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On anonymous and weighted voting systems

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Abstract Many bodies around the world make their decisions through voting systems in which voters have several options and the collective result also has several options. Many of these voting systems are anonymous, i.e., all voters have an identical role in voting. Anonymous simple voting games, a binary vote for voters and a binary collective decision, can be represented by an easy weighted game, i.e., by means of a quota and an identical weight for the voters. Widely used voting systems of this type are the majority and the unanimity decision rules.

In this article we analyze the case in which voters have two or more voting options and the collective result of the vote has also two or more options. We prove that anonymity implies being representable through a weighted game if and only if the voting options for voters are binary. As a consequence of this result several significant enumerations are obtained.

Keywords Decision making · Multichoice games · Anonymous decision systems · Weighted decision systems · Pseudo-Boolean functions · Enumerations

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1 Introduction

In the context of simple voting games a single alternative, such as a bill or an amendment, is pitted against the status quo. In this setting, a collection of voters is said to form a winning coalition if passage of the issue at hand is guaranteed by yes votes from precisely those voters in the collection. Some few simple games are anonymous and many simple games are weighted, i.e., representable by a weight for each voter and a quota or threshold which serves to separate winning coalitions from losing coalitions. A seminal work that studies which simple games are weighted is Taylor and Zwicker (1992). Several works have continued the study, one of the most recent is Freixas et al. (2017).

Trivially, each anonymous simple game is weighted and admits a representation which assigns a weight of 1 to each voter, while the quota is a given integer number. Most of the real voting simple games are weighted and some examples of them already appear in the seminal book Von Neumann and Morgenstern (1944) but also in some other books, see e.g. Taylor and Zwicker (1999); Taylor and Pacelli (2008).

A class called (j, k)-simple games, Freixas and Zwicker (2003), extends the class of simple games which is met for $j = k = 2$. In (j, k)-simple games, voters are allowed to choose among a finite number of ordered alternatives, say j, expressing, for example, the degree of support to an amendment, and the output set is formed by several, say k, ordered aggregated decisions. An output is assigned to each combination of inputs for voters in a (j, k)-simple game. An example of a (j, k)-simple game is the case in which voters can abstain, an alternative in between voting yes and voting no, giving rise to $(3, 2)$-simple games. Games with abstention have been intensively studied, see e.g. Felsenthal and Machover (1997, 1998); Pongou et al. (2011); Freixas et al. (2014a,b). Quite surprisingly, the class of anonymous $(3, 2)$-simple games, in which all voters play an equivalent role in the voting rule, become very large and complex Freixas and Zwicker (2009) and there exist many anonymous $(3, 2)$-simple games which are not weighted. Thus, the link between to be anonymous and to be weighted for a (j, k)-simple game is not obvious as it is in simple games, and becomes a challenging problem since these structures arise in many contexts.

The problem of aggregating ordinal preferences on a set of ordered alternatives into a consensus has been the subject of study for more than two centuries. These problems have been studied from the perspectives of Social Choice, see e.g., May (1952); Gibbard (1973); Satterthwaite (1975); Fishburn (1973); Moulin (1983); Cato (2011); Freixas and Parker (2015) and to a lesser extent in Multiple-criteria decision-making (MCDM) or Multiple-Criteria Decision Analysis (MCDA), see Cook (2006) and even in Risk Analysis, see e.g., Aven (1992); Kuo and Zhu (2012). In most of these situations the actors or decision-makers are anonymous.

To a lesser extent, anonymous and weighted structures also appear in other disciplines. In Reliability, multistate monotone systems extend binary monotone systems where components and systems are allowed to have an arbitrary (finite) number of states/levels. Multistate monotone systems are used to model, e.g., production and transportation systems for oil and gas, and power transmission systems. Some multistate monotone systems are anonymous in the sense that all components play and identical role in the system. Some multistate monotone systems are weighted in the sense that they are representable by quotas and vectors of weights for each component. We refer the interested reader to Aven and Jensen (1999); Rausand and Høyland (2004); Levitin (2003); Levitin et al. (2003). In Neural Networks both multi-anonymous and multi-threshold logic gates are considered. In multi-anonymous logic gates all the inputs that intervene play a symmetrical role so that they are replaceable. Multi-threshold logic gates work in exactly the same way as a single output threshold gate, except that there are several outputs. Each output is delimited between two thresholds, so that the weighted sum is compared to these values. Multi-threshold logic gates are very versatile. We refer to Roychowdhury et al. (1994); Picton (2000).

The aim of this paper is to provide a unified formalization, useful for all the contexts described above, from which the link between anonymity (to be anonymous) and weightedness (to be weighted) is studied.

Boolean functions assign an aggregate binary output to any vector of binary components. We extend these functions to (j,k)-functions, which assign an aggregate output (among k possible values) to any vector with j possible inputs for each component. Both sets of possible choices, for the input indices and for the aggregate output, are assumed to be ordered from the lowest level to the highest level. The binary case achieved for $j = k = 2$ reduces to the well-known Boolean functions or simple games. Monotonic (j,k)-functions are considered in this paper. We focus in two significant subclasses: weighted and anonymous (j,k)-functions, and the relationship between them is studied. We prove that the property of anonymity for a function of this type implies that the function is weighted *if and only if* $j = 2$. Thus, the implication does not depend at all on the number of outputs. Models very close to the (j,k) functions can be found in some recent studies, Courtin et al. (2016) and Kurz et al. (2019).

As a consequence of this result several enumerations are obtained concerning the number of: anonymous $(2,k)$-functions of n variables, the number of exhaustive $(2,k)$-functions of n variables, and the number of pseudo-Boolean functions in a given quotient set.

The rest of the paper is organized as follows. Section 2 formally introduces the functions that are object of study of this paper. Section 3 introduces the most relevant subclasses of these functions. Section 4 contains the main result of the paper which characterizes the anonymous functions that are weighted in terms of the number of inputs for the indices. As a consequence, several enumerations are deduced in Section 5 for some anonymous functions and some pseudo-Boolean functions. A brief Conclusion ends the paper in Section 6.

2 (j,k)-functions

In this section we introduce several types of what we call (j,k)-functions. We start with some basic preliminaries.

Let $N = \{1, 2, \ldots, n\}$ denote a finite set of *indices*.[1] Let $\mathcal{J} = \{0, 1, \ldots, j - 1\}$ with $j \geq 2$ be the set of inputs for the indices for a given task, i.e., levels of performance for the indices or levels of support to a given proposal. The implicit ordering of the elements in $\mathcal{J}$ varies from the lowest level 0 to the greatest level $j - 1$; these numbers may have a qualitative meaning. Let $\mathcal{J}^n = \{0, 1, \ldots, j - 1\}^n$ be the n-Cartesian product of $\mathcal{J}$. Let $\mathcal{K} = \{0, 1, \ldots, k - 1\}$ with $k \geq 2$ be the set of outputs for the aggregated performance of the indices. Similarly, the implicit ordering of these numbers varies from the lowest level 0 to the greatest one $k - 1$.

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathcal{J}^n$ be a *vector*. The component $x_i \in \mathcal{J}$ indicates the level of performance of the index $i \in N$ for a given task, or the input level of index i.

A (j, k)*-function* of n variables $f : \mathcal{J}^n \to \mathcal{K}$ assigns to each vector $\mathbf{x} \in \mathcal{J}^n$ an aggregated output $f(\mathbf{x}) \in \mathcal{K}$.

An index i is *irrelevant* for a (j, k)-function f if for any $\mathbf{x} \in \mathcal{J}^n$ it is $f(\mathbf{x}) = f(\mathbf{y})$ for all $\mathbf{y} \in \mathcal{J}^n$ such that $y_\ell = x_\ell$ for any $\ell \neq i$.

In many contexts it is natural to add the property of monotonicity. The order relation we consider in $\mathcal{J}$ is componentwise, i.e., given two vectors $\mathbf{x}, \mathbf{y} \in \mathcal{J}^n$ we write $\mathbf{x} \leq \mathbf{y}$ when $x_i \leq y_i$ for every index $i = 1, \ldots, n$.

Definition 1 Monotonic (j, k)-function
Let f be a (j, k)-function.

$$f \text{ is } \textit{monotonic} \iff \text{For all } \mathbf{x}, \mathbf{y} \in \mathcal{J}^n,\ \mathbf{x} \leq \mathbf{y} \ \Rightarrow\ f(\mathbf{x}) \leq f(\mathbf{y})$$

Hereafter the monotonicity of the (j, k)-functions considered in this paper will be assumed and will not be necessarily stated.

Note that if $j = 2$ a $(2, k)$-function f is a particular case of a pseudo-Boolean function, see e.g. Hammer et al. (1988); Crama and Hammer (2011), which associates with every n-tuple $\mathbf{x} \in \{0, 1\}^n$ a real number $f(\mathbf{x})$.

3 Two significant subclasses of monotonic (j, k)-functions

3.1 Anonymous (j, k)-functions

Definition 2 Anonymous (j, k)-function
Let $\mathit{\Pi}$ be the set of permutations of N and let f be a monotonic (j, k)-function.

$$f \text{ is } \textit{anonymous} \iff f(\pi(\mathbf{x})) = f(\mathbf{x}) \ \text{ for all } \ \pi \in \mathit{\Pi} \ \text{ and for all } \ \mathbf{x} \in \mathcal{J}^n.$$

Let $A_{j,k}(n)$ be the number of anonymous (j, k)-functions of n variables. We collect some known results about these numbers in the following proposition. The

[1] The term index covers a broad spectrum of possibilities from physical components to persons or entities.

first one is a result of an easy check and the other two were proved in Freixas and Zwicker (2009).

Proposition 1

a. $A_{2,2}(n) = n + 2$
b. $A_{3,2}(n) = 2^{n+1}$
c. $A_{j,2}(n) = A_{n+1,2}(j-1)$

As a consequence of the main result in this paper, in Section 4, we deduce a closed formula for the number $A_{2,k}(n)$ of anonymous $(2,k)$-functions of n variables.

3.2 Weighted (j,k)-functions

Definition 3 Weighted (j,k)-function
f is a *weighted function* if for every $i \in N$ there exist a sequence of integer[2] weights:

$$w_i(j-1) \geq \cdots \geq w_i(1) \geq w_i(0) = 0 \tag{1}$$

and there exists a sequence of integer quotas $q_{k-1} \geq q_{k-2} \geq \cdots \geq q_1 \geq 0$ such that for each $\mathbf{x} \in \mathcal{J}^n$:

$$f(\mathbf{x}) = b \in \mathcal{K} \qquad \text{if and only if} \qquad q_{b+1} > w(\mathbf{x}) \geq q_b, \quad \text{where} \quad w(\mathbf{x}) := \sum_{i \in N} w_i(x_i) \tag{2}$$

where, for convenience, it is assumed that:

a. $q_0 = 0$ and $q_k = 1 + \sum_{i \in N} w_i(j-1)$, and
b. q_k is strictly greater than $w(\mathbf{x})$ for any $\mathbf{x}$.

As $w_i(0) = 0$ for all $i \in N$ in Definition 3, there is no need to consider this weight anymore.

The set of weights $\overline{w} = \{w_i(\ell) \mid 1 \leq i \leq n, 1 \leq \ell \leq j-1\}$ together with the set of quotas $\overline{q} = \{q_b \mid 1 \leq b \leq k-1\}$ constitute the *representation* $[\overline{w}, \overline{q}]$ for the weighted (j,k)-function f. Two different representations in weights and quotas are *tantamount* if they represent the same weighted (j,k)-function f. The number of representations of a given weighted (j,k)-function f is unbounded. Indeed, if $[\overline{w}, \overline{q}]$ is a representation for f, then for any $t > 0$, $[t \cdot \overline{w}, t \cdot \overline{q}]$ is also a representation for f, where $t \cdot \overline{w} = \{t \cdot w_i(\ell) \mid w_i(\ell) \in \overline{w}\}$ and $t \cdot \overline{q} = \{t \cdot q_b \mid q_b \in \overline{q}\}$, since the inequalities in Definition 3 are preserved. Thus, all these representations obtained for $t > 0$ are tantamount.

The following properties of weighted (j,k)-functions can be easily stated.

[2] An apparent more general definition would allow weights and quotas to be real numbers. However, it could be proved that any such representation with real weights and quotas has an equivalent representation in non-negative integer weights and quotas, so there is no reason here to add unnecessary complexity by allowing weights and quotas to be real numbers.

Proposition 2
Let f be a weighted (j,k)-function with representation $[\overline{w},\overline{q}]$, and $\mathbf{x},\mathbf{y}\in\mathcal{J}^n$. Then,

$i)$ $\mathbf{x}\le\mathbf{y} \implies w(\mathbf{x})\le w(\mathbf{y})$
$ii)$ $w(\mathbf{x})\le w(\mathbf{y}) \implies f(\mathbf{x})\le f(\mathbf{y})$,
$iii)$ $f(\mathbf{x})>f(\mathbf{y}) \implies w(\mathbf{x})>w(\mathbf{y})$.

Proof Part $i)$ is an immediate consequence of (1). To prove part $ii)$ let $\mathbf{x},\mathbf{y}\in\mathcal{J}^n$ be such that $w(\mathbf{x})\le w(\mathbf{y})$ and set $f(\mathbf{x})=b$. From (2) it is $f(\mathbf{x})=b$ if and only if $q_{b+1}>w(\mathbf{x})\ge q_b$. Now, if $w(\mathbf{y})<q_{b+1}$ it is $f(\mathbf{y})=f(\mathbf{x})$, and if $w(\mathbf{y})\ge q_{b+1}$ then $f(\mathbf{y})\ge b+1>f(\mathbf{x})$. Thus, in any case it is $f(\mathbf{x})\le f(\mathbf{y})$. Part $iii)$ comes immediately from $ii)$.

An immediate consequence of this proposition is the following:

Corollary 1 *Every weighted (j,k)-function is monotonic.*

4 Relationship between anonymous and weighted (j,k)-functions

This section contains two results. The first establishes that any anonymous and weighted (j,k)-function admits a representation in which the vectors of weights associated to the indices can be the same. The second constitutes the main result of the paper and characterizes which anonymous (j,k)-functions are weighted in terms of the number of inputs.

Proposition 3 *If f is an anonymous weighted (j,k)-function then there exist a representation of f that assigns, for each input level $\ell\in\mathcal{J}$, the same weight $w_\cdot(\ell)$ to all indices $i\in N$.*

Proof Let f be a weighted (j,k)-function with a representation $[\overline{w},\overline{q}]$. We will start by unifying the weights corresponding to the level 1. Set $w'(1)=\frac{1}{n}\sum_{i\in N}w_i(1)$ and define a new set of weights $\overline{w}'$ in the following way: for any level $\ell\in\{1,\dots,j-1\}$ and any index $i\in N$,

$$w'_i(\ell)=\begin{cases} w'(1), & \text{if } \ell=1\\ w_i(\ell), & \text{if } \ell\ne 1\end{cases}$$

It is clear that $w'(j-1)\ge\cdots\ge w'(1)\ge 0$. We will prove that $[\overline{w}',\overline{q}]$ is also a representation of f. Let $\mathbf{x}=(x_1,\dots,x_n)\in\mathcal{J}^n$ with $f(\mathbf{x})=b\in\mathcal{K}$, so that $q_{b+1}>w(\mathbf{x})=\sum_{i\in N}w_i(x_i)\ge q_b$. Let s be the number of components of $\mathbf{x}$ equal to 1, i.e., s is the cardinal of the set $\{i\in N|x_i=1\}$. Assume, without loss of generality, that $w_1(1)\ge w_2(1)\ge\cdots\ge w_n(1)$. Then, if $\mathbf{y}$ is a vector obtained from $\mathbf{x}$ by allocating all its components equal to 1 in the last places we have $f(\mathbf{y})=b$, because f is anonymous, and therefore $q_{b+1}>w(\mathbf{y})\ge q_b$. Thus,

$$w'(\mathbf{x})=\sum_{x_i\ne 1}w_i(x_i)+sw'(1)\ge\sum_{x_i\ne 1}w_i(x_i)+\sum_{i=n-s+1}^{n}w_i(1)=w(\mathbf{y})\ge q_b$$

On the other hand, if $\mathbf{z}$ is a vector obtained from $\mathbf{x}$ by allocating all its components equal to 1 in the first places we have $f(\mathbf{z}) = b$ and therefore $q_{b+1} > w(\mathbf{z}) \geq q_b$. Thus,

$$w'(\mathbf{x}) = \sum_{x_i \neq 1} w_i(x_i) + sw'(1) \leq \sum_{x_i \neq 1} w_i(x_i) + \sum_{i=1}^{s} w_i(1) = w(\mathbf{z}) < q_{b+1}$$

Hence, it is $q_{b+1} > w'(\mathbf{x}) \geq q_b$. Conversely, if $q_{b+1} > w'(\mathbf{x}) \geq q_b$ then, from $q_{b+1} > w'(\mathbf{x}) \geq w(\mathbf{y})$ we deduce that $f(\mathbf{x}) = f(\mathbf{y}) \leq b$ and, from $w(\mathbf{z}) \geq w'(\mathbf{x}) \geq q_b$ we deduce that $f(\mathbf{x}) = f(\mathbf{y}) \geq b$. Thus, $f(\mathbf{x}) = b$.

The same procedure can be applied recursively to the remaining levels $2, \ldots j-1$, and we end up with a set of common weights $w'(j-1) \geq \cdots \geq w'(1) \geq 0$ to all indices in such a way that $[\overline{w}', \overline{q}]$ is a representation of f.

An anonymous (j,k)-function is not necessarily weighted. Although, remarkably, this is true for $j = 2$, as the next theorem shows.

Theorem 1 *Every anonymous* (j,k)*-function is weighted* if and only if $j = 2$.

Proof Assume $j = 2$. If f is anonymous (and monotonic) then

$$\begin{array}{l} \sum\limits_{i=1}^{n} x_i = \sum\limits_{i=1}^{n} y_i \Rightarrow f(x) = f(y) \\ \sum\limits_{i=1}^{n} x_i < \sum\limits_{i=1}^{n} y_i \Rightarrow f(x) \leq f(y) \\ f(x) > f(y) \Rightarrow \sum\limits_{i=1}^{n} x_i > \sum\limits_{i=1}^{n} y_i \end{array} \tag{3}$$

To prove that f is weighted we assign weights $w_i(1) = 1$ for all $i \in N$, and we will find appropriate thresholds $q_{k-1} \geq q_k \geq \cdots \geq q_1 \geq 0$. Notice that, for all $\mathbf{x} \in \mathcal{J}^n$ it is $w(\mathbf{x}) = \sum\limits_{i=1}^{n} x_i$. Since $w(1, \ldots, 1) = n$ is the maximum possible weight for a vector in $\mathcal{J}^n$ we will set $q_k = n + 1$.

If f takes only one value, say $f(\mathbf{x}) = a \in \mathcal{K}$ for all $\mathbf{x} \in \mathcal{J}^n$, then we can take the thresholds $q_b = 0$ if $b \leq a$ and $q_b = n + 1$ if $b > a$. Clearly $f(\mathbf{x}) = a$ if and only if $q_{a+1} = n + 1 > w(\mathbf{x}) \geq q_a = 0$.

If f takes more than one value, then, let $\mathcal{K}' \subseteq \mathcal{K}$ be the image of f. Let us assume that the elements in $\mathcal{K}'$ are ordered as in $\mathcal{K}$ and write $\mathcal{K}' = \{\widetilde{1}, \ldots, \widetilde{k}\}$. In this way, $\widetilde{1}$ is the minimum value in $\mathcal{K}'$ and $\widetilde{k}$ is the maximum value in $\mathcal{K}'$. For $b \in \mathcal{K}'$ we denote by $s(b)$ its successor in $\mathcal{K}'$ and by $p(b)$ its predecessor in $\mathcal{K}'$. Obviously, $p(\widetilde{1})$ and $s(\widetilde{k})$ do not exist.

Now, for each $b \in \mathcal{K}'$ with $\widetilde{1} \leq b \leq p(\widetilde{k})$ let

$$M_b = \max\{w(\mathbf{x}) \mid f(\mathbf{x}) = b\} \quad \text{and} \quad m_b = \min\{w(\mathbf{x}) \mid f(\mathbf{x}) = b\}.$$

It is clear that $m_{s(b)} > M_b$ because of (3).

Define the following thresholds, for $b \in \{1, \ldots, k-1\}$:

$$q_b = \begin{cases} 0 & \text{if } b \leq \widetilde{1} \\ m_t & \text{if } p(t) < b \leq t \text{with} t \in \mathcal{K}' \\ n+1 & \text{if } b > \widetilde{k} \end{cases}$$

We need to prove that

$$f(\mathbf{x}) = b \in \mathcal{K} \quad \text{if and only if} \quad q_b \leq w(\mathbf{x}) < q_{b+1}.$$

Assume that $f(\mathbf{x}) = b$. Then $b \in \mathcal{K}'$, and there are three possible cases:

(i) If $b = \widetilde{1}$ then $q_b = 0$ and $q_{b+1} = m_{s(b)}$ so that $q_b = 0 \leq w(\mathbf{x}) \leq M_b < m_{s(b)} = q_{b+1}$.

(ii) If $\widetilde{1} < b < \widetilde{k}$ then $q_b = m_b$ and $q_{b+1} = m_{s(b)}$ so that $q_b = m_b \leq w(\mathbf{x}) \leq M_b < m_{s(b)} = q_{b+1}$.

(iii) If $b = \widetilde{k}$ then $q_b = m_b$ and $q_{b+1} = n+1$ so that $m_b \leq w(\mathbf{x}) < n+1 = q_{b+1}$.

Conversely, assume that $q_b \leq w(\mathbf{x}) < q_{b+1}$. If $b \notin \mathcal{K}'$ then either $b < \widetilde{1}$, or $b > \widetilde{k}$ or there is some $t \in \mathcal{K}'$ such that $p(t) < b < t$. In any of these possibilities it would be $q_b = q_{b+1}$ and we are assuming that $q_b < q_{b+1}$.

Thus, $b \in \mathcal{K}'$ and there are three cases to consider:

(i) If $b = \widetilde{1}$ then $q_b = 0$ and $q_{b+1} = m_{s(b)}$ so that $0 \leq w(\mathbf{x}) < m_{s(b)}$. This implies $f(\mathbf{x}) < s(b)$ and, since $b = \widetilde{1}$, the only possibility is $f(\mathbf{x}) = b$.

(ii) If $\widetilde{1} < b < \widetilde{k}$ then $q_b = m_b$ and $q_{b+1} = m_{s(b)}$ so that $m_b \leq w(\mathbf{x}) < m_{s(b)}$. From the inequality on the left hand side we have $b \leq f(\mathbf{x})$. From the inequality on the right hand side we have $f(\mathbf{x}) < s(b)$. Thus, $b \leq f(\mathbf{x}) < s(b)$ and the only possibility is $f(\mathbf{x}) = b$.

(iii) If $b = \widetilde{k}$ then $q_b = m_b$ and $q_{b+1} = n+1$ so that $m_b \leq w(\mathbf{x}) < n+1$. But $m_b \leq w(\mathbf{x})$ implies $b \leq f(x)$, and the only possibility is $f(\mathbf{x}) = b$.

It has been proved that for $j = 2$ any anonymous (j,k)-function is weighted. Conversely, we see that for $j > 2$ there are anonymous (j,k)-functions which are not weighted with the following example:

Let f be the anonymous $(j,2)$-function with $j > 2$ of 4 variables defined as: $f(\mathbf{x}) = 1$ if x is either one the seven following vectors:

$$(2,2,0,0),\ (2,0,2,0),\ (2,0,0,2),\ (1,1,1,1),\ (0,2,2,0),\ (0,2,0,2),\ (0,0,2,2)$$

or $\mathbf{x}$ is a vector greater componentwise of any of them, and $f(\mathbf{x}) = 0$ otherwise. Thus, $f(2,1,1,0) = f(0,1,1,2) = 0$. As f is anonymous then, by Proposition 3, it admits a representation with the same weights assigned to all the indices for each input level. Let $w.(2)$ and $w.(1)$ the weights for the input levels 2 and 1 respectively.

By considering the vectors $(2,0,0,2)$, $(1,1,1,1)$ and $(2,1,1,0)$ twice, any representation of f should fulfill at least the next inequalities:

$$\begin{array}{rr} 2w.(2) \geq q & 4w.(1) \geq q \\ w.(2) + 2w.(1) < q & w.(2) + 2w.(1) < q \end{array}$$

By adding the two inequalities in each of the two rows we obtain

$$2q < 2w.(2) + 4w.(1) \geq 2q$$

which leads to a contradiction.

From the procedure followed in the proof of Theorem 1 we may derive some enumerations for monotonic anonymous functions with two inputs which are stated in next section.

5 Some enumerations of anonymous (j,k)-functions and anonymous pseudo-Boolean functions

The enumerations provided in this section concern the classes of monotonic and anonymous: $i)$ $(2,k)$-functions, $ii)$ exhaustive $(2,k)$-functions, and $iii)$ pseudo-Boolean functions.

5.1 $(2,k)$-functions

Let $A_{2,k}(n)$ be the number of monotonic anonymous $(2,k)$-functions of n variables.

Corollary 2 *For all $n \geq 1$ and $k \geq 2$ it holds*

$$A_{2,k}(n) = \binom{n+k}{k-1}$$

Proof From Theorem 1, $A_{2,k}(n)$, coincides with the number of quotas $(q_{k-1}, q_{k-2}, \ldots, q_1)$ satisfying the condition:

$$n+1 \geq q_{k-1} \geq q_{k-2} \geq \cdots \geq q_1 \geq 0. \tag{4}$$

Thus, the result follows from combinations with repetition.

From this corollary, interesting properties about the numbers $A_{2,k}(n)$ can be easily deduced. Some of them are summarized in the next corollary.

Corollary 3 *For all $n \geq 1$ and $k \geq 2$ it yields*

a. $A_{2,k+1}(n) = \sum_{t=-1}^{n} A_{2,k}(t)$, *with* $A_{2,k}(-1) = 1$ *and* $A_{2,k}(0) = k$.
b. $A_{2,k}(n+1) = \sum_{t=1}^{k} A_{2,t}(n)$, *with* $A_{2,1}(n) = 1$ *and* $A_{2,2}(n) = n+2$.
c. $A_{2,k+1}(n+1) = A_{2,k}(n+1) + A_{2,k+1}(n)$.
d. $A_{2,k}(n) = A_{2,n+2}(k-2)$.

Note the similarity of Corollary 3-d with Proposition 1-c, result obtained in Freixas and Zwicker (2009).

5.2 Exhaustive $(2, k)$-functions

Let $\mathcal{A}_{j,k}(n)$ be the number of *exhaustive* anonymous (j, k)-functions of n variables. If f is exhaustive every image for f must be achieved and therefore all the inequalities in (4) must be strict. From this it easily follows that $\mathcal{A}_{2,2}(n) = A_{2,2}(n) - 2$ because the number of anonymous $(2, 2)$-functions of n variables, $A_{2,2}(n)$, coincides with the number of integer values for the threshold q such that $n + 1 \geq q \geq 0$, but only $q = n + 1$ and $q = 0$ lead to a failure for f of being exhaustive.

Note that non-exhaustive anonymous $(2, 2)$-functions are avoided in the voting context because coalitions are not allowed to be either all winning or all losing.

Corollary 4 *For all* $n \geq 1$ *and* $k \geq 2$ *it holds*

a. $\mathcal{A}_{2,k}(n) = 0 \quad$ *if* $\quad k > n + 1$,
b. $\mathcal{A}_{2,k}(n) = \binom{n}{k-1} \quad$ *if* $\quad k \leq n + 1$.

Proof From Theorem 1 for $j = 2$ it exists a representation w such that $w_i(1) = 1$, for each integer i such that $0 \leq i \leq n$, and quotas $(q_{k-1}, q_{k-2}, \ldots, q_1)$ satisfying the condition: $n+1 \geq q_{k-1} \geq q_{k-2} \geq \cdots \geq q_1 \geq 0$. Thus f is exhaustive if and only if

$$n + 1 > q_{k-1} > q_{k-2} > \cdots > q_1 > 0.$$

The previous condition is never achieved if $k > n+1$, while it is in the opposite case. The number of elections for the quotas follows from combinations without repetition.

Let $\mathcal{A}_j(n)$ be the number of exhaustive anonymous (j, k)-functions for all $k \geq 2$. The next result easily follows from the previous Corollary 4.

Corollary 5 *For all* $n \geq 1$ *it holds*

$$\mathcal{A}_2(n) = 2^n - 1$$

Proof It holds $\mathcal{A}_2(n) = \sum_{k=2}^{\infty} \mathcal{A}_{2,k}(n)$ and from the previous Corollary

$$\mathcal{A}_2(n) = \sum_{k=2}^{n+1} \binom{n}{k-1} = 2^n - 1.$$

5.3 Pseudo-Boolean functions

Let $f : \{0,1\}^n \to \mathbb{R}$ be a monotonic anonymous function of n variables (pseudo-Boolean function). Let m be an integer verifying $0 \leq m \leq n$. From the anonymity of f it holds that $f(\mathbf{x})$ is constant for any vector $\mathbf{x}$ such that $\sum_{i=1}^{n} x_i = m$. Thus, we can identify $f(\mathbf{x})$ with $f(m)$ where the domain of f is taken as the set of integer numbers $\{0, 1, \ldots, n\}$. From now on, we think only in this domain for f. As f is monotonic it additionally holds

$$f(0) \leq f(1) \leq f(2) \leq \cdots \leq f(n) \tag{5}$$

Being a pseudo-Boolean function, f cannot be exhaustive, by definition, but regarded in the domain $\{0, 1, \ldots, n\}$ it can be injective. This only happens if all the inequalities in (5) are strict.

An interpretation for an injective monotonic anonymous pseudo-Boolean function is that the more support for an option (if m is strictly greater than m') the more profit people get (then $f(m)$ is strictly greater than $f(m')$). Instead, a non–injective monotonic anonymous pseudo-Boolean function models a situation where more support for an option can lead to the same collective profit.

Let $\mathrm{B}(n)$ be the infinite set of monotonic anonymous pseudo-Boolean functions of n variables and $\mathcal{B}(n)$ be the infinite set of injective monotonic anonymous pseudo-Boolean functions of n variables. On $\mathrm{B}(n)$, we define an equivalence relation $\sim$ by: $f \sim g$ if for all m ($0 \leq m \leq n-1$) it holds

$$f(m) < f(m+1) \qquad \text{if and only if} \qquad g(m) < g(m+1).$$

Thus, $f \sim g$ means that f and g show the same ranking, no matter what values take the two functions.

Let's consider the quotient set of $\mathrm{B}(n)$ by $\sim$, i.e. $\mathrm{B}(n)/\sim$, and denote it by $\mathrm{C}(n)$. The elements in the quotient set, $\mathrm{C}(n)$, constitute a partition of the set of monotonic anonymous pseudo-Boolean functions of n variables, $\mathrm{B}(n)$. In particular, $\overline{f} \in \mathrm{C}(n)$ means that $\overline{f} = \{g \in \mathrm{B}(n) \ : \ g \sim f\}$.

Analogously, let's consider the quotient set of $\mathcal{B}(n)$ by $\sim$, i.e. $\mathcal{B}(n)/\sim$, and denote it by $\mathcal{C}(n)$. The elements in the quotient set, $\mathcal{C}(n)$, constitute a partition of the set of injective monotonic anonymous pseudo-Boolean functions of n variables, $\mathcal{B}(n)$.

The following result is an enumeration for the finite sets $\mathrm{C}(n)$ and $\mathcal{C}(n)$, whose respective cardinalities are denoted by $|\mathrm{C}(n)|$ and $|\mathcal{C}(n)|$. Of course, $|\mathcal{C}(n)| \leq |\mathrm{C}(n)|$.

Corollary 6 *For all $n \geq 1$ it holds:*

a.

$$|\mathrm{C}(n)| = \binom{2n+1}{n}$$

b.

$$|\mathcal{C}(n)| = 1$$

Proof For the first part note that the maximum number of different images for any element $\overline{f} \in \mathrm{C}(n)$ is $n+1$. Corollary 2 gives $A_{2,k}(n) = \binom{n+k}{k-1}$, which is the number of anonymous $(2,k)$-functions of n variables. Then, observe that $|\mathrm{C}(n)| = A_{2,k}(n)$ for $k = n+1$, and from this equality it follows $|\mathrm{C}(n)| = \binom{2n+1}{n}$.

The injectivity of $\overline{f} \in \mathcal{C}(n)$ implies that $f(0) < f(1) < \cdots < f(n)$. Thus, $|\mathcal{C}(n)|$ coincides with the number of exhaustive monotonic anonymous $(2,n)$-functions. By Corollary 4 this number is $\mathcal{A}_{2,k}(n) = \binom{n}{k-1}$ for $k = n+1$. Thus, $|\mathcal{C}(n)| = 1$.

All the combinatorial numbers in the Corollaries of this section appear in OEIS (1964) under different interpretations. We have provided a new interpretation for them.

6 Conclusion

In this paper we have provided a unified framework for the class of (j,k)-functions in which j ordered inputs are allowed to indices or components and k aggregated ordinal outputs are feasible. We have considered monotonic functions of this type and studied the relationship between anonymous and weighted (j,k)-functions. The main result of the paper establishes that the anonymity of f implies that f is weighted *if and only if* the number of allowed inputs for indices is two. Thus, the implication does not depend at all on the number of outputs. The procedure followed in the proof allows to easily enumerate different types of anonymous $(2,k)$-functions and pseudo-Boolean functions.

A challenging problem consists of enumerating the class of anonymous and weighted (j,k)-functions for different combinations of the numbers, j (the number of inputs), k (the number of outputs) and n (the number of indices). Particularly interesting would be the combination for $j > 2$ and $k = 2$, whose simplest case corresponds to enumerating anonymous and weighted $(3,2)$-functions. Any progress in this line would be a valuable result.

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