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Title: The Gap between an Automorphism and its Inverse

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The Gap between an Automorphism and its Inverse

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Abstract

We introduce the function α_G (and β_G), defined in [11], that measures the gap between an (outer) automorphism of G and its inverse. We give an alternative proof of the lower bound for α_{F_r} of the free groups, and give an improvement for the lower bound of β_{F_r}. Furthermore, for the first time, a study of the function $\alpha_{\mathrm{BS}(1,N)}$ for the Baumslag-Solitar groups $\mathrm{BS}(1, N)$, $|N| > 1$, is made, and we prove that it grows linearly. Finally, in an independent way, we define the same concept over the virtual automorphisms and prove that the equivalent function for the free groups has an exponential lower bound.

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Contents

1 Introduction

In this work we analyse and explain a way to measure the difficulty of inverting an automorphism of a finitely generated group G, following the definition first introduced by M. Ladra, P. V. Silva and E. Ventura in [11]. This article serves as main source and motivation to a great part of our work.

We reintroduce the concept of norm of an automorphism, which gives an idea of its complexity, and the auto-gap function $\alpha_G(n)$ that measures the maximal difference between a norm of an automorphism and that of its inverse, intuitively telling us how difficult it is to invert the automorphism. The same construction can be done for the outer automorphisms of a group, with the analogous outer-gap function $\beta_G(n)$.

In this work, we dedicate the majority of our time in studying these concepts for two important families of groups: the free groups and the Baumslag-Solitar groups $\mathrm{BS}(1,N)$ for $|N|>1$.

For the free groups, we give an alternative proof of the results given in [11]. We try to improve the bounds described in the article, by giving a description of all its automorphism from a different point of view. Even though we are unable to prove stricter bounds for $\alpha_{F_r}(n)$, we expose some results that might serve as foundations for a future proof and make some new observations. We also study the growth of $\beta_{F_r}(n)$ and we are capable of giving a slight improvement to its lower bound for the free groups of rank $r>2$, from n^{r-1} given in [11] to n^r. We also give a complete description of these functions for the free group of rank 2, giving a new version of the proof in [11].

For the first time, the function $\alpha_{\mathrm{BS}(1,N)}(n)$ is studied for a new non-trivial case: the Baumslag-Solitar groups $\mathrm{BS}(1,N)$ for $|N|>1$. We make an exhaustive description of its automorphisms that allow us to prove that $\alpha_{\mathrm{BS}(1,N)}(n)$ grows linearly. This is an interesting result, as it shows that the task of finding these functions is quite difficult and complex in general.

We end this work with a new development motivated by an open question from [11]: in this paper it was asked whether there exists a group G whose auto-gap function α_G is exponential. We are not able to answer this question (which, as far as we know, remains open) but we define a variation of α_G, namely $\nu_{G,\mathcal{S}}$, and prove that it is certainly exponential for free groups of rank $r \geq 2$. This new function $\nu_{G,\mathcal{S}}$, called virtual-gap function and associated to each finitely generated group G together with a fixed ordered finite generating set $\mathcal{S}$, measures similarly the worst case difference between the norm of a virtual automorphism and its inverse. It generalises α_G in the sense that the automorphism of G are special cases of virtual automorphisms, while the set of these latter ones is much bigger; in particular, $\alpha_{G,S}(n) \leq \nu_{G,\mathcal{S}}(n)$. While far from being an answer to the question posted in [11], our result that $\nu_{F_r,\mathcal{S}}$ is at least exponential (where F_r is the free group of rank $r \geq 2$ and $\mathcal{S}$ is a free basis for it), shows that functions similar in spirit to α_G can certainly be exponential.

To define this function, we need to introduce the concept of Schreier graphs and some related results. A technical problem here is that we have not been able to show that $\nu_{G,\mathcal{S}}$ is independent of the chosen set of generators, even up to multiplicative constants (as it is nicely the case for the α_G function).

Finally, we would like to remark that in this thesis, we have also tried to prove many results which apparently seem true, without success, which includes trying to prove that $\alpha_G(n) \preceq \beta_G(n)n$ or improving the bounds of the free groups.

Key words: group, automorphism, inverse automorphism, auto-gap function, norm of an automorphism, outer automorphism, outer-gap function, free group, Baumslag-Solitar group, virtual automorphisms, Schreier graph, virtual-gap function.

2 First definitions

Let's introduce the notation and a few basic observations we are going to use throughout this thesis: G will denote a finitely generated group with a finite generating set $S = \{x_1, \ldots, x_s\}$ such that $1_G \notin S$. By definition of generating set, all elements g of G can be written as

$$g = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_l}^{\varepsilon_l}$$

with $i_1, \ldots, i_l \in \{1, \ldots, s\}$, $\varepsilon_i = \pm 1$ and $0 \leq l < \infty$. We say this way of writing $g \in G$ is an *expression of g over S*, and the set of all such expressions are called the *words in S*. It is not unusual to find two different expressions for the same element, such as $(0,1) + (1,0)$ and $(1,0) + (0,1)$ for $(1,1)$ in the group $\mathbb{Z}^2$ over the generating set $S = \{(1,0), (0,1)\}$ (these expressions are different, but when viewed in $\mathbb{Z}^2$, they correspond to the same element). We can also have an expression with a trivial relation, that is, the expression containing $x_i x_i^{-1}$ or $x_i^{-1} x_i$ for any $x_i \in S$. These expressions are trivial when viewed in G, so given any expression, we can apply a reduction, that recursively removes all $x_i x_i^{-1}$ or $x_i^{-1} x_i$ from the expression, until no more are left. Any expression without trivial relations is said to be *reduced*.

Given a fixed finite generating set, we can define a metric over G with respect to S, called the *word metric*, such that for all $g \in G$, $|g|_S$ equals the length l of the shortest expression of g over S. This is a well-defined norm and satisfies the following properties: $|1_G|_S = 0$, $|g|_S = |g^{-1}|_S$, $|gg'|_S \leq |g|_S + |g'|_S$ and $|g^n| \leq |n| \cdot |g|_S$ for all $g, g' \in G$ and $n \in \mathbb{Z}$ [8].

By taking S to be finite, we obtain that the balls in this metric (in particular, those centred at the origin, the trivial element) are finite, that is,

$$\mathrm{B}_S(n) = \{g \in G \mid |g|_S \leq n\}$$

is finite for all non-negative integers n. This allows us to define the notion of growth in G by studying how $|\mathrm{B}_S(n)|$ grows when $n \to \infty$. The first problem we encounter, is that this growth depends on the generating set we picked as, in general, $|\mathrm{B}_S(n)| \neq |\mathrm{B}_{S'}(n)|$ for different finite generating sets $S \neq S'$. But this can be solved if we introduce the following equivalence [8].

Definition 2.1. Let $f, g : \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$ be two non-decreasing functions. We say that *f is dominated by g* (or that *g dominates f*), denoted $f \preceq g$, if there exists a constant $C \geq 1$ such that $f(n) \leq Cg(Cn)$ for all $n > 0$. We say that f and g are equivalent, denoted $f \sim g$, if $f \preceq g$ and $f \succeq g$. Likewise, we'll write that $f \prec g$ if $f \preceq g$ and $f \not\sim g$.

The relation $\sim$ is a well-defined equivalence, and it can be easily verified that the following results are met: polynomials dominate those of lower degree: $0 \prec 1 \prec n \prec n^2 \prec \cdots \prec n^k \prec \cdots$, and any positive polynomial $p(n)$ is equivalent to n^k if and only if it has degree k. A function that is equivalent to a polynomial is said to have *polynomial growth*, and this notion is extended to include those of *constant growth*, *linear growth*, *quadratic growth*, etc. Any function that dominates all polynomials is said to have *superpolynomial growth*. Some examples of functions of superpolynomial growth include e^{n^α} for all $\alpha > 0$. Note that $\mathrm{e}^{n^\alpha} \prec \mathrm{e}^{n^\beta}$ if and only if $\alpha < \beta$ and $a^{n^\alpha} \sim b^{n^\alpha}$ for all $a, b > 1$. If $f \sim \mathrm{e}^n$, then we say that f has *exponential growth*, and if $f \prec \mathrm{e}^n$, we say that f has *subexponential growth*. A function that has both superpolynomial and subexponential growth is said to have *intermediate growth*, like, for example, e^{n^α} for $0 < \alpha < 1$. This list is not exhaustive, since there are other functions dominating all the previous growths, such as $n!$ or n^n, and others in between, such as $1 \prec \sqrt{n} \prec n$ [2, 8].

Going back to our explanation, let us define the *growth function of G relative to S* as the non-decreasing function $f_S : \mathbb{N} \to \mathbb{N}$ defined as $f_S(n) = |\mathrm{B}_S(n)|$ (usually it is denoted as β_S, but we are dropping this notation to avoid confusion with definition 2.2). This depends on S, but it can be proven that $f_S(n) \sim f_{S'}(n)$, for any two finite generating sets S and S' of G. This allows us to define the growth rate of finitely generated groups, independently of the choice of S. It is easy to see that, for any finitely generated group G and any finite set of generators S, f_S is of at most exponential growth [2, 8]. We are not going to enter in more detail into the growth rate of groups.

Since all the calculations and results we are going to state in this work depend in some way on the generating set we consider, it is necessary we find an equivalence such that the results don't depend on the chosen generating set. In particular, our functions will be equivalent using this equivalence relation we defined.

2.1 Inversion of automorphisms

In this subsection, we introduce the notions on the automorphisms and outer automorphisms that are needed to understand the results we will give in the thesis. The automorphism group $\mathrm{Aut}(G)$ of a group G is defined as the set of all automorphisms of G equipped with the conjugation as its operation. We let automorphisms act on the right, $g \mapsto g\varphi = (g)\varphi$; and compositions work accordingly: for all $\varphi_1, \varphi_2 \in \mathrm{Aut}(G)$, $\varphi_1\varphi_2$ acts as $g \mapsto g\varphi_1 \mapsto g\varphi_1\varphi_2$.

We define the subgroup of inner automorphism group $\mathrm{Inn}(G)$ as the (normal) subgroup of conjugations of $\mathrm{Aut}(G)$, which is isomorphic to the quotient of G by its centre. In this work, we are denoting the conjugations as γ_g ($h\gamma_g = g^{-1}hg$).

We define the outer automorphism group as the quotient

$$\mathrm{Out}(G) = {}^{\mathrm{Aut}(G)}/_{\mathrm{Inn}(G)},$$

where we are denoting the representative of $\varphi \in \mathrm{Aut}(G)$ in $\mathrm{Out}(G)$ as $[\varphi]$ [7].

Now we introduce the definitions of norm of an automorphism and an outer automorphism, that were first given in [11]. We know that any automorphism φ is determined by the image of φ over the elements of S. We define the *norm of φ with respect to $S = \{x_1, \ldots, x_s\}$* (or, the *S-norm* for short) as

$$\|\varphi\|_S = \sum_{i=1}^{s} |x_i\varphi|_S;$$

and the *norm of* $\Phi \in \mathrm{Out}(G)$ *with respect to* S as

$$\|\Phi\|_S = \min_{\phi \in \Phi}\{\|\phi\|\}.$$

Once again, the finiteness of S guarantees that $\|\varphi\|_S$ is always defined and that there is a finite number of automorphisms of a certain bounded norm, and consequently, there's also a finite number of outer automorphisms of a bounded norm. Note also that since $x_i\varphi$ is not trivial for any i, $\|\varphi\|_S \geq s$ and $\|[\varphi]\|_S \geq s$ for all $\varphi \in \mathrm{Aut}(G)$. Also, we have that $|g\varphi|_S \leq |g|_S\|\varphi\|_S$: if $|g|_S = l$, then there exists a word of length l on the elements in S such that $g = x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}\cdots x_{i_l}^{\varepsilon_l}$ and so

$$|g\varphi|_S = |(x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}\cdots x_{i_l}^{\varepsilon_l})\varphi|_S \leq \sum_{j=1}^{l}|x_{i_j}^{\varepsilon_j}\varphi|_S \leq \sum_{j=1}^{l}\|\varphi\|_S = l\|\varphi\|_S = |g|_S\|\varphi\|_S.$$

As the paper states, it is natural to compare $\|\varphi\|_S$ and $\|\varphi^{-1}\|_S$, and if one happens to be significantly larger than the other, we can intuitively say that inverting the automorphism φ is hard, and respectively for an outer automorphism. The following definition arises with the purpose of measuring the worst case difference between $\|\varphi\|_S$ and $\|\varphi^{-1}\|_S$ (and $\|\Phi\|_S$ and $\|\Phi^{-1}\|_S$).

Definition 2.2 (Auto-gap and outer-gap functions [11])**.** We define the *automorphism inversion gap function of* G *with respect to* S (or *auto-gap function*, for short) as the function $\alpha_{G,S} : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\alpha_{G,S}(n) = \max\{\|\varphi^{-1}\|_S \mid \varphi \in \mathrm{Aut}(G), \|\varphi\|_S \leq n\}.$$

Analogously, we define the *outer automorphism inversion gap function of* G *with respect to* S (or *outer-gap function*) as the function $\beta_{G,S} : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\beta_{G,S}(n) = \max\{\|\Phi^{-1}\|_S \mid \Phi \in \mathrm{Out}(G), \|\Phi\|_S \leq n\}.$$

In both definitions, the maximum of the empty set is defined as 0.

We will more frequently use the notation α_S and β_S when the G is clear from the context. Observe first of all that both functions are well-defined and non-decreasing. Also, if G is not trivial, $\alpha_S(n) = \beta_S(n) = 0$ if and only if $n < s$, since the identity has always norm s.

The first result we have is a direct comparison between both functions:

Proposition 2.3. *For any group* G *and any generating set* S, $\beta_S(n) \leq \alpha_S(n)$.

Proof. Let Φ be the outer automorphisms such that $\|\Phi\|_S \leq n$ and $\|\Phi^{-1}\| = \beta(n)$. Let φ be a minimal element in Φ (such that $\varphi \in \Phi$ and $\|\varphi\|_S = \|\Phi\|_S$). Then, $\varphi^{-1} \in \Phi^{-1}$. Therefore,

$$\beta_S(n) = \|\Phi^{-1}\| \leq \|\varphi^{-1}\| \leq \alpha_S(n),$$

because $\|\varphi\|_S = \|\Phi\|_S \leq n$. ☐

Once again, this function depends on the choice of S, but one can see that for any pair S, T of finite generating sets of G, we have that $\alpha_S \sim \alpha_T$, with $\sim$ as in Definition 2.1. This result requires the following lemma from [11].

Lemma 2.4. *Let G be a finitely generated group and let $S = \{x_1, \dots, x_s\}$ and $T = \{y_1, \dots, y_t\}$ be two generating sets of G. Then there exists a constant $C \geq 1$ such that, for all $\varphi \in \mathrm{Aut}(G)$ and $\Phi \in \mathrm{Out}(G)$,*

$$\frac{1}{C}\|\varphi\|_T \leq \|\varphi\|_S \leq C\|\varphi\|_T,$$
$$\frac{1}{C}\|\Phi\|_T \leq \|\Phi\|_S \leq C\|\Phi\|_T.$$

Proof. We take $M = \max\{|y_j|_S \mid j = 1, \dots, t\}$ and $N = \max\{|x_i|_T \mid i = 1, \dots, s\}$, and we let $C = MNst \geq 1$. We can easily verify that $|g|_S \leq |g|_T M$ and $|g|_T \leq |g|_S N$ for all $g \in G$. Then, for every $\varphi \in \mathrm{Aut}(G)$, we have

$$\begin{aligned}
\|\varphi\|_S &= |x_1\varphi|_S + \dots + |x_s\varphi|_S \\
&\leq |x_1\varphi|_T M + \dots + |x_s\varphi|_T M \\
&\leq M(|x_1|_T\|\varphi\|_T + \dots + |x_s|_T\|\varphi\|_T) \\
&= M(|x_1|_T + \dots + |x_s|_T)\|\varphi\|_T \\
&\leq MNs\|\varphi\|_T \\
&\leq C\|\varphi\|_T.
\end{aligned}$$

Furthermore, if ψ is an automorphism in Φ such that $\|\Phi\|_T = \|\psi\|_T$, then

$$\|\Phi\|_S = \min_{\phi \in \Phi}\{\|\phi\|_S\} \leq \|\psi\|_S \leq C\|\psi\|_T = C\|\Phi\|_T.$$

By symmetry, we have $\|\varphi\|_T \leq C\|\varphi\|_S$ and $\|\Phi\|_T \leq C\|\Phi\|_S$, so the statement follows. □

Theorem 2.5. *Let G be a finitely generated group and let $S = \{x_1, \dots, x_s\}$ and $T = \{y_1, \dots, y_t\}$ be two generating sets of G. Then, $\alpha_S \sim \alpha_T$ and $\beta_S \sim \beta_T$.*

Proof. We need to see there exists a constant $C \geq 1$ such that

$$\frac{1}{C}\alpha_T\left(\left\lfloor\frac{n}{C}\right\rfloor\right) \leq \alpha_S(n) \leq C\alpha_T(Cn).$$

We use the constant C from the previous lemma. If $n < s$, then $\alpha_T(\lfloor\frac{n}{C}\rfloor) = \alpha_S(n) = 0$ so the inequality holds. If $n \geq s$, then

$$\begin{aligned}
\alpha_S(n) &= \max\{\|\varphi^{-1}\|_S \mid \varphi \in \mathrm{Aut}(G), \|\varphi\|_S \leq n\} \\
&\leq \max\{\|\varphi^{-1}\|_S \mid \varphi \in \mathrm{Aut}(G), \|\varphi\|_T \leq Cn\} \\
&\leq \max\{C\|\varphi^{-1}\|_T \mid \varphi \in \mathrm{Aut}(G), \|\varphi\|_T \leq Cn\} \\
&= C\max\{\|\varphi^{-1}\|_T \mid \varphi \in \mathrm{Aut}(G), \|\varphi\|_T \leq Cn\} \\
&= C\alpha_T(Cn).
\end{aligned}$$

By symmetry, $\alpha_T(n) \leq C\alpha_S(Cn)$ and therefore,

$$\frac{1}{C}\alpha_T\left(\left\lfloor\frac{n}{C}\right\rfloor\right) \leq \alpha_S\left(C\left\lfloor\frac{n}{C}\right\rfloor\right) \leq \alpha_S(n)$$

proving the result for α. Changing α for β will give us the other result. □

This theorem allows us to define a class of functions for each group, which is independent of the choice of generating set S. We'll denote them as α_G and β_G.

Let's give some small useful results before we start with the main examples.

Proposition 2.6. *Let $\varphi \in \mathrm{Aut}(G)$ and $x_i \in S$ where S is a finite generator set of G. Then,*

$$\big| \|\varphi\gamma_{x_i}\|_S - \|\varphi\|_S \big| \leq 2|S|.$$

Proof. The proof is direct if we consider the extreme cases. It is clear that if we apply γ_{x_i} to $(x_j)\varphi$, we add two letters, which might or might not cancel out with the start or end of $(x_j)\varphi$. So the maximum difference between $|(x_j)\varphi|_S$ and $|(x_j)\varphi\gamma_{x_i}|_S$ is 2. Now, since $\|\varphi\|_S$ is the sum of the length of $(x_j)\varphi$ for all $x_j \in S$, we have that the maximum difference is $2|S|$, with the maximal values corresponding to all or none of letters cancelling out. □

Applying the same reasoning, we can prove:

Corollary 2.7. *Let $\varphi \in \mathrm{Aut}(G)$ and $g \in G$. Then,*

$$\big| \|\varphi\gamma_g\|_S - \|\varphi\|_S \big| \leq 2|S||g|_S.$$

Note that we could have a group without automorphisms of a certain norm. The following proposition tells us that there cannot exist arbitrarily big intervals I such that no automorphism has norm in I, if certain conditions are met.

Proposition 2.8. *Let G be a finitely generated group such that $\mathrm{Inn}(G) \cong {}^{G}/_{Z(G)}$ is infinite, and let S be a finite generating set for G. Then $N = \{\|\varphi\|_S \mid \varphi \in \mathrm{Aut}(G)\}$ intersects non-trivially with all positive intervals of length larger than $2|S|$.*

Proof. Since $\mathrm{Inn}(G)$ is infinite and there's only a finite number of automorphisms of any fixed norm, it is clear that we have conjugations of norm as large as we want.

Let $I = (a, b)$ be a positive interval such that $|b - a| > 2|S|$. Take $g \in G$ such that $\|\gamma_g\|_S \geq b$, and consider a reduced expression of $g = x_{i_1}^{\varepsilon_1} \cdots x_{i_l}^{\varepsilon_l}$. Let $g_k = x_{i_1}^{\varepsilon_1} \cdots x_{i_k}^{\varepsilon_k}$ for all $k \leq l$. By Proposition 2.6,

$$\big| \|\varphi\gamma_{g_{k+1}}\|_S - \|\varphi\gamma_{g_k}\|_S \big| \leq 2|S|,$$

so the sequence $\{\|\varphi\gamma_{g_k}\|_S\}_{k=0}^{l}$ goes from $\|\mathrm{id}\|_S = |S|$ to $\|\gamma_g\|_S$ with a maximum difference of $2|S|$ between two consecutive terms. This implies that $\{\|\varphi\gamma_{g_k}\|_S\}_{k=0}^{l}$ intersects non-trivially with I. □

Theorem 2.9. *The group $\mathrm{Aut}(G)$ is finite if and only if $\alpha_G(n) \sim 1$, except when G is trivial, where $\alpha_G(n) \sim 0$.*

If $\mathrm{Inn}(G) \cong {}^{G}/_{Z(G)}$ is infinite, then $n \preceq \alpha_G(n)$.

Proof. If $\mathrm{Aut}(G)$ is finite, there exists $N \geq 0$ such that $\|\varphi\| \leq N$ for all $\varphi \in \mathrm{Aut}(G)$. If N is the smallest value such that the previous verifies, then it implies that $\alpha_S(n) = N$ for all $n \geq n_0$, so $\alpha_S(n) \sim N$. Observe that we have $N = 0$ only in the trivial group, since the trivial group is generated by the empty set and therefore, is the only group with no automorphisms of positive norm (the identity has always norm s). Conversely, if $\alpha_S(n) \sim 1$, then there exists a value N such that $\alpha_S(n) = N$ for all $n > n_0$, which implies that $\mathrm{Aut}(G)$ is finite.

Now suppose $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ is infinite. We know that α_S for any S tends to infinity, since there's only a finite number of automorphisms of a bounded norm. Suppose $n \not\preceq \alpha_S(n)$, so for every $C \geq 1$ there exists an $m > 0$, such that $C\alpha_S(Cm) < m$.

Choose $C = (2|S|)^2$, and let $m > 0$ be an integer such that $C\alpha_S(Cm) < m$. By Proposition 2.8, there exists an automorphism ψ such that $Cm - 2|S| \leq \|\psi\|_S \leq Cm$, and its inverse has norm $\|\psi^{-1}\|_S \leq \lfloor \frac{m}{C} \rfloor$ because $C\alpha_S(Cm) < m$. Then,

$$\begin{aligned}\alpha_S(\lfloor \tfrac{m}{C} \rfloor) &= \max\{\|\varphi^{-1}\|_S \mid \varphi \in \mathrm{Aut}(G), \|\varphi\|_S \leq \lfloor \tfrac{m}{C} \rfloor\} \\ &= \max\{\|\varphi^{-1}\|_S \mid \varphi \in \mathrm{Aut}(G), \|\varphi\|_S \leq \tfrac{m}{C}\} \\ &= \max\{\|\varphi\|_S \mid \varphi \in \mathrm{Aut}(G), \|\varphi^{-1}\|_S \leq \tfrac{m}{C}\} \\ &\geq \|\psi\|_S \geq Cm - 2|S|,\end{aligned}$$

and

$$(2|S|)^2 m - 2|S| = Cm - 2|S| \leq \alpha_S(\lfloor \tfrac{m}{C} \rfloor) \leq C\alpha_S(Cm) < m,$$

which is a contradiction for all $m \geq 1$. Hence, $n \preceq \alpha_G(n)$. □

This proposition doesn't give a result for the cases where $\mathrm{Aut}(G)$ is infinite and $\mathrm{Inn}(G)$ is finite, such as some infinite abelian groups, which brings up the question of whether there exists a group G such that $1 \prec \alpha_G(n) \prec n$. If it were to exist, then it would be virtually abelian on one hand, but it also means that there would exist positive intervals as large as we want that don't intersect with $\{\|\varphi\|_S \mid \varphi \in \mathrm{Aut}(G)\}$, because if not, we could recreate the proof of the theorem, and we would be done.

The same question can be made for $\mathrm{Out}(G)$, on whether there exists a group such that $1 \prec \beta_G(n) \prec n$. Note that we can see that the analogous result of the theorem holds whenever $\mathrm{Out}(G)$ is finite.

3 Free groups and its auto-gap function

In this section we are giving an alternative construction for the results of [11] relevant to the free groups F_r of rank r. It is known that the automorphism group of the free group $F_1 \cong \mathbb{Z}$ is $\mathbb{Z}_2$, so $\alpha_{F_1} \sim 1$ by Theorem 2.9. For the non-abelian free groups ($r \geq 2$), we are using the following classical theorem to give structure to $\mathrm{Aut}(F_r)$, that will allow us to make some interesting constructions.

Theorem 3.1 (Nielsen [9])**.** *Let $S = \{x_1, \ldots, x_r\}$ be a basis of the free group F_r, $r \geq 2$. Then its automorphism group is generated by these four automorphisms (three in the case of $r = 2$)*

$$\begin{gathered}\alpha_1 = \{x_1 \mapsto x_2,\, x_2 \mapsto x_3,\, \ldots,\, x_n \mapsto x_1\} \\ \alpha_2 = \{x_1 \mapsto x_2,\, x_2 \mapsto x_1,\, x_i \mapsto x_i \text{ for all } i \neq 1, 2\} \\ \alpha_3 = \{x_1 \mapsto x_1^{-1},\, x_i \mapsto x_i \text{ for all } i \neq 1\} \\ \alpha_4 = \{x_1 \mapsto x_1 x_2,\, x_i \mapsto x_i \text{ for all } i \neq 1\}.\end{gathered}$$

This theorem gives us a minimal generating set for $\mathrm{Aut}(F_r)$. Observe that $\mathrm{Sym}_r := \langle \alpha_1, \alpha_2 \rangle$ is the subgroup of all permutations of the elements of S. We'll abuse the notation and denote every $\sigma \in \mathrm{Sym}_r$ with the same symbol as the corresponding permutation of $\{1, \dots, r\}$, since they are in bijection. We will also denote the transposition $x_i \leftrightarrow x_j$ as $\tau_{i,j}$, which is an element of order 2.

We also define, for all $i \neq j$, the following automorphisms

$$\begin{aligned}
\iota_i &= \{x_i \mapsto x_i^{-1},\, x_k \mapsto x_k \text{ for all } k \neq i\}, \\
\mu_{i,j} &= \{x_i \mapsto x_i x_j,\, x_k \mapsto x_k \text{ for all } k \neq i\}, \\
\lambda_{i,j} &= \{x_i \mapsto x_j x_i,\, x_k \mapsto x_k \text{ for all } k \neq i\},
\end{aligned}$$

which are well-defined automorphisms of the free group because they correspond to the following combinations of elements from Theorem 3.1:

$$\begin{aligned}
\iota_i &= \tau_{1,i}\alpha_3\tau_{1,i}, \\
\mu_{i,j} &= \tau_{2,j}\tau_{1,i}\alpha_4\tau_{1,i}\tau_{2,j}, \\
\lambda_{i,j} &= \iota_i\mu_{i,j}^{-1}\iota_i.
\end{aligned}$$

We can also compute easily their inverses: $\iota_i^{-1} = \iota_i$ and

$$\begin{aligned}
\mu_{i,j}^{-1} &= \{x_i \mapsto x_i x_j^{-1},\, x_k \mapsto x_k \text{ for all } k \neq i\}, \\
\lambda_{i,j}^{-1} &= \{x_i \mapsto x_j^{-1} x_i,\, x_k \mapsto x_k \text{ for all } k \neq i\}.
\end{aligned}$$

We will say that ι_i is an *automorphism of type* ι, $\mu_{i,j}$ and $\mu_{i,j}^{-1}$ are *automorphisms of type* μ, and $\lambda_{i,j}$ and $\lambda_{i,j}^{-1}$ are *automorphisms of type* λ. We are also referring to the collection of automorphisms of type μ and λ, as *automorphisms of type* κ. As a consequence of these definitions, the following result arises

Lemma 3.2. *Let $\sigma \in \mathrm{Sym}_r$ and $i \neq j$. Then the following relations are satisfied*

$$\begin{aligned}
\iota_i\sigma &= \sigma\iota_{\sigma(i)} \\
\mu_{i,j}\sigma &= \sigma\mu_{\sigma(i),\sigma(j)} \\
\lambda_{i,j}\sigma &= \sigma\lambda_{\sigma(i),\sigma(j)}.
\end{aligned}$$

Proof. Checking step-by-step, the result is clear: the automorphism $\iota_i\sigma$ corresponds to

$$\begin{array}{ccccc}
x_i & \longmapsto & x_i^{-1} & \longmapsto & x_{\sigma(i)}^{-1} \\
x_k & \longmapsto & x_k & \longmapsto & x_{\sigma(k)}
\end{array}$$

for all $k \neq i$, so clearly $\iota_i\sigma = \sigma\iota_{\sigma(i)}$. Meanwhile, $\mu_{i,j}\sigma$ corresponds to

$$\begin{array}{ccccc}
x_i & \longmapsto & x_i x_j & \longmapsto & x_{\sigma(i)}x_{\sigma(j)} \\
x_k & \longmapsto & x_k & \longmapsto & x_{\sigma(k)}
\end{array}$$

for all $k \neq i$, so clearly $\mu_{i,j}\sigma = \sigma\mu_{\sigma(i),\sigma(j)}$. Analogously, $\lambda_{i,j}\sigma = \sigma\lambda_{\sigma(i),\sigma(j)}$. □

Lemma 3.3. *Let $i, j, k \in \{1, \dots, r\}$ be pairwise different. Then, the following verifies:*

i) ι_i *commutes with* $\mu_{j,k}$ *and* $\lambda_{j,k}$.

ii) $\mu_{j,i}\iota_i = \iota_i\mu_{j,i}^{-1}$ *and* $\lambda_{j,i}\iota_i = \iota_i\lambda_{j,i}^{-1}$.

iii) $\mu_{i,k}\iota_i = \iota_i\lambda_{i,k}^{-1}$.

Proof. Note the third point is direct from the definition we gave: $\lambda_{i,k} = \iota_i \mu_{i,k}^{-1} \iota_i$. Note also that if i, j and k are all different, then it is direct that ι_i commutes with $\mu_{j,k}$ and $\lambda_{j,k}$, which is the first point. So we only need to prove the second one, which will be done step-by-step as before. Observe that $\mu_{j,i}\iota_i$ only changes

$$\begin{array}{ccccc} x_i & \longmapsto & x_i & \longmapsto & x_i^{-1} \\ x_j & \longmapsto & x_j x_i & \longmapsto & x_j x_i^{-1}, \end{array}$$

while $\iota_i \mu_{j,i}^{-1}$ only changes

$$\begin{array}{ccccc} x_i & \longmapsto & x_i^{-1} & \longmapsto & x_i^{-1} \\ x_j & \longmapsto & x_j & \longmapsto & x_j x_i^{-1}, \end{array}$$

so immediately, they are equal. Analogously, we obtain that $\lambda_{j,i}\iota_i = \iota_i \lambda_{j,i}^{-1}$. □

With these lemmas, the following result is direct.

Theorem 3.4. *All automorphism φ of the F_r can be written as*

$$\varphi = \sigma \iota \kappa_{i_1,j_1} \kappa_{i_2,j_2} \cdots \kappa_{i_m,j_m}$$

where $m \geq 0$, $\sigma \in \mathrm{Sym}_r$, ι is a composition of automorphism of type ι, and $\kappa_{i_k,j_k} \in \{\mu_{i_k,j_k}, \mu_{i_k,j_k}^{-1}, \lambda_{i_k,j_k}, \lambda_{i_k,j_k}^{-1}\}$ such that $i_k, j_k \in \{1, \dots, r\}$, $i_k \neq j_k$.

Proof. By Theorem 3.1, any automorphism can be written as a finite combination of α_1, α_2, α_3 and α_4. Remember that $\langle \alpha_1, \alpha_2 \rangle = \mathrm{Sym}_r$, $\iota_1 = \alpha_3$ and $\mu_{1,1} = \alpha_4$.

Lemma 3.2 allows us to move all permutations that appear in the expression of φ to the left-most part of the word, simply by changing the automorphism of type ι, μ and λ for other automorphisms of the same type. Therefore, we obtain that the left-most part of φ is a permutation σ.

Now we do the same operation but with the automorphism of type ι using Lemma 3.3. This allows us to place these automorphisms just right of σ by changing automorphism of type κ for automorphisms of the type κ. □

This expression of the automorphisms is going to be very useful to estimate the value of the norm $\|\varphi\|_S$ for the automorphism φ of the free group. Let's start with this proposition that motivates the expression.

Proposition 3.5. *Let $\varphi \in \mathrm{Aut}(F_r)$, $\sigma \in \mathrm{Sym}_r$ and ι be a composition of functions of type ι. Then*

$$\|\sigma\varphi\|_S = \|\iota\varphi\|_S = \|\varphi\sigma\|_S = \|\varphi\iota\|_S = \|\varphi\|_S.$$

Proof. We have that

$$\|\varphi\|_S = \sum_{x \in S} |x\varphi|_S = \sum_{x \in S} |x\sigma\varphi|_S$$

for all permutations $\sigma \in \mathrm{Sym}_r = \mathrm{Sym}(S)$, so $\|\sigma\varphi\|_S = \|\varphi\|_S$. We also have that $|x^{-1}\varphi|_S = |(x\varphi)^{-1}|_S = |x\varphi|_S$, so $\|\iota\varphi\|_S = \|\varphi\|_S$.

Let $x \in S$ and consider its minimal reduced expression of $x\varphi$ over S:

$$x\varphi = x_{i_1}^{\varepsilon_1} \cdots x_{i_l}^{\varepsilon_l}$$

with $i_1, \ldots, i_l \in \{1, \ldots, r\}$ and $\varepsilon_i = \pm 1$, such that $|x\varphi|_S = l$. Observe that composing with σ doesn't change the length since it is just a permutation of the letters in S; and observe also that composing with ι just changes the sign of all the powers of some certain letters, but never reduces the word length. Thus,

$$\|\varphi\sigma\|_S = \|\varphi\iota\|_S = \|\varphi\|_S. \qquad \square$$

So this allows us to understand the structure of all automorphisms of the free groups from another point of view. From Theorem 3.4 and Proposition 3.5 we deduce that to find the auto-gap and outer-gap functions we must only check the norm of every product of automorphisms of type κ. If we were to find an algorithm that calculates the norm of an automorphism in function of these κ's, then we would easily compute the auto-gap and outer-gap functions, since $\varphi = \kappa_{i_1,j_1}\kappa_{i_2,j_2} \ldots \kappa_{i_m,j_m}$ has inverse $\varphi^{-1} = \kappa^{-1}_{i_m,j_m} \ldots \kappa^{-1}_{i_1,j_1}$, which follows the same structure.

Finding an algorithm that works in general may be quite complex, but we can easily prove the following theorem, which gives the upper bound for the norm of such elements. It is based on the observation that every automorphism $\kappa_{i,j}$ adds one x_j (or x_j^{-1}) for every x_i we have.

The theorem requires a previous definition: For any $\varphi = \kappa_{i_1,j_1} \cdots \kappa_{i_m,j_m}$, we consider the sequence $(\kappa_{i_1,j_1}, \ldots, \kappa_{i_m,j_m})$ and define a *chain C in φ* as a subsequence $(\kappa_{i'_1,j'_1}, \ldots, \kappa_{i'_t,j'_t})$ such that $j'_k = i'_{k+1}$ for all $k < t$. We say that this chain ends with j'_t. We also define the set of chains of φ as $\mathcal{C}^\varphi$, and the subset of those that end with i as $\mathcal{C}^\varphi_i$.

Let's also give a quick example. The automorphism $\kappa_{1,2}\kappa_{2,1}\kappa_{2,3}\kappa_{4,5}\kappa_{3,5}$, has five chains of length 1 (because $m = 5$), three chains of length 2: $(\kappa_{1,2}, \kappa_{2,1})$, $(\kappa_{1,2}, \kappa_{2,3})$ and $(\kappa_{2,3}, \kappa_{3,5})$; and one of length 3: $(\kappa_{1,2}, \kappa_{2,3}, \kappa_{3,5})$.

Theorem 3.6. *Let $\varphi = \kappa_{i_1,j_1}\kappa_{i_2,j_2} \cdots \kappa_{i_m,j_m}$. Then*

$$\|\varphi\|_S \leq r + \#\mathcal{C}^\varphi.$$

We have an equality if and only if φ doesn't make any cancellations.

Proof. For this proof, we are considering $(x_j)\varphi$ as words in S rather than elements in F_r, that is, we aren't removing any trivial expression ($x_i x_i^{-1}$ and $x_i^{-1}x_i$) and we are leaving them as such each step of the way. We are defining $\#_i\varphi$ as the number of x_i or x_i^{-1} in the words $(x_j)\varphi$ for all x_j. Then, it is clear that

$$\|\varphi\|_S \leq \sum_{i=1}^{r} \#_i\varphi,$$

where the inequality comes when we consider the trivial reductions in F_r. Clearly, if no cancellations are done throughout the construction of φ, the equality holds.

We claim that $\#_i\varphi = 1 + \#\mathcal{C}^\varphi_i$. If this holds, then the result is direct. We will prove it by induction over m. Case $m = 0$ is immediate. If $m = 1$ and $\varphi = \kappa_{i,j}$, then we have only one chain and $\|\varphi_S\| = r + 1$. So, the equality follows.

Now suppose that the result holds for $m - 1$, and we prove it for m. We have that $\varphi = \psi\kappa_{i,j}$, where ψ is a product of $m - 1$ automorphisms of type κ. By construction of φ, $\#_j\varphi = \#_j\psi + \#_i\psi$ and $\#_k\varphi = \#_k\psi$ for $k \neq j$, since $\kappa_{i,j}$ only adds one x_j for every x_i, and we are not considering cancellations.

Observe that the number of chains ending at $k \neq j$ is the same in both ψ and φ, so

$$\#_k\varphi = \#_k\psi = 1 + \#\mathcal{C}_k^{\psi} = 1 + \#\mathcal{C}_k^{\varphi},$$

by the induction hypothesis.

Now let's see that $\#_j\varphi = 1 + \#\mathcal{C}_j^{\varphi}$. We can see that the chains of φ that end at j are: the chains that already start at j in ψ, the chains $(C, \kappa_{i,j})$ where C is a chain that starts at i in ψ, and $(\kappa_{i,j})$. So, $\#\mathcal{C}_j^{\varphi} = 1 + \#\mathcal{C}_j^{\psi} + \#\mathcal{C}_i^{\psi}$ and, by the induction hypothesis,

$$\begin{aligned}
\#_j\varphi &= \#_j\psi + \#_i\psi \\
&= 1 + \#\mathcal{C}_j^{\psi} + 1 + \#\mathcal{C}_i^{\psi} \\
&= 1 + \#\mathcal{C}_j^{\varphi}.
\end{aligned}$$

Thus, the claim is proven. □

By the theorem, it seems that choosing a combination of κ's such that there are a lot of chains in one direction, but very few in the other, we can construct an automorphism with a big gap. Note also that a chain of length t implies the existence of at least $t - k + 1$ chains of length k for every $k \leq t$.

The only problem we have to solve is the case of the cancellations, since they can reduce this gap by an undetermined size. Finding a way to solve this problem has resulted to be a very complex computational exercise, since there are multiple different expressions with several types of cancellations, and many of them are not replaceable by simpler expressions.

Immediately, if φ is the product of positive automorphisms of type κ (that is, a product of either $\mu_{i,j}$ or $\lambda_{i,j}$), then φ doesn't have any cancellations, and therefore its norm is given by the equality in Theorem 3.6. It is not true that a product of negative automorphism of type κ doesn't have cancellations, but if we were to find a positive product φ with a lot of chains, and whose inverse had very few, then we would have a big gap between $\|\varphi\|_S$ and $\|\varphi^{-1}\|_S$. This would give a lower bound for the auto-gap function of the group.

Intuitively, the following example makes use of our reasoning: the product

$$\varphi = \mu_{1,2}\mu_{2,3}\cdots\mu_{r-1,r}$$

is positive and corresponds to a chain of maximum length $(r-1)$, and its inverse

$$\varphi^{-1} = \mu_{r-1,r}^{-1}\mu_{r-2,r-1}^{-1}\cdots\mu_{1,2}^{-1}$$

has no chains of length larger than 1. More precisely, φ has $r - k$ chains of length k for every $k < r$, so $\|\varphi\|_S = r + \sum_{k=1}^{r-1}(r-k) = r + \frac{r(r-1)}{2} = \frac{r(r+1)}{2}$; and φ^{-1} has only $r-1$, so $\|\varphi\|_S \leq 2r-1$. We can then compute directly φ^{-1}, that is

$$(x_i)\varphi^{-1} = \begin{cases} x_i x_{i+1}^{-1} & \text{if } i < r \\ x_r & \text{if } i = r, \end{cases}$$

which certainly doesn't have any cancellations and has norm $2r - 1$. We also describe φ for later:

$$(x_i)\varphi = \begin{cases} x_i(x_{i+1})\varphi = x_{i+1}x_{i+2}\cdots x_r & \text{if } i < r \\ x_r & \text{if } i = r. \end{cases}$$

Observe also that adding a final term $\mu_{r,1}$ adds a chain of longer length in both φ and its inverse φ^{-1}.

The problem we must bear in mind is that there's a finite number of generators, and we want results to hold for n arbitrarily large. With this example in mind, we can replicate the lower bound for the auto-gap function of the free group F_r given in [11]. We are also denoting α_r and β_r as the respective functions of F_r with respect to the basis S, as in the article.

Theorem 3.7. *Let $r \geq 2$, then the auto-gap function of the free group dominates the polynomial of degree r: $n^r \preceq \alpha_r(n)$.*

Proof. For any $m > 0$, consider the automorphism

$$\varphi = \mu_{r-1,r}^{-m}\mu_{r-2,r-1}^{-m}\cdots\mu_{1,2}^{-m},$$

and its inverse,

$$\varphi^{-1} = \mu_{1,2}^{m}\mu_{2,3}^{m}\cdots\mu_{r-1,r}^{m}.$$

Observe that there are $(r-1)m$ chains in φ; and $m^k(r-k)$ chains of length k in φ for all $k < r$, and since it is positive

$$\|\varphi^{-1}\|_S = r + \sum_{k=1}^{r-1} m^k(r-k),$$

which is a polynomial of degree $r-1$ over m.

These automorphisms are

$$(x_i)\varphi = \begin{cases} x_i x_{i+1}^{-m} & \text{if } i < r \\ x_r & \text{if } i = r, \end{cases}$$

$$(x_i)\varphi^{-1} = \begin{cases} x_i (x_{i+1}^{m})\varphi^{-1} & \text{if } i < r \\ x_r & \text{if } i = r, \end{cases}$$

where (for $i < r-1$)

$$(x_{i+1}^m)\varphi^{-1} = (x_{i+1}(x_{i+2}(\cdots(x_{r-1}(x_r)^m)^m\cdots)^m)^m$$

starts with the letter x_{i+1}, ends with x_r and has length $m + m^2 + \cdots + m^{r-i}$ in S. In particular, no cancellations occur in φ, so $\|\varphi\|_S = r + m(r-1)$.

Thus, if $\|\varphi\|_S = n$, then $m = \frac{n-1}{r-1}$, and $\|\varphi^{-1}\|_S$ is a polynomial of degree $r-1$ over n, so $n^{r-1} \preceq \alpha_r(n)$.

A simple observation will let us improve the lower bound. Let $\gamma_{i,j}$ denote the conjugation of x_i by x_j (only changes $x_i \mapsto x_j^{-1}x_i x_j$, and leaves the other generators fixed). It is a well-defined automorphism in F_r because $\gamma_{i,j} = \lambda_{i,j}^{-1}\mu_{i,j}$.

Now consider $\psi = \varphi\gamma_{r,1}^{-n}$, which has inverse, $\psi^{-1} = \gamma_{r,1}^{n}\varphi^{-1}$. Then,

$$(x_i)\psi = \begin{cases} x_i x_{i+1}^{-m} & \text{if } i < r-1 \\ x_{r-1}x_1^m x_r^{-m} x_1^{-m} & \text{if } i = r-1 \\ x_1^m x_r x_1^{-m} & \text{if } i = r, \end{cases}$$

$$(x_i)\psi^{-1} = \begin{cases} x_i (x_{i+1}^{m})\varphi^{-1} & \text{if } i < r \\ [(x_1^m)\varphi^{-1}]^{-1} x_r [(x_1^m)\varphi^{-1}] & \text{if } i = r. \end{cases}$$

Observe now that ψ has chains of length 2, and ψ^{-1} has chains of length r, so it has a polynomial over m of degree r number of chains by a same argument as before.

They aren't positive, so we must check their norm manually. It is easy to see that $m(m-1)$ cancellations occur in ψ, and that $\|\psi\|_S = r + m(r-1) + 4m$, which is a polynomial of degree 1 over m.

On the other hand, if a cancellation occurs in φ^{-1}, it is in the expression of $(x_r)\psi^{-1}$, since it is the only word with both positive and negative letters, but as we have stated before, $[(x_1^m)\varphi^{-1}]^{-1}$ and $[(x_1^m)\varphi^{-1}]$ ends with x_1^{-1} and starts with x_1, respectively. Since $x_1 \neq x_r$ for $r \geq 2$, there are no reductions in $[(x_1^m)\varphi^{-1}]^{-1} x_r [(x_1^m)\varphi^{-1}]$. Also, we already know that $[(x_1^m)\varphi^{-1}]$ has length $1 + m + \cdots + m^r$ in S, and the image of the other generators has lower length (for m big enough), thus $\|\psi^{-1}\|_S$ is a polynomial of degree r over m.

So taking $\|\varphi\|_S = n$, we deduce $n^r \preceq \alpha_r(n)$. □

Observe two details from this proof: first that both the automorphism and its inverse which provide the lower bound have cancellations, and that by using the technique of the positive word we described before, we obtained the lower bound n^{r-1}. Even though a proof cannot be provided, some experimentation seems to indicate that we cannot improve this bound using a positive automorphism; and it seems that by using the conjugation "trick", we cannot improve the lower bound by more than one degree.

The question on whether this bound can be improved, or an upper bound can be given, remains still open, but it seems that an automorphism of norm n with a bigger gap than $\sim n^r$, if it were to exist, must be of a very complex nature and with carefully placed cancellations.

We make a quick remark before ending the subsection:

Remark 3.8. Let's talk a bit more on some simple cancellations that might occur. For example, we might have $\mu_{i,j}\lambda_{j,i}^{-1}$ that only changes

$$\begin{array}{ccccccc} x_i & \longmapsto & x_i x_j & \longmapsto & x_i x_i^{-1} x_j & = & x_j \\ x_j & \longmapsto & x_j & \longmapsto & x_i^{-1} x_j, & & \end{array}$$

which essentially corresponds to $\tau_{i,j}\iota_i\lambda_{i,j}^{-1}$, which doesn't have cancellations and therefore could be easily replaced. But if we take $\mu_{i,j}\mu_{j,i}\lambda_{j,i}^{-1}$ (with a new automorphism of type κ in between), that only changes

$$\begin{array}{ccccccccc} x_i & \longmapsto & x_i x_j & \longmapsto & x_i x_j x_i & \longmapsto & x_i x_i^{-1} x_j x_i & = & x_j x_i \\ x_j & \longmapsto & x_j & \longmapsto & x_j x_i & \longmapsto & x_i^{-1} x_j x_i, & & \end{array}$$

we have that it cannot be (easily) replaced by an automorphism without cancellations.

Similarly, $\mu_{i,j}\mu_{i,j}^{-1}$ is trivial, but if we place the automorphism $\mu_{j,i}$ in between: $\mu_{i,j}\mu_{j,i}\mu_{i,j}^{-1}$, we have that it changes

$$\begin{array}{ccccccccc} x_i & \longmapsto & x_i x_j & \longmapsto & x_i x_j x_i & \longmapsto & x_i x_j^{-1} x_j x_i x_j^{-1} & = & x_i^2 x_j^{-1} \\ x_j & \longmapsto & x_j & \longmapsto & x_j x_i & \longmapsto & x_j x_i x_j^{-1}, & & \end{array}$$

which has norm $r+3$ and some cancellations. Once again, it cannot be (easily) replaced by an automorphism without cancellations.

One observation to make is that in the majority of cases, a cancellation in φ implies a cancellation in φ^{-1}, for example, $(\mu_{i,j}\mu_{j,i}\lambda_{j,i}^{-1})^{-1} = \lambda_{j,i}\mu_{j,i}^{-1}\mu_{i,j}^{-1}$ only changes

$$\begin{array}{ccccccccc} x_i & \longmapsto & x_i & \longmapsto & x_i & \longmapsto & x_i x_j^{-1} & & \\ x_j & \longmapsto & x_i x_j & \longmapsto & x_i x_j x_i^{-1} & \longmapsto & x_i x_j^{-1} x_j x_j x_i^{-1} & = & x_i x_j^2 x_i^{-1}, \end{array}$$

which contains a similar cancellation of that of its inverse, but has a slighter larger norm (one unit more).

3.1 Outer-gap function of the free groups

Let's give once again an alternative proof of the results on the outer-gap function given in [11], that has an improvement to the given lower bound. As always, some previous results and definitions are needed.

Definition 3.9. Let $\varphi \in \mathrm{Aut}(G)$, $x_i \in S$ and $\varepsilon \in \{\pm 1\}$. We define x_i^ε-$\alpha_j(\varphi)$ as the value -1 if the reduced expression of $x_j\varphi$ starts with x_i^ε, and 1 otherwise. Similarly, we define x_i^ε-$\omega_j(\varphi)$ as the value -1 if the reduced expression of $x_i\varphi$ ends with $x_i^{-\varepsilon}$, and 1 otherwise.

Then, we define the *x_i^ε-evaluation of* φ, x_i^ε-val(φ), as the sum of the values of α_j and ω_j.

We can see that the possibles values of x_i^ε-val(φ) are the even integers in $[-2r, 2r]$. The following result arises directly from the definition:

Proposition 3.10. *Let $\varphi \in \mathrm{Aut}(G)$, $x_i \in S$ and $\varepsilon \in \{\pm 1\}$. Then,*

$$\|\varphi\gamma_{x_i^\varepsilon}\|_S - \|\varphi\|_S = a_i^\varepsilon\text{-val}(\varphi).$$

Proof. By construction, for every generator x_j, $|x_j\varphi\gamma_{x_i^\varepsilon}|_S = |x_i^{-\varepsilon}(x_j\varphi)x_i^\varepsilon|_S$, and then

i) If $x_j\varphi$ starts with x_i^ε and ends with $x_i^{-\varepsilon}$, then $|x_j\varphi\gamma_{x_i^\varepsilon}|_S = |x_j\varphi|_S - 2$.

ii) If $x_j\varphi$ verifies only one of the previous conditions, then $|x_j\varphi\gamma_{x_i^\varepsilon}|_S = |x_j\varphi|_S$.

iii) If $x_j\varphi$ doesn't verify either condition, then $|x_j\varphi\gamma_{x_i^\varepsilon}|_S = |x_j\varphi|_S + 2$.

More precisely, in every case this corresponds to $|x_j\varphi\gamma_{x_i^\varepsilon}|_S = |x_j\varphi|_S + x_i^\varepsilon\text{-}\alpha_j(\varphi) + x_i^\varepsilon\text{-}\omega_j(\varphi)$, and so

$$\begin{aligned} \|\varphi\gamma_{x_i^\varepsilon}\|_S &= \sum_{j=1}^r |x_j\varphi\gamma_{x_i^\varepsilon}|_S \\ &= \sum_{j=1}^r |x_j\varphi|_S + \sum_{j=1}^r (x_i^\varepsilon\text{-}\alpha_j(\varphi) + x_i^\varepsilon\text{-}\omega_j(\varphi)) \\ &= \|\varphi\|_S + x_i^\varepsilon\text{-val}(\varphi). \end{aligned}$$ □

Lemma 3.11. *Let $\varphi \in \mathrm{Aut}(G)$, $x_i \in S$ and $\varepsilon \in \{\pm 1\}$. If x_i^ε-val$(\varphi) \geq 0$, then x_j^δ-val$(\varphi\gamma_{x_i^\varepsilon}) \geq x_i^\varepsilon$-val$(\varphi) \geq 0$ for all $x_j^\delta \in S \cup S^{-1}$ such that $x_j^\delta \neq x_i^{-\varepsilon}$.*

Proof. By Proposition 3.10, $\|\varphi\gamma_{x_i^\varepsilon}\|_S \geq \|\varphi\|_S$, so when conjugating by x_i^ε, we have exactly $r - \frac{1}{2}x_i^\varepsilon\text{-val}(\varphi) \leq r$ cancellations, which means that the number j such that $(x_j)\varphi\gamma_{x_i^\varepsilon}$ ends with x_i^ε, plus the number of j such that $(x_j)\varphi\gamma_{x_i^\varepsilon}$ starts with $x_i^{-\varepsilon}$, is exactly $r + \frac{1}{2}x_i^\varepsilon\text{-val}(\varphi)$. In particular, since $x_j^\delta \neq x_i^{-\varepsilon}$, we obtain $x_j^\delta\text{-val}(\varphi\gamma_{x_i^\varepsilon}) \geq x_i^\varepsilon\text{-val}(\varphi) \geq 0$. □

In particular, this lemma tells us that

$$\|\varphi\gamma_{x_i^\varepsilon x_j^\delta}\|_S = \|\varphi\gamma_{x_i^\varepsilon}\gamma_{x_j^\delta}\|_S \geq \|\varphi\gamma_{x_i^\varepsilon}\|_S.$$

Corollary 3.12. *Let $\varphi \in \mathrm{Aut}(G)$, $x_i \in S$ and $\varepsilon \in \{\pm 1\}$, and let $g \in F_r$. If $x_i^\varepsilon\text{-val}(\varphi) \geq 0$ and $\|\varphi\gamma_g\|_S < \|\varphi\|_S$, then the reduced expression of g doesn't start with x_i^ε.*

Proof. Let $g = x_{i_1}^{\varepsilon_1} \cdots x_{i_l}^{\varepsilon_l}$ be the reduced expression of g. Suppose it starts with $x_i^{\varepsilon_i}$, then, applying Lemma 3.11 recursively, we obtain

$$0 \leq x_{i_1}^{\varepsilon_1}\text{-val}(\varphi) \leq x_{i_2}^{\varepsilon_2}\text{-val}(\varphi\gamma_{x_{i_1}^{\varepsilon_1}}) \leq \cdots \leq x_{i_l}^{\varepsilon_l}\text{-val}(\varphi\gamma_{x_{i_1}^{\varepsilon_1}\cdots x_{i_{l-1}}^{\varepsilon_{l-1}}}),$$

and applying Proposition 3.10,

$$\|\varphi\|_S \leq \|\varphi\gamma_{x_{i_1}^{\varepsilon_1}}\|_S \leq \cdots \leq \|\varphi\gamma_{x_{i_1}^{\varepsilon_1}\cdots x_{i_l}^{\varepsilon_l}}\|_S = \|\varphi\gamma_g\|_S,$$

which contradicts the initial hypothesis. □

Immediately from this result we obtain:

Corollary 3.13. *Let φ be an automorphism and let Φ be its class in $\mathrm{Out}(F_r)$. If $x_i^\varepsilon\text{-val}(\varphi) \geq 0$ for all generators x_i and signs ε, then φ is minimal in Φ (that is, $\|\Phi\|_S = \|\varphi\|_S$).*

Now, this result will allow us to prove the lower bound for the outer-gap function. This is a slight improvement of the lower bound for the case $r > 2$ given in [11], which was n^{r-1}.

Proposition 3.14. *Let $r > 2$. Then the outer-gap function of the free group F_r verifies $n^r \preceq \beta_r(n)$.*

Proof. We take the automorphisms φ and $\gamma_{r,1} := \lambda_{r,1}^{-1}\mu_{r,1}$ defined in Theorem 3.7, and we consider

$$\psi = \varphi\gamma_{r,1} = \mu_{r-1,r}^{-m}\mu_{r-2,r-1}^{-m}\cdots\mu_{1,2}^{-m}\gamma_{r,1}.$$

If we check the image of its generators, we obtain

$$(x_i)\psi = \begin{cases} x_i x_{i+1}^{-m} & \text{if } i < r-1 \\ x_{r-1}x_1^{-m}x_r^{-m}x_1^m & \text{if } i = r-1 \\ x_1^{-m}x_r x_1^m & \text{if } i = r, \end{cases}$$

and observe that since $r > 2$, x_1, x_{r-1} and x_r are pairwise different, so the expressions above are reduced. Therefore, $\|\varphi\|_S = r + m(r-1) + 4m$.

We can compute easily its inverse, that is

$$\psi^{-1} = \gamma_{r,1}^{-1}\varphi^{-1} = \gamma_{r,1}^{-1}\mu_{1,2}^m\mu_{2,3}^m\cdots\mu_{r-1,r}^m,$$

such that

$$(x_i)\psi^{-1} = \begin{cases} x_i(x_{i+1}^m)\varphi^{-1} & \text{if } i < r \\ [(x_1^m)\varphi^{-1}]x_r[(x_1^m)\varphi^{-1}]^{-1} & \text{if } i = r, \end{cases}$$

where

$$(x_{i+1}^m)\varphi^{-1} = (x_{i+1}(x_{i+2}(\cdots(x_{r-1}(x_r)^m)^m\cdots)^m)^m.$$

Hence, $[(x_1^m)\varphi^{-1}]$ ends with $x_{r-1}x_r^m$ and $[(x_1^m)\varphi^{-1}]^{-1}$ starts with $x_r^{-m}x_{r-1}^{-1}$, so in $[(x_1^m)\varphi^{-1}]x_r[(x_1^m)\varphi^{-1}]^{-1}$ only m cancellations occur. Since $|(x_1^m)\varphi^{-1}|_S$ is a polynomial of degree r in m, and only m cancellations occur in the expression for $(x_r)\psi^{-1}$, $|(x_r)\psi^{-1}|_S$ and $\|\psi\|_S$ are both polynomials of degree r in m.

Now let's compute the evaluations of both automorphisms to see that they are minimal.

We easily find the first and last letters of the reduced expressions of $(x_i)\psi$ and $(x_i)\psi^{-1}$, which are summarized in the following table

	starts with	ends with	
$(x_i)\psi$	x_i	x_{i+1}^{-1}	for all $i < r-1$
$(x_{r-1})\psi$	x_{r-1}	x_1	
$(x_r)\psi$	x_1^{-1}	x_1	
$(x_i)\psi^{-1}$	x_i	x_r	for all $i < r$
$(x_r)\psi^{-1}$	x_1	x_1^{-1}	

So, the evaluations of ψ and ψ^{-1} are the following (for $1 < i < r$)

$$\begin{aligned} x_1\text{-val}(\psi) &= 2r-2 & x_1\text{-val}(\psi^{-1}) &= 2r-6 \\ x_1^{-1}\text{-val}(\psi) &= 2r-6 & x_1^{-1}\text{-val}(\psi^{-1}) &= 2r \\ x_i\text{-val}(\psi) &= 2r-4 & x_i\text{-val}(\psi^{-1}) &= 2r-2 \\ x_i^{-1}\text{-val}(\psi) &= 2r & x_i^{-1}\text{-val}(\psi^{-1}) &= 2r \\ x_r\text{-val}(\psi) &= 2r & x_r\text{-val}(\psi^{-1}) &= 2r \\ x_r^{-1}\text{-val}(\psi) &= 2r & x_r^{-1}\text{-val}(\psi^{-1}) &= 2r-2(r-1) = 2. \end{aligned}$$

Observe that all of them are non-negative for all $r \geq 3$. By Corollary 3.13, we obtain that both expressions are minimal, so if $\psi \in \Phi \in \mathrm{Out}(G)$, then $\|\Phi\|_S$ is a polynomial of degree 1 in m, and $\|\Phi^{-1}\|_S$ is a polynomial of degree r in m, proving that $n^r \preceq \beta_r(n)$. □

In the following section we will study in detail the case $r = 2$ and observe that the theorem doesn't hold for this case.

The article [11] proves also an upper polynomial bound of degree M_r. The degree of this bound is considerably big and requires understanding the outer space, which we are not covering in this thesis.

Intuitively, it seems that $\alpha_r(n) \preceq \beta_r(n)f(n)$ where $f(n)$ is polynomial, but we have been unable to construct a proof nor counterexample. If this were to be true, then $\alpha_r(n)$ would also be of polynomial growth.

3.2 Upper bound for the free group of rank 2

Now let's prove the upper bound for the free group of rank 2. We will need to introduce some previous results. We defined in all non-abelian free groups $\gamma_{i,j} := \lambda_{i,j}^{-1}\mu_{i,j}$ as the conjugation of the generator x_i by x_j (with the image of the other generators unchanged), but, only in the free group of rank 2 does this coincide with the conjugation by x_j, $\gamma_{x_j} \in \mathrm{Inn}(F_2)$. This, together, with the fact that for any $g \in G$ and $\varphi \in \mathrm{Aut}(G)$,

$$\varphi\gamma_g = \gamma_{g\varphi^{-1}}\varphi \quad \text{and} \quad \gamma_g\varphi = \varphi\gamma_{g\varphi},$$

allows us to assume that $\mu_{i,j} = \lambda_{i,j}\gamma_{x_j}$ in the free group F_2, and immediately:

Proposition 3.15. *In the free group F_2, there exists $g, h \in F_2$ such that, for any product of positive automorphisms κ_{i_k,j_k} of type κ,*

$$\kappa_{i_1,j_1}^{\varepsilon_1} \cdots \kappa_{i_l,j_l}^{\varepsilon_l} = \mu_{i_1,j_1}^{\varepsilon_1} \cdots \mu_{i_l,j_l}^{\varepsilon_l}\gamma_g = \lambda_{i_1,j_1}^{\varepsilon_1} \cdots \lambda_{i_l,j_l}^{\varepsilon_l}\gamma_h.$$

So essentially, there's no difference between $\lambda_{i,j}$ and $\mu_{i,j}$ if we can add conjugations. So by considering our calculations in $\mathrm{Out}(F_2)$, we can greatly simplify them, because $[\mu_{i,j}] = [\lambda_{i,j}]$. In this section, since we are going to work mainly in $\mathrm{Out}(F_2)$, we are dropping the $[\,\cdot\,]$ to simplify the notation.

Before that, we state these two results on the elements of $\mathrm{Aut}(F_2)$. We are denoting τ as the non-trivial permutation of Sym_2, and we are naming the subgroup generated by Sym_2 and the automorphisms of type ι as the letter permuting group LP_2.

Lemma 3.16. *Let $\{i, j\} = \{1, 2\}$. Then the following equalities verify*

$$\begin{aligned}\lambda_{i,j} &= \tau\iota_i\mu_{i,j}^{-1}\mu_{j,i},\\ \mu_{i,j} &= \tau\iota_i\lambda_{i,j}^{-1}\lambda_{j,i}.\end{aligned}$$

Proof. It can be proven that the equalities hold step-by-step. For example, $\tau\iota_i\mu_{i,j}^{-1}\mu_{j,i}$ equals

$$\begin{array}{ccccccccc} x_i & \longmapsto & x_j & \longmapsto & x_j & \longmapsto & x_j & \longmapsto & x_jx_i \\ x_j & \longmapsto & x_i & \longmapsto & x_i^{-1} & \longmapsto & x_jx_i^{-1} & \longmapsto & x_j, \end{array}$$

which is clearly $\lambda_{i,j}$. Analogously, we have the other equality. □

Corollary 3.17. *Let $\{i, j\} = \{1, 2\}$ and $k, m > 0$. Then, the following equality verifies*

$$\mu_{i,j}^{-k}\mu_{j,i}^{m} = \tau\iota_j\lambda_{j,i}^{k-1}\lambda_{i,j}\mu_{j,i}^{m-1}.$$

Furthermore, we have that $\mu_{i,j}^{-k}\mu_{j,i}^{m} = \tau\iota_j\mu_{j,i}^{k-1}\mu_{i,j}\mu_{j,i}^{m-1}$ in $\mathrm{Out}(F_2)$.

Proof. Substituting the previous lemma, we obtain

$$\mu_{i,j}^{-k}\mu_{j,i}^{m} = \mu_{i,j}^{-k+1}\iota_i\tau\lambda_{i,j}\mu_{j,i}^{m-1} = \tau\lambda_{j,i}^{-k+1}\iota_j\lambda_{i,j}\mu_{j,i}^{m-1} = \tau\iota_j\lambda_{j,i}^{k-1}\lambda_{i,j}\mu_{j,i}^{m-1}.$$

The relation in $\mathrm{Out}(F_2)$ is immediate from this. □

So now let's suppose we are in $\mathrm{Out}(F_2)$, so every product of automorphisms of type κ can be written as a product M of automorphisms of type μ. Also, since there are only four different automorphisms of this type in F_2 ($\mu_{1,2}$, $\mu_{2,1}$, and their inverses), we can write every M as $M_1^{a_1}M_2^{a_2}\cdots M_m^{a_m}$ where $a_k \in \mathbb{Z}\backslash\{0\}$, and M_k is $\mu_{i,j}$ for k odd, and $\mu_{j,i}$ for k even. We say such expression has m *syllables.* We also denote M_k^* as the opposite, that, is $\mu_{i,j}$ if $M_k = \mu_{j,i}$ and $\mu_{j,i}$ if $M_k = \mu_{i,j}$.

Lemma 3.18. *Suppose we are in* $\mathrm{Out}(F_2)$*. Let* $\varphi = M_1^{a_1} M_2^{a_2} \cdots M_m^{a_m}$ *be a product of automorphisms of type* μ *as previously described. Then,* φ *can be written as* $\xi P \xi'$*, where* P *is a product of positive automorphisms of type* μ *and* $\xi, \xi' \in LP_2$*.*

Proof. Let's prove it by induction on m. If $m = 1$, then $a_1 > 0$ and we are done; else if $a_1 < 0$,

$$\varphi = M^{a_1} = \iota M^{-a_1} \iota$$

where ι is an automorphism of type ι, which is also of the wanted form.

Suppose it is now true for values smaller than m, and we prove it for m. Observe that if $a_m < 0$, then

$$\varphi = M_1^{a_1} M_2^{a_2} \cdots M_m^{a_m} = \iota M_1^{-a_1} M_2^{-a_2} \cdots M_m^{-a_m} \iota$$

for some ι automorphism of type ι and $\psi := M_1^{-a_1} M_2^{-a_2} \cdots M_m^{-a_m}$, with the last exponent positive, allowing us to reduce to the following case.

Let $a_m > 0$. If there are no negative exponents in the expression of φ, we are done; otherwise, let a_k be the last negative exponent. Since $k < m$, a_{k+1} exists and it is positive. By Corollary 3.17,

$$M_k^{a_k} M_{k+1}^{a_{k+1}} = \tau\iota [M_k^*]^{-a_k - 1} M_k M_{k+1}^{a_{k+1}-1},$$

with $-a_k - 1, a_{k+1} - 1 \geq 0$ and ι an automorphism of type ι. Therefore, by Lemmas 3.2 and 3.3,

$$\begin{aligned} \varphi &= M_1^{a_1} \cdots \tau\iota [M_k^*]^{-a_k - 1} M_k M_{k+1}^{a_{k+1}-1} \cdots M_m^{a_m} \\ &= \tau\iota [M_1^*]^{-a_1} \cdots [M_{k-1}^*]^{-a_{k-1}} [M_k^*]^{-a_k - 1} M_k M_{k+1}^{a_{k+1}-1} \cdots M_m^{a_m}. \end{aligned}$$

Now we are going to express φ as $\tau\iota\varphi' P'$ where φ' and P' are defined as follows:

i) If $-a_k - 1 > 0$, then we define them as

$$\begin{aligned} \varphi' &:= [M_1^*]^{-a_1} \cdots [M_{k-1}^*]^{-a_{k-1}} [M_k^*]^{-a_k - 1} \\ P' &:= M_k M_{k+1}^{a_{k+1}-1} \cdots M_m^{a_m}. \end{aligned}$$

ii) If $-a_k - 1 = 0$ and $a_{k+1} - 1 > 0$, then

$$[M_{k-1}^*]^{-a_{k-1}} [M_k^*]^0 M_k = [M_{k-1}^*]^{-a_{k-1}+1},$$

and we define

$$\begin{aligned} \varphi' &:= [M_1^*]^{-a_1} \cdots [M_{k-1}^*]^{-a_{k-1}+1} M_{k+1}^{a_{k+1}-1}, \\ P' &:= M_{k+2}^{a_{k+2}} \cdots M_m^{a_m}. \end{aligned}$$

iii) If $-a_k - 1 = a_{k+1} - 1 = 0$ and $m \geq k + 3$, then we define

$$\begin{aligned} \varphi' &:= [M_1^*]^{-a_1} \cdots [M_{k-1}^*]^{-a_{k-1}+1+a_{k+2}} M_{k+3}^{a_{k+3}}, \\ P' &:= M_{k+4}^{a_{k+4}} \cdots M_m^{a_m}. \end{aligned}$$

iv) If $-a_k - 1 = a_{k+1} - 1 = 0$ and $m \in \{k+1, k+2\}$, then we define

$$\varphi' := [M_1^*]^{-a_1} \cdots [M_{k-1}^*]^{-a_{k-1}+1+a_{k+2}},$$
$$P' := \mathrm{id},$$

where $a_{k+2} := 0$ if $m = k+1$.

Observe that in all cases, P' is a positive product of automorphisms of type μ (or the identity) and φ' is a product of automorphisms of type μ (or the identity). Observe also that in cases i), ii) and iii), the last exponent of φ' is positive (but not necessarily in case iv)). In these cases, we take $\varphi' P'$ written as described, and apply the same construction recursively until the remaining expression has all positive exponents, or we arrive to a case iv).

If we end with all positive elements, then φ is of the form ξP, with P a product of positive automorphisms of type μ and $\xi \in LP_2$.

Otherwise, if at some point we arrive to case iv), then the corresponding $\varphi' P' = \varphi'$ has fewer than m syllables, so we can apply induction to φ', and we are done. □

We need to introduce a definition for the next result. We say that a word in S is *cyclically reduced* if it doesn't both start with c^{-1} and end with c for any $c \in S \cup S^{-1}$; and that φ is *cyclically reduced* if both $(x_1)\varphi$ and $(x_2)\varphi$ are cyclically reduced. A consequence of being cyclically reduced is that all its evaluations are positive, so by Corollary 3.13, every cyclically reduced expression in φ, verifies $\|\varphi\|_S = \|[\varphi]\|_S$.

Theorem 3.19. *All automorphisms φ of the free group of rank 2 can be written as*

$$\varphi = \xi P \gamma_g \xi',$$

where $\xi, \xi' \in LP_2$, P is a product of positive automorphisms of type μ and $g \in F_2$ such that $\|P\|_S + 2|g|_S \leq \|\varphi\|_S$.

Proof. Here we are following directly the proof of [11]. We are also using the notation from the article and denoting φ as $\eta_{u,v}$, where $u = (x_1)\varphi$ and $v = (x_2)\varphi$. Remember that every φ is completely determined by these two elements.

It is direct from Lemma 3.18, Theorem 3.4 and the definition of outer group, that there exists $\xi, \xi' \in LP_2$, P a product of positive automorphisms of type μ and $g \in F_2$ such that $\varphi = \xi P \gamma_g \xi'$. The only condition we need to see is that $\|P\|_S + 2|g|_S \leq \|\varphi\|_S$.

We prove it by induction over $m = \|\varphi\|_S$. If $\|\varphi\|_S = 2$, then the result is direct from Lemma 3.11 and Proposition 3.5, since φ is cyclically reduced. Suppose true for norms smaller than m and let's prove it for m. If u and v are cyclically reduced, then once again the result is clear. So suppose, without lost of generality, that u is not cyclically reduced, that is, there exists $c \in S \cup S^{-1}$ and $u' \in F_2$ such that $u = c^{-1} u' c$ and $|u|_S = |u'|_S + 2 \geq 3$.

It is then not possible that v neither begins with c^{-1} or ends with c in its reduced form. Suppose by contradiction that this is the case, then the reduced form of $u^m = c^{-1}(u')^m c$ always starts with c and ends with c^{-1} for all $m \in \mathbb{Z} \backslash \{0\}$, and the reduced form of v^n neither starts with c^{-1} or ends with c, for all $n \in \mathbb{Z} \backslash \{0\}$. We also can check one-by-one that $uv^{\pm 1}$, $v^{\pm 1}u$, $u^{-1}v^{\pm 1}$ and $v^{\pm 1}u^{-1}$ don't have any cancellations for this the same reason. Since $\eta_{u,v}$

is a well-defined automorphism of F_2, $\langle u, v\rangle = F_2$, and so every element can be written as $u^{m_1}v^{n_1}u^{m_2}v^{n_2}\cdots u^{m_t}v^{n_t}$ with $m_i, n_i \in \mathbb{Z}\backslash\{0\}$, except for possibly m_1 and n_t which could be 0. Since no cancellations in-between the powers of u and v occur in the previous expression, and $|v^n|_S \geq 1$ and $|u^m|_S = 2 + |(u')^m| \geq 3$ for all $n, m \in \mathbb{Z}\backslash\{0\}$, every word of length 1 (the element of S) must be a power of v, and so the only remaining possibility is that $c = v^n$ for some n, but this is a contradiction because v^n doesn't end with c for any n.

So v starts with c^{-1} or ends with c, and therefore $|cvc^{-1}|_S \leq |v|_S$. Now, we factor $\eta_{u,v} = \eta_{u',cvc^{-1}}\gamma_c$, and observe that

$$\|\eta_{u',cvc^{-1}}\|_S = |u'|_S + |cvc^{-1}|_S \leq |u|_S - 2 + |v|_S = \|\eta_{u,v}\|_S - 2.$$

Therefore, by the induction hypothesis, there exists $\xi, \xi' \in LP_2$, P a product of positive automorphisms of type μ and $h \in F_2$ such that $\varphi = \xi P\gamma_h\xi'$ and $\|P\|_S + 2|h|_S \leq \|\eta_{u',cvc^{-1}}\|_S$. Note finally, that $|(h)\xi'|_S = |h|_S$ since ξ' cannot produce any cancellation (it just permutes letters and changes all signs of some generators), so $\eta_{u,v} = \xi P\gamma_h\xi'\gamma_c = \xi P\xi'\gamma_{(h)\xi'c}$, with

$$\|P\|_S + 2|(h)\xi'c|_S \leq \|P\|_S + 2|h|_S + 2 \leq \|\eta_{u',cvc^{-1}}\|_S + 2 \leq \|\eta_{u,v}\|_S,$$

which proves the result. □

Now we can prove the following big results.

Theorem 3.20. *The auto-gap function of the free group F_2 is $\alpha_2(n) \sim n^2$.*

Proof. By Theorem 3.7 we have that $n^2 \preceq \alpha_2(n)$, so we just need to see that $\alpha_2(n) \preceq n^2$.

By Theorem 3.19, every automorphism of the free group is of the form $\varphi = \xi P\gamma_g\xi'$ where $\xi, \xi' \in LP_2$, P is a product of positive automorphisms of type μ and $g \in F_2$ such that $\|P\|_S + 2|g|_S \leq \|\varphi\|_S$. Its inverse is

$$\varphi^{-1} = \xi'^{-1}\gamma_g^{-1}P^{-1}\xi^{-1} = \xi'^{-1}P^{-1}\gamma_h\xi^{-1},$$

with $h = [(g^{-1})P^{-1}]$.

Let's compute their norms in S. By Proposition 3.5,

$$\|\varphi\|_S = \|P\gamma_g\|_S \quad \text{and} \quad \|\varphi^{-1}\|_S = \|P^{-1}\gamma_h\|_S$$

and by Corollary 2.7,

$$\|\varphi^{-1}\|_S \leq \|P^{-1}\|_S + 4|h|_S \leq \|P^{-1}\|_S + 4|g^{-1}|_S\|P^{-1}\|_S = \|P^{-1}\|_S(1 + 4|g|_S).$$

Observe that since P is positive, no cancellations occur and its norm is equal to 2 plus the number of chains (on its expression as product of automorphisms of type κ), by Theorem 3.6. Observe also that in F_2, any φ and φ^{-1} have the same number of chains, since there are only two elements. Therefore, $\|P^{-1}\|_S \leq \|P\|_S$.

Hence, applying this inequality and $\|P\|_S + 2|g|_S \leq \|\varphi\|_S$, we obtain

$$\|\varphi^{-1}\|_S \leq \|P\|_S(1 + 4|g|_S) \leq \|\varphi\|_S(1 + 2\|\varphi\|_S).$$

In conclusion,

$$\begin{aligned}\alpha_2(n) &= \max\{\|\varphi^{-1}\|_S \mid \varphi \in \mathrm{Aut}(F_2), \|\varphi\|_S \leq n\} \\ &\leq \max\{\|\varphi\|_S(1 + 2\|\varphi\|_S) \mid \varphi \in \mathrm{Aut}(F_2), \|\varphi\|_S \leq n\} \\ &= n(1 + 2n),\end{aligned}$$

and so $\alpha_2(n) \sim n^2$. □

Proposition 3.21. *The outer-gap function of the free group F_2 is $\beta_2(n) = n$.*

Proof. We are going to see that $\|[\varphi]\|_S = \|[\varphi]^{-1}\|_S$, which proves the result. By Theorem 3.19, every outer automorphism can be written as $[\varphi] = [\xi P \xi']$ where $\xi \in LP_2$ and P is a product of positive automorphisms of type μ. Since P is positive, it is cyclically reduced, and we can easily see then that $\xi P \xi'$ is also cyclically reduced.

Finally, we see that P^{-1} is also cyclically reduced. This is because the (reduced) expression $(x_i)P^{-1}$ over S uses only the letters x_i and x_j^{-1} for $j \neq i$. Consequently, there are also no cancellations in its construction, so $\|P\|_S = \|P^{-1}\|_S$, by the same argument from the proof of Theorem 3.20.

In conclusion, by Proposition 3.5,

$$\begin{aligned} \|[\varphi]\|_S = \|[\xi P \xi']\|_S = \|\xi P \xi'\|_S = \|P\|_S \\ = \|P^{-1}\|_S = \|\xi'^{-1} P^{-1} \xi^{-1}\|_S = \|[\xi'^{-1} P^{-1} \xi^{-1}]\|_S = \|[\varphi]^{-1}\|_S. \end{aligned}$$ □

4 Baumslag-Solitar group BS(1, N) and its auto-gap function

Now let's study the $\alpha_S(n)$ function over a new family of groups: the Baumslag-Solitar groups $\mathrm{BS}(1,N) = \langle a, t \mid tat^{-1} = a^N \rangle$ for $N \in \mathbb{Z}$, $|N| > 1$, with respect to the generating set $S = \{a, t\}$. Firstly, we are going to mention some of this group's basic properties and define its automorphism group.

Note that manipulating the relation $tat^{-1} = a^N$, we obtain $ta = a^N t$ and $at^{-1} = t^{-1}a^N$. Inverting the equations gives us that $ta^{-1} = a^{-N}t$ and $a^{-1}t^{-1} = t^{-1}a^{-N}$, so we can easily verify then that

$$t^\alpha a^\beta = a^{\beta N^\alpha} t^\alpha \qquad a^\beta t^{-\alpha} = t^{-\alpha} a^{\beta N^\alpha}$$

for all $\alpha > 0$ and $\beta \in \mathbb{Z}$. Given any expression over S of any element of $\mathrm{BS}(1,N)$ we make the following observations. First, we can see that we can move the positive powers of t to the right-most side of the word, and also move the negative powers of t to the left-most side by raising (the necessary times) a to the N-th power (or by the above formulas). For example,

$$t^3 a t^{-1} a^2 t^{-3} = a^{N^3} t^3 t^{-1} a^2 t^{-3} = a^{N^3} a^{2N^2} t^2 t^{-3} = a^{2N^2 + N^3} t^{-1}.$$

Now, given a finite expression of an element $w \in \mathrm{BS}(1,N)$, we define the total x-exponent of an expression of w, denoted $|w|_x$, as the sum of all exponents of all $x \in S$ appearing in an expression of the element. We can easily see that the total t-exponent is an invariant of the expression of the element, while the total a-exponent is not well-defined for elements in $\mathrm{BS}(1,N)$.

We will be using Britton's Lemma, that is applicable because $\mathrm{BS}(1,N)$ are particular cases of HNN extensions:

Lemma 4.1 (Britton's Lemma [5])**.** *Let w be a word over $S = \{a, t\}$. If w equals the trivial element in $\mathrm{BS}(1,N)$, then it is the empty word, or it contains a subword of the type $ta^\alpha t^{-1}$ or $t^{-1}a^{\alpha N}t$, for some $\alpha \in \mathbb{Z}$.*

Corollary 4.2. *In $\mathrm{BS}(1,N)$, $a^n = a^m$ if and only if $n = m$.*

Proof. The left implication is direct. Suppose that $a^n = a^m$ in $\mathrm{BS}(1,N)$, then $a^{n-m} = 1_{\mathrm{BS}(1,N)}$ and by Britton's Lemma (4.1), $n = m$. $\square$

These observations allow us to introduce this proposition given in [3],

Proposition 4.3. *Every element in* $\mathrm{BS}(1,N)$ *can be uniquely written as a word of the form* $t^{-p}a^q t^r$ *with* $p, r \geq 0$ *and* $q \in \mathbb{Z}$*, and such that if* $N|q$ *then* $pr = 0$*. This expression is called the* normal form *of the element.*

Proof. By the previous observations it is clear that any element in $\mathrm{BS}(1,N)$ can be written as $t^{-p}a^q t^r$ with $p, r \geq 0$ and $q \in \mathbb{Z}$. Note that if $N|q$ (say, $q'N = q$) and $p, r > 0$, then $t^{-p}a^{q'N}t^r = t^{-p+1}a^{q'}t^{r-1}$ which is also of the previous form. This transformation can be done recursively as long as N divides the exponent of a and both of the exponents of t are non-zero, allowing us to conclude that all elements in $\mathrm{BS}(1,N)$ can be written as $t^{-p}a^q t^r$ with $p, r \geq 0$ and $q \in \mathbb{Z}$ such that $pr = 0$ if $N|q$.

Now let's see that this expression is unique. Let $t^{-p}a^q t^r = t^{-p'}a^{q'}t^{r'}$ be two such expressions representing the same element in $\mathrm{BS}(1,N)$. Since the total t-exponent is an invariant, $r - p = r' - p'$. Suppose without lost of generality that $p' \leq p$, so $p - p' = r - r' =: s \geq 0$. Then,

$$t^{-p}a^q t^r = t^{-p'}a^{q'}t^{r'} \iff a^q = t^{p-p'}a^{q'}t^{r'-r} = t^s a^{q'} t^{-s} = a^{q'N^s},$$

and therefore $q = q'N^s$ by Corollary 4.2. If $N \not| q$, then $q = q'$ and $s = 0$ ($p = p'$ and $r = r'$). If instead $N|q$, then by hypothesis $pr = p'r' = 0$, and together with $p - p' = r - r' = s \geq 0$ we obtain that $s = 0$. In conclusion, the expression is unique. $\square$

We are also going to study the powers of elements of the form $t^{-p}a^q t^{p+1}$, for $p \geq 0$ and $q \in \mathbb{Z}$. Clearly, for all $m > 0$

$$\begin{aligned}
(t^{-p}a^q t^{p+1})^m &= (t^{-p}a^q t^{p+1})(t^{-p}a^q t^{p+1}) \overset{m)}{\cdots} (t^{-p}a^q t^{p+1}) \\
&= t^{-p}a^q (ta^q)(ta^q) \overset{m-1)}{\cdots} (ta^q) t^{p+1} \\
&= t^{-p}a^q (a^{qN}t)(a^{qN}t) \overset{m-1)}{\cdots} (a^{qN}t) t^{p+1} \\
&= t^{-p}a^{q+qN} (ta^{qN}) \overset{m-2)}{\cdots} (ta^{qN}) t t^{p+1} \\
&= \cdots \\
&= t^{-p}a^{q(1+N+\cdots+N^{m-1})} t^{m-1} t^{p+1},
\end{aligned}$$

and we'll also denote $1 + N + \cdots + N^{m-1}$ as Σ_m.

4.1 The automorphism group of BS(1, N)

Now let's understand the automorphism group of $\mathrm{BS}(1,N)$. We'll construct and see the structure of all automorphisms and their inverses following the explanation in [15], but giving the complete description of the group's automorphism. Clearly, any endomorphism of $\mathrm{BS}(1,N)$ is completely determined by its image of a and t. Also, any endomorphism φ must verify

$$a\varphi = t^{-p}a^q t^r \quad \text{and} \quad t\varphi = t^{-\alpha}a^\beta t^\gamma$$

with $p, r, \alpha, \gamma \geq 0$ and $q, \beta \in \mathbb{Z}$. If we are interested in having these in normal form, we'll impose: if $N|q$, then $pr = 0$, and if $N|\beta$, then $\alpha\gamma = 0$. To ensure that φ is well-defined, the relation must be preserved, that is $(tat^{-1})\varphi = a^N\varphi$. Since the total exponent sum of t is invariant, we have that $|(tat^{-1})\varphi|_t = -|t\varphi|_t + |a\varphi|_t + |t\varphi|_t = r - p$ and $|a^N\varphi|_t = N|a\varphi|_t = N(r-p)$, thus $r = p$ since $N \neq 1$.

We assume therefore that $p = r$ and compute the images of both tat^{-1} and a^N by φ. We have that

$$\begin{aligned}(tat^{-1})\varphi &= (t^{-\alpha}a^{\beta}t^{\gamma})(t^{-p}a^{q}t^{p})(t^{-\gamma}a^{-\beta}t^{\alpha}) = t^{-\alpha}a^{\beta}t^{-p}t^{\gamma}a^{q}t^{-\gamma}t^{p}a^{-\beta}t^{\alpha} \\ &= t^{-\alpha}t^{-p}a^{\beta N^p}a^{qN^{\gamma}}a^{-\beta N^p}t^{p}t^{\alpha} = t^{-p-\alpha}a^{qN^{\gamma}}t^{p+\alpha},\end{aligned}$$

and $a^N\varphi = t^{-p}a^{qN}t^p$, so conjugating both expressions by $t^{-p-\alpha}$,

$$a^{qN^{\gamma}} = t^{-p+p+\alpha}a^{qN}t^{p-p-\alpha} = a^{qN^{\alpha+1}}$$

and $qN^{\gamma} = qN^{\alpha+1}$, so either $q = 0$ or $\gamma = \alpha + 1$. Thus, using the previous notation, φ is an endomorphism if and only if $p = r$ and either $q = 0$ or $\gamma = \alpha+1$. Note also that if $q = 0$, then $a\varphi = 1_{\mathrm{BS}(1,N)}$, which tells us that φ is not an automorphism. So we are just studying the other case, since this one doesn't interest us. Summing up, any automorphisms in $\mathrm{BS}(1, N)$ must verify that

$$a\varphi = t^{-p}a^{q}t^{p} \quad \text{and} \quad t\varphi = t^{-\alpha}a^{\beta}t^{\alpha+1}$$

with $p, \alpha \geq 0$ and $q, \beta \in \mathbb{Z}$, $q \neq 0$.

Now we want to study the sufficient and necessary extra conditions we must impose to guarantee invertibility. We will use the well-known property that $\mathrm{BS}(1, N)$ groups are Hopfian, that allows us to conclude that φ is invertible if there exists an endomorphism ϕ such that $\varphi\phi = \mathrm{id}$. As a consequence, we will have that $\phi = \varphi^{-1}$. Clearly, since ϕ is going to be invertible we suppose $q \neq 0$ to obtain that

$$a\phi = t^{-r}a^{s}t^{r} \quad \text{and} \quad t\phi = t^{-\delta}a^{\varepsilon}t^{\delta+1}$$

with $r, \delta \geq 0$ and $s, \varepsilon \in \mathbb{Z}$ such that $s \neq 0$. Now, we impose that $\varphi\phi = \mathrm{id}$, so, using the relation we computed before,

$$\begin{aligned}a = a(\varphi\phi) &= (t^{-p}a^{q}t^{p})\phi = (t^{-\delta}a^{a}t^{\delta+1})^{-p}(t^{-r}a^{s}t^{r})^{q}(t^{-\delta}a^{\varepsilon}t^{\delta+1})^{p} \\ &= (t^{-\delta}a^{\varepsilon\Sigma_p}t^{p+\delta})^{-1}(t^{-r}a^{qs}t^{r})(t^{-\delta}a^{\varepsilon\Sigma_p}t^{p+\delta}) \\ &= (t^{-p-\delta}a^{-\varepsilon\Sigma_p}t^{\delta})(t^{-r}a^{qs}t^{r})(t^{-\delta}a^{\varepsilon\Sigma_p}t^{p+\delta}) \\ &= t^{-p-\delta}a^{-\varepsilon\Sigma_p}t^{-r}t^{\delta}a^{qs}t^{-\delta}t^{r}a^{\varepsilon\Sigma_p}t^{p+\delta} \\ &= t^{-p-\delta-r}a^{-\varepsilon\Sigma_p N^r}a^{qsN^{\delta}}a^{\varepsilon\Sigma_p N^r}t^{p+\delta+r} \\ &= t^{-p-\delta-r}a^{qsN^{\delta}}t^{p+\delta+r} \\ &= t^{-p-r}a^{qs}t^{p+r}.\end{aligned}$$

Therefore,

$$a^{qs} = t^{p+r}at^{-p-r} = a^{N^{p+r}}$$

and $qs = N^{p+r}$. In particular, q and s must divide a power of N.

We also need to see what happens when $t = t(\varphi\phi)$, which makes use of the previous calculation

$$\begin{aligned} t = t(\varphi\phi) &= (t^{-\alpha}a^{\beta}t^{\alpha+1})\phi = (t^{-\delta}a^{\varepsilon}t^{\delta+1})^{-\alpha}(t^{-r}a^{s}t^{r})^{\beta}(t^{-\delta}a^{\varepsilon}t^{\delta+1})^{\alpha+1} \\ &= [(t^{-\delta}a^{\varepsilon}t^{\delta+1})^{-\alpha}(t^{-r}a^{s}t^{r})^{\beta}(t^{-\delta}a^{\varepsilon}t^{\delta+1})^{\alpha}](t^{-\delta}a^{\varepsilon}t^{\delta+1}) \\ &= [t^{-\alpha-r}a^{\beta s}t^{\alpha+r}](t^{-\delta}a^{\varepsilon}t^{\delta+1}) \\ &= t^{-\alpha-r-\delta}a^{\beta s N^{\delta}}a^{\varepsilon N^{\alpha+r}}t^{\alpha+r+\delta+1} \\ &= t^{-\alpha-r-\delta}a^{\beta s N^{\delta}+\varepsilon N^{\alpha+r}}t^{\alpha+r+\delta+1}, \end{aligned}$$

so, $a^{\beta s N^{\delta}+\varepsilon N^{\alpha+r}} = 1_{\mathrm{BS}(1,N)}$ and hence $\varepsilon N^{\alpha+r} = -\beta s N^{\delta}$. This relation allows us to define $t\phi$ depending on the other constants, that is:

$$\begin{aligned} t\phi &= t^{-\delta}a^{\varepsilon}t^{\delta+1} = t^{-\delta-\alpha-r}a^{\varepsilon N^{\alpha+r}}t^{\delta+\alpha+r+1} \\ &= t^{-\delta-\alpha-r}a^{-\beta s N^{\delta}}t^{\delta+\alpha+r+1} = t^{-\alpha-r}a^{-\beta s}t^{\alpha+r+1}. \end{aligned}$$

In conclusion, we have that

Theorem 4.4. *Let φ be an endomorphism of* $\mathrm{BS}(1,N)$*. Then $\varphi \in \mathrm{Aut}(\mathrm{BS}(1,N))$ if and only if φ verifies*

$$a\varphi = t^{-p}a^{q}t^{p} \quad \text{and} \quad t\varphi = t^{-\alpha}a^{\beta}t^{\alpha+1}$$

for some $p, \alpha \geq 0$ and $q, \beta \in \mathbb{Z}$, such that $q \neq 0$ divides a power of N. We will call this automorphism $\varphi_{p,q,\alpha,\beta}$.

The inverse of $\varphi_{p,q,\alpha,\beta}$ is $\varphi_{r,s,\alpha+r,-\beta s}$, where $r \geq 0$ and $s \in \mathbb{Z}$ verify $qs = N^{p+r}$.

Observe that any choice of $r \geq 0$ and $s \in \mathbb{Z}$ such that $qs = N^{p+r}$ is satisfied, gives a unique $\varphi_{r,s,\alpha+r,-\beta s}$ by the group's relation, so the inverse is well-defined.

Let's make a quick note about having $|a\varphi|_S$ and $|t\varphi|_S$ in normal form, since it is always preferable, and necessary for later. If we suppose that they are, that is, if $N|q$, then $p = 0$, and if $N|\beta$, then $\alpha = 0$; then we can ensure that $|a\phi|_S$ is also in normal form, by choosing $r \geq 0$ and $s \in \mathbb{Z}$ as the smallest absolute values satisfying $qs = N^{p+r}$. On the other hand, $|t\phi|_S$ is much harder to reduce into normal form, since we might have $N|\beta s$ and $\alpha + r > 0$, and any representation in normal form must take into consideration whether N, or any of its powers, divide βs or not, which is not direct nor easy.

By imposing normal form, we obtain that this analogous result that ensures some uniqueness

Theorem 4.5. *Let φ be an endomorphism of* $\mathrm{BS}(1,N)$*. Then $\varphi \in \mathrm{Aut}(\mathrm{BS}(1,N))$ if and only if φ verifies*

$$a\varphi = t^{-p}a^{q}t^{p} \quad \text{and} \quad t\varphi = t^{-\alpha}a^{\beta}t^{\alpha+1}$$

for some $p, \alpha \geq 0$ and $q, \beta \in \mathbb{Z}$, such that $q \neq 0$ divides a power of N, if $N|q$ then $p = 0$, and if $N|\beta$, then $\alpha = 0$.

The inverse of $\varphi_{p,q,\alpha,\beta}$ is $\varphi_{r,s,\alpha+r,-\beta s}$ where $r \geq 0$ and $s \in \mathbb{Z}$ are the smallest absolute values such that $qs = N^{p+r}$ (in particular, if $N|s$ then $r = 0$).

Note that with these restrictions, $|a\varphi|_S$, $|t\varphi|_S$ and $|a\varphi^{-1}|_S$ are in normal form, but not necessarily $|t\varphi^{-1}|_S$. In this case we say φ is in normal form.

Note also that each choice of values satisfying the hypothesis gives a unique automorphism and inverse, and that s and r are completely determined by p and q, and even though, we cannot express implicitly r and s in terms of p and q in a general way, we can bound them, as we see in the following lemma. In this lemma, we are directly stating the inequalities that are needed for the final result.

Lemma 4.6. *Using the previous notation for φ in normal form, we have that the inequalities $r \leq \log_2 |q|$ and $\log_{|N|} |s| \leq p + \log_2 |q|$ verify.*

Proof. Let's prove that $r \leq \log_2 |q|$. Let $m \geq 0$ be the integer such that $q \not| N^{m-1}$ and $q|N^m$ (take $m = 0$ if $q = \pm 1$). By construction, there exists $q' \in \mathbb{Z}$ such that $qq' = N^m$ and $N \not| q'$. We are going to see the result in two parts, first $r \leq m$ and then $m \leq \log_2 |q|$.

For the first inequality, observe that we have $q'N^{r+p} = q'qs = sN^m$, and so

i) If $m < p$, then $N^{r+p-m}|s$, and therefore $r = 0 \leq m$.

ii) If $m = p$, then $q'N^r = s$ and since r, s are minimal, $q' = s$ and $r = 0 \leq m$.

iii) If $m > p$, then $q'N^r = sN^{m-p}$ and $r \neq 0$, so $N \not| s$. Therefore, $s = q'$ and $r = m - p \leq m$.

Let see now that $m \leq \log_2 |q|$, which is equivalent to $2^m \leq |q|$, so we prove this one. Being clear for $m = 0$, suppose $m > 0$. Let $|N| = p_1 \cdots p_k$ be the decomposition of $|N|$ into primes (with $p_1 \leq p_2 \leq \cdots \leq p_k$). We see that there exists $l \in \{1, \ldots, k\}$ such that $p_l^m | q$. Note that if this wasn't the case, then $q|(p_1 \cdots p_k)^{m-1} = |N|^{m-1}$ because $q|N^m$ and $p_i^m \not| q$ for all i, which is a contradiction with the minimality of m. Therefore, there exists l such that $p_l^m|q$, and thus $2^m \leq p_l^m \leq |q|$. In conclusion, $r \leq m \leq \log_2 |q|$.

Now let's prove $\log_{|N|} |s| \leq p + \log_2 |q|$. From the relation $qs = N^{p+r}$ we obtain the inequality

$$\log_{|N|} |s| \leq \log_{|N|} |s| + \log_{|N|} |q| = p + r,$$

and applying the first inequality, we obtain

$$\log_{|N|} |s| \leq p + \log_2 |q|.$$

□

4.2 Auto-gap function of BS(1, N)

To prove the value of the auto-gap function of $\mathrm{BS}(1, N)$, we will need just one more essential result, that comes from [3].

Proposition 4.7. *Let g be an element in $\mathrm{BS}(1, N)$ of the form $w = t^{-p}a^q t^r$ with $p, r \geq 0$ and $q \in \mathbb{Z}$, $q \neq 0$. Then*

$$|g|_S \leq D(p + r + \log_{|N|} |q| + 1).$$

where D is the constant $D = |N| + 1 > 1$.

Additionally, if the previous expression for g is in normal form, then

$$|g|_S \geq C(p + r + \log |q|),$$

where C is the constant $0 < \frac{1}{2(\log |N| + 1)} < 1$ and the logarithm is taken in any well-defined base.

Proof. To prove the first inequality, we will need to write g in a short enough way. Note that the present expression $g = t^{-p}a^q t^r$ is too long because $|q| > \log_{|N|}|q|$. We need to find an alternative, more clever way.

We write $|q|$ in base $|N|$ as

$$|q| = \sum_{i=0}^{n} q_i |N|^i$$

with $0 \leq q_i < |N|$ and $q_n \neq 0$. Depending on the signs of q and N, there exists a choice of signs $\varepsilon_i \in \{\pm 1\}$ such that $q = \sum_{i=0}^{n} \varepsilon_i q_i N^i$. We have that $n = \lfloor \log_{|N|}|q| \rfloor$ and that

$$\begin{aligned}
t^{-p}a^q t^r &= t^{-p}a^{\varepsilon_0 q_0 + \varepsilon_1 q_1 N + \cdots + \varepsilon_n q_n N^n} t^r \\
&= t^{-p}a^{\varepsilon_0 q_0} a^{\varepsilon_1 q_1 N} \cdots a^{\varepsilon_n q_n N^n} t^r \\
&= t^{-p}a^{\varepsilon_0 q_0} t a^{\varepsilon_1 q_1} t^{-1} t^2 \cdots t^{-n-1} t^n a^{\varepsilon_n q_n} t^{-n} t^r \\
&= t^{-p}(a^{\varepsilon_0 q_0} t a^{\varepsilon_1 q_1} t \cdots t a^{\varepsilon_n q_n} t^{-n}) t^r.
\end{aligned}$$

Observe that $|a^{\varepsilon_i q_i} t|_S \leq |N|$, so

$$\begin{aligned}
|t^{-p}a^q t^r|_S &= |t^{-p}(a^{\varepsilon_0 q_0} t a^{\varepsilon_1 q_1} t \cdots t a^{\varepsilon_n q_n} t^{-n}) t^r|_S \\
&\leq |t^{-p}|_S + |a^{\varepsilon_0 q_0} t|_S + \cdots + |a^{\varepsilon_{n-1} q_{n-1}} t|_S + |a^{\varepsilon_n q_n}|_S + |t^{-n}|_S + |t^r|_S \\
&\leq p + (n+1)|N| + n + r \\
&\leq p + r + (|N|+1)(n+1) \\
&\leq (|N|+1)(p + r + n + 1) \\
&\leq (|N|+1)(p + r + \log_{|N|}|q| + 1).
\end{aligned}$$

For the lower bound, let $|g|_S = k$ and let $b_1 \cdots b_k$ be a minimal expression of g over S (where each $b_i \in S \cup S^{-1}$). We define a sequence of elements $g_0, g_1, \ldots, g_k$ such that $g_0 = 1_{\mathrm{BS}(1,N)}$, $g_i = b_1 \cdots b_i$ and $g_k = g$. Let $t^{-p_i}a^{q_i}t^{r_i}$ be the normal form of g_i for all i. Since g_1 is the identity, the sequence $\{p_i + r_i\}_{i=0}^k$ must go from 0 to $p + r$ in k steps. Since the t-exponent in g is invariant under any expression, the sequence might increase or decrease by at most one unit every step, and therefore $p + r \leq k$.

Now consider the sequence $\{|q_i|\}_{i=0}^k$, that changes as follows:

i) If $b_{i+1} = t$, then $g_{i+1} = g_i t$ and $|q_i| = |q_{i+1}|$.

ii) If $b_{i+1} = t^{-1}$, then $g_{i+1} = g_i t^{-1}$ and $|q_i| = |q_{i+1}|$, except when $r_i = 0$, where we have that

$$g_{i+1} = g_i t^{-1} = t^{-p_i}a^{q_i}t^{-1} = t^{-p_i-1}a^{Nq_i},$$

and so $\log|q_{i+1}| = \log|N| + \log|q_i| > \log|q_i|$. Therefore, $|q_{i+1}|$ stays constant with respect to $|q_i|$ or increases by a multiplication by $|N|$.

iii) If $b_{i+1} = a^{\pm 1}$, then $g_{i+1} = g_i a^{\pm 1}$ and

$$g_{i+1} = g_i a^{\pm 1} = t^{-p_i}a^{q_i}t^{r_i}a^{\pm 1} = t^{-p_i}a^{q_i \pm N^{r_i}} t^{r_i}$$

and so $|q_{i+1}| = |q_i \pm N^{r_i}|$. Therefore, $|q_{i+1}|$ varies at most by $|N|^{r_i}$ with respect to $|q_i|$.

Consequently, since $r_i \leq p_i + r_i \leq k$,

$$\begin{aligned}|q_{i+1}| &\leq \max\{|Nq_i|, |q_i| + |N|^{r_i}\}\\ &\leq \max\{|Nq_i|, |q_i| + |N|^{k}\}\\ &\leq |Nq_i| + |N|^k,\end{aligned}$$

and recursively (starting with $|q_0| = 1$), we obtain

$$\begin{aligned}|q| = |q_k| &\leq |N||q_{k-1}| + |N|^k\\ &\leq |N|(|N||q_{k-2}| + |N|^k) + |N|^k\\ &= |N|^2|q_{k-2}| + |N|^{k+1} + |N|^k \leq \cdots\\ &\leq |N|^k|q_0| + \sum_{i=1}^{k-1} |N|^{k+i} = \sum_{i=0}^{k-1} |N|^{k+i}\\ &< k|N|^{2k-1} < k|N|^{2k}.\end{aligned}$$

Therefore, $\log|q| \leq \log k + 2k\log|N| < k + 2k\log|N| = k(1 + 2\log|N|)$ and

$$p + r + \log|q| \leq k + k(1 + 2\log|N|) = 2k(1 + \log|N|).$$

And finally, dividing by $2(1 + \log|N|)$, we obtain

$$C(p + r + \log|q|) = \frac{1}{2(1 + \log|N|)}(p + r + \log|q|) \leq k = |g|_S.$$

Note that all operations for this last inequality work with the logarithm taken in any basis, so we did denote it without the subscript in its notation: log. □

Now, will use these bounds to prove the following ones for the norms of an automorphism and its inverse.

Lemma 4.8. *Let $\varphi = \varphi_{p,q,\alpha,\beta}$ be an automorphism of $\mathrm{BS}(1, N)$ in normal form. Then,*

$$\begin{aligned}\|\varphi\|_S &\geq C(2p + 2\alpha + 1 + \log_2|\beta| + \log_2|q|),\\ \|\varphi^{-1}\|_S &\leq D(2p + 2\alpha + 3 + \log_2|\beta| + 6\log_2|q|),\end{aligned}$$

where $C^{-1} = 2(\log_2|N| + 1)$ and $D = |N| + 1$ (in the case $\beta = 0$, we substitute the undefined value $\log_2|\beta|$ for 0 in the previous expressions).

Proof. In this proof we are using the notation of Theorem 4.5. By definition of norm we have that $\|\varphi\|_S = |a\varphi|_S + |t\varphi|_S$ and $\|\varphi^{-1}\|_S = |a\varphi^{-1}|_S + |t\varphi^{-1}|_S$, so we will only need to compute the respective lower and upper bounds for the length of those images of the generators. We start with the case $\beta \neq 0$.

Since φ is in normal form, we obtain a lower bound for $|a\varphi|_S = |t^{-p}a^qt^p|_S$ and $|t\varphi|_S = |t^{-\alpha}a^\beta t^{\alpha+1}|_S$ using Proposition 4.7 (where we choose the logarithm in base 2):

$$\begin{aligned}|a\varphi|_S &\geq C(2p + \log_2|q|)\\ |t\varphi|_S &\geq C(2\alpha + 1 + \log_2|\beta|),\end{aligned}$$

and therefore,

$$\|\varphi\|_S \geq C(2p + 2\alpha + 1 + \log_2|\beta| + \log_2|q|).$$

Applying now the upper bound from Proposition 4.7 for $|a\varphi^{-1}|_S = |t^{-r}a^s t^r|_S$ and $|t\varphi^{-1}|_S = |t^{-\alpha-r}a^{-\beta s}t^{\alpha+r+1}|_S$ we obtain

$$\begin{aligned}
|a\varphi^{-1}|_S &\leq D(2r + \log_{|N|}|s| + 1) \\
|t\varphi^{-1}|_S &\leq D(2r + 2\alpha + 1 + \log_{|N|}|\beta s| + 1) \\
&= D(2r + 2\alpha + 2 + \log_{|N|}|\beta| + \log_{|N|}|s|).
\end{aligned}$$

Note that the expression of $|t\varphi^{-1}|_S$ is not in general in normal form, but this was not required for the upper bound in the proposition.

Now, applying the inequalities from Lemma 4.6, we get

$$|a\varphi^{-1}|_S \leq D(2\log_2|q| + p + \log_2|q| + 1)$$
$$|t\varphi^{-1}|_S \leq D(2\log_2|q| + 2\alpha + 2 + \log_{|N|}|\beta| + p + \log_2|q|).$$

Finally, using the fact that $\log_{|N|} \leq \log_2$ as functions over $[1,\infty)$, we obtain the last result:

$$\|\varphi^{-1}\|_S \leq D(2p + 2\alpha + 3 + \log_2|\beta| + 6\log_2|q|).$$

Now we just need to make a comment about the case $\beta = 0$. Remember that by the definition of normal form, this implies that $\alpha = 0$ and so $|t\varphi|_S = |t^{-1}\varphi|_S = |t|_S = 1$. We can still use the bounds for $|a\varphi|_S$ and $|a\varphi^{-1}|_S$ to obtain that

$$\|\varphi\|_S \geq C(2p + \log_2|q|) + 1 \geq C(2p + 1 + \log_2|q|),$$
$$\|\varphi^{-1}\|_S \leq D(p + 3\log_2|q| + 1) + 1 \leq D(2p + 3 + 6\log_2|q|),$$

which are the wanted inequalities. $\square$

Theorem 4.9. *The auto-gap function of the Baumslag-Solitar group* $\mathrm{BS}(1,N)$ *with* $|N| > 1$ *verifies that* $\alpha(n) \sim n$*.*

Proof. Observe the value $m_\varphi := 2p + 2\alpha + 1 + \log_2|\beta| + \log_2|q|$ is completely determined by $\varphi = \varphi_{p,q,\alpha,\beta} \in \mathrm{Aut}(\mathrm{BS}(1,N))$ in normal form (taking into consideration the substitution for the case $\beta = 0$ as before). Therefore,

$$2p + 2\alpha + 3 + \log_2|\beta| + 6\log_2|q| \leq 12p + 12\alpha + 6 + 6\log_2|\beta| + 6\log_2|q| = 6m_\varphi,$$

so, by Lemma 4.8, we have that $\|\varphi\|_S \geq Cm_\varphi$ and

$$\|\varphi^{-1}\|_S \leq 6Dm_\varphi \leq \frac{6D}{C}\|\varphi\|_S.$$

This is valid for every $\varphi \in \mathrm{Aut}(\mathrm{BS}(1,N))$, so

$$\begin{aligned}
\alpha(n) &= \max\{\|\varphi^{-1}\|_S \mid \varphi \in \mathrm{Aut}(\mathrm{BS}(1,N)), \|\varphi\|_S \leq n\} \\
&\leq \max\{\tfrac{6D}{C}\|\varphi\|_S \mid \varphi \in \mathrm{Aut}(\mathrm{BS}(1,N)), \|\varphi\|_S \leq n\} \\
&= \frac{6D}{C}n,
\end{aligned}$$

proving that $\alpha(n) \preceq n$.

Finally, we get the other direction, $n \preceq \alpha(n)$, because $\mathrm{Inn}(\mathrm{BS}(1,N))$ is indeed infinite (2.9). $\square$

Observe that we end up with a function of linear growth, which is one of the simplest cases of growth, but that the proof is actually quite complex. This tells us that finding the auto-gap function of a group, in general, is a very difficult task to complete.

5 Bounding the gap between a virtual automorphism and its inverse

Let us introduce the concept of virtual automorphism of a group and some of its properties, as described in [14].

Definition 5.1 (Virtual automorphism)**.** Let G be a group. A virtual automorphism ϕ of G (also denoted $\phi : G \dashrightarrow G$) is a group isomorphism $\phi : H \longrightarrow K$, where H and K are subgroups of G of finite index. We refer to H as the domain of ϕ, and we denote it by $\operatorname{Dom}\phi$; and, as usual, we refer to $K = \operatorname{Im}\phi$ as the image of ϕ. We denote the set of virtual automorphisms of G as $\operatorname{VAut}(G)$.

For every virtual automorphism ϕ we can consider inverse, $\phi^{-1} : K \longrightarrow H$. Also, for every subgroup H of G of finite index, we can define the identical virtual automorphism over H as $\mathrm{id}_H : H \longrightarrow H$ such that $h \longmapsto h$ for all $h \in H$. Note also that $\operatorname{Aut}(G) \subseteq \operatorname{VAut}(G)$.

This set has an operation that gives it the structure of a monoid. Let $\phi_1 : H_1 \longrightarrow K_1$ and $\phi_2 : H_2 \longrightarrow K_2$ be two virtual automorphisms of G, then we define its product as

$$\begin{aligned} \phi_1\phi_2 : (H_2 \cap K_1)\phi_1^{-1} &\longrightarrow (H_2 \cap K_1)\phi_2 \\ h\phi_1^{-1} &\longmapsto h\phi_2 \end{aligned}$$

and clearly $\operatorname{VAut}(G)$ is closed by this operation, since the intersection of subgroups of finite index is a subgroup of finite index:

$$[G : H_2 \cap K_1] \leq [G : H_2][G : K_1] < \infty$$

and we know that $(H \cap K)\phi = (H)\phi \cap (K)\phi$ for all ϕ group isomorphisms and H, K subgroups. This operation is also associative, and id_G is the identity element of this operation, so $\operatorname{VAut}(G)$ is a monoid, but it isn't a group, since the property of invertibility isn't satisfied (we have for $\phi : H \longrightarrow K$ that $\phi\phi^{-1} = \mathrm{id}_H$ and $\phi^{-1}\phi = \mathrm{id}_K$, but generally $\mathrm{id}_H, \mathrm{id}_K \neq \mathrm{id}_G$).

It is not important for our work whether $\operatorname{VAut}(G)$ is a group, a monoid or none of the former; the only necessary condition is that there is a notion of an inverse for every virtual automorphism. We can check in the previous sections that the property that $\operatorname{Aut}(G)$ and $\operatorname{Out}(G)$ are groups is not used in the definition of the auto-gap and outer-gap functions, or in main results (2.4 and 2.5); and that the only places where the group axioms are used is for specific calculations of some bounds.

Even though we are not going to work with it, there's a group that derives from $\operatorname{VAut}(G)$ to whom we must dedicate at least a paragraph. The definition of this group requires the introduction of the following relation.

Definition 5.2 (Commensurable virtual automorphisms)**.** We say that two virtual automorphisms $\phi_1 : H_1 \longrightarrow K_1$ and $\phi_2 : H_2 \longrightarrow K_2$ are *commensurable* if there exists a subgroup $H \leq H_1 \cap H_2$ of finite index in G, such that $\phi_1|_H = \phi_2|_H$. In this case we write $\phi_1 \approx \phi_2$.

We can easily verify that the relation $\approx$ is an equivalence over $\mathrm{VAut}(G)$. Note that all identical virtual automorphisms are commensurable to id_G, so in particular $\phi\phi^{-1} \approx \mathrm{id}_G$ and $\phi^{-1}\phi \approx \mathrm{id}_G$ for all virtual automorphism ϕ, and thus ${}^{\mathrm{VAut}(G)}/_{\approx}$ with the induced operation has the structure of group. This group is called the *abstract commensurator* of G, denoted $\mathrm{Comm}(G)$.

More information about the abstract commensurator can be found in [1, 14], but we are working instead with $\mathrm{VAut}(G)$ because it is simpler and because, as we said, the condition of being a group is not needed.

We are now interested in searching a way to understand the gap between a virtual automorphism ϕ and its inverse ϕ^{-1} in similar sense. The main idea is to find analogous definitions to those of Subsection 2.1.

Since deducing a set of generators for a subgroup is a non-trivial task that can be done in multiple ways, we need to find a way to canonically assign a finite generating set to every subgroup of finite index in G. Hence, we introduce the concept of Schreier graphs.

5.1 Schreier graphs

The definition of Schreier coset graph and the proof of Schreier's Lemma, from where we are basing this section, can be found in [4, 6, 12].

Definition 5.3 (Schreier coset graph)**.** Given a finitely generated group G, a generating set $S = \{x_1, \dots, x_s\}$ of G and a subgroup $H \leq G$ of finite index m, we define Schreier's coset graph $\bar{\Gamma} = \bar{\Gamma}(G, H, S)$ as the finite directed graph whose vertices are the right cosets $\{H = Hg_1, \dots, Hg_m\}$ of H in G, and whose edges are (Hg_i, Hg_ix_j) (the edge from Hg_i to Hg_ix_j) for all $i = 1, \dots, m$ and $j = 1, \dots, s$. These edges are labelled canonically, that is, the edge from Hg_i to Hg_ix_j is labelled as x_j.

We must make some quick comments and observations from this definition. As we said, the graph is directed, with s different outgoing edges labelled with $x_1, \dots, x_s$ for each vertex Hg_i, but there are also s incoming edges, corresponding to the edges $(Hg_ix_j^{-1}, Hg_i)$ for all $j = 1, \dots, s$, also labelled with $x_1, \dots, x_s$. Therefore, Schreier coset graphs are regular, but bear in mind that the graph can contain loops and multiple edges.

Since Schreier graph is directed, we are denoting its set of edges as $\mathrm{E}\,\Gamma^+$, and we are defining the set

$$\mathrm{E}\,\Gamma^- = \{e^{-1} := (v, u) \in \mathrm{V}\,\bar{\Gamma}^2 \mid e = (u, v) \in \mathrm{E}\,\Gamma^+\},$$

that can be thought as the edges running in the opposite direction. They are also labelled: the edge $(v, u) \in \mathrm{E}\,\Gamma^-$ is labelled with the inverse of the label of $(u, v) \in \mathrm{E}\,\Gamma^+$ (if (u, v) is labelled as x_j, then (v, u) is labelled as x_j^{-1}). We let $\mathrm{E}\,\Gamma = \mathrm{E}\,\Gamma^+ \sqcup \mathrm{E}\,\Gamma^-$ denote the disjoint union of both edge sets, and we define a new graph $\Gamma := (\mathrm{V}\,\bar{\Gamma}, \mathrm{E}\,\Gamma)$.

Note that if we are in Hg_i, and we follow an edge from $\mathrm{E}\,\Gamma^+$ labelled with x_j (respecting the direction of the edge), we end up at Hg_ix_j, that is, we end up to the right multiplication of Hg_i by x_j, when we view them as cosets. Analogously, if we decide to follow the edge from $\mathrm{E}\,\Gamma^-$ labelled with x_j^{-1}, then we end up at $Hg_ix_j^{-1}$, which corresponds to the right multiplication of Hg_i by x_j^{-1}.

We label all walks γ over Γ by concatenation. Formally, if γ goes through the edges $e_1, e_2, \dots, e_l$ in that order (and respecting the direction), then its label $\ell(\gamma)$ is the word in S of length l such that the i-th letter is the label of e_i. As always, $\ell(\gamma)$ can be seen as an element of G, and every element of G can be written as the label of a walk on the graph starting at any vertex of the graph.

If we follow the walk labelled with $\ell(\gamma) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_l}^{\varepsilon_l}$ that starts in Hg_i, we will end up at $Hg_i x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_l}^{\varepsilon_l}$. This proves that the graph is connected, because every non-trivial element $g_i \in G$ can be expressed as $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_l}^{\varepsilon_l}$, and therefore H is connected to Hg_i by a walk labelled with $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_l}^{\varepsilon_l}$.

Note also that if we consider a closed walk γ in H (starting and finishing at H), then we have that $H = H \cdot \ell(\gamma)$, which happens if and only if $\ell(\gamma) \in H$, so actually $H = \{\ell(\gamma) \mid \gamma \text{ is a closed walk in } H\}$. Clearly, the words in S are in bijection with the walks starting at H (or at any other point), and even more precisely, the reduced words in S are in bijection with the walks in H without any backtracking (so that at no point does it go through the same edge twice in a row).

In Γ, for every incoming edge from u to v, there's an outgoing edge from u to v, that allows us to think that Γ is an undirected graph (by viewing every pair of these edges as a unique non-directed edge).

Let us introduce some new notation needed to prove Schreier's Lemma: for any tree $\mathfrak{T}$ and any two distinct vertices u, v, we denote the unique walk from u to v along $\mathfrak{T}$ as $\mathfrak{T}[u,v]$, and we define $\mathfrak{T}[v,v]$ as the empty walk. By definition of tree, these walks are well-defined and unique. Also, if γ and γ' are walks that, respectively, end and start at the same vertex, we can define its composition, $\gamma \cdot \gamma'$, in the usual way. Moreover, $\ell(\gamma \cdot \gamma') = \ell(\gamma)\,\ell(\gamma')$ in G.

With all these observations, Schreier's Lemma can be proven

Lemma 5.4 (Schreier's lemma)**.** *Let $\mathfrak{T}$ be a spanning tree of $\Gamma = \Gamma(G,H,S)$. For all $e = (Hg_i, Hg_j) \in \mathrm{E}\,\Gamma^+$, we define $X_e = \ell(\mathfrak{T}[H, Hg_i] \cdot e \cdot \mathfrak{T}[Hg_j, H])$, which is trivial in G if $e \in \mathrm{E}\,\mathfrak{T}$. Then,*

$$H = \langle X_e \mid e \in \mathrm{E}\,\Gamma^+ \setminus \mathrm{E}\,\mathfrak{T} \rangle,$$

and more precisely, $\{X_e \mid e \in \mathrm{E}\,\Gamma^+ \setminus \mathrm{E}\,\mathfrak{T}\}$ is a generating set for H with at most $m(s-1)+1$ elements. Also, $X_{e^{-1}} = X_e^{-1}$.

Proof. Since Γ is connected, $\mathfrak{T}$ is well-defined and so is every $\mathfrak{T}[Hg_i, Hg_j]$. Clearly, if $e = (Hg_i, Hg_j) \in \mathrm{E}\,\Gamma^+$, then $\mathfrak{T}[H, Hg_i] \cdot e \cdot \mathfrak{T}[Hg_j, H]$ is a closed walk in Γ. In particular, $X_e \in H$ for all $e \in \mathrm{E}\,\Gamma^+$. If $e \in \mathrm{E}\,\mathfrak{T}$, then this closed walk is over the tree $\mathfrak{T}$, so it cannot have a cycle and its label must be trivial in G, so we can exclude them as generators of H.

Now let's see that $H \subseteq \langle X_e \mid e \in \mathrm{E}\,\Gamma^+ \setminus \mathrm{E}\,\mathfrak{T} \rangle$: we consider a non-trivial $h \in H$ with a reduced expression $h = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_l}^{\varepsilon_l}$, and the closed walk in Γ starting at H with label $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_l}^{\varepsilon_l}$. Say this walk corresponds to $\gamma = e_1^{\varepsilon_1} \cdot e_2^{\varepsilon_2} \cdot \cdots \cdot e_l^{\varepsilon_l}$, where each $e_k \in \mathrm{E}\,\Gamma^+$ labelled with x_{i_k}. If $e_{k_1}^{\varepsilon_{k_1}}, \dots, e_{k_r}^{\varepsilon_{k_r}}$ correspond to all the edges (in order) that are not in the tree, then γ can be written as

$$\gamma = \gamma_1 \cdot e_{k_1}^{\varepsilon_{k_1}} \cdot \gamma_2 \cdot e_{k_2}^{\varepsilon_{k_2}} \cdot \cdots \cdot \gamma_r \cdot e_{k_r}^{\varepsilon_{k_r}} \cdot \gamma_{r+1}$$

where $\gamma_1 = e_1^{\varepsilon_1} \cdot \cdots \cdot e_{k_1-1}^{\varepsilon_{k_1-1}}$ (or γ_1 is the empty walk when $e_1^{\varepsilon_1} = e_{k_1}^{\varepsilon_{k_1}}$), $\gamma_t = e_{k_{t-1}+1}^{\varepsilon_{k_{t-1}+1}} \cdot \cdots \cdot e_{k_t-1}^{\varepsilon_{k_t-1}}$ for all $t = 2, \ldots, r$ (or empty when $e_{k_{t-1}}^{\varepsilon_{k_{t-1}}} = e_{k_t-1}^{\varepsilon_{k_t-1}}$) and $\gamma_{r+1} = e_{k_r+1}^{\varepsilon_{k_r+1}} \cdot \cdots \cdot e_l^{\varepsilon_l}$ (or empty if $e_{k_r}^{\varepsilon_{k_r}} = e_l^{\varepsilon_l}$). Observe that all these walks are completely contained in $\mathfrak{T}$.

Now we are going to prove that $h = \ell(\gamma)$ can be written in G as a product of elements in $\{X_e \mid e \in \mathrm{E}\,\Gamma^+ \setminus \mathrm{E}\,\mathfrak{T}\}$ and their inverses. Using the previous notation, we have that, for all $t = 2, \ldots, r$,

$$\begin{gathered}
\gamma_1 = \mathfrak{T}[H, Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_1-1}}^{\varepsilon_{k_1-1}}] \\
\gamma_t = \mathfrak{T}[Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_{t-1}}}^{\varepsilon_{k_{t-1}}}, Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_t-1}}^{\varepsilon_{k_t-1}}] \\
\gamma_{r+1} = \mathfrak{T}[Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_r}}^{\varepsilon_{k_r}}, Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_l}^{\varepsilon_l}] = \mathfrak{T}[Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_r}}^{\varepsilon_{k_r}}, H]
\end{gathered}$$

and since γ_t are walks over the tree,

$$\begin{aligned}
\ell(\gamma_t) &= \ell(\mathfrak{T}[Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_{t-1}}}^{\varepsilon_{k_{t-1}}}, Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_t-1}}^{\varepsilon_{k_t-1}}]) \\
&= \ell(\mathfrak{T}[Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_{t-1}}}^{\varepsilon_{k_{t-1}}}, H])\, \ell(\mathfrak{T}[H, Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_t-1}}^{\varepsilon_{k_t-1}}]).
\end{aligned}$$

In conclusion,

$$\begin{aligned}
\ell(\gamma) &= [\ell(\gamma_1)\, \ell(e_{k_1}^{\varepsilon_{k_1}})][\ell(\gamma_2)\, \ell(e_{k_2}^{\varepsilon_{k_2}})] \cdots [\ell(\gamma_r)\, \ell(e_{k_r}^{\varepsilon_{k_r}})]\, \ell(\gamma_{r+1}) \\
&= [\ell(\mathfrak{T}[H, Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_1-1}}^{\varepsilon_{k_1-1}}])\, \ell(e_{k_1}^{\varepsilon_{k_1}})][\ell(\mathfrak{T}[Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_1}}^{\varepsilon_{k_1}}, H]) \\
&\quad \ell(\mathfrak{T}[H, Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_2-1}}^{\varepsilon_{k_2-1}}])\, \ell(e_{k_2}^{\varepsilon_{k_2}})][\ell(\mathfrak{T}[Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_2}}^{\varepsilon_{k_2}}, H]) \cdots \\
&\quad \ell(\mathfrak{T}[H, Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_r-1}}^{\varepsilon_{k_r-1}}])\, \ell(e_{k_r}^{\varepsilon_{k_r}})]\, \ell(\mathfrak{T}[Hx_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_{k_r}}^{\varepsilon_{k_r}}, H]) \\
&= X_{e_{k_1}}^{\varepsilon_{k_1}} X_{e_{k_2}}^{\varepsilon_{k_2}} \cdots X_{e_{k_r}}^{\varepsilon_{k_r}}.
\end{aligned}$$

This proves that every element $h \in H$ can be written as the product of elements of $\{X_e^{\pm 1} \mid e \in \mathrm{E}\,\Gamma^+ \setminus \mathrm{E}\,\mathfrak{T}\}$. Thus, $H = \langle X_e \mid e \in \mathrm{E}\,\Gamma^+ \setminus \mathrm{E}\,\mathfrak{T}\rangle$.

Finally, since Γ and $\mathfrak{T}$ are finite graphs, $\{X_e \mid e \in \mathrm{E}\,\Gamma^+ \setminus \mathrm{E}\,\mathfrak{T}\}$ is a finite set of cardinality at most $|\mathrm{E}\,\Gamma^+| - |\mathrm{E}\,\mathfrak{T}|$ (equality holds if and only if all X_e are pairwise different as elements of G). We have that $\bar{\Gamma}$ is a s-regular finite graph with m vertices, and $\mathfrak{T}$ is a spanning tree of Γ, so $|\mathrm{E}\,\mathfrak{T}| + 1 = |\mathrm{V}\,\mathfrak{T}| = |\mathrm{V}\,\Gamma|$, and

$$|\{X_e \mid e \in \mathrm{E}\,\Gamma^+ \setminus \mathrm{E}\,\mathfrak{T}\}| \leq sm - m + 1 = m(s-1) + 1. \qquad \square$$

Observe that Schreier's Lemma serves as direct proofs of the following two important results in group theory.

Corollary 5.5. *Every subgroup of finite index of a finitely generated group is finitely generated.*

Corollary 5.6. *Let G be a finitely generated group and $m \geq 1$ fixed. Then, there are a finite number of subgroups of G of finite index m.*

We also know that the number of subgroups of index n might be very big, as shown in [10].

Note also that this generating set is not necessarily minimal, and might even contain repeated elements and trivial ones. This won't cause any problems, but is a thing to bear in mind. This bound cannot be improved, as shown by this well-known result.

Theorem 5.7 (Schreier's index formula)**.** *Let F_r be the group of rank r and H a subgroup of finite index m. Then,*

$$\operatorname{rank}(H) = m(r-1) + 1.$$

Here we are not proving this result, but we can find a proof in [16].

As previously stated, we are interested in defining a set of generators of H that only depend on G and S, and this lemma gives us a transversal that only depends on the spanning tree $\mathfrak{T}$. This set is called as Schreier's transversal, which is defined as $\mathcal{T} = \{\ell(\mathfrak{T}[H, v]) \mid v \in \mathrm{V}\,\mathfrak{T}\}$. Observe that the elements of the $\mathcal{T}$ verify the fact that if $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_l}^{\varepsilon_l} \in \mathcal{T}$, then $x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_j}^{\varepsilon_j} \in \mathcal{T}$ for all $j \leq l$.

As we said, the generating set of H depends on the choice we make of the spanning tree, so what we need to find now is a way of canonically constructing such a tree, in a way that depends only on the original group G and its generator set S.

Let's explain how we can do this: We define an order $\leq$ in S,

$$x_1 < x_2 < \cdots < x_s$$

that induces an order in $S \cup S^{-1}$ as follows:

$$x_1 < x_2 < \cdots < x_s < x_1^{-1} < x_2^{-1} < \cdots < x_s^{-1}$$

where x_i^{-1} is omitted if it has appeared before (for example, when x_i is order 2). We denote this generating set with a fixed order as $\mathcal{S} = (S, \leq)$. This allows us to define the spanning tree $\mathfrak{T} = \mathfrak{T}(\Gamma, \mathcal{S})$ step-by-step using a recursive algorithm.

Proposition 5.8 (Algorithm for the construction of $\mathfrak{T}$)**.** *The following algorithm allows us to construct a canonical spanning tree $\mathfrak{T} = \mathfrak{T}(\Gamma, \mathcal{S})$ over $\Gamma = \Gamma(G, H, S)$:*

1. *We start with $\mathfrak{T}$ being just the vertex H, and we initiate an ordered sequence of vertices $\{\sigma_i\}_i$ that for now only includes H.*
2. *Then the recursive algorithm is applied, until i) cannot be satisfied:*
   - *i) We take the first element Hg_i in the sequence $\{\sigma_i\}_i$ that has a neighbour in Γ that is not already a vertex of $\mathfrak{T}$.*
   - *ii) We get the smallest element $x_j^{\varepsilon_j}$ in $S \cup S^{-1}$ (with respect to $\leq$) such that $Hg_i x_j^{\varepsilon_j} \notin \mathrm{V}\,\mathfrak{T}$.*
   - *iii) We add this vertex to the end of the ordered sequence, and also to $\mathfrak{T}$ together with the edge that connects it to Hg_i.*
3. *The algorithm ends after the m-th step (considering the initial step as one), when $\mathfrak{T}$ is a spanning tree of Γ.*

Proof. We just need to see it is well-defined. The algorithm only depends on $\mathcal{S}$ and the original Γ, so if we see that $\mathfrak{T}$ is a well-defined tree in every step, and that the algorithm correctly stops when $\mathfrak{T}$ is a spanning tree and not before, then we are done.

Let's see that it is a tree every step of the way. The initial step corresponds to a tree, so let's prove the recurring part. If i) is satisfied, then ii) must be also satisfiable. Choosing a vertex that is not already in $\mathrm{V}\,\mathfrak{T}$, guarantees that when we add it to the tree together with the connecting edge, that it won't be already connected to any other edge, so the resulting graph is a well-defined tree.

Now let's see the algorithm stops after the m-th step, that is when $\mathfrak{T}$ is a spanning tree. If it stops before, then that means that we have a proper subset of $\mathrm{V}\,\Gamma$ such that it is not connected to any of the other vertices outside the set, which directly contradicts connectivity of Γ. □

In conclusion, we have described a generating set for a subgroup H of finite index that only depends on the Schreier graph and the order we give to S. In the following subsection we are denoting the canonical generating set of $H \leq G$ with respect to $\mathcal{S}$ we just described as $T = T(G, H, \mathcal{S})$.

5.2 Norm of a virtual automorphism and the virtual-gap function

We define the norm of a virtual automorphism $\phi : H \longrightarrow K$ of G with respect to an ordered finite generating set $\mathcal{S}$ as

$$\|\phi\|_{\mathcal{S}}^{\mathrm{v}} = [G : H] \sum_{t \in T} |t\phi|_S$$

where $T = T(G, H, \mathcal{S})$. Two important observations go behind this definition: Corollary 5.10 and Proposition 5.11.

Proposition 5.9. *Let $\phi \in \mathrm{VAut}(G)$ be such that $\mathrm{Dom}\,\phi = G$, $S = \{x_1, \dots, x_s\}$ be a finite generator set of G and $\mathcal{S}$ any order of S. Then,*

$$\|\phi\|_{\mathcal{S}}^{\mathrm{v}} = \sum_{s \in S} |s\phi|_S.$$

Proof. We have that $\mathrm{Dom}\,\phi := H = G$, and by construction, $\Gamma(G, G, S)$ is a bouquet with $|S| = s$ loops. The associated tree $\mathfrak{T}$ is simply a single point and the generating set T is

$$T = \{X_e \mid e \in \mathrm{E}\,\Gamma^+ \setminus \mathrm{E}\,\mathfrak{T}\} = \{1_G \cdot x \cdot 1_G \mid x \in S, x \neq 1_G\} = S,$$

where the order of $\mathcal{S}$ doesn't play any part, since there are no edges in $\mathfrak{T}$. In conclusion,

$$\|\phi\|_{\mathcal{S}}^{\mathrm{v}} = [G : G] \sum_{t \in T} |t\phi|_S = \sum_{s \in S} |s\phi|_S. \qquad \square$$

Corollary 5.10. *Let $\varphi \in \mathrm{Aut}(G) \subseteq \mathrm{VAut}(G)$, $S = \{x_1, \dots, x_s\}$ be a finite generator set of G and $\mathcal{S}$ any order of S. Then,*

$$\|\varphi\|_S = \|\varphi\|_{\mathcal{S}}^{\mathrm{v}}.$$

Proposition 5.11. *For every $n \geq 0$, there exists a finite number of virtual automorphisms ϕ of G such that $\|\phi\|_{\mathcal{S}}^{\mathrm{v}} \leq n$.*

Proof. Since $[G : \mathrm{Dom}\,\phi] \leq \|\phi\|_{\mathcal{S}}^{\mathrm{v}}$, every virtual automorphism φ such that $\|\phi\|_{\mathcal{S}}^{\mathrm{v}} \leq n$ must also verify $[G : \mathrm{Dom}\,\phi] \leq n$, so we have a finite number of choices for the possible domains for these virtual automorphism, by Corollary 5.6.

Now, for every possible fixed domain H, we have that all virtual automorphisms $\phi : H \longrightarrow K \leq G$ are completely determined by the image of the generators of H, but since S and T are finite, we have only a finite number of possibilities for these images to ensure they have a bounded norm. □

Remark 5.12. The addition of the term $[G : H]$ in the definition of the norm is vital in ensuring that there's no infinite number of virtual automorphisms of a bounded norm, in any finitely generated group.

Let's see this. Suppose $\| \cdot \|_{\mathcal{S}}^{\mathrm{v}\,\prime}$ defined as before but without the factor $[G : H]$, and consider the group $\mathbb{Z}$ with generating set $\mathcal{S} = \{1\}$ (it has only one order, so we don't need to refer to it). For every odd $m > 0$, consider the virtual automorphisms $\phi_m : m\mathbb{Z} \longrightarrow \mathbb{Z}$ such that $m\phi_m = 1$. Its Schreier graph $\Gamma(\mathbb{Z}, m\mathbb{Z}, \mathcal{S})$ is the cycle graph C_m, and after applying the Algorithm 5.8, we obtain the tree formed by removing the edge $e = (\frac{m-1}{2}, \frac{m+1}{2})$. From Schreier's Lemma (5.4), we obtain that the only element of T is m. Therefore,

$$\|\phi_m\|_{\mathcal{S}}^{\mathrm{v}\,\prime} = \sum_{t \in T} |t\phi_m|_S = |m\phi_m|_S = |1|_S = 1.$$

Its inverse, on the other hand, is $\phi_m^{-1} : \mathbb{Z} \longrightarrow m\mathbb{Z}$. By Proposition 5.9, $\|\phi_m^{-1}\|_{\mathcal{S}}^{\mathrm{v}\,\prime} = m$, so

$$\max\{\|\phi^{-1}\|_{\mathcal{S}}^{\mathrm{v}\,\prime} \mid \phi \in \mathrm{VAut}(G), \|\phi\|_{\mathcal{S}}^{\mathrm{v}\,\prime} \leq 1\} \geq \max_{\substack{m \in \mathbb{N} \\ m \text{ odd}}} \{\|\phi_m^{-1}\|_{\mathcal{S}}^{\mathrm{v}\,\prime} = m\} = \infty.$$

We see that this missing factor, $[\mathbb{Z} : m\mathbb{Z}] = m$ solves this problem.

We can now define the following function that, analogously to the auto-gap and outer-gap functions, pretends to measure the maximum gap between a virtual automorphism and its inverse, or intuitively, the worst case difference between $\|\phi\|_S$ and $\|\phi^{-1}\|_S$, for all $\phi \in \mathrm{VAut}(G)$.

Definition 5.13 (Virtual-gap function)**.** We define the *virtual automorphism inversion gap function of G with respect to $\mathcal{S}$* (or *virtual-gap function*, for short) as the function $\nu_{G,\mathcal{S}} : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\nu_{G,\mathcal{S}}(n) = \max\{\|\phi^{-1}\|_{\mathcal{S}}^{\mathrm{v}} \mid \phi \in \mathrm{VAut}(G),\ \|\phi\|_{\mathcal{S}}^{\mathrm{v}} \leq n\}$$

Once again, the maximum of the empty set is defined as 0. We will also denote it simply as $\nu_{\mathcal{S}}$ whenever possible.

Note that it is well-defined by Proposition 5.11. Also, immediately from Corollary 5.10, we obtain the following result:

Proposition 5.14. *For any group G and ordered generating set $\mathcal{S}$, $\beta_{\mathcal{S}}(n) \leq \alpha_{\mathcal{S}}(n) \leq \nu_{\mathcal{S}}(n)$.*

Despite the analogous nature of the definition of the virtual-gap function to the auto-gap and outer-gap functions, a problem arises. We cannot give analogous versions of Lemma 2.4 and Theorem 2.5, since the bounding constant of the lemma directly depends on the virtual automorphism, more concretely, on the index of its domain. We are going to prove that the analogous constant given in the lemma cannot be bounded for all virtual automorphism.

We need a previous proposition to prove this

Proposition 5.15. *Let Γ be a finite s-regular directed graph, with a selected vertex v_0, such that each vertex has only one incoming and only one outgoing edge labelled with x_i for each x_i in a finite set S of size s. Then Γ is the Schreier graph $\bar{\Gamma}(F(S), H, S)$, where $F(S)$ denotes the free group generated by S, in particular, of rank $|S|$; and H is a subgroup of $F(S)$ of finite index $|\mathrm{V}\,\Gamma|$.*

Proof. Let Γ be such graph, and observe that we can construct a walk starting at this vertex labelled with any word in S, by the hypothesis of this graph. Now take any closed walk in v_0 in Γ labelled with w, then it is clear that the set of all these labels generates a subgroup H of $F(S)$, since a composition of elements corresponds to a closed walk in v_0. We name v_0 as H from now on.

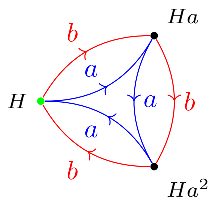
Let g_i be the label of a walk from H to another vertex v, then name it Hg_i. Repeating this for all the vertices, we obtain that $\mathrm{V}\,\Gamma = \{H = Hg_1, Hg_2, \ldots, Hg_m\}$ with $g_1 = 1_{F(S)}$. Observe now that every $w \in F(S)$ corresponds to a reduced word in S and therefore, there's a walk over Γ starting at H and ending at some Hg_i labelled with w, and so wg_i^{-1} is the label of a closed walk in H. Thus,

$$wg_i^{-1} \in H \iff w \in Hg_i.$$

In particular, every element in F_2 belongs to one and only one of the finite right cosets $H = Hg_1, Hg_2, \ldots, Hg_m$, which proves that H is a subgroup of $F(S)$ of finite index m. Now, it is easy to see that Γ is indeed $\bar{\Gamma}(F(S), H, S)$ following the definition. □

Another proof of this proposition is given in [13]. Using this result, we can see the following:

Remark 5.16. Let's see that the virtual-gap function depends on $\mathcal{S}$. Consider the following finite directed 2-regular labelled graph $\bar{\Gamma}$ with $m \geq 2$ vertices $\{0, \ldots, m-1\}$, and with an edge labelled a and a parallel edge labelled b from i to $i+1 \pmod m$, for all $i \in \{0, \ldots, m-1\}$. The image below corresponds to the case $m = 3$:



By Proposition 5.15, we have that this graph corresponds to $\bar{\Gamma}(F_2, H_m, S)$ for $S = \{a, b\}$ a base of F_2 and H_m a subgroup of finite index m.

Consider the two different orders of S: $\mathcal{S} = \{a < b\}$ and $\mathcal{S}' = \{b < a\}$, then, by Schreier's Lemma (5.4) and the Algorithm 5.8, the respective generating sets for H, T and T' verify the following two conditions:

i) $a^m \in T$ and $b^m \in T'$.

ii) No word in T has more than 1 b's, and no word in T' has more than 1 a's.

Consequently, $|b^m|_{T'} \geq m$ and $|a^m|_T \geq m$, and if we take a look at the constant C in Lemma 2.4, we see it must be larger than m^2.

Since we can construct this for all $m \geq 2$, we cannot bound all norms between all virtual automorphisms by the same constant C, since it must be larger than m^2.

It is possible that the result still holds, but we have been unable to prove so. Therefore, it is necessary to specify over which ordered generating set we are talking about. It may be possible to define the virtual-gap function in such a way that $\nu_{\mathcal{S}} \sim \nu_{\mathcal{S}'}$, or weaken Definition 2.1 to ensure this.

5.3 Virtual-gap function of the free groups

In this subsection we give a lower bound for virtual-gap function for the free group F_r, $r \geq 1$, with respect to an ordered basis $\mathcal{S}$. Let $\nu_{r,\mathcal{S}}$ denote such function, then

Theorem 5.17. *In the free group $F_1 \cong \mathbb{Z}$, $\nu_{\mathcal{S}}(n) = n$.*

Proof. We just need to see that $\|\phi\|_{\mathcal{S}}^{\mathrm{v}} = \|\phi^{-1}\|_{\mathcal{S}}^{\mathrm{v}}$ for all $\phi \in \mathrm{VAut}(G)$. We know that the subgroups of finite index of $\mathbb{Z}$ are $m\mathbb{Z}$ for all $m \in \mathbb{Z}\backslash\{0\}$, and that $\mathrm{Aut}(\mathbb{Z}) \cong \mathrm{Aut}(m\mathbb{Z}) \cong \mathbb{Z}_2$. So, the only isomorphisms such that $\phi : m\mathbb{Z} \longrightarrow m'\mathbb{Z}$ are those verifying $m\phi = \pm m'$.

Following Remark 5.12, we see that $\bar{\Gamma}(\mathbb{Z}, m\mathbb{Z}, \mathcal{S})$ is a directed cycle, and therefore, the generating set obtained from the Algorithm 5.8 is the label of the full cycle, which is m. So $T = \{m\}$, and by definition of norm

$$\|\phi\|_{\mathcal{S}}^{\mathrm{v}} = [\mathbb{Z} : m\mathbb{Z}] \sum_{t \in T} |t\phi|_S = m|m\phi|_S = m|\pm m'|_S = mm'.$$

Symmetrically, we also obtain that

$$\|\phi^{-1}\|_{\mathcal{S}}^{\mathrm{v}} = [\mathbb{Z} : m'\mathbb{Z}] \sum_{t \in T'} |t\phi^{-1}|_S = m'|m'\phi^{-1}|_S = m'|\pm m|_S = m'm,$$

so in conclusion, $\|\phi\|_{\mathcal{S}}^{\mathrm{v}} = \|\phi^{-1}\|_{\mathcal{S}}^{\mathrm{v}}$. □

Now let's prove a lower bound for the non-abelian free groups.

Theorem 5.18. *In the free group F_r, $r \geq 2$, we have that*

$$\nu_{r,\mathcal{S}}(n) \geq \max_{m \in \mathbb{N}} \left\{ m\alpha_{m(r-1)+1,S}\left(\left\lfloor \frac{n}{m} \right\rfloor \right) \right\}.$$

Proof. We fix $r \geq 2$. By Theorem 5.7, we have that the subgroups H of finite index m of F_r have rank

$$\mathrm{rank}(H) = m(\mathrm{rank}(F_r) - 1) + 1 = m(r-1) + 1,$$

but since H is a subgroup of F_r, it must be free by Nielsen-Schreier's Theorem [16]. So, all subgroups of index m are isomorphic to $F_{m(r-1)+1}$. Furthermore, if $\phi : H \longrightarrow K \leq F_r$ is a virtual automorphism with $[F_r : H] = m$, then $H \cong K \cong F_{m(r-1)+1}$, because the rank of a group is an invariant by isomorphism (and thus, $[F_r : K] = m$). So all virtual automorphism of index m correspond to an automorphism of the free group $F_{m(r-1)+1}$, allowing us to write $\nu_{r,\mathcal{S}}(n)$ as

$$\nu_{r,\mathcal{S}}(n) = \max\{\|\phi^{-1}\|_{\mathcal{S}}^{\mathrm{v}} \mid \phi \in \mathrm{VAut}(F_r), \|\phi\|_{\mathcal{S}}^{\mathrm{v}} \leq n\}$$

$$
\begin{aligned}
&\geq \max\{m\|\phi^{-1}\|_S \mid m \geq 1, \phi \in \mathrm{Aut}(F_{m(r-1)+1}), m\|\phi\|_S \leq n\} \\
&= \max_{m\in\mathbb{N}}\{\max\{m\|\phi^{-1}\|_S \mid \phi \in \mathrm{Aut}(F_{m(r-1)+1}), m\|\phi\|_S \leq n\}\} \\
&= \max_{m\in\mathbb{N}}\left\{m \max\left\{\|\phi^{-1}\|_S \mid \phi \in \mathrm{Aut}(F_{m(r-1)+1}), \|\phi\|_S \leq \left\lfloor \frac{n}{m} \right\rfloor\right\}\right\} \\
&= \max_{m\in\mathbb{N}}\left\{m\alpha_{m(r-1)+1,S}\left(\left\lfloor \frac{n}{m} \right\rfloor\right)\right\},
\end{aligned}
$$

where the inequality comes from the fact that we need to fix the basis. □

Let's quickly observe that the result of previous value is well-defined, because $m\alpha_{m(r-1)+1,S}(\lfloor \frac{n}{m} \rfloor) = 0$ for all m such that $\frac{n}{m} < m(r-1)+1$ (in particular, when $m > \sqrt{\frac{n}{r-1}} =: n'$), and therefore,

$$\nu_{r,\mathcal{S}}(n) \geq \max_{m\in\mathbb{N}}\left\{m\alpha_{m(r-1)+1,S}\left(\left\lfloor \frac{n}{m} \right\rfloor\right)\right\} = \max_{m\leq n'}\left\{m\alpha_{m(r-1)+1,S}\left(\left\lfloor \frac{n}{m} \right\rfloor\right)\right\}.$$

Now let's give a lower bound using the bound we have for α_r.

Theorem 5.19. *The virtual-gap function $\nu_{r,\mathcal{S}}(n)$ for the free group F_r, $r \geq 2$, has exponential growth.*

Proof. By Theorems 3.7 and 5.18,

$$\nu_S(n) \geq \max_{m\in\mathbb{N}}\left\{m\alpha_{m(r-1)+1,S}\left(\left\lfloor \frac{n}{m} \right\rfloor\right)\right\} \succeq \max_{m\in\mathbb{N}}\left\{m\left\lfloor \frac{n}{m} \right\rfloor^{m(r-1)+1}\right\}$$

and we will obtain the result by studying in more detail the equation $f_n(m) := m\lfloor \frac{n}{m} \rfloor^{m(r-1)+1}$ over $m \in \mathbb{R}_{>0}$ with n fixed.

First observe that this function is not continuous. We have a discontinuity point in every point such that $\frac{n}{m}$ is an integer, so in $m = \frac{n}{k}$ for all $k \in \mathbb{N}$ (and only in these). It is an increasing function in the continuous intervals defined by these discontinuity points, that is, it is continuous and increasing in $(\frac{n}{k+1}, \frac{n}{k}]$ for all $k \in \mathbb{N}$, and also in (n, ∞). The maximum of f_n over the intervals $(\frac{n}{k+1}, \frac{n}{k}]$ is $f_n(\frac{n}{k}) = \frac{n}{k}k^{\frac{n}{k}(r-1)+1} = nk^{\frac{n}{k}(r-1)}$. Observe also, that the maximum value in each of these intervals coincides with the function $g_n(m) := m(\frac{n}{m})^{m(r-1)+1}$.

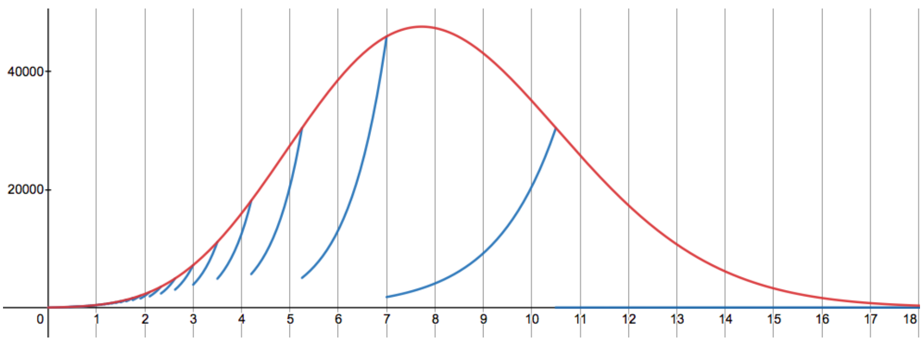


Figure 1: Plot of $f_n(m)$ (in blue) and $g_n(m)$ (in red) for the case $n = 21$ and $r = 2$.

Studying $g_n(m)$ and its derivative $g_n'(m) = n(r-1)(\ln(\frac{n}{m})-1)(\frac{n}{m})^{(r-1)m}$ allows us to conclude that it is a positive bell-shaped function with a unique maximum in $\frac{n}{\mathrm{e}}$ and limit 0 when $m \to 0$ and when $m \to \infty$, so it is clear that we have the maximum of $f_n(m)$ is either when $m = \frac{n}{2}$ or $m = \frac{n}{3}$. A simple algebraic manipulation allows us to see that

$$f_n(\tfrac{n}{2}) = n2^{\frac{n}{2}(r-1)} < n3^{\frac{n}{3}(r-1)} = f_n(\tfrac{n}{3}), \quad \text{because} \quad n^6 8^{n(r-1)} < n^6 9^{n(r-1)}$$

for all $n > 0$, so the maximum is at $m = \frac{n}{3}$.

But here we are interested in the maximum over the positive integers and not the positive real numbers. That's why we impose that n is a multiple of 3 to unsure that $\frac{n}{3}$ is an integer. So, changing $n \mapsto 3n$ we obtain

$$\nu_S(3n) \succeq \max_{m\in\mathbb{N}} \left\{ m \left\lfloor \frac{3n}{m} \right\rfloor^{m(r-1)+1} \right\} = n \left\lfloor \frac{3n}{n} \right\rfloor^{n(r-1)+1} = n3^{n(r-1)+1}$$

and by definition of function domination (2.1), we can see that $\nu_S(3n) \sim \nu_S(n)$ and that $n3^{n(r-1)+1} \sim [3^{r-1}]^n$, since $3^{n(r-1)} \leq n3^{n(r-1)+1}$ and $n3^{n(r-1)+1} \leq \frac{r}{r-1}3^{rn}$. In conclusion, $\nu_{r,\mathcal{S}}$ is of exponential growth. □

One of the most interesting questions the article [11] asks is whether there exist a group with an auto-gap function of exponential growth. We cannot give a direct answer to this question, but we can contrast it with the result we just have seen: there exists a group with a virtual-gap function of exponential growth, showing that there are functions similar nature to α_G can be exponential.

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