P(\nu_j) = 0$ for some $j \in \mathbb{N}$, which completes the proof of the Lemma. $\qquad \square$

Now, using the previous Lemma, the strong stability of system (1)-(3) is given for the following result, for all $\alpha \geq 0$.

Proposition 2.6. *Let $\varrho(\mathcal{B}_\alpha)$ the resolvent set of operator $\mathcal{B}_\alpha$. Therefore, assuming $n = 0$ and $\gamma \neq 0$, then $i\mathbb{R} \subset \varrho(\mathcal{B}_\alpha)$ without any extra condition. On the other hand, for $n \neq 0$, supposing that*

$$\mu + \frac{\gamma d}{n} < 0 \quad \text{or} \quad P(\nu_j) \neq 0 \quad \text{for all} \quad j \in \mathbb{N}, \tag{11}$$

then we have again $i\mathbb{R} \subset \varrho(\mathcal{B}_\alpha)$. This is, system (1)-(3) is strongly stable in both situations.

Proof. By contradiction, let us suppose that $i\mathbb{R} \subset \varrho(\mathcal{B}_\alpha)$ is not true, then there exists $\lambda \in \mathbb{R}$ such that $i\lambda \in \sigma(\mathcal{B}_\alpha)$, with $\lambda \neq 0$, this because $0 \in \varrho(\mathcal{B}_\alpha)$. Then, using the compact embedding $D(\mathcal{B}_\alpha) \hookrightarrow \mathcal{M}_\alpha$, we have that $i\lambda$ is an eigenvalue of $\mathcal{B}_a$. Therefore, there exists $U \in D(\mathcal{B}_a)$, $U \neq 0$, satisfying

$$i\lambda U - \mathcal{B}_\alpha U = 0 \quad \text{in} \quad \mathcal{M}_\alpha.$$

So, multiplying this equation by $U \in \mathcal{M}_\alpha$ and taking the real part, we can deduce that

$$-k^* ||\nabla \theta||_{L^2}^2 = \text{Re}(\mathcal{B}_\alpha U, U)_{\mathcal{M}_\alpha} = 0 \qquad \text{where} \qquad U = (u, v, \varphi, \phi, \theta)^T,$$

which implies $\theta \equiv 0$. So, using this condition in the resolvent equation, we obtain the same system (8)-(10), this is

$$-\lambda^2 \rho u + \lambda^2 \alpha \Delta u + \mu \Delta^2 u - \gamma \Delta \varphi = 0 \quad \text{in} \quad L^2(\Omega) \qquad (\text{or } H^{-1}(\Omega))$$

$$-\lambda^2 J \varphi - b \Delta \varphi + \xi \varphi - \gamma \Delta u = 0 \quad \text{in} \quad L^2(\Omega)$$

$$n\varphi + d\Delta u = 0, \quad \text{in} \quad L^2(\Omega).$$

Therefore, in the case $n = 0$, the third equation implies $u \equiv 0$ which implies $\varphi \equiv 0$ in the first equation because $\gamma \neq 0$ by hypotheses. Consequently we obtain $U \equiv 0$ in $\mathcal{M}_\alpha$, which is a contradiction. On the other hand, for $n \neq 0$ and assuming condition (11), the Lemma 2.5 implies $i\mathbb{R} \cap \sigma(\mathcal{B}_\alpha) = \emptyset$ implying that $i\mathbb{R} \subset \varrho(\mathcal{B}_\alpha)$, which completes the proof of the Proposition. $\qquad \square$

Remark 2.7. *The previous Proposition is used to exclude undamped solutions, which is the particular case when $n = \gamma = 0$. It is important to identify this kind of solutions because they do not appear in classical second order porous-elastic systems, see for example [5, 6]. On the other hand, the study of the behavior of damped solutions, which is the main purpose of this paper, is given in the following Subsection.*

Now, for damped solutions, we will use the following characterization of exponential stable semigroups.

Lemma 2.8. *Let* $T(t) = e^{\mathcal{A}t}$ $(t \geq 0)$, *a semigroup of contractions on the Hilbert space* $\mathcal{H}$ *with generator* $\mathcal{A}$ *and associated norm* $||\cdot||_{\mathcal{H}}$. *Then* $\{T(t)\}_{t\geq 0}$ *is exponentially stable if and only if*

$$i\mathbb{R} \subset \varrho(\mathcal{A}) \tag{12}$$

and

$$\limsup_{|\lambda|\to\infty} ||(i\lambda I - \mathcal{A})^{-1}||_{\mathcal{L}(\mathcal{H},\mathcal{H})} < \infty. \tag{13}$$

Proof. See [24, 35]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.2. **Exponential stability.** In this subsection, excluding undamped solutions, we prove that condition (5) is sufficient to stabilize exponentially system (1)-(3). In fact, assuming that

$$\mu - \frac{\alpha b}{J} = 0 \qquad \text{and} \qquad \gamma \neq 0,$$

we will prove exponential stability of solutions. Here, note that the interesting situation is $\alpha > 0$, because $\alpha = 0$ in the previous condition implies $\mu = 0$ which is not possible. The case $\alpha = 0$ is also included in the next subsection (non-exponential stability).

So, for the case $\alpha > 0$, in order to prove exponential stability, we use Lemma 2.8. In fact, note that condition (12) is a direct consequence of Proposition 2.6. Now, in order to show (13), we will prove that

$$||U||_{\mathcal{M}_\alpha} \leq M||F||_{\mathcal{M}_\alpha},$$

where $F \in \mathcal{M}_\alpha$, with $M > 0$ independent of λ, F, and U is the solution of the resolvent system

$$(i\lambda I - \mathcal{B}_\alpha)U = F, \quad \text{in} \quad \mathcal{M}_\alpha. \tag{14}$$

System (14) can be written, in its components, by

$$i\lambda u - v = f_1 \qquad \text{in} \quad H^2(\Omega) \cap H_0^1(\Omega)$$

$$i\lambda \rho v - (I - \frac{\alpha}{\rho}\Delta)^{-1}\Delta(-\mu\Delta u + d\theta + \gamma\varphi) = \rho f_2 \qquad \text{in} \quad H_0^1(\Omega)$$

$$i\lambda\varphi - \phi = f_3 \qquad \text{in} \quad H_0^1(\Omega)$$

$$i\lambda J\phi - b\Delta\varphi + \xi\varphi - n\theta - \gamma\Delta u = Jf_4 \qquad \text{in} \quad L^2(\Omega)$$

$$i\lambda a^*\theta - k^*\Delta\theta + n\phi + d\Delta v = a^*f_5 \qquad \text{in} \quad L^2(\Omega),$$

which is equivalent to

$$i\lambda u - v = f_1 \qquad \text{in} \quad H^2(\Omega) \cap H_0^1(\Omega) \tag{15}$$

$$i\lambda\rho v - i\lambda\alpha\Delta v + \Delta(\mu\Delta u - d\theta - \gamma\varphi) = \rho f_2 - \alpha\Delta f_2 \qquad \text{in} \quad H^{-1}(\Omega) \tag{16}$$

$$i\lambda\varphi - \phi = f_3 \qquad \text{in} \quad H_0^1(\Omega) \tag{17}$$

$$i\lambda J\phi - b\Delta\varphi + \xi\varphi - n\theta - \gamma\Delta u = Jf_4 \qquad \text{in} \quad L^2(\Omega) \tag{18}$$

$$i\lambda a^*\theta - k^*\Delta\theta + n\phi + d\Delta v = a^*f_5 \qquad \text{in} \quad L^2(\Omega), \tag{19}$$

Now, let us define the multipliers z, y and w as solutions of elliptic equations

$$-\Delta z = v \quad \text{in} \quad \Omega, \qquad z = 0 \quad \text{in} \quad \partial\Omega, \tag{20}$$

$$-\Delta y = \theta \quad \text{in} \quad \Omega, \qquad y = 0 \quad \text{in} \quad \partial\Omega, \tag{21}$$

$$-\Delta w = \phi \quad \text{in} \quad \Omega, \qquad w = 0 \quad \text{in} \quad \partial\Omega. \tag{22}$$

Then, there exists $C > 0$ such that

$$||z||_{H_0^1} \leq C||v||_{L^2}, \tag{23}$$

$$||y||_{H_0^1} \leq C||\theta||_{L^2}, \tag{24}$$

$$||w||_{H_0^1} \leq C||\phi||_{L^2}, \tag{25}$$

hold. Now, multiplying (14) by U in $\mathcal{M}_\alpha$ and taking the real part we have

$$||\nabla\theta||^2_{L^2} \leq C||U||_{\mathcal{M}_\alpha}||F||_{\mathcal{M}_\alpha}. \tag{26}$$

Now, applying (16) in $y \in H^1_0(\Omega)$ and using (21) we obtain

$$\underbrace{i\lambda\rho(v,y)_{L^2}}_{J_1} - \underbrace{i\lambda\alpha(\Delta v,y)_{L^2}}_{J_2} - \mu(\Delta u,\theta)_{L^2} + d||\theta||^2_{L^2} + \underbrace{\gamma(\varphi,\theta)_{L^2}}_{J_3} = \rho(f_2,y)_{L^2} + \alpha(f_2,\theta)_{L^2}. \tag{27}$$

For J_1, using (20) and (19), we obtain

$$
\begin{aligned}
J_1 &= \rho(z, -i\lambda\theta)_{L^2} \\
&= \frac{\rho}{a^*}(z, -k^*\Delta\theta + n\phi + d\Delta v - a^* f_5)_{L^2} \\
&= \frac{\rho k^*}{a^*}(v,\theta)_{L^2} + \frac{\rho n}{a^*}(z,\phi)_{L^2} - \frac{\rho d}{a^*}||v||^2_{L^2} - \rho(z,f_5)_{L^2} \\
&= \frac{\rho k^*}{a^*}(v,\theta)_{L^2} - \frac{\rho d}{a^*}||v||^2_{L^2} - \rho(z,f_5)_{L^2} + \frac{\rho n b}{i\lambda a^* J}(v,\varphi)_{L^2} \\
&\quad + \frac{\rho n\xi}{i\lambda a^* J}(z,\varphi)_{L^2} - \frac{\rho n^2}{i\lambda a^* J}(z,\theta)_{L^2} - \frac{\rho n\gamma}{i\lambda a^* J}(z,\Delta u)_{L^2} - \frac{\rho n}{i\lambda a^*}(z,f_4)_{L^2},
\end{aligned} \tag{28}
$$

where (18) was also used in the last equality. Also, using (19) and (18) again, we deduce

$$
\begin{aligned}
J_2 &= \alpha(v, i\lambda\theta)_{L^2} \\
&= \frac{\alpha}{a^*}(v, k^*\Delta\theta - n\phi - d\Delta v + a^* f_5)_{L^2} \\
&= -\frac{\alpha k^*}{a^*}(\nabla v, \nabla\theta)_{L^2} - \frac{\alpha n}{a^*}(v,\phi)_{L^2} + \frac{\alpha d}{a^*}||\nabla v||^2_{L^2} + \alpha(v,f_5)_{L^2} \\
&= -\frac{\alpha k^*}{a^*}(\nabla v, \nabla\theta)_{L^2} + \frac{\alpha d}{a^*}||\nabla v||^2_{L^2} + \alpha(v,f_5)_{L^2} - \frac{\alpha n b}{i\lambda a^* J}(\nabla v, \nabla\varphi)_{L^2} \\
&\quad - \frac{\alpha n\xi}{i\lambda a^* J}(v,\varphi)_{L^2} + \frac{\alpha n^2}{i\lambda a^* J}(v,\theta)_{L^2} + \frac{\alpha n\gamma}{i\lambda a^* J}(v,\Delta u)_{L^2} + \frac{\alpha n}{i\lambda a^*}(v,f_4)_{L^2}.
\end{aligned} \tag{29}
$$

Also, using (17) we have

$$J_3 = \gamma(\varphi,\theta)_{L^2} = \frac{\gamma}{i\lambda}(\phi + f_3,\theta)_{L^2} = \frac{\gamma}{i\lambda}(\phi,\theta)_{L^2} + \frac{\gamma}{i\lambda}(f_3,\theta)_{L^2}. \tag{30}$$

Then, substituting (28)-(30) into (27) we obtain

$$
\begin{aligned}
\frac{\rho d}{a^*}||v||^2_{L^2} + \frac{\alpha d}{a^*}||\nabla v||^2_{L^2} &= \frac{\rho k^*}{a^*}(v,\theta)_{L^2} - \rho(z,f_5)_{L^2} + \frac{\rho n b}{i\lambda a^* J}(v,\varphi)_{L^2} + \frac{\rho n\xi}{i\lambda a^* J}(z,\varphi)_{L^2} \\
&\quad - \frac{\rho n^2}{i\lambda a^* J}(z,\theta)_{L^2} - \frac{\rho n\gamma}{i\lambda a^* J}(z,\Delta u)_{L^2} - \frac{\rho n}{i\lambda a^*}(z,f_4)_{L^2} + \frac{\alpha k^*}{a^*}(\nabla v, \nabla\theta)_{L^2} \\
&\quad - \alpha(v,f_5)_{L^2} + \frac{\alpha n b}{i\lambda a^* J}(\nabla v, \nabla\varphi)_{L^2} + \frac{\alpha n\xi}{i\lambda a^* J}(v,\varphi)_{L^2} - \frac{\alpha n^2}{i\lambda a^* J}(v,\theta)_{L^2} \\
&\quad - \frac{\alpha n\gamma}{i\lambda a^* J}(v,\Delta u)_{L^2} - \frac{\alpha n}{i\lambda a^*}(v,f_4)_{L^2} - \mu(\Delta u,\theta)_{L^2} + d||\theta||^2_{L^2} \\
&\quad + \frac{\gamma}{i\lambda}(\phi,\theta)_{L^2} + \frac{\gamma}{i\lambda}(f_3,\theta)_{L^2} - \rho(f_2,y)_{L^2} - \alpha(f_2,\theta)_{L^2},
\end{aligned}
$$

which implies (remembering that $d \neq 0$)

$$
\begin{aligned}
\rho||v||^2_{L^2} + \alpha||\nabla v||^2_{L^2} &\leq C||v||_{L^2}||\theta||_{L^2} + C||\nabla v||_{L^2}||\nabla\theta||_{L^2} + \frac{C}{|\lambda|}||U||^2_{\mathcal{M}_\alpha} + \frac{C}{|\lambda|}||U||_{\mathcal{M}_\alpha}||F||_{\mathcal{M}_\alpha} \\
&\quad + C||U||_{\mathcal{M}_\alpha}||F||_{\mathcal{M}_\alpha} + C||\Delta u||_{L^2}||\theta||_{L^2} + C||\theta||^2_{L^2}.
\end{aligned}
$$

Then, using (26) we deduce

$$\frac{\rho}{2}||v||^2_{L^2} + \frac{\alpha}{2}||\nabla v||^2_{L^2} \leq C\left(1 + \frac{1}{|\lambda|}\right)||U||_{\mathcal{M}_\alpha}||F||_{\mathcal{M}_\alpha} + \frac{C}{|\lambda|}||U||^2_{\mathcal{M}_\alpha} + C||\theta||_{L^2}||U||_{\mathcal{M}_a}. \tag{31}$$

Similarly, applying (16) in u we obtain

$$\underbrace{i\lambda\rho(v,u)_{L^2}}_{J_4} - \underbrace{i\lambda\alpha(\Delta v, u)_{L^2}}_{J_5} + \mu||\Delta u||^2_{L^2} - d(\theta, \Delta u)_{L^2} - \underbrace{\gamma(\varphi, \Delta u)_{L^2}}_{J_6} = \rho(f_2, u)_{L^2} - \alpha(f_2, \Delta u)_{L^2}.$$

Then, using (15) into J_4, J_5 and J_6, we deduce

$$\mu||\Delta u||^2_{L^2} = \rho||v||^2_{L^2} + \alpha||\nabla v||^2_{L^2} + \rho(v, f_1)_{L^2} - \alpha(v, \Delta f_1)_{L^2} + d(\theta, \Delta u)_{L^2}$$
$$+ \frac{\gamma}{i\lambda}(\nabla\varphi, \nabla v)_{L^2} - \frac{\gamma}{i\lambda}(\varphi, \Delta f_1)_{L^2} + \rho(f_2, u)_{L^2} - \alpha(f_2, \Delta u)_{L^2},$$

which implies, using (26), that

$$\mu||\Delta u||^2_{L^2} \leq \rho||v||^2_{L^2} + \alpha||\nabla v||^2_{L^2} + C\left(1 + \frac{1}{|\lambda|}\right)||U||_{\mathcal{M}_\alpha}||F||_{\mathcal{M}_\alpha} + \frac{C}{|\lambda|}||U||^2_{\mathcal{M}_\alpha} + C||\theta||_{L^2}||U||_{\mathcal{M}_a}. \tag{32}$$

Therefore, doing 3(31)+(32), we obtain

$$\mu||\Delta u||^2_{L^2} + \frac{\rho}{2}||v||^2_{L^2} + \frac{\alpha}{2}||\nabla v||^2_{L^2} \leq C\left(1 + \frac{1}{|\lambda|}\right)||U||_{\mathcal{M}_\alpha}||F||_{\mathcal{M}_\alpha} + \frac{C}{|\lambda|}||U||^2_{\mathcal{M}_\alpha} + C||\theta||_{L^2}||U||_{\mathcal{M}_a}. \tag{33}$$

On the other hand, applying (16) in φ we obtain

$$\underbrace{i\lambda\rho(v,\varphi)_{L^2}}_{J_7} - \underbrace{i\lambda\alpha(\Delta v, \varphi)_{L^2}}_{J_8} + \underbrace{\mu(\Delta u, \Delta\varphi)_{L^2}}_{J_9} + d(\nabla\theta, \nabla\varphi)_{L^2} + \gamma||\nabla\varphi||^2_{L^2} = \rho(f_2, \varphi)_{L^2} + \alpha(\nabla f_2, \nabla\varphi)_{L^2}.$$

Then, using (17) into J_7, we obtain

$$J_7 = -\rho(v, \phi)_{L^2} - \rho(v, f_3)_{L^2}.$$

Similarly, using (17) and (15) into J_8, we obtain

$$J_8 = \alpha(\Delta v, \phi + f_3)_{L^2} = \alpha(\Delta v, \phi)_{L^2} - \alpha(\nabla v, \nabla f_3)_{L^2}$$
$$= \alpha(i\lambda\Delta u - \Delta f_1, \phi)_{L^2} - \alpha(\nabla v, \nabla f_3)_{L^2}$$
$$= i\lambda\alpha(\Delta u, \phi)_{L^2} - \alpha(\Delta f_1, \phi)_{L^2} - \alpha(\nabla v, \nabla f_3)_{L^2}.$$

Also, using (18) into J_9 we obtain

$$J_9 = \frac{\mu}{b}(\Delta u, i\lambda J\phi + \xi\varphi - n\theta - \gamma\Delta u - Jf_4)_{L^2}$$
$$= -i\lambda\frac{J\mu}{b}(\Delta u, \phi)_{L^2} + \frac{\mu\xi}{b}(\Delta u, \varphi)_{L^2} - \frac{\mu n}{b}(\Delta u, \theta)_{L^2} - \frac{\mu\gamma}{b}||\Delta u||^2_{L^2} - \frac{J\mu}{b}(\Delta u, f_4)_{L^2}$$
$$= -i\lambda\frac{J\mu}{b}(\Delta u, \phi)_{L^2} - \frac{\mu\xi}{i\lambda b}(\Delta u, \phi)_{L^2} - \frac{\mu\xi}{i\lambda b}(\Delta u, f_3)_{L^2} - \frac{\mu n}{b}(\Delta u, \theta)_{L^2} - \frac{\mu\gamma}{b}||\Delta u||^2_{L^2} - \frac{J\mu}{b}(\Delta u, f_4)_{L^2},$$

where (17) was also used in the last equality. Then, substituting J_7, J_8 and J_9 we deduce

$$\gamma||\nabla\varphi||^2_{L^2} = \rho(v, \phi)_{L^2} + \rho(v, f_3)_{L^2} - i\lambda\left(\alpha - \frac{J\mu}{b}\right)(\Delta u, \phi)_{L^2} + \alpha(\Delta f_1, \phi)_{L^2} + \alpha(\nabla v, \nabla f_3)_{L^2}$$
$$+ \frac{\mu\xi}{i\lambda b}(\Delta u, \phi)_{L^2} + \frac{\mu\xi}{i\lambda b}(\Delta u, f_3)_{L^2} + \frac{\mu n}{b}(\Delta u, \theta)_{L^2} + \frac{\mu\gamma}{b}||\Delta u||^2_{L^2} + \frac{J\mu}{b}(\Delta u, f_4)_{L^2}$$
$$+ \rho(f_2, \varphi)_{L^2} + \alpha(\nabla f_2, \nabla\varphi)_{L^2},$$

which implies, using conditions (5) and (26), that

$$||\nabla\varphi||^2_{L^2} \leq C\left(1 + \frac{1}{|\lambda|}\right)||U||_{\mathcal{M}_\alpha}||F||_{\mathcal{M}_\alpha} + \frac{C}{|\lambda|}||U||^2_{\mathcal{M}_\alpha} + C||\nabla\theta||_{L^2}||U||_{\mathcal{M}_\alpha} + C||v||_{L^2}||\phi||_{L^2} + C||\Delta u||^2_{L^2}. \tag{34}$$

Similarly, multiplying (18) by φ we have

$$\underbrace{i\lambda J(\phi, \varphi)_{L^2}}_{J_{10}} + b\|\nabla\varphi\|_{L^2}^2 + \xi\|\varphi\|_{L^2}^2 - \underbrace{n(\theta, \varphi)_{L^2}}_{J_{11}} - \underbrace{\gamma(\Delta u, \varphi)_{L^2}}_{J_{12}} = J(f_4, \varphi)_{L^2}. \tag{35}$$

Substituting (17) into J_{10}, J_{11} and J_{12}, we obtain

$$J_{10} = -J\|\phi\|_{L^2}^2 - J(\phi, f_3)_{L^2} \;, \qquad J_{11} = -\frac{n}{i\lambda}(\theta, \phi)_{L^2} - \frac{n}{i\lambda}(\theta, f_3)_{L^2}$$

and

$$J_{12} = -\frac{\gamma}{i\lambda}(\Delta u, \phi)_{L^2} - \frac{\gamma}{i\lambda}(\Delta u, f_3)_{L^2}.$$

So, substituting J_{10}, J_{11} and J_{12} into (35) we deduce

$$J\|\phi\|_{L^2}^2 = -J(\phi, f_3)_{L^2} + b\|\nabla\varphi\|_{L^2}^2 + \xi\|\varphi\|_{L^2}^2 + \frac{n}{i\lambda}(\theta, \phi)_{L^2} + \frac{n}{i\lambda}(\theta, f_3)_{L^2}$$
$$+ \frac{\gamma}{i\lambda}(\Delta u, \phi)_{L^2} + \frac{\gamma}{i\lambda}(\Delta u, f_3)_{L^2} - J(f_4, \varphi)_{L^2},$$

which implies, using (26), that

$$\|\phi\|_{L^2}^2 \leq C\left(1 + \frac{1}{|\lambda|}\right)\|U\|_{\mathcal{M}_\alpha}\|F\|_{\mathcal{M}_\alpha} + C_2\|\nabla\varphi\|_{L^2}^2 + \frac{C}{|\lambda|}\|U\|_{\mathcal{M}_\alpha}^2. \tag{36}$$

So, doing $2C_2(34)+(36)$ we obtain

$$C_2\|\nabla\varphi\|_{L^2}^2 + \|\phi\|_{L^2}^2 \leq C\left(1 + \frac{1}{|\lambda|}\right)\|U\|_{\mathcal{M}_\alpha}\|F\|_{\mathcal{M}_\alpha} + \frac{C}{|\lambda|}\|U\|_{\mathcal{M}_\alpha}^2$$
$$+ C\|\nabla\theta\|_{L^2}\|U\|_{\mathcal{M}_\alpha} + C\|v\|_{L^2}\|\phi\|_{L^2} + C\|\Delta u\|_{L^2}^2,$$

which implies, using the Young inequality, that

$$C_2\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{L^2}^2 \leq C\left(1 + \frac{1}{|\lambda|}\right)\|U\|_{\mathcal{M}_\alpha}\|F\|_{\mathcal{M}_\alpha} + \frac{C}{|\lambda|}\|U\|_{\mathcal{M}_\alpha}^2$$
$$+ C\|\nabla\theta\|_{L^2}\|U\|_{\mathcal{M}_\alpha} + C_3\left[\frac{\mu}{2}\|\Delta u\|_{L^2}^2 + \frac{\rho}{4}\|v\|_{L^2}^2\right]. \tag{37}$$

Therefore, doing $C_3(33)+(37)$, we have

$$C_3\left[\frac{\mu}{2}\|\Delta u\|_{L^2}^2 + \frac{\rho}{4}\|v\|_{L^2}^2 + \frac{\alpha}{2}\|\nabla v\|_{L^2}^2\right] + C_2\|\nabla\varphi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{L^2}^2 \leq C\left(1 + \frac{1}{|\lambda|}\right)\|U\|_{\mathcal{M}_\alpha}\|F\|_{\mathcal{M}_\alpha}$$
$$+ \frac{C_4}{|\lambda|}\|U\|_{\mathcal{M}_\alpha}^2 + C\|\nabla\theta\|_{L^2}\|U\|_{\mathcal{M}_\alpha},$$

which implies, using (26), that

$$\|U\|_{\mathcal{M}_\alpha}^2 \leq C\left(1 + \frac{1}{|\lambda|}\right)\|U\|_{\mathcal{M}_\alpha}\|F\|_{\mathcal{M}_\alpha} + \frac{C_5}{|\lambda|}\|U\|_{\mathcal{M}_\alpha}^2.$$

Finally, choosing $|\lambda| > M$ with M large enough, we obtain

$$\frac{1}{2}\|U\|_{\mathcal{M}_\alpha}^2 \leq C\|U\|_{\mathcal{M}_\alpha}\|F\|_{\mathcal{M}_\alpha},$$

which implies

$$\|U\|_{\mathcal{M}_\alpha} \leq C_6\|F\|_{\mathcal{M}_\alpha} \qquad \text{for all} \qquad |\lambda| > M,$$

where C_6 is independent of λ, U and F. Additionally, using that resolvent operators $R(i\lambda; \mathcal{B}) := (i\lambda I - \mathcal{B})^{-1}$ are bounded on bounded domains, then $\|U\|_{\mathcal{M}_\alpha} \leq C_7\|F\|_{\mathcal{M}_\alpha}$ for all $\lambda \in [-M, M]$, which completes the proof of (13).

2.3. **Non-exponential stability.** In this subsection, for damped solutions, we prove that exponential stability implies condition (5). Equivalently, we will show that

$$\mu - \frac{\alpha b}{J} \neq 0 \quad \text{or} \quad \gamma = 0, \tag{38}$$

implies non-exponential stability of solutions. In fact, in order to prove non-exponential stability, using Lemma 2.8, we will see that there exists a sequence $\{F_j\}_{j\in\mathbb{N}} \subset \mathcal{M}_\alpha$ (bounded), and $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{R}^+$, such that U_j is the solution of

$$(i\lambda_j I - \mathcal{B}_\alpha)U_j = F_j \quad \text{in} \quad \mathcal{M}_\alpha, \quad (\alpha \geq 0) \tag{39}$$

satisfying

$$\lim_{j\to\infty} ||U_j||_{\mathcal{M}_\alpha} = \infty.$$

Here, we will use the eigenvalues ν_j with corresponding eigenvectors e_j defined in (7). As F_n we choose

$$F_n := (0,0,0,\frac{1}{J}e_j,0)' \in \mathcal{B}_\alpha.$$

Note that $\{F_j\}_{j\in\mathbb{N}}$ is bounded in $\mathcal{B}_\alpha$. Moreover, the solution $U_j = (u_j, v_j, \varphi_j, \phi_j, \theta_j)'$ of (39) should satisfy

$$i\lambda u_j - v_j = 0 \quad \text{in} \quad H^2(\Omega) \cap H_0^1(\Omega)$$

$$i\lambda\rho v_j + (I - \frac{\alpha}{\rho}\Delta)^{-1}\Delta(\mu\Delta u_j - d\theta_j - \gamma\varphi_j) = 0 \quad \text{in} \quad L^2(\Omega) \quad (\text{or } H_0^1(\Omega))$$

$$i\lambda\varphi_j - \phi_j = 0 \quad \text{in} \quad H_0^1(\Omega)$$

$$i\lambda J\phi_j - b\Delta\varphi_j - m\Delta\psi_j + \xi\varphi_j - n\theta_j - \gamma\Delta u_j = e_j \quad \text{in} \quad L^2(\Omega)$$

$$i\lambda\psi_j - \theta_j = 0 \quad \text{in} \quad H_0^1(\Omega)$$

$$i\lambda a^*\theta_j - \Delta(k\psi_j + m\varphi_j + k^*\theta_j) + n\phi_j + d\Delta v_j = 0 \quad \text{in} \quad L^2(\Omega),$$

where the second equation is formulated in $L^2(\Omega)$ if $\alpha = 0$ or $H_0^1(\Omega)$ if $\alpha > 0$. Simplifying the previous system we obtain

$$-\lambda^2\rho u_j + \lambda^2\alpha\Delta u_j + \Delta(\mu\Delta u_j - d\theta_j - \gamma\varphi_j) = 0 \quad \text{in} \quad L^2(\Omega) \quad (\text{or } H^{-1}(\Omega)) \tag{40}$$

$$-\lambda^2 J\varphi_j - b\Delta\varphi_j + \xi\varphi_j - n\theta_j - \gamma\Delta u_j = e_j \quad \text{in} \quad L^2(\Omega) \tag{41}$$

$$i\lambda a^*\theta_j - k^*\Delta\theta_j + i\lambda n\varphi_j + i\lambda d\Delta u_j = 0, \quad \text{in} \quad L^2(\Omega), \tag{42}$$

which can be solved by

$$u_j = A_j e_j, \qquad \varphi_j = B_j e_j, \qquad \theta_j = C_j e_j,$$

where A_j, B_j, C_j are depending of λ and will be defined explicitly. Here, note that u_j, φ_j, θ_j are compatible with boundary conditions (2). Using (7), system (40)-(42) is equivalent to

$$\left[-\lambda^2(\rho + \alpha\nu_j) + \mu\nu_j^2\right]A_j + \gamma\nu_j B_j + d\nu_j C_j = 0$$

$$\left[-\lambda^2 J + b\nu_j + \xi\right]B_j - nC_j + \gamma\nu_j A_j = 1$$

$$\left[i\lambda a^* + k^*\nu_j\right]C_j + i\lambda n B_j - i\lambda d\nu_j A_j = 0,$$

which can be written as

$$\begin{bmatrix} p_1(\lambda) & \gamma\nu_j & d\nu_j \\ \gamma\nu_j & p_2(\lambda) & -n \\ -i\lambda d\nu_j & i\lambda n & p_3(\lambda) \end{bmatrix} \begin{bmatrix} A_j \\ B_j \\ C_j \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{43}$$

where

$$p_1(\lambda) := -\lambda^2(\rho + \alpha\nu_j) + \mu\nu_j^2$$

$$p_2(\lambda) := -\lambda^2 J + b\nu_j + \xi$$

$$p_3(\lambda) := -i\lambda a^* + k^*\nu_j.$$

In this point, we define the sequence $\lambda := \lambda_j$ such that $p_2(\lambda_j) = 0$, this is

$$\lambda_j := \sqrt{\frac{b\nu_j + \xi}{J}} \approx O(\nu_j^{1/2}), \qquad \text{for all} \qquad j \in \mathbb{N}. \tag{44}$$

In this case $p_1(\lambda_j)$ is given by

$$p_1(\lambda_j) = \left(\mu - \frac{b\alpha}{J}\right)\nu_j^2 - \left(\frac{b\rho + \xi\alpha}{J}\right)\nu_j - \frac{\xi\rho}{J}, \tag{45}$$

and system (43) can be written as

$$\underbrace{\begin{bmatrix} p_1(\lambda_j) & \gamma\nu_j & d\nu_j \\ \gamma\nu_j & 0 & -n \\ -i\lambda_j d\nu_j & i\lambda_j n & p_3(\lambda_j) \end{bmatrix}}_{M} \begin{bmatrix} A_j \\ B_j \\ C_j \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \tag{46}$$

with

$$\det(M) = -\gamma^2\nu_j^2 p_3(\lambda_j) + i\lambda_j n^2 p_1(\lambda_j) + 2i\lambda_j nd\gamma\nu_j^2. \tag{47}$$

Our analysis now will be divided in two cases: $\gamma = 0$ and $\gamma \neq 0$.

- CASE $\gamma = 0$. In this case the interesting situation is $n \neq 0$. Otherwise $\gamma = n = 0$ implies that the second equation of (1) is given by

$$J\varphi_{tt} - b\Delta\varphi + \xi\varphi = 0,$$

which is a conservative wave equation with stationary solutions. Consequently, system (1) will be non-exponentially stable. So, assuming $n \neq 0$, then

$$\det(M) = i\lambda_j n^2 p_1(\lambda_j) \neq 0, \qquad \text{for } j \in \mathbb{N} \text{ large enough.}$$

So, using the Cramer's rule, we obtain

$$B_j = \frac{p_1(\lambda_j)p_3(\lambda_j) + id^2\lambda_j\nu_j^2}{\det(M)} = \frac{p_1(\lambda_j)p_3(\lambda_j) + id^2\lambda_j\nu_j^2}{i\lambda_j n^2 p_1(\lambda_j)} = \frac{d^2\nu_j^2}{n^2 p_1(\lambda_j)} - i\frac{p_3(\lambda_j)}{n^2\lambda_j},$$

which implies

$$|B_j| \geq \frac{1}{n^2}\frac{|p_3(\lambda_j)|}{|\lambda_j|} \approx \frac{O(\nu_j)}{O(\nu_j^{1/2})} = O(\nu_j^{1/2}) \longrightarrow +\infty \quad (j \to \infty).$$

Therefore

$$\|U_j\|_{\mathcal{M}_\alpha} \geq C\|\varphi_j\|_{L^2} = Ce_0|B_j| \longrightarrow +\infty \quad (j \to \infty),$$

where e_0 is defined from (7). Then the proof of (13) is complete.

- CASE $\gamma \neq 0$. In this case, by (38), we have

$$\mu - \frac{\alpha b}{J} \neq 0,$$

which implies in (45) that

$$\frac{p_1(\lambda_j)}{\nu_j^2} \longrightarrow \left(\mu - \frac{\alpha b}{J}\right) \neq 0 \quad (j \to \infty). \tag{48}$$

On the other hand, using (47) and the definition of $p_3(\lambda_j)$, we have

$$\det(M) = -\gamma^2 k^*\nu_j^3 + i\left[-a^*\lambda_j + \lambda_j n^2 p_1(\lambda_j) + 2\lambda_j nd\gamma\nu_j^2\right] \neq 0.$$

So, using again Cramer's rule and the definition of $p_3(\lambda_j)$, we obtain

$$B_j \;=\; \frac{p_1(\lambda_j)p_3(\lambda_j) + id^2\lambda_j\nu_j^2}{\det(M)} \;=\; \frac{k^*\nu_j p_1(\lambda_j) + i\left[-a^*\lambda_j p_1(\lambda_j) + d^2\lambda_j\nu_j^2\right]}{-\gamma^2 k^*\nu_j^3 + i\left[-a^*\lambda_j + \lambda_j n^2 p_1(\lambda_j) + 2\lambda_j nd\gamma\nu_j^2\right]}$$

$$=\; \frac{k^*\dfrac{p_1(\lambda_j)}{\nu_j^2} + i\left[-a^*\dfrac{\lambda_j}{\nu_j}\dfrac{p_1(\lambda_j)}{\nu_j^2} + d^2\dfrac{\lambda_j}{\nu_j}\right]}{-\gamma^2 k^* + i\left[-a^*\dfrac{\lambda_j}{\nu_j^3} + n^2\dfrac{\lambda_j}{\nu_j}\dfrac{p_1(\lambda_j)}{\nu_j^2} + 2nd\gamma\dfrac{\lambda_j}{\nu_j}\right]}.$$

Then, using convergence (48) and noting that $\dfrac{\lambda_j}{\nu_j} \to 0$, we deduce

$$B_j \;\longrightarrow\; -\frac{\left(\mu - \dfrac{\alpha b}{J}\right)}{\gamma^2} \neq 0.$$

Finally, using that $\phi_j = i\lambda_j\varphi_j = i\lambda_j B_j e_j$ and the definition (44), we have

$$\|U_j\|_{\mathcal{M}_\alpha} \;\geq\; C\|\phi_j\|_{L^2} \;=\; Ce_0\sqrt{\frac{b\nu_j + \xi}{J}}|B_j| \;\longrightarrow\; +\infty, \quad (j \to \infty),$$

which completes the proof of (13). Here again e_0 is defined from (7).

Remark 2.9. *From our analysis, it is interesting to remark that $\alpha = 0$ implies non-exponential stability. This is, system*

$$\rho u_{tt} + \mu\Delta^2 u - d\Delta\theta - \gamma\Delta\varphi \;=\; 0 \quad in \quad \Omega \times \mathbb{R}^+$$

$$J\varphi_{tt} - b\Delta\varphi + \xi\varphi - n\theta - \gamma\Delta u \;=\; 0 \quad in \quad \Omega \times \mathbb{R}^+$$

$$a^*\theta_t - k^*\Delta\theta + n\varphi_t + d\Delta u_t \;=\; 0 \quad in \quad \Omega \times \mathbb{R}^+,$$

is non-exponentially stable. On the other hand, under conditions of Proposition 2.6, the previous system is strongly stable which implies that $E(t) \to 0$ with some rate of decay. We expect that polynomial rates of decay can be obtained by standard arguments, like semigroup characterizations given by [4].

3. Type III's Thermal Effects

In this section we consider thermal effects of Type III. Here the stability results do not depend on $\alpha > 0$ or $\alpha = 0$. Note the difference with results obtained in Section 2, where the presence of the inertia rotational term $\alpha\Delta u_{tt}$ play an important role for the exponential stability. See also [17] where this term is also important for the stability.

In our case, the system is given by

$$\rho u_{tt} - \alpha\Delta u_{tt} + \mu\Delta^2 u - d\Delta\theta - \gamma\Delta\varphi \;=\; 0 \quad in \quad \Omega \times \mathbb{R}^+$$

$$J\varphi_{tt} - b\Delta\varphi - m\Delta\psi + \xi\varphi - n\theta - \gamma\Delta u \;=\; 0 \quad in \quad \Omega \times \mathbb{R}^+ \qquad (49)$$

$$a^*\psi_{tt} - k\Delta\psi - m\Delta\varphi - k^*\Delta\theta + n\varphi_t + d\Delta u_t \;=\; 0 \quad in \quad \Omega \times \mathbb{R}^+,$$

where $\psi_t = \theta$, with boundary conditions

$$u(x,t) = \Delta u(x,t) \;=\; 0 \quad in \quad \partial\Omega \times \mathbb{R}^+$$

$$\varphi(x,t) = \psi(x,t) \;=\; 0 \quad in \quad \partial\Omega \times \mathbb{R}^+ \qquad (50)$$

and initial conditions

$$u(\cdot,0) = u_0(\cdot) \quad , \quad u_t(\cdot,0) = u_1(\cdot) \quad in \quad \Omega$$

$$\varphi(\cdot,0) = \varphi_0(\cdot) \quad , \quad \varphi_t(\cdot,0) = \varphi_1(\cdot) \quad in \quad \Omega \qquad (51)$$

$$\psi(\cdot,0) = \psi_0(\cdot) \quad , \quad \psi_t(\cdot,0) = \psi_1(x) \quad in \quad \Omega.$$

Here, note that ψ describes the thermal displacement. The hypotheses on the constitutive constants are the following

$$\mu > 0, \quad \xi\mu - \gamma^2 > 0, \quad bk - m^2 > 0, \quad \rho, J, a^*, b, k^* > 0, \quad d, m \neq 0, \quad n, \gamma \in \mathbb{R}, \qquad (52)$$

with $\alpha \geq 0$. Similarly to conditions (4), conditions (52) are motivated to guarantee that a bilinear form proposed later in $\mathcal{H}_\alpha$ (considered shortly) defines an inner product.

3.1. **Well-posedness.** We define the family of Hilbert spaces

$$\mathcal{H}_0 := [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega), \quad (\alpha = 0)$$

$$\mathcal{H}_\alpha := [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega), \quad (\alpha > 0)$$

with associated inner products for each $\alpha \geq 0$ and for $U_i = (u_i, v_i, \varphi_i, \phi_i, \psi_i, \theta_i)^T$, $i = 1, 2$ given by

$$
\begin{aligned}
(U_1, U_2)_{\mathcal{H}_\alpha} = & \ \mu(\Delta u_1, \Delta u_2)_{L^2} + \rho(v_1, v_2)_{L^2} + \alpha(\nabla v_1, \nabla v_2)_{L^2} + b(\nabla \varphi_1, \nabla \varphi_2)_{L^2} + \xi(\varphi_1, \varphi_2)_{L^2} \\
& + J(\phi_1, \phi_2)_{L^2} + k(\nabla \psi_1, \nabla \psi_2)_{L^2} + a^*(\theta_1, \theta_2)_{L^2} \\
& + m(\nabla \varphi_1, \nabla \psi_2)_{L^2} + m(\nabla \psi_1, \nabla \varphi_2)_{L^2} - \gamma(\varphi_1, \Delta u_2)_{L^2} - \gamma(\Delta u_1, \varphi_2)_{L^2},
\end{aligned}
$$

which implies the family of norms, for $U = (u, v, \varphi, \phi, \psi, \theta)^T$

$$
\begin{aligned}
||U||_{\mathcal{H}_\alpha}^2 := & \ \mu||\Delta u||_{L^2}^2 + \rho||v||_{L^2}^2 + \alpha||\nabla v||_{L^2}^2 + b||\nabla \varphi||_{L^2}^2 + \xi||\varphi||_{L^2}^2 + J||\phi||_{L^2}^2 + k||\nabla \psi||_{L^2}^2 \\
& + a^*||\theta||_{L^2}^2 + 2m \mathrm{Re}(\nabla \varphi, \nabla \psi)_{L^2} - 2\gamma \mathrm{Re}(\varphi, \Delta u)_{L^2}.
\end{aligned}
$$

Additionally, let $\mathcal{A}_\alpha$ be the associated operators

$$
\mathcal{A}_\alpha \begin{pmatrix} u \\ v \\ \varphi \\ \phi \\ \psi \\ \theta \end{pmatrix} = \begin{pmatrix} v \\ \dfrac{1}{\rho}(I - \dfrac{\alpha}{\rho}\Delta)^{-1}\Delta\left(-\mu\Delta u + d\theta + \gamma\varphi\right) \\ \phi \\ \dfrac{1}{J}\left[b\Delta\varphi + m\Delta\psi - \xi\varphi + n\theta + \gamma\Delta u\right] \\ \theta \\ \dfrac{1}{a^*}\Delta\left(k\psi + m\varphi + k^*\theta\right) - \dfrac{n}{a^*}\phi - \dfrac{d}{a^*}\Delta v \end{pmatrix}, \tag{53}
$$

for each $\alpha \geq 0$, which are defined in the corresponding domains

$$
\begin{aligned}
D(\mathcal{A}_0) = \Big\{ & (u, v, \varphi, \phi, \psi, \theta)^T \in \mathcal{H}_0 \ : \quad v \in H^2(\Omega) \cap H_0^1(\Omega); \ \phi, \ \theta, \ \varphi, \ \psi \in H_0^1(\Omega); \\
& (\mu\Delta u - d\theta - \gamma\varphi) \in H^2(\Omega) \cap H_0^1(\Omega); \ (b\Delta\varphi + m\Delta\psi - \xi\varphi + \gamma\Delta u) \in L^2(\Omega) \quad \text{and} \\
& (k\psi + m\varphi + k^*\theta) \in H^2(\Omega) \cap H_0^1(\Omega) \Big\},
\end{aligned}
$$

$$
\begin{aligned}
D(\mathcal{A}_\alpha) = \Big\{ & (u, v, \varphi, \phi, \psi, \theta)^T \in \mathcal{H}_\alpha \ : \quad v \in H^2(\Omega) \cap H_0^1(\Omega); \ \phi, \ \theta, \ \varphi, \ \psi \in H_0^1(\Omega); \\
& (\mu\Delta u - d\theta - \gamma\varphi) \in H_0^1(\Omega) \quad \text{and} \quad (k\psi + m\varphi + k^*\theta) \in H^2(\Omega) \cap H_0^1(\Omega) \Big\},
\end{aligned}
$$

respectively.

Remark 3.1. *Note that*

(1) *In the domains $D(\mathcal{A}_\alpha)$ we used that the operators $(I - \frac{\alpha}{\rho}\Delta)$ are isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$.*

(2) *Using hypothesis (52), by the definitions of $D(\mathcal{A}_0)$ and $D(\mathcal{A}_\alpha)$, we can deduce the regularity $\varphi, \psi \in H_0^1(\Omega)$ which also implies that $\Delta u \in H_0^1(\Omega)$, satisfying the boundary condition (50).*

In this context, the well-posedness is a direct consequence of the following Theorem.

Theorem 3.2. *The operators $\mathcal{A}_\alpha$ associated to system (49)-(51) when $\alpha \geq 0$ are infinitesimal generators of C_0-semigroups on the Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_\alpha$, respectively.*

Proof. For the proof, it is sufficient to see that $\overline{D(\mathcal{A}_\alpha)} = \mathcal{H}_\alpha$, $\mathcal{A}_\alpha$ are dissipative and $0 \in \varrho(\mathcal{A}_\alpha)$, for all $\alpha \geq 0$, see [24, 34].

In fact, note that the density results are standard. Also, for straightforward calculations we have

$$\mathrm{Re}(\mathcal{A}_\alpha U, U)_{\mathcal{H}_\alpha} = -k^*||\nabla\theta||_{L^2}^2, \qquad \forall U = (u, v, \varphi, \phi, \psi, \theta)^T \in \mathcal{H}_\alpha,$$

implying that the operators $\mathcal{A}_\alpha$ are dissipative. Additionally, in order to show that $0 \in \varrho(\mathcal{A}_\alpha)$, let $F = (f_1, f_2, \cdots, f_6) \in \mathcal{H}_\alpha$, then the equation

$$\mathcal{A}_\alpha U = F \quad \text{in} \quad \mathcal{H}_\alpha,$$

implies $v = f_1 \in H^2 \cap H_0^1$, $\phi = f_3 \in H_0^1$, $\theta = f_5 \in H_0^1$ and (u, φ, ψ) should satisfy

$$\Delta(\mu\Delta u - \gamma\varphi) = g_1 \quad \text{in} \quad H^{-1}$$

$$-b\Delta\varphi - m\Delta\psi + \xi\varphi - \gamma\Delta u = g_2 \quad \text{in} \quad L_2$$

$$-k\Delta\psi - m\Delta\varphi = g_3 \quad \text{in} \quad H^{-1},$$

where $g_1 := \rho f_2 - \alpha\Delta f_2 + d\Delta f_5$, $g_2 := Jf_4 - nf_5$ and $g_3 := a^* f_6 + nf_3 + d\Delta f_1 + k^*\Delta f_5$. So, using conditions (52) and the Lax-Milgram Theorem we obtain a unique solution (u, φ, ψ) for the previous system with regularity $(u, \varphi, \psi) \in [H^2 \cap H_0^1] \times H_0^1 \times H_0^1$ and satisfying the conditions of $D(\mathcal{A}_\alpha)$, for each $\alpha \geq 0$. Then $\mathcal{A}_\alpha$ is bijective. Also, working with the components of $\mathcal{A}_\alpha U = F$, it is not difficult to prove that $||U||_{\mathcal{H}_\alpha} \leq C||F||_{\mathcal{H}_\alpha}$, which implies that $\mathcal{A}_\alpha^{-1}$ is bounded. Therefore $0 \in \varrho(\mathcal{A}_\alpha)$ for all $\alpha \geq 0$, which completes the proof of the Theorem. $\qquad\square$

In order to exclude undamped solutions, similarly to previous Section, we establish now a strong stability result.

Proposition 3.3. *Let $\varrho(\mathcal{A}_\alpha)$ the resolvent set of operator $\mathcal{A}_\alpha$. Assuming $\gamma \neq 0$, then $i\mathbb{R} \subset \varrho(\mathcal{A}_\alpha)$ without any extra conditions. On the other hand, assuming $\gamma = 0$ and supposing that $P_0(\nu_j) \neq 0$ for all $j \in \mathbb{N}$, where*

$$P_0(X) := (J\mu - \alpha b)X^2 - (\rho b + \alpha\xi)X - \rho\xi,$$

then we have $i\mathbb{R} \subset \varrho(\mathcal{A}_\alpha)$. This is, system (49)-(51) is strongly stable in both situations.

Proof. We proceed with similar arguments used in Lemma 2.5. In fact, supposing that $i\mathbb{R} \subset \varrho(\mathcal{A}_\alpha)$ is not true, then there exists $\lambda \in \mathbb{R}$ such that $i\lambda \in \sigma(\mathcal{A}_\alpha)$, with $\lambda \neq 0$, this because $0 \in \varrho(\mathcal{A}_\alpha)$. Then, using the compact embedding $D(\mathcal{A}_\alpha) \hookrightarrow \mathcal{H}_\alpha$, we have that $i\lambda$ is an eigenvalue of $\mathcal{A}_a$. Therefore, there exists $U \in D(\mathcal{A}_a)$, $U \neq 0$, satisfying

$$i\lambda U - \mathcal{A}_\alpha U = 0 \quad \text{in} \quad \mathcal{H}_\alpha.$$

So, multiplying this equation by $U \in \mathcal{H}_\alpha$ and taking the real part, we can deduce that

$$-k^*||\nabla\theta||_{L^2}^2 = \text{Re}(\mathcal{A}_\alpha U, U)_{\mathcal{M}_\alpha} = 0 \qquad \text{where} \qquad U = (u, v, \varphi, \phi, \psi, \theta)^T,$$

which implies $\theta \equiv 0$. Additionally, in the previous equation, $\theta \equiv 0$ implies $\psi \equiv 0$. So, using these conditions in the resolvent equation, we obtain

$$-\lambda^2\rho u + \lambda^2\alpha\Delta u + \mu\Delta^2 u - \gamma\Delta\varphi = 0 \quad \text{in} \quad L^2(\Omega) \qquad (\text{or } H^{-1}(\Omega))$$

$$-\lambda^2 J\varphi - b\Delta\varphi + \xi\varphi - \gamma\Delta u = 0 \quad \text{in} \quad L^2(\Omega) \qquad\qquad (54)$$

$$-m\Delta\varphi + i\lambda n\varphi + i\lambda d\Delta u = 0, \quad \text{in} \quad L^2(\Omega).$$

Therefore, substituting Δu given by $(54)_3$ into $(54)_2$, we deduce

$$-\left(b + \frac{\gamma m}{i\lambda d}\right)\Delta\varphi = \left(\lambda^2 J - \xi + \frac{\gamma n}{d}\right)\varphi. \qquad\qquad (55)$$

So, assuming $\gamma \neq 0$, the unique possible solution of (55) is $\varphi \equiv 0$ (remembering that $m, d \neq 0$), which implies from $(54)_3$ that $u \equiv 0$, implying the contradiction $U \equiv 0$ in $\mathcal{H}_\alpha$. On the other hand, assuming $\gamma = 0$, system (54) is written as

$$-\lambda^2\rho u + \lambda^2\alpha\Delta u + \mu\Delta^2 u = 0 \quad \text{in} \quad L^2(\Omega) \qquad (\text{or } H^{-1}(\Omega))$$

$$-\lambda^2 J\varphi - b\Delta\varphi + \xi\varphi = 0 \quad \text{in} \quad L^2(\Omega) \qquad\qquad (56)$$

$$-m\Delta\varphi + i\lambda n\varphi + i\lambda d\Delta u = 0, \quad \text{in} \quad L^2(\Omega),$$

which can be solved by $u = A_j e_j$ and $\varphi = B_j e_j$, where e_j is defined in (7), with $A_j \neq 0$ and $B_j \neq 0$. So, substituting into (56) and using the same arguments used in Lemma 2.5, we conclude that $P_0(\nu_j) = 0$ for some $j \in \mathbb{N}$ which is also a contradiction. Consequently, $i\mathbb{R} \subset \varrho(\mathcal{A}_\alpha)$, which completes the proof. $\qquad\square$

Remark 3.4. *Note that $P_0(X)$ is the same polynomial function $P(X)$ defined in Lemma 2.5 assuming $\gamma = 0$. Also note that, in Proposition 2.6 the constants $n, \gamma \in \mathbb{R}$ play an important role in order to exclude undamped solutions. In the case of Proposition 3.3, in order to avoid undamped solutions, the constant $n \in \mathbb{R}$ is not relevant, which is expected by the influence of $m \neq 0$. Finally, as mentioned in Remark 2.7, it is important to identify this kind of solutions because they do not appear in classical second order thermo-porous-elastic systems, see for example [23, 30].*

3.2. **Stability.** The main result of this Section is given by the following Theorem.

Theorem 3.5. *Assuming the hypothesis of Proposition 3.3 and assuming $\alpha = 0$ or even $\alpha > 0$, then the semigroup of contractions associated to the system (49)-(51) is exponentially stable.*

Proof. Using Lemma 2.8, it is sufficient to show conditions (12)-(13). In fact, note that the first condition (12) is given by Proposition 3.3. In order to prove (13) we will show that

$$||U||_{\mathcal{H}_\alpha} \leq M||F||_{\mathcal{H}_\alpha},$$

for all $\alpha \geq 0$ and for all $F \in \mathcal{H}_\alpha$, with $M > 0$ is independent of λ, F, and U solution of the resolvent system

$$(i\lambda I - \mathcal{A}_\alpha)U = F, \quad \text{in} \quad \mathcal{H}_\alpha. \tag{57}$$

System (57) can be written, in its components, by

$$i\lambda u - v = f_1 \quad \text{in} \quad H^2(\Omega) \cap H_0^1(\Omega)$$

$$i\lambda \rho v - (I - \frac{\alpha}{\rho}\Delta)^{-1}\Delta(-\mu\Delta u + d\theta + \gamma\varphi) = \rho f_2 \quad \text{in} \quad L^2(\Omega) \quad (\text{or } H_0^1(\Omega))$$

$$i\lambda\varphi - \phi = f_3 \quad \text{in} \quad H_0^1(\Omega)$$

$$i\lambda J\phi - b\Delta\varphi - m\Delta\psi + \xi\varphi - n\theta - \gamma\Delta u = Jf_4 \quad \text{in} \quad L^2(\Omega)$$

$$i\lambda\psi - \theta = f_5 \quad \text{in} \quad H_0^1(\Omega)$$

$$i\lambda a^*\theta - \Delta(k\psi + m\varphi + k^*\theta) + n\phi + d\Delta v = a^* f_6 \quad \text{in} \quad L^2(\Omega),$$

where the second equation is formulated in $L^2(\Omega)$ if $\alpha = 0$ or $H_0^1(\Omega)$ in the case $\alpha > 0$. The previous system is equivalent to

$$i\lambda u - v = f_1 \quad \text{in} \quad H^2(\Omega) \cap H_0^1(\Omega) \tag{58}$$

$$i\lambda\rho v - i\lambda\alpha\Delta v + \Delta(\mu\Delta u - d\theta - \gamma\varphi) = \rho f_2 - \alpha\Delta f_2 \quad \text{in} \quad L^2(\Omega) \quad (\text{or } H^{-1}(\Omega)) \tag{59}$$

$$i\lambda\varphi - \phi = f_3 \quad \text{in} \quad H_0^1(\Omega) \tag{60}$$

$$i\lambda J\phi - b\Delta\varphi - m\Delta\psi + \xi\varphi - n\theta - \gamma\Delta u = Jf_4 \quad \text{in} \quad L^2(\Omega) \tag{61}$$

$$i\lambda\psi - \theta = f_5 \quad \text{in} \quad H_0^1(\Omega) \tag{62}$$

$$i\lambda a^*\theta - \Delta(k\psi + m\varphi + k^*\theta) + n\phi + d\Delta v = a^* f_6 \quad \text{in} \quad L^2(\Omega). \tag{63}$$

Eventually, we will use the same multipliers z, y, w given by (20)-(22) with their respective estimates (23)-(25). Also, by similar arguments used in the previous section, multiplying (57) by U in $\mathcal{H}_\alpha$ and taking the real part, we obtain

$$||\nabla\theta||_{L^2}^2 \leq C||U||_{\mathcal{H}_\alpha}||F||_{\mathcal{H}_\alpha}. \tag{64}$$

Also, (62) implies

$$||\nabla\psi||_{L^2} \leq \frac{1}{|\lambda|}||\nabla\theta||_{L^2} + \frac{C}{|\lambda|}||F||_{\mathcal{H}_\alpha}. \tag{65}$$

Now, multiplying (59) by $y \in L^2(\Omega)$ if $\alpha = 0$ (or applying in $y \in H_0^1(\Omega)$ if $\alpha > 0$), and using (21) we obtain

$$\underbrace{i\lambda\rho(v, y)_{L^2}}_{J_1} - \underbrace{i\lambda\alpha(\Delta v, y)_{L^2}}_{J_2} - \mu(\Delta u, \theta)_{L^2} + d||\theta||_{L^2}^2 + \gamma(\varphi, \theta)_{L^2} = \rho(f_2, y)_{L^2} + \alpha(f_2, \theta)_{L^2}. \tag{66}$$

For J_1, using (20) and (63), we obtain

$$
\begin{aligned}
J_1 &= \rho(z, -i\lambda\theta)_{L^2} \\
&= \frac{\rho}{a^*}(z, -\Delta(k\psi + m\varphi + k^*\theta) + n\phi + d\Delta v - af_6)_{L^2} \\
&= \frac{\rho k}{a^*}(v, \psi)_{L^2} + \frac{\rho m}{a^*}(v, \varphi)_{L^2} + \frac{\rho k^*}{a^*}(v, \theta)_{L^2} + \frac{\rho n}{a^*}(z, \phi)_{L^2} - \frac{\rho d}{a^*}\|v\|_{L^2}^2 - \rho(z, f_6)_{L^2} \\
&= \frac{\rho k}{a^*}(v, \psi)_{L^2} + \frac{\rho m}{a^*}(v, \varphi)_{L^2} + \frac{\rho k^*}{a^*}(v, \theta)_{L^2} - \frac{\rho d}{a^*}\|v\|_{L^2}^2 - \rho(z, f_6)_{L^2} + \frac{\rho n b}{i\lambda a^* J}(v, \varphi)_{L^2} \\
&\quad + \frac{\rho n m}{i\lambda a^* J}(v, \psi)_{L^2} + \frac{\rho n \xi}{i\lambda a^* J}(z, \varphi)_{L^2} - \frac{\rho n^2}{i\lambda a^* J}(z, \theta)_{L^2} - \frac{\rho n \gamma}{i\lambda a^* J}(z, \Delta u)_{L^2} - \frac{\rho n}{i\lambda a^*}(z, f_4)_{L^2}, \quad (67)
\end{aligned}
$$

where (61) was also used in the last equality. Additionally, for J_2 in the case $\alpha > 0$, using (63) and (61) again, we deduce

$$
\begin{aligned}
J_2 &= \alpha(v, i\lambda\theta)_{L^2} \\
&= \frac{\alpha}{a^*}(v, \Delta(k\psi + m\varphi + k^*\theta) - n\phi - d\Delta v + a^* f_6)_{L^2} \\
&= -\frac{\alpha k}{a^*}(\nabla v, \nabla \psi)_{L^2} - \frac{\alpha m}{a^*}(\nabla v, \nabla \varphi)_{L^2} - \frac{\alpha k^*}{a^*}(\nabla v, \nabla \theta)_{L^2} - \frac{\alpha n}{a^*}(v, \phi)_{L^2} + \frac{\alpha d}{a^*}\|\nabla v\|_{L^2}^2 + \alpha(v, f_6)_{L^2} \\
&= -\frac{\alpha k}{a^*}(\nabla v, \nabla \psi)_{L^2} - \frac{\alpha m}{a^*}(\nabla v, \nabla \varphi)_{L^2} - \frac{\alpha k^*}{a^*}(\nabla v, \nabla \theta)_{L^2} + \frac{\alpha d}{a^*}\|\nabla v\|_{L^2}^2 + \alpha(v, f_6)_{L^2} - \frac{\alpha n b}{i\lambda a^* J}(\nabla v, \nabla \varphi)_{L^2} \\
&\quad - \frac{\alpha n m}{i\lambda a^* J}(\nabla v, \nabla \psi)_{L^2} - \frac{\alpha n \xi}{i\lambda a^* J}(v, \varphi)_{L^2} + \frac{\alpha n^2}{i\lambda a^* J}(v, \theta)_{L^2} + \frac{\alpha n \gamma}{i\lambda a^* J}(v, \Delta u)_{L^2} + \frac{\alpha n}{i\lambda a^*}(v, f_4)_{L^2}. \quad (68)
\end{aligned}
$$

Then, substituting (67)-(68) into (66) we obtain for all $\alpha \geq 0$,

$$
\begin{aligned}
\frac{\rho d}{a^*}\|v\|_{L^2}^2 + \frac{\alpha d}{a^*}\|\nabla v\|_{L^2}^2 &= \frac{\rho k}{a^*}(v, \psi)_{L^2} + \frac{\rho m}{a^*}(v, \varphi)_{L^2} + \frac{\rho k^*}{a^*}(v, \theta)_{L^2} - \rho(z, f_6)_{L^2} + \frac{\rho n b}{i\lambda a^* J}(v, \varphi)_{L^2} \\
&\quad + \frac{\rho n m}{i\lambda a^* J}(v, \psi)_{L^2} + \frac{\rho n \xi}{i\lambda a^* J}(z, \varphi)_{L^2} - \frac{\rho n^2}{i\lambda a^* J}(z, \theta)_{L^2} - \frac{\rho n \gamma}{i\lambda a^* J}(z, \Delta u)_{L^2} \\
&\quad - \frac{\rho n}{i\lambda a^*}(z, f_4)_{L^2} + \frac{\alpha k}{a^*}(\nabla v, \nabla \psi)_{L^2} + \frac{\alpha m}{a^*}(\nabla v, \nabla \varphi)_{L^2} + \frac{\alpha k^*}{a^*}(\nabla v, \nabla \theta)_{L^2} \\
&\quad - \alpha(v, f_6)_{L^2} + \frac{\alpha n b}{i\lambda a^* J}(\nabla v, \nabla \varphi)_{L^2} + \frac{\alpha n m}{i\lambda a^* J}(\nabla v, \nabla \psi)_{L^2} + \frac{\alpha n \xi}{i\lambda a^* J}(v, \varphi)_{L^2} \\
&\quad - \frac{\alpha n^2}{i\lambda a^* J}(v, \theta)_{L^2} - \frac{\alpha n \gamma}{i\lambda a^* J}(v, \Delta u)_{L^2} - \frac{\alpha n}{i\lambda a^*}(v, f_4)_{L^2} - \mu(\Delta u, \theta)_{L^2} \\
&\quad + d\|\theta\|_{L^2}^2 + \gamma(\varphi, \theta)_{L^2} - \rho(f_2, y)_{L^2} - \alpha(f_2, \theta)_{L^2},
\end{aligned}
$$

which implies

$$
\begin{aligned}
\rho\|v\|_{L^2}^2 + \alpha\|\nabla v\|_{L^2}^2 &\leq C\|v\|_{L^2}\|\psi\|_{L^2} + C\|v\|_{L^2}\|\varphi\|_{L^2} + C\|v\|_{L^2}\|\theta\|_{L^2} + \frac{C}{|\lambda|}\|U\|_{\mathcal{H}_\alpha}^2 \\
&\quad + \frac{C}{|\lambda|}\|U\|_{\mathcal{H}_\alpha}\|F\|_{\mathcal{H}_\alpha} + C\alpha\|\nabla v\|_{L^2}\|\nabla \psi\|_{L^2} + C\alpha\|\nabla v\|_{L^2}\|\nabla \varphi\|_{L^2} \\
&\quad + C\|U\|_{\mathcal{H}_\alpha}\|F\|_{\mathcal{H}_\alpha} + \frac{C}{|\lambda|}\|F\|_{\mathcal{H}_\alpha}^2 + C\alpha\|\nabla v\|_{L^2}\|\nabla \theta\|_{L^2} \\
&\quad + C\|\Delta u\|_{L^2}\|\theta\|_{L^2} + C\|\theta\|_{L^2}^2 + C\|\varphi\|_{L^2}\|\theta\|_{L^2}.
\end{aligned}
$$

Then, using (64) and (65) we deduce

$$\frac{\rho}{2}||v||_{L^2}^2 + \frac{\alpha}{2}||\nabla v||_{L^2}^2 \;\leq\; C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_\alpha}||F||_{\mathcal{H}_\alpha} + \frac{C}{|\lambda|}\left(||U||_{\mathcal{H}_\alpha}^2 + ||F||_{\mathcal{H}_\alpha}^2\right)$$

$$+C\alpha||\nabla\varphi||_{L^2}^2 + C||\theta||_{L^2}||U||_{\mathcal{H}_a}. \tag{69}$$

On the other hand, multiplying (59) by $u \in L^2(\Omega)$ if $\alpha = 0$ (or applying in $u \in H_0^1(\Omega)$ if $\alpha > 0$), we obtain

$$\underbrace{i\lambda\rho(v,u)_{L^2}}_{J_3} - \underbrace{i\lambda\alpha(\Delta v,u)_{L^2}}_{J_4} + \mu||\Delta u||_{L^2}^2 - d(\theta,\Delta u)_{L^2} - \underbrace{\gamma(\varphi,\Delta u)_{L^2}}_{J_5} = \rho(f_2,u)_{L^2} - \alpha(f_2,\Delta u)_{L^2}.$$

Then, using (58) into J_3, J_4 and (60) into I_5, we deduce

$$\mu||\Delta u||_{L^2}^2 \;=\; \rho||v||_{L^2}^2 + \alpha||\nabla v||_{L^2}^2 + \rho(v,f_1)_{L^2} - \alpha(v,\Delta f_1)_{L^2} + d(\theta,\Delta u)_{L^2}$$

$$+\frac{\gamma}{i\lambda}(\phi,\Delta u)_{L^2} + \frac{\gamma}{i\lambda}(f_3,\Delta u)_{L^2} + \rho(f_2,u)_{L^2} - \alpha(f_2,\Delta u)_{L^2},$$

which implies, using (64), that

$$\mu||\Delta u||_{L^2}^2 \;=\; \rho||v||_{L^2}^2 + \alpha||\nabla v||_{L^2}^2 + C\left(1 + \frac{1}{|\lambda|}\right)||U||_{\mathcal{H}_\alpha}||F||_{\mathcal{H}_\alpha} + \frac{C}{|\lambda|}||U||_{\mathcal{H}_\alpha}^2 + C||\theta||_{L^2}||U||_{\mathcal{H}_a}. \tag{70}$$

So, doing $\frac{1}{4}$(70)+(69) we have

$$\frac{\mu}{4}||\Delta u||_{L^2}^2 + \frac{\rho}{2}||v||_{L^2}^2 + \frac{\alpha}{2}||\nabla v||_{L^2}^2 \;\leq\; C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_\alpha}||F||_{\mathcal{H}_\alpha} + \frac{C}{|\lambda|}\left(||U||_{\mathcal{H}_\alpha}^2 + ||F||_{\mathcal{H}_\alpha}^2\right)$$

$$+C_3\alpha||\nabla\varphi||_{L^2}^2 + C||\nabla\theta||_{L^2}||U||_{\mathcal{H}_a}. \tag{71}$$

Now, multiplying (61) by φ we have

$$\underbrace{i\lambda J(\phi,\varphi)_{L^2}}_{J_6} + b||\nabla\varphi||_{L^2}^2 + m(\nabla\psi,\nabla\varphi)_{L^2} + \xi||\varphi||_{L^2}^2 - n(\theta,\varphi)_{L^2} - \underbrace{\gamma(\Delta u,\varphi)_{L^2}}_{J_7} = J(f_4,\varphi)_{L^2}. \tag{72}$$

Substituting (60) into J_6 and J_7, we obtain

$$J_6 \;=\; -J||\phi||_{L^2}^2 - J(\phi,f_3)_{L^2} \qquad \text{and} \qquad J_7 \;=\; -\frac{\gamma}{i\lambda}(\Delta u,\phi)_{L^2} - \frac{\gamma}{i\lambda}(\Delta u,f_3)_{L^2}.$$

So, substituting J_6 and J_7 into (72) we deduce

$$J||\phi||_{L^2}^2 \;=\; -J(\phi,f_3)_{L^2} + b||\nabla\varphi||_{L^2}^2 + m(\nabla\psi,\nabla\varphi)_{L^2} + \xi||\varphi||_{L^2}^2 - n(\theta,\varphi)_{L^2}$$

$$+\frac{\gamma}{i\lambda}(\Delta u,\phi)_{L^2} + \frac{\gamma}{i\lambda}(\Delta u,f_3)_{L^2} - J(f_4,\varphi)_{L^2},$$

which implies, using (64) and (65), that

$$||\phi||_{L^2}^2 \;\leq\; C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_\alpha}||F||_{\mathcal{H}_\alpha} + C_2||\nabla\varphi||_{L^2}^2 + \frac{C}{|\lambda|}||U||_{\mathcal{H}_\alpha}^2. \tag{73}$$

Also, multiplying (63) by φ we obtain

$$\underbrace{i\lambda a^*(\theta,\varphi)_{L^2}}_{J_8} + k(\nabla\psi,\nabla\varphi)_{L^2} + m||\nabla\varphi||_{L^2}^2 + k^*(\nabla\theta,\nabla\varphi)_{L^2} + \underbrace{n(\phi,\varphi)_{L^2}}_{J_9} + \underbrace{d(\Delta v,\varphi)_{L^2}}_{J_{10}} = a^*(f_6,\varphi)_{L^2}. \tag{74}$$

Then, using (60) into J_6 and J_7 we deduce

$$J_8 \;=\; -a^*(\theta,\phi)_{L^2} - a^*(\theta,f_3)_{L^2} \qquad \text{and} \qquad J_9 \;=\; -\frac{n}{i\lambda}||\phi||_{L^2}^2 - \frac{n}{i\lambda}(\phi,f_3)_{L^2}.$$

Now, for J_{10}, we need two type of estimates depending on $\alpha = 0$ or $\alpha > 0$. In fact, for $\alpha = 0$, using (58) and (60) into J_{10} we obtain

$$J_{10} \;=\; d(i\lambda\Delta u - \Delta f_1,\varphi)_{L^2} \;=\; d(\Delta u,-i\lambda\varphi)_{L^2} - d(\Delta f_1,\varphi)_{L^2}$$

$$=\; d(\Delta u,-\phi - f_3)_{L^2} - d(\Delta f_1,\varphi)_{L^2}$$

$$=\; -d(\Delta u,\phi)_{L^2} - d(\Delta u,f_3)_{L^2} - d(\Delta f_1,\varphi)_{L^2}.$$

Consequently, substituting the new formulas of J_8, J_9 and J_{10} into (74) we obtain (for $\alpha = 0$)

$$m||\nabla\varphi||_{L^2}^2 = a^*(\theta,\phi)_{L^2} + a^*(\theta,f_3)_{L^2} - k(\nabla\psi,\nabla\varphi)_{L^2} - k^*(\nabla\theta,\nabla\varphi)_{L^2} + \frac{n}{i\lambda}||\phi||_{L^2}^2 + \frac{n}{i\lambda}(\phi,f_3)_{L^2}$$

$$+d(\Delta u,\phi)_{L^2} + d(\Delta u,f_3)_{L^2} + d(\Delta f_1,\varphi)_{L^2} + a^*(f_6,\varphi)_{L^2},$$

which implies, applying (64) and (65), that (for $\alpha = 0$)

$$||\nabla\varphi||_{L^2}^2 \leq C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_0}||F||_{\mathcal{H}_0} + C||\nabla\theta||_{L^2}||U||_{\mathcal{H}_0} + \frac{C_1}{|\lambda|}||U||_{\mathcal{H}_0}^2 + C||\Delta u||_{L^2}||\phi||_{L^2}. \quad (75)$$

So, doing $2C_2$(75)+(73), we can deduce (for $\alpha = 0$) that

$$C_2||\nabla\varphi||_{L^2}^2 + \frac{1}{2}||\phi||_{L^2}^2 \leq C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_0}||F||_{\mathcal{H}_0} + C||\nabla\theta||_{L^2}||U||_{\mathcal{H}_0}$$

$$+\frac{C}{|\lambda|}||U||_{\mathcal{H}_0}^2 + C_3||\Delta u||_{L^2}^2. \quad (76)$$

On the other hand, for the case $\alpha > 0$, using (60) into J_{10} we obtain

$$J_{10} = d\langle v, \Delta\varphi\rangle_{H_0^1 \times H^{-1}}$$

$$= \frac{d}{\gamma}\langle v, i\lambda\rho v - i\lambda\alpha\Delta v + \mu\Delta^2 u - d\Delta\theta - \rho f_2 + \alpha\Delta f_2\rangle_{H_0^1 \times H^{-1}}$$

$$= -\frac{i\lambda d\rho}{\gamma}||v||_{L^2}^2 - \frac{i\lambda d\alpha}{\gamma}||\nabla v||_{L^2}^2 + \frac{d\mu}{\gamma}(\Delta v,\Delta u)_{L^2} + \frac{d^2}{\gamma}(\nabla v,\nabla\theta)_{L^2}$$

$$-\frac{d\rho}{\gamma}(v,f_2)_{L^2} - \frac{d\alpha}{\gamma}(\nabla v,\nabla f_2)_{L^2}$$

$$= -\frac{i\lambda d\rho}{\gamma}||v||_{L^2}^2 - \frac{i\lambda d\alpha}{\gamma}||\nabla v||_{L^2}^2 + \frac{i\lambda d\mu}{\gamma}||\Delta u||_{L^2}^2 - \frac{d\mu}{\gamma}(\Delta f_1,\Delta u)_{L^2}$$

$$+\frac{d^2}{\gamma}(\nabla v,\nabla\theta)_{L^2} - \frac{d\rho}{\gamma}(v,f_2)_{L^2} - \frac{d\alpha}{\gamma}(\nabla v,\nabla f_2)_{L^2},$$

where (58) was also used in the last equality. Then substituting J_8, J_9 and J_{10} into (74) we have (for $\alpha > 0$)

$$m||\nabla\varphi||_{L^2}^2 = a^*(\theta,\phi)_{L^2} + a^*(\theta,f_3)_{L^2} - k(\nabla\psi,\nabla\varphi)_{L^2} - k^*(\nabla\theta,\nabla\varphi)_{L^2} + \frac{n}{i\lambda}||\phi||_{L^2}^2$$

$$+\frac{n}{i\lambda}(\phi,f_3)_{L^2} + \frac{i\lambda d\rho}{\gamma}||v||_{L^2}^2 + \frac{i\lambda d\alpha}{\gamma}||\nabla v||_{L^2}^2 - \frac{i\lambda d\mu}{\gamma}||\Delta u||_{L^2}^2 + \frac{d\mu}{\gamma}(\Delta f_1,\Delta u)_{L^2}$$

$$-\frac{d^2}{\gamma}(\nabla v,\nabla\theta)_{L^2} + \frac{d\rho}{\gamma}(v,f_2)_{L^2} + \frac{d\alpha}{\gamma}(\nabla v,\nabla f_2)_{L^2} + a^*(f_6,\varphi)_{L^2}.$$

Consequently, taking the real part we obtain

$$m||\nabla\varphi||_{L^2}^2 = \mathfrak{Re}\Big\{ a^*(\theta,\phi)_{L^2} + a^*(\theta,f_3)_{L^2} - k(\nabla\psi,\nabla\varphi)_{L^2} - k^*(\nabla\theta,\nabla\varphi)_{L^2} + \frac{n}{i\lambda}(\phi,f_3)_{L^2}$$

$$+\frac{d\mu}{\gamma}(\Delta f_1,\Delta u)_{L^2} - \frac{d^2}{\gamma}(\nabla v,\nabla\theta)_{L^2} + \frac{d\rho}{\gamma}(v,f_2)_{L^2} + \frac{d\alpha}{\gamma}(\nabla v,\nabla f_2)_{L^2} + a^*(f_6,\varphi)_{L^2}\Big\},$$

which implies, using (64) and (65), that (for $\alpha > 0$)

$$||\nabla\varphi||_{L^2}^2 \leq C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_\alpha}||F||_{\mathcal{H}_\alpha} + C||\nabla\theta||_{L^2}||U||_{\mathcal{H}_\alpha}. \quad (77)$$

Consequently, doing $\frac{1}{2C_2}$(73)+(77), we obtain (for $\alpha > 0$)

$$\frac{1}{2}||\nabla\varphi||_{L^2}^2 + \frac{1}{2C_2}||\phi||_{L^2}^2 \leq C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_\alpha}||F||_{\mathcal{H}_\alpha} + C||\nabla\theta||_{L^2}||U||_{\mathcal{H}_\alpha} + \frac{1}{|\lambda|}||U||_{\mathcal{H}_\alpha}^2. \quad (78)$$

Finally, let us combine all the estimates to prove condition (13) in each case $\alpha = 0$ and $\alpha > 0$. In fact, for the case $\alpha = 0$ doing (71)$+\dfrac{\mu}{8C_3}$(76), we have

$$
\frac{\mu}{8}||\Delta u||_{L^2}^2 + \frac{\rho}{2}||v||_{L^2}^2 + C_4||\nabla\varphi||_{L^2}^2 + C_5||\phi||_{L^2}^2 \leq C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_0}||F||_{\mathcal{H}_0} + C||\nabla\theta||_{L^2}||U||_{\mathcal{H}_0}
$$
$$
+ \frac{C}{|\lambda|}\left(||U||_{\mathcal{H}_0}^2 + ||F||_{\mathcal{H}_0}^2\right),
$$

which implies, using (64) and (65), that (for $\alpha = 0$)

$$
||U||_{\mathcal{H}_0}^2 \leq C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_0}||F||_{\mathcal{H}_0} + \frac{C}{|\lambda|}||U||_{\mathcal{H}_0}^2 + C\left(\frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||F||_{\mathcal{H}_0}^2. \tag{79}
$$

Similarly, for the case $\alpha > 0$, doing (71)$+4C_3\alpha$(78) we can deduce (after calculations)

$$
\frac{\mu}{4}||\Delta u||_{L^2}^2 + \frac{\rho}{2}||v||_{L^2}^2 + \frac{\alpha}{2}||\nabla v||_{L^2}^2 + C_3\alpha||\nabla\varphi||_{L^2}^2 + C_6||\phi||_{L^2}^2 \leq C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_\alpha}||F||_{\mathcal{H}_\alpha}
$$
$$
+ \frac{C}{|\lambda|}\left(||U||_{\mathcal{H}_\alpha}^2 + ||F||_{\mathcal{H}_\alpha}^2\right)
$$
$$
+ C||\nabla\theta||_{L^2}||U||_{\mathcal{H}_a},
$$

which implies, combining with (64) and (65), that (for $\alpha = 0$)

$$
||U||_{\mathcal{H}_a}^2 \leq C\left(1 + \frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||U||_{\mathcal{H}_\alpha}||F||_{\mathcal{H}_\alpha} + \frac{C}{|\lambda|}||U||_{\mathcal{H}_\alpha}^2 + C\left(\frac{1}{|\lambda|} + \frac{1}{|\lambda|^2}\right)||F||_{\mathcal{H}_\alpha}^2. \tag{80}
$$

Therefore, for all $\alpha \geq 0$, choosing $|\lambda| > M$ with M large enough, we obtain from (79) and (80) that

$$
\frac{1}{2}||U||_{\mathcal{H}_\alpha}^2 \leq C_7||U||_{\mathcal{H}_\alpha}||F||_{\mathcal{H}_\alpha} + C_7||F||_{\mathcal{H}_\alpha}^2,
$$

which implies

$$
||U||_{\mathcal{H}_\alpha} \leq C_8||F||_{\mathcal{H}_\alpha} \qquad \text{for all} \qquad |\lambda| > M,
$$

where C_8 is independent of λ, U and F. Additionally, using that resolvent operators $R(i\lambda; \mathcal{A}_\alpha) := (i\lambda I - \mathcal{A}_\alpha)^{-1}$ are bounded on bounded domains, then $||U||_{\mathcal{H}_\alpha} \leq C_8||F||_{\mathcal{H}_\alpha}$ for all $\lambda \in [-M, M]$, which completes the proof of (13). $\qquad\qquad\square$

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