# SPATIAL BEHAVIOUR OF SOLUTIONS OF THE MOORE-GIBSON-THOMPSON EQUATION

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SUMMARY: In this note we study the spatial behaviour of the Moore-Gibson-Thompson equation. As it is a hyperbolic equation, we prove that the solutions do not grow along certain spatial-time lines. Given the presence of dissipation, we show that the solutions also decay exponentially in certain directions.

KEY WORDS: Moore-Gibson-Thompson equation, spatial estimates, domain of influence, exponential decay.

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## 1. INTRODUCTION

The Moore-Gibson-Thompson (MGT) equation has attracted much attention in recent years, see e.g. [5, 7, 8, 9, 17, 18, 21, 23, 24]. While the model was originally introduced in the context of fluid mechanics, this equation can be also obtained by introducing a relaxation parameter into the type III heat conduction [12, 13, 14][1]. For this reason, it has recently been also adopted as the heat conduction equation. Furthermore, in [6, 22, 25] it is the heat equation for a so called MGT thermoelasticity. At the same time, it can also be obtained as a particular case of the three-phase-lag theory proposed by Choudhuri [4].

The study of the spatial behaviour of solutions was motivated by a desire to understand the Saint-Venant principle from a mathematical point of view. With that goal in mind, considerable mathematical analysis has been developed to clarify the spatial behaviour of solutions for a general class of partial differential equations and systems. In particular, some studies on the high-order partial differential equations have been developed in recent years [26, 20].

[1]Note that the introduction of a relaxation parameter is natural in view that the waves for the type III heat conduction propagates instantaneously and therefore this theory violates the principle of causality (see [11, 27]). Then, we use a similar argument to the one involved in the Cattaneo-Maxwell heat conduction and the thermoelasticity with one relaxation time [16].

In this short note, we study the spatial behaviour of solutions of the problem determined by the MGT equation on a three-dimensional semi-infinite cylinder $R = D \times [0, \infty)$ where $D$ is a two-dimensional bounded domain. As the MGT equation is an hyperbolic linear equation, in this work we follow the typical approach to study the spatial behaviour of hyperbolic linear equations and systems [3, 10, 15]. It is worth recalling that the results obtained in [26] can be applied to the MGT equation. However, the approach we propose here improves the kind of results which we specify with our equation. In fact, we will be able to give a more accurate description of the spatial behaviour. Moreover the results of the last section do not have a counterpart in the reference [26]. The end face of the finite cylinder is in the plane $x_3 = 0$, while the boundary of its cross-section, $\partial D$, is supposed to be regular enough to allow the use of the divergence theorem. By $R(z)$ we denote the set of points of the cylinder $R$ such that $x_3$ is greater than $z$, and by $D(z)$ the cross-section of the points such that $x_3 = z$.

In the next section we recall the problem we want to study as well as several easy calculations which are relevant in our study. Later, we show the spatial behaviour of solutions in the sense that the solutions are not increasing on several space-time lines. In particular a *domain of influence result* will be obtained. We complete the paper by noting the exponential decay of solutions along certain space-time lines. We note that the results that we will obtain in section 3 are the usual ones in the study of the spatial behaviour for linear hyperbolic equations and systems. However the results proposed in section 4 are not usual in the literature, because, to the best of the authors' knowledge, the only similar contribution of this kind was proposed in [15].

## 2. PRELIMINARIES

We consider solutions $u(\mathbf{x}, t)$ for the MGT equation which can be proposed by

$$\triangle \hat{u} = \dot{\tilde{u}}, \tag{1}$$

where $\Delta$ is the three-dimensional Laplace operator and the superposed dot denotes the derivative with respect to time $t$. The variables $\mathbf{x}$ and $t$ have been suitably non-dimensionalized. We have used the notation

$$\hat{u} = k^* u + k\dot{u}, \quad \tilde{u} = \dot{u} + \tau \ddot{u}, \tag{2}$$

where $\tau > 0$, $k > \tau k^*$[2]are the dimensionless parameters. Our equation for the dimensionless $u(\mathbf{x}, t)$ can be written

$$\tau \dddot{u} + \ddot{u} - k\triangle \dot{u} - k^* \triangle u = 0. \tag{3}$$

We study the spatial behaviour of the solutions $u(\mathbf{x}, t)$ of our equation subject to the initial conditions

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \;\; \dot{u}(\mathbf{x}, 0) = \vartheta^0(\mathbf{x}), \;\; \ddot{u}(\mathbf{x}, 0) = \phi^0(\mathbf{x}), \;\; \mathbf{x} \in D \times [0, \infty), \tag{4}$$

and the boundary conditions

$$\begin{aligned} u(\mathbf{x}, t) &= 0, \;\; \mathbf{x} \in \partial D \times [0, \infty), \;\; t > 0, & (5)\\ u(x_1, x_2, 0, t) &= g(x_\alpha, t) \quad \text{on } D \times \{0\} \times [0, \infty), & (6) \end{aligned}$$

[2]This assumptioin is fundamental in the studies related with the MGT equation. It guarantees the stability of the solutions.

where the proposed boundary data $g$ on the finite end $x_3 = 0$ is such that $g(x_\alpha, 0, 0) = u^0(x_\alpha, 0)$, $\dot{g}(x_\alpha, 0, 0) = \vartheta^0(x_\alpha, 0)$, $\ddot{g}(x_\alpha, 0, 0) = \phi^0(x_\alpha, 0)$ and $g$ is assumed to vanish on $\partial D\times$ $[0, \infty)$ for every $t > 0$. We do not impose any *a priori* assumption about the spatial behaviour of solutions[3].

In the course of our calculations, we will make use of the fact that the eigenvalues of the real symmetric positive definite matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}, \tag{7}$$

with $a, b, c$ real numbers, are

$$\nu^{\pm} = \frac{1}{2}\Big(a + c \pm \sqrt{(a-c)^2 + 4b^2}\Big). \tag{8}$$

So that the smallest eigenvalue is:

$$\nu^{-} = \frac{1}{2}\Big(a + c - \sqrt{(a-c)^2 + 4b^2}\Big). \tag{9}$$

We will use the constants in two particular cases.

(i) When

$$a = k^*, \ b = \tau k^*, \ c = \tau k, \tag{10}$$

it can be easily verified that the matrix is positive definite and so its smallest positive eigenvalue, denoted by $\nu_0$, is given by

$$\nu_0 = \frac{1}{2}\Big(k^* + \tau k - \sqrt{(k^* - \tau k)^2 + 4\tau^2 (k^*)^2}\Big). \tag{11}$$

(ii) When

$$a = \delta k^*, \quad b = \tau\delta k^* \quad c = \delta\tau k + 2(k - \tau k^*), \quad \tau \ > \ 0, \tag{12}$$

the matrix (7) is positive definite with the smallest eigenvalue, denoted by $\nu_\delta$, given by

$$\nu_\delta = \frac{1}{2}\Big(\delta(k^* + k\tau) + 2(k - k^*\tau) - \sqrt{[\delta(\tau k - k^*) + 2(k - k^*\tau)]^2 + 4\tau^2\delta^2(k^*)^2}\Big). \tag{13}$$

To understand the analysis of the last section it will be relevant that the sign of $\nu_\delta - \nu_0\delta$ is positive. We have

$$\begin{gathered} 2(\nu_\delta - \nu_0\delta) = 2(k - k^*\tau) + \sqrt{\delta^2(k^* - \tau k)^2 + 4\tau^2(k^*)^2\delta^2} \\ -\sqrt{(\delta(k^* - \tau k) + 2(k - k^*\tau))^2 + 4\tau^2(k^*)^2\delta^2} = A - B. \end{gathered} \tag{14}$$

To prove that $A - B > 0$ it will be sufficient to show that $A^2 > B^2$. But

$$A^2 - B^2 = 4(k - k^*\tau)\left(\sqrt{\delta^2(k^* - \tau k)^2 + 4\tau^2(k^*)^2\delta^2} - \delta(\tau k - k^*)\right) > 0,$$

whenever $\delta > 0$.

[3]A description of the way to prove the existence, uniqueness and regularity of the solutions to this problem can be found in the appendix of this paper.

## 3. NON-ZERO INITIAL CONDITIONS

In this Section we establish results on the spatial evolution of solutions of our problem, provided that the initial data is assumed to be bounded in a certain norm and the boundary conditions are determined by (5,6).

We start by considering the function

$$G(z,t) = -\int_0^t \int_{D(z)} \hat{u}_{,3}\tilde{u} dA ds. \tag{15}$$

We find that

$$\frac{\partial G(z,t)}{\partial t} = -\int_{D(z)} \hat{u}_{,3}\tilde{u} dA, \tag{16}$$

and, upon the use of the divergence theorem on the cylinder determined by the sections $D(z)$ and $D(z+h)$, we obtain

$$\begin{aligned} G(z+h,t) - G(z,t) &= -\textstyle\int_0^t \int_z^{z+h} \int_{D(\eta)} (\hat{u}_{,i}\tilde{u})_{,i} dV ds \\ &= -\textstyle\int_0^t \int_z^{z+h} \int_{D(\eta)} (\hat{u}_{,ii}\tilde{u} + \hat{u}_{,i}\tilde{u}_{,i}) dV ds \\ -\textstyle\int_0^t \int_z^{z+h} \int_{D(\eta)} \Big(\tilde{u}_s\tilde{u} + \frac{1}{2}\frac{d}{ds}&(k^*(u_{,i}+\tau\dot{u}_{,i})(u_{,i}+\tau\dot{u}_{,i}) + \tau K \dot{u}_{,i}\dot{u}_{,i}) + K\dot{u}_{,i}\dot{u}_{,i}\Big) dV ds, \end{aligned} \tag{17}$$

where from now on we use the notation $K = k - \tau k^*$.

Therefore

$$\begin{aligned} \tfrac{\partial G(z,t)}{\partial z} = -\tfrac{1}{2}\textstyle\int_{D(z)} \Big((\tilde{u})^2 + k^*(u_{,i}+\tau\dot{u}_{,i})(u_{,i}+\tau\dot{u}_{,i}) + \tau K\dot{u}_{,i}\dot{u}_{,i}\Big) dA \\ -\textstyle\int_0^t \int_{D(z)} K\dot{u}_{,i}\dot{u}_{,i} dA ds + E_1(z), \end{aligned}$$

where

$$E_1(z) = \frac{1}{2}\int_{D(z)} \Big((\tilde{u}^0)^2 + k^*(u^0_{,i} + \tau\vartheta^0_{,i})(u^0_{,i} + \tau\vartheta^0_{,i}) + \tau K \vartheta^0_{,i}\vartheta^0_{,i}\Big) dA. \tag{18}$$

Here and from now on $\tilde{u}^0 = \vartheta^0 + \tau\phi^0$.

It is worth noting that $E_1(z)$ is explicitly defined in terms of the initial data. On integration with respect to the first parameter from 0 to $z$, we get

$$\begin{aligned} &G(z,t) - G(0,t) = \\ -\frac{1}{2}\int_0^z \int_{D(\eta)} \Big((\tilde{u})^2 + k^*(u_{,i}+\tau\dot{u}_{,i})&(u_{,i}+\tau\dot{u}_{,i}) + \tau K\dot{u}_{,i}\dot{u}_{,i}\Big) dV - \int_0^t\int_0^z\int_{D(\eta)} K\dot{u}_{,i}\dot{u}_{,i} dV ds \\ +\frac{1}{2}\int_0^z \int_{D(\eta)} \Big((\tilde{u}^0)^2 + k^*&(u^0_{,i}+\tau\vartheta^0_{,i})(u^0_{,i}+\tau\vartheta^0_{,i}) + \tau K\vartheta^0_{,i}\vartheta^0_{,i}\Big) dV. \end{aligned} \tag{19}$$

Our next step is to establish an inequality between the time and spatial derivatives of $G(z,t)$. We have that

$$\frac{\partial G(z,t)}{\partial z} \leq -\frac{1}{2}\int_{D(z)} \Big((\tilde{u})^2 + \nu_0(|\nabla u|^2 + |\nabla \dot{u}|^2)\Big) dA + E_1(z). \tag{20}$$

where $\nu_0$ is defined at (11).

On applying Schwarz's inequality, we get

$$\left|\frac{\partial G}{\partial t}\right| \leq \Big(\int_{D(z)} (\tilde{u})^2 dA\Big)^{1/2} \Big(\int_{D(z)} ((k^*)^2 u_{,3}^2 + k^2\dot{u}_{,3}^2 + 2kk^* u_{,3}\dot{u}_{,3}) dA\Big)^{1/2} \tag{21}$$

$$\leq k^*\sqrt{1+(\frac{k}{k^*})^2}\Big(\int_{D(z)}(\tilde{u})^2dA\Big)^{1/2}\Big(\int_{D(z)}(|\nabla u|^2+|\nabla\dot{u}|^2)dA\Big)^{1/2}$$
$$\leq \frac{k^*}{2}\sqrt{(1+(\frac{k}{k^*})^2}\nu_0{}^{-1/2}\Big(\int_{D(z)}\Big((\tilde{u})^2+\nu_0(|\nabla u|^2+|\nabla\dot{u}|^2)\Big)dA\Big),$$

where the weighted arithmetic-geometric mean inequality has been employed to obtain each of the last two steps of (21).

We can write

$$\left|\frac{\partial G}{\partial t}\right|+\beta\frac{\partial G}{\partial z}\leq\beta E_1(z), \tag{22}$$

where

$$\beta=k^*\sqrt{(1+(\frac{k}{k^*})^2}\nu_0{}^{-1/2}. \tag{23}$$

This inequality implies that

$$\frac{\partial G}{\partial t}+\beta\frac{\partial G}{\partial z}\leq\beta E_1(z), \tag{24}$$

and

$$\frac{\partial G}{\partial t}-\beta\frac{\partial G}{\partial z}\geq-\beta E_1(z). \tag{25}$$

On integrating and taking into account the definition of $E_1(z)$ we obtain

$$G(z,\beta^{-1}(z-z^*))\leq\frac{1}{2}\int_{z^*}^{z}\int_{D(\eta)}\Big((\tilde{u}^0)^2+k^*(u^0_{,i}+\tau\vartheta^0_{,i})(u^0_{,i}+\tau\vartheta^0_{,i})+\tau K\vartheta^0_{,i}\vartheta^0_{,i}\Big)dV, \tag{26}$$

where $z\geq z^*$. Similarly, on integrating we obtain

$$G(z,\beta^{-1}(z^{**}-z))\geq-\frac{1}{2}\int_{z}^{z^{**}}\int_{D(\eta)}\Big((\tilde{u}^0)^2+k^*(u^0_{,i}+\tau\dot{\vartheta}^0_{,i})(u^0_{,i}+\tau\vartheta^0_{,i})+\tau K\vartheta^0_{,i}\vartheta^0_{,i}\Big)dV, \tag{27}$$

where $z^{**}\geq z$.

If we now assume that the initial data satisfies

$$\mathcal{G}(0,0)=\frac{1}{2}\int_{R}\Big((\tilde{u}^0)^2+k^*(u^0_{,i}+\tau\vartheta^0_{,i})(u^0_{,i}+\tau\vartheta^0_{,i})+\tau K\vartheta^0_{,i}\vartheta^0_{,i}\Big)dV<\infty. \tag{28}$$

Then the previous inequalities imply that, for each finite time $t$,

$$\lim_{z\to\infty}G(z,t)=0. \tag{29}$$

Therefore, we have

$$G(z,t)=\frac{1}{2}\int_{R(z)}\Big((\tilde{u})^2+k^*(u_{,i}+\tau\dot{u}_{,i})(u_{,i}+\tau\dot{u}_{,i})+\tau K\dot{u}_{,i}\dot{u}_{,i}\Big)dV+\int_0^t\int_{R(z)}K\dot{u}_{,i}\dot{u}_{,i}dVds \tag{30}$$
$$-\frac{1}{2}\int_{R(z)}\Big((\tilde{u}^0)^2+k^*(u^0_{,i}+\tau\vartheta^0_{,i})(u^0_{,i}+\tau\vartheta^0_{,i})+\tau K\vartheta^0_{,i}\vartheta^0_{,i}\Big)dV.$$

Now we see that

$$\mathcal{G}(z,t)\leq\mathcal{G}(z^*,0), \tag{31}$$

where

$$\mathcal{G}(z,t)=\frac{1}{2}\int_{R(z)}\Big(((\tilde{u})^2+k^*(u_{,i}+\tau\dot{u}_{,i})(u_{,i}+\tau\dot{u}_{,i})+\tau K\dot{u}_{,i}\dot{u}_{,i}\Big)dV \tag{32}$$

$$+\int_0^t\int_{R(z)} K\dot{u}_{,i}\dot{u}_{,i}dVds,$$

and $z, z^*$ and $t$ are related by $t=\beta^{-1}(z-z^*)$. In a similar way, we get

$$\mathcal{G}(z,t)\geq\mathcal{G}(z^{**},0), \tag{33}$$

for $t=\beta^{-1}(z^{**}-z)$. From the previous inequalities we conclude that

$$\mathcal{G}(z,t)\leq\mathcal{G}(z^*,t^*), \tag{34}$$

for $|t-t^*|\leq\beta^{-1}(z-z^*)$. Thus, we have proved:

**Theorem 3.1.** *Let $u$ be a solution of our initial-boundary-value problem where $g$ satisfies the conditions (52-54) of the appendix. We also assume that $\tau k^*<k$. Then the energy function $\mathcal{G}(z,t)$ defined in (32) satisfies the inequality (34) whenever $|t-t^*|\leq\beta^{-1}(z-z^*)$, provided that the initial data satisfy (28).*

If one defines the measure

$$\mathcal{G}^*(z,t)=\int_0^t\mathcal{G}(z,s)ds, \tag{35}$$

the following inequalities can be obtained as in [1]:

$$\mathcal{G}^*(z,t)\leq\beta^{-1}\int_{z-\beta t}^{z}\mathcal{G}(\eta,0)d\eta,\ \ \beta t\leq z, \tag{36}$$

$$\mathcal{G}^*(z,t)\leq\beta^{-1}\int_0^{z}\mathcal{G}(\eta,0)d\eta+\Big(1-\frac{z}{\beta t}\Big)\mathcal{G}^*(0,t),\ \ \beta t\geq z. \tag{37}$$

Theorem 3.1 and estimates (36) and (37) are usual in the study of hyperbolic equations and systems. They allow to control the energy of the system contained in a sub-cylinder until a time $t$ in terms of the energy of the cylinder on a previous time.

It is worth noting that in case that the initial data vanish we will obtain that the solution also vanishes whenever $\beta t\leq z$. This is a result of the *domain of influence* type. To be precise we obtain that

$$u(x_1,x_2,x_3,t)=0,$$

whenever $\beta t\leq x_3$. That is $u$ vanishes in the subset of points $(x_1,x_2,x_3,t)\in R$ such that $\beta t\leq x_3$.

At the same when $\beta t>z$ we have

$$\mathcal{G}^*(z,t)=\int_{\beta^{-1}z}^t\mathcal{G}(z,s)ds\leq\int_0^{t-\beta^{-1}z}\mathcal{G}(0,s)ds=\mathcal{G}^*(0,t^*),$$

where $t^*=t-\beta^{-1}z$. This estimate is also usual for linear hyperbolic equations and systems, but in the general case we do not know how to obtain a decay estimate for hyperbolic problems which describes the decay along these lines. However as there exists dissipation in the system we consider, it seems possible to obtain some decay estimates. We will see this effect for the MGT in the next section.

# 4. ZERO INITIAL CONDITIONS

In this section we study the spatial behaviour of solutions of our problem subject to the boundary conditions (5,6) where $g$ satisfies the conditions (52-54) of the appendix, but in the case where we assume *homogeneous* initial conditions.

In this situation we define the function

$$G_\delta(z,t) = -\int_0^t \int_{D(z)} \exp(-\delta s)\hat{u}_{,3}\tilde{u}dAds, \tag{38}$$

where $\delta > 0$ is an arbitrary positive constant. We have

$$G_\delta(z,t) = G_\delta(0,t) - \frac{1}{2}\int_0^z \int_{D(\eta)} \exp(-\delta t)\Big((\tilde{u})^2 + k^*(u_{,i}+\tau\dot{u}_{,i})(u_{,i}+\tau\dot{u}_{,i}) + \tau K\dot{u}_{,i}\dot{u}_{,i}\Big)dV \tag{39}$$

$$-\frac{\delta}{2}\int_0^t\int_0^z \int_{D(\eta)} \exp(-\delta s)\Big((\tilde{u})^2 + k^*(u_{,i}+\tau\dot{u}_{,i})(u_{,i}+\tau\dot{u}_{,i}) + \tau K\dot{u}_{,i}\dot{u}_{,i}\Big)dV ds$$

$$-\int_0^t\int_0^z \int_{D(\eta)} \exp(-\delta s)K\dot{u}_{,i}\dot{u}_{,i}dV ds.$$

On using arguments similar to those of the previous Section, we obtain

$$\left|\frac{\partial G_\delta}{\partial t}\right| + \beta\frac{\partial G_\delta}{\partial z} \leq 0, \tag{40}$$

where $\beta$ is given as previously.

The inequality (40) implies that

$$\frac{\partial G_\delta}{\partial t} + \beta\frac{\partial G_\delta}{\partial z} \leq 0, \quad \frac{\partial G_\delta}{\partial t} - \beta\frac{\partial G_\delta}{\partial z} \geq 0. \tag{41}$$

Thus, we can write

$$G_\delta(z,t) = \frac{1}{2}\int_{R(z)} \exp(-\delta t)\Big((\tilde{u})^2 + k^*(u_{,i}+\tau\dot{u}_{,i})(u_{,i}+\tau\dot{u}_{,i}) + \tau K\dot{u}_{,i}\dot{u}_{,i}\Big)dV \tag{42}$$

$$+\frac{\tau}{2}\int_0^t \int_{R(z)} \exp(-\delta s)\Big((\tilde{u})^2 + k^*(u_{,i}+\tau\dot{u}_{,i})(u_{,i}+\tau\dot{u}_{,i}) + \tau K\dot{u}_{,i}\dot{u}_{,i}\Big)dV ds$$

$$+\int_0^t \int_{R(z)} \exp(-\delta s)K\dot{u}_{,i}\dot{u}_{,i}dV ds.$$

Next, we obtain the main spatial estimate of this section. To this end we estimate the absolute value of $G_\delta$ in terms of its derivative with respect to a space-time direction. On using Schwarz's inequality in (38), we get

$$|G_\delta| \leq \Big(\int_0^t \int_{D(z)} \exp(-\delta s)(\tilde{u})^2 dAds\Big)^{1/2}\Big(\int_0^t \int_{D(z)} \exp(-\delta s)((k^*)^2u_{,3}^2 + k^2\dot{u}_{,3}^2 + 2kk^*u_{,3}\dot{u}_{,3})dAds\Big)^{1/2}. \tag{43}$$

On twice using the weighted arithmetic-geometric mean inequality we find that

$$|G_\delta| \leq k^*\sqrt{(1+(\frac{k}{k^*})^2}\Big(\int_0^t \int_{D(z)} \exp(-\delta s)(\tilde{u})^2 dAds\Big)^{1/2}\Big(\int_0^t \int_{D(z)} \exp(-\delta s)(|\nabla u|^2 + |\nabla\dot{u}|^2)dAds\Big)^{1/2}$$

$$\leq \frac{1}{2}\sqrt{\frac{(k^2+(k^*)^2)}{\delta\nu_\delta}}\Big(\int_0^t\int_{D(z)}\exp(-\tau s)\Big(\delta(\tilde{u})^2+\nu_\delta(|\nabla u|^2+|\nabla\dot{u}|^2)\Big)dAds\Big).$$

Thus we find that

$$|\ G_\delta\ |\ \ \leq \sqrt{\frac{(k^2+(k^*)^2)}{\delta\nu_\delta}}\Big(\Big|\frac{\partial G_\delta}{\partial t}\Big|-\beta\frac{\partial G_\delta}{\partial z}\Big), \tag{44}$$

where the constant $\nu_\delta$ is given explicitly in (13), and $\beta$ is given in (23). On letting $\xi$ measure the distance along the line

$$\beta(t-t^*)=z, \tag{45}$$

the inequality (44) implies that

$$\frac{\partial G_\delta}{\partial \xi}+\sqrt{\frac{\delta\nu_\delta}{(k^2+(k^*)^2))(1+\beta^2)}}G_\delta\leq 0. \tag{46}$$

An integration from the point $(0,t^*)$ to the point $(z,t)$ along the line (45) leads to

$$G_\delta(z,t)\leq G_\delta(0,t^*)\exp\Big(-\sqrt{\frac{\delta\nu_\delta}{(k^2+(k^*)^2)}}z\Big). \tag{47}$$

On recalling the definition of $\mathcal{G}$, we may write

$$\mathcal{G}(z,t)\leq \exp(\delta t)G_\delta(z,t)\leq \exp(\delta t)G_\delta(0,t^*)\exp\Big(-\sqrt{\frac{\delta\nu_\delta}{(k^2+(k^*)^2)}}z\Big). \tag{48}$$

We thus obtain

$$\mathcal{G}(z,t)\leq \exp(\delta t^*)G_\delta(0,t^*)\exp\Big(\big(\beta^{-1}\delta-\sqrt{\frac{\delta\nu_\delta}{k^2+(k^*)^2}}\big)z\Big). \tag{49}$$

The constant $\delta>0$ is an arbitrary constant. We may now choose $\delta$ so that the quantity

$$\beta^{-1}\delta-\sqrt{\frac{\delta\nu_\delta}{(k^2+(k^*)^2))}}=\frac{\delta^{1/2}}{\sqrt{k^2+(k^*)^2}}(\nu_0^{1/2}\delta^{1/2}-\nu_\delta^{1/2}), \tag{50}$$

is negative. This implies that the energy decays exponentially along the line (45).

We have proved:

**Theorem 4.1.** *Let $u$ be a solution of our initial-boundary-value problem when the initial conditions vanish where $g$ satisfies the conditions (52-54) of the appendix. We also assume that $\tau k^*<k$. Then the energy function $\mathcal{G}(z,t)$ satisfies the inequality (49) whenever $t,t^*$ and $z$ are related by the condition (45).*

It is worth noting that the estimates of the type proposed by this theorem are not usual to find in the literature. In fact we obtain a spatial decay estimate along the line (45), which improves the estimate obtained at the end of the previous section. We do not know estimates of this kind apart the one proposed at [15].

We also note that estimate (49) implies that

$$\mathcal{G}^*(z,t)\leq\Big[\int_{t-\beta^{-1}z}^t\exp(\delta(\xi-\beta^{-1}z))G_\delta(0,\xi-\beta^{-1}z)d\xi\Big]\exp\Big(\big(\beta^{-1}\delta-\sqrt{\frac{\delta\nu_\delta}{k^2+(k^*)^2}}\big)z\Big).$$

After the change of variable $\eta = \xi - \beta^{-1}z$ we get

$$\mathcal{G}^*(z,t) \leq \Big[ \int_0^{t^*} \exp(\delta\eta) G_\delta(0,\eta) d\eta \Big] \exp\Big( (\beta^{-1}\delta - \sqrt{\frac{\delta\nu_\delta}{k^2+(k^*)^2}})z \Big),$$

where $t^* = t - \beta^{-1}z$. Therefore, we conclude

$$\mathcal{G}^*(z,t) \leq \exp(\delta t^*) \int_0^{t^*} G_\delta(0,\eta) d\eta \exp\Big( (\beta^{-1}\delta - \sqrt{\frac{\delta\nu_\delta}{k^2+(k^*)^2}})z \Big).$$

This is an estimate where we can see that the energy contained in the sub-cylinders decays in an exponential way.

## 5. CONCLUSION

In this note we have analyzed the spatial behaviour of solutions of the Moore-Gibson-Thompson equation. As this is an hyperbolic equation, we have obtained spatial estimates. In particular, the *domain of influence result* has been established. We have also demonstrated that the solutions decay exponentially along the lines determined by eq. (45). This alternative result is not well known in the literature.

## Conflicts of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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APPENDIX

In the development of the paper we have assumed the existence, uniqueness and regularity of the solutions to the proposed problem (3-6). To be precise we have used that

$$\hat{u} \in W^{2,2}(R);\;\; \tilde{u},\, u,\;\, \dot{u} \in W_0^{1,2}(R). \tag{51}$$

In this appendix we prove that these properties hold whenever we assume that

$$g \in C^4(0,T,W^{2,2}(D) \cap W_0^{1.2}(D)), \tag{52}$$

as well as

$$g(x_1,x_2,0,0) = u^0(x_1,x_2,0),\;\; \dot{g}(x_1,x_2,0,0) = \vartheta^0(x_1,x_2,0),\;\; \ddot{g}(x_1,x_2,0,0) = \phi^0(x_1,x_2,0), \tag{53}$$

and

$$\begin{gathered} k^*u^0 + k\vartheta^0 \in W^{2,2}(R),\quad u^0 - g(x_1,x_2,0)\exp(-\gamma x_3) \in W_0^{1,2}(R), \\ \vartheta^0 - \dot{g}(x_1,x_2,0)\exp(-\gamma x_3),\;\; \phi^0 - \ddot{g}(x_1,x_2,0)\exp(-\gamma x_3) \in W_0^{1,2}(R), \end{gathered} \tag{54}$$

where $\gamma$ is an arbitrary positive constant.

First, we will need the following result:

**Theorem 5.1.** *Let us consider the problem determined by the equation*

$$\tau\dddot{w} + \ddot{w} - k\triangle\dot{w} - k^*\triangle w = f(\mathbf{x},t), \tag{55}$$

*the initial conditions*

$$w(\mathbf{x},0) = a^0(\mathbf{x}),\;\; \dot{w}(\mathbf{x},0) = b^0(\mathbf{x}),\;\; \ddot{w}(\mathbf{x},0) = c^0(\mathbf{x}), \tag{56}$$

*and the boundary conditions*

$$w(\mathbf{x},t)=0,\ \ \mathbf{x}\in\partial R\ \times\ [0,\infty), \tag{57}$$

*where* $a^0(\mathbf{x})\in W_0^{1,2}(R)$, $b^0(\mathbf{x})\in W_0^{1,2}(R)$, $c^0(\mathbf{x})\in W_0^{1,2}(R)$ *and* $k^*a^0(\mathbf{x})+kb^0(\mathbf{x})\in L^2(R)$. *We also assume that* $f(\mathbf{x},t)\in C^1(0,T,L^2(R))$. *Then, there exists a unique solution* $w(\mathbf{x},t)\in W_0^{1,2}(R)$ *such that* $\dot{w}(\mathbf{x},t)\in W_0^{1,2}(R)$ , $\ddot{w}(\mathbf{x},t)\in L^2(R)$, $\hat{w}(\mathbf{x},t)\in W^{2,2}(R)$, $\tilde{w}(\mathbf{x},t)\in W_0^{1,2}(R)$

**Proof**. We consider the Hilbert space $\mathcal{H}=W_0^{1,2}(R)\times W_0^{1,2}(R)\times L^2(R)$ and we can write our problem as

$$\frac{d}{dt}\begin{pmatrix}w\\ \theta\\ \varphi\end{pmatrix}=\mathcal{A}\begin{pmatrix}w\\ \theta\\ \varphi\end{pmatrix}+\begin{pmatrix}0\\ 0\\ f\end{pmatrix},\ \ w(0)=a^0(\mathbf{x}),\ \theta(0)=b^0(\mathbf{x}),\ \varphi(0)=c^0(\mathbf{x}), \tag{58}$$

where

$$\mathcal{A}\begin{pmatrix}w\\ \theta\\ \varphi\end{pmatrix}=\begin{pmatrix}\theta\\ \varphi\\ \tau^{-1}[k^*\Delta w+k\Delta\theta-\varphi]\end{pmatrix}. \tag{59}$$

The domain of this operator is the subspace of $\mathcal{H}$ such that $k^*\Delta w+k\Delta\theta\in L^2(R)$ and $\varphi\in W_0^{1,2}(R)$. Following the arguments proposed in ([25], p. 4025), we can prove that $\mathcal{A}$ generates a $C^0$-semigroup of contractions. As $f(\mathbf{x},t)\in C^1(0,T,L^2(R))$ we can conclude the existence and uniqueness of solutions (see [2], p.117). We also know that the solutions belong to the domain of $\mathcal{A}$ and then the required regularity conditions are satisfied.

**Theorem 5.2.** *Let us assume that the boundary and initial datas satisfy the conditions proposed at the begin of the appendix (52-54). Then, there exists a unique solution to the problem (3-6). Moreover, this solution satisfies the regularity conditions.*

**Proof**. We define the function $H(\mathbf{x},t)=g(x_1,x_2,t)\exp(-\gamma x_3)$ where $\gamma$ is an arbitrary positive number. We denote by $w(t)$ the solution of the problem (55-57) when

$$f(\mathbf{x},t)=\tau\dddot{H}+\ddot{H}-k\Delta\dot{H}-k^*\Delta H,$$

and

$$a^0(\mathbf{x})=u^0(\mathbf{x})-H(\mathbf{x},0),\ \ b^0(\mathbf{x})=\vartheta^0(\mathbf{x})-\dot{H}(\mathbf{x},0),\ \ c^0(\mathbf{x})=\phi^0(\mathbf{x})-\ddot{H}(\mathbf{x},0).$$

We note that $f(\mathbf{x},t)$ and $a^0(\mathbf{x}),b^0(\mathbf{x}),c^0(\mathbf{x})$ satisfy the conditions of the previous theorem.

The solution to the problem (3-6) can be obtained by taking

$$u(\mathbf{x},t)=w(\mathbf{x},t)+H(\mathbf{x},t). \tag{60}$$

Existence, uniqueness and regularity of the solution are clearly checked in view of the previous theorem.

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