ON THE AXIOMATIC CHARACTERIZATION OF THE COALITIONAL MULTINOMIAL PROBABILISTIC VALUES *

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Abstract

The coalitional multinomial probabilistic values extend the notion of multinomial probabilistic value to games with a coalition structure, in such a way that they generalize the symmetric coalitional binomial semivalues and link and combine the Shapley value and the multinomial probabilistic values. By considering the property of balanced contributions within unions, a new axiomatic characterization is stated for each one of these coalitional values, provided that it is defined by a positive tendency profile, by means of a set of logically independent properties that univocally determine the value. Two applications are also shown: (a) to the Madrid Assembly in Legislature 2015–2019 and (b) to the Parliament of Andalucía in Legislature 2018–2022.

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1 Introduction

This paper focuses on *coalitional multinomial probabilistic values*, a family of coalitional values introduced by Carreras and Puente [8]. These coalitional values combine the *Shapley value* [19] and the corresponding *multinomial probabilistic value* [18] (see also [7], [9], [10], [13], [14]). They first apply this latter value to the quotient game and obtain a payoff for each union; next, they apply within each union the Shapley value to a reduced game, played in the union, for sharing that payoff efficiently.

These values form a n–parametric family (n being the number of players) since they depend on profiles $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ that supply information not included in the characteristic function of the game. We interpret each component p_i as the tendency of player i to form coalitions. By using coalitional multinomial probabilistic values one can take into account the influence of players' different personalities in the study of the coalition formation process. The political examples analyzed in Section 4 illustrate this idea and also the good behavior of these values as power indices (i.e., acting on simple games). In fact, these coalitional values look highly interesting for a voting setup since, once an alliance is formed —and, especially, if it supports a coalition government—, cabinet ministries, parliamentary and institutional positions, budget management, and other political responsibilities have to be distributed among the members of the coalition *efficiently*, so the use of the Shapley value is crucial here.

Coalitional multinomial probabilistic values widely generalize the *symmetric coalitional binomial semivalues* [6] (and, in particular, the *symmetric coalitional Banzhaf value* [1]) and they

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provide a promising framework for applications. One might think that we have to pay a price in terms of mathematical properties for introducing n parameters in our evaluation of games and games with a coalition structure. However, as it will be seen along this work, this is not true.

Summing up, the aim of this paper is to provide a new axiomatic characterization for each coalitional multinomial probabilistic value defined by a positive tendency profile, by just replacing the property of *symmetry within unions* with the property of *balanced contributions within unions*, thus contributing to a better understanding of these coalitional values as a consistent alternative or complement to classic coalitional values. The fact that they are based on tendency profiles provides new tools to encompass a large variety of situations that arise when playing a given game, as shown in the political examples studied in Section 4.

The organization of the paper is as follows. In Section 2, a minimum of preliminaries is given. Section 3 includes the new axiomatic characterization of these coalitional values based on the property of *balanced contributions within unions*, and translates this characterization to their restriction to simple games, where they are interpreted as coalitional power indices. Section 4 contains two political applications of the coalitional multinomial probabilistic values to the analysis of: (a) the Madrid Assembly (Legislature 2015–2019) and (b) the Parliament of Andalucía (Legislature 2018–2022).

2 Preliminaries

2.1 Games and values. Multinomial probabilistic values

Let N be a finite set of *players*, usually denoted as $N = \{1, 2, \ldots, n\}$, and 2^N be the set of *coalitions* (subsets of N). A *(cooperative) game* in N is a function $v : 2^N \to \mathbb{R}$ that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T \subseteq N$. Player $i \in N$ is a *dummy in* v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$, and *null in* v if, moreover, $v(\{i\}) = 0$. Two players $i, j \in N$ are *symmetric in* v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

Endowed with the natural operations for real–valued functions, i.e. $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set $\mathcal{G}_N$ of all games in N becomes a vector space. For every nonempty coalition $T \subseteq N$, the *unanimity game* u^T in N is defined by $u^T(S) = 1$ if $T \subseteq S$ and $u^T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$. Then, for each $v \in \mathcal{G}_N$,

$$v = \sum_{\emptyset \neq T \subseteq N} \alpha_T(v) u^T \quad \text{where} \quad \alpha_T(v) = \sum_{S \subseteq T} (-1)^{t-s} v(S), \quad t = |T|, \text{ and } s = |S|.$$

Let $\emptyset \neq R \subset N$ and w be a game in N. The *restriction* of w to R is the game w_R in R defined by

$$w_R(S) = w(S) \quad \text{for all} \quad S \subseteq R.$$

Now let w be a game in R. The *null extension* of w to N is the game w^* defined by

$$w^*(S) = w(S \cap R) \quad \text{for all} \quad S \subseteq N.$$

Then, the players in $N \backslash R$ become null players in w^*. The map from $\mathcal{G}_R$ to $\mathcal{G}_N$ defined by $w \mapsto w^*$ is linear and, in particular, u^T (in R) maps to $(u^T)^* = u^T$ (in N). We will apply these ideas later to the case where $R = N \backslash \{i\}$ for some $i \in N$, and will denote the restriction to $N \backslash \{i\}$ as w_{-i}.

By a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \to \mathbb{R}^N$, that assigns to every game v a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

In particular, the Banzhaf value [4] β, is given by

$$\beta_i[v] = \frac{1}{2^{n-1}} \sum_{S \subseteq N \backslash \{i\}} [v(S \cup \{i\}) - v(S)] \text{ for all } i \in N \text{ and all } v \in \mathcal{G}_N,$$

and the Shapley value [19] φ by

$$\varphi_i[v] = \frac{1}{n} \sum_{S \subseteq N \backslash \{i\}} \frac{1}{\binom{n-1}{s}} [v(S \cup \{i\}) - v(S)] \text{ for all } i \in N \text{ and all } v \in \mathcal{G}_N.$$

Notice that $\beta_i[v]$ is the average of the marginal contributions of player i to all coalitions to which it does not belong and $\varphi_i[v]$ is a weighted average of those contributions, where now the weights depend on the size of coalitions S.

The multinomial probabilistic values form a subfamily of *probabilistic values* [20]. They were introduced in reliability of systems by Puente [18] (see also [13]) as follows. Let $N = \{1, 2, \ldots, n\}$ and let $\mathbf{p} \in [0, 1]^n$, that is, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ with $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, n$, be given. Then the coefficients

$$p_S^i = \prod_{j \in S} p_j \prod_{\substack{k \in N \setminus S \\ k \neq i}} (1 - p_k) \quad \text{for all } i \in N \text{ and } S \subseteq N \setminus \{i\}$$

(where the empty product, arising if $S = \emptyset$ or $S = N \setminus \{i\}$, is taken to be 1) define a probabilistic value on $\mathcal{G}_N$ that is called the $\mathbf{p}$*–multinomial probabilistic value* and will be denoted here as $\lambda^{\mathbf{p}}$. Its action is then given, for all $i \in N$ and $v \in \mathcal{G}_N$, by

$$\lambda_i^{\mathbf{p}}[v] = \sum_{S \subseteq N \setminus \{i\}} \Big[\prod_{j \in S} p_j \prod_{\substack{k \in N \setminus S \\ k \neq i}} (1 - p_k) \Big] [v(S \cup \{i\}) - v(S)].$$

Remark 2.1 (a) For example, for $n = 2$ we have $\mathbf{p} = (p_1, p_2)$ and, if $i \neq j$,

$$\lambda_i^{\mathbf{p}}[v] = (1 - p_j)[v(\{i\} - v(\emptyset)] + p_j[v(N) - v(\{j\})].$$

Thus, the payoff allocated by $\lambda^{\mathbf{p}}$ to player i does not depend on p_i but only on p_j. If player j is not greatly interested in cooperating, and hence p_j is small, player i mainly receives his individual utility whereas, otherwise, if player j is interested in cooperating, and hence p_j is great, player i mainly receives his marginal contribution to the grand coalition.

(b) It is easy to check that the action of $\lambda^{\mathbf{p}}$ on a unanimity game u_T is given by:

$$\lambda_i^{\mathbf{p}}[u^T] = \prod_{j \in T \setminus \{i\}} p_j \quad \text{if } i \in T \quad \text{or else} \quad \lambda_i^{\mathbf{p}}[u^T] = 0. \tag{1}$$

This will be used in Example 4.1.

From now on, we will assign to parameter p_i the meaning of *generic tendency of player i to form coalitions*, assuming that p_i and p_j are independent of each other if $i \neq j$. We also assume that $0 \leq p_i \leq 1$ for each player i and collect all parameters in the *tendency profile* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.

2.2 Games with a coalition structure. The coalitional multinomial probabilistic value

Given $N = \{1, 2, \ldots, n\}$, we will denote by $B(N)$ the set of all partitions of N. Each $B \in B(N)$ is called a *coalition structure* in N, and each member of B is called a *union*. The so–called *trivial coalition structures* are $B^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ (singletons) and $B^N = \{N\}$ (grand coalition). A *(cooperative) game with a coalition structure* in N is a pair $[v; B]$, where $v \in \mathcal{G}_N$ and $B \in B(N)$ for a given N. Each partition B gives a pattern of cooperation among players. We denote by $\mathcal{G}_N^{cs} = \mathcal{G}_N \times B(N)$ the set of all games with a coalition structure and player set N.

If $[v; B] \in \mathcal{G}_N^{cs}$ and $B = \{B_1, B_2, \ldots, B_m\}$, the *quotient game* v^B is the game played by the unions or, rather, by the *quotient set* $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^B(R) = v\left(\bigcup_{r \in R} B_r\right) \quad \text{for all } R \subseteq M.$$

Definition 2.2 Given a tendency profile $\mathbf{p}$ and a coalition structure B in N, a tendency profile $\overline{\mathbf{p}}$ in M is a *tendency profile induced by* $\mathbf{p}$ iff: (i) each $\overline{p}_r$ depends only on those p_i such that $i \in B_r$; and (ii) if, for a given $B_r \in B$, there is some $q \in [0, 1]$ such that $p_i = q$ for all $i \in B_r$ then $\overline{p}_r = q$.[1] Of course, if $B_r = \{i\}$ then $\overline{p}_r = p_i$.

[1]Condition (ii) ensures consistency with symmetric coalitional binomial semivalues (cf. [5, 2, 6]).

The interpretation attached to $p_1, p_2, \ldots, p_n$ in Subsection 2.1 will be kept in passing to the quotient. Among the infinitely many possibilities to define an induced tendency profile $\overline{\mathbf{p}} = (\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_m)$ in terms of $\mathbf{p}$, let us suggest a few ones only as a matter of example:

(α) $\overline{p}_r = \min_{i \in B_r} \{p_i\}$

(β) $\overline{p}_r = p_i$ for some $i \in B_r$ arbitrarily chosen

(γ) $\overline{p}_r = \dfrac{1}{b_r} \sum_{i \in B_r} p_i$, where $b_r = |B_r|$

(δ) $\overline{p}_r = \max_{i \in B_r} \{p_i\}$

We will not try to discuss here which is the best option. It may happen that different situations require different ways to define $\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_m$. Even more, nothing prevents different unions to make different choices when defining their respective tendencies in the quotient game —this is the reason for having imposed conditions (i) and (ii) *for each union*. Fortunately, the great freedom in this choice will not affect the validity of the theoretical results: the theory developed in this paper will be of application provided that $\overline{\mathbf{p}}$ is a tendency profile induced by $\mathbf{p}$, no matter by which mechanism.[2]

By a *coalitional value* on $\mathcal{G}_N^{cs}$ we will mean a map $g : \mathcal{G}_N^{cs} \to \mathbb{R}^N$, which assigns to every pair $[v; B]$ a vector $g[v; B]$ with components $g_i[v; B]$ for each $i \in N$.

If f is a value on $\mathcal{G}_N$ and g is a coalitional value on $\mathcal{G}_N^{cs}$, it is said that g is a *coalitional value of* f (or a *coalitional f–value*, for short) iff $g[v; B^n] = f[v]$ for all $v \in \mathcal{G}_N$.

The coalitional $\mathbf{p}$–multinomial probabilistic values were introduced in [8]. They represent a two–step bargaining procedure where, first, each union obtains in the quotient game the payoff given by the $\mathbf{p}$–multinomial probabilistic value $\lambda^{\mathbf{p}}$ and, then, this payoff is efficiently shared within each union according to the Shapley value φ.[3]

Definition 2.3 Let $N = \{1, 2, \ldots, n\}$ be a finite player set and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ be a tendency profile in N. The *coalitional* $\mathbf{p}$*–multinomial probabilistic value* is the coalitional value $\Lambda^{\mathbf{p}} : \mathcal{G}_N^{cs} \to \mathbb{R}^N$ defined as follows. If $[v; B] \in \mathcal{G}_N^{cs}$ and $i \in B_k \in B$,

$$\Lambda_i^{\mathbf{p}}[v; B] = \sum_{R \subseteq M \backslash \{k\}} \Big[\prod_{j \in R} \overline{p}_j \prod_{\substack{h \in M \backslash R \\ h \neq k}} (1 - \overline{p}_h) \Big] \sum_{T \subseteq B_k \backslash \{i\}} \frac{v(Q \cup T \cup \{i\}) - v(Q \cup T)}{b_k \binom{b_k - 1}{t}} ,$$

where $\overline{\mathbf{p}}$ is a tendency profile induced by $\mathbf{p}$ in M, $Q = \bigcup_{r \in R} B_r$, $b_k = |B_k|$, and $t = |T|$.

As this has just been said, there are infinitely many possibilities to define an induced tendency profile $\overline{\mathbf{p}} = (\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_m)$ in terms of $\mathbf{p}$ but this does not affect the theoretical results of the paper. In order to simplify the notation and avoid any ambiguity, we will implicitly assume from now on that, for any given N, a unique mechanism has been chosen to induce, given $\mathbf{p}$ and B in N, a tendency profile $\overline{\mathbf{p}}$ in M.

[2] Although the results obtained in practice will depend in general on this mechanism.

[3] The reduced game to which the Shapley value is applied is as follows. First, if $S \subseteq B_k$, let $\overline{v}_S^B$ be the *pseudoquotient game* in M defined by

$$\overline{v}_S^B(R) = v\Big[(\bigcup_{r \in R} B_r) \backslash (B_k \backslash S) \Big] \quad \text{for each} \quad R \subseteq M.$$

This game is the modification of the standard quotient game v^B when S replaces union B_k, as if the players of $B_k \backslash S$ were temporarily inactive. The *reduced game* of v in B_k, denoted by w_k, is then given by

$$w_k(S) = \lambda_k^{\overline{\mathbf{p}}}[\overline{v}_S^B] \quad \text{for each} \quad S \subseteq B_k.$$

In particular, let u^T be a unanimity game in N. Let $\overline{T} = \{k \in M : T \cap B_k \neq \emptyset\}$, and, for each $k \in \overline{T}$, $T_k = T \cap B_k$. If $i \in T_k$, and $\overline{\mathbf{p}}$ is a tendency profile induced by $\mathbf{p}$ in M, then

$$\Lambda_i^{\mathbf{p}}[u^T; B] = \frac{1}{|T_k|} \prod_{r \in \overline{T} \setminus \{k\}} \overline{p}_r. \tag{2}$$

This will be used in Sections 3 and 4.

2.3 Properties and a first axiomatic characterization

In this section we will present an existing characterization result given in [8]. To do this, first of all, we consider standard properties for a generic coalitional value g on $\mathcal{G}_N^{cs}$:

- *linearity*: $g[\alpha v + \beta v'; B] = \alpha g[v; B] + \beta g[v'; B]$ for all $\alpha, \beta \in \mathbb{R}$, $v, v \in \mathcal{G}_N$ and $B \in B(N)$.
- *positivity*: if $v \in \mathcal{G}_N$ is monotonic, then $g[v; B] \geq 0$ for all $B \in B(N)$.
- *dummy player property*: if i is a dummy in $v \in \mathcal{G}_N$, then $g_i[v; B] = v(\{i\})$ for all $B \in B(N)$.
- *symmetry within unions:* if $i, j \in B_k$ are symmetric players in $v \in \mathcal{G}_N$ then

$$g_i[v; B] = g_j[v; B].$$

- *symmetry in the quotient game*: if $r, s \in M$ are symmetric players in $v^B \in \mathcal{G}_M$, $B \in B(N)$, then

$$\sum_{i \in B_r} g_i[v; B] = \sum_{j \in B_s} g_j[v; B].$$

- *quotient game property*: for all $[v; B] \in \mathcal{G}_N^{cs}$ and all $k \in M$,

$$\sum_{i \in B_k} g_i[v; B] = g_k[v^B; B^m].$$ [4]

In order to obtain a first axiomatic characterization of each coalitional multinomial probabilistic value, two additional nonstandard properties are considered.

Definition 2.4 Let $\mathbf{p}$ be a tendency profile in N. A coalitional value g on $\mathcal{G}_N^{CS}$ satisfies the *coalitional* $\mathbf{p}$*–multinomial total power property* iff, for all $[v; B] \in \mathcal{G}_N^{cs}$,

$$\sum_{i \in N} g_i[v; B] = \sum_{k \in M} \sum_{R \subseteq M \setminus \{k\}} \Big[\prod_{j \in R} \overline{p}_j \prod_{\substack{h \in M \setminus R \\ h \neq k}} (1 - \overline{p}_h) \Big] [v(Q \cup B_k) - v(Q)],$$

where $\overline{\mathbf{p}}$ is any tendency profile induced by $\mathbf{p}$ in M and $Q = \bigcup_{r \in R} B_r$.

Remark 2.5 This property is the natural extension of a *total power property* that was first stated for the Banzhaf value [15] (cf. also [12, 11]) and gave rise later, among others, to the coalitional q–binomial total power property [5, 2, 6] and the $\mathbf{p}$–multinomial total power property [7]. It reduces to this latter if $B = B^n$ but it also extends *efficiency*, to which it reduces if $B = B^N$.

Moreover, as $\Lambda^{\overline{\mathbf{p}}}$ is a coalitional $\lambda^{\overline{\mathbf{p}}}$–value, that property is a consequence of the quotient game property—maybe more compelling at first glance—and can be simply written as

$$\sum_{i \in N} \Lambda_i^{\mathbf{p}}[v; B] = \sum_{k \in M} \lambda_k^{\overline{\mathbf{p}}}[v^B],$$

thus establishing that the total amount shared according to $\Lambda^{\mathbf{p}}$ in $[v; B]$ coincides with the amount shared according to $\lambda^{\overline{\mathbf{p}}}$ in the quotient game v^B.

[4] In principle, this property makes sense only for coalitional values defined for all N; in such a case, one generally abuses the notation and uses a unique symbol g on both $\mathcal{G}_N^{CS}$ and $\mathcal{G}_M^{CS}$. However, the property also makes sense for a coalitional value g on a given $\mathcal{G}_N^{cs}$ provided, at least, that it induces a coalitional value $\overline{g}$ on $\mathcal{G}_M^{cs}$ for each $B \in B(N)$. And this is precisely the case of the coalitional multinomial probabilistic values.

Definition 2.6 Let $\mathbf{p}$ be a tendency profile in N. A coalitional value g on $\mathcal{G}_N^{CS}$ satisfies the *property of* $\mathbf{p}$*–weighted payoffs for quotients of unanimity games* iff, for any $B \in B(N)$ and any nonempty $T \subseteq N$,

$$\overline{p}_k \sum_{i \in B_k} g_i[u^T; B] = \overline{p}_\ell \sum_{j \in B_\ell} g_j[u^T; B] \quad \text{for all } B_k, B_\ell \in B \text{ intersecting } T,$$

where $\overline{\mathbf{p}}$ is any tendency profile induced by $\mathbf{p}$ in M.

Remark 2.7 The coalitional $\mathbf{p}$–multinomial probabilistic value *fails to satisfy* the standard property of *symmetry in the quotient game*. It should be clear that the failure is essentially due to the fact that, in general, neither $\lambda^{\overline{\mathbf{p}}}$ nor $\lambda^{\mathbf{p}}$ satisfy anonymity. But this is precisely the positive reason by which we are considering here multinomial probabilistic values instead of binomial semivalues: the possibility, offered by profiles $\mathbf{p}$ and $\overline{\mathbf{p}}$, to discriminate among players and unions, respectively.

Remark 2.8 The property of $\mathbf{p}$*–weighted payoffs for quotients of unanimity games* does not hold for general quotients. The corresponding statement, that could be called "$\mathbf{p}$–weighted symmetry in the quotient game" would look as follows: if $k, \ell \in M$ are symmetric players in v^B then

$$\overline{p}_k \sum_{i \in B_k} \Lambda_i^{\mathbf{p}}[v; B] = \overline{p}_\ell \sum_{j \in B_\ell} \Lambda_j^{\mathbf{p}}[v; B].$$

Again it is easy to see (even for $n = 2$) that this is not true in general. Also it can be easily shown that *the* $\mathbf{p}$*–coalitional multinomial probabilistic value satisfies* $\mathbf{p}$*–weighted symmetry in the quotient game iff it is a symmetric coalitional binomial semivalue*, but then the property becomes just symmetry in the quotient game (and $p_1 = p_2 = \cdots = p_n$).

Our axiomatic characterizations hold for any coalitional $\mathbf{p}$–multinomial probabilistic value with a *positive* tendency profile $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, that is, a profile $\mathbf{p}$ such that $p_i > 0$ for $i = 1, 2, \ldots, n$.

Theorem 2.9 *([8]) (First axiomatic characterization of each coalitional multinomial probabilistic value with positive profile) Let* $\mathbf{p}$ *be a positive profile in a given player set* N*. Then there is a unique coalitional value on* $\mathcal{G}_N^{cs}$ *that satisfies linearity, the dummy player property, symmetry within unions, the coalitional* $\mathbf{p}$*–multinomial total power property, and the property of* $\mathbf{p}$*–weighted payoffs for quotients of unanimity games. It is the coalitional* $\mathbf{p}$*–multinomial probabilistic value.*
□

3 Second axiomatic characterization

3.1 Cooperative games

A new axiomatic characterization of each coalitional $\mathbf{p}$–multinomial probabilistic value with positive tendency profile can be obtained by just replacing the property of symmetry within unions in Theorem 2.9 with the property of balanced contributions within unions. Initially, we consider:

- *balanced contributions within unions*: for all $[v; B] \in \mathcal{G}_N^{cs}$, $B_k \in B$ and $i \neq j$ in $B_k \in B$,

$$g_i[v; B] - g_i[v; B_{-j}] = g_j[v; B] - g_j[v; B_{-i}],$$

where B_{-i} is the coalition structure that results when player i disappears, i.e.,

$$B_{-i} = \{B_1, \ldots, B_{k-1}, B_k \backslash \{i\}, B_{k+1}, \ldots, B_m\},$$

and B_{-j} is defined analogously.

This property refers to players belonging to the same union and it states that the loss (resp., gain) of a player $i \in B_k$ when a distinct player $j \in B_k$ leaves the game is the same as the loss (resp., gain) of player j when player i leaves the game.

A problem of this statement of the balanced contributions property for a generic coalitional value is that it should be restricted to coalitional values g defined not only on $\mathcal{G}_N^{cs}$ but also for restricted games v_{-i} with coalition structure B_{-i} in other player sets of the form $N_{-i} = N\backslash\{i\}$ for some $i \in N$, and this would disturb the precision and reduce the scope of our axiomatization.

With the aim of going back to the initial setup, and to remain there, we consider in detail all elements that intervene in the case of a coalitional multinomial probabilistic value $\Lambda^{\mathbf{p}}$. Initially, we have the player set $N = \{1, 2, \ldots, n\}$, the coalition structure $B = \{B_1, B_2, \ldots, B_m\}$, the tendency profile $\mathbf{p} = (p_1, p_2, \ldots, p_n)$, and the game v, all of them defined in N. If $i \neq j$ in $B_k \in B$, a way to describe what happens if j leaves the game consists in considering $N_{-j} = N\backslash\{j\}$ as new player set, $B_{-j} = \{B_1, B_2, \ldots, B_k\backslash\{j\}, \ldots, B_m\}$ as restricted coalition structure in N_{-j}, $\mathbf{p}_{-j} = (p_1, p_2, \ldots, p_i, \ldots, \widehat{p_j}, \ldots, p_n)$ as restricted tendency profile in N_{-j}, and v_{-j} as restricted game, defined by $v_{-j}(S) = v(S)$ for all $S \subseteq N_{-j}$. A similar description holds when player i leaves the game. Notice that, for unanimity games u^T, if $j \in T \subseteq N$ then $u^T_{-j} = 0$.

Then, the property of *balanced contributions within unions* for the coalitional multinomial probabilistic value $\Lambda^{\mathbf{p}}$ states that

$$\Lambda_i^{\mathbf{p}}[v; B] - \Lambda_i^{\mathbf{p}_{-j}}[v_{-j}; B_{-j}] = \Lambda_j^{\mathbf{p}}[v; B] - \Lambda_j^{\mathbf{p}_{-i}}[v_{-i}; B_{-i}].$$

(This property will be checked in the existence part of Theorem 3.2.) The crucial fact here is given in the following result, where $(v_{-j})^*$ is the null extension of v_{-j} to N, defined by $(v_{-j})^*(S) = v_{-j}(S \cap N_{-j})$ for all $S \subseteq N$. We recall that j is a null player in this game.

Lemma 3.1 *For any game v in N, any coalition structure B, any tendency profile $\mathbf{p}$, all defined in N, and any pair $i \neq j$ in $B_k \in B$,*

$$\Lambda_i^{\mathbf{p}_{-j}}[v_{-j}; B_{-j}] = \Lambda_i^{\mathbf{p}}[(v_{-j})^*; B]$$

and, in particular, if $\emptyset \neq T \subseteq N$ then

$$\Lambda_i^{\mathbf{p}_{-j}}[u^T_{-j}; B_{-j}] = \Lambda_i^{\mathbf{p}}[(u^T_{-j})^*; B].$$

Proof: By linearity, we just need to check the second equation, concerned with unanimity games only. This is easy if we split it into the following three cases:

(a) If $i, j \in T$, then
$$\Lambda_i^{\mathbf{p}_{-j}}[u^T_{-j}; B_{-j}] = 0 = \Lambda_i^{\mathbf{p}}[(u^T_{-j})^*; B]$$
since $u^T_{-j} = 0$ and hence $(u^T_{-j})^* = 0$. Something similar for j.

(b) If, e.g., $i \in T$ and $j \notin T$, then, on one hand, using Eq. (2),
$$\Lambda_i^{\mathbf{p}_{-j}}[u^T_{-j}; B_{-j}] = \frac{1}{|T_k|} \prod_{h \in M\backslash\{k\}} \overline{p}_h = \Lambda_i^{\mathbf{p}}[(u^T_{-j})^*; B]$$
since $u^T_{-j} = u^T$ in N_{-j} and $(u^T_{-j})^* = u^T$ in N. And, on the other hand,
$$\Lambda_j^{\mathbf{p}_{-i}}[u^T_{-i}; B_{-i}] = 0 = \Lambda_j^{\mathbf{p}}[(u^T_{-i})^*; B]$$
since $u^T_{-i} = 0$ and hence $(u^T_{-i})^* = 0$.

(c) If $i, j \notin T$
$$\Lambda_i^{\mathbf{p}_{-j}}[u^T_{-j}; B_{-j}] = 0 = \Lambda_i^{\mathbf{p}}[(u^T_{-j})^*; B]$$
since $u^T_{-j} = u^T$ in N_{-j}, $(u^T_{-j})^* = u^T$ in N, and i is a null player in both games. Something similar for j. $\square$

Using this result we can provide an *alternative and definitive expression* for the ***property of balanced contributions within unions***: for all $[v; B] \in \mathcal{G}_N^{cs}$ and $i \neq j$ in $B_k \in B$,

$$g_i[v; B] - g_i[(v_{-j})^*; B] = g_j[v; B] - g_j[(v_{-i})^*; B]. \tag{3}$$

Theorem 3.2 *(Second axiomatic characterization of each coalitional multinomial probabilistic value with positive tendency profile) Let* $\mathbf{p}$ *be a positive tendency profile in a given player set* N*. Then there is a unique coalitional value on* $\mathcal{G}_N^{cs}$ *that satisfies linearity, the dummy player property, balanced contributions within unions, the coalitional* $\mathbf{p}$*–multinomial total power property, and the property of* $\mathbf{p}$*–weighted payoffs for quotients of unanimity games. It is the coalitional* $\mathbf{p}$*–multinomial probabilistic value.*

Proof: (a) (Existence) Taking into account the results already obtained in Theorem 2.9, it suffices to show that the coalitional $\mathbf{p}$–multinomial probabilistic value $\Lambda^{\mathbf{p}}$ satisfies the property of balanced contributions within unions as stated in Eq. (3). And, by linearity, we only need to deal with the unanimity game u^T with a coalition structure B in N for an arbitrary nonempty $T \subseteq N$, thus it suffices to check that

$$\Lambda_i^{\mathbf{p}}[u^T; B] - \Lambda_i^{\mathbf{p}}[(u_{-j}^T)^*; B] = \Lambda_j^{\mathbf{p}}[u^T; B] - \Lambda_j^{\mathbf{p}}[(u_{-i}^T)^*; B] \quad \text{for any } i \neq j \text{ in } B_k \in B.$$

Using the notation introduced for Eq. (2), we distinguish three cases:

(a) If $i, j \in T$ then we obtain

$$\Lambda_i^{\mathbf{p}}[u^T; B] - \Lambda_i^{\mathbf{p}}[(u_{-j}^T)^*; B] = \frac{1}{|T_k|} \prod_{r \in \overline{T} \setminus \{k\}} \overline{p}_r - 0 = \frac{1}{|T_k|} \prod_{r \in \overline{T} \setminus \{k\}} \overline{p}_r.$$

since $(u_{-j}^T)^* = 0$. Similarly, since $(u_{-i}^T)^* = 0$,

$$\Lambda_j^{\mathbf{p}}[u^T; B] - \Lambda_j^{\mathbf{p}}[(u_{-i}^T)^*; B] = \frac{1}{|T_k|} \prod_{r \in \overline{T} \setminus \{k\}} \overline{p}_r - 0 = \frac{1}{|T_k|} \prod_{r \in \overline{T} \setminus \{k\}} \overline{p}_r.$$

(b) If, e.g., $i \in T$ and $j \notin T$ then, on one hand,

$$\Lambda_i^{\mathbf{p}}[u^T; B] - \Lambda_i^{\mathbf{p}}[(u_{-j}^T)^*; B] = \frac{1}{|T_k|} \prod_{r \in \overline{T} \setminus \{k\}} \overline{p}_r - \frac{1}{|T_k|} \prod_{r \in \overline{T} \setminus \{k\}} \overline{p}_r = 0,$$

since $(u_{-j}^T)^* = u^T$; on the other hand,

$$\Lambda_j^{\mathbf{p}}[u^T; B] - \Lambda_j^{\mathbf{p}}[(u_{-i}^T)^*; B] = 0 - 0 = 0,$$

since j is null in u^T and u_{-i}^T.

(c) Finally, if $i, j \notin T$ then

$$\Lambda_i^{\mathbf{p}}[u^T; B] - \Lambda_i^{\mathbf{p}}[(u_{-j}^T)^*; B] = 0 - 0 = 0,$$

since i is null in both u^T and $(u_{-j}^T)^* = u^T$. Analogously,

$$\Lambda_j^{\mathbf{p}}[u^T; B] - \Lambda_j^{\mathbf{p}}[(u_{-i}^T)^*; B] = 0 - 0 = 0.$$

(b) (Uniqueness) Since, according to Theorem 2.9, $\Lambda^{\mathbf{p}}$ satisfies the property of coalitional $\mathbf{p}$–multinomial total power property, this property can be rewritten for an arbitrary coalitional value g as

$$\sum_{i \in N} g_i[v; B] = \sum_{i \in N} \Lambda_i^{\mathbf{p}}[v; B],$$

which will be used here. Let g be a coalitional value on $\mathcal{G}_N^{cs}$ that satisfies the stated properties for a given tendency profile $\mathbf{p}$. We will show that g is uniquely determined on all $[v; B] \in \mathcal{G}_N^{cs}$, so it must coincide with Λ^p. Using linearity we only need to prove that g is uniquely determined on each unanimity game u^T.

Let again $\overline{T} = \{k \in M; T \cap B_k \neq \emptyset\}$ and $T_k = T \cap B_k$ for each $k \in \overline{T}$. By the dummy player property, $g_i[u^T; B] = 0$ if $i \notin T$. This leaves us with the players of T. By the coalitional **p**–multinomial total power property,

$$\sum_{k \in \overline{T}} \sum_{i \in T_k} g_i[u^T; B] = \sum_{i \in T} g_i[u^T; B] = \sum_{i \in T} \Lambda_i^{\mathbf{p}}[u^T; B]. \tag{4}$$

By the property of **p**–weighted payoffs for unanimity games, for all $k, h \in \overline{T}$ we have

$$\overline{p}_k \sum_{i \in T_k} g_i[u^T; B] = \overline{p}_h \sum_{j \in T_h} g_j[u^T; B]. \tag{5}$$

Now we assume that the induced tendency profile $\overline{\mathbf{p}}$ is positive too. This is not a restrictive assumption, as the final result will not depend on the induced tendency profile and, e.g., possibilities (α), (β), (γ) and (δ) suggested after Definition 2.2 are positive if so is tendency profile **p**. Thus, setting $\alpha_k = \sum_{i \in T_k} g_i[u^T; B]$ for all $k \in \overline{T}$ and using Eqs. (4) and (5), we obtain a linear system of $\overline{t}$ equations with $\overline{t}$ unknowns α_k. The determinant of the matrix of coefficients of this system is inductively shown to be

$$D = (-1)^{\overline{t}-1} \sum_{k \in \overline{T}} \overline{p}_1 \overline{p}_2 \ldots \overset{\wedge}{\overline{p}_k} \ldots \overline{p}_{\overline{t}} \neq 0.$$

This implies that the system has a unique solution, i.e., $\sum_{i \in T_k} g_i[u^T; B]$ is uniquely determined for each $k \in \overline{T}$. Finally, by the property of balanced contributions within unions, if $i, j \in T_k$ then

$$g_i[u^T; B] - g_i[(u^T_{-j})^*; B] = g_j[u^T; B] - g_j[(u^T_{-i})^*; B].$$

Given that $u^T_{-j} = 0 = u^T_{-i}$, it follows that, for all $i, j \in T_k$,

$$g_i[u^T; B] = g_j[u^T; B]$$

and hence, for any $k \in \overline{T}$ and any $i \in T_k$,

$$g_i[u^T; B] = \frac{1}{|T_k|} \sum_{j \in T_k} g_j[u^T; B]$$

is uniquely determined. $\square$

3.2 Discussion

In this section we compare the characterizations of the coalitional multinomial probabilistic value with parallel axiomatizations of other coalitional values such as the Owen value [16], the Banzhaf–Owen value [17] or the symmetric coalitional Banzhaf value [1].

The Owen value can be viewed as a two–step allocation rule. First, each union B_k receives its payoff in the quotient game according to the Shapley value; then, each B_k splits this amount among its players by applying the Shapley value to a game played in B_k as follows: the worth of each subcoalition T of B_k is the Shapley value that T would get in a "pseudoquotient game" played by T and the remaining unions on the assumption that $B_k \backslash T$ leaves the game, *i.e.* the quotient game after replacing B_k with T. This is the way to bargain within the union: each subcoalition T claims the payoff it would obtain when dealing with the other unions in absence of its partners in B_k. The Owen value is characterized uniquely by efficiency, the null player property, symmetry within unions, symmetry in the quotient game, and additivity.

The Owen–Banzhaf value [17] follows a similar scheme. The resulting formula parallels that of the Owen value by replacing everywhere the Shapley value with the Banzhaf value. This value, which is a *coalitional value of the Banzhaf value* β, does not satisfy efficiency, but neither symmetry in the quotient game nor the quotient game property.

Alonso and Fiestras [1] introduced a modification of the Owen–Banzhaf value. This coalitional value applies the Banzhaf value in the quotient game and the Shapley value within unions. This *symmetric coalitional Banzhaf value* satisfies the same properties as the Owen value, with the sole exception of efficiency —replaced by a total power property—, as well as the quotient game property, and it is a coalitional value of the Banzhaf value.

Notice that the *coalitional multinomial probabilistic values* satisfy additivity, positivity, the dummy player property, balanced contributions within unions, symmetry within unions, and the quotient game property. Among these values, as we have said before, we find all symmetric coalitional binomial semivalues and, in particular, the symmetric coalitional Banzhaf value, that also satisfy symmetry in the quotient game. Instead, the Banzhaf–Owen value [17] and its counterpart [3] fall out of this class. $\Lambda^{\mathbf{p}}$ is a coalitional $\lambda^{\mathbf{p}}$–value for any tendency profile $\mathbf{p}$.

Moreover, the only distinction between the Owen value [16] and any coalitional $\mathbf{p}$–multinomial probabilistic value is that the former satisfies efficiency and symmetry in the quotient game, whereas the latter satisfies *local efficiency* (i.e., within unions) and the coalitional $\mathbf{p}$–multinomial total power property, in a way that parallels the distinction between the Shapley value and any $\mathbf{p}$–multinomial probabilistic value. Symmetry in the quotient game cannot be satisfied in general by the coalitional multinomial probabilistic values because the tendency profile usually breaks symmetry in the mere game. All these features may provide criteria to decide what coalitional multinomial probabilistic value to use.

Remark 3.3 The only difference between Theorem 2.9 and Theorem 3.2 is the substitution of symmetry within unions (SU, for short) by balanced contributions within unions (BCU). In general, these two properties are not related. However, in presence of the dummy player property and linearity, it is not difficult to see that BCU is stronger than SU, in the sense that BCU implies SU but the converse is not true.

3.3 Simple games

Finally, let us consider simple games, which form an especially interesting class of cooperative games, denoted by $\mathcal{S}_N$ for each player set N. A cooperative game v in N is a *simple game* iff it is monotonic, $v(S) \in \{0,1\}$ for every $S \subseteq N$, and $v(N) = 1$. A coalition $S \subseteq N$ is *winning in* v if $v(S) = 1$ (otherwise it is called *losing*), and $W(v)$ denotes the set of winning coalitions in v. Due to monotonicity, the subset $W^m(v)$ of all *minimal* winning coalitions determines $W(v)$ and hence the game. A simple game v is a *weighted majority game* iff there are nonnegative *weights* $w_1, w_2, \ldots, w_n$ allocated to the players and a positive *quota* $q \leq \sum_{i=1}^n w_i$, such that

$$v(S) = 1 \quad \text{iff} \quad \sum_{i \in S} w_i \geq q.$$

We then write $v \equiv [q; w_1, w_2, \ldots, w_n]$, although this representation is never unique. $\mathcal{S}_N$ (and even $\mathcal{G}_N$) becomes a lattice under the standard composition laws defined by $(v \vee v')(S) = \max\{v(S), v'(S)\}$ and $(v \wedge v')(S) = \min\{v(S), v'(S)\}$. In particular, $u^R \wedge u^T = u^{R \cup T}$. We denote as $\mathcal{S}_N^{cs} = \mathcal{S}_N \times B(N)$ the class of simple games with a coalition structure in N. When restricting a value (coalitional or not) to $\mathcal{S}_N$ or $\mathcal{S}_N^{cs}$ it is customary to speak of a "power index".

In the case of Theorem 3.2 we reach a "parallel" axiomatization on the class of simple games by just replacing additivity with the

- *transfer property*: $g[v \vee v'; B] = g[v; B] + g[v'; B] - g[v \wedge v'; B]$ for all v, v' and B.

The analogue of Theorem 3.2 for simple games is given below without proof (it is very similar to that of Theorem 3.2).

Theorem 3.4 *(Axiomatic characterization of each coalitional multinomial probabilistic power index with positive tendency profile) Let* $\mathbf{p}$ *be a positive tendency profile in a given player set* N*. Then there is a unique coalitional power index on* $\mathcal{SG}_N^{cs}$ *that satisfies the transfer property, the dummy player property, balanced contributions within unions, the coalitional* $\mathbf{p}$*–multinomial total* *power property, and the property of* $\mathbf{p}$*–weighted payoffs for quotients of unanimity games. It is (the restriction of) the coalitional* $\mathbf{p}$*–multinomial probabilistic value.* $\square$

Remark 3.5 The logical independence of the axiomatic systems used in Theorems 3.2 and 3.4 is shown in the Appendix.

4 Two applications to the political analysis

Example 4.1 We consider here the Madrid Assembly in Legislature 2015–2019. Four parties elected members to this regional parliamentary body (129 seats) in the elections held on 24 May 2015. The seat distribution among the parties was as follows.

1: PP (Partido Popular), conservative party: 48 seats.

2: PSOE (Partido Socialista Obrero Español), moderate left–wing party: 37 seats.

3: Podemos, radical left–wing party: 27 seats.

4: C's (Ciudadanos), liberal party: 17 seats.

Under the standard absolute majority rule, and assuming voting discipline within parties, the structure of this parliamentary body can be represented by the weighted majority game

$$v \equiv [65; 48, 37, 27, 17].$$

Therefore, the situation is described by the family of minimal winning coalitions

$$W^m(v) = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3,4\}\},$$

so players 2, 3 and 4 are symmetric in v, and the expression of v in terms of unanimity games is

$$v = u^{\{1,2\}} + u^{\{1,3\}} + u^{\{1,4\}} + u^{\{2,3,4\}} - u^{\{1,2,3\}} - u^{\{1,2,4\}} - u^{\{1,3,4\}}.$$

A main feature of the Madrid Assembly issued from the elections was the absence of a party enjoying absolute majority, so a coalition government was expected to form. We will not try to give here a full description of the political complexity in Madrid at regional level. We wish only to state that the politically most likely coalitions to form, and the corresponding coalition structures to which we are going to restrict our analysis, are clearly the following:

- PP + C's, the right–wing majority alliance: $B_R = \{\{1,4\}, \{2\}, \{3\}\}$.
- PSOE + Podemos + C's, the left–wing majority alliance: $B_L = \{\{1\}, \{2,3,4\}\}$.

We would like to analyze this situation. Of course, our main interest will center on the strategic possibilities of party 4 (C's), whose position is crucial in the two–alternative scenario we are considering.

1. Classic, non-parametric coalitional values.

A classic approach to study the problem would consist in using either (a) the Shapley value and the Owen value, (b) the Banzhaf value [15, 4] and the Banzhaf–Owen value, or (c) the Banzhaf value and the symmetric coalitional Banzhaf value [1], in order to evaluate the strategic possibilities of each party under both hypotheses. The results are given in Table 1, where (–) means no coalition formation, (R) means that PP + C's forms, and (L) means that PSOE + Podemos + C's forms.

According to (a), the a priori power of C's duplicates in both alliances. Instead, according to (b), C's is indifferent among the three options. Finally, according to (c), C's would strictly prefer joining PSOE and Podemos instead of PP.

2. Binomial semivalues and symmetric coalitional binomial semivalues.

	(a)			(b)			(c)		
	(–)	(R)	(L)	(–)	(R)	(L)	(–)	(R)	(L)
PP	0.5000	0.6667	0.0000	0.7500	0.7500	0.0000	0.7500	0.7500	0.0000
PSOE	0.1667	0.0000	0.3333	0.2500	0.0000	0.2500	0.2500	0.0000	0.3333
Podemos	0.1667	0.0000	0.3333	0.2500	0.0000	02500	0.2500	0.0000	0.3333
C's	0.1667	0.3333	03333	0.2500	0.2500	0.2500	0.2500	0.2500	0.3333

Table 1: classic measures of power in the Madrid Assembly 2015–2019

Now we will use binomial semivalues [18] (a generalization of the Banzhaf value and a particular case of the multinomial probabilistic values when $p_i = p$ for all $i \in N$), and symmetric coalitional binomial semivalues [6] (a generalization of the symmetric coalitional Banzhaf value and a particular case of the coalitional multinomial probabilistic values when $p_i = p$ for all $i \in N$) whenever a coalition structure exists.

The conclusion derived from the results of the theoretical analysis, given in Table 2, is as follows. Both alliances increase the a priori power of C's for all values of $p \in [0, 1]$. However, C's would strictly prefer the left–wing alliance if p belongs to the interval $(\frac{3-\sqrt{3}}{6}, \frac{3+\sqrt{3}}{6})$ (i.e., in 57.7% of cases) and the right–wing alliance otherwise.

	(–)	(R)	(L)
PP	$3p(1-p)$	$\frac{1+2p}{2} - p^2$	0.0000
PSOE	$p(1-p)$	0.0000	0.3333
Podemos	$p(1-p)$	0.0000	0.3333
C's	$p(1-p)$	$\frac{1-2p}{2} + p^2$	03333

Table 2: Binomial semivalues and symmetric coalitional binomial semivalues in the Madrid Assembly 2015–2019

3. Multinomial probabilistic values and coalitional multinomial probabilistic values.

We first apply a multinomial probabilistic value $\lambda^{\mathbf{p}}[v]$ for a tendency profile $\mathbf{p}$ using Eq. (1):

$$\begin{aligned}
\lambda_1^{\mathbf{p}}[v] &= p_2 + p_3 + p_4 - p_2p_3 - p_2p_4 - p_3p_4, \\
\lambda_2^{\mathbf{p}}[v] &= p_1 + p_3p_4 - p_1p_3 - p_1p_4 = p_1(1 - p_3 - p_4) + p_3p_4, \\
\lambda_3^{\mathbf{p}}[v] &= p_1 + p_2p_4 - p_1p_2 - p_1p_4 = p_1(1 - p_2 - p_4) + p_2p_4, \\
\lambda_4^{\mathbf{p}}[v] &= p_1 + p_2p_3 - p_1p_2 - p_1p_3 = p_1(1 - p_2 - p_3) + p_2p_3.
\end{aligned}$$

These allocations reflect the a priori power distribution. (a) The payoff to each party depends on the tendencies of the remaining parties as to coalition formation, and not on its own tendency. (b) The payoff to party 1 increases if p_2 increases, provided that $p_3+p_4 \leq 1$; otherwise, it decreases. (c) The payoff to party 2 is improved by either the interest of party 1 combined with the apathy of parties 3 and 4 or, alternatively, by the interest of these parties combined with the unconcern of party 1. (d) The same is true, *mutatis mutandis*, for parties 3 and 4.

Now, we use the coalitional multinomial probabilistic value $\Lambda^{\mathbf{p}}$. The calculation of $\Lambda^{\mathbf{p}}[v; B]$ follows from Eq. (2). The analysis is divided into two stages.

First stage. We apply $\Lambda^{\mathbf{p}}$ in cases $B = B_R$ and $B = B_L$. Then,

- For the right–wing majority alliance PP + C's:

$$\Lambda_1^{\mathbf{p}}[v;B_R] = \frac{1}{2}(1+p_2+p_3) - p_2p_3 \quad \text{and} \quad \Lambda_4^{\mathbf{p}}[v;B_R] = \frac{1}{2}(1-p_2-p_3) + p_2p_3.$$

  Notice that the maximum of $\Lambda_4^{\mathbf{p}}[v;B_R]$ is $\frac{1}{2}$ for $p_2 = 1 = p_3$.

- For the left–wing majority alliance PSOE + Podemos + C's:

$$\Lambda_2^{\mathbf{p}}[v;B_L] = \Lambda_3^{\mathbf{p}}[v;B_L] = \Lambda_4^{\mathbf{p}}[v;B_L] = \frac{1}{3}\ .$$

It is not true that all players get profit, with respect to the a priori power distribution, from entering a coalition. There always exist suitable values of the p_i's that produce a damage to a given player when joining.

Incidentally, let us notice that tendency profile $\overline{\mathbf{p}}$ does not appear in these expressions for several reasons: (a) the payoffs to the members of a union depend only on the tendencies of the remaining unions and not on the individual tendencies of its members because the multinomial value λ^p is applied in the quotient game played by the unions; (b) if a union B_r reduces to a singleton $\{j\}$ then $\overline{p}_r = p_j$; (c) in both B_R and B_L, only one coalition with more than one member forms, so the payoffs to its players can be expressed, according to (b), only in terms of tendency profile $\mathbf{p}$ for any induced tendency profile $\overline{\mathbf{p}}$; (d) since the coalition that forms is winning, the quotient game is a dictatorship, and hence the outside players become null and get 0. In both scenarios, the payoffs sum up to 1 by local efficiency.

Of course, the property of balanced contributions within unions is well illustrated in this example. (a) For parties 1 and 4, which form B_1 in the coalition structure B_R, we have

$$\begin{aligned}
&\Lambda_1^{\mathbf{p}}[v;B_R] - \Lambda_1^{\mathbf{p}}[(v_{-4})^*;B_R] = \big[\frac{1}{2}(p_1+p_2+p_3) - p_2p_3\big] - \big[p_2+p_3-p_2p_3\big] \\
&= \frac{1}{2}(1-p_2-p_3), \\
&\Lambda_4^{\mathbf{p}}[v;B_R] - \Lambda_4^{\mathbf{p}}[(v_{-1})^*;B_R] = \big[\frac{1}{2}(p_1-p_2-p_3) + p_2p_3\big] - \big[p_2p_3\big] = \frac{1}{2}(1-p_2-p_3).
\end{aligned}$$

(b) For e.g. parties 2 and 4, which belong to B_2 in the coalition structure B_L, we find

$$\Lambda_2^{\mathbf{p}}[v;B_L] - \Lambda_2^{\mathbf{p}}[(v_{-4})^*;B_L] = \frac{1}{3} - \frac{1}{2}p_1 = \Lambda_4^{\mathbf{p}}[v;B_L] - \Lambda_4^{\mathbf{p}}[(v_{-2})^*;B_L]$$

Second stage. Finally, we discuss the strategic possibilities of party 4 (C's). The power of party 4 in B_R and B_L is, respectively,

$$\frac{1}{2}(1-p_2-p_3) + p_2p_3 \quad \text{and} \quad \frac{1}{3}\ .$$

The coincidence arises when

$$3p_2 + 3p_3 - 6p_2p_3 - 1 = 0.$$

If $p_2 \neq \frac{1}{2}$, it follows that

$$p_3 = \frac{1}{2}\left(1 - \frac{1}{3}\frac{1}{1-2p_2}\right).$$

Consider p_3 as a function of $p_2 \in [0,1] \setminus \{\frac{1}{2}\}$. This function is decreasing and concave if $p_2 < \frac{1}{2}$ and decreasing and convex if $p_2 > \frac{1}{2}$, tends to $+\infty$ when p_2 tends to $\frac{1}{2}^+$, and tends to $-\infty$ when p_2 tends to $\frac{1}{2}^-$ (see Fig. 1).

- For $p_2 < \frac{1}{2}$, the open region S_1, limited by lines $p_2 = \frac{1}{2}$, $p_2 = 0$, $p_3 = 0$ and the curve, represents the cases where party 4 strictly prefers B_R. Its area is

$$A(S_1) = \int_0^{\frac{1}{3}} \frac{1}{2}\left(1 - \frac{1}{3}\frac{1}{1-2x}\right)dx = \frac{1}{6} - \frac{1}{12}\log 3 = 0.0751$$

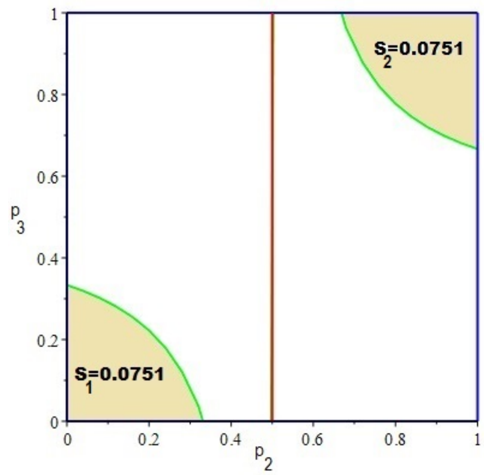

Figure 1: p_3 as a function of p_2

- For $p_2 > \frac{1}{2}$ the open region S_2, limited by lines $p_2 = \frac{1}{2}$, $p_2 = 1$, $p_3 = 1$ and the curve, represents the cases where party 4 strictly prefers B_R. Its area is

$$A(S_2) = \frac{1}{3} - \int_{\frac{2}{3}}^{1} \frac{1}{2}\left(1 - \frac{1}{3}\frac{1}{1-2x}\right) dx = \frac{1}{6} - \frac{1}{12}\log 3 = 0.0751$$

The sum of these areas, that is, 0.1502, can be taken as a measure of the probability that C's finally decides joining PP instead of PSOE and Podemos. If p_2 and p_3 are small enough (say, roughly speaking, near to 0) or, instead, if they are great enough (say, near to 1), then party 4 would prefer the right-wing coalition, where it would obtain a power between $\frac{1}{3}$ and $\frac{1}{2}$ (maximum power for this party). Hence there are "many" cases (15%) where party 4 should prefer party 1 instead of parties 2 and 3. The increase of strategic options for C's, cannot be discovered by merely using the traditional coalitional values: it follows only from the possibility to let n parameters vary, which is just one of the main features of the coalitional multinomial probabilistic values. The conclusion is that *it is not unreasonable that C's decided to join PP and VOX, **as it actually happened**.*

Example 4.2 We consider here the Andalusian Parliament in Legislature 2018–2022. In this case, five parties elected members for 109 seats in the elections held on 2 December 2018. The seat distribution among the parties is as follows.

1: PSOE (Partido Socialista Obrero Español), moderate left–wing party: 33 seats.

2: PP (Partido Popular), conservative party: 26 seats.

3: C's (Ciudadanos), liberal party: 21 seats.

4: AA (Adelante Andalucía), radical left–wing party: 17 seats.

5: VOX, far–right party: 12 seats.

Under the standard absolute majority rule, and assuming voting discipline within parties, the structure of this parliamentary body can be represented by the weighted majority game

$$v \equiv [55; 33, 26, 21, 17, 12].$$

Therefore, the situation is described by the family of minimal winning coalitions

$$W^m(v) = \{\{1,2\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}, \{2,4,5\}\},$$

so, on one hand, players 1 and 2 are symmetric in v and, on the other hand, players 3, 4 and 5 are also symmetric in v. The expression of v in terms of unanimity games is

$$
\begin{aligned}
v &= u^{\{1,2\}} + u^{\{1,3,4\}} + u^{\{1,3,5\}} + u^{\{1,4,5\}} + u^{\{2,3,4\}} + u^{\{2,3,5\}} + u^{\{2,4,5\}} \\
&- 2u^{\{1,2,3,4\}} - 2u^{\{1,2,3,5\}} - 2u^{\{1,2,4,5\}} - 2u^{\{1,3,4,5\}} - 2u^{\{2,3,4,5\}} + 4u^{\{1,2,3,4,5\}}.
\end{aligned}
$$

As in the case of the Madrid Assembly, there is not a party enjoying absolute majority, so a coalition government was expected to form. The corresponding coalition structures to the analysis of which we will limit ourselves, are the following:

- PP + C's + VOX, the right–wing majority alliance: $B_R = \{\{1\}, \{2,3,5\}, \{4\}\}$.
- PSOE + C's + AA, the left–wing majority alliance: $B_L = \{\{1,3,4\}, \{2\}, \{5\}\}$.

Our main interest will center on the strategic possibilities of party 3 (C's), whose position is again crucial in the two–alternative scenario we are considering.

Now, we use the coalitional multinomial probabilistic value $\Lambda^{\mathbf{p}}$. Again, the calculation of $\Lambda^{\mathbf{p}}[v;B]$ follows from Eq. (2). We divide the study into two stages.

First stage. We apply $\Lambda^{\mathbf{p}}$ in cases $B = B_R$ and $B = B_L$ for any tendency profile $\mathbf{p}$. Then,

- For the right–wing majority alliance PP + C's + VOX:

$$
\begin{aligned}
\Lambda_2^{\mathbf{p}}[v;B_R] &= \frac{1}{3}(1 + p_1 + p_4 - 2p_1p_4), \\
\Lambda_3^{\mathbf{p}}[v;B_R] &= \frac{1}{6}(2 - p_1 - p_4 + 2p_1p_4), \\
\Lambda_5^{\mathbf{p}}[v;B_R] &= \frac{1}{6}(2 - p_1 - p_4 + 2p_1p_4), \\
\Lambda_1^{\mathbf{p}}[v;B_R] &= \Lambda_4^{\mathbf{p}}[v;B_R] = 0.
\end{aligned}
$$

- For the left–wing majority alliance PSOE + C's + AA:

$$
\begin{aligned}
\Lambda_1^{\mathbf{p}}[v;B_L] &= \frac{1}{3}(1 + p_2 + p_5 - 2p_2p_5), \\
\Lambda_3^{\mathbf{p}}[v;B_L] &= \frac{1}{6}(2 - p_2 - p_5 + 2p_2p_5), \\
\Lambda_4^{\mathbf{p}}[v;B_L] &= \frac{1}{6}(2 - p_2 - p_5 + 2p_2p_5), \\
\Lambda_2^{\mathbf{p}}[v;B_L] &= \Lambda_5^{\mathbf{p}}[v;B_L] = 0.
\end{aligned}
$$

Notice that tendency profile $\overline{\mathbf{p}}$ does not appear in these expressions for the same reasons as in the previous example. And, again, the property of balanced contributions within unions is illustrated by this example. (a) For parties 2 and 5, which belong to B_2 in the coalition structure B_R, we have

$$
\begin{aligned}
&\Lambda_2^{\mathbf{p}}[v;B_R] - \Lambda_2^{\mathbf{p}}[(v_{-5})^*;B_R] = \Big[\frac{1}{3}(1 + p_1 + p_4 - 2p_1p_4)\Big] - \Big[p_1 + \frac{1}{2}p_4 - p_1p_4\Big] \\
&= \frac{1}{6}(2 - 4p_1 - p_4 + 2p_1p_4), \\
&\Lambda_5^{\mathbf{p}}[v;B_R] - \Lambda_5^{\mathbf{p}}[(v_{-2})^*;B_R] = \Big[\frac{1}{6}(2 - p_1 - p_4 + 2p_1p_4)\Big] - \Big[\frac{1}{2}p_1\Big] \\
&= \frac{1}{6}(2 - 4p_1 - p_4 + 2p_1p_4).
\end{aligned}
$$

(b) For parties 1 and 4, which belong to B_1 in the coalition structure B_L, we find

$$
\begin{aligned}
&\Lambda_1^{\mathbf{p}}[v;B_L] - \Lambda_1^{\mathbf{p}}[(v_{-4})^*;B_L] = \Big[\frac{1}{3}(1 + p_2 + p_5 - 2p_2p_5)\Big] - \Big[p_2 + \frac{1}{2}p_5 - p_2p_5\Big] \\
&= \frac{1}{6}(2 - 4p_2 - p_5 + 2p_2p_5), \\
&\Lambda_4^{\mathbf{p}}[v;B_L] - \Lambda_4^{\mathbf{p}}[(v_{-1})^*;B_L] = \Big[\frac{1}{6}(2 - p_2 - p_5 + 2p_2p_5)\Big] - \Big[\frac{1}{2}p_2\Big] \\
&= \frac{1}{6}(2 - 4p_2 - p_5 + 2p_2p_5).
\end{aligned}
$$

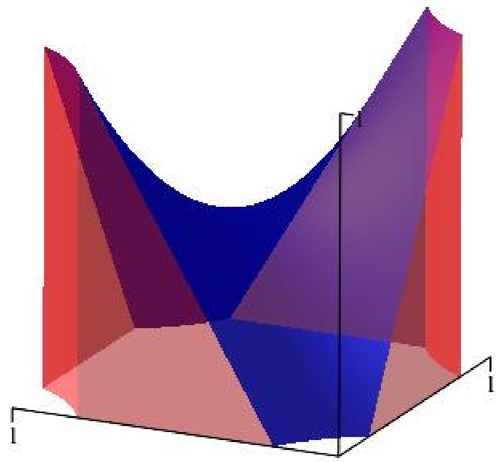

Figure 2: The hyperbolic paraboloid $p_4 = \dfrac{p_1 + 2p_2p_5 - p_2 - p_5}{2p_1 - 1}$

A study of the payoffs obtained by the parties under the coalition structures B_R or B_L, in the 16 vertices of the hypercube $[0,1]^4$, the domain of (p_1, p_2, p_4, p_5) —the tendency of party 3 does not matter, as it belongs to both winning coalitions—, gives only two different results for the three members of each winning coalition: $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$ or $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Party 1 gets its maximum payoff in 8 vertices: those where only party 2 is fully cooperative ($p_2 = 1$) and only party 5 is not at all cooperative ($p_5 = 0$), or conversely. Analogously, party 2 gets its maximum payoff in 8 vertices: those where only party 1 is fully cooperative ($p_1 = 1$) and only party 4 is not at all cooperative ($p_4 = 0$), or conversely. The maximum payoff for parties 3, 4 and 5 appears in 8 of the 16 vertices. The inner point $p_1 = p_2 = p_4 = p_5 = \frac{1}{2}$ is only a saddle point and gives $\frac{1}{3}$ to all parties that form one of the winning coalitions.

Second stage. Finally, we will discuss the strategic possibilities of party 3 (C's). The power of party 3 in B_R and B_L is, respectively,

$$\frac{1}{6}(2 - p_1 - p_4 + 2p_1p_4) \quad \text{and} \quad \frac{1}{6}(2 - p_2 - p_5 + 2p_2p_5).$$

The coincidence arises when

$$p_1 + p_4 - p_2 - p_5 - 2p_1p_4 + 2p_2p_5 = 0, \tag{6}$$

and we distinguish two cases.

(a) If $p_1 = \frac{1}{2}$ then $p_2 = \frac{1}{2}$ or $p_5 = \frac{1}{2}$ for any value of p_4. In this case, the sharing in B_R gives $\frac{1}{2}$ to party 2 and $\frac{1}{4}$ each to parties 3 and 5; and the sharing in B_L gives $\frac{1}{2}$ to party 1 and $\frac{1}{4}$ each to parties 3 and 4.

(b) If $p_1 \neq \frac{1}{2}$, from Eq.(6) it follows that

$$(2p_1 - 1)p_4 = p_1 + 2p_2p_5 - p_2 - p_5 \quad \text{and hence} \quad p_4 = \frac{p_1 + 2p_2p_5 - p_2 - p_5}{2p_1 - 1},$$

so party 3 strictly prefers B_L if

$$(2p_1 - 1)p_4 < p_1 + 2p_2p_5 - p_2 - p_5.$$

Let us take $p_1 \in [0,1] \setminus \{\frac{1}{2}\}$ as a parameter and consider p_4 as a function of $(p_2, p_5) \in [0,1]^2$.

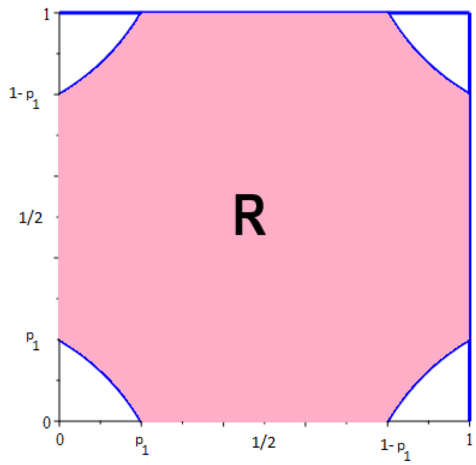

Figure 3: Projection of D_1 in the plane p_2p_5 if $0 \leq p_1 < 1/2$

- For $0 \leq p_1 < \frac{1}{2}$, party 3 strictly prefers B_L if

$$p_4 > \frac{p_1 + 2p_2p_5 - p_2 - p_5}{2p_1 - 1},$$

and hence the points of the solid D_1 limited by the plane $p_4 = 1$ and the surface

$$p_4 = \frac{p_1 + 2p_2p_5 - p_2 - p_5}{2p_1 - 1} \quad \text{for} \quad (p_2, p_5) \in [0,1]^2$$

(the hyperbolic paraboloid described in Fig. 2) represent the cases where party 3 strictly prefers B_L for a given p_1. Its volume can be taken as a measure of the probability that C's finally decides to join PSOE and Podemos instead of PP and VOX.

Notice that the intersection between the surface

$$p_4 = \frac{p_1 + 2p_2p_5 - p_2 - p_5}{2p_1 - 1}$$

and the plane $p_4 = 0$ is the curve $p_2 + p_5 - 2p_2p_5 = p_1$ in this plane (recall that p_1 is a parameter), and the intersection between the same surface and the plane $p_4 = 1$ is the curve $p_2 + p_5 - 2p_2p_5 = 1 - p_1$ in this second plane. Then, the projection R of solid D_1 in the plane p_2p_5 is given in Fig. 3, and the volume of D_1 is

$$\begin{aligned} V = V(D_1) = & \int_0^{p_1} \int_{\frac{p_1 - x}{1-2x}}^{\frac{1-p_1-x}{1-2x}} \left[1 - \frac{p_1 + 2xy - x - y}{(2p_1 - 1)} \right] dy\, dx + \\ & \int_{p_1}^{1-p_1} \int_0^1 \left[1 - \frac{p_1 + 2xy - x - y}{(2p_1 - 1)} \right] dy\, dx + \\ & \int_{1-p_1}^{1} \int_{\frac{1-p_1-x}{1-2x}}^{\frac{p_1 - x}{1-2x}} \left[1 - \frac{p_1 + 2xy - x - y}{(2p_1 - 1)} \right] dy\, dx = \\ & \frac{1 - 2p_1}{2} [1 - \log(1 - 2p_1)]. \end{aligned}$$

Notice that, for $p_1 = 0$, the projection of the solid D_1 is the square $[0,1] \times [0,1]$, and $V = \frac{1}{2}$ is the maximum of V because

$$\frac{dV}{dp_1} = \log(1 - 2p_1) \leq 0 \text{ for all } p_1 \in [0, 1/2).$$

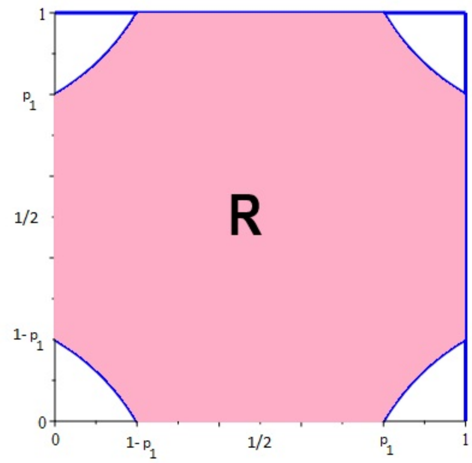

Figure 4: Projection of D_2 in the plane p_2p_5 if $1/2 < p_1 \leq 1$

- For $\frac{1}{2} < p_1 \leq 1$, party 3 strictly prefers B_L if

$$p_4 < \frac{p_1 + 2p_2p_5 - p_2 - p_5}{2p_1 - 1},$$

and hence the points of the solid D_2 limited by the plane $p_4 = 1$ and the same surface as in the previous case represent the cases where party 3 strictly prefers B_L for a given p_1. Its volume can be taken as a measure of the probability that C's finally decides joining PSOE and Podemos instead of PP and VOX.

In this case, the projection R of the solid D_2 is given in Fig. 4 and, by symmetry, its volume, that is, the probability that C's finally decides joining PSOE and Podemos instead of PP and VOX, is

$$\begin{aligned} V = V(D_2) = & \int_0^{1-p_1} \int_{\frac{1-p_1-x}{1-2x}}^{\frac{p_1-x}{1-2x}} \left[\frac{p_1 + 2xy - x - y}{(2p_1 - 1)}\right] dy\, dx + \\ & \int_{1-p_1}^{p_1} \int_0^1 \left[\frac{p_1 + 2xy - x - y}{(2p_1 - 1)}\right] dy\, dx + \\ & \int_{p_1}^{1} \int_{\frac{p_1-x}{1-2x}}^{\frac{1-p_1-x}{1-2x}} \left[\frac{p_1 + 2xy - x - y}{(2p_1 - 1)}\right] dy\, dx = \\ & \frac{2p_1 - 1}{2}[1 - \log(2p_1 - 1)]. \end{aligned}$$

Notice that, for $p_1 = 1$, the projection of the solid D_2 is the square $[0, 1] \times [0, 1]$, and $V = \frac{1}{2}$ is the maximum of V because

$$\frac{dV}{dp_1} = -\log(2p_1 - 1) \geq 0 \text{ for all } p_1 \in (1/2, 1].$$

Then, summing up, for $p_1 \neq \frac{1}{2}$ the probability that C's finally decides joining PSOE and Podemos instead of PP and VOX, is

$$P = P(\text{party 3 chooses } B_L) = \frac{|2p_1 - 1|}{2}(1 - \log|2p_1 - 1|).$$

Table 3 displays P for different values of p_1 (for $p_1 = 0.5$ the value of P is the limit when p_1 tends to 0.5 from either the left and the right). The conclusion is that *C's would be more inclined to join PP and VOX,* ***as it actually happened.***

p_1	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
P	0.50	0.49	0.45	0.38	0.26	0	0.26	0.38	0.45	0.49	0.50

Table 3: P for different values of p_1

If we had taken in (6) $p_2 \in [0,1] \backslash \{\frac{1}{2}\}$ as a parameter and consider p_5 as a function of $(p_1, p_4) \in [0,1]^2$ then,

$$p_5 = \frac{p_2 + 2p_1p_4 - p_1 - p_4}{2p_2 - 1} \quad \text{for} \quad (p_1, p_4) \in [0,1]^2$$

and the results obtained for $P = P(\text{party 3 chooses } B_L)$ would be the same as the previous case, by just replacing p_1 with p_2 in P. The discussion for $p_2 = 1/2$ is analogous to that for $p_1 = 1/2$.

Following a suggestion of a referee, we include here a study of the effects of a third coalition structure:

$$B_C = \{\{1,4\},\{2,5\},\{3\}\}.$$

In this case no winning coalition is formed. However, the interest of this coalition structure lies in the fact that it highlights the strategic position of C's, a partner desired by the two coalitions already formed: PSOE + AA on one hand and PP + VOX on the other.

We first recall in Table 4 the power distributions under B_L and B_R for subsequent comparisons.

coalition structure	$B_L = \{\{1,3,4\},\{2\},\{5\}\}$	$B_R = \{\{1\},\{2,3,5\},\{4\}\}$
1	$\frac{1+p_2+p_5-2p_2p_5}{3}$	0
2	0	$\frac{1+p_1+p_4-2p_1p_4}{3}$
3	$\frac{2-p_2-p_5+2p_2p_5}{6}$	$\frac{2-p_1-p_4+2p_1p_4}{6}$
4	$\frac{2-p_2-p_5+2p_2p_5}{6}$	0
5	0	$\frac{2-p_1-p_4+2p_1p_4}{6}$

Table 4: Power distribution in $[v; B_L]$ and $[v; B_R]$

First stage. We apply $\Lambda^{\mathbf{p}}$ for player 3 in case $B = B_C$ for any tendency profile $\mathbf{p}$ and obtain

$$\Lambda_3^{\mathbf{p}}[v; B_C] = \overline{p}_1 + \overline{p}_2 - 2\overline{p}_1\overline{p}_2.$$

Now, $\Lambda_3[v; B_C]$ depends on $\overline{p}_1$ and $\overline{p}_2$ and ranges from 0 when $(\overline{p}_1, \overline{p}_2) = (0,0)$ or $(1,1)$, to 1, when $(\overline{p}_1, \overline{p}_2) = (1,0)$ or $(0,1)$. Nevertheless, this structure of non–winning *protocoalitions* (following Owen's nomenclature) represents only an intermediate step addressed to form a winning coalition in a subsequent bargaining.

Second stage. In order to form a winning coalition, the three main coalition structures of second order that can arise from B_C are

$$B'_L = \{\{\{1,4\},3\},\dots\}, \quad B'_R = \{\{\{2,5\},3\},\dots\}, \quad B'_M = \{\{\{1,4\},\{2,5\},3\}\},$$

described, in an informal but appealing way, as

$$B'_L : (1+4)+3,\dots; \quad B'_R : (2+5)+3,\dots; \quad \text{and} \quad B'_M : (1+4)+(2+5)+3.$$

In the two first cases, the dots ... mean that the players not mentioned may remain together or alone since this does not have influence on the power distribution among the players that form the winning coalition in each case.

The power distribution under the second order coalition structures can be determined taking $\overline{14}$ and $\overline{25}$ as players in the quotient set $M = \{\overline{14},\overline{25},3\}$, the quotient game v^M defined by $v^M(S) = 0$ if $|S| \leq 1$ and $v^M(S) = 1$ if $|S| \leq 2$, and the coalition structures B'_L, B'_R, and B'_M defined above. The payoffs obtained by the players when the coalitional multinomial value $\Lambda^{\mathbf{p}}$ is applied to these coalition structures are given in Table 5.

coalition structure	$B'_L = \{\overline{143},\overline{25}\}$	$B'_R = \{\overline{14},\overline{253}\}$	$B'_M = \{\overline{14},\overline{25},3\}$
$\overline{14}$	1/2	0	1/3
$\overline{25}$	0	1/2	1/3
3	1/2	1/2	1/3

Table 5: Power distributions in M

Now all is ready to summarize and discuss this additional analysis of the Andalusian Parliament.

We first adopt the viewpoint of player 3 and recall the result of comparing the coalition structures $B_L = \{\{1,3,4\},2,5\}$ and $B_R = \{1,\{2,3,5\},4\}$. We found that, on one hand,

$$\Lambda_3[v;B_L] = \frac{2-p_2-p_5+2p_2p_5}{6},$$

which depends on p_2 and p_5 and ranges from $1/6 = 0.1667$, when $(p_2,p_5) = (1,0)$ or $(0,1)$, to $1/3 = 0.3333$, when $(p_2,p_5) = (0,0)$ or $(1,1)$.

On the other hand,

$$\Lambda_3[v;B_R] = \frac{2-p_1-p_4+2p_1p_4}{6},$$

which depends on p_1 and p_4 and also ranges from $1/6 = 0.1667$, when $(p_1,p_4) = (1,0)$ or $(0,1)$, to $1/3 = 0.3333$, when $(p_1,p_4) = (0,0)$ or $(1,1)$.

In spite of the apparent structural symmetry between B_L and B_R, from Table 3 it follows that *player 3 prefers in general B_R whatever is the value of parameter p_1* because $\Lambda_3[v;B_R] \geq \Lambda_3[v;B_L]$. This player is indifferent in the extreme cases $p_1 = 0$ and $p_1 = 1$; otherwise, the probability that this player chooses B_R is at least 0.55 and tends to 1 when p_1 tends to 1/2.

When considering the coalition structure $B_C = \{\{1,4\},\{2,5\},3\}$, at first glance player 3 might think that, once the protocoalitions $\{1,4\}$ and $\{2,5\}$ are formed, each one of them would be very interested in opening a further negotiation with this player to form a winning coalition (of second order): either $\{\{1,4\},3\}$ or $\{\{2,5\},3\}$. Player 3 might think that entering one of these coalitions would imply getting a payoff greater than in $\{1,3,4\}$ or $\{2,3,5\}$, respectively. As it is shown in Table 5, the payoff obtained by player 3 in both cases would be 1/2.

Following Table 5 and comparing the payoff to player 3 under each coalition structure, it is easy to find that this player is indifferent between B'_L and B'_R. Moreover, it strictly prefers B'_L to B_L and B'_R to B_R for any values of the involved parameters p_2 and p_5 or p_1 and p_4, respectively. From Table 5 again, this player also prefers B'_L and B'_R to B'_M, which in turn is preferred to B_L

and B_R with the sole exception of a few cases where indifference occurs. With a symbolic notation, the preferences of player 3 can be described by

$$B'_L \equiv B'_R > B'_M \geq B_L \equiv B_R.$$

However, we should take into account the viewpoint of the other players, especially the main ones. We can see that players 1 and 4 prefer, as a single player, B_L instead of B'_L, and players 2 and 5 prefer B_R instead of B'_R.

Table 6 and Table 7 display the payoffs obtained by the players for different values of $\mathbf{p}$ and different coalition structures.

	B_L					B'_L				
$(p_1, p_2, p_3, p_4, p_5)$	1	2	3	4	5	1	2	3	4	5
$(0.6, 0.6, 0.2, 0, 0)$	0.533	0	0.233	0.233	0	0.250	0	0.500	0.250	0
$(0.2, 0.2, 0.4, 0.2, 0.2)$	0.440	0	0.280	0.280	0	0.250	0	0.500	0.250	0
$(0.1, 0.3, 0.5, 0.4, 0.1)$	0.447	0	0.277	0.277	0	0.250	0	0.500	0.250	0
$(0.1, 0.5, 0.5, 0.4, 0.2)$	0.500	0	0.250	0.250	0	0.250	0	0.500	0.250	0
$(1, 1, 1, 1, 1)$	0.333	0	0.333	0.333	0	0.250	0	0.500	0.250	0
$(0, 1, 0.5, 1, 0)$	0.667	0	0.167	0.167	0	0.250	0	0.500	0.250	0

Table 6: Power distribution in $[v; B_L]$ and $[v; B'_L]$ for different values of $\mathbf{p}$

	B_R					B'_R				
$(p_1, p_2, p_3, p_4, p_5)$	1	2	3	4	5	1	2	3	4	5
$(0.6, 0.6, 0.2, 0, 0)$	0	0.533	0.233	0	0.233	0	0.250	0.500	0	0.250
$(0.2, 0.2, 0.4, 0.2, 0.2)$	0	0.440	0.280	0	0.280	0	0.250	0.500	0	0.250
$(0.1, 0.3, 0.5, 0.4, 0.1)$	0	0.473	0.263	0	0.263	0	0.250	0.500	0	0.250
$(0.1, 0.5, 0.5, 0.2, 0.2)$	0	0.500	0.250	0	0.250	0	0.250	0.500	0	0.250
$(1, 1, 1, 1, 1)$	0	0.333	0.333	0	0.333	0	0.250	0.500	0	0.250
$(0, 1, 0.5, 1, 0)$	0	0.677	0.167	0	0.167	0	0.250	0.500	0	0.250

Table 7: Power distribution in $[v; B_R]$ and $[v; B'_R]$ for different values of $\mathbf{p}$

Summing up, the assumption that player 3 would prefer that, first, the protocoalitions form and then this party chooses joining one of them is rather illusory. The main parties 1 and 2, with their respective minor partners 4 and 5, absolutely and strictly prefer B_L and B_R, respectively. Therefore player 3 prefers B_R according to Table 3. This means that *our initial conclusion is the most plausible and, incidentally,* ***it fits well the actual behavior of the parties***.

5 Conclusions

The paper is essentially self–contained, and provides in Section 3 a new axiomatic characterization of the coalitional multinomial probabilistic values for games with a coalition structure on the full domain of all cooperative games. It is the second axiomatization of these values obtained from [8] by just replacing the property of symmetry within unions with the balanced contributions property, a property of different nature very appreciated in cooperative game theory. The elaboration on the balanced contributions property, which follows from Lemma 3.1, leads to an equivalent statement of this property exclusively referred to games in the player set and not to any subgame.

In Section 3.2 we have compared our axiomatic characterization with parallel characterizations of the classic, non-parametric coalitional values, as well as the symmetric coalitional binomial semivalues. A replication of the axiomatic characterization for coalitional power indices on simple games can be achieved by just replacing linearity with the classic transfer property. Moreover, the logical independence of the two corresponding sets of axioms has been checked in the Appendix.

The intensive use of unanimity games in the theoretical part gives rise to easier proofs. They are also used for the computation of values in practice (numerical examples), thus avoiding the recourse to the more cumbersome multilineal extension procedure.

Finally, in Section 4 the application to two actual examples of different difficulty degree illustrates the analysis of a game in terms of coalitional multinomial probabilistic values, showing that this greatly enlarge the set of strategic options of the players with respect to classic values.

Acknowledgments

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Appendix: logical independence

Proposition A.1 *The axiomatic system used in Theorem 3.2 is logically independent.*

Proof: We will assume that a player set N (with $n = |N| \geq 2$) and a positive tendency profile $\mathbf{p}$ in N are given. We will abbreviate the properties as follows: LI = linearity, DP = dummy player property, BC = balanced contributions within unions, TP = coalitional $\mathbf{p}$–multinomial total power property, and WU = property of $\mathbf{p}$–weighted payoffs for quotients of unanimity games.

1. LI is logically independent of DP, BC, TP and WU.

We define a coalitional value g for all $[v; B] \in \mathcal{G}_N^{cs}$ as follows.

(a) Unanimity games. If $v = u^T$ with $\emptyset \neq T \subseteq N$ then

$$g_i[v; B] = \Lambda_i^{\mathbf{p}}[v; B] \quad \text{for each } i \in N.$$

(b) Otherwise, that is, if v is not a unanimity game,

$$g_i[v; B] = \begin{cases} \Lambda_i^{\mathbf{p}}[v; B] = v(\{i\}) & \text{if } i \text{ is a dummy in } v, \\ \dfrac{1}{b'_k} \displaystyle\sum_{j \in B'_k} \Lambda_j^{\mathbf{p}}[v; B] & \text{if } i \in B_k \in B \text{ is not a dummy in } v, \end{cases}$$

where B'_k denotes the set of players of B_k that are not dummies in v, and $b'_k = |B'_k|$.

DP, BC, TP and WU are clearly satisfied by g. However, this value fails to satisfy LI for e.g. game $v = u^{\{1\}} + 2u^{\{1,2\}}$ and the trivial coalition structure $B = B^N$.

2. DP is logically independent of LI, BC, TP and WU.

We define a coalitional value g for all $[v; B] \in \mathcal{G}_N^{cs}$ as follows. For each $i \in N$,

$$g_i[v; B] = \frac{v(N)}{n} \quad \text{if } B = B^N,$$

and otherwise

$$g_i[v;B] = \Lambda_i^{\mathbf{p}}[v;B].$$

LI, BC, TP and WU are clearly satisfied by g. However, this value fails to satisfy DP for the unanimity game $v = u^{\{1\}}$ and the trivial coalition structure $B = B^N$.

3. BC is logically independent of LI, DP, TP and WU.

We define a coalitional value g for all $[v;B] \in \mathcal{G}_N^{cs}$ as follows.
(a) If $n = 2$ and $B = B^N$, then

$$\begin{cases} g_1[v;B] = v(\{1\}), \\ g_2[v;B] = v(N) - v(\{1\}). \end{cases}$$

(b) Otherwise, that is, if some of the above conditions does not hold,

$$g_i[v;B] = \Lambda_i^{\mathbf{p}}[v;B] \quad \text{for each } i \in N.$$

LI, DP, TP and WU are clearly satisfied by g. However, this value fails to satisfy BC for the unanimity game $v = u^{\{1,2\}}$ and the trivial coalition structure $B = B^N$.

4. TP is logically independent of LI, DP, BC and WU.

We define a coalitional value g for all $[v;B] \in \mathcal{G}_N^{cs}$ as follows.
(a) If $n = 2$ and $B = B^n$, then

$$\begin{cases} g_1[v;B] = v(\{1\}), \\ g_2[v;B] = v(\{2\}). \end{cases}$$

(b) Otherwise, that is, if some of the above conditions does not hold,

$$g_i[v;B] = \Lambda_i^{\mathbf{p}}[v;B] \quad \text{for each } i \in N.$$

LI, DP, BC and WU are clearly satisfied by g. However, this value fails to satisfy TP for the unanimity game $v = u^{\{1,2\}}$ and the trivial coalition structure $B = B^n$.

5. WU is logically independent of LI, DP, BC and TP.

Let α, β be real numbers such that $\alpha + \beta = p_1 + p_2$ and $\alpha p_1 \neq \beta p_2$. We define a coalitional value g for all $[v;B] \in \mathcal{G}_N^{cs}$ as follows.
(a) If $n = 2$ and $B = B^n$, then

$$\begin{cases} g_1[v;B] = v(\{1\}) + \alpha[v(N) - v(\{1\}) - v(\{2\})], \\ g_2[v;B] = v(\{2\}) + \beta[v(N) - v(\{1\}) - v(\{2\})]. \end{cases}$$

(b) Otherwise, that is, if some of the above conditions does not hold,

$$g_i[v;B] = \Lambda_i^{\mathbf{p}}[v;B] \quad \text{for each } i \in N.$$

LI, DP, BC and TP are clearly satisfied by g. However, this value fails to satisfy WU for the unanimity game $v = u^{\{1,2\}}$ and the trivial coalition structure $B = B^n$. $\square$

Proposition A.2 *The axiomatic system used in Theorem 3.4 is logically independent.*

Proof: We continue to assume that a player set N (with $n = |N| \geq 2$) and a positive tendency profile $\mathbf{p}$ in N are given. We will continue to abbreviate the properties as follows: LI = linearity, DP = dummy player property, BC = balanced contributions within unions, TP = coalitional

$\mathbf{p}$–multinomial total power property, and WU = property of $\mathbf{p}$–weighted payoffs for quotients of unanimity games. We will also abbreviate TR = transfer property.

Let us consider now the restriction to simple games of each coalitional value g defined for Proposition A.1 in parts 2, 3, 4 and 5. In each case, this restriction is a coalitional power index that satisfies the desired properties with the sole exception of linearity, which does not make sense for simple games. But the transfer property is also satisfied by all these coalitional values. The argument is as follows.

The composition laws $\vee$ and $\wedge$ make sense and satisfy

$$u \vee v + u \wedge v = u + v,$$

for all cooperative games. If u and v are simple games and, for a while, we think of them as cooperative games then, using linearity, we obtain for each g used in Proposition A.1

$$g[u \vee v; B] + g[u \wedge v; B] = g[u; B] + g[v; B] \quad \text{for all } B,$$

which is precisely TR.

Moreover, all counterexamples provided in cases 2, 3, 4 and 5 for Proposition A.1 are simple games, so they can be used also for this Proposition A.2. Thus, it only remains to check the following.

1bis. TR is logically independent of DP, BC, TP and WU.

We define a coalitional power index g for all $[v; B] \in \mathcal{SG}_N^{cs}$ as follows.
(a) If $n = 2$, $v = u^{\{1\}} \vee u^{\{2\}}$ and $B = B^n$, then

$$\begin{cases} g_1[v; B] = 2, \\ g_2[v; B] = -p_1 - p_2. \end{cases}$$

(b) Otherwise

$$g_i[v; B] = \Lambda_i^{\mathbf{p}}[v; B] \quad \text{for each } i \in N.$$

DP, BC, TP and WU are clearly satisfied by g. However, this power index fails to satisfy TR for game $v = u^{\{1\}} \vee u^{\{2\}}$ and the trivial coalition structure $B = B^n$. $\square$

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