INVOLUTORY AUTOMORPHISMS OF GROUPS
OF ODD ORDER

A thesis submitted for the degree

of

DOCTOR OF PHILOSOPHY

by

James N. Ward

based on research done under the supervision of

Professor G.E. Wall.

Department of Mathematics
(Pure Mathematics).

UNIVERSITY OF SYDNEY.

INVOLUTORY AUTOMORPHISMS OF GROUPS OF ODD ORDER

A thesis submitted for the degree

of

Doctor of Philosophy

by

J. N. Ward

SUMMARY

One of the most important and far reaching concepts in mathematics is that of a group. It is to this branch of mathematics that the work in this thesis belongs.

An automorphism of a group is an expression of the symmetry of the group structure. If there is sufficient symmetry then it is to be expected that the structure will be restricted. The purpose of this thesis is to make a precise statement of this relationship for the case in which the symmetry is given by an involutory automorphism.

Although the investigation of this problem dates back to Burnside at the end of the nineteenth century, it is only during the last ten years that any reasonably general results have been obtained. A survey of recent work on the problem for the case of finite groups with arbitrary automorphisms is given in the introduction. There is also an explicit statement of the results proved in the thesis together with an indication of the method of proof.

Included in the second chapter is a summary of the theorems needed for the proofs in later chapters. These results are either group theoretic or belong to the theory of representations of finite groups. It is interesting to notice that it is not necessary to use any of the more recent deep theorems of group theory in an essential way.

Chapters 3 and 4 deal with preliminary results which are needed for the later theorems. The division into chapters corresponds to group theoretic results and results depending on the theory of group representations. At the end of chapter 4 there is an application of the earlier results to prove a theorem on the structure of groups expressible as a product of subgroups of a particular kind.

The remaining two chapters are devoted to the proof of the main theorems. Some examples are also included to indicate the strength of the theorems.

PREFACE

All the theorems proved in this thesis are original. The lemmas are also all original with the exception of the following lemmas in Chapter III. Lemma 3 is a generalization of a result proved by Dr. Kovacs and Professor Wall in [14]. Lemma 5 and its corollaries have appeared in various forms, the method of proof being essentially due to Burnside. Lemma 4 is probably well known and whilst lemma 2 does not appear to have been proved in the literature, the method of proof is well known.

The first two lemmas in the proof of the theorem in Chapter V follow Kovacs and Wall [14] fairly closely.

I wish to thank my supervisor, Professor G.E. Wall, for his guidance and for the simplifications which he has suggested in several of the proofs.

CHAPTER I

INTRODUCTION

A problem which frequently arises in finite group theory is the interpretation of information given about the action of an automorphism ω on a finite group G in terms of the structure of G. The object of this thesis is to examine some special cases of this problem in which the automorphisms are of order 2. As an example of the kind of result which is obtained we have[1]: if the finite group G of odd order is acted on by an automorphism ω of order 2 such that G_ω is a Hall subgroup of G then there exists a normal abelian complement of G_ω in G.

The earliest problems of this kind arose out of the study of Frobenius groups; the groups which can arise as the regular subgroup of a Frobenius group are precisely those groups with a regular[2] automorphism of prime order. Burnside proved in the first edition of his "Theory of Groups of Finite Order" that the finite groups with a regular automorphism of order 2 are precisely the abelian groups of odd order. General results about the structure of a group with a regular automorphism of order p have only been obtained comparatively recently.

---

[1]The notation used in this Chapter is explained in Chapter II.

[2]An automorphism ω of a group G is said to be regular if $x^\omega = x$ for $x \in G$ implies $x = 1$.

Higman [13][3] has shown that if such a group is soluble then it is nilpotent of bounded class, the bound being a function of p. Thompson [15] succeeded in showing that the group must be soluble.

Some other conditions for solubility involving the action of automorphisms on a finite group have been obtained. Fischer [7] has shown that if the finite group G has a regular automorphism ω of order $2p$, where p is a prime, then G is soluble if G_{ω^p} either is a 2-group or contains a Sylow 2-subgroup of G. Fischer's proof depends on the Feit-Thompson theorem that groups of odd order are soluble, the Higman-Thompson result already mentioned and the theorem of Brauer and Suzuki on groups whose Sylow 2-subgroup is a quaternion group. Earlier Gorenstein and Hernstein [9] had proved independently of the Feit-Thompson theorem that a finite group with a regular automorphism of order 4 is soluble. They also showed that for such a group G' is nilpotent.

It is also of interest to suppose at the outset that the group is soluble and then to seek additional properties. In this vein, Alperin [1] has generalized Higman's result by proving that if a finite soluble group G has an automorphism ω of prime order p and if $|G_\omega| = p^n$, then the derived length of G is bounded by a number depending only on p and n.

---

[3] Numbers in square brackets refer to the references.

However the most general results on automorphisms of soluble groups are due to Thompson [16]. If p is a prime, let $O_p(G)$ denote the largest normal p-subgroup of G and $O_{p'}(G)$ the largest normal subgroup of G of order prime to p. Define $O_{p_1p_2\ldots p_r}(G)$ inductively as the inverse image in G of $O_{p_r}(G/O_{p_1\ldots p_{r-1}}(G))$. Then Thompson's theorem may be stated as follows: if G is a finite soluble group with an automorphism ω of prime order not dividing the order of G, then for any prime q

(a) $O_q(G_\omega) \leq F_3(G)$ if $q \neq 2$.

(b) $O_2(G_\omega) \leq F_4(G)$

(c) $O_q(G_\omega) \leq O_{q,q',q}(G)$.

Suppose that G is a group of odd order with an automorphism ω of order 2 such that G_ω is nilpotent. Such a group is soluble by the Feit-Thompson theorem. Kovacs and Wall [14] have shown that in these circumstances $G/F(G)$ is nilpotent provided that all the Sylow subgroups of G_ω are regular and that whenever $G/F(G)$ is nilpotent, $G/F(G)$ is contained in the variety generated by G_ω and the class of abelian groups. The principal theorem proved in this thesis gives information when $G/F(G)$ is not nilpotent. We show that $G = F_3(G)$ and that if $(G_\omega)^{(r)} = 1$ then $G^{(r)}$ is nilpotent. These results are the best possible in the sense that there do exist groups G of the kind under consideration for which $G \neq F_2(G)$ or $G^{(r-1)}$ is not nilpotent when G_ω

has derived length r. The requirement that G_ω be nilpotent is essential. Indeed given any integer n there exist groups G with involutory automorphisms possessing metabelian fixed point groups for which $G^{(n)}$ is not nilpotent.

The principal theorem is proved by induction on the order of the group and by way of contradiction. The Kovacs-Wall theorem is assumed for the special case in which G_ω is abelian but this could be avoided. An initial group theoretical reduction provides a faithful irreducible representation of $G/F(G)$. This representation is used together with Clifford's theory on the restriction of an irreducible representation to a normal subgroup to find further group properties. An extensive group theoretical analysis then leads to the final contradiction. Some results of P.Hall on the system normalizers of a soluble group are employed in the final stages of the proof.

We also consider groups of odd order which possess several automorphisms of order 2. If a group G of odd order is operated on by a group of automorphisms of order 8 and exponent 2 such that for each non-trivial automorphism belonging to the group the fixed point group is nilpotent, then G is nilpotent. This is not true in general if the group of automorphisms is only of order 4, the other conditions all being satisfied; however in this case we can say that G' is nilpotent. It is perhaps of interest to note that G need not be supersoluble.

In conclusion, we mention a result which arose out of the proof of the main theorem: if the finite group G contains two complementary Hall subgroups, one of which is abelian and the other nilpotent of derived length r then $G^{(r)}$ is nilpotent. The solubility of G is obtained from a theorem of Wielandt. Results of this kind seem to be scarce in the literature.

CHAPTER II

NOTATION AND SUMMARY OF KNOWN RESULTS

The following notation is, for the most part, well known. All groups in this thesis are finite.

Let G denote a finite group.

$\|G\|$	the order of G
$\phi(G)$	the Frattini subgroup of G
$Z(G)$	the centre of G
G'	the derived group of G
$G^{(r)}$	the r-th derived group of G, defined inductively by $G^{(o)} = G$, $G^{(r+1)} = (G^{(r)})'$
$F(G)$	the Fitting subgroup of G, the largest normal nilpotent subgroup of G
$F_n(G)$	the n-th term of the upper Fitting series of G defined inductively by $F_1(G) = F(G)$, $F_{n+1}(G) =$ the inverse image in G of $F(G/F_n(G))$
1	the unit of a group, or the subgroup of a group containing only the unit, according to the context.

Let $x, y, \ldots$ denote elements of G and $H, K, \ldots$ subgroups of G.

$\|G:H\|$	the index of H in G
$H \cap K$	the intersection of H and K
$\{H,K\}$	the least subgroup of G containing all the elements of both H and K. If K is normalized by H, $\{K,H\} = HK$.

H is disjoint from K — $H \cap K = 1$.

$\{x, y, \ldots ; H, K, \ldots\}$ — the least subgroup of G containing all the elements $x, y, \ldots$ of G and all the elements of each of the subgroups $H, K \ldots$ of G.

$H \leq K$ — H is a subgroup of K, $H = 1$ and $H = K$ are allowed.

$H < K$ — H is a proper subgroup of K, $H = 1$ is allowed but not $H = K$.

$H \triangleleft K$ — H is a normal subgroup of K, $H = 1$ and $H = K$ are allowed.

H is a non-trivial subgroup of G. $H < K$, $H \neq 1$.

If $K \triangleleft H$ we call H/K a section of G.

$C_H(K)$ — the centralizer of K in H.

$N_H(K)$ — the normalizer of K in H.

If L and M are groups, $L \times M$ denotes the direct product of L and M.

Let ω denote an automorphism of G and A a group of automorphisms of G.

If $x \in G$ we denote the image of x under the automorphism ω by x^{ω}.

H^{ω} — The collection of elements x^{ω} where $x \in H$ forms a subgroup denoted by H^{ω}.

ω is a regular automorphism of G : $x^{\omega} = x$ only if $x = 1$.

H/K is a ω-section of G: H/K is a section of G, $H^{\omega} = H$ and $K^{\omega} = K$.

H/K is an A-section of G: H/K is a ω-section of G for all automorphisms ω in A.

G_ω The centralizer of ω in G.

Let $\mathcal{L}$ be a field and G a group. Then $\mathcal{L}(G)$ denotes the group algebra of G over $\mathcal{L}$. If V is an $\mathcal{L}(G)$-module, we write scalars as left operators on V and elements of $\mathcal{L}(G)$ as right operators on V. The statement "V is an $\mathcal{L}(G)$-module" may be abbreviated to "V is a G-module" when it is clear from the context that we are working over the field $\mathcal{L}$. If W_1 and W_2 are $\mathcal{L}(G)$-submodules of V, $W_1 + W_2$ is the smallest $\mathcal{L}(G)$-submodule of V containing W_1 and W_2. We write $W_1 \oplus W_2$ for $W_1 + W_2$ if $W_1 \cap W_2 = 0$, the submodule of V containing only the zero element, 0, of V. If $\mathcal{L}_1$ is an extension field of $\mathcal{L}$, $V^{\mathcal{L}_1}$ denotes the $\mathcal{L}_1(G)$-module obtained by extending the scalar field of V from $\mathcal{L}$ to $\mathcal{L}_1$.

$\mathcal{F}$ denotes the algebraic closure of $GF(p)$, the Galois field with p-elements.

Some further definitions are contained in the summary of known results.

The following results are assumed to be known but are not explicitly mentioned each time that they are used.

The modular law.

Let L, M and N be subgroups of G such that $L \leq N$. Then $L(M \cap N) = LM \cap N$.

([10] Theorem 8,4.1.)

The fundamental property of the Frattini subgroup.

If $L \leq G$ and $L\phi(G) = G$ then $L = G$.

([10] Theorem 10.4.1)

The Frattini subgroup of a p-group contains the derived group and the p-th power of any element in the group.

([10] Theorem 10.4.3)

Nilpotent groups.

The following statements are equivalent.

a) The group G is nilpotent.

b) Some term of the upper central series of G is equal to G .

c) G is a direct product of its Sylow subgroups.

d) Elements of relatively prime order in G commute.

e) All the maximal subgroups of G are normal in G .

([10] section 10.3)

Hall subgroups of soluble groups.

Let G be a soluble group of order mn where m and n are relatively prime. Then

1) G possesses at least one subgroup of order m .

2) Any two subgroups of order m are conjugate.

3) Any subgroup whose order m' divides m is contained in a subgroup of order m .

4) The number h_m of subgroups of order m may be expressed as a product of factors, each of which is a power of a prime dividing the order of G .

([10] Theorem 9,3,1.)

Subgroups of a group whose order and index are relatively prime are called Hall subgroups. If p is a prime, a Hall p'-subgroup of the group G is a subgroup H whose index is a power of p and whose order is prime to p.

The Feit-Thompson Theorem.

Groups of odd order are soluble. [6] .

The Schur-Zassenhaus Theorem.

Let G be a finite group of order mn containing a normal subgroup K of order m, where m and n are relatively prime. Then there exists a complement of K in G and all such complements are conjugate.

([10] Theorem 15.2.2 and [18] p. 162, Theorem 27)

If G is a finite group with a regular automorphism of order 2 then G is abelian of odd order and the automorphism maps each element onto its inverse. ([3] p.90)

If G is a soluble group then $C_G(F(G)) \leq F(G)$.([2]p.646)

$F(G/\phi(G)) = F(G)/\phi(G)$. ([2] p. 647).

A group G is called the splitting extension of K by H if K is a normal subgroup of G and H is a subgroup of G whose elements may be taken as the coset representatives of K. Given two groups H and K and for every element $h \in H$ an automorphism of K, $k \longleftrightarrow k^h$ all $k \in K$, such that $k^{h_1h_2} = (k^{h_1})^{h_2}$, $h_1, h_2 \in H$ there exists a group G which is the splitting extension of K by H. ([10], section 6.5.)

The remaining results are numbered and will be referred to by means of these numbers each time that they are used.

1) If G contains two nilpotent complementary Hall subgroups then G is soluble. [17]

2) If N is a normal subgroup of the group G then $\phi(N) \leq \phi(G)$. ([8] page 162.)

3) A set of pairwise permutable Sylow subgroups of a group G, one for each prime dividing the order of G, is called a Sylow system for G. If G is a soluble group then there exists a Sylow system for G and all such systems are conjugate. The intersection of the normalizers of the subgroups of a Sylow system is called the system normalizer. The set of system normalizers forms a characteristic class of conjugate subgroups of G. Let D be a system normalizer of G. If $G = G_o > G_1 > \dots > G_r = 1$ is a chief series for G then $G_{i+1} D \geq G_i$ if G centralizes G_i/G_{i+1} (the covering theorem) whilst $G_i \cap D \leq G_{i+1}$ if $(G,G_i)G_{i+1} = G_i$ (the avoiding theorem). These results may all be found in [11] and [12].

4) If $G = F_2(G)$ then the system normalizers of G are their own normalizers ([4] theorem 5.6).

5) Let G be a non nilpotent soluble group and suppose that $F(G)$ is the unique minimal normal subgroup of G. Then $F(G)$ is complemented in G. ([2] page 651.)

6) If ω is an automorphism, of order 2, of G and the order of G is odd then $F(G_\omega) \leq F_3(G)$. [16].

7) Let G be a group of odd order with an automorphism ω of order 2. Suppose that G_ω is abelian. Then $G' \leq F(G)$. ([14] page 114).

8) Clifford's Theorem and Related Results.

Let G denote a group, H a normal subgroup of G and $\mathcal{L}$ an arbitrary field. Suppose W is a right $\mathcal{L}(H)$-module. For a fixed $g \in G$ let $W^{(g)}$ be the right $\mathcal{L}(H)$-module whose underlying vector space is W and on which H acts according to the rule $w \circ h = w\, g^{-1}hg$ for each $w \in W$, $h \in H$, where $w \circ h$ denotes the module operation in $W^{(g)}$, and wh the operation in W. $W^{(g)}$ is called a conjugate of W.

Now let V be an irreducible $\mathcal{L}(G)$-module, then V considered as an $\mathcal{L}(H)$-module is completely reducible and the irreducible $\mathcal{L}(H)$-submodules of V are all conjugates of each other. Therefore as an $\mathcal{L}(H)$-module $V = \bigoplus_{i=1}^{s} Wg_i$ where W is an irreducible $\mathcal{L}(H)$-submodule of V and the Wg_i are certain conjugates of W. Suppose that $Wg_1, \ldots, Wg_r$ form a maximal set of non-isomorphic conjugates of W and for each i, $1 \leq i \leq r$, let V_i be the sum of all the conjugates Wg_j, $1 \leq j \leq s$, such that $Wg_j \cong Wg_i$ as right $\mathcal{L}(H)$-modules. Then

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r .$$

The V_i are uniquely determined and are called the homogeneous components of V as an $\mathcal{L}(H)$-module. G acts as a transitive permutation group on the V_i. The number r of distinct homogeneous components of V is equal to the index $|G:H^*|$ where H^* is the subgroup of G consisting of all $g \in G$ such that $Wg \cong W$. ([5], section 49.)

9) Let G be a finite group and k a perfect field. Then there exists a splitting field $\mathcal{E}$, containing k , for G such that $|\mathcal{E} : k|$ is finite. ([5], theorem 69.11).

10) Let G be a finite group, k a perfect field and $\mathcal{E}$ a splitting field for G which is a finite normal extension of k . Then if V is an irreducible $k(G)$-module, $V^{\mathcal{E}}$ is a completely reducible $\mathcal{E}(G)$-module and the irreducible submodules of $V^{\mathcal{E}}$ are algebraically conjugate. ([5], theorem 70.15.)

CHAPTER III

PRELIMINARY LEMMAS

Lemma 1. Let P be a p-group and H a proper subgroup of P. Then

$$|P:H| > |P':P' \cap H| .$$

Proof. Since $|P:P' \cap H| = |P:H||H:P' \cap H| = |P:P'||P':P' \cap H|$ it is sufficient to prove that $|P:P'| > |H:P' \cap H|$. Now P' is a normal subgroup of P so by the isomorphism theorem $P'H/P' \cong H/P' \cap H$. Therefore we need only prove that $|P:P'| > |P'H:P'|$. But this is true unless $P'H = P$.

Since P is a p-group, $P' \leq \phi(P)$, so that $P'H = P$ only if $H = P$ by the fundamental property of the Frattini subgroup. This contradicts the assumption that H is a proper subgroup of P and therefore the lemma is proved.

Lemma 2. Let G be a group with an automorphism ω of prime order p where $p \nmid |G|$. Suppose that $F(G)$ is abelian and that $Z(F_2(G)) = 1$. Then there exists an ω-complement of $F(G)$ in G.

Proof. There exists a Sylow system of $F_2(G)$ since $F_2(G)$ is soluble. (See Chapter II (3)). Consider the set of system normalizers of $F_2(G)$. They form a characteristic class of conjugate subgroups of $F_2(G)$. Since $p \nmid |F_2(G)|$ it follows that we can find an ω-invariant system normalizer D of $F_2(G)$.

By the covering theorem $DF(G) = F_2(G)$. On the other hand by the avoiding theorem if K is a minimal normal subgroup of $F_2(G)$ contained in $D \cap F(G)$ then $(D,K) = 1$. But $(F(G),K) = 1$ since $F(G)$ is abelian so $K \leq Z(DF(G)) = Z(F_2(G)) = 1$. Thus $D \cap F(G) = 1$ and D is an ω-complement of $F(G)$ in $F_2(G)$.

We now show that $R = N_G(D)$ is an ω-complement of $F(G)$ in G. Since D is its own normalizer in $F_2(G)$ (Chapter II (4)) $R \cap F(G) = N_{F_2(G)}(D) \cap F(G) = D \cap F(G) = 1$. Since the system normalizers of $F_2(G)$ form a characteristic system of conjugate subgroups of $F_2(G)$ if $x \in G$ then $x^{-1}Dx$ is also a system normalizer of $F_2(G)$ so that $x^{-1}Dx = y^{-1}Dy$ for some $y \in F_2(G)$. Since $F_2(G) = DF(G)$ we may suppose that $y \in F(G)$. Then $xy^{-1} \in N_G(D) = R$ so $x \in Ry$. Since x was an arbitrary element of G, we have that $G = RF(G)$. D is ω-invariant so $R = N_G(D)$ is also ω-invariant and complements $F(G)$ in G.

Lemma 3. Let G be a soluble group operated on by a group A of automorphisms. Suppose that G satisfies the following condition:

for some pair of non-negative integers (m,n) $(n > 0)$ $G^{(m)}$ is not contained in $F_n(G)$; but if H/K is any A-section of G different from $G/1$, then $(H/K)^{(m)} \leq F_n(H/K)$. Then if H is a normal A-subgroup of G different from 1, $F(G) \leq H$ and $F(G)$ is an elementary abelian p-group for some prime p.

<u>Proof</u>. Suppose that there exist two disjoint proper normal A-subgroups of G, say H and K. Then G is isomorphic to a subgroup of $G/H \times G/K$ so that $G^{(m)} \leq F_n(G)$. This is a contradiction so that the intersection M of all the non-trivial normal A-subgroups of G is a non-trivial normal A-subgroup of G. Since $n > 0$, M is a proper normal A-subgroup of G. G is soluble and M is characteristically simple so M is an elementary abelian p-group for some prime p. Clearly $F(G), \geq M$, is also a p-group.

Let Φ denote the Frattini subgroup of G. Since $F(G/\Phi) = F(G)/\Phi$, $F_n(G/\Phi) = F_n(G)/\Phi$. Now if $\Phi > 1$, then $(G/\Phi)^{(m)} \leq F_n(G/\Phi)$ so $G^{(m)}\Phi/\Phi \leq F_n(G)/\Phi$. Therefore $G^{(m)} \leq F_n(G)$ contradicting the hypothesis. Hence $\Phi = 1$. It follows, by Chapter II(2), that $F(G)$ is an elementary abelian p-group.

Write $H/M = F(G/M)$. Then $F(G) \leq H$ and if P/M is the Sylow p-subgroup of H/M then, since M is a p-group, P is a normal p-subgroup of G so that $P \leq F(G)$. Hence $P = F(G)$. Now by hypothesis $(G/M)^{(m)} \leq F_n(G/M)$ whilst $G^{(m)}$ is not contained in $F_n(G)$. Therefore, as $G^{(m)}/M = (G/M)^{(m)}$, H properly contains $F(G)$.

Since $F(G)$ is an elementary abelian normal Sylow p-subgroup of H, $F(G)$ is a completely reducible $H/F(G)$-module. But M is an $H/F(G)$-submodule of $F(G)$ so $F(G) = M \times N$ where N is also an $H/F(G)$-submodule of $F(G)$.

Since H/M is nilpotent and $F(G)$ is abelian, $N \leq Z(H)$. Now suppose that $Z(H) > 1$. Then $Z(H)$ is a characteristic subgroup of H, a normal A-subgroup of G. Hence $Z(H)$ is a normal A-subgroup of G and therefore contains M. Now H/M is nilpotent so the upper central series of H terminates at H. But this implies that H is nilpotent so that $H \leq F(G)$ contradicting the conclusion of the previous paragraph. Therefore $N \leq Z(H) = 1$ and $M = F(G)$ proving the lemma.

<u>Lemma 4</u>. Let H be an arbitrary finite group and q a prime which does not divide the order of H. Let A denote the group algebra of H over $GF(q)$, the field with q elements, and let G denote the set of mappings

$$x \rightarrow ax + b$$

of A into itself where a runs over H and b runs over A. If we take the composition of maps as a product in G then G is a group. The Fitting subgroup of G consists of the maps of the form $x \rightarrow x + b$ where $b \in A$. $G/F(G) \cong H$.

<u>Proof</u>. The element $x \rightarrow ax + b$ may be denoted by the pair (a,b) where $a \in H$ and $b \in A$. Then the product operation in G is given by

$$(a,b) \circ (c,d) = (ac, ad + b) .$$

It is now easy to verify that G is a group and that the set N of elements of G of the form $(1,b)$ where $b \in A$ is a normal abelian subgroup of G.

The map $\phi : G \to H$ defined by

$$\phi((a,b)) = a$$

is a homomorphism of G onto H with kernel N. Thus $G/N \cong H$.

Finally we prove that $N = F(G)$. Since N is a normal abelian subgroup of G, $N \leq F(G)$. For $b \in A$, $(1,b)^q = (1,qb) = (1,0)$ since A is the group algebra of H over $GF(q)$. Thus every non-trivial element of N has order equal to the prime q. Since $|G:N| = |H|$ is prime to q, N is the Sylow q-subgroup of G. N is normal and abelian so $N \leq Z(F(G))$. Suppose that $(a,b) \in F(G)$. Then since $N \leq F(G)$, $(a,0) \in F(G)$. But then $(a,0)$ must commute with all the elements of N so

$$ac = c$$

for all $c \in A$. This implies that $a = 1$ and hence that $F(G) = N$. The lemma is now established.

<u>Lemma 5</u>. Let G be a group of odd order with an automorphism ω of order 2. Then there exists precisely one element of G, which is inverted by ω, in each left (right) coset of G_ω.

<u>Proof</u>. Consider the elements of G which can be written in the form $x^{-1}x^\omega$ for some x G. Since $\omega^2 = 1$, any such element is inverted by ω. Now suppose that

$$x^{-1}x^\omega = y^{-1}y^\omega .$$

Then $x^{\omega}(y^{-1})^{\omega} = xy^{-1}$ so that $xy^{-1} \in G_{\omega}$. Hence x and y belong to the same left coset of G_{ω}. Thus the number of elements of G inverted by ω is at least as many as the index of G_{ω} in G.

Now suppose that x and y are two elements of G which are both inverted by ω and $xy^{-1} \in G_{\omega}$. Then applying ω we obtain $x^{-1}y = xy^{-1}$ so that $x^2 = y^2$. Since the order of G is odd we deduce that $x = y$. This proves the lemma for left cosets. A similar proof applies to right cosets.

Corollary 1. Let G be a group of odd order with an automorphism ω of order 2. Every element of G may be expressed as the product of an element fixed by ω and an element inverted by ω.

Corollary 2. Let G be a group of odd order with an automorphism ω of order 2. Let H be a subgroup of G containing G_{ω}. Then $H^{\omega} = H$.

Proof. Let $x \in H$. We require to prove that $x^{\omega} \in H$. Now $x = yz$ where $y^{\omega} = y^{-1}$ and $z^{\omega} = z$ by corollary 1. Since $G_{\omega} \leq H$, $z \in H$ so $y = xz^{-1} \in H$. Thus $x^{\omega} = y^{-1}z \in H$ as required.

Corollary 3. Let G be an abelian group of odd order with an automorphism ω of order 2. Then if N is the set of elements of G which are inverted by ω, N is a subgroup of G and $G = N \times G_{\omega}$.

Theorem 1. Let G be a group of odd order with an automorphism ω of order 2. Suppose that G_ω is a Hall subgroup of G. Then there exists a normal abelian complement of G_ω in G.

Note. If we omit the condition that the group is of odd order, then the conclusion is no longer valid. For let

$$G = \{x,y,z \mid x^2 = y^2 = z^3 = 1,\ z^{-1}xz = y,\ z^{-1}yz = xy, xy=yx\},$$

and take ω to be the inner automorphism of G induced by x. Then $G_\omega = \{x,y\} = F(G)$. Since G is soluble there cannot exist a normal complement of G_ω in G.

Proof. Consider the set of Hall subgroups of G which complement G_ω. Since the order of G is odd, and all the groups in the set are conjugate in G, the set contains an odd number of groups. The groups in the set form a characteristic system of subgroups of G. Since ω is of order 2 and the number of groups in the set is odd, at least one group, say H, of the set is mapped onto itself by ω. Since $H_\omega = H \cap G_\omega$, ω acts as a regular automorphism of H so that ω inverts all the elements of H and N is abelian.

Since $H \cap G_\omega = 1$ whilst $\{H, G_\omega\} = G$, the elements of H form a complete set of representatives of the left cosets of G_ω. Thus H consists of precisely those elements of G which are inverted by ω, by lemma 5. Now let $x \in H$ and $y \in G_\omega$. Then $(y^{-1}xy)^\omega = (y^\omega)^{-1}x^\omega y^\omega = y^{-1}x^{-1}y = (y^{-1}xy)^{-1}$ so that $y^{-1}xy \in H$. Therefore G_ω normalizes H. Therefore H is a normal abelian complement of G_ω in G, and the theorem is proved.

CHAPTER IV

On Groups Expressible as a Product of Nilpotent Hall Subgroups.

We denote by $GF(p)$ the field containing p elements and by $\mathfrak{F}$ the algebraic closure of $GF(p)$.

Lemma 1. Let P be a Sylow p-subgroup of the group G. Suppose that

(i) P has derived length r

and

(ii) there exists a faithful irreducible $\mathfrak{F}(G)$-module V such that for all $v \in V$ and $x_i \in P^{(i)}$ $(i=0,1,\ldots,r-1)$

$$v(1-x_o)(1-x_1) \ldots (1-x_{r-1}) = 0 .$$

Then $P^{(r-1)} \leq C_G(A)$ for any normal abelian subgroup A of G .

Proof. The lemma is certainly true if $r = 0$ so we may suppose $r \geq 1$. Let A denote a normal abelian subgroup of G .

Consider V as an $\mathfrak{F}(A)$-module . Since V is an irreducible $\mathfrak{F}(G)$-module it follows from Chapter II (8) that

$$V = V_1 \oplus \ldots \oplus V_n$$

where $V_1,\ldots,V_n$ are the homogeneous components of V as an $\mathfrak{F}(A)$-module. Since A is abelian and $\mathfrak{F}$ is algebraically closed, the irreducible $\mathfrak{F}(A)$-submodules of V are one dimensional. But each V_i is a sum of isomorphic irreducible $\mathfrak{F}(A)$-submodules of V so that we may describe the action of

$x \in A$ on $w \in V_i$ by

$$wx = \chi_i(x)w \ .$$

Since A is a normal subgroup of G, G acts as a permutation group on the V_i. For each $i=1,\dots,n$ let H_i be the subgroup of G which fixes V_i.

We now prove that $P^{(r-1)} \leq H_i$ for all i. For suppose that $P^{(r-1)} \nleq H_j$. Then $H_j \cap P^{(k)}$ is a proper subgroup of $P^{(k)}$ for $k = 0,1,\dots,r-1$. Thus by Chapter III, lemma 1

$$|P^{(i)}: P^{(i)} \cap H_j| > |P^{(i+1)}: P^{(i+1)} \cap H_j|$$

for $i = 0,1,\dots,r-2$. Now $|P^{(i)}:P^{(i)} \cap H_j|$ is the number of V_k in the same system of transitivity as V_j under $P^{(i)}$. Thus we may choose $x_i \in P^{(i)}$ such that V_jx_i is not in the same system of transitivity as V_j under $P^{(i+1)}$ $(i=0,\dots,r-2)$. Let $x \in P^{(r-1)}$. Then for all $v \in V_j$

$$v(1-x_0)(1-x_1) \dots (1-x_{r-2})(1-x) = 0 \ .$$

Since $V_j x_0$ is not in the same system of transitivity as V_j under P' and since $V = V_1 \oplus \dots \oplus V_n$ as an $\mathcal{F}(A)$-module we can conclude that

$$v(1-x_1)(1-x_2) \dots (1-x_{r-2})(1-x) = 0 \ .$$

Proceeding in this way we finally deduce that

$$v(1-x) = 0 \ .$$

But v was arbitrary in V_j so it follows that $x \in H_j$. Since x was arbitrary in $P^{(r-1)}$ we have $P^{(r-1)} \leq H_j$, a contradiction. Thus $P^{(r-1)} \leq H_i$ for all i.

Let $x \in P^{(r-1)}$ and $y \in A$. If $v \in V$ then

$$v = v_1 + v_2 + \dots + v_n$$

where $v_i \in V_i$ $(i=1,\dots,n)$. Now

$$\begin{aligned} vxy &= (v_1x)y + (v_2x)y + \dots + (v_nx)y \\ &= \chi_1(y)\, v_1x + \chi_2(y) v_2x + \dots + \chi_n(y)\, v_nx \end{aligned}$$

since $v_ix \in V_i$ for each i. Thus

$$\begin{aligned} vxy &= (\chi_1(y)v_1)x + (\chi_2(y)v_2)x + \dots + (\chi_n(y)v_n)x \\ &= (v_1y)x + (v_2y)x + \dots + (v_ny)x \\ &= (v_1 + v_2 + \dots + v_n)yx \\ &= vyx . \end{aligned}$$

But v was arbitrary in V so xy and yx are represented in the same way on V. Since V is a faithful $\mathcal{F}(G)$-module

$$xy = yx$$

or as y was an arbitrary element of A and x an arbitrary element of $P^{(r-1)}$, $P^{(r-1)} \leq C_G(A)$. This completes the proof.

<u>Lemma 2</u>. Let G be a finite soluble group with Sylow p-subgroup P. Suppose that

a) $F(G)$ is a minimal normal subgroup of G.

b) $F(G)$ is a p-group.

c) P has derived length r.

Then $P^{(r-1)}$ centralizes every abelian normal subgroup of $G/F(G)$.

<u>Proof</u>. Since the lemma is trivially true if $G = F(G)$ we may assume that G is not nilpotent. Set $F = F(G)$. It follows from (a) and Chapter II (5) that there exists a complement M of F in G. We must show that if R is a Sylow p-subgroup of M then $R^{(r-1)}$ centralizes every abelian normal subgroup of M.

M acts as a group of automorphisms of F since F is a normal subgroup of G. Since F is the minimal normal subgroup of G, F is an elementary abelian p-group. Considering F as a vector space over $GF(p)$ we obtain an irreducible $GF(p)\,(M)$-module. $C_G(F) \leq F$ so the representation of M corresponding to the module is faithful.

Since R is a Sylow p-subgroup of M, RF is a Sylow p-subgroup of G, and is conjugate to P. Thus $(RF)^{(r)} = 1$. Letting $x_i \in R^{(i)}$ for each $i = 0,1,\ldots,r-1$ and using the additive notation we deduce that

(1) $$f(1-x_0)(1-x_1)\ \ldots\ (1-x_{r-1}) = 0$$

for all $f \in F$.

We now construct from F an irreducible $\mathfrak{F}(M)$-module V which induces a faithful representation of M and for which (1) is satisfied for all $f \in V$.

Since M is a finite group and $GF(p)$ is a perfect field, there exists, by Chapter II (9), a finite extension field of $GF(p)$ which is a splitting field for M. Since an extension field of a splitting field is still a splitting field, there exists a finite normal extension field of $GF(p)$, which is a splitting field for M. This enables us to apply Chapter II (10). However the conclusion of Chapter II (10) will still apply if we replace the finite normal extension field of $GF(p)$ by $\mathfrak{J}$, the algebraic closure of $GF(p)$. Form the $\mathfrak{J}(M)$-module $F^{\mathfrak{J}}$ by extending the scalar field from $GF(p)$ to $\mathfrak{J}$ and let V be an irreducible $\mathfrak{J}(M)$-submodule.

$F^{\mathfrak{J}}$ is obtained from F by extending the field of scalars so as F gives rise to a faithful representation of M so does $F^{\mathfrak{J}}$. By Chapter II (10) the irreducible $\mathfrak{J}(M)$-submodules of $F^{\mathfrak{J}}$ are all algebraically conjugate so they all induce representations with the same kernel. Since $F^{\mathfrak{J}}$ is completely reducible, it is a direct sum of irreducible $\mathfrak{J}(M)$-submodules, each of which induces a representation of M with the same kernel as the representation induced by V. Thus as $F^{\mathfrak{J}}$ induces a faithful representation of M, V represents M faithfully.

Since (1) is true for $f \in F$ and $F^{\mathfrak{J}}$ is obtained from F by extending the field of scalars (1) holds for $f \in F^{\mathfrak{J}}$. But V is an $\mathfrak{J}(M)$-submodule of $F^{\mathfrak{J}}$ so (1) is true if we take $f \in V$.

Now replacing G by M and P by R in the statement of lemma 1 we see that the hypothesis of the lemma is satisfied so it follows that $R^{(r-1)} \leq C_M(A)$ for any normal abelian subgroup A of M. This is what we set out to prove.

<u>Theorem 1</u>. Let the group G contain two complementary Hall subgroups H and K such that H is abelian and K is nilpotent with $K^{(r)} = 1$. Then the r-th derived group of G is nilpotent.

<u>Proof</u>. Suppose that the theorem is false and let G be a counterexample of minimum order. It follows from Chapter II (1) that G is soluble. Therefore every proper subgroup and factor group of G satisfies the hypothesis of the theorem, and so by the minimality of G, also satisfies the conclusion. Let A be the group containing just the identity automorphism of G, and apply Chapter III lemma 3. Since all the conditions of the lemma are satisfied we deduce that $F(G)$ is the unique minimal normal subgroup of G.

It follows that $F(G)$ is a p-group for some prime p. Since $F(G)$ is a normal subgroup of G, $F(G) \leq H$ or $F(G) \leq K$. Suppose $F(G) \leq H$. Since G is soluble $C_G(F(G)) \leq F(G)$, so that as H is abelian, $H \leq F(G)$. Thus $H = F(G)$ and $G/F(G) \cong K$. But $K^{(r)} = 1$ so $G^{(r)} \leq F(G)$. Thus we may assume that $F(G) \leq K$.

Since $F(G)$ is a p-group, K is nilpotent and $C_G(F(G)) \leq F(G)$, K is a p-group. Taking K for P in lemma 2 we deduce that $K^{(r-1)}$ centralizes every abelian normal subgroup of $G/F(G)$.

On the other hand since $F(G)$ is a p-group, $F_2(G)/F(G)$ is a p'-group. Therefore $F_2(G)/F(G) \leq HF(G)/F(G)$ and so is abelian. Now $K^{(r-1)}F(G)/F(G) \leq C_{G/F(G)}(F_2(G)/F(G)) \leq F_2(G)/F(G)$ so as $K^{(r-1)}F(G)$ is a p-group whilst $F_2(G)/F(G)$ is a p'-group, $K^{(r-1)} \leq F(G)$.

We now apply the theorem to the group $G/F(G)$. $G/F(G)$ is the product of the two Hall subgroups $KF(G)/F(G)$ and $HF(G)/F(G)$, the first of which is nilpotent of derived length $r-1$ and the second is abelian. Thus $(G/F(G))^{(r-1)} \leq F_2(G)/F(G)$ and therefore $G^{(r-1)} \leq F_2(G)$. Since as we have already seen $F_2(G)/F(G)$ is abelian, $G^{(r)} \leq F(G)$. This contradiction to the definition of G proves the theorem.

A group of interest in connection with theorem 1 is the group G of 2×2 matrices over $GF(3)$. This group is of order 48 and so may be written as the product of a Sylow 3-subgroup H and a Sylow 2-subgroup K . H is abelian and K has derived length 2 . Thus by the theorem G'' is nilpotent. However G'' is not abelian and G' is not nilpotent.

CHAPTER V

THE MAIN THEOREM

Theorem 1. Let G be a group of odd order with an automorphism ω of order 2. Suppose that G_ω is nilpotent and that $(G_\omega)^{(r)} = 1$. Then $G^{(r)}$ is nilpotent and $G = F_3(G)$.

Kovacs and Wall [14] have exhibited a group G satisfying the conditions of theorem 1 and for which $G \neq F_2(G)$. Before proving the theorem we give some other examples.

Example 1. Given any group H of odd order we construct a group G with an automorphism ω of order 2 such that $G_\omega \cong H$ but for which $G^{(r-1)}$ is not nilpotent where r is the derived length of H.

Indeed let q be any odd prime not dividing the order of H and construct a group G as described in Chapter III lemma 4. For this group, $G/F(G) \cong H$ so that $G^{(r)}$ is not nilpotent. It remains to construct an automorphism ω of G of order 2 and such that $G_\omega \cong H$. We define the mapping $\omega : G \to G$ by letting the element $x \to ax + b$ of G be mapped onto the mapping $x \to ax - b$ of G. It is easy to check that this mapping is an automorphism of G of order 2. Since q is odd the fixed point group G_ω consists of the mappings $x \to ax$ and therefore is isomorphic to H.

Example 2. Let n be an arbitrary integer. We construct a group G of odd order with an automorphism ω of order 2, such that G_ω is metabelian (but not nilpotent). However $G^{(n)}$ is not nilpotent.

To construct such a group choose a p-group P, for p an odd prime, with an automorphism ω of order 2 such that P_ω is cyclic and $P^{(n)} \neq 1$. Kovacs and Wall [14] have given examples of groups with these properties. Form H, the splitting extension of P by ω and choose an odd prime $q \neq p$. We may now construct the group G described in lemma 4 of Chapter III. Let M be the normal subgroup of G of index 2 in G. It is clear that $F(M) = F(G)$ and that $M/F(M) \cong P$. Thus $M^{(n)}$ is not nilpotent. On the other hand if θ is an involution of G then $(F(M))_\theta$ is abelian since $F(M)$ is abelian and $M_\theta/(F(M))_\theta \cong P_\omega$ is cyclic. Thus M_θ is metabelian and we have a group with the required properties.

The theorem is proved by induction on $|G|$ and by way of contradiction. Suppose therefore that G is a group of minimal order satisfying the hypotheses of the theorem but not the conclusion. Since the theorem is known for $r=1$ we may suppose that $r > 1$.

The order of G is odd by hypothesis so G is soluble.

<u>Lemma 1</u>. $F(G)$ is the unique minimal normal ω-subgroup of G. Therefore $F(G)$ is an elementary abelian p-group for some prime p.

<u>Proof</u>. Let H/K be a ω-section of G. Then $(H/K)_\omega = G_\omega K \cap H/K$ since the order of ω is a prime not dividing

the order of G. Thus H/K satisfies the hypotheses of the theorem and if $H/K \neq G/1$, it follows from the minimality of G that H/K satisfies the conclusion of the theorem. Let A denote the group of automorphisms of G consisting of ω and the identity automorphism. Then an A-section of G is just a ω-section of G so G satisfies the hypothesis of Chapter III lemma 3. This completes the proof of lemma 1.

<u>Notation</u>. For each positive integer n, set $F_n = F_n(G)$. Let Γ denote the splitting extension of G by ω.

<u>Lemma 2</u>. (i) $(G/F_1)_\omega \neq G/F_1$,

(ii) F_1 is a faithful irreducible Γ/F_1-module,

(iii) $(F_1)_\omega > 1$. Therefore $p \mid |G_\omega|$.

<u>Proof</u>. (i) If $(G/F_1)_\omega = G/F_1$, then G/F_1 is isomorphic to a section of G_ω and therefore is nilpotent of derived length less than or equal to r. It follows that G satisfies the conclusion of the theorem. Hence $(G/F_1)_\omega \neq G/F_1$.

(ii) Lemma 1 implies that F_1 is an irreducible Γ/F_1-module. To prove that F_1 is a faithful Γ/F_1-module we need to prove that $C_\Gamma(F_1) = F_1$. Since F_1 is the Fitting subgroup of G, and since G is soluble, $C_G(F_1) = F_1$. Hence if $C_\Gamma(F_1) > F_1$, $|C_\Gamma(F_1):F_1| = 2$. In this case Γ/F_1 has a normal Sylow 2-subgroup so that $\Gamma/F_1 = G/F_1 \times gp\{\omega F_1\}$ from which it follows that $(G/F_1)_\omega = G/F_1$, contradicting (i). This proves (ii).

(iii) $(F_1)_\omega > 1$ for if $(F_1)_\omega = 1$, ω must invert all the elements of F_1. Then, since Γ/F_1 is faithfully represented by its action on F_1, ωF_1 lies in the centre of Γ/F_1. But this again implies that $(G/F_1)_\omega = G/F_1$, contradicting (i).

<u>Lemma 3</u>. F_2/F_1 is a p'-group. G/F_1 has no non-trivial normal p-subgroups.

<u>Proof</u>. Suppose that P/F_1 is the Sylow p-subgroup of F_2/F_1. Then as F_1 is a p-group, P is a normal p-subgroup of G. Hence $P \leq F_1$. The second statement follows from the first.

<u>Lemma 4</u>. If G_ω is a p-group then $G = F_2G_\omega$ and $(G_\omega)^{(r-1)}$ is not contained in F_1. F_2/F_1 is abelian.

<u>Proof</u>. We know, by lemma 2, that F_1 is a p-group and, by lemma 3, that F_2/F_1 is a p'-group. Since G/F_1 is soluble and F_2/F_1 is a normal subgroup of G/F_1, F_2/F_1 is contained in every Hall p'-subgroup of G/F_1. Now the Hall p'-subgroups of G/F_1 are all conjugate and the order of G is odd so the number of Hall p'-subgroups is odd. Clearly the automorphism ω permutes these Hall p'-subgroups and since the number of them is odd, at least one is fixed by ω. Thus we can choose a Hall p'-subgroup H/F_1 such that $H^\omega = H$.

Now $H_\omega = H \cap G_\omega$ is a p-group so that $H_\omega \leq F_1$. Thus ω acts as a regular automorphism on H/F_1 so that H/F_1 is abelian. Since G/F_1 is a soluble group, $C_{G/F_1}(F_2/F_1) < F_2/F_1$. But $F_2/F_1 \leq H/F_1$ so that as H/F_1 is abelian, $H/F_1 \leq C_{G/F_1}(F_2/F_1) \leq F_2/F_1 \leq H/F_1$. Thus F_2/F_1 is the unique Hall p'-subgroup of G/F_1. It follows that G/F_2 is a p-group. Therefore $G = F_3$.

Since $G = F_3$ and G does not satisfy the conclusion of the theorem, $G^{(r)}$ is not nilpotent.

Suppose by way of contradiction that $G_\omega{}^{(r-1)} \leq F_1$. Then $(G/F_1)_\omega$ has derived length at most $r-1$, so by the minimality of G, $(G/F_1)^{(r-1)}$ is nilpotent. Thus $G^{(r-1)} \leq F_2$ and since, as we have already seen, F_2/F_1 is abelian, $G^{(r)} \leq F_1$. This contradiction proves that $G_\omega{}^{(r-1)}$ is not contained in F_1.

Finally we show that if $G_\omega F_2 < G$, $G_\omega{}^{(r-1)}$ is contained in F_1. It then follows from the conclusion of the previous paragraph that $G_\omega F_2 = G$. Suppose then that $G_\omega F_2 < G$, and let K be a maximal subgroup of G containing $G_\omega F_2$. Since K is a maximal subgroup of G containing F_2 and since G/F_2 is nilpotent, K is a normal subgroup of G. By Chapter III lemma 5, corollary 2, as $G_\omega \leq G_\omega F_2 \leq K$, K is an ω-subgroup of G. Therefore, by the minimality of G, $K^{(r)}$ is nilpotent. But $K^{(r)}$ is a characteristic subgroup of K, a normal subgroup of G, and therefore $K^{(r)}$ is a normal subgroup of G.

Hence $K^{(r)} \leq F_1$ so that $K^{(r-1)} \leq F_2$. Now $G_\omega^{(r-1)} \leq K^{(r-1)} \leq F_2$. But G_ω is a p-group and F_1 is the Sylow p-subgroup of F_2 so $G_\omega^{(r-1)} \leq F_1$. This completes the proof of the lemma.

We have shown that F_1 is the unique minimal normal ω-subgroup of G. Since G is a normal subgroup of Γ, $F(G) \leq F(\Gamma)$. If $F(\Gamma) \neq F(G)$ then $|F(\Gamma):F(G)| = 2$ so that $\omega \in F(\Gamma)$. But in this case, since $F(\Gamma)$ is nilpotent, $(F_1)_\omega = F_1$ contradicting lemma 2(ii). Thus $F_1 = F(\Gamma)$ is the unique minimal normal subgroup of Γ. $|\Gamma:G| = 2$ so the solubility of Γ follows from that of G. Therefore we can apply Chapter II (5) to deduce the existence of a complement N of F_1 in Γ. By Sylow's theorem we can suppose, by taking a suitable conjugate of N if necessary, that $\omega \in N$.

Let $M = G \cap N$. Then, by the modular law, $G = G \cap \Gamma = G \cap NF_1 = (G \cap N)F_1 = MF_1$ so M is a complement of F_1 in G.

Since the elements of N form a complete set of coset representatives of F_1 in Γ, we may consider F_1 as an $GF(p)(N)$-module. We now summarize the results of the lemmas and the hypotheses of the theorem in module notation.

(1) F_1 is a faithful irreducible N-module over $GF(p)$.

(2) $(F_1)_\omega > 0$.

(3) If $f \in (F_1)_\omega$ and $x_i \in (M_\omega)^{(i)}$ $(i=0,1,\ldots,r-1)$ then $f(1-x_0)(1-x_1)\ldots(1-x_{r-1}) = 0$.

(4) If $f \in (F_1)_\omega$ and $x \in M_\omega$ is of order prime to p, then since G_ω is nilpotent,

$$fx = f .$$

It also follows from lemma 2 that $M_\omega \neq M$.

Using the same method as in the proof of Chapter IV, lemma 2, we obtain an N-module V, over $\mathfrak{J}$, the algebraic closure of $GF(p)$, with the following properties:

(1) V is a faithful irreducible N-module over $\mathfrak{J}$,

(2) $V_\omega = \{v \in V | v\omega = v\} > 0$,

(3) If $v \in V_\omega$ and $x_i \in (M_\omega)^{(i)}$ $(i=0,1,\ldots,r-1)$ then $v(1-x_0)(1-x_1),\ldots,(1-x_{r-1}) = 0$.

(4) If $v \in V_\omega$ and $x \in M_\omega$ is of order prime to p, then $vx = v$.

<u>Notation</u>. $$Q = F(M) .$$

<u>Lemma 5</u>. V is an irreducible M-module.

<u>Proof</u>. By way of contradiction suppose that there exists an irreducible $\mathfrak{J}(M)$-submodule W of V such that $0 < W < V$. Since $W\omega$ is also an irreducible M-submodule of V and since $W + W\omega$ is an $\mathfrak{J}(N)$-module, it follows from the first property of V that $V = W + W\omega$. Therefore as an $\mathfrak{J}(M)$-module

$$V = W \oplus W\omega .$$

Suppose that G_ω is not a p-group. Then there exists an element $x \neq 1$ in M_ω of order prime to p. Let $w \in W$ be arbitrary. Then $w + w\omega \in V_\omega$ so, by the property (4) of

V, $(w + w\omega)x = w + w\omega$. Equating the W and $W\omega$ components of both sides, we deduce that x acts trivially on both W and $W\omega$. Hence x acts trivially on $W + W\omega = V$. But this contradicts the first property of V. Thus we may assume that G_ω is a p-group.

Let $x \in Q$ and suppose that for all $w \in W$, $wx = w$. Since Q is a p'-group and G_ω is a p-group $x^\omega = x^{-1}$. Hence if $w \in W$, $(w\omega)x = wx^{-1}\omega = w\omega$ so that x acts trivially on $W + W\omega = V$. But then by the first property of V, $x = 1$, so that W is a faithful $\mathfrak{F}(Q)$-module. Since $Q = F(M)$ and M is soluble, any normal subgroup of M has a non-trivial intersection with Q. Hence if W were not a faithful $\mathfrak{F}(M)$-module, W would not be a faithful $\mathfrak{F}(Q)$-module. Thus W is a faithful $\mathfrak{F}(M)$-module.

If $w \in W$ then $w + w\omega \in V_\omega$ so that if $x_i \in (M_\omega)^{(i)}$ $(i=0,1,2,\dots,r-1)$ it follows from the third property of V that

$$(w+w\omega)(1-x_0)(1-x_1)\dots(1-x_{r-1}) = 0.$$

Equating the W-component of the left hand side to 0 we obtain

$$w(1-x_0)(1-x_1)\dots(1-x_{r-1}).$$

From lemma 3 and lemma 4 it follows that M_ω is the Sylow p-subgroup of M and has derived length r. Thus applying Chapter IV, lemma 1 to the group M, taking M_ω as the Sylow p-subgroup and W as the $\mathfrak{F}(M)$-module we deduce that $M_\omega^{(r-1)} \leq C_M(A)$ for any abelian normal subgroup A of M. In particular taking $A = F(M) = Q$ which is abelian by lemma 3 we obtain $M_\omega^{(r-1)} \leq C_M(Q) = Q$. But $M_\omega^{(r-1)}$ is a p-group whilst Q is a p'-group. Thus $M_\omega^{(r-1)} = 1$ contrary to lemma 4 . This contradiction completes the proof of lemma 5.

<u>Lemma 6</u>. If $L \neq 1$ is a normal ω-subgroup of M , then L_ω is a non-trivial proper subgroup of L .

<u>Proof</u>. Since M is soluble and every soluble group contains a characteristic subgroup which is abelian, it is sufficient to prove the lemma for abelian L . Therefore L is supposed to be a normal abelian subgroup of M . Now L is contained in $F(M) = Q$. It follows from lemma 3 that L is a p'-group. Write

$$V = V_1 \oplus V_2 \oplus \ldots \oplus V_s$$

where V is considered as an $\mathfrak{F}(L)$-module and the V_i are the homogeneous components. Since L is an abelian p'-group whilst $\mathfrak{F}$ is algebraically closed of characteristic p , the action of $x \in L$ on $v \in V_i$ may be described by

$$vx = \chi_i(x)v \ .$$

It follows from Chapter II, (8), that the characters χ_i are all conjugate and that the number, s , of homogeneous components divides the order of M . Thus none of the characters χ_i $(i=1,2,\ldots,s)$ is the trivial character since V is a faithful module. Also s is odd.

We complete the proof of the lemma by showing that if $L_\omega = 1$ or $L_\omega = L$ then we can choose an i such that χ_i is the trivial character.

Since ω has order 2 and V is an $\mathcal{F}(N)$-module, for each i $(=1,2,\ldots,s)$ there exists j such that $V_i\omega = V_j$ and $V_j\omega = V_i$. Since s is odd there exists at least one i for which $V_i\omega = V_i$. Suppose $v \in V_i$ and $x \in L$. Then $\chi_i(x)v_i\omega = v_i\omega x = v_i x^\omega \omega = \chi_i(x^\omega)v_i\omega$ so that $\chi_i(x) = \chi_i(x^\omega)$ for all $x \in L$. Now if $L_\omega = 1$, then for all $x \in L$, $x^\omega = x^{-1}$ so that $\chi_i(x) = \chi_i(x^{-1})$ or $\chi_i(x^2) = 1$. Since L has odd order, it follows that χ_i is the trivial character. Thus $L_\omega > 1$.

Now suppose that $L_\omega = L$. By the second property of V, $V_\omega > 0$ so that there exists $0 \neq v = v\omega \in V$. Since $L = L_\omega$ is a p'-group, it follows from the fourth property of V that $v = vx$ for all $x \in L$. Thus $\{kv \mid k \in \mathcal{F}\}$ is a trivial L-submodule of V and therefore is contained in some V_j. For this V_j, $\chi_j = 1$ clearly. This contradiction proves the lemma.

<u>Remark</u>. In lemma 6, L_ω cannot be a normal subgroup of M, for if it were we would obtain a contradiction by applying lemma 6 to L_ω. But $(Z(M))_\omega$ is a normal subgroup of M so $Z(M) = 1$. Therefore we can now assume that $Q = F(M)$ is a proper subgroup of M.

<u>Lemma 7</u>. Q is abelian.

Proof. We consider V as a Q-module and write

$$V = V_1 \oplus V_2 \oplus \dots \oplus V_s$$

where the V_i are the homogeneous components of V. Let Q_i be the kernel of the representation of Q obtained on V_i for each $i = 1,\dots,s$. Then the Q_i are all conjugate by Chapter II (8), so that if $Q' \leq Q_i$ for some i then $Q' \leq Q_i$ for all i. Therefore in this case Q' is contained in the kernel of $V_1 \oplus V_2 \oplus \dots \oplus V_s = V$. But V is a faithful M-module so that this implies that $Q' = 1$, and proves the lemma.

Now suppose that Q_ω is contained in one of the groups Q_i $(i=1,\dots,s)$, say Q_j. Then by Chapter III, lemma 5, corollary 2, $Q_j^{\omega} = Q_j$. Therefore ω induces a regular automorphism on Q/Q_j so that Q/Q_j is abelian. Consequently $Q' \leq Q_j$. Thus it is sufficient to prove that for some i, Q_ω is contained in Q_i.

Suppose that there exists an i such that $V_i\omega \neq V_i$. Let $v \in V_i$. Then $v + v\omega \in V_\omega$ so that as Q_ω is a p'-group if $x \in Q_\omega$, by the fourth property of V, $(v+v\omega)x = v + v\omega$. Equating the V_i components of both sides, we see that $vx = v$ so that Q_ω is contained in Q_i.

It remains only to show that we cannot have $V_i\omega = V_i$ for all i. Suppose by way of contradiction that for all i $V_i\omega = V_i$ and fix i. Considering V_i as a $Z(Q)$-module, we may write

$$V_i = W_{i1} \oplus W_{i2} \oplus \dots \oplus W_{iu}$$

where for each j , W_{ij} is a homogeneous component of V_i . Since $V_i\omega = V_i$ we find, as we have done previously in similar circumstances, that there exists a j such that

$$W_{ij}\omega = W_{ij} .$$

Since $Z(Q)$ is an abelian p'-group, the elements of $Z(Q)$ act as scalar multipliers on the W_{ij} . Suppose that if $x \in Z(Q)$ and $w \in W_{ij}$, $wx = \chi_{ij}(x)w$. Then $\chi_{ij}(x)w\omega = w\omega x = wx^{\omega}\omega = \chi_{ij}(x^{\omega})w\omega$ so that $\chi_{ij}(x) = \chi_{ij}(x^{\omega})$. Since $Z(Q)$ is a non-trivial normal abelian subgroup of M , it follows that $(Z(Q))_{\omega} < Z(Q)$ by lemma 6. Therefore, by Chapter III, lemma 5, corollary 3, the set H of elements of $Z(Q)$ inverted by ω forms a non-trivial subgroup of $Z(Q)$. Since H is a subgroup of $Z(Q)$, H is normal in Q . Now if $x \in H$, $\chi_{ij}(x) = \chi_{ij}(x^{\omega}) = \chi_{ij}(x^{-1})$. Since H is of odd order, for all $x \in H$, $\chi_{ij}(x) = 1$. Thus H is contained in the kernel of the representation of $Z(Q)$ given by W_{ij} . Since for $k \neq j$, the kernel of W_{ik} is conjugate to that of W_{ij} in Q and since H is a normal subgroup of Q , H is contained in the kernel of W_{ik} for all k . Thus H is contained in the kernel of $W_{i1} \oplus \cdots \oplus W_{iu} = V_i$. But this is true for all i so that H is contained in the kernel of $V_1 \oplus V_2 \oplus \ldots \oplus V_s = V$. Since V is a faithful $\mathcal{F}(M)$-module, this implies that $H = 1$. This contradiction to the fact that H is a non-trivial subgroup of $Z(Q)$ completes the proof of the lemma.

<u>Lemma 8</u>. $G_\omega \leq F_3(G)$.

<u>Proof</u>. Since G_ω is nilpotent this follows immediately from Chapter II, (6). However it is of interest to give a proof which does not depend on Thompson's theorem.

Suppose by way of contradiction that G_ω is not contained in $F_3(G)$. Thus $G \neq F_3(G)$. By the minimality of G , if H is any normal ω-subgroup of G , $H = F_3(H)$. It follows that $G/F_3(G)$ is cyclic of prime order, say t . We may also choose $x \in M_\omega$ such that $M = \{x, F_2(M)\}$.

We begin by showing that $Z(F_2(M)) = 1$. For if $K = Z(F_2(M)) > 1$ then it is a non-trivial normal ω-subgroup of M . By lemma 6 it follows that $K_\omega > 1$. Now $K_\omega = K \cap M_\omega \lhd M_\omega$ and $K_\omega \leq Z(F_2(M))$ so $K_\omega \lhd F_2(M)$. It follows that $1 < K_\omega \lhd \{x, F_2(M)\} = M$, contradicting lemma 6. Thus $Z(F_2(M)) = 1$. Lemma 2 of Chapter III now implies the existence of an ω-complement H of $Q = F(M)$ in M .

We now prove that Q is a q-group for some prime q . Suppose by way of contradiction that Q_1 and Q_2 are two non-trivial complementary Hall subgroups of Q . Let $i = 1$ or 2 . Since Q_i is a characteristic subgroup of M , F_1Q_iH is a proper ω-subgroup of G . Therefore $F_3(F_1Q_iH) = F_1Q_iH$. Let $F_2(F_1Q_iH) = F_1Q_iR_i$ where $R_i \leq H$. Then $H/R_i \cong F_1Q_iH/F_1Q_iR_i$ is nilpotent. Since $H/R_1 \cap R_2$ is isomorphic to a subgroup of $H/R_1 \times H/R_2$ and H is not nilpotent, $R_1 \cap R_2 \neq 1$. Let S denote a minimal normal ω-subgroup of H contained in $R_1 \cap R_2$. Then S is an s-group for some prime s. If $s \nmid |Q_i|$ then since $F_2(F_1Q_iS) = F_1Q_iS$, Q_iS is nilpotent and therefore $S \leq C_M(Q_i)$. If $s \mid |Q_i|$ then S centralizes the Sylow s'-subgroups of Q_i by the same argument. Thus $SQ_1Q_2 = SQ$ is a nilpotent normal subgroup of M. Thus $SQ \leq F(M) = Q$ or $S \leq Q \cap H = 1$. This contradiction proves that Q is a q-group for some prime q.

Let R denote a Sylow p'-subgroup of $F(H)$. We will prove that $R = 1$ and so conclude that $F(H)$ is a p-group. If $R \neq 1$, then for some integer $\ell \geq 1$, $R^{(\ell)} = 1$ and $R^{(\ell-1)} \neq 1$. Since $Q = F(M)$ is a q-group, $F(H) \cong F_2(M)/F(M)$ is a q'-group, so R is of order prime to q. Consider the group F_1QR. Since R is a characteristic subgroup of H, F_1QR is a proper normal ω-subgroup of G. Also $(F_1QR)_\omega = (F_1)_\omega Q_\omega R_\omega = (F_1)_\omega \times Q_\omega \times R_\omega$ since G_ω is nilpotent and F, Q, R have relatively prime orders. Thus $((F_1QR)_\omega)^{(\ell)} \leq (F_{1\omega})' \times (Q_\omega)' \times (R_\omega)^{(\ell)} \leq R^{(\ell)} = 1$. By the minimality of G it follows that $(F_1QR)^{(\ell)}$ is nilpotent. Since $F_1QR \lhd G$, $(F_1QR)^{(\ell)} \leq F_1$. Therefore $(F_1QR)^{(\ell-1)} \leq FQ$ so that $R^{(\ell-1)} \leq R \cap FQ = 1$. This contradiction to the definition of ℓ proves that $R = 1$ and hence that $F(H)$ is a p-group. Write $P = F(H)$.

Since H is not nilpotent and $|H:P| = |G:F_3(G)| = t$, a prime, $t \neq p$. Let S denote a Sylow t-subgroup of H which is normalized by ω. Then S complements P in H and $S_\omega = S$.

We now show that any proper ω-subgroup of P which is normalized by S is centralized by S. For if R is a proper ω-subgroup of P which is normalized by S then, by the minimality of G, $F_3(F_1QRS) = F_1QRS$. Since $F_1 \leq F(F_1QRS)$ and F_1 is a p-group which is its own centralizer in G, $F(F_1QRS)$ is a p-group. Also $M \cap F(F_1QRS) \leq C_M(Q) = Q$, a q-group. Thus $F_1 = F(F_1QRS)$. Similarly since Q is a q-group, $F_2(F_1QRS)/F_1$ is a q-group. If $F_2(F_1QRS) > F_1Q$ then since R is a p-group, $F_2(F_1QRS) = F_1QS$. But then R normalizes S so that $(R,S) \leq R \cap S = 1$. If on the other hand $F_2(F_1QRS) = F_1Q$ then $F_1QRS/F_1Q \cong RS$ is nilpotent so again $(R,S) = 1$. Thus any proper ω-subgroup of P which is normalized by S is centralized by S.

Since $F_3(G) \neq G$, S does not centralize P. Also $P/\Phi(P)$ is completely reducible under the action of $\{\omega,S\}$ so $P/\Phi(P)$ is acted on irreducibly by $\{\omega,S\}$. Suppose that $P_\omega \nleq \Phi(P)$. Then $P_\omega\Phi(P)/\Phi(P)$ is an $\{\omega,S\}$-sub-moduel of $P/\Phi(P)$. Thus $P_\omega\Phi(P) = P$ and therefore by the fundamental property of the Frattini subgroup, $P_\omega = P$. But then since G_ω is nilpotent S centralizes P which is false. Thus $P_\omega \leq \Phi(P)$. It follows that ω inverts all the elements of $P/\Phi(P)$ and so it also inverts all the elements of P/P'. Thus $P_\omega \leq P'$.

Suppose P has derived length ℓ. Then $\ell \geq 1$, $P^{(\ell)} = 1$ and $P^{(\ell-1)} \neq 1$. Consider $F_3(G) = F_1QP$.

$(F_3(G))_\omega = (F_1)_\omega Q_\omega P_\omega = Q_\omega \times ((F_1)_\omega P_\omega)$. Since $P_\omega \leq P'$, $P_\omega^{(\ell-1)} = 1$ and hence as $(F_1)_\omega \triangleleft (F_1)_\omega P_\omega$, $((F_1)_\omega P_\omega)^{(\ell)} = 1$. It follows that $(F_3(G))_\omega$ has derived length at most ℓ . Thus $(F_3(G))^{(\ell)} \leq F_1$ and therefore $(F_3(G))^{(\ell-1)} \leq F_2(G)$. Thus $P^{(\ell-1)} \leq P \cap F_1Q = 1$, a contradiction. This contradiction completes the proof of lemma 8.

Lemma 9. $$G = F_3(G).$$

Proof. Suppose by way of contradiction that $G > F_3(G)$. Since $G_\omega \leq F_3(G)$ by lemma 8, ω induces a regular automorphism on G/F_3 so that G/F_3 is abelian. If H is any subgroup of G containing F_3 then since $G_\omega \leq H$, $H^\omega = H$ (Chapter III, lemma 5, corollary 2) and $H \triangleleft G$ since G/F_3 is abelian. Thus if $H \neq G$, by the minimality of G , $H = F_3(H)$. Since H is normal in G , $F_3(H) \leq F_3$. Thus $F_3 \leq H = F_3(H) \leq F_3$ and therefore $H = F_3$. It follows that G/F_3 is a cyclic group of prime order.

Since $G_\omega < F_3 < G$, $M_\omega \leq F_2(M) < M$ and by Chapter III, lemma 5, we can choose an element $x \in M$ such that $\{x, F_2(M)\} = M$ and $x^\omega = x^{-1}$. Now consider the ω-subgroup of G , $K = \{x; Q, F_1\}$. Since $x^\omega = x^{-1}$, whilst $F_2 = QF_1$ is a normal subgroup of K , $K_\omega \leq (QF_1)_\omega$. But Q is an abelian p'-group, F_1 is an abelian p-group, and G_ω is nilpotent; therefore K_ω is abelian. Thus, as the theorem is true for $r = 1$, K' is nilpotent.

Write $K' = A \times B$ where A is a Sylow p-subgroup of K'. Then B is a normal p'-subgroup of K and since F_1 is a p-group, $B \cap F_1 = 1$. Since F_1 is also a normal subgroup of K and G is soluble, $B \leq C_K(F_1) \leq C_G(F_1) \leq F_1$. Thus $B = 1$ and therefore $K' \leq QF_1$ is a p-group. Therefore $K' \leq F_1$. Let $L = \{x, Q\}$. Then L is a subgroup of M and $K = F_1L$. Now $L \cong K/F_1$ is abelian so that $x \in C_M(Q)$. But $Q = F(M)$ and M is soluble, so this implies that $x \in Q$. This contradiction to the choice of x proves the lemma.

Corollary. Since $G = F_3(G)$ by lemma 9, whilst G does not satisfy the conclusion of the theorem, it follows that $G^{(r)} > F_1$. Thus $M^{(r)} > 1$.

Lemma 10. There exists an ω-complement D of Q in M. Q is a q-group for some prime $q \neq p$ and M/Q is a q'-group.

Proof. Since $G = F_3(G)$, $M = F_2(M)$. We have shown in the remark at the end of the proof of lemma 6 that $Z(M) = 1$. Also by lemma 7, $Q = F(M)$ is abelian. Thus we may apply Chapter III, lemma 2 to deduce the existence of an ω-subgroup D of M which complements Q in M.

We next show that if K is a proper ω-subgroup of M then $K^{(r)} = 1$. For if K is a proper ω-subgroup of M,

F_1K is a proper ω-subgroup of G. Since $(G_\omega)^{(r)} = 1$, $((F_1K)_\omega)^{(r)} = 1$, and therefore the minimality of G implies that $(F_1K)^{(r)}$ is nilpotent. Let A denote the Hall p'-subgroup of $(F_1K)^{(r)}$. Then A is a normal subgroup of F_1K and so also is F_1. Therefore $(F_1,A) \leq F_1 \cap A = 1$ since F_1 is a p-group whilst A is a p'-group. It follows that $A \leq C_G(F_1) = F_1$ so that $A = 1$. Thus $(F_1K)^{(r)}$ is a p-group. We also have that $K^{(r)} \leq M^{(r)} \leq Q$, a p'-group so it follows that $K^{(r)} = 1$, as we set out to prove.

Now suppose that Q is not a q-group for any prime q. Then we may write $Q = Q_1Q_2$ where Q_1 and Q_2 are Hall subgroups of Q of relatively prime orders. Since $Q = F(M)$, the Q_i are normal ω-subgroups of M. Thus for each i DQ_i is a proper ω-subgroup of M and so $(DQ_i)^{(r)} = 1$. Since Q_i is abelian, it follows that

$$(Q_i, D, D', \ldots, D^{(r-1)}) = 1 \qquad (i=1,2).$$

Also D is a proper ω-subgroup of G so that $D^{(r)} = 1$. Now

$$\begin{aligned} M^{(r)} &= (DQ_1Q_2)^{(r)} \\ &= D^{(r)}(Q_1,D,D',\ldots,D^{(r-1)})(Q_2,D,\ldots,D^{(r-1)}) \\ &= 1 \end{aligned}$$

using, in addition to the above results, the fact that $Q = Q_1Q_2$ is an abelian group. But this contradicts the corollary to lemma 9. Thus Q is a q-group for some prime $q \neq p$.

Since M/Q is nilpotent, Q is a q-group, and $Q = F(M)$ it follows that $D \cong M/Q$ is a q'-group.

<u>Lemma 11</u>. $D_\omega = D$.

<u>Proof</u>. Suppose that $D_\omega < D$. Then, since $D \cong M/Q$ is nilpotent by lemma 9, there exists a proper normal subgroup K of D containing D_ω . Form KQF_1 , a proper normal subgroup of G . Since $G_\omega = (F_1)_\omega Q_\omega D_\omega$ is contained in KQF_1 , KQF_1 is an ω-subgroup of G by Chapter III lemma 5, corollary 2. Hence by the minimality of G , $(KQF_1)^{(r)} \leq F_1$ and therefore $(KQF_1)^{(r-1)} \leq F_2 = F_1Q$. Thus $D_\omega^{(r-1)} \leq K^{(r-1)} \leq D \cap F_1Q = 1$. Since $r > 1$, G_ω is nilpotent and D is a q'-group whilst Q is a q-group, $M_\omega = D_\omega Q_\omega$ has derived length at most $r-1$. Thus $M^{(r-1)} \leq F(M) = Q$ and since Q is abelian $M^{(r)} = 1$. But this contradicts the corollary to lemma 9. Thus $D_\omega = D$.

Now Q_ω is normalized by Q since Q is abelian and is normalized by D , since $Q_\omega = Q \cap M_\omega$ and $D = D_\omega \leq M_\omega$. Thus Q_ω is normalized by $DQ = M$, contrary to lemma 6.

This last contradiction completes the proof of the theorem.

CHAPTER VI

FURTHER THEOREMS

The theorems to be proved in this chapter are

Theorem 1. Let G be a group of odd order with a group of automorphisms, A, of order 4 and exponent 2, such that for each $\omega \in A$, $\omega \neq 1$, G_ω is nilpotent. Then $G' \leq F(G)$.

Theorem 2. Let G be a group of odd order with a group of automorphisms, A, of order 8 and exponent 2, such that for each $\omega \in A$, $\omega \neq 1$, G_ω is nilpotent. Then G is nilpotent.

A group satisfying the hypothesis of theorem 1 need not be supersoluble. To see this take H to be the group $\{x,y \mid x^2 = y^{13} = 1, x^{-1}yx = y^{-1}\}$, let $q = 3$ and form the group G described in Chapter III, lemma 4. Then there exists a normal subgroup L of G of odd order and index 2 in G. $F(L) = F(G)$ is an elementary abelian 3-group whilst $|L:F(L)| = 13$. Thus L is not supersoluble. We show that L possesses a group of automorphisms satisfying the hypothesis of theorem 1. Choose an element ω_1 of G of order 2 and an element $x \in L$ of order 13 such that $\omega_1 x \omega_1 = x^{-1}$. Then ω_1 induces an automorphism of L of order 2 with $L_{\omega_1} \leq F(L)$ and so nilpotent. A second automorphism ω_2 of

L may be defined by $x^{\omega_2} = x$ and $z^{\omega_2} = z^{-1}$ for all $z \in F(L)$. Then ω_2 is also of order 2 and L_{ω_2} is nilpotent. It is easily verified that ω_1 and ω_2 commute and that $L_{\omega_1\omega_2}$ is nilpotent. Hence L is a non-supersoluble group satisfying the hypothesis of theorem 1.

Each of the theorems is to be proved by induction on the order of the group G, and by way of contradiction. Let G be a group of minimal order not satisfying the hypothesis of the theorem in question. For theorem 1 we take $(m,n) = (1,1)$ and for theorem 2 we take $(m,n) = (0,1)$. Now if $H/K \neq G/1$ is an A-section of G, either A is represented faithfully as a group of automorphisms of H/K in which case by induction $(H/K)^{(m)} \leq F_n(H/K)$ or for some automorphism $\omega \in A$, $\omega \neq 1$, $(H/K)_\omega = (H/K)$ so that H/K is nilpotent being isomorphic to a section of G_ω. Thus as $|G|$ is odd, the Hypothesis of Chapter III, lemma 3 is satisfied and we conclude that $F(G)$ is the unique minimal normal A-subgroup of G.

The proofs of each of the theorems will now be completed independently.

Proof of Theorem 1. Suppose that the theorem is false and let G be a counterexample of minimal order. Then $F(G)$ is the unique minimal normal A-subgroup of G. $F(G)$ is an elementary abelian p-group for some prime p.

Let Γ denote the splitting extension of G by A, and write $F = F(G)$.

Suppose that $(G/F)_\omega = G/F$ for some $\omega \in A$, $\omega \neq 1$. Then since G_ω is nilpotent, G/F is nilpotent. It is now an easy consequence of the minimality of G that G/F is a q-group for some prime $q \neq p$. Therefore we can choose a Sylow q-subgroup Q of G to complement F in G. Since $N_\Gamma(Q)F = \Gamma$, by taking a suitable conjugate of Q if necessary, we may assume A normalizes Q. Since $(G/F)_\omega = G/F$, $Q_\omega = Q$. Now $Z(G) = 1$ for if $Z(G) > 1$, $Z(G) \geq F$ which is false since G is soluble. Since G_ω is nilpotent and $Q = Q_\omega$ is a group of order prime to p, whilst F is an abelian p-group, $F_\omega = G_\omega \cap F \leq Z(FQ) = Z(G)=1$. Therefore $F_\omega = 1$. Now we may write $\omega = \omega_1\omega_2$ where ω_1 and ω_2 are non-trivial elements of A. Since $Q_{\omega_1\omega_2} = Q$, it follows that $Q_{\omega_1} = Q_{\omega_2}$. Now form F_{ω_1} and F_{ω_2}. Since $F_{\omega_1\omega_2} = 1$, it follows from Chapter III, lemma 5 that $F_{\omega_1}F_{\omega_2} = F$. Now G_{ω_1} and G_{ω_2} are nilpotent so as before $Q_{\omega_1} = Q_{\omega_2}$ is centralized by F_{ω_1} and F_{ω_2}. Therefore $Q_{\omega_1} \leq C_G(F_{\omega_1}F_{\omega_2}) = C_G(F) = F$. Thus ω_1 induces a regular automorphism of Q which implies that Q is abelian. Since $G = FQ$, we conclude that $G' \leq F$ contrary to the definition of G. Therefore for no $\omega \in A$, $\omega \neq 1$, is $(G/F)_\omega = G/F$.

If $F_\omega = 1$ or F for some $\omega \in A$, $\omega \neq 1$ then ω either inverts or fixes all the elements of F. Since G/F is represented faithfully as a group of automorphisms of F, it follows that $(G/F)_\omega = G/F$. Since this has already been shown to be false, we conclude that for each $\omega \in A$, $\omega \neq 1$, $F > F_\omega > 1$. Also, since $C_G(F) = F$, $C_\Gamma(F) = F$.

Since $C_\Gamma(F) = F$, $F = F(\Gamma)$ is the unique minimal normal subgroup of Γ. Thus we may deduce from Chapter II (5), that there exists a complement N of F in Γ. By Sylow's theorem we may suppose that $A \leq N$. Let $M = G \cap N$ and $F(M) = Q$. The modular law implies that M is a complement of F in G.

For convenience we now summarize the properties of F which we have obtained.

(a) F is the unique minimal normal subgroup of Γ.

(b) $C_\Gamma(F) = F$.

(c) If $\omega \in A$, $\omega \neq 1$ then $Q_\omega \leq C_G(F_\omega)$. This follows since F is a p-group, $Q = F_2(G)/F$ and G_ω is nilpotent.

(d) for each $\omega \in A$, $\omega \neq 1$, $F_\omega > 1$.

Because of properties (a) and (b), F may be considered as a faithful irreducible (Γ/F)-module over $GF(p)$. Let $\mathfrak{F}$ denote the algebraic closure of $GF(p)$. Applying the same method as in the proof of Chapter IV, lemma 2 we deduce the existence of an $\mathfrak{F}(N)$-module V with the properties:

(1) V is a faithful irreducible N-module over $\mathcal{F}$,

(2) for each $\omega \in A$, $V_\omega = \{v \in V \mid v\omega = v\} > 0$,

(3) for each $\omega \in A$, $\omega \neq 1$, if $v \in V_\omega$ and $x \in Q_\omega$ then $vx = v$.

The next step in the proof is to show that

(4) V is an irreducible $\mathcal{F}(M)$-module.

Suppose by way of contradiction that V is not an irreducible $\mathcal{F}(M)$-module. Let W be an irreducible $\mathcal{F}(M)$-submodule of V . Then for at least two elements $\omega_1, \omega_2 \in A$, $(\omega_1, \omega_2 \neq 1)$ we have $W\omega_1 \neq W$ and $W\omega_2 \neq W$. Let $w \in W$ so that $\omega + \omega\omega_i \in V_{\omega_i}$ $(i=1,2)$. Now if $y \in Q_{\omega_i}$ then by property (3) $(w + w\omega_i)y = w + w\omega_i$.

Equating the W components of each side we deduce that Q_{ω_i} acts trivially on W and so on V . But V is a faithful N-module over $\mathcal{F}$ so it follows that $Q_{\omega_i} = 1$ for $i=1,2$. By Chapter III, lemma 4 ω_1 and ω_2 each invert all the elements of Q so $Q_{\omega_1\omega_2} = Q$. By properties (c) and (d) of F and since F is abelian it follows that

$$1 < F_{\omega_1\omega_2} \leq Z(Q_{\omega_1\omega_2}F) = Z(FQ) = Z(F_2(G)) .$$

Since F is the unique minimal normal subgroup of Γ and $Z(F_2(G))$ is normal in Γ , $F \leq Z(F_2(G))$. But this implies that $F_2(G)$ is nilpotent, contrary to the fact that G is soluble and non-nilpotent. This contradiction proves (4) .

If we now follow the proof of Chapter V, lemma 6 then we deduce

(A) if $\omega \in A$, $\omega \neq 1$, and L is a non-trivial normal ω-subgroup of M , then $1 < L_\omega < L$.

It follows from (A) that $Z(M) = 1$, or else since $(Z(M))_\omega$ is a normal ω-subgroup of M , $((Z(M))_\omega)_\omega \neq (Z(M))_\omega$, a contradiction.

Since $Z(M) = 1$, $F_2(G)$ is a proper subgroup of G , so by the minimality of G , $Q = F_2(G)/F$ is abelian. We may also deduce from the minimality of G that M/Q is characteristically simple. Therefore M/Q is an elementary abelian r-group for some prime r .

Suppose that r divides the order of Q . Let R be a Sylow r-subgroup of M . Since Q is a normal subgroup of M and $r \mid |Q|$, $Z(R) \cap Q > 1$. But $Z(R) \cap Q \leq Z(RQ) = Z(M)$ since Q is abelian. Thus $Z(M) > 1$, a contradiction. It follows that r does not divide the order of Q .

Let R be a Sylow r-subgroup of M . Since Q is of order prime to r , $R \cap Q = 1$. Clearly $RQ = M$. Now form $N_N(R)$. It is easily shown that $N_N(R)Q = N$ so, by taking a suitable conjugate of R if necessary, we may suppose that A normalizes R . Thus R is an A-complement of Q in M . We could also have obtained the existence of this complement by using Chapter III, lemma 2.

It is an easy consequence of the minimality of G that the representation of A on R is irreducible. But an irreducible representation over a field of characteristic not equal to two, of the non-cyclic group of order 4 is one-dimensional. Therefore for at least one $\omega \in A$,

$\omega \neq 1$, we have $R_\omega = R$. Now since $M_\omega \leq G_\omega$ is nilpotent, r does not divide the order of Q and Q is abelian, $Q_\omega \leq Z(RQ) = Z(M) = 1$. But by (A) , $Q_\omega > 1$. This contradiction completes the proof of theorem 1 .

Proof of Theorem 2. Suppose that the theorem is false and choose a counterexample G of minimal order. Then $F = F(G)$ is the unique minimal normal A-subgroup of G .

If L is a proper normal A-subgroup of G then L is nilpotent. Therefore F is the unique maximal normal A-subgroup of G and G/F is an elementary abelian r-group for some prime r since G is soluble. Thus G/F is an irreducible A-module over $GF(r)$. Since A is of exponent two, any representation of A over a field of characteristic not equal to two is one dimensional. Therefore the kernel of the representation of A on G/F must have order at least 4. Let ω_1 and ω_2 be two distinct non-unit elements of A in the kernel. Then

$$G/F = (G/F)_{\omega_1} = (G/F)_{\omega_2} = (G/F)_{\omega_1\omega_2} .$$

Suppose that $\omega \in A$, $\omega \neq 1$, and $(G/F)_\omega = G/F$. Since F is the unique minimal normal A-subgroup of the soluble group G , F is an elementary abelian p-group. By definition G is not nilpotent, so G/F is not a p-group. Therefore $r \neq p$. Now $(G/F)_\omega = G/F$ is isomorphic to a section of G_ω so the Sylow r-subgroup R of G_ω is a complement of F in G . Since G_ω is nilpotent and F is abelian, $F_\omega = F \cap G_\omega$ is centralized by $RF = G$. Now if $Z(G) > 1$, since F is the unique minimal normal A-subgroup of G , $Z(G) \geq F = F(G)$, a contradiction since G is soluble. Therefore $F_\omega \leq Z(G) = 1$. It follows that ω inverts all the elements of F .

Combining these results we have for $x \in F$,

$$x^{\omega_1} = x^{\omega_2} = x^{\omega_1\omega_2} = x^{-1} .$$

Thus $x^{-1} = x^{\omega_1\omega_2} = (x^{-1})^{\omega_2} = x$,

contrary to the assumption that the order of G is odd. This proves the theorem.

REFERENCES

[1] Alpenin, J.L., "Automorphisms of Solvable Groups", Proc. Amer. Math. Soc. 13 (1962),175-180.

[2] Baer, R., "Nilpotent Characteristic Subgroups of Finite Groups", Amer.J.Math. 75 (1953), 633-664.

[3] Burnside, W., "Theory of Groups of Finite Order", Second Edition, Dover, 1955.

[4] Carter, R.W., "On a Class of Finite Soluble Groups", Proc. London Math. Soc. (3) 9 (1959), 623-640.

[5] Curtis, C. and Reiner, I., "Representation Theory of Finite Groups and Associative Algebras", Interscience, (1962).

[6] Feit, W. and Thompson, J., "Solvability of Groups of Odd Order", Pacific J. Math. 13 (1963) 775-1029.

[7] Fischer, B., "Finite Groups Admitting a Fixed-Point-Free Automorphism of Order 2p", J. Algebra 3 (1966),99-114.

[8] Gaschutz, W., "Über die Φ-Untergruppe endlicher Gruppen", Math. Zeitschrift, 58 (1953), 160-170.

[9] Gorenstein, D. and Hernstein, I.N., "Finite Groups Admitting a Fixed-Point-Free Automorphism of Order 4", Amer. J. Math. 83 (1961), 71-78.

[10] Hall, M. Jr., "The Theory of Groups", New York, 1959.

[11] Hall, P., "On the Sylow Systems of a Soluble Group", Proc. London Math. Soc., (2) 43 (1937), 316-323.

[12] Hall, P., "On the System Normalizers of a Soluble Group", Proc. London Math. Soc., (2) 43 (1937), 507-528.

[13] Higman, G., "On Groups and Rings which Possess Automorphisms without Non-trivial Fixed Elements", J. London Math. Soc., 32 (1957), 321-334.

[14] Kovacs, L. and Wall, G.E., "Involutary Automorphisms of Groups of Odd Order and their Fixed-Point Groups", Nagoya Math. J., 27 (1966), 113-119.

[15] Thompson, J., "Finite Groups with Fixed-Point-Free Automorphisms of Prime Order", Proc. Nath. Acad. Sci., 45 (1959), 578-581.

[16] Thompson, J., "Automorphisms of Soluble Groups", J. Algebra, 1 (1964), 259-267.

[17] Wielandt, H., "Über Produkte von nilpotenten Gruppen", Illinois J. Math., 2 (1959), 611-618.

[18] Zassenhaus, H., "The Theory of Groups", Second Edition, Chelsea, New York, 1958.