INDUCED REPRESENTATIONS OF LIE ALGEBRAS

a thesis submitted for the degree

of

DOCTOR OF PHILOSOPHY

by

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PREFACE

All the results proved in this thesis after chapter 3 are original except where I have indicated otherwise in the text. It should be noted that several of the results were inspired by papers of Nolan R. Wallach [16], [17]. In particular, section (8.2) presents relevant parts of Wallach's work, and interprets his work in terms of chapters 4 - 7, while section (8.3) is a more precise and categorical description of another part of Wallach's work. Section (8.4) is original, but section (8.5) uses techniques developed by Hochschild and Mostow in their paper [7] to obtain similar results to theirs. My construction in section (8.5) is, unlike theirs, functorial and natural.

The material of chapter 3 and of sections (4.3), (4.4a), (4.5), (4.6a) is probably well-known, but I have been unable to locate proofs in the literature.

I wish to thank my supervisor, Doctor D.W. Barnes, for his guidance and for numerous suggestions about the presentation of the material in this thesis. I also wish to thank Doctor James N. Ward for undertaking the onerous task of reading early drafts of this thesis and suggesting many improvements and corrections. Finally, I should like to thank members of the Sydney Category Theory Group for their time and help in clarifying several points for me.

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Contents:

Chapter 0 - Introduction

Notation is explained in chapter 1. The symbol (m.n) refers to the n^{th} section of chapter m.

(0.1) Comparison of Induced Representations of Groups and Lie Algebras

In the theory of representations of finite groups, a construction which often proves useful is that of the induced representation. Given finite groups $H \leq G$, a field k, and a right kH-module M, one forms the induced module $M \otimes_{kH} kG$ and the coinduced module $\mathrm{Hom}_{kH}(kG, M)$, both of which may be given the structure of right kG-modules in a natural way. Some of the important properties of induced and coinduced modules for finite groups are:

(1) (Frobenius reciprocity isomorphisms) If M is a right kH-module and N is a right kG-module, then

$$\mathrm{Hom}_{kG}(M \otimes_{kH} kG, N) \simeq \mathrm{Hom}_{kH}(M, N)$$

and

$$\mathrm{Hom}_{kG}(N, \mathrm{Hom}_{kH}(kG, M)) \simeq \mathrm{Hom}_{kH}(N, M);$$

(2) $\dim_k(M \otimes_{kH} kG) = [G \div H] \,.\, \dim_k M$;

(3) M may be embedded in $M \otimes_{kH} kG$, regarded as a kH-module, by a naturally split, natural kH-monomorphism;

(4) $M \otimes_{kH} kG \simeq \mathrm{Hom}_{kH}(kG, M)$ as kG-modules.

These four properties of induced representations are among the reasons why induced representations form a useful tool in the study of finite groups and their representations.

When we attempt a parallel construction for finite-dimensional Lie algebras $\mathfrak{h} \leq \mathfrak{g}$ over a field k and a right $\mathfrak{h}$-module W, difficulties arise. The Lie-algebra analogue of the group algebra is the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ (defined in section (1.2)). One can construct the $U\mathfrak{g}$-modules $W \otimes_{U\mathfrak{h}} U\mathfrak{g}$ and $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$ as before - details are given in chapter 3. The analogues of properties (1) and (3) above hold. However the analogues of properties (2) and (4) above fail except when $W = (0)$ or $\mathfrak{h} = \mathfrak{g}$. This difficulty destroys most of the usefulness of the constructions.

The aim of this thesis is to look for alternative constructions and to determine what properties such alternative constructions may possess.

(0.2) Suitable Properties for an Induced Module

The isomorphisms of property (1) of section (0.1) determine $M \otimes_{kH} kG$ and $\mathrm{Hom}_{kH}(kG, M)$ in an (essentially) unique way. The remarks in the latter part of section (0.1) then show that we cannot expect our alternative constructions to satisfy such isomorphism properties.

Let us denote by R the obvious restriction functor,

$$R : \text{Mod-}\mathfrak{g} \to \text{Mod-}\mathfrak{h}$$

for Lie algebras $\mathfrak{h} \leq \mathfrak{g}$.

Bearing property (3) of section (0.1) in mind, we should like to find, for every finite-dimensional right $U\mathfrak{h}$-module W, a finite dimensional right $U\mathfrak{g}$-module V and a $U\mathfrak{h}$-monomorphism

$$j_W : W \to RV.$$

Even this turns out to be impossible in general. We shall produce two examples, in section (0.3), which demonstrate this fact.

Thus, instead, we shall try to associate with each (finite-dimensional or infinite-dimensional) $U\underline{\underline{h}}$-module W, a $U\underline{\underline{g}}$-module V and a $U\underline{\underline{h}}$-monomorphism

$$j_W : W \to RV.$$

Later, we shall investigate conditions for finite-dimensionality.

We shall make three other demands on our "induced module" V and the associated injection j_W:

(i) we require that V depend functorially on W; that is, we suppose that there exists a functor $I : \text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{g}}$ and for each right $\underline{\underline{h}}$-module an $\underline{\underline{h}}$-monomorphism $j_W : W \to RIW$;

(ii) we require that j_W be natural in W;

(iii) we make a requirement which ensures that IW is not unnecessarily large; we require that

$$(\text{im } j_W) \,.\, U\underline{\underline{g}} = IW.$$

One of the central results of this thesis (theorem (5.8)) will be to show that these three conditions imply an important part of the analogue of the Frobenius reciprocity isomorphisms (see property (1) of section (0.1)).

All of the remarks about j_W may be dualized. If this is done, we find ourselves discussing a natural $U\underline{\underline{h}}$-epimorphism $k_W : RIW \to W$; theorem (5.16) is a dual characterization of another part of the Frobenius reciprocity isomorphisms.

In fact, theorem (5.16) implies that if the natural map $k_W : RIW \to W$ satisfies the condition that ker k_W contains no nonzero

$\mathfrak{g}$-modules, then there is a natural injection

$$\mathrm{Hom}_{U\mathfrak{g}}(V, IW) \to \mathrm{Hom}_{U\mathfrak{h}}(RV, W)$$

for all $\mathfrak{g}$-modules V and $\mathfrak{h}$-modules W. Compare this with the second Frobenius isomorphism of section (0.1), number (1).

The development, in chapters 4, 5 and 6, of the ideas outlined above, will be carried out for a pair of abstract categories $\mathbf{H}$ and $\mathbf{G}$ together with functors

$$R : \mathbf{G} \to \mathbf{H}$$

and

$$I : \mathbf{H} \to \mathbf{G}.$$

It will sometimes be necessary to assume that $\mathbf{H}$ and $\mathbf{G}$ have certain properties of Abelian categories. Further, in chapters 2 to 6, the development will be carried out in a way converse to that outlined above. That is, we shall start with properties <u>like</u> the Frobenius reciprocity isomoprhisms and show that they are equivalent to certain properties of the maps j_W and k_W mentioned above.

In chapter 7, we shall discuss ways of constructing a functor I and maps j_W and k_W in the particular case where $\mathbf{H}$ = Mod-$\mathfrak{h}$ and $\mathbf{G}$ = Mod-$\mathfrak{g}$ and $\mathfrak{h} \leq \mathfrak{g}$ are Lie algebras. The discussion in chapter 7 is, however, still theoretical. We also prove, in this theoretical setting, a simplicity criterion for induced modules, based on a result of Wallach [16].

In chapter 8, we discuss <u>models</u> of the theory developed in chapters 4 - 7, including constructions of Wallach [15,16] and a modification of a construction of Hochschild and Mostow [7].

Chapter 9 contains a report on some results in Lie structure of rings which arose as an offshoot of the work described above:- an important part of the theory of induced representations of groups is Clifford's theory of induced representations of group extensions. We prove analogues of some of Clifford's theorems for Lie ideal subrings[1] of rings. These results are also analogous to those of Zassenhaus [17] and Barnes and Newell [1] for Lie algebras.

(0.3) Two Examples of Lie Algebras in which Induction is, in General, Impossible

In this section we prove a claim, made in section (0.2), that there exist Lie algebras $\mathfrak{h} \leq \mathfrak{g}$ and finite-dimensional right $\mathfrak{h}$-modules W which cannot be embedded in any finite-dimensional $\mathfrak{G}$-module.

We shall use the following interesting theorem of Zassenhaus:

<u>Theorem</u> ([17], page 252): Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic zero and let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$. Then every finite-dimensional representation of $\mathfrak{g}$ restricts to a nilpotent representation of[2] $[\mathfrak{g},\mathfrak{g}] \cap \mathrm{rad}(\mathfrak{h})$.

We shall now produce a (finite-dimensional) $U\mathfrak{h}$-module on which $[\mathfrak{g},\mathfrak{g}] \cap \mathrm{rad}(\mathfrak{h})$ does <u>not</u> act nilpotently.

<u>Example A</u>: Let $\mathfrak{g}$ be a 2-dimensional Lie algebra over the field $\mathbb{C}$ of complex numbers, with basis $\{e,f\}$ and multiplication determined by the relation $[e,f] = e$. Let $\mathfrak{h}$ be the subspace of $\mathfrak{g}$ spanned by $\{e\}$. It is easily verified that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and that $[\mathfrak{g},\mathfrak{g}] \cap \mathrm{rad}(\mathfrak{h}) = \mathfrak{h}$.

Let W be a one-dimensional vector space over $\mathbb{C}$. Determine a $U\mathfrak{h}$-module structure on W by choosing a non-zero $w \in W$ and setting

---

1 defined in chapter 9.

2 notation is explained in section (1.4).

$$w \,.\, e = \lambda w$$

for some chosen $\lambda \in \mathbb{C}$.

If $\lambda \neq 0$, then $\underline{\underline{h}} = [\underline{\underline{g}},\underline{\underline{g}}] \cap \mathrm{rad}(\underline{\underline{h}})$ does not act nilpotently on W.

$\square$

Remark: Because of the importance of Cartan subalgebras, that is, self-normalising nilpotent subalgebras, in the study of semisimple Lie algebras, and since the subalgebra $\underline{\underline{h}}$ of Example A above is not a Cartan subalgebra, we present an extra example with a Cartan subalgebra in it.

Example B: Let $\underline{\underline{g}} = sl(2,\mathbb{C})$ - the Lie algebra of 2×2 matrices over $\mathbb{C}$ with trace zero. It is well-known that $\underline{\underline{g}}$ is simple and has a Cartan subalgebra of dimension one - spanned by $\{h\}$, say.

Define a one-dimensional $\underline{\underline{h}}$-module W by choosing a non-zero $w \in W$ and a non-integer $\lambda \in \mathbb{C}$ and setting

$$w \,.\, h = \lambda w.$$

Then the following result (quoted from Humphreys [8], Corollary 7.2 page 33) shows that W cannot be embedded in a finite-dimensional $\underline{\underline{g}}$-module.

Proposition: Let V be any finite-dimensional $\underline{\underline{g}}$-module ($\underline{\underline{g}} = sl(2,\mathbb{C})$). Then the eigenvalues of the Cartan subalgebra $\underline{\underline{h}}$ on V are all integers.

$\square$

Chapter 1 - Notation and Assumed Results

(1.0) Linearity.

Many of the results and constructions of this thesis require a check that a map is linear. Without exception, these checks are trivial. They will therefore be omitted without further comment.

(1.1) Categorical Conventions, Assumptions, and Definitions

The basic notions of category, object, morphism, domain and co-domain, functor, natural transformation, left and right adjoint and adjunction, isomorphism, (commutative) diagram, full and faithful functors, and duality will be assumed to be known. The notation used for these concepts is set out in section (1.4). See MacLane [12] for definitions.

We shall also require the notion of preadditive category and additive functor (see MacLane [12], pages 28-29): any functor between preadditive categories will be tacitly assumed to be additive. Similarly, if the morphism sets in a category carry a vector space structure, all functors and natural transformations will be assumed to be linear.

Zero Object. All categories will be assumed to contain a zero object, that is, an object, denoted 0, such that, for every other object A in the category, there is exactly one morphism $0 \to A$ and exactly one morphism $A \to 0$. Both these morphisms will be denoted by the symbol 0.

Composition of Morphisms. Morphisms will be composed on the left. In particular categories where the morphisms are functions, they will be written on the left. Thus, if $f : A \to B$ and $g : B \to C$ are morphisms in some category, then their composition is written $gf : A \to C$, or simply gf, or sometimes, $g\circ f$. In particular, functors are written and composed on the left.

"Factoring Through". Suppose $f : A \to C$ and $g : B \to C$ are morphisms. We say that f factors through B via g if there exists a morphism $h : A \to B$ such that $gh = f$.

The expression "factors through" is also used in the dual situation: if $f : A \to C$ and $h : A \to B$ are morphisms, we say f factors through B via h if there exists $g : B \to C$ such that $gh = f$.

The next dozen or so definitions follow Mitchell [13], pages 5-18.

Monomorphism or monic. A morphism $\alpha : A \to B$ is called a monomorphism or a monic if, for all pairs f, g of morphisms with codomain A, $\alpha f = \alpha g$ implies $f = g$.

Epimorphism or epi. A morphism $\alpha : A \to B$ is called an epimorphism or an epi if, for all pairs f, g of morphisms with domain B, $f\alpha = g\alpha$ implies $f = g$.

Subobjects. If $\alpha : A' \to A$ is a monic, we shall call (A',α) a subobject of A, and shall refer to α as the (natural) inclusion of A' in A. If it is clear from context which monic $\alpha : A' \to A$ is being referred to, we may refer to A' as a subobject of A.

Isomorphic subobjects. Suppose $\alpha_1 : A_1 \to A$ and $\alpha_2 : A_2 \to A$ are subobject inclusions. A_1 and A_2 are called isomorphic subobjects of A if there is an isomorphism $\iota : A_1 \to A_2$ such that $\alpha_2\iota = \alpha_1$.

Quotient objects. If $\alpha : A \to A'$ is epi, we shall refer to (A',α) (or sometimes just A') as a quotient object of A, and shall refer to α as the (natural) projection of A onto A'.

Isomorphic quotient objects are defined in a manner dual to the definition of isomorphic subobjects.

Image of a morphism. An image of a morphism $f : A \to B$ is defined to be a subobject (I,u) of B such that

(i) f factors through I via u; and

(ii) if (J,v) is any other subobject of B such that f factors through (J,v), then there is a monic $w : I \to J$ such that the following diagram commutes:

$$\begin{tikzcd} I \arrow[rr, "u"] \arrow[dr, "w"'] & & B \\ & J \arrow[ur, "v"'] & \end{tikzcd}$$

That is, $vw = u$.

Coimage. A coimage of a morphism is defined in a manner dual to the definition of "image".

In general, a morphism f need have neither image nor coimage. An image of f, if it exists, is denoted by $\mathrm{im}\, f$, similarly $\mathrm{coim}\, f$.

Kernel of a morphism. A kernel of a morphism $f : A \to B$ is a subobject (K,i) of A such that

(i) $fi = 0$; and

(ii) if (J,l) is any subobject of A such that $fl = 0$, then there is a unique monic $m : J \to K$ such that $im = l$, that is, the following diagram commutes:

$$\begin{tikzcd} K \arrow[rr, "i"] & & A \arrow[r, "f"] & B \\ & J \arrow[ul, "m"] \arrow[ur, "l"'] & & \end{tikzcd}$$

Cokernel. The definition of a cokernel is dual to that of a kernel.

In general, a morphism need have neither a kernel nor a cokernel. If, in a category $\underline{\underline{A}}$, every morphism has a kernel (respectively, a cokernel), we say that $\underline{\underline{A}}$ has kernels (respectively, has cokernels). The kernel of a morphism f, if there is one, is denoted by $\ker f$, similarly $\mathrm{coker}\, f$.

Exact Sequence. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of morphisms and objects in a category is called exact at B if im f and ker g exist, and are isomorphic subobjects of B.

Exact Category. A category $\underline{\underline{A}}$ is called exact (cf Mitchell [13], page 18) if the following four conditions hold:

(i) $\underline{\underline{A}}$ has kernels and cokernels;

(ii) every monic in $\underline{\underline{A}}$ is a kernel;

(iii) every epi in $\underline{\underline{A}}$ is a cokernel;

(iv) every morphism $\alpha : A \to B$ in $\underline{\underline{A}}$ can be written as the composition of a monic i and an epi p, so that $\alpha = ip$; that is, so that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, "\alpha"] \arrow[dr, "p"'] & & B \\ & I \arrow[ur, "i"'] & \end{tikzcd}$$

(1.2) Universal Enveloping Algebras

The definition and elementary properties of tensor products will be assumed to be known (cf Curtis and Reiner [2] (12.1) - (12.6)). The notation is explained in section (1.4).

Construction: Let $\underline{\underline{g}}$ be a Lie algebra over a field k. We are going to construct an associative algebra $U\underline{\underline{g}}$ called the universal enveloping algebra of $\underline{\underline{g}}$.

First we form the tensor algebra $T\underline{\underline{g}}$ on the vector space underlying $\underline{\underline{g}}$. Define $T^o\underline{\underline{g}} = k$ and $T^{i+1}\underline{\underline{g}} = T^i\underline{\underline{g}} \otimes_k \underline{\underline{g}}$ for $i \geq 0$ and set

$$T\underline{\underline{g}} = \bigoplus_{i=0}^{\infty} T^i\underline{\underline{g}} = k \oplus \underline{\underline{g}} \oplus (\underline{\underline{g}} \otimes \underline{\underline{g}}) \oplus \ldots$$

The tensor algebra is endowed with an associative k-algebra structure in an obvious way - see Hilton and Stammbach [6] page 230 for details.

Next, we form the two-sided ideal R of $T\underline{\underline{g}}$ generated by all elements of the form $x \otimes y - y \otimes x - [x,y]$ (where $x,y \in \underline{\underline{G}}$ and $[x,y]$ denotes Lie multiplication in $\underline{\underline{g}}$.

Finally, we form the quotient algebra

$$U\underline{\underline{g}} = T\underline{\underline{g}}/R.$$

Definition: Suppose A is an associative k-algebra. We define a _Lie algebra LA_ as follows. Let the underlying vector space of LA be the same as that of A. Suppose $\cdot$ denotes the multiplication on A. We define a Lie multiplication [,] on LA by setting

$$[x,y] = x \cdot y - y \cdot x \quad \text{for } x,y \in A = LA.$$

It can be checked that [,] is indeed a Lie multiplication, and that L is a functor from the category of all associative k-algebras to the category of all Lie algebras over k.

Remarks on Universal Enveloping Algebras

(1) $U\underline{\underline{g}}$ is an associative k-algebra with 1. The map $\underline{\underline{g}} \to LU\underline{\underline{g}}$ defined by $g \mapsto g + R \in T\underline{\underline{g}}/R = U\underline{\underline{g}}$ (for $g \in \underline{\underline{g}}$) is a (natural) monomorphism of Lie algebras. That the map so defined is injective is an immediate consequence of the Poincaré-Birkhoff-Witt theorem - see section (1.3).

(2) Despite the construction using tensor products and quotient by an ideal, multiplication in $U\underline{\underline{g}}$ will usually be denoted either by $\cdot$ or by juxtaposition.

(3) $U\underline{\underline{g}}$ will frequently be regarded as a (right and/or left) $U\underline{\underline{g}}$-module via the regular representation(s) (cf Curtis and Reiner [2], page 48).

(4) There is a natural isomorphism between the category of right $\mathfrak{g}$-modules and the category of right $U\mathfrak{g}$-modules which preserves the underlying vector spaces and respects the natural embedding of $\mathfrak{g}$ in $LU\mathfrak{g}$, mentioned in (1) above.

Accordingly, we use the terms "$\mathfrak{g}$-module" and "$U\mathfrak{g}$-module" interchangeably.

All modules will be either right modules or bimodules.

(5) If $\mathfrak{h} \leq \mathfrak{g}$ are Lie algebras, then there is an obvious embedding $T\mathfrak{h} \to T\mathfrak{g}$. The theorem described in the next section (section (1.3)) allows us to deduce that this embedding $T\mathfrak{h} \to T\mathfrak{g}$ induces an embedding $U\mathfrak{h} \to U\mathfrak{g}$. Thus, in particular, $U\mathfrak{g}$ may be regarded as a (left and/or right) $U\mathfrak{h}$-module, by restriction of the regular representations of $U\mathfrak{g}$ on $U\mathfrak{g}$.

(1.3) The Poincaré-Birkhoff-Witt Theorem.

Let $\mathfrak{g}$ be a Lie algebra over k. We retain in this section the notation of section (1.2) above. The structure of the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ is elucidated by a theorem of Poincaré, Birkhoff and Witt. To state this, we must make a definition.

Definition - Standard Monomial. Let $\{e_i : i \in J\}$ be a basis of $\mathfrak{g}$ over k, and let J be totally ordered. For each nondecreasing sequence $S = (i_1, \ldots, i_l)$ of elements of J, we define an element e_S of $U\mathfrak{g}$ by

$$e_S = e_{i_1} \otimes \ldots \otimes e_{i_l} + R.$$

This element is, in fact, usually written as

$$e_S = e_{i_1} e_{i_2} e_{i_3} \ldots e_{i_l},$$

omitting the $\otimes$-signs and the ideal R, as explained in Remark (2) of section (1.2). Any element of $U\mathfrak{g}$ so constructed is called a *standard monomial* (with respect to the totally ordered basis $\{e_i : i \in J\}$ of $\mathfrak{g}$.)

Note that the empty sequence $S = \phi$ is allowed, and that e_ϕ is the identity element of $U\mathfrak{g}$; it will be denoted by $1_{U\mathfrak{g}}$.

Theorem (Poincaré-Birkhoff-Witt): The standard monomials, with respect to any ordered basis of $\mathfrak{G}$, form a basis for the underlying vector space of $U\mathfrak{g}$.

For a proof, see, for example, Humphreys [8], pages 93ff.

Corollary: Let $\mathfrak{h}$ be a subalgebra of the Lie algebra $\mathfrak{g}$. Then $U\mathfrak{g}$ is free as a (left *or* right) $U\mathfrak{h}$-module.

Proof of corollary (taken from Hilton and Stammbach [6], page 232): Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{x}$ as a vector space - that is, choose a vector space complement $\mathfrak{x}$ for $\mathfrak{h}$ in $\mathfrak{g}$. Let H be a totally ordered basis of $\mathfrak{h}$, and let X be a totally ordered basis of $\mathfrak{x}$. These orderings may be extended to a total ordering of the basis $H \cup X$ of $\mathfrak{g}$, such that if $h \in H$ and $x \in X$ then $h < x$, in exactly one way.

With respect to this total ordering of $H \cup X$, the standard monomials which involve only elements of X form a basis of $U\mathfrak{g}$ as left $U\mathfrak{h}$-module, by the Poincaré-Birkhoff-Witt theorem. That is

(Equation (A)) $$U\mathfrak{g} \simeq \bigoplus_{\substack{x_S \text{ a standard} \\ \text{monomial in } X}} U\mathfrak{h} \,.\, x_S \quad \text{as } \textit{left } U\mathfrak{h}\text{-modules.}$$

The orderings of H and X may also be extended to a total ordering of $H \cup X$ such that if $h \in H$ and $x \in X$ then $h > x$, again in exactly one way. This time, we deduce from the Poincaré-Birkhoff-Witt theorem that, as a right $U\underline{\underline{h}}$-module,

(Equation (B)) $$U\underline{\underline{g}} \simeq \bigoplus_{\substack{x'_S \text{ a standard} \\ \text{monomial in } X}} x'_S \,.\, U\underline{\underline{h}} \qquad \text{as a right } U\underline{\underline{h}}\text{-module.}$$

The corollary now follows from the facts that $U\underline{\underline{h}} \,.\, x_S$ is isomorphic to $U\underline{\underline{h}}$ as left $U\underline{\underline{h}}$-module, and $x'_S \,.\, U\underline{\underline{h}}$ is isomorphic to $U\underline{\underline{h}}$ as right $U\underline{\underline{h}}$-module.

Corollary: If $\underline{\underline{h}}$ is a subalgebra of the Lie algebra $\underline{\underline{g}}$ such that there exists a subalgebra $\underline{\underline{x}}$ of $\underline{\underline{g}}$ such that $\underline{\underline{g}} = \underline{\underline{h}} \oplus \underline{\underline{x}}$ as a vector space, then

$$U\underline{\underline{g}} = U\underline{\underline{h}} \oplus U\underline{\underline{h}} \,.\, \underline{\underline{x}} \,.\, U\underline{\underline{x}} \quad \text{as a left } U\underline{\underline{h}}\text{-module}$$

and $$U\underline{\underline{g}} = U\underline{\underline{h}} \oplus U\underline{\underline{x}} \,.\, \underline{\underline{x}} \,.\, U\underline{\underline{h}} \quad \text{as a right } U\underline{\underline{h}}\text{-module.}$$

Proof. Note that $\underline{\underline{x}} \,.\, U\underline{\underline{x}}$ may be thought of as the subspace of $U\underline{\underline{x}}$ spanned by all standard monomials e_S for which $S \neq \phi$. The corollary now follows from the proof of the previous corollary. $\square$

(1.4) Notation

(a) Categorical Notation

Let $\underline{\underline{A}}$, $\underline{\underline{B}}$ be categories. Then

(i) by $A \in \underline{\underline{A}}$ we shall mean that A is an object of A; (Let $A_1, A_2, A \in \underline{\underline{A}}$. The notations $f : A_1 \to A_2$ and $A_1 \xrightarrow{f} A_2$ will suggest that f is a morphism with domain A_1 and codomain A_2. This notation serves largely as a reminder about domains and codomains.)

(ii) 1_A and $1_{\underline{\underline{A}}}$ denote, respectively, the identity morphism on A and the identity functor $\underline{\underline{A}} \to \underline{\underline{A}}$;

(iii) $\underline{\underline{A}}(A_1, A_2)$ means the set of all morphisms $A_1 \to A_2$ in $\underline{\underline{A}}$;

(iv) $\underline{\underline{A}}(A, f)$ denotes the induced map

$$\underline{\underline{A}}(A,A_1) \to \underline{\underline{A}}(A,A_2)$$

defined by $\underline{\underline{A}}(A,f)(\phi) = f \circ \phi$ for $\phi \in \underline{\underline{A}}(A,A_1)$;

(v) $\underline{\underline{A}}(f,A)$ denotes the induced map

$$\underline{\underline{A}}(A_2,A) \to \underline{\underline{A}}(A_1,A)$$

defined by $\underline{\underline{A}}(f,A)(\phi) = \phi \circ f$ for $\phi \in \underline{\underline{A}}(A_2,A)$;

(vi) let $A_1', A_2' \in \underline{\underline{A}}$ and choose $\alpha \in \underline{\underline{A}}(A_1',A_1)$ and $\beta \in \underline{\underline{A}}(A_2,A_2')$. Then $\underline{\underline{A}}(\alpha,\beta)$ denotes the induced map

$$\underline{\underline{A}}(A_1,A_2) \to \underline{\underline{A}}(A_1',A_2')$$

defined by $\underline{\underline{A}}(\alpha,\beta)(\phi) = \beta \circ \phi \circ \alpha$ for $\phi \in \underline{\underline{A}}(A_1,A_2)$.

(vii) let F, G be functors $\underline{\underline{A}} \to \underline{\underline{B}}$. Then the notation $\eta : F \dot{\to} G$ will mean that η is a natural transformation from F to G. The <u>A-component</u> of a natural transformation $\eta : F \dot{\to} G$ will be denoted by $\eta_A : FA \to GA$, or just η_A.

(viii) Gf will denote the image of f under the morphism function of the functor G.

(b) Set-Theoretic Notation

An elementary knowledge of set theory will be assumed. Let G, H and K be sets. Then $g \in G$ means that g is an element of G, and

(i) $G \times H$ denotes the Cartesian product of G and H;

(ii) let $\alpha : G \to H$ be a function and suppose K is a subset of G: then $\alpha|_K$ denotes the function α with domain restricted to be K;

(iii) $K \subseteq G$ means K is a subset of G;

(iv) $K \subset G$ means K is a proper subset of G;

(v) $G \cup H$ means the union of G and H;

$G \cap H$ means the intersection of G and H;

(vi) the notations $f : a \mapsto b$ and $a \overset{f}{\mapsto} b$ indicate that $f(a) = b$; (these notations are often convenient when defining a particular function; for example, (iv) of section (a) above could have been written as "$\underline{\underline{A}}(A,f)$ is defined by

$$\phi \mapsto f \circ \phi \text{ for } \phi \in \underline{\underline{A}}(A,A_1)."\text{)}$$

(vii) a function $\alpha : G \to H$ is <u>injective</u> if for all pairs $g_1, g_2 \in G$, $\alpha(g_1) = \alpha(g_2)$ implies $g_1 = g_2$; a function $\alpha : G \to H$ is <u>surjective</u> if, for all $h \in H$, there exists $g \in G$ such that $\alpha(g) = h$.

(c) Lie Algebra Notation

An elementary knowledge of Lie algebras will be assumed.

All Lie algebras will be over a field k unless otherwise specified. k is also used to denote a certain natural transformation in the second half of the thesis, but, with this warning, no confusion should arise. Let $\underline{\underline{h}}$ and $\underline{\underline{g}}$ be Lie algebras. Then

(i) if $x,y \in \underline{\underline{g}}$, the (Lie) product of x and y will be written $[x,y]$;

(ii) $\underline{\underline{h}} \leq \underline{\underline{g}}$ means $\underline{\underline{h}}$ is a subalgebra of $\underline{\underline{g}}$;
$\underline{\underline{h}} < \underline{\underline{g}}$ means $\underline{\underline{h}}$ is a proper subalgebra of $\underline{\underline{g}}$;
$\underline{\underline{h}} \trianglelefteq \underline{\underline{g}}$ means $\underline{\underline{h}}$ is a (Lie) ideal of $\underline{\underline{g}}$;

(iii) $[\underline{\underline{g}},\underline{\underline{g}}]$ denotes the derived subalgebra of $\underline{\underline{g}}$;

(iv) $\mathrm{rad}(\underline{\underline{h}})$ denotes the solvable radical of $\underline{\underline{h}}$;

(v) $\mathrm{Aut}(\underline{\underline{g}})$ denotes the automorphism group of $\underline{\underline{g}}$;

(vi) Mod-$\underline{\underline{g}}$, Mod-$U\underline{\underline{g}}$, $\mathrm{Hom}_{\underline{\underline{g}}}$ and $\mathrm{Hom}_{U\underline{\underline{g}}}$ all denote the category of right $U\underline{\underline{g}}$-modules.

(d) Group-theoretic Notation

An elementary knowledge of group theory will be assumed. Let G and H be groups and let k be a field. Then

(i) $H \leq G$ means H is a subgroup of G;

$H < G$ means H is a proper subgroup of G;

$H \trianglelefteq G$ means H is a normal subgroup of G;

(ii) Aut G means automorphism group of G;

(iii) kG means group algebra of G over k;

(iv) Mod-kG, Hom_{kG} both mean the category of all right kG-modules;

(v) $[G \div H]$ means index of H in G (assuming $H \leq G$).

(e) Notation for Associative Rings and Algebras

An elementary knowledge of associative rings and algebras will be assumed. Let A, B be either associative rings with 1 or associative algebras with 1 over a field k. Let $x,y \in B$. Then

(i) 1_B denotes the identity element of B;

(ii) $A \leq B$ means A is a subring (subalgebra) of B and $1_B \in A$;

$A < B$ means A is a proper subring (subalgebra) of B and $1_B \in A$;

(iii) $A \trianglelefteq B$ means A is an ideal of B;

B/A means quotient of B by A;

(iv) $[x,y]$ denotes the commutator $xy - yx$ of x and y;

(v) Mod-A and A-Mod denote, respectively, the categories of right and left A-modules.

(f) Notation and Assumed Results for Module Theory and Vector Space Theory

We shall assume a fair amount of module theory: say the relevant parts of Hersteins "Topics in Algebra" (Blaisdell, 1964), together with some knowledge of products and coproducts, composition series, tensor products, semisimplicity, which can be found in Curtis and Reiner [2], and Rotman [15]. All modules will be unitary.

Let A, C be right modules over a ring or algebra R, and let B, B_1, B_2 be left R-modules. Let $\{S_\lambda : \lambda \in \Lambda\}$ be a family of R-modules (left or right but not a mixture), and let T be an Abelian group. Then

(i) $\mathrm{Hom}_R(A,C)$ means the set (Abelian group, or vector space) of all R-homomorphisms from A to C;

(ii) $A \otimes B$ denotes tensor product of A and B over R;

(iii) $\mathrm{End}_R(A)$ denotes the endomorphism ring of A as R-module;

(iv) an R-balanced map $\phi : A \times B \to T$ means a bilinear map ϕ, such that for all $a \in A$, $b \in B$ and $r \in R$, $\phi(ar,b) = \phi(a,rb)$;

(v) if $Y \subseteq A$, then $Y . R$ denotes the R-submodule of A generated by Y;

(vi) if $X \subseteq R$, then $\mathrm{Ann}_X(A) = \{r \in X : \text{for all } a \in A,\ a . r = 0\}$;

(vii) $\bigoplus_{\lambda \in \Lambda} S_\lambda$ means direct sum of the modules S_λ;

(viii) $\prod_{\lambda \in \Lambda} S_\lambda$ means direct product of the modules S_λ;

(ix) $- \otimes_R B$ means the functor $A \mapsto A \otimes_R B$;

(x) $A \otimes_R B$ means the functor $B \mapsto A \otimes_R B$;

(xi) $A \leq C$ means A is a submodule of C;

$A < C$ means A is a proper submodule of C;

C/A denotes the quotient of C by A (assumes $A \leq C$);

(xii) A <u>subquotient</u> of C is a submodule of a quotient module of C.

Let $V \leq W$ be vector spaces over a field k. Much of the above notation applies to vector spaces.

(xiii) $\dim_k V$ and $\dim V$ denote the dimension of V over k;

(xiv) the <u>codimension</u> of V in W is defined to be $\dim_k(W/V)$.

Chapter 2 - Green's Axiomatic Approach to Induced Representations

(2.1) Axiomatization of Induction, Restriction and Conjugation

Let $\underline{\underline{h}} \leq \underline{\underline{g}}$ be Lie algebras.

In section (0.3), we say that it is, in general, impossible to embed each finite-dimensional $\underline{\underline{h}}$-module in a finite-dimensional $\underline{\underline{g}}$-module. Furthermore, the pairs of Lie algebras used in examples A and B of section (0.3) were by no means contrived or pathological.

In this chapter, therefore, we shall change our aim. We shall discuss <u>well-behaved</u> ways of embedding $\underline{\underline{h}}$-modules in (not necessarily finite-dimensional) $\underline{\underline{g}}$-modules. By "well-behaved ways", we mean ways that obey axioms, (which we shall specify in section (2.2)) and which have the properties outlined in section (0.2), (such as functoriality and naturality).[1]

Our model of behaviour comes from the theory of induced representations of finite groups. J.A. Green, in [3], showed that the operations of induction, restriction and conjugation among the character rings of subgroups of a finite group can be characterized by a list of "axioms" relating induction and restriction and conjugation.

(2.2) Green's Axiom Scheme

We shall describe Green's axiomatization for categories of <u>modules</u> over finite groups rather than character rings, since we are

---

1 given a pair of Lie algebras $\underline{\underline{h}} \leq \underline{\underline{g}}$ and a rule for embedding $\underline{\underline{h}}$-modules in $\underline{\underline{g}}$-modules, we could pose the question: "how large is the class of $\underline{\underline{h}}$-modules which are embedded in finite-dimensional $\underline{\underline{g}}$-modules by the given rule?" We shall return to this question in chapters 7 and 8 (mainly section (8.2), parts (ix) and (x)).

interested in modules for Lie algebras.

Let G be a finite group and k a field. Let K and H be subgroups of G. If K is a subgroup of H, we define an <u>induction functor</u>

$$I_K^H : \text{Mod-}kK \to \text{Mod-}kH$$

by

$$I_K^H M = M \otimes_{kK} kH \quad \text{for } M \in \text{Mod-}kK.$$

Secondly, we define a <u>restriction functor</u>

$$R_K^H : \text{Mod-}kH \to \text{Mod-}kK$$

by

$$R_K^H N = N$$

as a vector space, but with algebra of operations restricted to be kK, for $N \in$ Mod-kK.

Finally, we define, for $\alpha \in$ Aut G and $M \in$ Mod-kK, a <u>conjugation functor</u>

$$C_{K,\alpha} : \text{Mod-}kK \to \text{Mod-}kK^{\alpha}$$

by demanding that the underlying vector space of $C_{K,\alpha}M$ be the same as that of M, and defining the K^{α}-module product $*$ on $C_{K,\alpha}M$ by

$$m * t = m \,.\, t^{\alpha^{-1}}$$

where $m \in C_{K,\alpha}M$, $t \in K^{\alpha}$ and . denotes the module product in M.

We shall use $C_{K,g}$ to denote the conjugation functor determined by the subgroup K and the inner automorphism $l \mapsto g^{-1}lg$ $(l \in G)$ of G.

We can now state Green's 9 axioms relating the functors I, R, and C.

(1) Transitivity: $I_K^K \simeq$ identity functor and, if $K \leq H \leq L$, then $I_H^L\, I_K^H \simeq I_K^L$.

(2) Transitivity: $R_K^K \simeq$ identity functor and, if $K \le H \le L$, then $R_K^H R_H^L \simeq R_K^L$.

(3) Transivity: $C_{K,\alpha}$ = identity functor if $\alpha|_K = 1_K$ and if $\alpha,\beta \in \text{Aut } G$, then $C_{K^\alpha,\beta}C_{K,\alpha} \simeq C_{K,\alpha\beta}$.

(4) $I_{K^\alpha}^{H^\alpha}C_{K,\alpha} \simeq C_{H,\alpha}\, I_K^H$ (for $K \le H$ and $\alpha \in \text{Aut } G$).

(5) $R_{K^\alpha}^{H^\alpha}C_{H,\alpha} \simeq C_{K,\alpha}\, R_K^H$ (for $K \le H$ and $\alpha \in \text{Aut } G$).

(6) Let T be a transversal of (H,K)-double cosets in G. $(H,K \le G)$. Then

$$R_K^G\, I_H^G \simeq \sum_{g \in T} I_{H^g \cap K}^K\, R_{H^g \cap K}^{H^g}\, C_{H,g}\ .$$

(This is referred to as the Mackey axiom. Cf. Huppert [9], page 553.)

(7) $\text{Hom}_{kH}(I_K^H\, M,N) \simeq \text{Hom}_{kK}(M,\, R_K^H\, N)$ and $\text{Hom}_{kH}(N,I_K^H M) \simeq \text{Hom}_{kK}(R_K^H N, M)$ for $N \in$ Mod-kH and $M \in$ Mod-kK.

This result is known as Frobenius reciprocity, or Nakayama's lemma (cf. Huppert [9], page 556). It may also be expressed by saying that I_K^H is a simultaneous left and right adjoint for R_K^H.

(8) If $A,B \in$ Mod-kK, and $\alpha \in \text{Aut } G$, then

$$C_{K,\alpha}(A \otimes_k B) \simeq C_{K,\alpha}A \otimes_k C_{K,\alpha}B.$$

(9) If $A,B \in$ Mod-kH and $K \le H$, then

$$R_K^H(A \otimes_k B) \simeq R_K^H A \otimes_k R_K^H B.$$

We add another property of interest: the "cohomological axiom": if $A \in$ Mod-kK, and $K \le H$, then

$$\dim_k R_K^H\, I_K^H\, A = [H:K]\dim_k A.$$

Finally, we note a fact that seems to have no counterpart in

Green's considerations:

if $K \leq H$ and $\alpha \in \mathrm{Aut}\, G$, then

$$I^H_K,\ R^H_K \text{ and } C_{K,\alpha} \text{ are faithful functors.}$$

(2.3) Axioms for Induction-Restriction-Conjugation in Lie Algebras

Let $\underline{\underline{h}} \leq \underline{\underline{g}}$ be Lie algebras over a field k, and let $V \in \text{Mod-}\underline{\underline{g}}$.

Definition: We define a functor

$$R^{\underline{\underline{g}}}_{\underline{\underline{h}}} : \text{Mod-}\underline{\underline{g}} \to \text{Mod-}\underline{\underline{h}}$$

by

$$R^{\underline{\underline{g}}}_{\underline{\underline{h}}} V = \begin{cases} \text{underlying vector space of } V \text{ with} \\ \text{operators restricted to } U\underline{\underline{h}} \end{cases}$$

$R^{\underline{\underline{g}}}_{\underline{\underline{h}}}$ will be called a restriction functor, and often abbreviated to R.

Definition: We define the conjugation functor

$$C_{\underline{\underline{h}},\alpha} : \text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{h}}^\alpha$$

($\alpha \in \mathrm{Aut}\, \underline{\underline{G}}$) on the module $W \in \text{Mod-}\underline{\underline{h}}$ by

$$C_{\underline{\underline{h}},\alpha} W = \begin{cases} \text{underlying vector space of } W \text{ with } \underline{\underline{h}}^\alpha\text{-} \\ \text{module multiplication } * \text{ given by} \\ \qquad w * h^\alpha = w \,.\, h \\ \text{where } w \in W,\ h \in \underline{\underline{h}} \text{ and "." is the} \\ \underline{\underline{h}}\text{-module multiplication for } W. \end{cases}$$

If $I^{\underline{\underline{g}}}_{\underline{\underline{h}}}$ is an arbitrary functor system

$$I^{\underline{\underline{g}}}_{\underline{\underline{h}}} : \text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{g}} \quad (\underline{\underline{h}} \leq \underline{\underline{g}})$$

then it makes sense to ask if Green's axioms, (except (6), and modified where necessary), hold for the functors $I^{\underline{\underline{g}}}_{\underline{\underline{h}}}$, $R^{\underline{\underline{g}}}_{\underline{\underline{h}}}$ and $C_{\underline{\underline{h}},\alpha}$ ($\underline{\underline{h}} \leq \underline{\underline{g}}$, $\alpha \in \mathrm{Aut}\, \underline{\underline{g}}$).

It is easy to check that all the axioms which involve only R and $C_{\underline{\underline{h}},\alpha}$ functors do in fact hold, so that interest centres on the remaining axioms, viz. (1), (4) and (7).

It turns out that (1) and (4) are implied by (7): this will be proved in chapter 3 (results (3.8), (3.9)).

Thus we shall devote most of the rest of this thesis to a study of axiom (7) and weakened forms of it.

Chapter 3 - Preliminary Study of $W \otimes_{U\mathfrak{h}} U\mathfrak{g}$ and $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$

(3.1) Module Structure on $W \otimes_{U\mathfrak{h}} U\mathfrak{g}$ and $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$

Let $\mathfrak{h} \leq \mathfrak{g}$ be Lie algebras over a field k and let W be a right $\mathfrak{h}$-module.

We can construct a vector space $W \otimes_{U\mathfrak{h}} U\mathfrak{g}$ by regarding $U\mathfrak{g}$ as a left $U\mathfrak{h}$-module. We shall define a right $U\mathfrak{g}$-module structure on $W \otimes_{U\mathfrak{h}} U\mathfrak{g}$. For $w \in W$, $u \in U\mathfrak{g}$ and $g \in U\mathfrak{g}$, we set

$$(w \otimes u).g = w \otimes (ug).$$

This uniquely determines a $U\mathfrak{g}$-module product on $W \otimes_{U\mathfrak{h}} U\mathfrak{g}$.

We can also construct a vector space $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$ by regarding $U\mathfrak{g}$ as a right $U\mathfrak{h}$-module.

We shall define a right $U\mathfrak{g}$-module structure on $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$. For $f \in \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$, $u \in U\mathfrak{g}$ and $g \in U\mathfrak{g}$, we define $f^g \in \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$ by

$$f^g(u) = f(gu).$$

This pairing $(f,g) \mapsto f^g$ is a $U\mathfrak{g}$-module multiplication on $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, W)$.

(3.2) The embedding of W in $W \otimes_{U\mathfrak{h}} U\mathfrak{g}$ and a dual map.

Lemma A: Let $\mathfrak{h} \leq \mathfrak{g}$ be Lie algebras and let W be a $U\mathfrak{h}$-module. Then the map $i_W \colon W \to W \otimes_{U\mathfrak{h}} U\mathfrak{g}$ defined by

$$i_W(w) = w \otimes 1_{U\mathfrak{g}} \qquad \text{for } w \in W$$

is an embedding of $U\mathfrak{h}$-modules.

Proof: We use equation (A) of section (1.3) of this thesis: this equation tells us that, as left $U\mathfrak{h}$-modules,

$$U\mathfrak{g} \simeq \bigoplus_{x \in X} U\mathfrak{h}.x$$

where X is a certain set of "standard monomials" containing the identity monomial $1_{U\underline{\underline{g}}}$. Thus

$$U\underline{\underline{g}} \simeq U\underline{\underline{h}} \oplus T$$

as left $U\underline{\underline{h}}$-modules, where $T = \bigoplus_{x \in X \setminus \{1_{U\underline{\underline{g}}}\}} U\underline{\underline{h}}.x$.

Since $W \otimes_{U\underline{\underline{h}}} -$ preserves direct sums (see Curtis and Reiner [2] (12.12)), it follows that, as vector spaces

$$W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}} \simeq W \otimes_{U\underline{\underline{h}}} U\underline{\underline{h}} \oplus W \otimes_{U\underline{\underline{h}}} T.$$

It is easy to see that the direct sum injection from the left hand summand in the isomorphism above is given by $w \otimes h \mapsto w \otimes \varepsilon(h)$, where $\varepsilon : U\underline{\underline{h}} \to U\underline{\underline{g}}$ is the natural injection of enveloping algebras. Further, by (12.14) of Curtis and Reiner [2], the map $w \mapsto w \otimes 1_{U\underline{\underline{h}}}$ is an isomorphism of vector spaces (even of $U\underline{\underline{h}}$-modules) from W to $W \otimes_{U\underline{\underline{h}}} U\underline{\underline{h}}$. Composing these two maps, we see that the map $i_W : W \to W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$, defined in the statement of this lemma, is injective. It is easy to check that i_W is a $U\underline{\underline{h}}$-homomorphism. □

Lemma B: Let $\underline{\underline{h}} \leq \underline{\underline{g}}$ be Lie algebras and let W be a right $U\underline{\underline{h}}$-module. The map $q_W : \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}, W) \to W$ defined by $q_W(f) = f(1_{U\underline{\underline{g}}})$ for $f \in \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}, W)$ is an epimorphism of $U\underline{\underline{h}}$-modules.

Proof: First we prove that q_W is a $U\underline{\underline{h}}$-homomorphism. Let $h \in U\underline{\underline{h}}$ and $f \in \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}, W)$, then

$$\begin{aligned} q_W(f^h) &= f^h(1_{U\underline{\underline{g}}}) \\ &= f(h.1_{U\underline{\underline{g}}}) \\ &= f(h) \\ &= f(1_{U\underline{\underline{g}}}).h \qquad \text{since } f \in \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}, W) \\ &= q_W(f).h, \end{aligned}$$

so k_W is indeed a $U\underline{\underline{h}}$-homomorphism.

It now remains to prove surjectivity. That is, we must exhibit, for each $w \in W$, a map $f_w \in Hom_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$ such that $k_W(f_w) = f_w(1_{U\underline{\underline{g}}}) = w$.

We shall use equation (B) of section (1.3) of this thesis: this tells us that, as right $U\underline{\underline{h}}$-modules,

$$U\underline{\underline{g}} \simeq \bigoplus_{x \in X} x.U\underline{\underline{h}}$$

where X is a certain set of "standard monomials", containing the empty monomial $1_{U\underline{\underline{g}}}$.

Now, given $w \in W$, we define a linear map

$$f_w : U\underline{\underline{g}} \to W$$

by setting (for $x \in X$, $h \in U\underline{\underline{h}}$), $f_w(x.h) = \begin{cases} 0 & \text{if } x \neq 1_{U\underline{\underline{g}}} \\ w.h & \text{if } x = 1_{U\underline{\underline{g}}} \end{cases}$.

Certainly $f_w(1_{U\underline{\underline{g}}}) = w$.

By the defining property of a direct sum, this f_w extends uniquely to a vector space homomorphism $U\underline{\underline{g}} \to W$. Suppose $h,\bar{h} \in U\underline{\underline{h}}$, and $x \in X$; then

$$\begin{aligned} f_w((x.h)\bar{h}) &= f_w(x.(h\bar{h})) \\ &= \begin{cases} 0 & \text{if } x \neq 1_{U\underline{\underline{g}}} \\ w.(h\bar{h}) & \text{if } x = 1_{U\underline{\underline{g}}} \end{cases} \\ &= \begin{cases} 0 & \text{if } x \neq 1_{U\underline{\underline{g}}} \\ (wh).\bar{h} & \text{if } x = 1_{U\underline{\underline{g}}} \end{cases} \\ &= (f_w(x.h)).\bar{h}. \end{aligned}$$

Thus f_w is a $U\underline{\underline{h}}$-homomorphism. □

The remainder of this chapter is devoted to proving that, unless $W = (0)$ or $\underline{\underline{h}} = \underline{\underline{g}}$, $W \otimes_{\underline{\underline{h}}} U\underline{\underline{g}}$ and $Hom_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$ are infinite-dimensional;

that axioms (1) and (4) of chapter 2 hold for the functors $-\otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$ and $\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},-)$; and that $-\otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$ and $\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},-)$ are respectively the left and right adjoints to the restriction functor $R : \text{Mod-}\underline{\underline{g}} \to \text{Mod-}\underline{\underline{h}}$.

The proofs are straightforward but tedious.

(3.3) Proposition: If $\underline{\underline{h}} < \underline{\underline{g}}$ are Lie algebras over a field k, and W is a nonzero finite-dimensional right $\underline{\underline{h}}$-module, then $\dim_k \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W) = \infty$.

Proof: By formula (B), page 14 section (1.3) above, we can write

$$U\underline{\underline{g}} \simeq \bigoplus_{x\in X} X.U\underline{\underline{h}}$$

as vector spaces where X is a certain set of "standard monomials", noting as well that the set X is infinite since $\underline{\underline{h}} < \underline{\underline{g}}$.

Thus, as vector spaces,

$$\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W) \simeq \mathrm{Hom}_{U\underline{\underline{h}}}\left(\bigoplus_{x\in X} x.U\underline{\underline{h}},W\right)$$

$$\simeq \prod_{x\in X} \mathrm{Hom}_{U\underline{\underline{h}}}(x.U\underline{\underline{h}},W)$$

since $\mathrm{Hom}_{U\underline{\underline{h}}}(-,W)$ turns sums into products, and for each $x \in X$ $\mathrm{Hom}_{U\underline{\underline{h}}}(x.U\underline{\underline{h}},W) \simeq W$ as vector spaces, under the map $f \mapsto f(x)$ ($f \in \mathrm{Hom}_{U\underline{\underline{h}}}(x.U\underline{\underline{h}},W)$. Thus unless $\dim_k W = 0$,

$$\dim_k \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W) = \dim\left(\prod_{x\in X} W\right)$$

$$= \infty \qquad \text{since } \underline{\underline{h}} < \underline{\underline{g}}. \qquad \square$$

(3.4) Proposition: If $\underline{\underline{h}} < \underline{\underline{g}}$ are Lie algebras over a field k, and W is a non-zero right $\underline{\underline{h}}$-module, then

$$\dim_k (W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}) = \infty.$$

Proof: By formula (A), page 13, section (1.3) of this thesis, we can write

$$\underline{\underline{Ug}} \simeq \bigoplus_{x \in X} \underline{\underline{Uh}}.x$$

where X is an infinite of "standard monomials". Since $W \otimes_{\underline{\underline{Uh}}} -$ preserves direct sums, we can deduce the following vector space isomorphism:

$$W \otimes_{\underline{\underline{Uh}}} \underline{\underline{Ug}} \simeq \bigoplus_{x \in X} W \otimes_{\underline{\underline{Uh}}} \underline{\underline{Uh}}.x.$$

Also, $W \otimes_{\underline{\underline{Uh}}} \underline{\underline{Uh}}.x \simeq W$ as vector spaces. Hence

$$\begin{aligned} \dim_k(W \otimes_{\underline{\underline{Uh}}} \underline{\underline{Ug}}) &= \sum_{x \in X} \dim W \\ &= \infty \qquad \text{since } X \text{ is infinite and } \dim W \neq 0. \end{aligned}$$

□

<u>(3.5) Theorem:</u> $\mathrm{Hom}_{\underline{\underline{Uh}}}(\underline{\underline{Ug}},-)$ is a right adjoint to the restriction functor.

That is, if $\underline{\underline{h}} \leq \underline{\underline{g}}$ are Lie algebras, $W \in$ Mod-$\underline{\underline{h}}$, $V \in$ Mod-$\underline{\underline{g}}$ and if R : Mod-$\underline{\underline{g}} \to$ Mod-$\underline{\underline{h}}$ is the restriction functor, then there is an isomorphism

$$\mathrm{Hom}_{\underline{\underline{Ug}}}(V, \mathrm{Hom}_{\underline{\underline{Uh}}}(\underline{\underline{Ug}},W)) \to \mathrm{Hom}_{\underline{\underline{Uh}}}(RV,W)$$

which is natural in V and W.

<u>Proof:</u> Define the map

$$J_{VW} : \mathrm{Hom}_{\underline{\underline{Ug}}}(V, \mathrm{Hom}_{\underline{\underline{Uh}}}(\underline{\underline{Ug}},W)) \to \mathrm{Hom}_{\underline{\underline{Uh}}}(RV,W)$$

by, for $\phi \in \mathrm{Hom}_{\underline{\underline{Ug}}}(V, \mathrm{Hom}_{\underline{\underline{Uh}}}(\underline{\underline{Ug}},W))$ and $v \in RV$,

$$J_{VW}(\phi)(v) = (\phi(v))(1_{\underline{\underline{Ug}}}).$$

Thus $J_{VW}(\phi)$ is a map $RV \to W$.

We must show (1) $J_{VW}(\phi)$ is a $\underline{\underline{Uh}}$-homomorphism;

(2) J_{VW} is injective;

(3) J_{VW} is surjective;

and (4) J is natural in V and W.

(1) <u>$J_{VW}(\phi)$ is a $U\underline{\underline{h}}$-homomorphism.</u>

Let $v \in RV$ and $h \in U\underline{\underline{h}}$. Then

$$\begin{aligned} J_{VW}(\phi)(v.h) &= (\phi(v.h)(1_{U\underline{\underline{g}}}) \\ &= \phi(v)^h(1_{U\underline{\underline{g}}}) \quad \text{since } \phi \text{ is a } U\underline{\underline{h}}\text{-homomorphism} \\ &= \phi(v)(h.1_{U\underline{\underline{g}}}) \\ &= \phi(v)(1_{U\underline{\underline{g}}}.h) \\ &= ((\phi(v))(1_{U\underline{\underline{g}}})).h \quad \text{since } \phi(v) \text{ is a } U\underline{\underline{h}}\text{-homomorphism} \\ &= ((J_{VW}(\phi))(v)).h. \end{aligned}$$

(2) <u>Injectivity of J_{VW}.</u>

If $J_{VW}(\phi) = 0$, then for all $v \in RV$,

$$0 = (J_{VW}(\phi))(v) = (\phi(v))(1_{U\underline{\underline{g}}}).$$

Thus, for all $x \in U\underline{\underline{g}}$,

$$\begin{aligned} (\phi(v))(x) &= ((\phi(v))^x)(1_{U\underline{\underline{g}}}) \\ &= (\phi(v.x))(1_{U\underline{\underline{g}}}) \quad \text{since } \phi \text{ is } U\underline{\underline{g}}\text{-homomorphism} \\ &= 0 \text{ since } vx \in RV. \end{aligned}$$

That is, for all $v \in Rv$, $\phi(v) = 0$.

That is, $\phi = 0$.

(3) <u>Surjectivity of J_{VW}.</u>

For each $\phi \in \mathrm{Hom}_{U\underline{\underline{h}}}(RV,W)$, we must find a $\phi \in \mathrm{Hom}_{U\underline{\underline{g}}}(V, \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W))$ such that $J_{VW}(\phi) = \psi$.

Given ψ, we define $\phi : V \to \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$ by

$$(\phi(v))(u) = \psi(v.u) \quad (\text{for } v \in V,\ u \in U\underline{\underline{g}}).$$

We must check that (a) $\phi(v)$ is an $\mathfrak{h}$-homomorphism and that (b) ϕ is a $\mathfrak{g}$-homomorphism.

(a) Let $u \in U\mathfrak{g}$, $h \in U\mathfrak{h}$. Then

$$\begin{aligned}(\phi(v))(u.h) &= \psi(v.uh) \\ &= (\psi(v.u)).h \quad \text{since } \psi \text{ is an } \mathfrak{h}\text{-homomorphism} \\ &= ((\phi(v))(u)).h.\end{aligned}$$

(b) If $v \in V$, $x,u \in U\mathfrak{g}$, then

$$\begin{aligned}(\phi(v.x))(u) &= \psi(vxu) \\ &= (\phi(v))(xu) \\ &= (\phi(v))^{x}(u) \\ \therefore \quad \phi(v.x) &= \phi(v)^{x}.\end{aligned}$$

Thus is a well-defined map in $\mathrm{Hom}_{U\mathfrak{g}}(V, \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g},W))$, and, for $v \in RV$

$$\begin{aligned}(J_{VW}(\phi))(v) &= (\phi(v))(1_{U\mathfrak{g}}) \\ &= \psi(v.1_{U\mathfrak{g}}) \\ &= \psi(v),\end{aligned}$$

so $$J_{VW}(\phi) = \psi \text{ , as required.}$$

(4) <u>J_{VW} is natural in V and W.</u>

(a) <u>Naturality in V.</u> Let $f : V_2 \to V_1$ be a $\mathfrak{g}$-homomorphism between $V_1, V_2 \in \text{Mod-}\mathfrak{g}$. We must show that the following diagram commutes for all $W \in \text{Mod-}\mathfrak{h}$:

$$\begin{tikzcd}
\mathrm{Hom}_{U\mathfrak{g}}(V_1, \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g},W)) \arrow[r, "J_{V_1W}"] \arrow[d, "{\mathrm{Hom}_{U\mathfrak{g}}(f,\, \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g},W))}"'] & \mathrm{Hom}_{U\mathfrak{h}}(RV_1, W) \arrow[d, "{\mathrm{Hom}_{U\mathfrak{h}}(Rf,W)}"] \\
\mathrm{Hom}_{U\mathfrak{g}}(V_2, \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g},W)) \arrow[r, "J_{V_2W}"] & \mathrm{Hom}_{U\mathfrak{h}}(RV_2, W)
\end{tikzcd}$$

Suppose $\phi \in \mathrm{Hom}_{U\underline{\underline{g}}}(V_1, \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W))$. Then for $v \in RV_2$,

$$[\mathrm{Hom}_{U\underline{\underline{h}}}(Rf,W)(J_{V_1W}(\phi))](v) = (\phi(f(v)))(1_{U\underline{\underline{g}}})$$

while $$[J_{V_2W}(\mathrm{Hom}_{U\underline{\underline{g}}}(f, \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W))(\phi))](v) = (\phi(f(v)))(1_{U\underline{\underline{g}}})$$

so the diagram commutes.

(b) <u>Naturality in W.</u> Let $V \in$ Mod-$\underline{\underline{g}}$, let $W_1, W_2 \in$ Mod-$\underline{\underline{h}}$ and let $g : W_1 \to W_2$ be an $\underline{\underline{h}}$-homomorphism. We must show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{U\underline{\underline{g}}}(V, \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W_1)) \arrow[r, "J_{VW_1}"] \arrow[d, "\mathrm{Hom}_{U\underline{\underline{g}}}(V, \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},g))"'] & \mathrm{Hom}_{U\underline{\underline{h}}}(RV,W_1) \arrow[d, "\mathrm{Hom}_{U\underline{\underline{h}}}(RV,g)"] \\
\mathrm{Hom}_{U\underline{\underline{g}}}(V, \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W_2)) \arrow[r, "J_{VW_2}"] & \mathrm{Hom}_{U\underline{\underline{h}}}(RV,W_2)
\end{tikzcd}$$

Suppose $\phi \in \mathrm{Hom}_{U\underline{\underline{g}}}(V, \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W_1))$. Note that, for $v \in V$

$$[\mathrm{Hom}(V, \mathrm{Hom}(U\underline{\underline{g}},g))(\phi)](v) = g \circ \phi(v).$$

Thus, for $v \in V$, $J_{VW_2}(\mathrm{Hom}(V, \mathrm{Hom}(U\underline{\underline{g}},g))(\phi))(v) = (g \circ \phi(v))(1_{U\underline{\underline{g}}})$ while on the other hand, for $v \in V$,

$$(J_{VW_1}(\phi))(v) = \phi(v)(1_{U\underline{\underline{g}}})$$

so
$$\begin{aligned}(\mathrm{Hom}\ (RV,g)(J_{VW_1}(\phi)))(v) &= g((\phi(v))(1_{U\underline{\underline{g}}})) \\ &= (g \circ \phi(v))(1_{U\underline{\underline{g}}})\end{aligned}$$

so the diagram commutes.

This completes the proof of theorem (3.5). □

<u>(3.6) Theorem:</u> $- \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$ is a left adjoint to the restriction functor.

That is, if $\underline{\underline{h}} \le \underline{\underline{g}}$ are Lie algebras, $V \in$ Mod-$\underline{\underline{g}}$ and $W \in$ Mod-$\underline{\underline{h}}$, then there is an isomorphism

$$K_{WV} : \mathrm{Hom}_{U\underline{\underline{g}}}(W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}, V) \to \mathrm{Hom}_{U\underline{\underline{h}}}(W, RV)$$

which is natural in W and V.

<u>Proof:</u> We define the map K_{WV} as follows: for $\phi \in \mathrm{Hom}_{U\underline{\underline{g}}}(W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}, V)$, $w \in W$, set

$$(K_{WV}(\phi))(W) = \phi(W \otimes 1_{U\underline{\underline{g}}}).$$

Clearly $K_{WV}(\phi)$ is a linear map $W \to RV$; we must show

(1) $K_{WV}(\phi)$ is an h-homomorphism;

(2) K_{WV} is injective;

(3) K_{WV} is surjective;

and (4) K_{WV} is natural in W and V.

(1) <u>$K_{WV}(\phi)$ is an $\underline{\underline{h}}$-homomorphism.</u>

If $w \in W$ and $h \in U\underline{\underline{h}}$, then

$$\begin{aligned}(K_{WV}(\phi))(w.h) &= \phi(wh \otimes 1_{U\underline{\underline{g}}}) \\ &= \phi(w \otimes h) \\ &= \phi(w \otimes 1_{U\underline{\underline{g}}}).h \quad \text{since } \phi \text{ is a } \underline{\underline{g}}\text{-homomorphism} \\ &= ((K_{WV}(\phi))(w)).h \ .\end{aligned}$$

(2) <u>K_{WV} is injective.</u>

Suppose $K_{WV}(\phi) = 0$. That is, for $w \in W$

$$0 = (K_{WV}(\phi))(w) = \phi(w \otimes 1_{U\underline{\underline{g}}}).$$

Since ϕ is a $\underline{\underline{g}}$-homomorphism, this implies that for all $x \in U\underline{\underline{g}}$, $w \in W$

$$0 = \phi(w \otimes 1_{U\underline{\underline{g}}}).x = \phi(w \otimes x).$$

So $\phi = 0$, hence K_{WV} is injective.

(3) <u>K_{WV} is surjective.</u>

Suppose $\psi \in \mathrm{Hom}_{U\underline{\underline{h}}}(W,RV)$. We want to find a map $\phi \in \mathrm{Hom}_{U\underline{\underline{g}}}(W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}, V)$ such that $K_{WV}(\phi) = \psi$.

We construct such a map using the definition of the tensor product $W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$ (cf. Curtis and Reiner [2] section (12.1)-(12.6)). Consider the map $\hat{\phi} : W \times U\underline{\underline{g}} \to V$ defined by

$$\hat{\phi}(w,u) = \psi(w).u \qquad \text{for } w \in W,\ u \in U\underline{\underline{g}}.$$

This is easily seen to be a $U\underline{\underline{h}}$-balanced bilinear map, so $\hat{\phi}$ factors through $W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$ by a unique, well-defined map

$$\phi : W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}} \to V$$

given by $\phi(w \otimes u) = \psi(w).u$ for $w \in W$ and $u \in U\underline{\underline{g}}$. We shall check that ϕ is a $\underline{\underline{g}}$-homomorphism. For $g \in U\underline{\underline{g}}$, $u \in U\underline{\underline{g}}$, and $w \in W$,

$$\begin{aligned} \phi((w \otimes u).g) &= \psi(w).ug \\ &= (\psi(w).u).g \\ &= \phi(w \otimes u).g. \end{aligned}$$

So $\phi \in \mathrm{Hom}_{U\underline{\underline{g}}}(W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}, V)$. Finally, for any $w \in W$

$$\begin{aligned} (K_{WV}(\phi))(w) &= \phi(W \otimes 1_{U\underline{\underline{g}}}) \\ &= \psi(w).1_{U\underline{\underline{g}}} \\ &= \psi(w). \end{aligned}$$

So $K_{WV}(\phi) = \psi$.

(4) <u>(a) K_{WV} is natural in W.</u>

Let W_1, W_2 be $U\underline{\underline{h}}$-modules, and let $g : W_2 \to W_1$ be an $\underline{\underline{h}}$-homomorphism. We must show that for all $V \in \text{Mod-}\underline{\underline{g}}$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{U\mathfrak{g}}(W_1 \otimes_{U\mathfrak{h}} U\mathfrak{g}.V) \arrow[r, "K_{W_1V}"] \arrow[d, "\mathrm{Hom}(g \otimes U\mathfrak{g},V)"'] & \mathrm{Hom}_{U\mathfrak{h}}(W_1,RV) \arrow[d, "\mathrm{Hom}(g,RV)"] \\
\mathrm{Hom}_{U\mathfrak{g}}(W_2 \otimes_{U\mathfrak{h}} U\mathfrak{g},V) \arrow[r, "K_{W_2V}"] & \mathrm{Hom}_{U\mathfrak{h}}(W_2,RV)
\end{tikzcd}$$

Let $\phi \in \mathrm{Hom}_{\mathfrak{g}}(W_1 \otimes_{U\mathfrak{h}} U\mathfrak{g},V)$. Then, for $w_1 \in W_1$

$$K_{W_1V}(\phi)(w_1) = \phi(w_1 \otimes 1_{U\mathfrak{g}})$$

so for $w_2 \in W_2$,

$$\begin{aligned}(\mathrm{Hom}(g,RV)(K_{W_1V}(\phi)))(w_2) &= (K_{W_1V}(\phi) \circ g)(w_2) \\ &= \phi(g(w_2) \otimes 1_{U\mathfrak{g}})\end{aligned}$$

while $\mathrm{Hom}(g \otimes U\mathfrak{g},V)(\phi) = \phi \circ (g \otimes U\mathfrak{g})$, so for $w_2 \in W_2$,

$$\begin{aligned}K_{W_2V}(\mathrm{Hom}(g \otimes U\mathfrak{g},V)(\phi))(w_2) &= \phi \circ (g \otimes U\mathfrak{g})(w_2 \otimes 1_{U\mathfrak{g}}) \\ &= \phi(g(w_2) \otimes 1_{U\mathfrak{g}})\end{aligned}$$

so the diagram commutes as required.

(b) $\underline{K_{WV} \text{ is natural in } V.}$

Let V_1, V_2 be $\mathfrak{g}$-modules and let $f : V_1 \to V_2$ be a $\mathfrak{g}$-homomorphism. We need to show that for any $\mathfrak{h}$-module W, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{U\mathfrak{g}}(W \otimes_{U\mathfrak{h}} U\mathfrak{g},V_1) \arrow[r, "K_{WV_1}"] \arrow[d, "\mathrm{Hom}(W \otimes_{U\mathfrak{h}} U\mathfrak{g},f)"'] & \mathrm{Hom}_{U\mathfrak{h}}(W,RV_1) \arrow[d, "\mathrm{Hom}(W,Rf)"] \\
\mathrm{Hom}_{U\mathfrak{g}}(W \otimes_{U\mathfrak{h}} U\mathfrak{g},V_2) \arrow[r, "K_{WV_2}"] & \mathrm{Hom}_{U\mathfrak{h}}(W,RV_2)
\end{tikzcd}$$

Choose $\phi \in \mathrm{Hom}_{U\mathfrak{g}}(W \otimes_{U\mathfrak{h}} U\mathfrak{g},V_1)$. Then for $w \in W$,

$$K_{WV_1}(\phi)(w) = \phi(w \otimes 1),$$

so $(\mathrm{Hom}(W,Rf)(K_{WV}\ (\phi)))(w) = f(\phi(w \otimes 1))$ while, for $w \in W$ and $u \in U\mathfrak{g}$,

$$(\mathrm{Hom}(W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}, f)(\phi))(w \otimes u) = f(\phi(w \otimes u))$$

so for $w \in W$,

$$K_{WV_2}(\mathrm{Hom}(W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}, f)(\phi))(w) =$$

$$= (\mathrm{Hom}(w \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}, f)(\phi))(W \otimes 1_{U\underline{\underline{g}}})$$

$$= f(\phi(w \otimes 1_{U\underline{\underline{g}}})).$$

Thus the diagram commutes.

This completes the proof of theorem (3.6). □

(3.7) Unsuitability of Frobenius Reciprocity as an Axiom.

By corollary 1, page 83 of MacLane [12], any two left adjoints to a functor are naturally isomorphic, and dually for right adjoints. Hence any left adjoint to the restriction functor $R : \text{Mod-}\underline{\underline{g}} \to \text{Mod-}\underline{\underline{h}}$ ($\underline{\underline{h}} < \underline{\underline{g}}$ Lie algebras) is naturally isomorphic to the functor $- \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$, and any right adjoint to R is naturally isomorphic to $\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}, -)$. Thus, in both cases, such an adjoint functor takes finite-dimensional nonzero $\underline{\underline{h}}$-modules to infinite dimensional $\underline{\underline{g}}$-modules (by propositions (3.3) and (3.4)).

For this reason, we shall discontinue our study of Green's Frobenius reciprocity axioms (axiom (7) of section (2.2)) at the end of this chapter, and study, instead, modified forms of the Frobenius reciprocity axioms. First, however, we shall indicate how either Frobenius reciprocity axiom may be used to prove the Lie algebra analogues of "axioms" (1) and (4) (of section (2.2)).

(3.8) Frobenius Reciprocity Implies the Transitivity of Induction

Let $\underline{\underline{h}} \le \underline{\underline{g}} \le \underline{\underline{f}}$ be Lie algebras. Let us denote the functors

$$- \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}} \quad \text{and} \quad - \otimes_{U\underline{\underline{g}}} U\underline{\underline{f}}$$

by the symbols I_1 and I_2 respectively, and let

$$R_2 \ : \ \text{Mod-}\underline{\underline{f}} \to \text{Mod-}\underline{\underline{g}}$$

$$R_1 \ : \ \text{Mod-}\underline{\underline{g}} \to \text{Mod-}\underline{\underline{h}}$$

be the obvious restriction functors.

Note that $R_1R_2 \ : \ \text{Mod-}\underline{\underline{f}} \to \text{Mod-}\underline{\underline{h}}$ coincides with the natural restriction functor $\text{Mod-}\underline{\underline{f}} \to \text{Mod-}\underline{\underline{h}}$.

Let $W \in \text{Mod-}\underline{\underline{h}}$, $U \in \text{Mod-}\underline{\underline{f}}$. Then, using the natural isomorphisms of theorem (3.6), we have

$$\begin{aligned} \text{Hom}_{U\underline{\underline{f}}}(I_2I_1W,U) &\simeq \text{Hom}_{U\underline{\underline{g}}}(I_1W,R_2U) \\ &\simeq \text{Hom}_{U\underline{\underline{h}}}(W,R_1R_2U). \end{aligned}$$

Hence I_2I_1 is a left adjoint to the natural restriction functor $R_1R_2 : \text{Mod-}\underline{\underline{f}} \to \text{Mod-}\underline{\underline{h}}$. But by theorem (3.6) of this thesis, the functor $- \otimes_{U\underline{\underline{h}}} U\underline{\underline{f}}$ is <u>another</u> left adjoint to R_1R_2. Hence, by MacLane [12] p.83 Corollary 1, I_2I_1 is naturally isomorphic to $- \otimes_{U\underline{\underline{h}}} U\underline{\underline{f}}$. Hence the fact that the induction functors I_1, I_2 are left adjoints implies transitivity of induction. □

Also $- \otimes_{U\underline{\underline{h}}} U\underline{\underline{h}}$ is naturally isomorphic to the identity functor; (see Curtis and Reiner [2] (12.14) , or use (3.6) of this thesis together with the obvious fact that the identity functor is self-adjoint).

Similarly axiom (1), of section (2.2), follows from the right adjointness part of the Frobenius reciprocity axiom.

(3.9) Frobenius Reciprocity Implies that Conjugation Commutes with Induction.

Let $\mathfrak{h} \leq \mathfrak{g} \leq \mathfrak{f}$ be Lie algebras, and let $\alpha \in \operatorname{Aut} \mathfrak{f}$. Let $C_{\mathfrak{h},\alpha}$, $C_{\mathfrak{g},\alpha}$ be conjugation functors as defined in section (2.3):

$$C_{\mathfrak{h},\alpha} : \text{Mod-}\mathfrak{h} \to \text{Mod-}\mathfrak{h}^{\alpha}$$

$$C_{\mathfrak{g},\alpha} : \text{Mod-}\mathfrak{g} \quad \text{Mod-}\mathfrak{g}^{\alpha} .$$

Let $R^{\mathfrak{g}}_{\mathfrak{h}}$, $R^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha}$ be restriction functors as defined in section (2.3):

$$R^{\mathfrak{g}}_{\mathfrak{h}} : \text{Mod-}\mathfrak{g} \to \text{Mod-}\mathfrak{h}$$

$$R^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} : \text{Mod-}\mathfrak{g}^{\alpha} \to \text{Mod-}\mathfrak{h}^{\alpha}.$$

Let $I^{\mathfrak{g}}_{\mathfrak{h}} : \text{Mod-}\mathfrak{h} \to \text{Mod-}\mathfrak{g}$ and $I^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} : \text{Mod-}\mathfrak{h}^{\alpha} \to \text{Mod-}\mathfrak{g}^{\alpha}$ be left adjoints to $R^{\mathfrak{g}}_{\mathfrak{h}}$ and $R^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha}$ respectively.

We shall show that $C_{\mathfrak{g},\alpha} I^{\mathfrak{g}}_{\mathfrak{h}} \simeq I^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} C_{h,\alpha}$.

We shall use the easy results that if $W \in \text{Mod-}\mathfrak{h}$, $V \in \text{Mod-}\mathfrak{g}$, then

$$\text{(A)} \qquad \operatorname{Hom}_{U\mathfrak{h}}(W, R^{\mathfrak{g}}_{\mathfrak{h}} V) = \operatorname{Hom}_{U\mathfrak{h}^\alpha}(C_{\mathfrak{h},\alpha}W,\ R^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} C_{\mathfrak{g},\alpha}V)$$

and

$$\text{(B)} \qquad \operatorname{Hom}_{U\mathfrak{g}^\alpha}(I^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} C_{\mathfrak{h},\alpha}W, C_{\mathfrak{g},\alpha}V) = \operatorname{Hom}_{U\mathfrak{g}}(C_{\mathfrak{g}^\alpha,\alpha^{-1}} I^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} W, V).$$

By (A),

$$\begin{aligned} \operatorname{Hom}_{U\mathfrak{h}}(W, R^{\mathfrak{g}}_{\mathfrak{h}} V) &= \operatorname{Hom}_{U\mathfrak{h}^\alpha}(C_{\mathfrak{h},\alpha}W, R^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} C_{\mathfrak{g},\alpha}V) \\ &\simeq \operatorname{Hom}_{U\mathfrak{g}^\alpha}(I^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} C_{\mathfrak{h},\alpha}W, C_{\mathfrak{g},\alpha}V) \qquad \text{(by left adjointness)} \\ &= \operatorname{Hom}_{U\mathfrak{g}}(C_{\mathfrak{g}^\alpha,\alpha^{-1}} I^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} C_{\mathfrak{h},\alpha}W, V) \qquad \text{by (B) above,} \end{aligned}$$

so $C_{\mathfrak{g}^\alpha,\alpha^{-1}} I^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} C_{\mathfrak{h},\alpha}$ is a left adjoint to $R^{\mathfrak{g}}_{\mathfrak{h}}$. Hence, by the uniqueness of left adjoints (MacLane [12] p.83 Corollary 1),

$$I^{\mathfrak{g}}_{\mathfrak{h}} \simeq C_{\mathfrak{g}^\alpha,\alpha^{-1}} I^{\mathfrak{g}^\alpha}_{\mathfrak{h}^\alpha} C_{h,\alpha}.$$

It follows that $C_{\underline{\underline{g}},\alpha} I_{\underline{\underline{h}}}^{\underline{\underline{g}}} \simeq I_{\underline{\underline{h}}^{\alpha}}^{\underline{\underline{g}}^{\alpha}} C_{\underline{\underline{h}},\alpha}$. □

(3.10) Remarks

Thus axioms (1) and (4) of section (2.2) follow from axiom (7) of section (2.2). The weakened forms of axiom (7) that we shall be studying from now on do not seem to imply axioms (1) and (4) (nor even weakened forms of axioms (1) and (4)!). We shall not, however, study axioms (1) and (4) any further in this thesis.

Chapter 4 - Partial Adjoints

(4.1) Introduction to the Concept of a Partial Adjoint

Since the remarks in 3.7 show that we cannot hope to find a finite-dimensional induced module functor which is either a left or right adjoint to the restriction functor $R : \text{Mod-}\underline{\underline{g}} \to \text{Mod-}\underline{\underline{h}}$ where $\underline{\underline{h}} \leq \underline{\underline{g}}$ are Lie algebras, it is natural to ask if we can find a similar but weaker property which an induction functor might satisfy.

Motivated by a paper of Wallach [17] (see Lemma 2.2), we consider eight possible weakenings of left and right adjointness.

Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be arbitrary categories and let $I : \underline{\underline{H}} \to \underline{\underline{G}}$ and $R : \underline{\underline{G}} \to \underline{\underline{H}}$ be functors between them. Consider the following axioms:

(i) For all $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$ there is a map

$$\underline{\underline{G}}(IW,V) \to \underline{\underline{H}}(W,RV)$$

which is *injective*, and natural in W and V.

(ii) For all $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, there is a map

$$\underline{\underline{G}}(IW,V) \to \underline{\underline{H}}(W,RV)$$

which is *surjective*, and natural in W and V.

(iii) For all $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, there is a map

$$\underline{\underline{H}}(W,RV) \to \underline{\underline{G}}(IW,V)$$

which is *injective* and natural in W and V.

(iv) For all $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, there is a map

$$\underline{\underline{H}}(W,RV) \to \underline{\underline{G}}(IW,V)$$

which is *surjective* and natural in W and V.

(i)' For all $V \in \underline{\underline{G}}$, $W \in \underline{\underline{H}}$, there is a map

$$\underline{\underline{G}}(V,IW) \to \underline{\underline{H}}(RV,W)$$

which is <u>injective</u> and natural in V and W.

(ii)' For all $V \in \underline{\underline{G}}$, $W \in \underline{\underline{H}}$, there is a map

$$\underline{\underline{G}}(V,IW) \to \underline{\underline{H}}(RV,W)$$

which is <u>surjective</u> and natural in V and W.

(This axiom has also been studied by Kainen in [11].)

(iii)' For all $V \in \underline{\underline{G}}$, $W \in \underline{\underline{H}}$, there is a map

$$\underline{\underline{H}}(RV,W) \to \underline{\underline{G}}(V,IW)$$

which is <u>injective</u> and natural in V and W.

(iv)' For all $V \in \underline{\underline{G}}$, $W \in \underline{\underline{H}}$, there is a map

$$\underline{\underline{H}}(RV,W) \to \underline{\underline{G}}(V,IW)$$

which is <u>surjective</u> and natural in V and W.

<u>(4.2)</u>

Let $\underline{\underline{h}} \leq \underline{\underline{g}}$ be Lie algebras. Put $\underline{\underline{H}}$ = Mod-$\underline{\underline{h}}$ and $\underline{\underline{G}}$ = Mod-$\underline{\underline{g}}$ and let $R : \underline{\underline{G}} \to \underline{\underline{H}}$ be the restriction functor. In the rest of this chapter, we shall obtain results which show that a functor $I : \underline{\underline{H}} \to \underline{\underline{G}}$ satisfying any of axioms (ii), (ii)', (iii), (iii)' has a representation which precludes it from being a finite-dimensional-induced-module functor. Thereafter, we shall concentrate our attention on the axioms (i) and (i)', (iv) and (iv)'.

We need four lemmas.

<u>(4.3) Lemma:</u> Suppose $A \xleftarrow{\alpha} B \xleftarrow{\beta} C$ is a sequence of modules and morphisms in a category $\underline{\underline{G}}$. Suppose that ker α, coker β and im β exist in $\underline{\underline{G}}$ and

that $\operatorname{im} \beta = \ker(\operatorname{coker} \beta)$. Finally, suppose that the induced sequence

$$\underline{\underline{G}}(A,V) \xrightarrow{\underline{\underline{G}}(\alpha,V)} \underline{\underline{G}}(B,V) \xrightarrow{\underline{\underline{G}}(\beta,V)} \underline{\underline{G}}(C,V)$$

is exact when $V = A$ and when $V = \operatorname{coker} \beta$.

Then the original sequence must have been exact, too.

<u>Proof:</u> From exactness of $\underline{\underline{G}}(A,A) \xrightarrow{\underline{\underline{G}}(\alpha,A)} \underline{\underline{G}}(B,A) \xrightarrow{\underline{\underline{G}}(\alpha,A)} \underline{\underline{G}}(C,A)$ it follows that

$$\begin{aligned} \alpha = \underline{\underline{G}}(\alpha,A)(1_A) &\in \operatorname{im}(\underline{\underline{G}}(\alpha,A)) \\ &= \ker(\underline{\underline{G}}(\beta,A)), \end{aligned}$$

so $0 = \underline{\underline{G}}(\beta,A)(\alpha) = \alpha \circ \beta$. Thus $\operatorname{im} \alpha \subseteq \ker \beta$. For the reverse inequality, consider the exact sequence

$$\underline{\underline{G}}(A,\operatorname{coker} \beta) \xrightarrow{\underline{\underline{G}}(\alpha,\operatorname{coker} \beta)} \underline{\underline{G}}(B,\operatorname{coker} \beta) \xrightarrow{\underline{\underline{G}}(\beta,\operatorname{coker} \beta)} \underline{\underline{G}}(C,\operatorname{coker} \beta).$$

If k denotes the canonical map $B \to \operatorname{coker} \beta$, then

$$\underline{\underline{G}}(\beta, \operatorname{coker} \beta)(k) = k \circ \beta = 0$$

so

$$k \in \ker \underline{\underline{G}}(\beta, \operatorname{coker} \beta) = \operatorname{im} \underline{\underline{G}}(\alpha, \operatorname{coker} \beta).$$

That is, there exists $\phi \in \underline{\underline{G}}(A, \operatorname{coker} \beta)$ such that

$$k = \underline{\underline{G}}(\alpha, \operatorname{coker} \beta)(\phi) = \phi \circ \alpha.$$

$$\begin{tikzcd} B \arrow[r, "\alpha"] \arrow[dr, "k"'] & A \arrow[d, "\phi"] \\ & \operatorname{coker} \beta \end{tikzcd}$$

$$\begin{aligned} \text{Clearly } \ker \alpha &\subseteq \ker(\phi \circ \alpha) \\ &= \ker k \\ &= \operatorname{im} \beta \text{ by hypothesis.} \end{aligned}$$

Thus $\ker \alpha$ and $\operatorname{im} \beta$ are equivalent subobjects; that is, the original sequence is exact. □

<u>(4.4) Lemma:</u> (Yoneda lemma). Let $\underline{\underline{G}}$ be a category and let $A, B \in \underline{\underline{G}}$. Suppose that

$$\eta : \underline{\underline{G}}(A,-) \dot{\to} \underline{\underline{G}}(B,-)$$

is a natural transformation. Then the morphism $\eta_A(1_A) : B \to A$ induces η, in the sense that for all $V \in \underline{\underline{G}}$,

$$\eta_V = \underline{\underline{G}}(\eta_A(1_A),V).$$

Proof: See MacLane [12], page 61. $\square$

(4.4a) Corollary: Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be categories and let $C,D : \underline{\underline{H}} \to \underline{\underline{G}}$ be functors. Suppose that

$$\eta_{WV} : \underline{\underline{G}}(CW,V) \to \underline{\underline{G}}(DW,V)$$

($W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$) is a natural transformation. Then the morphism $\eta_{W,CW}(1_{CW}) : DW \to CW$ induces η and is natural in W.

Proof: By lemma (4.4), $\eta_{W,CW}(1_{CW})$ induces η. Suppose $W,W' \in \underline{\underline{H}}$ and $f \in \underline{\underline{H}}(W,W')$. Then we know that the following diagram commutes:

$$\begin{tikzcd}
\underline{\underline{G}}(CW',CW') \arrow[r, "\eta_{W',CW'}"] \arrow[d, "\underline{\underline{G}}(Cf,CW')"'] & \underline{\underline{G}}(DW',CW') \arrow[d, "\underline{\underline{G}}(Df,CW')"] \\
\underline{\underline{G}}(CW,CW') \arrow[r, "\eta_{W,CW'}"] & \underline{\underline{G}}(DW,CW')
\end{tikzcd}$$

Hence, in particular,

$$\underline{\underline{G}}(Df,CW')(\eta_{W',CW'}(1_{CW'})) = \eta_{W,CW'}(\underline{\underline{G}}(Cf,CW')(1_{CW'})).$$

That is

$$\begin{aligned}
\eta_{W',CW'}(1_{CW'}) \circ Df &= \eta_{W,CW'}(Cf) \\
&= \underline{\underline{G}}(\eta_{W,CW}(1_{CW}),CW')(Cf) \\
&\qquad \text{by lemma (4.4)} \\
&= Cf \circ \eta_{W,CW}(1_{CW})
\end{aligned}$$

$$\begin{tikzcd}
DW \arrow[r, "\eta_{W,CW}(1_{CW})"] \arrow[d, "Df"'] & CW \arrow[d, "Cf"] \\
DW' \arrow[r, "\eta_{W',CW'}(1_{CW'})"'] & CW'
\end{tikzcd}$$

Thus $\eta_{W,CW}(1_{CW})$ is natural in W. $\square$

(4.5) Lemma: Suppose $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is a sequence of modules and morphisms in a category $\underline{\underline{G}}$. Suppose that ker β, coker α and im α exist in $\underline{\underline{G}}$, and that the induced sequence

$$\underline{\underline{G}}(V,A) \xrightarrow{\underline{\underline{G}}(V,\alpha)} \underline{\underline{G}}(V,B) \xrightarrow{\underline{\underline{G}}(V,\beta)} \underline{\underline{G}}(V,C)$$

is exact when $V = A$ and when $V = \ker \beta$. Then the original sequence must have been exact.

Proof: Let k denote the inclusion morphism $\ker \beta \to B$. Since

$$\underline{\underline{G}}(\ker \beta,A) \xrightarrow{\underline{\underline{G}}(\ker \beta,\alpha)} \underline{\underline{G}}(\ker \beta ,B) \xrightarrow{\underline{\underline{G}}(\ker \beta,\beta)} \underline{\underline{G}}(\ker \beta ,C)$$

is exact, and $k \in \ker(\underline{\underline{G}}(\ker \beta ,B))$, we may deduce that

$$k \in \operatorname{im} \underline{\underline{G}}(\ker \beta ,\alpha),$$

hence there exists $\phi \in \underline{\underline{G}}(\ker \beta ,A)$ such that

$$k = \alpha \circ \phi.$$

Thus $\ker \beta = \operatorname{im} k = \operatorname{im}(\alpha \circ \phi) \subseteq \operatorname{im} \alpha$.

Now we prove the reverse inequality. Since the sequence

$$\underline{\underline{G}}(A,A) \xrightarrow{\underline{\underline{G}}(A,\alpha)} \underline{\underline{G}}(A,B) \xrightarrow{\underline{\underline{G}}(A,\beta)} \underline{\underline{G}}(A,C)$$

is exact,

$$\alpha = \alpha \circ 1_A = \underline{\underline{G}}(A,\alpha)(1_A) \in \operatorname{im} \underline{\underline{G}}(A,\alpha)$$

and $\operatorname{im} \underline{\underline{G}}(A,\alpha) = \ker \underline{\underline{G}}(A,\beta)$, hence

$$\beta \circ \alpha = 0.$$

That is, $\operatorname{im} \alpha \subseteq \ker \beta$. Thus $\operatorname{im} \alpha = \ker \beta$ as required. $\square$

(4.6) Lemma: (Yoneda lemma). Let $\underline{\underline{G}}$ be a category and let $A,B \in \underline{\underline{G}}$. Suppose that

$$\eta : \underline{\underline{G}}(-,A) \to \underline{\underline{G}}(-,B)$$

is a natural transformation. Then the morphism $\eta_A(1_A) \in \underline{\underline{G}}(A,B)$ induces η in the sense that for all $V \in \underline{\underline{G}}$

$$\eta_V = \underline{\underline{G}}(V, \eta_A(1_A)).$$

<u>Proof:</u> Dual to that of (4.4). $\square$

<u>(4.6a) Corollary:</u> Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be categories and let $C, D : \underline{\underline{H}} \to \underline{\underline{G}}$ be functors. Suppose that for $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$,

$$\eta_{VW} : \underline{\underline{G}}(V, CW) \to \underline{\underline{G}}(V, DW)$$

are the components of a natural transformation η. Then the morphism $\eta_{CW,C}(1_{CW}) \in \underline{\underline{G}}(CW, DW)$ induces η and is natural in W.

<u>Proof:</u> By lemma (4.6), $\eta_{CW,W}(1_{CW})$ induces η. Thus it remains to prove naturality. Suppose $W, W' \in \underline{\underline{H}}$ and $f \in \underline{\underline{H}}(W, W')$. Then, by assumption, the following diagram commutes:

$$\begin{tikzcd} G(CW,CW) \arrow[r, "\eta_{CW,W}"] \arrow[d, "\underline{\underline{G}}(CW{,}Cf)"'] & \underline{\underline{G}}(CW,DW) \arrow[d, "\underline{\underline{G}}(CW{,}Df)"] \\ \underline{\underline{G}}(CW,CW') \arrow[r, "\eta_{CW,W'}"'] & \underline{\underline{G}}(CW,DW') \end{tikzcd}$$

Hence, in particular,

$$\underline{\underline{G}}(CW, Df)(\eta_{CW,W}(1_{CW})) = \eta_{CW,W'}(\underline{\underline{G}}(CW, Cf)(1_{CW}))$$

i.e.

$$\begin{aligned} Df \circ \eta_{CW,W}(1_{CW}) &= \eta_{CW,W'}(Cf) \\ &= \eta_{CW',W'}(1_{CW'}) \circ Cf \end{aligned}$$

since $\eta_{CW',W'}(1_{CW'})$ induces $\eta_{CW,W'}$ by the first part of this corollary.

That is, the following diagram commutes:

$$\begin{tikzcd} CW \arrow[r, "\eta_{CW,W}(1_{CW})"] \arrow[d, "Cf"'] & DW \arrow[d, "Df"] \\ CW' \arrow[r, "\eta_{CW',W'}(1_{CW'})"'] & DW' \end{tikzcd}$$

that is, $\eta_{CW,W}(1_{CW})$ is natural in W. □

We shall now apply these lemmas to obtain some consequences of axioms (ii) and (iii), (ii)' and (iii)' (of section (4.1)) in theorems (4.7), (4.8), (4.9) and (4.10) respectively.

<u>(4.7) Theorem:</u> Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be categories and let $I : \underline{\underline{H}} \to \underline{\underline{G}}$, $R : \underline{\underline{G}} \to \underline{\underline{H}}$ be functors. Suppose that there exists a natural surjection

$$\underline{\underline{G}}(IW,V) \to \underline{\underline{H}}(W,RV)$$

for each $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$ and that R has a left adjoint $L : \underline{\underline{H}} \to \underline{\underline{G}}$. Then there is a split natural monomorphism $\theta_W \in \underline{\underline{G}}(LW,IW)$ for each $W \in \underline{\underline{H}}$.

<u>Corollary (4.7a):</u> Let $\underline{\underline{h}} \leq \underline{\underline{g}}$ be Lie algebras and set $\underline{\underline{H}} = \text{Mod-}\underline{\underline{h}}$, $G = \text{Mod-}\underline{\underline{g}}$. Let $R : \underline{\underline{G}} \to \underline{\underline{H}}$ be the restriction functor, and suppose that there exists a functor $I : \underline{\underline{H}} \to \underline{\underline{G}}$, and, for every $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, a linear surjection $\underline{\underline{G}}(IW,V) \to \underline{\underline{H}}(W,RV)$, natural in W and V.. Then there is a natural $U\underline{\underline{g}}$-monomorphism

$$W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}} \to IW \qquad \text{for each } W \in \underline{\underline{H}}.$$

In particular, $\dim IW = \infty$ unless $W = \{0\}$ or $\underline{\underline{h}} = \underline{\underline{g}}$.

<u>Proof of Corollary:</u> By (3.6), $- \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$ is the left adjoint to R. So theorem (4.7) applies, and guarantees the existence of the natural monomorphism $W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}} \to IW$ for each $W \in \underline{\underline{H}}$. By proposition (3.4), $\dim IW = \infty$ unless $W = \{0\}$ or $\underline{\underline{h}} = \underline{\underline{g}}$. □

<u>Proof of theorem (4.7):</u> By the hypotheses, there is, for every $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, a natural surjective composition map

$$\begin{tikzcd} \theta_{WV} : \underline{\underline{G}}(IW,V) \arrow[rr] \arrow[dr] & & \underline{\underline{G}}(LW,V) \\ & \underline{\underline{H}}(W,RV) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

where the righthand map is the adjunction. Set $\theta_W = \theta_{W,IW}(1_{IW})$. Then by Yoneda lemma, (4.4) and (4.4a), θ_W is natural in W and for any $\alpha \in \underline{\underline{G}}(IW,V)$, $\theta_{WV}(\alpha) = \alpha \circ \theta_W$. Put V = LW. Since $\theta_{W,LW}$ is surjective, there exists $\theta_W \in \underline{\underline{G}}(IW,LW)$ such that $\theta_{W,LW}(\phi_W) = 1_{LW}$. That is, $\phi_W \circ \theta_W = 1_{LW}$:

$$LW \underset{\phi_W}{\overset{\theta_W}{\rightleftarrows}} IW$$

Hence θ_W is a split, natural monomorphism. $\square$

<u>(4.8) Theorem:</u> Let $\underline{\underline{G}}$ and $\underline{\underline{H}}$ be categories and let $R : \underline{\underline{G}} \to \underline{\underline{H}}$ and $I : \underline{\underline{H}} \to \underline{\underline{G}}$ be functors. Suppose that R has a left adjoint $L : \underline{\underline{H}} \to \underline{\underline{G}}$ and that there is a natural injection

$$\underline{\underline{H}}(W,RV) \to \underline{\underline{G}}(IW,V)$$

for every $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$. Then, for each $W \in \underline{\underline{H}}$, there is an epimorphism

$$\theta_W \in \underline{\underline{G}}(IW,LW)$$

which is natural in W.

<u>Proof:</u> Let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$. Let θ_{WV} denote the composition map

$$\underline{\underline{G}}(LW,V) \overset{\simeq}{\to} \underline{\underline{H}}(W,RV) \to \underline{\underline{G}}(IW,V)$$

where the lefthand map is the adjunction map and the righthand map is the natural injection whose existence was supposed in the statement of the theorem. Then θ_{WV} is injective and natural, and so for any

$\alpha \in \underline{\underline{G}}(LW,V)$, the following diagram commutes:

$$\begin{tikzcd} \underline{\underline{G}}(LW,LW) \arrow[r, "\theta_{W,LW}"] \arrow[d, "\underline{\underline{G}}(LW{,}\alpha)"'] & \underline{\underline{G}}(IW,LW) \arrow[d, "\underline{\underline{G}}(IW{,}\alpha)"] \\ \underline{\underline{G}}(LW,V) \arrow[r, "\theta_{WV}"'] & \underline{\underline{G}}(IW,V) \end{tikzcd}$$

Define $\theta_W \in \underline{\underline{G}}(IW,LW)$ by $\theta_W = \theta_{W,LW}(1_{LW})$. By corollary (4.4a)

$$\theta_{WV}(\alpha) = \alpha \circ \theta_W \tag{*}$$

and θ_W is natural in W. To see that θ_W is epi, consider

$$IW \xrightarrow{\theta_W} LW \overset{\alpha}{\underset{\beta}{\rightrightarrows}} V.$$

If $\alpha \circ \theta_W = \beta \circ \theta_W$, then, by (*) above, it follows that $\theta_{WV}(\alpha) = \theta_{WV}(\beta)$, and so, since θ_{WV} is injective, we see $\alpha = \beta$. Thus θ_W is epi. □

Corollary (4.8a): Let $\underline{\underline{G}}$ = Mod-$\underline{\underline{g}}$ and $\underline{\underline{H}}$ = Mod-$\underline{\underline{h}}$ where $\underline{\underline{h}} \leq \underline{\underline{g}}$ are Lie algebras. Let $R : \underline{\underline{G}} \to \underline{\underline{H}}$ be the restriction functor and suppose that there exists a functor $I : \underline{\underline{H}} \to \underline{\underline{G}}$, and, for every $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, a linear injection $\underline{\underline{H}}(W,RV) \to \underline{\underline{G}}(IW,V)$ natural in W and V. Then, for all $W \in \underline{\underline{H}}$ there is a $U\underline{\underline{g}}$-epimorphism $IW \to W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$, natural in W. In particular $\dim IW = \infty$ unless $W = \{0\}$ or $\underline{\underline{h}} = \underline{\underline{g}}$.

Proof: By (3.6), $- \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$ is the left adjoint to R. Applying (4.8) and proposition (3.4), we obtain the conclusions of the corollary. □

(4.9) Theorem: Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be categories and let $R : \underline{\underline{G}} \to \underline{\underline{H}}$, $I : \underline{\underline{H}} \to \underline{\underline{G}}$ be functors. Suppose that R possesses a right adjoint F, and that for all $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, there is a natural surjection

$$\underline{\underline{G}}(V,IW) \to \underline{\underline{H}}(RV,W).$$

Then for all $W \in \underline{\underline{H}}$, there is a split natural epimorphism $\theta_W : IW \to FW$.

Proof: Let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$. We have a natural surjective composition map

$$\underline{\underline{G}}(V,IW) \to \underline{\underline{H}}(RV,W) \xrightarrow{\cong} \underline{\underline{G}}(V,FW)$$

which we shall denote by θ_{VW}. Set $\theta_W = \theta_{IW,W}(1_{IW})$. Then by Yoneda lemma (4.6) and (4.6a), θ_W is natural in W and for any $\alpha \in \underline{\underline{G}}(V,IW)$,

$$\theta_{VW}(\alpha) = \theta_W \circ \alpha.$$

Put V = FW. Since $\theta_{FW,W}$ is surjective, there exists $\phi_W \in \underline{\underline{G}}(FW,IW)$ such that $\theta_{FW,W}(\phi_W) = 1_{FW}$. That is $\theta_W \circ \phi_W = 1_{FW}$. Thus θ_W is a split epimorphism $IW \to FW$. $\square$

Corollary: Let $\underline{\underline{h}} \leq \underline{\underline{g}}$ be Lie algebras, set $\underline{\underline{H}}$ = Mod-$\underline{\underline{h}}$, $\underline{\underline{G}}$ = Mod-$\underline{\underline{g}}$, and let $R : \underline{\underline{G}} \to \underline{\underline{H}}$ be the restriction functors. Suppose that there is a functor $I : \underline{\underline{H}} \to \underline{\underline{G}}$ and, for every $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, a surjection

$$\underline{\underline{G}}(V,IW) \to \underline{\underline{H}}(RV,W)$$

natural in V and W. Then, for every $W \in \underline{\underline{H}}$ there is a $U\underline{\underline{g}}$-epimorphism

$$IW \to \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W).$$

In particular, dim $IW = \infty$ unless $\underline{\underline{h}} = \underline{\underline{g}}$ or $W = \{0\}$.

Proof: By (3.5), $\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},-)$ is a right adjoint to R. Applying (4.9) and proposition (3.3), we obtain the conclusion of the corollary. $\square$

(4.10) Theorem: Let $\underline{\underline{G}}$ and $\underline{\underline{H}}$ be categories. Let $R : \underline{\underline{G}} \to \underline{\underline{H}}$ and $I : \underline{\underline{H}} \to \underline{\underline{G}}$ be functors, and suppose that for every $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$ there is a natural injection

$$\underline{\underline{H}}(RV,W) \to \underline{\underline{G}}(V,IW).$$

If R possesses a right adjoint $F : \underline{\underline{H}} \to \underline{\underline{G}}$, then for every $W \in \underline{\underline{H}}$ there is a monomorphism $\theta_W \in \underline{\underline{G}}(FW,IW)$ natural in W.

Proof: Let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$. We have a natural, injective composition map

$$\underline{\underline{G}}(V,FW) \overset{\simeq}{\to} \underline{\underline{H}}(RV,W) \to \underline{\underline{G}}(V,IW)$$

which we shall denote by θ_{VW}. Set $\theta_W = \theta_{FW,W}(1_{FW})$. By Yoneda lemma ((4.6) and (4.6a)), θ_W is natural in W and for every $\alpha \in \underline{\underline{G}}(V,FW)$,

$$\theta_{VW}(\alpha) = \theta_W \circ \alpha \ .$$

Consider the following diagram: $V \underset{\beta}{\overset{\alpha}{\rightrightarrows}} FW \xrightarrow{\theta_W} IW$. If $\theta_W \circ \alpha = \theta_W \circ \beta$, then the equation above tells us that $\theta_{VW}(\alpha) = \theta_{VW}(\beta)$, so, since θ_{VW} is injective, it follows that $\alpha = \beta$. Thus θ_W is monic. □

Corollary (4.10a): Let $\underline{\underline{h}} \leq \underline{\underline{g}}$ be Lie algebras and let $\underline{\underline{H}}$ = Mod-$\underline{\underline{h}}$ and $\underline{\underline{G}}$ = Mod-$\underline{\underline{g}}$. Let $R : \underline{\underline{G}} \to \underline{\underline{H}}$ be the restriction functor and suppose that for every $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$ there is a natural linear injection

$$\underline{\underline{H}}(RV,W) \to \underline{\underline{G}}(V,IW).$$

Then, for all $W \in \underline{\underline{H}}$ there is a $U\underline{\underline{g}}$-monomorphism

$$\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W) \to IW.$$

In particular, dim IW = ∞ unless W = {0} or $\underline{\underline{h}} = \underline{\underline{g}}$.

Proof: By (3.5), $\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},-)$ is the right adjoint to R. Applying theorem (4.10) and proposition (3.3), we obtain the conclusions of the corollary. □

Remark: The corollaries to theorems (4.7) - (4.10) show that functors satisfying any of axioms (ii), (iii), (ii)' or (iii)' are unsuitable for producing finite-dimensional induced modules. We shall study only axioms (i), (iv), (i)' and (iv)' in the rest of this thesis.

Chapter 5 - The Injectivity Axioms

(5.1) This chapter studies consequences of the axioms (i) and (i)' of section (4.1); these axioms are restated below for ease of reference. They will dominate chapters 7 and 8.

Notation: Throughout this chapter, $\underline{\underline{H}}$ and $\underline{\underline{G}}$ will be categories (with zero objects) and

$$I : \underline{\underline{H}} \to \underline{\underline{G}}$$

$$R : \underline{\underline{G}} \to \underline{\underline{H}}$$

will be functors.

The Left* Injectivity Axiom (Axiom (i) of (4.1)) holds for I and R if, for all $W \in \underline{\underline{H}}$, all $V \in \underline{\underline{G}}$, there exists a natural injection

$$\theta_{WV} : \underline{\underline{G}}(IW,V) \to \underline{\underline{H}}(W,RV).$$

The Right Injectivity Axiom ((i)' of (4.1)) holds for I and R if, for all $W \in \underline{\underline{H}}$ and all $V \in \underline{\underline{G}}$, there exists a natural injection

$$\eta_{VW} : \underline{\underline{G}}(V,IW) \to \underline{\underline{H}}(RV,W).$$

Remarks: In chapter 8, we will produce functors I and R which satisfy both the left and right injectivity axioms simultaneously. This is something which we can't do with the left and right adjoints to the restriction functor Mod-$\underline{\underline{g}}$ $\to$ Mod-$\underline{\underline{h}}$ (where $\underline{\underline{h}} < \underline{\underline{g}}$ are Lie algebras). For we showed, in the proofs of (3.3) and (3.4) that $W \otimes_{\underline{\underline{h}}} U\underline{\underline{g}}$ is isomorphic to a direct sum of $|X|$ copies of W as a vector space, while $\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$ is isomorphic as a vector space to the direct product of $|X'|$ copies of W, where X and X' are infinite sets with the same cardinality. Thus, if $W \neq (0)$, $\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$ and $W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}$ cannot be isomorphic as vector

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* "left" since I resembles a left adjoint to R.

spaces, let alone as $\underline{\underline{g}}$-modules.

In chapter 7, we shall study the special consequences of both left and right injectivity axioms holding simultaneously. In this chapter, we shall study these axioms individually.

Convention: In sections (5.2) to (5.6) we shall suppose that the left injectivity axiom holds with respect to I and R.

(5.2) Definition of j_W. Let $W \in \underline{\underline{H}}$. Define a morphism

$$j_W : W \to RIW \quad \text{by} \quad j_W = \theta_{W,IW}(1_{IW})$$

noting that

$$\theta_{W,IW} : \underline{\underline{G}}(IW,IW) \to \underline{\underline{H}}(W,RIW).$$

Remark: MacLane [12] (page 81) would probably call j_W the unit of the (weakened) adjunction θ_{WV}. Note that in the terminology of MacLane again, θ_{WV} has no counit, hence no "triangular identities" in the sense of MacLane [12], page 83.

(5.3) Lemma: Let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$. Then j_W induces θ_{WV}, in the sense that if $\phi \in \underline{\underline{G}}(IW,V)$ then

$$\theta_{WV}(\phi) = R\phi \circ j_W.$$

Proof: Suppose $\phi \in \underline{\underline{G}}(IW,V)$. Then certainly $R\phi \circ j_W \in \underline{\underline{H}}(W,RV)$, and we have the following commutative naturality diagram:

$$\begin{tikzcd}
\underline{\underline{G}}(IW,IW) \arrow[r, "\theta_{W,IW}"] \arrow[d, "\underline{\underline{G}}(IW{,}\phi)"'] & \underline{\underline{H}}(W,RIW) \arrow[d, "\underline{\underline{H}}(W{,}R\phi)"] \\
\underline{\underline{G}}(IW,V) \arrow[r, "\theta_{WV}"] & \underline{\underline{H}}(W,RV)
\end{tikzcd}$$

Hence $\underline{\underline{H}}(W,R\phi)(\theta_{W,IW}(1_{IW})) = R\phi \circ j_W$, and $\theta_{WV}(\underline{\underline{G}}(IW,\phi)(1_{IW})) = \theta_{WV}(\phi)$ are equal. $\square$

(5.4) Lemma: j is a natural transformation from the identity functor on $\underline{\underline{H}}$ to the functor $RI : \underline{\underline{H}} \to \underline{\underline{H}}$. That is, $j_W : W \to RIW$ is natural in W, for $W \in \underline{\underline{H}}$.

Proof: Let $W_1, W_2 \in \underline{\underline{H}}$ and choose any $\psi \in \underline{\underline{H}}(W_2, W_1)$. By naturality of θ_{WV}, the following diagram commutes for all $V \in \underline{\underline{G}}$:

$$\begin{array}{ccc} \underline{\underline{G}}(IW_1, V) & \xrightarrow{\theta_{W_1V}} & \underline{\underline{H}}(W_1, RV) \\ {\scriptstyle \underline{\underline{G}}(I\psi, V)}\downarrow & & \downarrow{\scriptstyle \underline{\underline{H}}(\psi, RV)} \\ \underline{\underline{G}}(IW_2, V) & \xrightarrow{\theta_{W_2V}} & \underline{\underline{H}}(W_2, RV) \end{array}$$

Putting $V = IW_1$ and chasing 1_{IW_1} around the diagram, we find that

$$\begin{aligned} \underline{\underline{H}}(\psi, RIW_1)(\theta_{W_1, IW_1}(1_{IW_1})) &= \underline{\underline{H}}(\psi, RIW_1)(j_{W_1}) \\ &= j_{W_1} \circ \psi \end{aligned}$$

while

$$\begin{aligned} \theta_{W_2, IW_1}(\underline{\underline{G}}(I\psi, IW_1)(1_{IW_1})) &= \theta_{W_2, IW_1}(I\psi) \\ &= RI\psi \circ j_{W_2}. \end{aligned}$$

Thus, by commutativity of the diagram above,

$$j_{W_1} \circ \psi = RI\psi \circ j_{W_2},$$

that is, the diagram

$$\begin{array}{ccc} W_2 & \xrightarrow{j_{W_2}} & RIW_2 \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle RI\psi} \\ W_1 & \xrightarrow{j_{W_1}} & RIW_1 \end{array}$$

commutes. □

(5.5) Proposition: I is epi-preserving.

Proof: Let $W_1, W_2 \in \underline{\underline{H}}$. We must show that if $\psi \in \underline{\underline{H}}(W_1, W_2)$ is epi, then $I\psi$ is epi.

Suppose $V \in \underline{\underline{G}}$ and $\gamma, \delta \in \underline{\underline{G}}(IW_2, V)$ are such that $\gamma \circ I\psi = \delta \circ I\psi$

$$\begin{tikzcd} IW_1 \arrow[r, "I\psi"] & IW_2 \arrow[r, shift left, "\gamma"] \arrow[r, shift right, "\delta"'] & V \end{tikzcd}$$

Then, by functoriality of R, it follows that

$$R\gamma \circ RI\psi = R\delta \circ RI\psi,$$

hence $$R\gamma \circ RI\psi \circ j_{W_1} = R\delta \circ RI\psi \circ j_{W_1}.$$

Consider the following diagram:

$$\begin{tikzcd} RIW_1 \arrow[r, "RI\psi"] & RIW_2 \arrow[r, shift left, "R\gamma"] \arrow[r, shift right, "R\delta"'] & RV \\ W_1 \arrow[u, "j_{W_1}"] \arrow[r, "\psi"] & W_2 \arrow[u, "j_{W_2}"'] & \end{tikzcd}$$

which commutes, by lemma (5.4). From the commutativity of the diagram, and the equation above it, we deduce that

$$R\gamma \circ j_{W_2} \circ \psi = R\delta \circ j_{W_2} \circ \psi.$$

Since ψ is epi, it follows that $R\gamma \circ j_{W_2} = R\delta \circ j_{W_2}$. That is, $\theta_{W_2V}(\gamma) = \theta_{W_2V}(\delta)$, by lemma (5.3). Hence $\gamma = \delta$ since θ_{W_2V} is injective. So $\gamma \circ I\psi = \delta \circ I\psi$ entails that $\gamma = \delta$. That is, $I\psi$ is epi. □

(5.6) Proposition: Let $\underline{\underline{H}}$ be a category with cokernels, let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$. If $\phi \in \underline{\underline{G}}(IW,V)$ is a morphism with the property that $R\phi$ factors through coker j_W via its natural projection, then $\phi = 0$.

Proof: It is easy to check that since $\underline{\underline{H}}$ has cokernels, $R\phi$ factors through coker j_W via the natural projection only if $R\phi \circ j_W = 0$.

$$\begin{tikzcd} W \arrow[r, "j_W"] & RIW \arrow[r, "R\phi"] \arrow[dr, "\text{projection}"'] & RV \\ & & \text{coker } j_W \arrow[u] \end{tikzcd}$$

But $R\phi \circ j_W = \theta_{WV}(\phi)$ and θ_{WV} is injective. Hence, if $R\phi$ factors through coker j_W via the natural projection, then $\phi = 0$. □

Corollary: Let $\underline{\underline{h}} \leq \underline{\underline{g}}$ be Lie algebras. If $\underline{\underline{H}}$ = Mod-$\underline{\underline{h}}$ and $\underline{\underline{G}}$ = Mod-$\underline{\underline{g}}$ and R : Mod-$\underline{\underline{g}} \to$ Mod-$\underline{\underline{h}}$ is the restriction functor, then for all $W \in$ Mod-$\underline{\underline{h}}$,

$$IW = (\mathrm{im}\ j_W).U\underline{\underline{g}}.$$

Proof: Let $V = IW/(\mathrm{im}\ j_W.U\underline{\underline{g}})$, and let ϕ be the canonical projection $IW \to V$.

Clearly $R\phi$ factors through coker j_W via the natural projection:

$$\begin{tikzcd}
RIW \arrow[r] \arrow[dr, "R\phi"'] & (RIW/\mathrm{im}\ j_W) = \mathrm{coker}\ j_W \arrow[d] \\
& \dfrac{(RIW/\mathrm{im}\ j_W)}{(R(\mathrm{im}\ j_W.U\underline{\underline{g}})/\mathrm{im}\ j_W)} \simeq RIW/R(\mathrm{im}\ j_W.U\underline{\underline{g}}).
\end{tikzcd}$$

Hence, by proposition (5.6), $\phi = 0$. But ϕ is surjective, so

$$0 = \mathrm{im}\ \phi = IW/(\mathrm{im}\ j_W).U\underline{\underline{g}}.$$

Hence $$IW = (\mathrm{im}\ j_W).U\underline{\underline{g}}.$$ □

(5.7) Proposition: Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be categories and let $R : \underline{\underline{G}} \to \underline{\underline{H}}$, $I : \underline{\underline{H}} \to \underline{\underline{G}}$ be functors for which the left injectivity axiom holds. Then I is faithful if and only if for each $W \in \underline{\underline{H}}$ the morphism j_W defined in section (5.2) is monic.

Proof: (a) I faithful implies j_W monic. Suppose that I is faithful, let $W_1, W_2 \in \underline{\underline{H}}$, and let $f, g \in \underline{\underline{H}}(W_1, W_2)$ and consider the diagram

$$W_1 \underset{g}{\overset{f}{\rightrightarrows}} W_2 \xrightarrow{j_{W_2}} RIW_2 .$$

We must show that if $j_{W_2} \circ f = j_{W_2} \circ g$, then $f = g$.

By the naturality of j (lemma (5.4)) and lemma (5.3),

$$j_{W_2} \circ f = RIf \circ j_{W_1} = \theta_{W_1, IW_2}(If)$$

and $$j_{W_2} \circ g = RIg \circ j_{W_1} = \theta_{W_1, IW_2}(Ig).$$

Hence $j_{W_2} \circ f = j_{W_2} \circ g$ implies $\theta_{W_1, IW_2}(If) = \theta_{W_1, IW_2}(Ig)$.

Since θ_{W_1, IW_2} is injective, this implies If = Ig.
Since I is faithful, this implies f = g.

(b) <u>j_W monic implies I faithful.</u> Suppose that j_W is monic for all $W \in \underline{\underline{H}}$. Let $W_1, W_2 \in \underline{\underline{H}}$ and choose $f,g \in \underline{\underline{H}}(W_1, W_2)$ such that If = Ig. We must show that f = g.

Now If = Ig implies RIf = RIg

which implies $RIg \circ j_{W_1} = RIg \circ j_{W_1}$

which implies $j_{W_2} \circ f = j_{W_2} \circ g$

by lemma (5.4), and, since j_{W_2} is monic, this last equation implies f = g. □

We sum up (5.2) - (5.4) and (5.6) in the next theorem.

<u>(5.8) Main Theorem:</u> Suppose that $\underline{\underline{H}}$ and $\underline{\underline{G}}$ are preadditive categories, that $I : \underline{\underline{H}} \to \underline{\underline{G}}$ and $R : \underline{\underline{G}} \to \underline{\underline{H}}$ are functors, and that $\underline{\underline{H}}$ has cokernels.

Then the left injectivity axiom of (5.1) is equivalent to the following two conditions:

(a) there exists a natural transformation $j : 1_{\underline{\underline{H}}} \dot{\to} RI$

<u>and</u> (b) for all $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$ and all $\phi \in \underline{\underline{G}}(IW,V)$, if $R\phi$ factors through coker j_W via the natural projection, then $\phi = 0$.

<u>Proof</u>: (5.2) and (5.4) tell us that the left injectivity axiom implies condition (a), and (5.6) tells us that the left injectivity axiom implies condition (b). Thus it remains to show that (a) and (b) tojether imply the left injectivity axiom.

Assume that (a) and (b) hold. For all $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, we must define a map

$$\theta_{WV} : \underline{\underline{G}}(IW,V) \to \underline{\underline{H}}(W,RV).$$

Suppose $\phi \in \underline{\underline{G}}(IW,V)$. We define $\theta_{WV}(\phi) = R\phi \circ j_W$. It is easy to check that $\theta_{WV}(\phi) \in \underline{\underline{H}}(W,RV)$.

Next we must show that θ_{WV} is natural in W and V, and injective.

(1) Naturality of θ_{WV}

Let $W,W' \in \underline{\underline{H}}$ and $V,V' \in \underline{\underline{G}}$. Choose any $\alpha \in \underline{\underline{H}}(W',W)$ and any $\beta \in \underline{\underline{G}}(V,V')$. We shall show that the following diagram commutes:

$$\begin{tikzcd}
\underline{\underline{G}}(IW,V) \arrow[r, "\theta_{WV}"] \arrow[d, "\underline{\underline{G}}(I\alpha{,}\beta)"'] & \underline{\underline{H}}(W,RV) \arrow[d, "\underline{\underline{H}}(\alpha{,}R\beta)"] \\
\underline{\underline{G}}(IW',V') \arrow[r, "\theta_{W'V'}"'] & \underline{\underline{H}}(W',RV')
\end{tikzcd}$$

That is, for all $\phi \in \underline{\underline{G}}(IW,V)$, we want to show

$$\underline{\underline{H}}(\alpha,R\beta)(\theta_{WV}(\phi)) = \theta_{W'V'}(\underline{\underline{G}}(I\alpha,\beta)(\phi)).$$

That is, for all $\phi \in \underline{\underline{G}}(IW,V)$, we want to show

$$R\beta \circ (R\phi \circ j_W) \circ \alpha = R(\beta \circ \phi \circ I\alpha) \circ j_{W'}$$

or $$R\beta \circ R\phi \circ (j_W \circ \alpha) = R\beta \circ R\phi \circ (RI\alpha \circ j_{W'})$$

using the functoriality of R.

But condition (a) tells us that

$$j_W \circ \alpha = RI\alpha \circ j_{W'},$$

and so, premultiplying both sides of this by $R\beta \circ R\phi$, we obtain the required commutativity condition.

(2) Injectivity. Let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$. We must show that for any $\phi \in \underline{\underline{G}}(IW,V)$, $\theta_{WV}(\phi) = 0$ implies $\phi = 0$. That is, that $R\phi \circ j_W = 0$ implies $\phi = 0$. It is easy to check that since $\underline{\underline{H}}$ has cokernels, $R\phi$ factors through coker j_W via the natural projection if $R\phi \circ j_W = 0$. Thus, using condition (b), we see that $R\phi \circ j_W = 0$ implies $\phi = 0$. □

Remark: Let $\underline{\underline{h}} \leq \underline{\underline{g}}$ be Lie algebras. If $\underline{\underline{H}}$ = Mod-$\underline{\underline{h}}$, $\underline{\underline{G}}$ = Mod-$\underline{\underline{g}}$ and R : Mod-$\underline{\underline{g}} \to$ Mod-$\underline{\underline{h}}$ is the restriction functor, then condition (b) of (5.8) can be replaced by: (b)' for all $W \in \underline{\underline{H}}$, $IW = (\mathrm{im}\ j_W).U\underline{\underline{g}}$.

Proof: We must show that if $W \in$ Mod-$\underline{\underline{h}}$, $V \in$ Mod-$\underline{\underline{g}}$, and $\phi \in \mathrm{Hom}_{\underline{\underline{g}}}(IW,V)$, then $R\phi \circ j_W = 0$ implies $\phi = 0$ when (b)' holds.

Let $w \in W$ and $u \in U\underline{\underline{g}}$. Then, since $R\phi \circ j_W = 0$, $\phi(j_W(w)).u = 0$. But ϕ is a $U\underline{\underline{g}}$-homomorphism, so

$$\phi(j_W(w).u) = 0.$$

That is $\phi((\mathrm{im}\ j_W).U\underline{\underline{g}}) = 0$. But then, by (b)', $\phi(IW) = 0$; that is, $\phi = 0$.

The converse - that the left injectivity axiom implies condition (b)' - is proved in the corollary to Proposition (5.6). □

The next result is another in the series begun with theorems (4.7) to (4.10). This time, the result gives us a representation of our functor $I : \underline{\underline{H}} \to \underline{\underline{G}}$ in terms of a (hypothetical) left adjoint to the functor $R : \underline{\underline{G}} \to \underline{\underline{H}}$.

(5.9) Theorem: Let $\underline{\underline{G}}$ and $\underline{\underline{H}}$ be categories. Let $R : \underline{\underline{G}} \to \underline{\underline{H}}$ and $I : \underline{\underline{H}} \to \underline{\underline{G}}$ be functors satisfying the left injectivity axiom of section (5.1) and suppose that R has a left adjoint $L : \underline{\underline{H}} \to \underline{\underline{G}}$. Then, for each $W \in \underline{\underline{H}}$ there exists an epimorphism $\mu_W \in \underline{\underline{G}}(LW,IW)$ which is natural in W.

Proof: Suppose $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$. Then there is a natural, injective composition map

$$\begin{tikzcd} \underline{\underline{G}}(IW,V) \arrow[rr] \arrow[dr] & & \underline{\underline{G}}(LW,V) \arrow[dl, "\simeq"] \\ & \underline{\underline{H}}(W,RV) & \end{tikzcd}$$

where the righthand map is the adjunction. Denote this composition map by μ_{WV}, and set $\mu_W = \mu_{W,IW}(1_{IW})$. By Yoneda lemma, (4.4) and (4.4a), for any $\alpha \in \underline{\underline{G}}(IW,V)$, $\mu_{WV}(\alpha) = \alpha \circ \mu_W$ and μ_W is natural in W. Consider the diagram $LW \xrightarrow{\mu_W} IW \underset{\beta}{\overset{\alpha}{\rightrightarrows}} V$, where $\alpha, \beta \in \underline{\underline{G}}(IW,V)$. $\alpha \circ \mu_W = \beta \circ \mu_W \Leftrightarrow \mu_{WV}(\alpha) = \mu_{WV}(\beta)$, which implies $\alpha = \beta$ since μ_{WV} is injective. Thus μ_W is epi. $\square$

<u>(5.9a) Corollary:</u> For $W \in \underline{\underline{H}}$, denote the adjunction isomorphism by $K_{W,IW} : \underline{\underline{G}}(LW,IW) \to \underline{\underline{H}}(W,RIW)$. If $j_W : W \to RIW$ is the morphism defined in section (5.2), then

$$K_{W,IW}(\mu_W) = j_W.$$

<u>Proof:</u> Let $W \in \underline{\underline{H}}$. In the notation of sections (5.2) and (5.9), $\mu_{W,IW} = K_{W,IW}{}^{-1} \circ \theta_{W,IW}$, $\mu_W = \mu_{W,IW}(1_{IW})$, and $j_W = \theta_{W,IW}(1_{IW})$. Hence

$$\begin{aligned} \mu_W &= K_{W,IW}{}^{-1}(\theta_{W,IW}(1_{IW})) \\ &= K_{W,IW}{}^{-1}(j_W) \end{aligned}$$

$$\therefore K_{W,IW}(\mu_W) = j_W. \qquad \square$$

<u>(5.9b) Corollary:</u> Let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, and let K_{WV} denote the adjunction $\underline{\underline{G}}(LW,V) \to \underline{\underline{H}}(W,RV)$. If $\psi \in \underline{\underline{H}}(W,RV)$ and there exists $\pi \in \underline{\underline{G}}(IW,V)$ such that $\theta_{WV}(\pi) = \psi$, then $K_{WV}{}^{-1}(\psi) = \pi \circ \mu_W$. In other words, a morphism $\psi \in \underline{\underline{H}}(W,RV)$ lifts through θ_{WV} to a morphism $\pi \in \underline{\underline{G}}(IW,V)$ only if $K_{WV}{}^{-1}(\psi)$ factors through IW via μ_W as shown in the commutative diagram below:

$$\begin{tikzcd} W \arrow[d, "\psi"] \\ RV \end{tikzcd} \qquad \begin{tikzcd} LW \arrow[d, "K_{WV}{}^{-1}(\psi)"'] \arrow[dr, "\mu_W"] & \\ V & IW \arrow[l, "\pi"] \end{tikzcd}$$

The converse is also true.

<u>Proof:</u> Suppose we are given $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, $\psi \in \underline{\underline{H}}(W,RV)$ and $\pi \in \underline{\underline{G}}(IW,V)$ such that $\theta_{WV}(\pi) = \psi$. The first thing we want to prove is that

$K_{WV}(\pi \circ \mu_W) = \psi$. Now,

$$K_{WV}(\pi \circ \mu_W) = R(\pi \circ \mu_W) \circ K_{W,IW}(1_{IW}), \text{ by MacLane [12], Theorem 1, page 80,}$$

$$= R\pi \circ R\mu_W \circ K_{W,IW}(1_{IW}), \text{ by functoriality of } R,$$

$$= R\pi \circ K_{W,IW}(\mu_W), \text{ by an argument like that of (5.3),}$$

$$= R\pi \circ j_W, \text{ by Corollary (5.9a),}$$

$$= \theta_{WV}(\pi), \text{ by lemma (5.3),}$$

$$= \psi, \text{ by hypothesis.}$$

To prove the converse, we must suppose that $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, $\psi \in \underline{\underline{H}}(W,RV)$ and that $\pi \in \underline{\underline{G}}(IW,V)$ satisfies

$$\pi \circ \mu_W = K_{WV}^{-1}(\psi),$$

and show that $\theta_{WV}(\pi) = \psi$.

The argument to show this is an obvious reversal of the steps of the proof of the first part of this corollary. □

The Right Injectivity Axiom

The study of the right injectivity axiom is dual to that of the left injectivity axiom. We carry out this dualization in detail.

Convention: In sections (5.10) to (5.14) we shall suppose that the right injectivity axiom holds for the functors I and R. (See section (5.1) for definition.)

(5.10) Definition of k_W: Let $W \in \underline{\underline{H}}$. We define a morphism

$$k_W : RIW \to W$$

by $k_W = \eta_{IW,W}(1_{IW})$, noting that

$$\eta_{IW,W} : \underline{\underline{G}}(IW,IW) \to \underline{\underline{H}}(RIW,W).$$

<u>(5.11) Lemma:</u> Let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$. Then k_W induces η_{VW}, in the sense that if $\phi \in \underline{\underline{G}}(V,IW)$ then

$$\eta_{VW}(\phi) = k_W \circ R\phi.$$

<u>Proof:</u> Suppose $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$ and $\phi \in \underline{\underline{G}}(V,IW)$.

Certainly $k_W \circ R\phi \in \underline{\underline{H}}(RV,W)$, and we have the following commutative naturality diagram:

$$\begin{tikzcd}
\underline{\underline{G}}(IW,IW) \arrow[r, "\eta_{IW,W}"] \arrow[d, "\underline{\underline{G}}(\phi{,}IW)"'] & \underline{\underline{H}}(RIW,W) \arrow[d, "\underline{\underline{H}}(R\phi{,}W)"] \\
\underline{\underline{G}}(V,IW) \arrow[r, "\eta_{VW}"'] & \underline{\underline{H}}(RV,W)
\end{tikzcd}$$

Hence $\underline{\underline{H}}(R\phi,W)(\eta_{(W,W}(1_{IW})) = k_W \circ R\phi$, on the one hand, is equal to $\eta_{VW}(\underline{\underline{G}}(\phi,IW)(1_{IW})) = \eta_{VW}(\phi)$, on the other hand. $\square$

<u>(5.12) Lemma:</u> k is a natural transformation from the functor $RI : \underline{\underline{H}} \to \underline{\underline{H}}$ to the identity functor on $\underline{\underline{H}}$. That is, k_W is natural in W, for $W \in \underline{\underline{H}}$.

<u>Proof:</u> Let $W_1, W_2 \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, and choose $\psi \in \underline{\underline{H}}(W_1,W_2)$. By the naturality of η_{VW}, the following diagram commutes:

$$\begin{tikzcd}
\underline{\underline{G}}(V,IW_1) \arrow[r, "\eta_{VW_1}"] \arrow[d, "\underline{\underline{G}}(V{,}I\psi)"'] & \underline{\underline{H}}(RV,W_1) \arrow[d, "\underline{\underline{H}}(RV{,}\psi)"] \\
\underline{\underline{G}}(V,IW_2) \arrow[r, "\eta_{VW_2}"'] & \underline{\underline{H}}(RV,W_2)
\end{tikzcd}$$

Put $V = IW_1$ and chase 1_{IW_1} around the diagram. We now find

$$\underline{\underline{H}}(RIW_1,\psi)(\eta_{IW_1,W_1}(1_{IW_1})) = \eta_{IW_1,W_2}(\underline{\underline{G}}(IW_1,I\psi)(1_{IW_1})).$$

That is

$$\underline{\underline{H}}(RIW_1,\psi)(k_{W_1}) = \eta_{IW_1,W_2}(I\psi)$$

or

$$\psi \circ k_{W_1} = k_{W_2} \circ RI\psi.$$

That is, the following diagram commutes:

$$\begin{tikzcd} RIW_1 \arrow[r, "k_{W_1}"] \arrow[d, "RI\psi"'] & W_1 \arrow[d, "\psi"] \\ RIW_2 \arrow[r, "k_{W_2}"'] & W_2 \end{tikzcd}$$

so k_W is natural in W. □

(5.13) <u>Proposition:</u> I is monic-preserving.

<u>Proof:</u> Suppose that $W_1, W_2 \in \underline{\underline{H}}$ and $\psi \in \underline{\underline{H}}(W_1,W_2)$. We must show that if ψ is monic then $I\psi$ is monic.

Suppose $V \in \underline{\underline{G}}$ and $\gamma, \delta \in \underline{\underline{G}}(V,IW_1)$ are such that

$$I\psi \circ \gamma = I\psi \circ \delta:$$

$$\begin{tikzcd} V \arrow[r, shift left, "\gamma"] \arrow[r, shift right, "\delta"'] & IW_1 \arrow[r, "I\psi"] & IW_2 \end{tikzcd}$$

Then, by functoriality of R, it follows that

$$RI\psi \circ R\gamma = RI\psi \circ R\delta,$$

hence $k_{W_2} \circ RI\psi \circ R\gamma = k_{W_2} \circ RI\psi \circ R\delta$.

Consider the following diagram:

$$\begin{tikzcd} RV \arrow[r, shift left, "R\gamma"] \arrow[r, shift right, "R\delta"'] & RIW_1 \arrow[r, "RI\psi"] \arrow[d, "k_{W_1}"'] & RIW_2 \arrow[d, "k_{W_2}"] \\ & W_1 \arrow[r, "\psi"'] & W_2 \end{tikzcd}$$

By lemma (5.12), the righthand square commutes. Hence the last equation implies

$$\psi \circ k_{W_1} \circ R\gamma = \psi \circ k_{W_1} \circ R\delta.$$

Since ψ is monic, this leads to

$$k_{W_1} \circ R\gamma = k_{W_2} \circ R\delta,$$

and by lemma (5.11), this is the same as saying

$$\eta_{VW_1}(\gamma) = \eta_{VW_1}(\delta).$$

Since η_{VW_1} is injective,

$$\gamma = \delta.$$

Thus $I\psi$ is monic. $\square$

<u>(5.14) Proposition:</u> Let $\underline{\underline{H}}$ be a category with kernels, and let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$. If $\phi \in \underline{\underline{G}}(V, IW)$ is a morphism with the property that $R\phi$ factors through $\ker k_W$ via its natural inclusion, then $\phi = 0$.

<u>Proof:</u> Since $\underline{\underline{H}}$ has kernels, $R\phi$ factors through $\ker k_W$ via the natural inclusion only if $k_W \circ R\phi = 0$. But $k_W \circ R\phi = \eta_{VW}(\phi)$ by lemma (5.11) and η_{VW} is injective.

$$\begin{tikzcd}
RV \arrow[r, "R\phi"] \arrow[dr] & RIW \arrow[r, "k_W"] & W \\
 & \ker k_W \arrow[u, "\text{inclusion}"'] &
\end{tikzcd}$$

Hence, if R factors through $\ker k_W$ via the natural inclusion, then $\phi = 0$. $\square$

<u>Corollary:</u> Let $\underline{\underline{h}} \leq \underline{\underline{g}}$ be Lie algebras. If $\underline{\underline{H}} = \text{Mod-}\underline{\underline{h}}$, $\underline{\underline{G}} = \text{Mod-}\underline{\underline{g}}$, and $R : \text{Mod-}\underline{\underline{g}} \to \text{Mod-}\underline{\underline{h}}$ is the restriction functor, then $\ker k_W$ contains no nonzero $\underline{\underline{g}}$-submodules of IW, for all $W \in \text{Mod-}\underline{\underline{h}}$.

<u>Proof:</u> Let $W \in \text{Mod-}\underline{\underline{h}}$, and let V be a $U\underline{\underline{g}}$-submodule of IW, such that RV is contained in $\ker k_W$, and let $\phi : V \to IW$ be the natural inclusion. Then certainly $R\phi$ factors through $\ker k_W$ via the inclusion of $\ker k_W$

in RIW. Hence $\phi = 0$. But ϕ is an inclusion map, so $V = 0$. That is, ker k_W contains no nonzero $\underline{\underline{g}}$-modules. □

<u>(5.15) Proposition:</u> Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be categories and $I : \underline{\underline{H}} \to \underline{\underline{G}}$, $R : \underline{\underline{G}} \to \underline{\underline{H}}$ functors for which the right injectivity axiom holds. Then I is faithful if and only if for all $W \in \underline{\underline{H}}$ the morphisms $k_W : RIW \to W$, defined in (5.10), are epi.

<u>Proof:</u> (a) <u>I faithful implies k_W epi.</u>

Suppose I is faithful, let $W_1, W_2 \in \underline{\underline{H}}$, let $f,g \in \underline{\underline{H}}(W_1, W_2)$ and consider the diagram

$$RIW_1 \xrightarrow{k_{W_1}} W_1 \underset{g}{\overset{f}{\rightrightarrows}} W_2 \ .$$

We must show that if $f \circ k_{W_1} = g \circ k_{W_1}$, then $f = g$.

By naturality of k (lemma (5.12)) and lemma (5.11),

$$f \circ k_{W_1} = k_{W_2} \circ RIf = \eta_{IW_1, W_2}(If)$$

$$g \circ k_{W_1} = k_{W_2} \circ RIg = \eta_{IW_1, W_2}(Ig).$$

Thus $f \circ k_{W_1} = g \circ k_{W_1}$ implies

$$\eta_{IW_1, W_2}(If) = \eta_{IW_1, W_2}(Ig)$$

which leads to $\qquad If = Ig$.

But then, since I is faithful,

$$f = g.$$

(b) <u>k_W epi implies I faithful.</u>

Suppose k_W is epi for all $W \in \underline{\underline{H}}$. Let $W_1, W_2 \in \underline{\underline{H}}$ and choose $f,g \in \underline{\underline{H}}(W_1, W_2)$ such that

$$If = Ig.$$

We must deduce that $f = g$.

Now $\qquad If = Ig$ implies $RIf = RIg$

which implies $k_{W_2} \circ RIf = k_{W_2} \circ RIg$

which implies $f \circ k_{W_1} = g \circ k_{W_1}$

which implies $f = g$

since k_{W_1} is epi. □

We sum up (5.10) - (5.12) and (5.14) in the next theorem.

(5.16) Main Theorem: Suppose that $\underline{\underline{H}}$ and $\underline{\underline{G}}$ are preadditive categories, that $I : \underline{\underline{H}} \to \underline{\underline{G}}$ and $R : \underline{\underline{G}} \to \underline{\underline{H}}$ are functors, and that $\underline{\underline{H}}$ has kernels.

Then the right injectivity axiom of (5.1) is equivalent to the following two conditions:

(a) there exists a natural transformation $k : RI \to 1_{\underline{\underline{H}}}$;

and (b) for all $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$ and $\phi \in \underline{\underline{G}}(V,IW)$, if $R\phi$ factors through $\ker k_W$ via the natural inclusion then $\phi = 0$.

Proof: (5.10) and (5.12) tell us that the right injectivity axiom implies condition (a), and (5.14) tells us that right injectivity axiom implies condition (b).

Thus, it remains to show that conditions (a) and (b) together imply the right injectivity axiom.

Assume that (a) and (b) hold. For all $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$ we must define a map

$$\eta_{VW} : \underline{\underline{G}}(V,IW) \to \underline{\underline{H}}(RV,W).$$

Suppose $\phi \in \underline{\underline{G}}(V,IW)$.

We define $\eta_{VW}(\phi) = k_W \circ R\phi$.

Clearly $\eta_{VW}(\phi) \in \underline{\underline{H}}(RV,W)$.

We must show that η_{VW} is natural in V and W, and injective.

(1) Naturality of η_{VW}.

Let $W, W' \in \underline{\underline{H}}$, $V, V' \in \underline{\underline{G}}$ and choose $\alpha \in \underline{\underline{H}}(W, W')$, $\beta \in \underline{\underline{G}}(V', V)$. We shall show that the following diagram commutes:

$$\begin{tikzcd} \underline{\underline{G}}(V, IW) \arrow[r, "\eta_{VW}"] \arrow[d, "\underline{\underline{G}}(\beta, I\alpha)"'] & \underline{\underline{H}}(RV, W) \arrow[d, "\underline{\underline{H}}(R\beta, \alpha)"] \\ \underline{\underline{G}}(V', IW') \arrow[r, "\eta_{V'W'}"'] & \underline{\underline{H}}(RV', W') \end{tikzcd}$$

We need to show that for all $\phi \in \underline{\underline{G}}(V, IW)$,

$$\underline{\underline{H}}(R\beta, \alpha)(\eta_{VW}(\phi)) = \eta_{V'W'}(\underline{\underline{G}}(\beta, I\alpha)(\phi)).$$

That is, we must show, for $\phi \in \underline{\underline{G}}(V, IW)$, that

$$\alpha \circ (k_W \circ R\phi) \circ R\beta = k_{W'} \circ R(I\alpha \circ \phi \circ \beta)$$

or
$$\alpha \circ k_W \circ R\phi \circ R\beta = k_{W'} \circ RI\alpha \circ R\phi \circ R\beta$$

using the functoriality of R.

But condition (a) tells us that

$$\alpha \circ k_W = k_{W'} \circ RI\alpha,$$

and so, postmultiplying by $R\phi \circ R\beta$, we obtain the required commutativity condition.

(2) Injectivity.

Let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$. We must show that for any $\phi \in \underline{\underline{G}}(V, IW)$, $\eta_{VW}(\phi) = 0$ implies $\phi = 0$.

It is easy to check that, since $\underline{\underline{H}}$ has kernels, $R\phi$ factors through $\ker k_W$ via the natural inclusion if $k_W \circ R\phi = 0$. Thus, using condition (b), we see that $k_W \circ R\phi = 0$ implies $\phi = 0$. □

Remark: Let $\mathbf{h} \leq \mathbf{g}$ be Lie algebras. If $\mathbf{H} = \text{Mod-}\mathbf{h}$, $\mathbf{G} = \text{Mod-}\mathbf{g}$ and $R : \text{Mod-}\mathbf{g} \to \text{Mod-}\mathbf{h}$ is the restriction functor, then condition (b) of (5.16) can be replaced by

(b)' for all $W \in \mathbf{H}$, $\ker k_W$ contains no nonzero $U\mathbf{g}$-modules.

Proof: We must show that if $W \in \text{Mod-}\mathbf{h}$, $V \in \text{Mod-}\mathbf{g}$ and $\phi \in \text{Hom}_{\mathbf{g}}(V, IW)$, then $k_W \circ R\phi = 0$ implies $\phi = 0$ when (b)' holds. Suppose (b)' holds, and $k_W \circ R\phi = 0$. Then, for all $v \in V$, $k_W(\phi(v)) = 0$, so

$$\text{im}\, \phi \subseteq \ker k_W .$$

But $\text{im}\, \phi$ is a $U\mathbf{g}$-submodule of IW. Thus, by assumption (b)', $\text{im}\, \phi = 0$, i.e. $\phi = 0$.

The converse - that the right injectivity axiom implies condition (b)' - was proved in the corollary to Proposition (5.14). $\square$

The next result is related to (4.7) - (4.10) and (5.9). It allows us to represent our functor $I : \mathbf{H} \to \mathbf{G}$ in terms of a right adjoint to $R : \mathbf{G} \to \mathbf{H}$ (if such a right adjoint exists).

(5.17) Theorem: Let $\mathbf{H}$ and $\mathbf{G}$ be categories. Let $R : \mathbf{G} \to \mathbf{H}$ and $I : \mathbf{H} \to \mathbf{G}$ be functors satisfying the right injectivity axiom of section (5.1) and suppose that R has a right adjoint $F : \mathbf{H} \to \mathbf{G}$. Then for each $W \in \mathbf{H}$, there is a monic $\nu_W \in \mathbf{G}(IW, FW)$ which is natural in W.

Proof: Let $W \in \mathbf{H}$ and $V \in \mathbf{G}$. Let $J_{VW} : \mathbf{G}(V, FW) \to \mathbf{H}(RV, W)$ denote the adjunction map, and let ν_{VW} denote the composite map which forms the top line of the following commutative diagram:

$$\begin{tikzcd}
\mathbf{G}(V, IW) \arrow[rr, "\nu_{VW}"] \arrow[dr, "\eta_{VW}"'] & & \mathbf{G}(V, FW) \\
& \mathbf{H}(RV, W) \arrow[ur, "\simeq", "J_{VW}^{-1}"'] &
\end{tikzcd}$$

In the diagram above, η_{VW} is the map explained in (5.1). Thus ν_{VW} is injective and natural in V and W. Set $\nu_W = \nu_{IW,W}(1_{IW})$. By Yoneda lemma, ((4.6) and (4.6a)), ν_W is natural in W and if $\alpha \in \underline{\underline{G}}(V,IW)$, then $\nu_{VW}(\alpha) = \nu_W \circ \alpha$.

Consider the diagram $V \underset{\beta}{\overset{\alpha}{\rightrightarrows}} IW \xrightarrow{\nu_W} FW$ where $\alpha,\beta \in \underline{\underline{G}}(V,IW)$. $\nu_W \circ \alpha = \nu_W \circ \beta \Longleftrightarrow \nu_{VW}(\alpha) = \nu_{VW}(\alpha)$ which implies $\alpha = \beta$ since ν_{VW} is injective. Thus ν_W is monic. $\square$

In the next two corollaries and their proofs, we maintain the notation of theorem (5.17) and its proof.

<u>(5.17a) Corollary:</u> For $W \in \underline{\underline{H}}$, recall that

$$J_{IW,W} : \underline{\underline{G}}(IW,FW) \to \underline{\underline{H}}(RIW,W)$$

is the adjunction isomorphism.

If $k_W : RIW \to W$ is the morphism defined in section (5.10), then

$$J_{IW,W}(\nu_W) = k_W.$$

<u>Proof:</u> Let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$. By definition,

$$\nu_{VW} = J_{VW}^{-1} \circ \eta_{VW}, \text{ and}$$

$$\nu_W = \nu_{IW,W}(1_{IW}).$$

Thus,
$$\nu_W = J_{IW,W}^{-1}(\eta_{IW,W}(1_{IW}))$$
$$= J_{IW,W}^{-1}(k_W) \text{ by definition (5.10).}$$

That is,
$$k_W = J_{IW,W}(\nu_W).$$
$\square$

<u>(5.17b) Corollary:</u> Let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$. Let $J_{VW}: \underline{\underline{G}}(V,FW) \to \underline{\underline{H}}(RV,W)$ denote the adjunction map. If $\psi \in \underline{\underline{H}}(RV,W)$ and there exists $\pi \in \underline{\underline{G}}(V,IW)$ such that $\eta_{VW}(\pi) = \psi$, then $J_{VW}^{-1}(\psi) = \nu_W \circ \pi$. In other words, a morphism $\psi \in \underline{\underline{H}}(RV,W)$ lifts through η_{VW} to a morphism $\pi \in \underline{\underline{G}}(V,IW)$ only

if the morphism $J_{VW}^{-1}(\psi)$ factors through IW via ν_W.

The converse is also true.

Proof: Suppose we are given $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, $\psi \in \underline{\underline{H}}(RV,W)$ and $\pi \in \underline{\underline{G}}(V,IW)$ such that $\eta_{VW}(\pi) = \psi$.

$$\begin{tikzcd} RV \arrow[d, "\psi"'] \\ W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[d, "J_{VW}^{-1}(\psi)"'] \arrow[r, "\pi"] & IW \arrow[dl, "\nu_W"] \\ LW & \end{tikzcd}$$

The first thing we want to prove is that $J_{VW}(\nu_W \circ \pi) = \psi$.

Now $J_{VW}(\nu_W \circ \pi) = J_{IW,W}(1_{IW}) \circ R(\nu_W \circ \pi)$ by MacLane, [12] Theorem 1 page 80,

$= J_{IW,W}(1_{IW}) \circ R\nu_W \circ R\pi$ by functoriality of R,

$= J_{IW,W}(\nu_W) \circ R\pi$ by an argument like that of lemma (5.11)

$= k_W \circ R\pi$ by Corollary (5.17a)

$= \eta_{VW}(\pi)$ by lemma (5.11)

$= \psi$ by hypothesis.

To prove the converse, we must suppose that $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, $\psi \in \underline{\underline{H}}(RV,W)$ and that $\pi \in \underline{\underline{G}}(V,IW)$ satisfies $\nu_W \circ \pi = J_{VW}^{-1}(\psi)$, and show that

$$\eta_{VW}(\pi) = \psi.$$

The argument to show this is an obvious reversal of the steps of the proof of the first part of this corollary. □

Remarks: In chapter 7, we shall return to study the properties of functors $I : \underline{\underline{H}} \to \underline{\underline{G}}$ and $R : \underline{\underline{G}} \to \underline{\underline{H}}$ which satisfy the right and left injectivity axioms simultaneously.

Chapter 6 - The Surjectivity Axioms

(6.1): Of the eight possible weak types of adjointness proposed in section (4.1), we have now studied all but types (iv) and (iv)'. This chapter is devoted to filling this gap.

We shall restate and rename axioms (iv) and (iv)' below for convenience.

Notation: Throughout this chapter, $\underline{\underline{H}}$ and $\underline{\underline{G}}$ will be categories (with zero objects) and

$$I : \underline{\underline{H}} \to \underline{\underline{G}}$$

$$R : \underline{\underline{G}} \to \underline{\underline{H}}$$

will be functors.

The Left Surjectivity Axiom (axiom (iv) of section (4.1)) holds for I and R if, for all $W \in \underline{\underline{H}}$, all $V \in \underline{\underline{G}}$, there exists a natural surjection

$$\beta_{WV} : \underline{\underline{H}}(W,RV) \to \underline{\underline{G}}(IW,V).$$

The Right Surjectivity Axiom (axiom (iv)' of section (4.1)) holds for I and R if, for all $W \in \underline{\underline{H}}$, all $V \in \underline{\underline{G}}$, there exists a natural surjection

$$\alpha_{VW} : \underline{\underline{H}}(RV,W) \to \underline{\underline{G}}(V,IW).$$

Convention: In sections (6.2) to (6.6), we shall suppose that the left surjectivity axiom holds with respect to I and R.

(6.2) Definition of b_V: Let $V \in \underline{\underline{G}}$. We define a morphism $b_V \in G(IRV,V)$ by

$$b_V = \beta_{RV,V}(1_{RV}),$$

noting that

$$\beta_{RV,V} : \underline{\underline{H}}(RV,RV) \to \underline{\underline{G}}(IRV,V).$$

<u>(6.3) Lemma:</u> Let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$. Then b_V <u>induces</u> β_{WV} in the sense that, if $\psi \in \underline{\underline{H}}(W,RV)$, then $\beta_{WV}(\psi) = b_V \circ I\psi$.

<u>Proof:</u> Let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, and suppose $\psi \in \underline{\underline{H}}(W,RV)$. Then certainly $b_V \circ I\psi \in \underline{\underline{G}}(IW,V)$. By the naturality of β_{WV} in W, the following diagram commutes:

$$\begin{tikzcd}
\underline{\underline{H}}(RV,RV) \arrow[r, "\beta_{RV,V}"] \arrow[d, "\underline{\underline{H}}(\psi{,}RV)"'] & \underline{\underline{G}}(IRV,V) \arrow[d, "\underline{\underline{G}}(I\psi{,}V)"] \\
\underline{\underline{H}}(W,RV) \arrow[r, "\beta_{WV}"'] & \underline{\underline{G}}(IW,V)
\end{tikzcd}$$

We chase 1_{RV} around this diagram, and find:

$$\underline{\underline{G}}(I\psi,V)(\beta_{RV,V}(1_{RV})) = \beta_{WV}(\underline{\underline{H}}(\psi,RV)(1_{RV}))$$

that is, $$b_V \circ I\psi = \beta_{WV}(\psi).$$ □

<u>(6.4) Lemma:</u> b is a natural transformation $IR \to 1_{\underline{\underline{G}}}$. That is, b_V is natural in V for $V \in \underline{\underline{G}}$.

<u>Proof:</u> Let $W \in \underline{\underline{H}}$, $V_1,V_2 \in \underline{\underline{G}}$ and let $\phi \in \underline{\underline{G}}(V_1,V_2)$. By the naturality of β_{WV} in V, the following diagram commutes:

$$\begin{tikzcd}
\underline{\underline{H}}(W,RV_1) \arrow[r, "\beta_{WV_1}"] \arrow[d, "\underline{\underline{H}}(W{,}R\phi)"'] & \underline{\underline{G}}(IW,V_1) \arrow[d, "\underline{\underline{G}}(IW{,}\phi)"] \\
\underline{\underline{H}}(W,RV_2) \arrow[r, "\beta_{WV_2}"'] & \underline{\underline{G}}(IW,V_2)
\end{tikzcd}$$

We now put $W = RV_1$ and chase 1_{RV_1} around the resulting commutative diagram, and find that:

$$\underline{\underline{G}}(IRV_1,\phi)(\beta_{RV_1,V_1}(1_{RV_1})) = \beta_{RV_1,V_2}(\underline{\underline{H}}(RV_1,R\phi)(1_{RV_1})).$$

That is,
$$\begin{aligned}\phi \circ b_{V_1} &= \beta_{RV_1,V_2}(R\phi)\\ &= b_{V_2} \circ IR\phi \qquad \text{by lemma (6.3).}\end{aligned}$$

In other words, the following diagram commutes:

$$\begin{tikzcd} IRV_1 \arrow[r, "b_{V_1}"] \arrow[d, "IR\phi"'] & V_1 \arrow[d, "\phi"] \\ IRV_2 \arrow[r, "b_{V_2}"'] & V_2 \end{tikzcd}$$

so b_V is natural in V for $V \in \underline{\underline{G}}$. □

<u>(6.5) Proposition:</u> Let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, and suppose $\phi \in \underline{\underline{G}}(IW,V)$. Then ϕ factors through IRV via b_V.

<u>Proof:</u> Let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, $\phi \in \underline{\underline{G}}(IW,V)$. By the left surjectivity axiom, there exists $\psi \in \underline{\underline{H}}(W,RV)$ such that $\beta_{WV}(\psi) = \phi$. But, by lemma (6.3),

$$\beta_{WV}(\psi) = b_V \circ I\psi.$$

So
$$\phi = b_V \circ I\psi,$$

i.e. the following diagram commutes:

$$\begin{tikzcd} IW \arrow[rr, "\phi"] \arrow[dr, "I\psi"'] & & V \\ & IRV \arrow[ur, "b_V"'] & \end{tikzcd}$$

so ϕ factors through IRV via b_V. □

<u>(6.6) Lemma</u>: If R is full, then for all $V,V' \in \underline{\underline{G}}$ and $\delta \in \underline{\underline{G}}(IRV,V')$, there exists $\phi \in \underline{\underline{G}}(V,V')$ such that $\delta = \phi \circ b_V$; that is, such that the following diagram commutes:

$$\begin{tikzcd} IRV \arrow[rr, "\delta"] \arrow[dr, "b_V"'] & & V' \\ & V \arrow[ur, "\phi"'] & \end{tikzcd}$$

<u>Proof</u>: Let $V,V' \in \underline{\underline{G}}$ and $\delta \in \underline{\underline{G}}(IRV,V')$. We have a surjective map

$$\beta_{RV,V'} : \underline{\underline{H}}(RV,RV') \to \underline{\underline{G}}(IRV,V'),$$

hence there exists $\psi \in \underline{\underline{H}}(RV,RV')$ such that $\beta_{RV,V'}(\psi) = \delta$. Since R is full,

$$R : \underline{\underline{G}}(V,V') \to \underline{\underline{H}}(RV,RV')$$

is surjective, so there exists $\phi \in \underline{\underline{G}}(V,V')$ such that

$$R\phi = \psi.$$

That is, $\beta_{RV,V'}(R\phi) = \delta$. But, by lemma (6.3),

$$\begin{aligned} \beta_{RV,V'}(R\phi) &= b_{V'} \circ IR\phi \\ &= \phi \circ b_V \quad \text{by naturality of } b_V \\ &\qquad \text{(lemma (6.4))} \end{aligned}$$

So $\delta = \phi \circ b_V$ as claimed. □

<u>Remark</u>: Suppose $\underline{\underline{h}} \leq \underline{\underline{g}}$ are Lie algebras. The natural restriction functor R : Mod-$\underline{\underline{g}}$ $\to$ Mod-$\underline{\underline{h}}$ is <u>not</u> full.

<u>(6.7) Theorem</u>: Let $I : \underline{\underline{H}} \to \underline{\underline{G}}$ be a full functor. Then the left surjectivity axiom of (6.1) is equivalent to the following pair of conditions:

(a) There is a natural transformation $b : IR \overset{\bullet}{\to} 1_{\underline{\underline{G}}}$

and (b) For $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, every morphism $\phi \in \underline{\underline{G}}(IW,V)$ factors through IRV via b_V.

Proof: (6.2) and (6.4) tell us that the left surjectivity axiom implies condition (a), and (6.5) tells us that the left surjectivity axiom implies condition (b). It remains to prove that conditions (a) and (b) together imply the left surjectivity axiom, provided I is full.

Assume that (a) and (b) hold, and let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$. We must define a map

$$\beta_{WV} : \underline{\underline{H}}(W,RV) \to \underline{\underline{G}}(IW,V)$$

Suppose $\psi \in \underline{\underline{H}}(W,RV)$. We define $\beta_{WV}(\psi) = b_V \circ I\psi$. It is easy to check that $\beta_{WV}(\psi) \in \underline{\underline{G}}(IW,V)$.

Now we must show that β_{WV} is natural in W and V, and surjective, provided I is full.

(1) Naturality of β_{WV}.

Let $W,\bar{W} \in \underline{\underline{H}}$ and $V,\bar{V} \in \underline{\underline{G}}$. Choose $\gamma \in \underline{\underline{H}}(\bar{W},W)$ and $\delta \in \underline{\underline{G}}(V,\bar{V})$. We need to show that the following diagram is commutative.

$$\begin{tikzcd}
\underline{\underline{H}}(W,RV) \arrow[r, "\beta_{WV}"] \arrow[d, "\underline{\underline{H}}(\gamma{,}R\delta)"'] & \underline{\underline{G}}(IW,V) \arrow[d, "\underline{\underline{G}}(I\gamma{,}\delta)"] \\
\underline{\underline{H}}(\bar{W},R\bar{V}) \arrow[r, "\beta_{\bar{W}\bar{V}}"'] & \underline{\underline{G}}(I\bar{W},\bar{V})
\end{tikzcd}$$

That is, we must show that for all $\psi \in \underline{\underline{H}}(W,RV)$

$$\underline{\underline{G}}(I\gamma,\delta)(\beta_{WV}(\psi)) = \beta_{\bar{W}\bar{V}}(\underline{\underline{H}}(\gamma,R\delta)(\psi)).$$

That is, for all $\psi \in \underline{\underline{H}}(W,RV)$

$$\delta \circ (b_V \circ I\psi) \circ I\gamma = b_{\bar{V}} \circ I(R\delta \circ \psi \circ \gamma)$$

or $$\delta \circ b_V \circ I\psi \circ I\gamma = b_{\bar{V}} \circ IR\delta \circ I\psi \circ I\gamma.$$

Now condition (a) tells us that

$$\delta \circ b_V = b_{\bar{V}} \circ IR\delta .$$

If we multiply on the right by $I\psi \circ I\gamma$, we obtain the desired conclusion.

(2) Surjectivity of β_{WV}

Let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, and suppose I is full. Suppose $\phi \in \underline{\underline{G}}(IW,V)$. We must find a morphism $\psi \in \underline{\underline{H}}(W,RV)$ such that $\beta_{WV}(\psi) = \phi$.

By condition (b), ϕ factorizes as $\phi = b_V \circ \chi$ for some $\chi \in \underline{\underline{G}}(IW,IRV)$:

$$\begin{tikzcd} IW \arrow[r, "\phi"] \arrow[d, "\chi"'] & V \\ IRV \arrow[ur, "b_V"'] & \end{tikzcd}$$

Since I is full, there exists $\psi \in \underline{\underline{H}}(W,RV)$ such that $I\psi = \chi$.

Then
$$\begin{aligned} \beta_{WV}(\psi) &= b_V \circ I\psi \\ &= b_V \circ \chi \\ &= \phi \text{ , as required.} \end{aligned}$$
□

Next we pay a visit to the sequence of results begun with (4.7)-(4.10), (5.9) and (5.17).

(6.8) Theorem: Let I and R satisfy the left surjectivity axiom, and suppose that R has a left adjoint $L : \underline{\underline{H}} \to \underline{\underline{G}}$. Then, for all $W \in \underline{\underline{H}}$, there is a split monomorphism $\beta_W \in \underline{\underline{G}}(IW,LW)$ such that β_W is natural in W.

Proof: Suppose $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, and let $K_{WV} : \underline{\underline{G}}(LW,V) \to \underline{\underline{H}}(W,RV)$ denote the adjunction isomorphism.

Let $\hat{\beta}_{WV}$ denote the composition map

$$\underline{\underline{G}}(LW,V) \xrightarrow{K_{WV}} \underline{\underline{H}}(W,RV) \xrightarrow{\beta_{WV}} \underline{\underline{G}}(IW,V).$$

$\hat{\beta}_{WV}$ is surjective since β_{WV} is surjective.

Hence, by Corollary (4.4a), the morphism $\beta_W \in \underline{\underline{G}}(IW,LW)$ defined by

$$\beta_W = \hat{\beta}_{W,LW}(1_{LW})$$

is natural in W. Also, since $\hat{\beta}_{W,IW}$ is surjective, there exists $\pi_W \in \underline{\underline{G}}(LW,IW)$ such that

$$\hat{\beta}_{W,IW}(\pi_W) = 1_{IW}.$$

By (4.4a) and (4.4), this means

$$\pi_W \circ \beta_W = 1_{IW}.$$

Thus β_W is a split monic. □

Convention: In sections (6.9)-(6.13) we shall assume that the right surjectivity axiom holds with respect to I and R.

(6.9) Definition of a_V: Let $V \in \underline{\underline{G}}$. We define a morphism $a_V \in \underline{\underline{G}}(V,IRV)$ by

$$a_V = \alpha_{V,RV}(1_{RV}),$$

noting that

$$\alpha_{V,RV} : \underline{\underline{H}}(RV,RV) \to \underline{\underline{G}}(V,IRV).$$

(6.10) Lemma: Let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$. Then a_V induces α_{VW} in the sense that, if $\psi \in \underline{\underline{H}}(RV,W)$, then $\alpha_{VW}(\psi) = I\psi \circ a_V$.

Proof: Suppose $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$ and $\psi \in \underline{\underline{H}}(RV,W)$. Then certainly $I\psi \circ a_V \in \underline{\underline{G}}(V,IW)$. By the naturality of α_{VW} in W, the following diagram commutes:

$$\begin{tikzcd}
\underline{\underline{H}}(RV,RV) \arrow[r, "\alpha_{V,RV}"] \arrow[d, "\underline{\underline{H}}(RV,\psi)"'] & \underline{\underline{G}}(V,IRV) \arrow[d, "\underline{\underline{G}}(V,I\psi)"] \\
\underline{\underline{H}}(RV,W) \arrow[r, "\alpha_{VW}"'] & \underline{\underline{G}}(V,IW)
\end{tikzcd}$$

We chase 1_{RV} around this diagram, and find:

$$\underline{\underline{G}}(V,I\psi)(\alpha_{V,RV}(1_{RV})) = \alpha_{VW}(\underline{\underline{H}}(RV,\psi)(1_{RV})).$$

That is, $$I\psi \circ a_V = \alpha_{VW}(\psi).$$ □

(6.11) Lemma: a is a natural transformation $1_{\underline{\underline{G}}} \to IR$. That is, a_V is natural in V ($V \in \underline{\underline{G}}$).

Proof: Let $W \in \underline{\underline{H}}$, $V_1, V_2 \in \underline{\underline{G}}$ and let $\phi \in \underline{\underline{G}}(V_2,V_1)$. By naturality of α_{VW} in V, the following diagram commutes:

$$\begin{tikzcd}
\underline{\underline{H}}(RV_1,W) \arrow[r, "\alpha_{V_1W}"] \arrow[d, "\underline{\underline{H}}(R\phi,W)"'] & \underline{\underline{G}}(V_1,IW) \arrow[d, "\underline{\underline{G}}(\phi,IW)"] \\
\underline{\underline{H}}(RV_2,W) \arrow[r, "\alpha_{V_2W}"'] & \underline{\underline{G}}(V_2,IW)
\end{tikzcd}$$

Now put $W = RV_1$ and chase 1_{RV_1} around the resulting diagram to obtain:

$$\underline{\underline{G}}(\phi,IRV_1)(\alpha_{V_1,RV_1}(1_{RV_1})) = \alpha_{V_2,RV_1}(\underline{\underline{H}}(R\phi,RV_1)(1_{RV_1})).$$

That is

$$\begin{aligned} a_{V_1} \circ \phi &= \alpha_{V_2,RV_1}(R\phi) \\ &= IR\phi \circ a_{V_2}, \qquad \text{by lemma (6.10)} \end{aligned}$$

So the following diagram commutes:

$$\begin{tikzcd}
V_1 \arrow[r, "a_{V_1}"] & IRV_1 \\
V_2 \arrow[u, "\phi"] \arrow[r, "a_{V_2}"'] & IRV_2 \arrow[u, "IR\phi"']
\end{tikzcd}$$

That is, a_V is natural in V for $V \in \underline{\underline{G}}$. □

(6.12) Proposition: Let $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$, and suppose $\phi \in \underline{\underline{G}}(V,IW)$. Then ϕ factors through IRV via a_V.

Proof: Let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$ and $\phi \in \underline{\underline{G}}(V,IW)$. By the right surjectivity axiom, there exists $\psi \in \underline{\underline{H}}(RV,W)$ such that $\alpha_{VW}(\psi) = \phi$.

But, by lemma (6.10), $\alpha_{VW}(\psi) = I\psi \circ a_V$. Thus

$$\phi = I\psi \circ a_V,$$

that is, the following diagram commutes:

$$\begin{tikzcd} V \arrow[rr, "\phi"] \arrow[dr, "a_V"'] & & IW \\ & IRV \arrow[ur, "I\psi"'] & \end{tikzcd}$$

So ϕ factors through IRV via a_V. $\square$

<u>(6.13) Lemma:</u> Suppose R is full, and let $V,V' \in \underline{\underline{G}}$. For all $\delta \in \underline{\underline{G}}(V',IRV)$, there exists $\phi \in \underline{\underline{G}}(V',V)$ such that $\delta = a_V \circ \phi$; that is, such that the following diagram commutes:

$$\begin{tikzcd} V' \arrow[rr, "\delta"] \arrow[dr, "\phi"'] & & IRV \\ & V \arrow[ur, "a_V"'] & \end{tikzcd}$$

<u>Proof:</u> Let $V,V' \in \underline{\underline{G}}$ and choose $\delta \in \underline{\underline{G}}(V',IRV)$. We have a surjective map

$$\alpha_{V',RV} : \underline{\underline{H}}(RV',RV) \to \underline{\underline{G}}(V',IRV)$$

so there exists $\psi \in \underline{\underline{H}}(RV',RV)$ such that

$$\alpha_{V',RV}(\psi) = \delta.$$

Since R is full,

$$R : \underline{\underline{G}}(V',V) \to \underline{\underline{H}}(RV',RV)$$

is surjective, so there exists $\phi \in \underline{\underline{G}}(V',V)$ such that $R\phi = \psi$. Hence

$$\alpha_{V',RV}(R\phi) = \delta.$$

But, by lemma (6.10), $\alpha_{V',RV}(R\phi) = IR\phi \circ a_V,$

$$= a_V \circ \phi$$

by the naturality of a_V in V (see lemma (6.11)).

Thus $\delta = a_V \circ \phi$. This completes the proof. □

(6.14) Theorem: Let $I : \underline{\underline{H}} \to \underline{\underline{G}}$ be a full functor. Then the right surjectivity axiom of (6.1) is equivalent to the following pair of conditions.

(a) There is a natural transformation $a : 1_{\underline{\underline{G}}} \dot{\to} IR$;

and (b) For $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, every morphism $\phi \in \underline{\underline{G}}(V,IW)$ factors through IRV via a_V.

Proof: (6.9) and (6.11) tell us that the right surjectivity axiom implies condition (a), and (6.12) tells us that the right surjectivity axiom implies condition (b). It remains to prove that the conditions (a) and (b) together imply the right surjectivity axiom, provided I is full.

Assume (a) and (b) hold, and let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$. We must define a map

$$\alpha_{VW} : \underline{\underline{H}}(RV,W) \to \underline{\underline{G}}(V,IW).$$

Suppose $\psi \in \underline{\underline{H}}(RV,W)$. We define

$$\alpha_{VW}(\psi) = I\psi \circ a_V.$$

It is easy to check that $\alpha_{VW}(\psi) \in \underline{\underline{G}}(V,IW)$.

Now we shall show that α_{VW} is natural in V and W and that, provided I is full, α_{VW} is surjective.

(1) Naturality of α_{VW}

Let $W,\bar{W} \in \underline{\underline{H}}$ and $V,\bar{V} \in \underline{\underline{G}}$. Choose $\gamma \in \underline{\underline{H}}(W,\bar{W})$ and $\delta \in \underline{\underline{G}}(\bar{V},V)$. We need to show that the following diagram is commutative:

$$\begin{tikzcd}
\underline{\underline{H}}(RV,W) \arrow[r, "\alpha_{VW}"] \arrow[d, "\underline{\underline{H}}(R\delta{,}\gamma)"'] & \underline{\underline{G}}(V,IW) \arrow[d, "\underline{\underline{G}}(\delta{,}I\gamma)"] \\
\underline{\underline{H}}(R\bar{V},\bar{W}) \arrow[r, "\alpha_{\bar{V}\bar{W}}"'] & \underline{\underline{G}}(\bar{V},I\bar{W})
\end{tikzcd}$$

To show that the diagram commutes, we must prove that for all $\psi \in \underline{\underline{H}}(RV,W)$,

$$\underline{\underline{G}}(\delta,\ I\gamma)(\alpha_{VW}(\psi)) = \alpha_{\bar{V}\bar{W}}(\underline{\underline{H}}(R\delta,\gamma)(\psi)).$$

That is, we must prove that for all $\psi \in \underline{\underline{H}}(RV,W)$,

$$I\gamma \circ (I\psi \circ a_V) \circ \delta = I(\gamma \circ \psi \circ R\delta) \circ a_{\bar{V}}$$

i.e. $$I\gamma \circ I\psi \circ a_V \circ \delta = I\gamma \circ I\psi \circ IR\delta \circ a_{\bar{V}}\ ,$$

by functoriality of I.

Now, condition (a) tells us that

$$a_V \circ \delta = IR\delta \circ a_{\bar{V}}\ ,$$

and premultiplying both sides of this equation by $I\gamma \circ I\psi$ (where $\psi \in \underline{\underline{H}}(RV,W)$), we obtain the desired conclusion.

(2) Surjectivity of α_{VW}

Let $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, and suppose I is full. For each $\phi \in \underline{\underline{G}}(V,IW)$, we must find a $\psi \in \underline{\underline{H}}(RV,W)$ such that $\alpha_{VW}(\psi) = \phi$.

By condition (b), each ϕ $\underline{\underline{G}}(V,IW)$ may be factorized as $\phi = \chi \circ a_V$, where $\chi \in \underline{\underline{G}}(IRV,IW)$:

$$\begin{tikzcd}
V \arrow[rr, "\phi"] \arrow[dr, "a_V"'] & & IW \\
& IRV \arrow[ur, "\chi"'] &
\end{tikzcd}$$

Since I is full, there exists $\psi \in \underline{\underline{H}}(RV,W)$ such that

$$I\psi = \chi.$$

Then $\alpha_{VW}(\psi) = I\psi \circ a_V = \chi \circ a_V = \phi$, so we have found a morphism ψ with the required property. $\square$

Next we conclude the series of results begun with (4.7)-(4.10), (5.9) and (5.17), and (6.8).

(6.15) Theorem: Let I and R satisfy the right surjectivity axiom, and suppose that R has a right adjoint $F : \underline{\underline{H}} \to \underline{\underline{G}}$. Then, for all $W \in \underline{\underline{H}}$, there is a split epimorphism $\alpha_W \in \underline{\underline{G}}(FW,IW)$ such that α_W is natural in W.

Proof: Suppose $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, and let

$$J_{VW} : \underline{\underline{G}}(V,IW) \to \underline{\underline{H}}(RV,W)$$

denote the adjunction isomorphism.

Let $\hat{\alpha}_{VW}$ denote the composition map

$$\underline{\underline{G}}(V,FW) \xrightarrow{J_{VW}} \underline{\underline{H}}(RV,W) \xrightarrow{\alpha_{VW}} \underline{\underline{G}}(V,IW).$$

$\hat{\alpha}_{VW}$ is surjective since α_{VW} is surjective.

Hence, by corollary (4.6a), the map $\alpha_W \in \underline{\underline{G}}(FW,IW)$ defined by

$$\alpha_W = \hat{\alpha}_{FW,W}(1_{FW})$$

is natural in W.

Also, since $\hat{\alpha}_{IW,W}$ is surjective, there exists $\mu_W \in \underline{\underline{G}}(IW,FW)$ such that

$$\hat{\alpha}_{IW,W}(\mu_W) = 1_{IW}.$$

But by (4.6) and (4.6a),

$$\hat{\alpha}_{IW,W}(\mu_W) = \alpha_W \circ \mu_W.$$

So $\alpha_W \circ \mu_W = 1_{IW}$, and α_W is a split epimorphism. $\square$

Chapter 7 - The Injectivity Axioms Revisited

(7.1) Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be categories, and let

$$I : \underline{\underline{H}} \to \underline{\underline{G}} \text{ be a faithful functor}$$

$$\text{and} \quad R : \underline{\underline{G}} \to \underline{\underline{H}} \quad \text{a functor.}$$

In this chapter, we shall investigate the consequences of assuming that I and R satisfy both the left <u>and</u> right injectivity axioms. These axioms, it will be recalled, were first investigated separately in chapter 5, and we shall maintain the notational conventions introduced there; in particular, we shall maintain the notation j_W, $(W \in \underline{\underline{H}})$ introduced in section (5.2) for a certain morphism in $\underline{\underline{H}}(W,RIW)$, and the notation k_W $(W \in \underline{\underline{H}})$ introduced in section (5.10) for a certain morphism in $\underline{\underline{H}}(RIW,W)$.

We shall use the reformulations of the injectivity axioms given in theorems (5.8) and (5.16).

In most of this chapter, we shall be dealing with specified categories $\underline{\underline{H}}$ and $\underline{\underline{G}}$.

(7.2) The Splitting Axiom.

<u>Definition</u>: Let $\underline{\underline{H}}$ be an exact preadditive category. An object $W \in \underline{\underline{H}}$ is said to be <u>simple</u>, if the only subobjects of W are 0 and W.

In the usual way, we have

<u>Schur's Lemma</u>: If $\underline{\underline{H}}$ is an exact preadditive category and W is a simple object of $\underline{\underline{H}}$, then $\underline{\underline{H}}(W,W)$ is a division ring.

We shall use the definition and lemma above to make plausible an axiom which we are going to state at the end of this section, and shall call the Splitting Axiom.

Suppose $\underline{\underline{H}}$, $\underline{\underline{G}}$ are categories, that $I : \underline{\underline{H}} \to \underline{\underline{G}}$ and $R : \underline{\underline{G}} \to \underline{\underline{H}}$ are functors, and that $j : 1_{\underline{\underline{H}}} \dot{\to} RI$ and $k : RI \dot{\to} 1_{\underline{\underline{H}}}$ are natural transformations. Choose $W, \bar{W} \in \underline{\underline{H}}$ and $f \in \underline{\underline{H}}(W,\bar{W})$, and consider the following diagram:

$$\begin{tikzcd} W \arrow[r, "j_W"] \arrow[d, "f"'] & RIW \arrow[r, "k_W"] \arrow[d, "RIf"'] & W \arrow[d, "f"] \\ \bar{W} \arrow[r, "j_{\bar{W}}"'] & RI\bar{W} \arrow[r, "k_{\bar{W}}"'] & \bar{W} \end{tikzcd}$$

The outside rectangle commutes because the inside squares commute, i.e. $k \circ j : 1_{\underline{\underline{H}}} \dot{\to} 1_{\underline{\underline{H}}}$ is a natural transformation.

Hence, if $\underline{\underline{H}}$ is an exact, preadditive category and W is a simple object in $\underline{\underline{H}}$, then $k_W \circ j_W$ is invertible or zero.

Suppose that for <u>all</u> $W \in \underline{\underline{H}}$, $k_W \circ j_W$ is invertible, and denote this composition map by ξ_W. It is easy to check that for $W \in \underline{\underline{H}}$, $j_W \circ \xi_W^{-1}$ is natural in W. It is also easy to check that, for all $W \in \underline{\underline{H}}$, if j_W has the property expressed in condition (b) of theorem (5.8), then so does $j_W \circ \xi_W^{-1}$.

Thus, by theorem (5.8), for all $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, $j_W \circ \xi_W^{-1}$ induces a natural injection

$$\underline{\underline{G}}(IW,V) \to \underline{\underline{H}}(W,RV).$$

Finally, for all $W \in \underline{\underline{H}}$,

$$k_W \circ (j_W \circ \xi_W^{-1}) = 1_W.$$

<u>Definition:</u> Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be categories, and let $I : \underline{\underline{H}} \to \underline{\underline{G}}$, $R : \underline{\underline{G}} \to \underline{\underline{H}}$ be functors. Suppose $j : 1_{\underline{\underline{H}}} \dot{\to} RI$ and $k : RI \dot{\to} 1_{\underline{\underline{H}}}$ are natural transformations.

We shall say that <u>j and k satisfy the splitting axiom</u> if, for each

$W \in \underline{\underline{H}}$,

$$k_W \circ j_W = 1_W.$$

If I and R satisfy both left and right injectivity axioms, and hence, by (5.4) and (5.12), give rise to natural transformations $j : 1_{\underline{\underline{H}}} \dot{\to} RI$, $k : RI \dot{\to} 1_{\underline{\underline{H}}}$, then we shall say that I and R satisfy the splitting axiom if, for each $W \in \underline{\underline{H}}$, $k_W \circ j_W = 1_W$.

(7.3) Proposition: Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be categories of modules, and let $I : \underline{\underline{H}} \to \underline{\underline{G}}$, $R : \underline{\underline{G}} \to \underline{\underline{H}}$ be functors satisfying the left and right injectivity axioms and the splitting axiom. Choose $W_1, W_2, W_3 \in \underline{\underline{H}}$ and $\alpha \in \underline{\underline{H}}(W_1,W_2)$, $\beta \in \underline{\underline{H}}(W_2,W_3)$. If

$$RIW_1 \xrightarrow[RI\alpha]{} RIW_2 \xrightarrow[RI\beta]{} RIW_3$$

is an exact sequence, then the original sequence

$$W_1 \xrightarrow{\alpha} W_2 \xrightarrow{\beta} W_3$$

must have been exact. That is, RI "reflects" exactness.

Proof: We know that $im(RI\alpha) = ker(RI\beta)$, and that the following diagram is commutative:

$$\begin{tikzcd} RIW_1 \arrow[r, "RI\alpha"] \arrow[d, shift left, "k_{W_1}"] & RIW_2 \arrow[r, "RI\beta"] \arrow[d, shift left, "k_{W_2}"] & RIW_3 \arrow[d, shift left, "k_{W_3}"] \\ W_1 \arrow[r, "\alpha"'] \arrow[u, shift left, "j_{W_1}"] & W_2 \arrow[r, "\beta"'] \arrow[u, shift left, "j_{W_2}"] & W_3 \arrow[u, shift left, "j_{W_3}"] \end{tikzcd}$$

We also know that $k_{W_i} \circ j_{W_i} = 1_{W_i}$ for $i = 1,2,3$.

(1) First, we shall show that $\ker \beta \subseteq \operatorname{im} \alpha$.

If $b \in \ker \beta$, then $\beta(b) = 0$, so $j_{W_3}(\beta(b)) = 0$, so $RI\beta(j_{W_2}(b)) = 0$ by naturality of j. Hence $j_{W_2}(b) \in \ker RI\beta = \operatorname{im}\ RI\alpha$. That is, there exists $a^* \in RIW_1$ such that $RI\alpha(a^*) = j_{W_2}(b)$.

$$\therefore \quad b = k_{W_2}(j_{W_2}(b))$$
$$= k_{W_2}(RI\alpha(a^*))$$
$$= \alpha(k_{W_1}(a^*)) \quad \text{by naturality of } k$$
$$\in \text{im } \alpha.$$

(2) Next we prove $\text{im } \alpha \subseteq \ker \beta$.

Let $b \in \text{im } \alpha$, so that there exists $a \in W_1$ such that $b = \alpha(a)$. Then

$$\beta(b) = \beta(\alpha(a)) = \beta(a(k_{W_1}(j_{W_1}(a)))$$
$$= \beta(k_{W_2}(RI\alpha(j_{W_1}(a))) \quad \text{by naturality of } k$$
$$= k_{W_3}(RI\beta(RI\alpha(j_{W_1}(a))) \quad \text{by naturality of } k$$

and $RI\beta \circ RI\alpha = 0$ since $\text{im } RI\alpha = \ker RI\beta$, so $\beta(b) = 0$. That is, $b \in \ker \beta$. □

Convention. The following conventions will be in force until the end of section (7.14). We shall suppose that $\underline{\underline{h}} \leq \underline{\underline{g}}$ are Lie algebras, and write $\underline{\underline{H}} = \text{Mod-}\underline{\underline{h}}$, $\underline{\underline{G}} = \text{Mod-}\underline{\underline{g}}$. We shall denote by R the restriction functor $\text{Mod-}\underline{\underline{g}} \to \text{Mod-}\underline{\underline{h}}$. $I : \text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{g}}$ will be a functor.

(7.4): Suppose that I and R satisfy the left injectivity axiom. Then theorems (5.9) and (3.6) guarantee the existence of a natural epimorphism of $\underline{\underline{g}}$-modules

$$\mu_W : W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}} \to IW.$$

The following proposition identifies this map quite precisely.

(7.4) Proposition: In the above notation, and for $W \in \underline{\underline{H}}$, $w \in \underline{\underline{W}}$, and $u \in U\underline{\underline{g}}$,

$$\mu_W(w \otimes u) = j_W(w).u$$

where the multiplication referred to on the righthand side is the module multiplication in IW.

Proof: By Corollary (5.9a), we have that

$$\mu_W = K_{W,IW}^{-1}(j_W).$$

From the definition of K (see proof of theorem (3.6)), it may be seen that for any $V \in \underline{\underline{G}}$ and $\psi \in \mathrm{Hom}_{U\underline{\underline{h}}}(W,RV)$,

$$(K_{WV}^{-1}(\psi))(w \otimes u) = \psi(w).u.$$

Hence, in the case $V = IW$, $\psi = j_W$,

$$\mu_W(w \otimes u) = (K_{W,IW}^{-1}(j_W))(w \otimes u) = j_W(w).u \ . \qquad \square$$

(7.5) Next, we shall identify the morphism ν_W of Theorem (5.17) in a similar fashion to section (7.4). Suppose I and R satisfy the right injectivity axiom. Let $W \in \underline{\underline{H}}$. Theorems (5.17) and (3.5) guarantee the existence of a natural injection ν_W: $IW \to \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$.

(7.5) Proposition: In the above notation, and for $W \in \underline{\underline{H}}$, $v \in IW$, $u \in U\underline{\underline{g}}$,

$$(\nu_W(v))(u) = k_W(vu),$$

where the multiplication referred to on the right hand side is the module multiplication in IW.

Proof: By Corollary (5.17a),

$$\nu_W = J_{IW,W}^{-1}(k_W).$$

From the definition of J (see the proof of theorem (3.5)) it may be seen that for any $V \in \underline{\underline{G}}$, and any $\psi \in \mathrm{Hom}_{\underline{\underline{h}}}(RV,W)$, $v \in V$ and $u \in U\underline{\underline{g}}$

$$((J_{VW}^{-1}(\psi))(v))(u) = \psi(v.u).$$

Thus, when $V = IW$ and $\psi = k_W$, we find that for $v \in IW$, $u \in U\underline{\underline{g}}$,

$$(\nu_W(v))(u) = k_W(v.u). \qquad \square$$

(7.6). Suppose $I : \underline{\underline{H}} \to \underline{\underline{G}}$ and $R : \underline{\underline{G}} \to \underline{\underline{H}}$ satisfy both left and right injectivity axioms. For $W \in \underline{\underline{H}}$, ν_W will denote the $\underline{\underline{g}}$-module monomorphism

$$\nu_W : IW \to \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$$

defined by the equation in Proposition (7.5).

<u>(7.6) Proposition:</u> With notation as above, and with $W \in \underline{\underline{H}}$, $w \in W$, $x \in U\underline{\underline{g}}$, $u \in U\underline{\underline{g}}$, the embedding

$$\nu_W : IW \to \mathrm{Hom}_{\underline{\underline{h}}}(U\underline{\underline{g}},W)$$

is completely determined by the equation

$$(\nu_W(j_W(w).x))(u) = k_W(j_W(w).xu)$$

where the multiplication on both sides of the equation is the module multiplication in IW.

<u>Proof:</u> By the corollary to proposition (5.6), $IW = (\mathrm{im}\ j_W).U\underline{\underline{g}}$. Thus, if $v \in IW$, v can be written as

$$v = \sum_{i=1}^{n} j_W(w_i).x_i$$

for suitable $w_1,\ldots,w_n \in W$ and $x_1,\ldots,x_n \in U\underline{\underline{g}}$. Thus, by (7.5) for any $u \in U\underline{\underline{g}}$,

$$\begin{aligned}(\nu_W(v))(u) &= k_W(v.u)\\ &= k_W((\sum_{i=1}^{n} j_W(w_i).x_i).u)\\ &= \sum_{i=1}^{n} k_W(j_W(w_i).x_iu).\end{aligned}$$

The proposition now follows from the fact that ν_W is linear in v.

<u>(7.7) Proposition:</u> Suppose that I and R satisfy the right injectivity axiom. Let $W \in \underline{\underline{H}}$, and let $e : R(\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)) \to W$ be the "evaluation" map, defined by

$$e(\psi) = \psi(1_{U\underline{\underline{g}}}) \quad \text{for } \psi \in R(\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)).$$

Then, with the notation of (7.5), the following diagram commutes:

$$\begin{tikzcd} & R(\mathrm{Hom}_{\underline{\underline{h}}}(U\underline{\underline{g}},W)) \arrow[dr, "e"] & \\ RIW \arrow[ur, "R\nu_W"] \arrow[rr, "k_W"'] & & W \end{tikzcd} .$$

Proof: If $v \in RIW$, then

$$\begin{aligned} e((R\nu_W)(v)) &= (\nu_W(v))(1_{U\underline{\underline{g}}}) \\ &= k_W(v.1_{U\underline{\underline{g}}}) \\ &= k_W(v). \end{aligned}$$

Thus $$e \circ R\nu_W = k_W. \qquad \square$$

(7.8) Proposition: (cf Wallach [16], theorem 3.1). Let I and R satisfy both left and right injectivity axioms. If $\bar{I}$: Mod-$\underline{\underline{h}}$ → Mod-$\underline{\underline{g}}$ is a functor, and $\bar{j} : 1_{\underline{\underline{H}}} \dot{\to} R\bar{I}$, $\bar{k} : R\bar{I} \dot{\to} 1_{\underline{\underline{H}}}$ are natural transformations satisfying

(i) for all $W \in \underline{\underline{H}}$, im $\bar{j}_W.U\underline{\underline{g}} = \bar{I}W$;

(ii) for all $W \in \underline{\underline{H}}$, ker $\bar{k}_W$ contains no non-zero $\underline{\underline{g}}$-modules,

and (iii) for all $W \in \underline{\underline{H}}$, $w \in W$, and $u \in U\underline{\underline{g}}$,

$$\bar{k}_W(\bar{j}_W(w).u) = k_W(j_W(w).u)$$

then for all $W \in \underline{\underline{H}}$

$$\bar{I}W \simeq IW \qquad \text{as } \underline{\underline{g}}\text{-modules.}$$

Proof: Let $W \in \underline{\underline{H}}$. Define $\nu_W : IW \to \mathrm{Hom}_{\underline{\underline{h}}}(U\underline{\underline{g}},W)$ as in (7.6). Define $\bar{\nu}_W : I\bar{W} \to \mathrm{Hom}_{\underline{\underline{h}}}(U\underline{\underline{g}},W)$ analogously, by $(\bar{\nu}_W(\bar{j}_W(w).x))(u) = \bar{k}_W(\bar{j}_W(w).xu)$ for $w \in W$, $x \in U\underline{\underline{g}}$, $u \in U\underline{\underline{g}}$. (We have implicitly used condition (i) in this definition.) By theorems (5.8), (5.16), (5.17) and proposition (7.6), $\bar{\nu}_W$ is a $\underline{\underline{g}}$-monomorphism.

Let $\hat{\nu}_W$ be ν_W with codomain restricted to im ν_W.

Let $\hat{\bar{\nu}}_W$ be $\bar{\nu}_W$ with codomain restricted to im $\bar{\nu}_W$.

Clearly, condition (iii) implies that

$$\text{im } \nu_W = \text{im } \bar{\nu}_W.$$

Hence $(\hat{\nu}_W)^{-1} \circ \hat{\bar{\nu}}_W : \bar{I}W \to IW$ is an isomorphism of $U\underline{\underline{g}}$-modules. □

(7.9) Proposition: (This result was originally proved by N.R. Wallach in [17] - Proposition 3.1, for a particular functor I, which will be described in the next chapter.) Let I and R satisfy the left and right injectivity axioms, and the splitting axiom. Let W be a simple $U\underline{\underline{h}}$-module with the property that ker k_W contains no subquotients $U\underline{\underline{h}}$-isomorphic to W. Then IW is a simple $U\underline{\underline{g}}$-module.

Proof: Suppose IW is not simple: let M be a proper non-trivial $\underline{\underline{g}}$-submodule of IW. Write $\ell : M \to IW$ for the inclusion map.

Then $R\ell - j_W \circ k_W \circ R\ell$ is an $\underline{\underline{h}}$-monomorphism. For certainly $R\ell - j_W \circ k_W \circ R\ell$ is an $\underline{\underline{h}}$-homomorphism, and if $m \neq 0$, $m \in M$ and $m - j_W(k_W(m)) = 0$, then $m \in \text{im } j_W$, so that, by the simplicity of W and the facts that j_W is an $\underline{\underline{h}}$-homomorphism, and $m \neq 0$,

$$\text{im } j_W = m.U\underline{\underline{h}}.$$

But then, by Corollary (5.6), which says that $IW = \text{im } j_W.U\underline{\underline{g}}$, we see that

$$m.U\underline{\underline{g}} = \text{im } j_W.U\underline{\underline{g}} = IW.$$

Hence $IW = m.U\underline{\underline{g}} \subseteq M \subset IW$, a contradiction. So $R\ell - j_W \circ k_W \circ R\ell$ is monic, as claimed.

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* Wallach's statement and proof of this result contain an error. His proof does not, of course, use the injectivity axioms.

By the corollary to (5.14), $M \not\subseteq \ker k_W$, so we can choose an element $v \in M$ such that $v \in \ker k_W$. Set $\tilde{W} = \{v - j_W(k_W(v))\}.U\underline{\underline{h}}$. $\tilde{W}$ is a $U\underline{\underline{h}}$-submodule of RM, and it is easy to check that $W \in \ker k_W$, using the splitting axiom.

Since every element of $\tilde{W}$ may be written in the form $v.h - j_W(k_W(v.h))$ for some $h \in U\underline{\underline{h}}$ and with the element v chosen as above, we may define a map $\xi : \tilde{W} \to W$ as follows:

Let $h \in U\underline{\underline{h}}$. Set $\xi(v.h - j_W(k_W(v.h))) = k_W(v.h)$.

We must check that ξ is well-defined. Since $R\ell - j_W \circ k_W \circ R\ell$ is monic,

$$v.h - j_W(k_W(v.h)) = 0$$

implies that $vh = 0$ which implies that $k_W(vh) = 0$. Thus ξ is well-defined, and obviously an $\underline{\underline{h}}$-homomorphism. Also, since $v \notin \ker k_W$, $\operatorname{im} \xi \neq \{0\}$. Hence, by the simplicity of W, $\operatorname{im} \xi = W$. That is,

$$\tilde{W} \;/\; \ker \xi \simeq W$$

as an $\underline{\underline{h}}$-module. But $\tilde{W} / \ker \xi$ is a subquotient of $\ker k_W$. This contradicts a hypothesis of the proposition. Thus the supposition that there existed a proper nonzero submodule of IW must have been false. □

<u>Remark:</u> This result is, in a sense, an analogue of the Mackey axiom - axiom (6) of chapter 2. For the Mackey isomorphism is used in the theory of induced representations of groups, to prove a rather similar simplicity criterion: see Huppert, [9], page 553 ff.

<u>(7.10) Discussion:</u> Let W be an $\underline{\underline{h}}$-module. Suppose that we have a natural $U\underline{\underline{h}}$-monomorphism

$$\nu_W^{\circ} : W \to R\,\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}, W).$$

Let $\bar{W}$ be another $U\underline{\underline{h}}$-module, let $f \in \mathrm{Hom}_{U\underline{\underline{h}}}(W,\bar{W})$, and let $\phi \in \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$. We can define a functor

$$I : \text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{g}}$$

by

$$IW = (\mathrm{im}\ \nu_W^\circ).U\underline{\underline{g}} \subseteq \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$$

and

$$(If)(\phi) = f \circ \phi \in \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},\bar{W}).$$

It is easy to verify that I is, in fact, a functor.

It is easy to check that there are natural $\underline{\underline{h}}$-homomorphisms

$$j_W : W \to RIW$$

and $\quad k_W : RIW \to W$

given by $\quad j_W(w) = \nu_W^\circ(w) \quad \text{for } w \in W$

and $\quad k_W(\phi) = \phi(1_{U\underline{\underline{g}}}) \quad \text{for } \phi \in RIW.$

Clearly j_W is injective, and $(\mathrm{im}\ j_W).U\underline{\underline{g}} = IW$. So by Theorem (5.8), I and R satisfy the left injectivity axiom. It is not clear, from the assumptions we have made so far, that k_W need be surjective, nor that $\ker k_W$ need contain no nonzero $\underline{\underline{g}}$-modules. Thus we don't know whether I satisfies the right injectivity axiom (cf Theorem (5.16)). We don't know whether j and k satisfy the splitting axiom, either.

However, we can clearly hope to derive some benefit from the study of natural $\underline{\underline{h}}$-monomorphisms

$$W \to \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$$

for $W \in \text{Mod-}\underline{\underline{h}}$.

Let $\bar{U}$ be a left $U\underline{\underline{g}}$-, right $U\underline{\underline{h}}$-module, and let $W \in \text{Mod-}\underline{\underline{h}}$. Then $\mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W)$ may be given the structure of a $\underline{\underline{g}}$-module as follows.

Let $\phi \in \mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W)$, let $\bar{u} \in \bar{U}$, let $x \in U\underline{\underline{g}}$. Define $\phi^x \in \mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W)$ by

$$\phi^x(\bar{u}) = \phi(x.\bar{u}).$$

Now let $\gamma : U\underline{\underline{g}} \to \bar{U}$ be a right $U\underline{\underline{h}}$-homomorphism. γ induces a linear map

$$\mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W) \xrightarrow{\mathrm{Hom}(\gamma,W)} \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$$

and it is of interest to know when $\mathrm{Hom}(\gamma,W)$ is an injective g-homomorphism, and when $\mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W)$ contains a copy of the $\underline{\underline{h}}$-module W in a natural way.

<u>(7.11) Lemma:</u> In the notation used above, $\mathrm{Hom}(\gamma,W)$ is a $\underline{\underline{g}}$-homomorphism if and only if γ is a left $U\underline{\underline{g}}$-homomorphism.

<u>Proof:</u> (1) <u>$\mathrm{Hom}(\gamma,W)$ a $U\underline{\underline{g}}$-homomorphism for all $W \in \text{Mod-}\underline{\underline{h}}$ implies γ a left $U\underline{\underline{g}}$-homomorphism.</u>

Let $W = \bar{U}$. We are supposing that $\mathrm{Hom}(\gamma,\bar{U})$ is a $U\underline{\underline{g}}$-homomorphism. Let $g \in U\underline{\underline{g}}$. Let us consider the action of $\mathrm{Hom}(\gamma,\bar{U})$ on $1_{\bar{U}}$, and on $1_{\bar{U}}^g$. For all $u \in U\underline{\underline{g}}$,

$$\begin{aligned}(\mathrm{Hom}(\gamma,\bar{U})(1_{\bar{U}}^g))(u) &= (1_{\bar{U}}^g)(\gamma(u)) \\ &= 1_{\bar{U}}(g.\gamma(u)) \\ &= g.\gamma(u)\end{aligned}$$

while

$$\begin{aligned}\{\mathrm{Hom}(\gamma,\bar{U})(1_{\bar{U}})\}^g(u) &= 1_{\bar{U}}(\gamma(gu)) \\ &= \gamma(gu).\end{aligned}$$

Since $\mathrm{Hom}(\gamma,\bar{U})$ is a $U\underline{\underline{g}}$-homomorphism, it follows that

$$g.\gamma(u) = \gamma(gu).$$

That is, γ is a left $U\underline{\underline{g}}$-homomorphism.

(2) γ a left $U\underline{\underline{g}}$-homomorphism implies $\mathrm{Hom}(\gamma,W)$ a $\underline{\underline{g}}$-homomorphism for all $W \in \mathrm{Mod}$-$\underline{\underline{h}}$.

Suppose γ is a left $U\underline{\underline{g}}$-homomorphism. Let $\phi \in \mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W)$, $g,u \in U\underline{\underline{g}}$, and $W \in \mathrm{Mod}$-$\underline{\underline{h}}$. Then

$$\begin{aligned}\{\mathrm{Hom}(\gamma,W)(\phi)\}^g(u) &= \{\mathrm{Hom}(\gamma,W)(\phi)\}(gu)\\ &= \phi(\gamma(gu))\\ &= \phi(g.\gamma(u))\\ &= \phi^g(\gamma(u))\\ &= \{\mathrm{Hom}(\gamma,W)(\phi^g)\}(u).\end{aligned}$$

Thus $$\mathrm{Hom}(\gamma,W)(\phi^g) = \{\mathrm{Hom}(\gamma,W)(\phi)\}^g.$$ □

(7.12) Lemma: In the notation explained just before lemma (7.11), $\mathrm{Hom}(\gamma,W)$ is injective for all $W \in \mathrm{Mod}$-$\underline{\underline{h}}$ if and only if $\gamma : U\underline{\underline{g}} \to \bar{U}$ is surjective.

Proof: Suppose γ is surjective, that W is any right $\underline{\underline{h}}$-module, and $\phi,\ \phi' \in \mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W)$ are such that the following diagram commutes:

$$U\underline{\underline{g}} \xrightarrow{\gamma} \bar{U} \underset{\phi'}{\overset{\phi}{\rightrightarrows}} W$$

Then, by the surjectivity of γ, $\phi = \phi'$. That is,

$$\phi \circ \gamma = \phi' \circ \gamma \quad \text{implies} \quad \phi = \phi'.$$

But $\phi \circ \gamma = \mathrm{Hom}(\gamma,W)(\phi)$

and $\phi' \circ \gamma = \mathrm{Hom}(\gamma,W)(\phi')$.

Thus, $\mathrm{Hom}(\gamma,W)(\phi) = \mathrm{Hom}(\gamma.W)(\phi')$ implies $\phi = \phi'$, so $\mathrm{Hom}(\gamma,W)$ is injective.

Conversely, suppose $\mathrm{Hom}(\gamma,W)$ is injective for all $W \in \mathrm{Mod}$-$\underline{\underline{h}}$.

Let $W \in \mathrm{Mod}$-$\underline{\underline{h}}$ and choose $\phi,\ \phi' \in \mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W)$. Then $\mathrm{Hom}(\gamma,W)(\phi) = \mathrm{Hom}(\gamma,W)(\phi')$ implies $\phi = \phi'$. That is, $\phi \circ \gamma = \phi' \circ \gamma$ implies $\phi = \phi'$. That is, γ is surjective. □

(7.13) Lemma: Let $\bar{U}$ be a fixed right $\underline{\underline{h}}$-module, and let W be a right $\underline{\underline{h}}$-module.

The natural $U\underline{\underline{h}}$-monomorphisms from W to $\mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W)$ are in bijective correspondence with the $U\underline{\underline{h}}$-epimorphisms from $\bar{U}$ to $U\underline{\underline{h}}$.

Proof: Let $\chi_W : W \to \mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W)$ be a natural $\underline{\underline{h}}$-monomorphism. There is a natural $\underline{\underline{h}}$-isomorphism

$$e_W : \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{h}},W) \to W$$

given by $e_W(\phi) = \phi(1_{U\underline{\underline{h}}})$ for $\phi \in \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{h}},W)$, so χ_W corresponds to a natural $U\underline{\underline{h}}$-monomorphism

$$\chi_W \circ e_W : \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{h}},W) \to \mathrm{Hom}_{U\underline{\underline{h}}}(\bar{U},W).$$

The result now follows from MacLane [12], p.89, Lemma. $\square$

Discussion: The gist of the last three lemmas is that one way to construct an induction functor from Mod-$\underline{\underline{h}}$ to Mod-$\underline{\underline{g}}$ is to look for a left $U\underline{\underline{g}}$-, right $U\underline{\underline{h}}$-module $\bar{U}$ and a pair of maps

$$\gamma : U\underline{\underline{g}} \to \bar{U},$$
$$\delta : \bar{U} \to U\underline{\underline{h}}$$

where γ is a left $U\underline{\underline{g}}$-, right $U\underline{\underline{h}}$-epimorphism and δ is a right $U\underline{\underline{h}}$-epimorphism.

It should perhaps be mentioned that the case where $\gamma = 1_{U\underline{\underline{g}}}$ and δ is a map constructed using the Poincare-Birkhoff-Witt theorem (of section (1.3)) has already been discussed in sections (3.5) and (3.3). At the other extreme, some progress can be made with the case $\bar{U} = U\underline{\underline{h}}$: see section (8.5).

Returning to the general case, a left $U\underline{\underline{g}}$-, right $U\underline{\underline{h}}$-epimorphism $\gamma : U\underline{\underline{g}} \to \bar{U}$ has for kernel a left $U\underline{\underline{g}}$-, right $U\underline{\underline{h}}$-submodule A of $U\underline{\underline{g}}$, by the homomorphism theorems: that is $\bar{U} = U\underline{\underline{g}}/A$.

Before embarking on a detailed study of these ideals, we shall conclude this discussion with a result which suggests yet another way of constructing induced module functors. An analogous result is well-known in the theory of group representations.

<u>(7.14) Theorem:</u> (cf Mitchell, p. 143, Theorem 3.1). Let $W, \bar{W} \in$ Mod-$\underline{\underline{h}}$. Then there is a natural isomorphism

$$\mathrm{Hom}_{U\underline{\underline{h}}}(\bar{W}, R(\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}, W))) \simeq \mathrm{Hom}_{U\underline{\underline{h}}}(R(\bar{W} \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}), W).$$

<u>Proof</u>: $\mathrm{Hom}_{U\underline{\underline{h}}}(\bar{W}, R\ \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}, W)) \simeq \mathrm{Hom}_{U\underline{\underline{g}}}(\bar{W} \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}, \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}, W))$

by theorem (3.6)

$\simeq \mathrm{Hom}_{U\underline{\underline{h}}}(R(\bar{W} \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}), W)$

by theorem (3.5).

<u>Discussion:</u> We are actually interested in the case $W = \bar{W}$. The result tells us that looking for $\underline{\underline{h}}$-homomorphisms

$$W \to R(\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}, W)) \qquad (W \in \text{Mod-}\underline{\underline{h}})$$

is equivalent to looking for $\underline{\underline{h}}$-homomorphisms

$$R(W \otimes_{U\underline{\underline{h}}} U\underline{\underline{g}}) \to W.$$

We now return to the study of left $U\underline{\underline{g}}$-, right $U\underline{\underline{h}}$-submodules.

<u>(7.15) Definition:</u> Let $\underline{\underline{h}} \le \underline{\underline{g}}$ be Lie algebras. The symbol <u>(Mod-$\underline{\underline{h}}$|monics)</u> will denote the category of all right $\underline{\underline{h}}$-modules and all $\underline{\underline{h}}$-monomorphisms between them. <u>(Mod-$\underline{\underline{g}}$|monics)</u> is similarly defined. The symbol <u>Sub($U\underline{\underline{g}}$,$U\underline{\underline{h}}$)</u> will denote the (lattice) category of all left $U\underline{\underline{g}}$-, right $U\underline{\underline{h}}$-submodules of $U\underline{\underline{g}}$, and all submodule inclusions between them.

Convention: For the rest of this chapter, $\underline{\underline{h}} \leq \underline{\underline{g}}$ will be Lie algebras. We shall set $\underline{\underline{H}}$ = (Mod-$\underline{\underline{h}}$|monics), $\underline{\underline{G}}$ = (Mod-$\underline{\underline{g}}$|monics), R : (Mod-$\underline{\underline{g}}$|monics) $\to$ (Mod-$\underline{\underline{h}}$|monics) the obvious restriction functor, and I will be a functor from (Mod-$\underline{\underline{h}}$|monics) to (Mod-$\underline{\underline{g}}$|monics) except where otherwise noted.

Discussion: The original aim of this thesis was to find finite-dimensional induced modules. With this in mind, suppose that $A \in \mathrm{Sub}(U\underline{\underline{g}}, U\underline{\underline{h}})$ has the property that $U\underline{\underline{g}}/A$ is of finite rank as a $U\underline{\underline{h}}$-module.

That is, there is an epimorphism $M \to U\underline{\underline{g}}/A$ of $\underline{\underline{h}}$-modules, where M is a free $U\underline{\underline{h}}$-module of finite rank. Let $W \in$ Mod-$\underline{\underline{h}}$. There is an induced $\underline{\underline{h}}$-monomorphism $\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/A, W) \to \mathrm{Hom}_{U\underline{\underline{h}}}(M, W)$ like that used in the proof of lemma (7.12).

We can deduce that
$$\begin{aligned}\dim \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/A, W) &\leq \dim \mathrm{Hom}_{U\underline{\underline{h}}}(M, W)\\ &\leq \dim W \times \mathrm{rank}\ M.\end{aligned}$$

Thus, if $\dim W < \infty$, then $\dim \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/A, W) < \infty$. If we could find a suitable functor

$$I : \text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{g}}$$

such that $IW \subseteq \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/A, W)$ whenever $\dim W < \infty$, then we would have achieved the original aim of this thesis.

However, the two examples in section (0.3) of this thesis show that such a functor I cannot be found.

We therefore modify our aim a little.

Aim: We shall seek a contravariant functor A : (Mod-$\underline{\underline{h}}$|monics) $\to$ $\mathrm{Sub}(U\underline{\underline{g}}, U\underline{\underline{h}})$, and a functor I : (Mod-$\underline{\underline{h}}$|monics) $\to$ (Mod-$\underline{\underline{g}}$|monics) such that for every $W \in$ (Mod-$\underline{\underline{h}}$|monics)

$$IW \subseteq \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/AW, W),$$

such that I and R satisfy the left injectivity axiom.

Now, the argument above leads to the conclusion that $\dim IW < \infty$ provided that $U\mathfrak{g}/AW$ is of finite rank as $U\mathfrak{h}$-module and $\dim W < \infty$.

<u>Remark</u>: It seems to be impossible to demand that A and I be defined on domains larger than (Mod-$\mathfrak{h}$|monics) and still prove the main result (7.21) below. The reason for this is embodied in the proof of the next proposition, and in lemma (7.18).

<u>(7.16) Proposition</u>: Let A : (Mod-$\mathfrak{h}$|monics) $\to$ Sub($U\mathfrak{g}$,$U\mathfrak{h}$) be a contravariant functor, and let $W,\bar{W} \in$ (Mod-$\mathfrak{h}$ monics). Let $\gamma_W : U\mathfrak{g} \to U\mathfrak{g}/AW$ denote the obvious projection map. Let $f : W \to \bar{W}$ be an $\mathfrak{h}$-monomorphism. Then it is possible to define a map

$$f_* : \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}/AW,W) \to \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}/A\bar{W},\bar{W})$$

so that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}/AW,W) \arrow[r, "\mathrm{Hom}(\gamma_W, W)"] \arrow[d, "f_*"'] & \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g},W) \arrow[d, "\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g},f)"] \\
\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}(AW,W) \arrow[r, "\mathrm{Hom}(\gamma_{\bar{W}}, \bar{W})"'] & \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g},\bar{W})
\end{tikzcd}$$

In fact, f_* can be defined so that it is also a $U\mathfrak{g}$-homomorphism.

<u>Proof</u>: Let $u,u' \in U\mathfrak{g}$, $\phi \in \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}/AW,W)$.

Define f_* by

$$(f_*(\phi))(u + A\bar{W}) = f(\phi(u + AW)).$$

There are several items to check.

<u>#1. Is f_* a well-defined mapping?</u>

Suppose $u + A\bar{W} = u' + A\bar{W}$.

Then $u - u' \in A\bar{W}$, and since there is a $U\mathfrak{h}$-monomorphism $f : W \to \bar{W}$, $Af : A\bar{W} \subseteq AW$ is an inclusion, by our hypothesis about A, so

$$u - u' \in AW .$$

Therefore $$u + AW = u' + AW,$$

so $$f(\phi(u + AW)) = (f(\phi(u' + AW)).$$

That is, $$(f_*(\phi))(u + A\bar{W}) = (f_*(\phi))(u' + A\bar{W})$$

and so f_* is a well-defined mapping.

#2. Is $f_*(\phi) \in \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/A\bar{W},\bar{W})$?

That is, is $f_*(\phi)$ a $U\underline{\underline{h}}$-homomorphism? Let $h \in U\underline{\underline{h}}$. Then

$$\begin{aligned}(f_*(\phi))(uh + A\bar{W}) &= f(\phi(uh + AW)) \\ &= f(\phi(u + AW)).h \\ &= ((f_*(\phi))(u + A\bar{W})).h \,.\end{aligned}$$

Thus $f_*(\phi)$ is an $\underline{\underline{h}}$-homomorphism.

#3. Is f_* a right $\underline{\underline{g}}$-module homomorphism?

Let $x \in U\underline{\underline{g}}$.

$$\begin{aligned}(f_*(\phi^x))(u + A\bar{W}) &= f(\phi^x(u + AW)) \\ &= f(\phi(xu + AW)) \\ &= (f_*(\phi))(xu + A\bar{W}) \\ &= (f_*(\phi))^x(u + A\bar{W})\end{aligned}$$

So f_* is a right $\underline{\underline{g}}$-module homomorphism.

#4. Does the diagram in the statement of the Proposition commute, with f_* defined as above?

$$\begin{aligned}&\mathrm{Hom}(U\underline{\underline{g}},f)((\mathrm{Hom}(\gamma_W,W)(\phi))(u)) \\ &\qquad = (f \circ \phi \circ \gamma_W)(u) \\ &\qquad = f(\phi(u + AW))\end{aligned}$$

while
$$\begin{aligned}&\mathrm{Hom}(\gamma_{\bar{W}},\bar{W})((f_*(\phi))(u)) \\ &\qquad = (f_*(\phi) \circ \gamma_{\bar{W}})(u) \\ &\qquad = f_*(\phi)(u + A\bar{W}) \\ &\qquad = f(\phi(u + AW)).\end{aligned}$$

Thus the diagram commutes. $\square$

Let W be a right $U\mathfrak{h}$-module, and let A be a contravariant functor from (Mod-$\mathfrak{h}$|monics) to Sub($U\mathfrak{g}$,$U\mathfrak{h}$). The next proposition answers the following question: What restriction does the condition that IW be embedded in $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}/AW,W)$ place on the submodule AW of $U\mathfrak{g}$?

<u>(7.17) Proposition:</u> Let A and γ_W be as in (7.16). Let V be any $\mathfrak{g}$-submodule of $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g},W)$. Then $V \subseteq \mathrm{im}\,\mathrm{Hom}(\gamma_W,W)$ if and only if

$$AW \subseteq \{u \in U\mathfrak{g} : \text{for all } v \in V,\ v(u) = 0\}.$$

<u>Proof:</u> It is easy to check that

$$\mathrm{Hom}(\gamma_W,W)(\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}/AW,W))$$

is the set of maps $\phi \in \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g},W)$ which factor through $U\mathfrak{g}/AW$ via γ_W. That is, $\phi \in \mathrm{im}\,\mathrm{Hom}(\gamma_W,W)$ if and only if there exists $\chi \in \mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}/AW,W)$ such that the following diagram commutes:

$$\begin{tikzcd}
U\mathfrak{g} \arrow[rr, "\phi"] \arrow[dr, "\gamma_W"'] & & W \\
& U\mathfrak{g}/AW \arrow[ur, "\chi"'] &
\end{tikzcd}$$

This condition holds if and only if the map

$$\chi : U\mathfrak{g}/AW \to W$$

"defined" by (for $u \in U\mathfrak{g}$)

$$\chi(u + AW) = \phi(u)$$

is well-defined and in fact χ is well-defined if and only if

$$u \in AW \quad \text{implies} \quad \phi(u) = 0.$$

Thus $V \subseteq \mathrm{Hom}(\gamma_W,W)(\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}/AW,W))$ if and only if

$$AW \subseteq \{u \in U\mathfrak{g} : \text{for all } v \in V,\ v(u) = 0\}. \quad \square$$

(7.17a) Corollary: Let $W \in (\text{Mod-}\underline{\underline{h}}|\text{monics})$. If I is an induction functor $(\text{Mod-}\underline{\underline{h}}|\text{monics}) \to (\text{Mod-}\underline{\underline{g}}|\text{monics})$ arising from natural transformations $j : 1 \dot{\to} RI$, $k : RI \dot{\to} 1$ as outlined in proposition (7.6), and if there is a contravariant functor $A : (\text{Mod-}\underline{\underline{h}}|\text{monics}) \to \text{Sub}(U\underline{\underline{g}}, U\underline{\underline{h}})$ such that

$$IW \subseteq \text{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/AW, W)$$

then $AW \subseteq \{u \in U\underline{\underline{g}} : \text{for all } w \in W,\ x \in U\underline{\underline{g}},\ k_W(j_W(w).xu) = 0\}$.

Proof: This follows from Proposition (7.17) above, together with Proposition (7.6). □

(7.18) Construction of the functor B: Suppose that there exist natural transformations $j : 1_{\underline{\underline{H}}} \dot{\to} RI$, $k : RI \dot{\to} 1_{\underline{\underline{H}}}$, and let W, $\bar{W}$ be right $\underline{\underline{h}}$-modules.

Define

$$\underline{BW} = \{u \in U\underline{\underline{g}} : \text{for all } w \in W,\ x \in U\underline{\underline{g}},\ k_W(j_W(w).xu) = 0\}.$$

Then BW is a left $U\underline{\underline{g}}$-, right $U\underline{\underline{h}}$-submodule of $U\underline{\underline{g}}$: this is easy to verify.

Suppose that there exists an $\underline{\underline{h}}$-monomorphism $f : W \to \bar{W}$. We shall show that in this case, $B\bar{W} \subseteq BW$, and we shall denote the inclusion map by the symbol Bf. It is here that we use the naturality of j and k.

Suppose $u \in B\bar{W}$. Then for all $x \in U\underline{\underline{g}}$ and all $w \in W$, $f(w) \in \bar{W}$, so

$$\begin{aligned} 0 &= k_{\bar{W}}(j_{\bar{W}}(f(w)).xu) \\ &= k_{\bar{W}}(RIf(j_W(w)).xu) && \text{by naturality of } j \\ &= k_{\bar{W}}(RIf(j_W(w).xu)) && \text{since RIf is an } U\underline{\underline{g}}\text{-homomorphism} \\ &= f(k_W(j_W(w).xu)) && \text{by naturality of } k. \end{aligned}$$

But f is monic, hence, for all $x \in U\underline{\underline{g}}$ and all $w \in W$,

$$0 = k_W(j_W(w).xu).$$

That is $u \in BW.$

So $B\bar{W} \subseteq BW.$ $\square$

Proposition: B is a contravariant functor (Mod-$\underline{\underline{h}}$|monics) $\rightarrow$ Sub($U\underline{\underline{g}}$,$U\underline{\underline{h}}$).

(7.20) Proposition: Let I and R satisfy left and right injectivity axioms. Let W be a right $\underline{\underline{h}}$-module. If dim $IW < \infty$, then dim $\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/BW, W) < \infty$.

Proof: Suppose dim $IW < \infty$.

1) $\mathrm{Ann}_{U\underline{\underline{g}}}(IW) \subseteq BW.$

For, let $u \in \mathrm{Ann}_{U\underline{\underline{g}}}(IW)$. Then for all $w \in W$, for all $x \in U\underline{\underline{g}}$, $j_W(w).xu = 0$ (since $j_W(w).x \in IW$). Hence, a fortiori, for all $w \in W$ and $x \in U\underline{\underline{g}}$

$$k_W(j_W(w).xu) = 0.$$

Thus $u \in BW$.

2) IW is a faithful $U\underline{\underline{g}}/\mathrm{Ann}_{U\underline{\underline{g}}}(IW)$-module. Thus $U\underline{\underline{g}}/\mathrm{Ann}_{U\underline{\underline{g}}}(IW)$ may be embedded, as a k-algebra, in the finite dimensional k-algebra $\mathrm{End}_k(IW)$. So $U\underline{\underline{g}}/\mathrm{Ann}_{U\underline{\underline{g}}}(IW)$ is of finite dimension over k.

3) Since $\mathrm{Ann}_{U\underline{\underline{g}}}(IW) \subseteq BW$, it follows from (2) that $U\underline{\underline{g}}/BW$ is also of finite dimension over k.

4) Since W is embedded in RIW (by j_W), W is finite-dimensional. Thus $\mathrm{Hom}_k(U\underline{\underline{g}}/BW, W)$ is finite-dimensional. Hence, a fortiori, $\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/BW, W)$ is finite-dimensional. $\square$

(7.21) Construction and Theorem: Let A : (Mod-$\underline{\underline{h}}$ monics) $\rightarrow$ Sub($U\underline{\underline{g}}$,$U\underline{\underline{h}}$) be a contravariant functor. Let W, $\bar{W}$ be right $\underline{\underline{h}}$-modules, and let $f : W \rightarrow \bar{W}$ be an $\underline{\underline{h}}$-monomorphism. Let $\tau_W : W \rightarrow R\,\mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/AW, W)$ be a $U\underline{\underline{h}}$-monomorphism such that, if f_* is defined as in Proposition (7.16), then the following diagram commutes:

$$\begin{tikzcd} W \arrow[r, "\tau_W"] \arrow[d, "f"'] & \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/AW, W) \arrow[d, "f_*"] \\ \bar{W} \arrow[r, "\tau_{\bar{W}}"'] & \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}}/A\bar{W}, \bar{W}) \end{tikzcd}$$

Suppose that $W = \sum_{w \in W} \mathrm{im}'\{\tau_W(w)\}$.

Define I : (Mod-$\underline{\underline{h}}$ monics)$\to$ (Mod-$\underline{\underline{g}}$ monics) by setting $IW = (\mathrm{im}\ \tau_W)U\underline{\underline{g}}$ and defining, for $w \in W$ and $x \in U\underline{\underline{g}}$,

$$(If)(\tau_W(w).x) = \tau_{\bar{W}}(f(w)).x$$

and extending this definition of If to all of IW by linearity.

Then I is a functor and I and R satisfy the left injectivity axiom. Furthermore, IW is finite-dimensional whenever AW is of finite codimension in $U\underline{\underline{g}}$ and W is finite-dimensional.

Proof: (1) Functoriality of I. First we shall check that If is well-defined. Suppose $x_1,\ldots,x_n \in U\underline{\underline{g}}$ and $w_1,\ldots,w_n \in W$, and that

$$\sum_{i=1}^{n} \tau_W(w_i).x_i = 0.$$

Then, for all $u \in U\underline{\underline{g}}$,

$$\begin{aligned} &((If)(\sum_{i=1}^{n} \tau_W(w_i).x_i))(u + A\bar{W}) \\ &= \sum_{i=1}^{n} (\tau_{\bar{W}}(f(w_i)).x_i)(u + A\bar{W}) && \text{by definition of If,} \\ &= \sum_{i=1}^{n} (f_*(\tau_W(w_i)).x_i)(u + A\bar{W}) && \text{by commutativity,} \\ &= \sum_{i=1}^{n} f((\tau_W(w_i).x_i)(u + AW)) && \text{by definition of } f_* \text{ (see Propn. (7.16))} \\ &= f((\sum_{i=1}^{n} \tau_W(w_i).x_i)(u + AW)) \\ &= f(0) = 0. \end{aligned}$$

It is also easy to verify that $(If)(\tau_W(w).x)$ is always a $U\underline{\underline{h}}$-homomorphism, for $w \in W$ and $x \in U\underline{\underline{g}}$. Further, If is a $U\underline{\underline{g}}$-homomorphism, since, if $w \in W$ and $x,y,u \in U\underline{\underline{g}}$, then

$$((If)(\tau_W(w).x))^y(u + A\bar{W})$$

$$= ((If)(\tau_W(w).x))(yu + A\bar{W})$$

$$= (\tau_{\bar{W}}(f(w)).x)(yu + A\bar{W})$$

$$= (\tau_{\bar{W}}(f(w))).(xyu + A\bar{W})$$

while $$((If)(\tau_W(w).xy))(u + A\bar{W})$$

$$= (\tau_{\bar{W}}(f(w)).xy)(u + A\bar{W})$$

$$= \tau_{\bar{W}}(f(w))(xyu + A\bar{W}).$$

Finally, If is injective, since, if $w_1,\ldots,w_n \in W$ and $x_1,\ldots,x_n \in U\underline{\underline{g}}$,

$$(If)(\sum_{i=1}^{n} \tau_W(w_i).x_i) = 0,$$

then for all $u \in U\underline{\underline{g}}$,

$$0 = \sum_{i=1}^{n} (\tau_{\bar{W}}(f(w_i)).x_i)(u + A\bar{W}) \qquad \text{by definition of If}$$

$$= \sum_{i=1}^{n} (f_*(\tau_W(w_i))(x_i u + A\bar{W}) \qquad \text{by commutativity hypothesis}$$

$$= \sum_{i=1}^{n} f(\tau_W(w_i)(x_i u + AW)) \qquad \text{by definition of } f_*$$

$$= f(\sum_{i=1}^{n} (\tau_W(w_i).x_i)(u + AW)).$$

Since f is injective, this forces

$$0 = \sum_{i=1}^{n} (\tau_W(w_i).x_i)(u + AW) \qquad \text{for all } u \in U\underline{\underline{g}}.$$

That is $$\sum_{i=1}^{n} \tau_W(w_i).x_i = 0.$$

Thus I is a well-defined functor from (Mod-$\underline{\underline{h}}$|monics) to (Mod-$\underline{\underline{g}}$|monics).

(2) <u>Left injectivity axiom.</u> We shall show that I and R satisfy the left injectivity axiom. By theorem (5.8), it is sufficient to show that there exists a morphism j_W of (Mod-$\underline{\underline{h}}$|monics) such that $j_W : W \to RIW$ is natural in W and $(\text{im } j_W).U\underline{\underline{g}} = IW$.

We define j_W to be τ_W with codomain restricted to RIW. That j_W is a morphism of (Mod-$\underline{\underline{h}}$|monics) and $(\text{im } j_W)U\underline{\underline{g}} = IW$ are trivial. It remains to prove that j_W is natural in W.

Recall that W, $\bar{W}$ are right $U\underline{\underline{h}}$-modules and f is a right $U\underline{\underline{h}}$-monomorphism. Thus, for $w \in W$,

$$\begin{tikzcd} W \arrow[r, "j_W"] \arrow[d, "f"'] & RIW \arrow[d, "RIf"] \\ \bar{W} \arrow[r, "j_{\bar{W}}"'] & RI\bar{W} \end{tikzcd}$$

$$\begin{aligned}
(RIf \circ j_W)(w) &= (RIf)(j_W(w)) \\
&= (RIf)(\tau_W(w)) && \text{by definition of } j_W \\
&= (If)(\tau_W(w)) \\
&= \tau_{\bar{W}}(f(w) && \text{by definition of If} \\
&= (j_W \circ f)(w) && \text{by definition of } j_W.
\end{aligned}$$

So j_W is natural in W.

(3) <u>Dimension of IW.</u> The proof that, if W is finite-dimensional and $U\underline{\underline{g}}/AW$ is finite-dimension, then IW is finite-dimensional, is trivial.

□

Chapter 8 - Models of the Axiom Systems

(8.1) Convention: In this chapter $\underline{\underline{h}} \leq \underline{\underline{g}}$ will be Lie algebras of various special types, and $R : \text{Mod-}\underline{\underline{g}} \to \text{Mod-}\underline{\underline{h}}$ will denote the obvious restriction functor.

(8.2) Induction from Cartan-type subalgebras I - Wallach's Functor

In his papers [16], [17], Wallach constructs an induction functor for a certain type of subalgebra $\underline{\underline{h}}$ of $\underline{\underline{g}}$, which in fact satisfies the left and right injectivity axioms. Wallach proves that his functor satisfies the injectivity claim of the right injectivity axiom. He ignores all questions of naturality.

We shall describe Wallach's functor and show that it satisfies both left and right injectivity axioms by verifying the conditions of theorems (5.8) and (5.16) of this thesis.

Recall that $\underline{\underline{h}} \leq \underline{\underline{g}}$.

Let $\underline{\underline{g}}$ have subalgebras $\underline{\underline{n}}_1$, $\underline{\underline{n}}_2$ such that $\underline{\underline{g}} = n_1 \oplus \underline{\underline{h}} \oplus n_2$ as a vector space, and such that $[\underline{\underline{n}}_1,\underline{\underline{h}}] \subseteq \underline{\underline{n}}_1$,

$$[\underline{\underline{n}}_2,\underline{\underline{h}}] \subseteq \underline{\underline{n}}_2.$$

Wallach calls such a Lie algebra a "Lie algebra with decomposition".

We shall write $\underline{\underline{t}} = \underline{\underline{n}}_1 \oplus \underline{\underline{h}}$.

(i) Definition of the functor I. Let $W \in \text{Mod-}\underline{\underline{h}}$. Wallach constructs his module[1] IW as follows. Define a functor[2]

$$\sim \; : \text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{t}}$$
$$W \mapsto \tilde{W}$$

1. Wallach uses the symbol $\hat{W}$ for what we have called $\tilde{W}$; Wallach uses the symbol ~ for another purpose.

2. Wallach calls his induced module W*, not IW.

by requiring that the underlying vector space of $\tilde{W}$ be the same as that of W, that $\underline{\underline{h}}$ acts on $\tilde{W}$ as it does on W, and that $\underline{\underline{n}}_1$ acts <u>trivially</u> on $\tilde{W}$. This functor acts as an identity map on morphisms, and we shall not distinguish between a morphism and its image under the functor.

Recall from section (1.3)(second corollary) that

$$U\underline{\underline{g}} = U\underline{\underline{t}} \oplus U\underline{\underline{n}}_2 . \underline{\underline{n}}_2 . U\underline{\underline{t}}$$

as a right $U\underline{\underline{t}}$-module. Since $[\underline{\underline{h}},\underline{\underline{n}}_2] \subseteq \underline{\underline{n}}_2$, it is easy to check that the summands are also left $U\underline{\underline{h}}$-modules. Hence, the projection map $\gamma : U\underline{\underline{g}} \to U\underline{\underline{t}}$ onto the first summand (above) is a left $U\underline{\underline{h}}$-, right $U\underline{\underline{t}}$-module homomorphism.

Thus we can define a map[3]

$$\hat{j}_W : W \to R(\mathrm{Hom}_{U\underline{\underline{t}}}(U\underline{\underline{g}},\tilde{W}))$$

by

$$\hat{j}_W(w)(u) = w.\gamma(u)$$

where $w \in W$ and $u \in U\underline{\underline{g}}$, and it is easy to verify that $\hat{j}_W$ is an $\underline{\underline{h}}$-monomorphism.

Next, Wallach sets

$$IW = (\mathrm{im}\ \hat{j}_W).U\underline{\underline{g}}.$$

Let $\bar{W} \in \text{Mod-}\underline{\underline{h}}$, and choose $\phi \in \mathrm{Hom}_{\underline{\underline{h}}}(W,\bar{W})$. Wallach defines a map $I\phi : IW \to I\bar{W}$ by

$$(I\phi)(f) = \phi \circ f$$

for $f \in IW \subseteq \mathrm{Hom}_{U\underline{\underline{t}}}(U\underline{\underline{g}},W)$. It is easy to verify that, with this definition, I is a functor $\text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{g}}$.

---

3. Wallach calls his $\underline{\underline{h}}$-monomorphism ω, not $\hat{j}_W$.

(ii) Proof that I is faithful. Suppose that $\phi \in \mathrm{Hom}_{\underline{\underline{h}}}(W,\bar{W})$, and $I\phi = 0$; then, in particular, for all $w \in W$, $(I\phi)(\hat{j}_W(w)) = 0$. That is, for all $w \in W$, $\phi \circ \hat{j}_W(w) = 0$, and so for all $w \in W$, $\phi(w) = (\phi \circ \hat{j}_W(w))(1_{U\underline{\underline{g}}}) = 0$. That is, $\phi = 0$.

(iii) Definition of the natural transformation $j : 1_{\mathrm{Mod}\text{-}\underline{\underline{h}}} \xrightarrow{\cdot} RI$. Let $W \in \mathrm{Mod}\text{-}\underline{\underline{h}}$. We define the map $j_W : W \to RIW$ to be $\hat{j}_W$ with codomain restricted to be RIW. I claim that j_W is natural in W. For, suppose that $\phi \in \mathrm{Hom}_{\underline{\underline{h}}}(W,\bar{W})$. Then, for $w \in W$ and $u \in U\underline{\underline{g}}$,

$$\begin{tikzcd} W \arrow[r, "j_W"] \arrow[d, "\phi"'] & RIW \arrow[d, "RI\phi"] \\ \bar{W} \arrow[r, "j_{\bar{W}}"'] & RI\bar{W} \end{tikzcd}$$

$$\begin{aligned} &RI\phi(j_W(w))(u) \\ &= (\phi \circ j_W(w))(u) \\ &= \phi(w.\gamma(u)) \\ &= \phi(w).\gamma(u) \\ &= (j_{\bar{W}}(\phi(w)))(u). \end{aligned}$$

Thus
$$RI\phi \circ j_W = j_{\bar{W}} \circ \phi.$$

(iv) I and R satisfy the left injectivity axiom. We use theorem (5.8) and the Remark following it. We have verified condition (a) of theorem (5.8) above, and condition (b)' (see the Remark) is a trivial consequence of the definition of j_W.

(v) Definition of the natural transformation $k : RI \xrightarrow{\cdot} 1_{\mathrm{Mod}\text{-}\underline{\underline{h}}}$. Let $f \in RIW$. Thus $f \in \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$, since $RIW \subseteq \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$. We define a map

$$k_W : RIW \to W$$

by

$$k_W(f) = f(1_{U\underline{\underline{g}}}).$$

It is easy to check that k_W is a $U\underline{\underline{h}}$-homomorphism. Note that for $w \in W$,

$$\begin{aligned} k_W(j_W(w)) &= (j_W(w))(1_{U\underline{\underline{g}}}) \\ &= w.\gamma(1_{U\underline{\underline{g}}}) \\ &= w.1 \\ &= w\,. \end{aligned}$$

Thus $$k_W \circ j_W = 1_W.$$

We deduce from this that k_W is an epimorphism of $U\underline{\underline{h}}$-modules. As a side-product:

(vi) I and R satisfy the splitting axiom of section (7.2)

(vii) k_W is natural in W, and $\ker k_W$ contains no nonzero $U\underline{\underline{g}}$-modules.

Suppose $\bar{W}$ is a $U\underline{\underline{h}}$-module and $\phi \in \mathrm{Hom}_{U\underline{\underline{h}}}(W,\bar{W})$. Let $f \in RIW$. Then

$$\begin{tikzcd} W & RIW \arrow[l, "k_W"'] \arrow[d, "RI\phi"] \\ \bar{W} \arrow[from=u, "\phi"'] & RI\bar{W} \arrow[l, "k_{\bar{W}}"] \end{tikzcd}$$

$$\begin{aligned} &(\phi \circ k_W)(f) \\ &\quad = \phi(f(1_{U\underline{\underline{g}}})) \end{aligned}$$

while $k_{\bar{W}}(RI\phi(f))$

$$\begin{aligned} &= k_{\bar{W}}(\phi \circ f) \\ &= (\phi \circ f)(1_{U\underline{\underline{g}}}) \\ &= \phi(f(1_{U\underline{\underline{g}}})). \end{aligned}$$

Thus k_W is natural in W.

Next we show that $\ker k_W$ contains no nonzero $U\underline{\underline{g}}$-modules. Suppose that $\ker k_W$ does contain a nonzero $\underline{\underline{g}}$-module: then $\ker k_W$ contains a cyclic $\underline{\underline{g}}$-module, $m.U\underline{\underline{g}}$, say, where $m \in \mathrm{Hom}_{U\underline{\underline{h}}}(U\underline{\underline{g}},W)$ and $m \neq 0$.

We shall show that $m.U\underline{\underline{g}} \subseteq \ker k_W$ implies $m = 0$, and this contradiction will establish what we want to prove.

If $m.U\underline{\underline{g}} \subseteq \ker k_W$, then for all $u \in U\underline{\underline{g}}$, $k_W(m^u) = 0$. That is, for all $u \in U\underline{\underline{g}}$, $m^u(1_{U\underline{\underline{g}}}) = 0$. That is, for all $u \in U\underline{\underline{g}}$, $m(u) = 0$. That is, $m = 0$.

(viii) <u>I and R satisfy the right injectivity axiom.</u> This follows from theorem (5.16), the remark following it, and (vii) above.

Wallach proves two results about his functor I which are interesting for us:

(ix) (Wallach [16] Theorem 3.1): Let $\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_2$ be a <u>triangular</u> decomposition of the Lie algebra $\mathfrak{g}$: - that is, a decomposition of the type explained at the start of section (8.2) of this thesis, but with the additional property that $\mathfrak{n}_1$ and $\mathfrak{n}_2$ must both act nilpotently on every finite-dimensional $\mathfrak{g}$-module.

If W is a simple $\mathfrak{h}$-module, then $\mathrm{Hom}_{U\mathfrak{t}}(U\mathfrak{g},\tilde{W})$ (see Section (i) above) contains at most one non-zero finite-dimensional simple $\mathfrak{g}$-module. Such a non-zero finite-dimensional simple $\mathfrak{g}$-module exists if and only if $\dim_k IW < \infty$, in which case IW <u>is</u> the simple $\mathfrak{g}$-module.

(x) (Wallach [17], Proposition (4.1)). Consider the case where $\mathfrak{g}$ is a semisimple Lie algebra over an algebraically closed field k of characteristic zero. Such a Lie algebra has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_2$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{n}_1$, $\mathfrak{n}_2$ are respectively the sums of weight spaces for the negative and for the positive roots in the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. (More details and references are given in Wallach [17] page 164.)

If W is a simple finite-dimensional (i.e. one-dimensional) $U\mathfrak{h}$-module, then IW is a simple (not necessarily finite-dimensional) $\mathfrak{g}$-module.

(8.3) Induction from Cartan-type subalgebras II - An adjunction related to Wallach's Functor.

Convention: In this section, I will be the functor defined in section (8.2), and $\underline{\underline{g}}$ will thus be a "Lie algebra with decomposition", as in (8.2). That is, $\underline{\underline{g}}$ has subalgebras $\underline{\underline{h}}$, $\underline{\underline{n}}_1$, $\underline{\underline{n}}_2$ such that

$$\underline{\underline{g}} = \underline{\underline{n}}_1 \oplus \underline{\underline{h}} \oplus \underline{\underline{n}}_2$$

as a vector space, and $[\underline{\underline{n}}_1,\underline{\underline{h}}] \subseteq \underline{\underline{n}}_1$ and $[\underline{\underline{n}}_2,\underline{\underline{h}}] \subseteq \underline{\underline{n}}_2$. We shall write $\underline{\underline{t}} = \underline{\underline{n}}_1 \oplus \underline{\underline{h}}$.

Introduction: Since the functor I is not (necessarily) a left adjoint to R, we can ask if I has a right adjoint (or a left adjoint, for that matter).

Wallach [17], in his Lemma 2.1 and Theorem 3.1, proved some results in this direction. In this section, we shall extend his work by defining a functor J, closely related to I, and describing a functor which is a two-sided adjoint to J.

(i) Definition of the functor $\tilde{C}$: Mod-$\underline{\underline{g}}$ $\to$ Mod-$\underline{\underline{h}}$. Let v $\in$ Mod-$\underline{\underline{g}}$. Define, first of all, the set

$$\tilde{C}V = \{ v \in V : v.\underline{\underline{n}}_2 = (0)\}.$$

Let $\bar{V} \in$ Mod-$\underline{\underline{g}}$ and suppose $f \in \mathrm{Hom}_{U\underline{\underline{g}}}(V,\bar{V})$. Then $f|_{\tilde{C}V} : \tilde{C}V \to \bar{V}$ is an $\underline{\underline{h}}$-homomorphism with image contained in $\tilde{C}\bar{V}$, since

. if $h \in U\underline{\underline{h}}$ and $v \in \tilde{C}V$, then for all $n \in \underline{\underline{n}}_2$,

$$\begin{aligned}(vh)n &= (vn)h + v.[h,n] \\ &= 0 \quad \text{since } vn = 0 \text{ and } [h,n] \in \underline{\underline{n}} \\ &\qquad \text{so } v.[h,n] = 0\end{aligned}$$

thus $vh \in \tilde{C}V$

and . if $v \in \tilde{C}V$ and $n \in \underline{\underline{n}}_2$,

$$f(v).n = f(v.n) = f(0) = 0$$

so $f(v) \in \tilde{C}\bar{V}$, i.e. $\operatorname{im} f\big|_{\tilde{C}V} \subseteq \tilde{C}\bar{V}$.

Thus, if we define $\tilde{C}f$ to be f with domain restricted to $\tilde{C}V$ and codomain restricted to $\tilde{C}\bar{V}$, then it is easy to check that $\tilde{C}$ is a functor from Mod-$\underline{\underline{g}}$ to Mod-$\underline{\underline{h}}$.

(ii) <u>Definition of the category $\underline{\underline{G}}$.</u> Let $\underline{\underline{G}}$ be the full subcategory of Mod-$\underline{\underline{g}}$ whose objects are all $V \in$ Mod-$\underline{\underline{g}}$ such that

(1) $(\tilde{C}V).U\underline{\underline{g}} = V$

(2) $V.\underline{\underline{n}}_1 \cap \tilde{C}V = (0)$

(3) $V.\underline{\underline{n}}_1$ contains no nonzero $\underline{\underline{g}}$-modules

where $V.\underline{\underline{n}}_1$ denotes the subspace of V (considered as a vector space) spanned by all elements of V of the form v.n where $v \in V$ and $n \in \underline{\underline{n}}_1$. It is easy to check that $V.\underline{\underline{n}}_1$ is an $\underline{\underline{h}}$-submodule of RV.

(iii) <u>Definition of the functor $C : \underline{\underline{G}} \to$ Mod-$\underline{\underline{h}}$.</u> We define C to be the restriction of the functor $\tilde{C}$ of part (i) to the category $\underline{\underline{G}}$.

(iv) <u>Definition of the functor $J :$ Mod-$\underline{\underline{h}} \to \underline{\underline{G}}$.</u> Wallach, in [17], lemma 2.1, proves that if $W \in$ Mod-$\underline{\underline{h}}$, then $IW \in \underline{\underline{G}}$. We shall write J for the functor I with codomain restricted to $\underline{\underline{G}}$.

(v) <u>Adjointness of J and C; definitions and calculations.</u> Wallach also states that $W \simeq CJW$. If $j_W : W \to RIW$ is the $U\underline{\underline{h}}$-monomorphism defined in section (8.2) above, then the map

$$s_W : W \to CJW,$$

defined to be j_W with codomain restricted to CJW, is such a $U\underline{\underline{h}}$-isomorphism. It is easy to check that s_W is natural in W, using the naturality of j_W.

Another result of Wallach [17], his theorem 3.1, shows that if $V \in \underline{\underline{G}}$, then $V \simeq JCV$. It is necessary to outline his proof, in order to specify the isomorphism. We shall then show that this isomorphism is natural in V.

Let $V \in \underline{\underline{G}}$. The conditions (1) and (2) of part (ii) above guarantee that

$$RV \simeq CV \oplus V.\underline{\underline{n}}_1$$

as an $\underline{\underline{h}}$-module.

Let $p_V : RV \to CV$ be the projection onto the first summand. We define a map

$$t_V : V \to JCV$$

by

$$(t_V(v))(u) = p_V(v.u) \qquad \text{for } v \in V,\ u \in U\underline{\underline{g}}.$$

(Recall that $JCV \subseteq \mathrm{Hom}_{U\underline{\underline{t}}}(U\underline{\underline{g}},\widehat{CV})$, from (8.2) part (i) and definitions of J and C.)

It is easily verified that for $v \in V$, $t_V(v)$ is a $U\underline{\underline{t}}$-homomorphism, that t_V is a $\underline{\underline{g}}$-homomorphism, and that, because of condition (3) of part (ii) above, t_V is injective.

We shall now prove that t_V is surjective.

Observe that $p_V(v.u) = v.\gamma(u)$ for $v \in CV$ and $u \in U\underline{\underline{g}}$, and where $\gamma : U\underline{\underline{g}} \to U\underline{\underline{t}}$ is the projection defined in part (i) of section (8.2). It follows that

$$t_V\big|_{CV} = j_{CV},$$

so that, using condition (1) of part (ii) above,

$$\begin{aligned} \mathrm{im}(t_V) &= (\mathrm{im}\ t_V\big|_{CV}).U\underline{\underline{g}} \\ &= (\mathrm{im}\ j_{CV}).U\underline{\underline{g}} \\ &= JCV \qquad \text{by definition of } J. \end{aligned}$$

Thus t_V is surjective and hence an isomorphism.

Next, we check the naturality of t_V in V. Let $V, \bar{V} \in \underline{\underline{G}}$, and let $\phi \in \underline{\underline{G}}(V,\bar{V})$. Consider the diagram

$$\begin{tikzcd} V \arrow[r, "t_V"] \arrow[d, "\phi"'] & JCV \arrow[d, "JC\phi"] \\ \bar{V} \arrow[r, "t_{\bar{V}}"'] & JC\bar{V} \end{tikzcd}$$

Let $v \in V$, $u \in \underline{\underline{Ug}}$: $(JC\phi \circ t_V)(v) = C\phi \circ (t_V(v))$, hence

$$(C\phi \circ (t_V(v)))(u) = C\phi(p_V(v.u))$$

while
$$t_{\bar{V}}(\phi(v))(u) = p_{\bar{V}}(\phi(v).u)$$
$$= p_{\bar{V}}(\phi(v.u))$$

So, the diagram above will commute if and only if

$$p_{\bar{V}} \circ R\phi = C\phi \circ p_V,$$

that is, if and only if the following diagram commutes for all $\phi \in \underline{\underline{G}}(V,\bar{V})$:

$$\begin{tikzcd} RV \arrow[r, "p_V"] \arrow[d, "R\phi"'] & CV \arrow[d, "C\phi"] \\ R\bar{V} \arrow[r, "p_{\bar{V}}"'] & C\bar{V} \end{tikzcd}$$

As was noted by Wallach [17], in the proof of this theorem 3.1, if $V \in \underline{\underline{G}}$ and $v \in V$, we can write

$$v = \tilde{v}.g$$

for some $\tilde{v} \in CV$ and some $g \in \underline{\underline{Ug}}$, because of condition (1) of part (ii) above.

Thus, with this notation

$$p_V(v) = \tilde{v}.\gamma(g).$$

A similar remark applies to $\bar{V}$. Thus, in our case, for $v \in V$ and $g \in U\underline{\underline{g}}$,

$$\phi(v) = \phi(\tilde{v}.g) = \phi(\tilde{v}).g$$

and so $p_{\bar{V}}(\phi(v)) = \phi(\tilde{v}).\gamma(g)$. But, on the other hand,

$$\begin{aligned}\phi(p_V(v)) &= \phi(\tilde{v}.\gamma(g)) \\ &= \phi(\tilde{v}).\gamma(g).\end{aligned}$$

Thus the diagrams above do commute, and t_V is natural in V.

(vi) <u>Adjointness of J and C; conclusions.</u> By part (v), there exist natural isomorphisms

$$s_W : W \to CJW \qquad \text{for } W \in \text{Mod-}\underline{\underline{h}}$$

$$t_V : V \to JCV \qquad \text{for } V \in \underline{\underline{G}}.$$

Thus, by MacLane [12] page 91, Theorem 1, C is both a left and a right adjoint for J, and the categories Mod-$\underline{\underline{h}}$ and $\underline{\underline{G}}$ are equivalent.

Since MacLane's proof is indirect, we shall write down the adjunction isomorphisms and their inverses:

Let $V \in \underline{\underline{G}}$, $W \in$ Mod-$\underline{\underline{h}}$.

(A) Define $\tau_{VW} : \text{Hom}_{U\underline{\underline{h}}}(CV,W) \to \underline{\underline{G}}(V,JW)$ as follows:

let $\psi \in \text{Hom}_{\underline{\underline{h}}}(CV,W)$, and set

$$\tau_{VW}(\psi) = J\psi \circ t_V.$$

The inverse τ_{VW} is defined as follows:

let $\phi \in \underline{\underline{G}}(V,JW)$, let $v \in CV$ and set

$$(\tau_{VW}^{-1}(\phi))(v) = ((C\phi)(v))(1_{U\underline{\underline{g}}}).$$

(B) Define σ_{WV}: $\underline{\underline{G}}(JW,V) \to \text{Hom}_{U\underline{\underline{h}}}(W,CV)$ as follows:

let $\phi \in \underline{\underline{G}}(JW,V)$ and set

$$\sigma_{WV}(\phi) = C\phi \circ s_W.$$

Define σ_{WV}^{-1} as follows:

let $\psi \in \mathrm{Hom}_{U\mathfrak{h}}(W, CV)$, let $w_1, \ldots, w_n \in W$,

let $x_1, \ldots, x_n \in U\mathfrak{g}$, so that $\sum_{i=1}^{n} j_W(w_i)x_i \in JW$, and set

$$(\sigma_{WV}^{-1}(\psi))\left(\sum_{i=1}^{n} j_W(w_i).x_i\right) = \sum_{i=1}^{n} t_V^{-1}(j_{CV}(\psi(w_i)).x_i).$$

<u>Theorem:</u> $\text{Mod-}\mathfrak{h} \underset{C}{\overset{J}{\rightleftarrows}} \mathfrak{G}$ is an equivalence of categories.

<u>(8.4) Induction from Cartan-type subalgebras III - a dualization of Wallach's construction.</u>

<u>Conventions:</u> In this section, $\mathfrak{g}$ will be a <u>"Lie algebra with decomposition"</u>, as in (8.2) and (8.3). That is, $\mathfrak{g}$ has subalgebras $\mathfrak{h}$, $\mathfrak{n}_1$, $\mathfrak{n}_2$ such that

$$\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{h} \oplus \mathfrak{n}_2$$

as a vector space, and $[\mathfrak{n}_1, \mathfrak{h}] \subseteq \mathfrak{n}_1$, $[\mathfrak{n}_2, \mathfrak{h}] \subseteq \mathfrak{n}_2$. We shall write $\mathfrak{t} = \mathfrak{n}_1 \oplus \mathfrak{h}$.

Recall from section (1.2) of this thesis that

$$U\mathfrak{g} = U\mathfrak{t} \oplus U\mathfrak{t}.\mathfrak{n}_2.U\mathfrak{n}_2$$

as a left $U\mathfrak{t}$-module. Since $[\mathfrak{n}_2, \mathfrak{h}] \subseteq \mathfrak{n}_2$, it is easy to check that the summands are right $U\mathfrak{h}$-modules. Let $\bar{\gamma} : U\mathfrak{g} \to U\mathfrak{t}$ denote the projection onto the first summand; $\bar{\gamma}$ is a left $U\mathfrak{t}$-, right $U\mathfrak{h}$-module homomorphism.

Note that $\bar{\gamma}$ is a different map from the map γ introduced in section (8.2).

<u>Introduction:</u> We are going to define a functor $\bar{I} : \text{Mod-}\mathfrak{h} \to \text{Mod-}\mathfrak{g}$ and show that it satisfies the left and right injectivity axioms. We shall define it in much the same fashion as we defined Wallach's functor (in section (8.2)) but we shall use the tensor product functor $- \otimes_{U\mathfrak{h}} U\mathfrak{g}$ in place of the functor $\mathrm{Hom}_{U\mathfrak{h}}(U\mathfrak{g}, -)$.

(i) Definition of the functor $\bar{I} : \text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{g}}$. Let $W \in \text{Mod-}\underline{\underline{h}}$.
Define a functor

$$\sim : \text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{t}}$$

$$W \mapsto \tilde{W}$$

as follows. The underlying vector space of $\tilde{W}$ is the same as that of W, and $\underline{\underline{h}}$ acts on $\tilde{W}$ as it does on W. $\underline{\underline{n}}_1$ acts trivially on $\tilde{W}$. Since $\underline{\underline{t}} = \underline{\underline{n}}_1 \oplus \underline{\underline{h}}$, this specifies the module $\tilde{W}$. This functor is defined to act trivially on morphisms (i.e. it does not change them) and we do not distinguish between a morphism and its image under the functor. That is, if $W_1, W_2 \in \text{Mod-}\underline{\underline{h}}$,

$$\text{Hom}_{U\underline{\underline{t}}}(\tilde{W}_1, \tilde{W}_2) = \text{Hom}_{U\underline{\underline{h}}}(W_1, W_2).$$

Next define an $\underline{\underline{h}}$-epimorphism

$$\hat{k}_W : R(\tilde{W} \otimes_{U\underline{\underline{t}}} U\underline{\underline{g}}) \to W$$

as follows: for $w \in \tilde{W}$ and $u \in U\underline{\underline{g}}$, set

$$\hat{k}_W(w \otimes u) = w.\bar{\gamma}(u)$$

and extend this definition to all of $\tilde{W} \otimes_{U\underline{\underline{t}}} U\underline{\underline{g}}$ by linearity. $\hat{k}_W$ is well-defined, because the map $\tilde{W} \times U\underline{\underline{g}} \to W$ defined by

$$(w,u) \to w.\bar{\gamma}(u) \qquad (\text{for } w \in \tilde{W},\ u \in U\underline{\underline{g}})$$

is $U\underline{\underline{t}}$-balanced and bilinear (cf Curtis and Reiner, sections (12.1) to (12.6).

Consider the $\underline{\underline{h}}$-submodule $\ker \hat{k}_W$ of $\tilde{W} \otimes_{U\underline{\underline{t}}} U\underline{\underline{g}}$. $\ker \hat{k}_W$ contains a unique largest $\underline{\underline{g}}$-module, namely the sum of <u>all</u> the $\underline{\underline{g}}$-modules contained in $\ker \hat{k}_W$. Call this unique largest $\underline{\underline{g}}$-module $Y(W) \subseteq \ker \hat{k}_W$. Clearly $\hat{k}_W$ factors through $R(\tilde{W} \otimes_{U\underline{\underline{t}}} U\underline{\underline{g}}/Y(W))$ via the natural projection - by a map

$$\bar{k}_W : R(\tilde{W} \otimes_{U\underline{\underline{t}}} U\underline{\underline{g}}/Y(W)) \to W,$$

say.

We define our functor

$$\bar{I} : \text{Mod-}\underline{\underline{h}} \to \text{Mod-}\underline{\underline{g}}$$

as follows: first, on objects.

Set $\bar{I}W = (\tilde{W} \otimes_{U\underline{\underline{t}}} U\underline{\underline{g}})/Y(W)$ and give it the quotient $\underline{\underline{g}}$-module structure. Now we must define the action of $\bar{I}$ on morphisms. Let $W_1, W_2 \in \text{Mod-}\underline{\underline{h}}$, and let $\psi \in \text{Hom}_{\underline{\underline{h}}}(W_1, W_2)$. Suppose $w \in W_1$, $u \in U\underline{\underline{g}}$, so that

$$w \otimes u + Y(W_1) \in \bar{I}W_1.$$

Define $(\bar{I}\psi)(w \otimes u + Y(W_1) = \psi(w) \otimes u + Y(W_2)$, and extend this definition to all of $\bar{I}W_1$ by linearity.

We must check that $\bar{I}\psi$ is well-defined. Note firstly that the map

$$w \otimes u \mapsto \psi(w) \otimes u \qquad (w \in \tilde{W}_1,\ u \in U\underline{\underline{g}})$$

$$\tilde{W}_1 \otimes_{U\underline{\underline{t}}} U\underline{\underline{g}} \to \tilde{W}_2 \otimes_{U\underline{\underline{t}}} U\underline{\underline{g}}$$

is well-defined by the functoriality of $- \otimes_{U\underline{\underline{t}}} U\underline{\underline{g}}$. Suppose that $w_1, \ldots, w_n \in \tilde{W}_1$ and $u_1, \ldots, u_n \in U\underline{\underline{g}}$, and that

$$\sum_{i=1}^{n} w_i \otimes u_i \in Y(W_1).$$

Then, since $Y(W_1)$ is a $\underline{\underline{g}}$-module contained in $\ker \hat{k}_{W_1}$, for all $x \in U\underline{\underline{g}}$

$$\begin{aligned}
0 &= \hat{k}_{W_1}\Big(\sum_{i=1}^{n} w_i \otimes u_i x\Big) \\
&= \sum_{i=1}^{n} w_i . \bar{\gamma}(u_i x) \\
\therefore \quad 0 &= \psi\Big(\sum_{i=1}^{n} w_i . \bar{\gamma}(u_i x)\Big) \\
&= \sum_{i=1}^{n} \psi(w_i) . \bar{\gamma}(u_i x) \\
&= \hat{k}_{W_2}\Big(\sum_{i=1}^{n} \psi(w_i) \otimes u_i x\Big).
\end{aligned}$$

That is, for all $x \in U\underline{\underline{g}}$, $\Big(\sum_{i=1}^{n} \psi(w_i) \otimes u_i\Big)x \in \ker \hat{k}_{W_2}$

or $(\sum_{i=1}^{n} \psi(w_i) \otimes u_i).U\underline{\underline{g}} \subseteq \ker \hat{k}_{W_2}$. But this forces $(\sum_{i=1}^{n} \psi(w_i) \otimes u_i)U\underline{\underline{g}}$ $\subseteq Y(W_2)$, so, in particular, $\sum_{i=1}^{n} \psi(w_i) \otimes u_i \in Y(W_2)$. Hence $\bar{I}\psi$ is well-defined. It is easy to check that $\bar{I}$ has the homomorphism property of a functor.

(ii) <u>Naturality of $\bar{k}_W$, and the Right Injectivity Axiom.</u> Let $W \in$ Mod-$\underline{\underline{h}}$. We now show that the $U\underline{\underline{h}}$-epimorphism $\bar{k}_W : R\bar{I}W \to W$, defined in part (i), is natural in W. Let $W_1, W_2 \in$ Mod-$\underline{\underline{h}}$, and let $\psi \in \mathrm{Hom}_{U\underline{\underline{h}}}(W_1, W_2)$. We must show that the following diagram commutes:

$$\begin{tikzcd} R\bar{I}W_1 \arrow[r, "\bar{k}_{W_1}"] \arrow[d, "R\bar{I}\psi"'] & W_1 \arrow[d, "\psi"] \\ R\bar{I}W_2 \arrow[r, "\bar{k}_{W_2}"'] & W_2 \end{tikzcd}$$

Let $w \in \tilde{W}_1$ and $u \in U\underline{\underline{g}}$. It is sufficient to check commutativity on a generator $w \otimes u + Y(W_1)$ of $R\bar{I}W_1$:

$$\begin{aligned} &\psi(\bar{k}_{W_1}(w \otimes u + Y(W_1)) \\ &= \psi(w.\bar{\gamma}(u)) \\ &= \psi(w).\bar{\gamma}(u) \end{aligned}$$

while

$$\begin{aligned} &\bar{k}_{W_2}(RI\psi(w \otimes u + Y(W_1))) \\ &= \bar{k}_{W_2}(\psi(w) \otimes u + Y(W_2)) \\ &= \psi(w).\bar{\gamma}(u). \end{aligned}$$

So the diagram commutes as required. By its definition, $\ker \bar{k}_W$ contains no nonzero $\underline{\underline{g}}$-modules. Thus theorem (5.16) of this thesis tells us that $\bar{I}$ and R satisfy the right injectivity axiom.

(iii) <u>Definition and naturality of the map $\bar{j}_W$.</u> Let $W \in$ Mod-$\underline{\underline{h}}$, let $w \in W$. We define a map

$$\bar{j}_W : W \to R\bar{I}W$$

by

$$\bar{j}_W(w) = w \otimes 1_{U\underline{\underline{g}}} + Y(W).$$

It is easy to check that $\bar{\bar{j}}_W$ is an $\underline{\underline{h}}$-homomorphism, and since $\bar{k}_W \circ \bar{\bar{j}}_W = 1_W$, $\bar{\bar{j}}_W$ is injective. Also, clearly, $(\text{im } \bar{\bar{j}}_W).U\underline{\underline{g}} = \bar{I}W$.

Let $W_1, W_2 \in \text{Mod-}\underline{\underline{h}}$ and suppose $\psi \in \text{Hom}_{U\underline{\underline{h}}}(W_1, W_2)$. I claim that $\bar{\bar{j}}_W$ is natural in W. To see this, we must check that the following diagram commutes:

$$\begin{tikzcd} W_1 \arrow[r, "\bar{\bar{j}}_{W_1}"] \arrow[d, "\psi"'] & R\bar{I}W_1 \arrow[d, "R\bar{I}\psi"] \\ W_2 \arrow[r, "\bar{\bar{j}}_{W_2}"'] & R\bar{I}W_2 \end{tikzcd}$$

Let $w \in W_1$. Then $(R\bar{I}\psi)(\bar{\bar{j}}_{W_1}(w)) = (R\bar{I}\psi)(w \otimes 1_{U\underline{\underline{g}}} + Y(W_1))$

$$= \psi(w) \otimes 1_{U\underline{\underline{g}}} + Y(W_2)$$

while $\bar{\bar{j}}_{W_2}(\psi(w)) = \psi(w) \otimes 1_{U\underline{\underline{g}}} + Y(W_2)$.

Thus the diagram does indeed commute, and so $\bar{\bar{j}}_W$ is natural in W.

(iv) Left Injectivity Axiom. By part (iii) above and theorem (5.8) of this thesis, $\bar{I}$ and R satisfy the left injectivity axiom.

Also, by part (iii) above, $\bar{I}$ satisfies the splitting axiom of section (7.2) of this thesis.

(8.5) Induction from Complemented Ideals.

Hochschild and Mostow, in their paper [7], pp. 937-939, described a way of inducing from a complemented ideal. Their induced module construction was not, however, on the face of it, functorial.

The modification of their construction described in this section is functorial, and the functor we shall construct satisfies the left injectivity axiom.

Conventions: In this section, $\underline{\underline{h}} \trianglelefteq \underline{\underline{g}}$ will be a complemented ideal - that is, an ideal $\underline{\underline{h}}$ of $\underline{\underline{g}}$ for which there exists a subalgebra $\underline{\underline{s}}$ in $\underline{\underline{g}}$ which is a vector space complement to $\underline{\underline{h}}$:

$$\underline{\underline{g}} = \underline{\underline{h}} \oplus \underline{\underline{s}} \qquad \text{as a vector space.}$$

Hochschild and Mostow define a left $\underline{\underline{g}}$-module structure on $U\underline{\underline{h}}$, which we shall presently describe. $U\underline{\underline{h}}$ will be assumed to bear this module structure "*" throughout this section.

(i) $\underline{\underline{g}}$-module structure on $U\underline{\underline{h}}$. Since $\underline{\underline{g}} = \underline{\underline{h}} \oplus \underline{\underline{s}}$, for any $g \in \underline{\underline{g}}$ there exist a unique $h \in \underline{\underline{h}}$ and a unique $s \in \underline{\underline{s}}$ such that

$$g = h + s.$$

Thus, if $u \in U\underline{\underline{h}}$, we can set

$$g * u = (h + s) * u = hu + (su - us).$$

It can be proved, by induction on the length of standard monomials, that $su - us \in U\underline{\underline{h}}$, and it is then easy to verify that the above equation(s) determine a $\underline{\underline{g}}$-module structure on $U\underline{\underline{h}}$.

(ii) Definition of the maps $\hat{j}_W$, j_W, and the object function of the functor I. Let $W \in \text{Mod-}\underline{\underline{h}}$. We can construct a vector space $\text{Hom}_k(U\underline{\underline{h}}, W)$ and we put a $\underline{\underline{g}}$-module structure on $\text{Hom}_k(U\underline{\underline{h}}, W)$ by defining, for $f \in \text{Hom}_k(U\underline{\underline{h}}, W)$ and $g \in \underline{\underline{g}}$, $u \in U\underline{\underline{h}}$

$$f^g(u) = f(g * u).$$

If we define a map

$$\hat{j}_W : W \to \text{Hom}_k(U\underline{\underline{h}}, W)$$

by (for $w \in W$ and $u \in U\underline{\underline{h}}$) setting

$$\hat{j}_W(w)(u) = w.u,$$

then it is easy to check that $\hat{j}_W$ is a well-defined $\underline{\underline{h}}$-monomorphism.

We define a $\underline{\underline{g}}$-submodule IW of $\mathrm{Hom}_k(U\underline{\underline{h}},W)$ by setting

$$IW = (\mathrm{im}\ \hat{j}_W).U\underline{\underline{g}}.$$

We define $j_W : W \to RIW$ to be $\hat{j}_W$ with codomain restricted to RIW. Clearly j_W is still an $\underline{\underline{h}}$-monomorphism and

$$(\mathrm{im}\ j_W)U\underline{\underline{g}} = IW.$$

(iii) Action of I on morphisms; naturality of j_W. Let $W,\bar{W} \in$ Mod-$\underline{\underline{h}}$, and let $\psi \in \mathrm{Hom}_{U\underline{\underline{h}}}(W,\bar{W})$. Let $f \in IW$, thus f may be written as a sum

$$f = \sum_{i=1}^{n} j_W(w_i).x_i$$

with $w_1,\ldots,w_n \in W$ and $x_1,\ldots,x_n \in U\underline{\underline{g}}$. Let $u \in U\underline{\underline{h}}$. We set

$$((I\psi)(f))(u) = (\psi \circ f)(u).$$

We shall show that $(I\psi)(f) \in I\bar{W}$.

$$\begin{aligned}((I\psi)(f))(u) &= (\psi \circ \sum_{i=1}^{n} j_W(w_i).x_i)(u)\\ &= \psi(\sum_{i=1}^{n} w_i.(x_i * u))\\ &= \sum_{i=1}^{n} \psi(w_i).(x_i * u)\\ &= (\sum_{i=1}^{n} j_{\bar{W}}(\psi(w_i)).x_i)(u).\end{aligned}$$

So $(I\psi)(f) \in I\bar{W}$. Thus $I\psi \in \mathrm{Hom}_{\underline{\underline{g}}}(IW,I\bar{W})$. Clearly, I satisfies the homomorphism property of a functor.

Next, I claim that j_W is natural in W. We must show that the following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow{j_W} & RIW \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle RI\psi} \\ \bar{W} & \xrightarrow[j_{\bar{W}}]{} & RI\bar{W} \end{array}$$

Choose $u \in U\underline{\underline{h}}$ and $w \in W$. Then

$$\begin{aligned}((RI\psi) \circ j_W)(w)(u) &= (\psi \circ j_W(w))(u) \\ &= \psi(w.u) \\ &= \psi(w).u \\ &= (j_{\overline{W}}(\psi(w))(u) \\ &= (j_{\overline{W}} \circ \psi)(w)(u).\end{aligned}$$

Thus $j_{\overline{W}} \circ \psi = RI\psi \circ j_W$ as required.

(iv) Conclusions: Injectivity Axiom and Faithfulness of I. By theorem (5.8) and the results of parts (ii) and (iii), it follows that I and R satisfy the left injectivity axiom.

Further I is a faithful functor.

For, if $I\psi = 0$, then, in particular, for all $w \in W$,

$$(I\psi)(j_W(w)) = 0.$$

So for all $w \in W$, $(I\psi)(j_W(w))(1_{U\underline{\underline{h}}}) = 0$.

That is $\psi(w) = 0$ for all $w \in W$. That is $\psi = 0$. Thus I is faithful.

(v) Theorem. (cf. Hochschild and Mostow [7] and Zassenhaus [18].)

Let $\underline{\underline{g}}$ be a finite-dimensional Lie algebra over a field k of characteristic zero, and let $\underline{\underline{h}}$ be a complemented ideal of $\underline{\underline{g}}$ with complementary subalgebra $\underline{\underline{s}} \leq \underline{\underline{g}}$.

Let W be a finite-dimensional $\underline{\underline{h}}$-module on which $[\underline{\underline{h}},\underline{\underline{s}}]$ acts nilpotently. Then IW, as defined in part (ii) above, is a finite-dimensional $\underline{\underline{g}}$-module.

Remark: It may be seen that this result is a form of converse to the theorem of Zassenhaus cited in section (0.3).

The proof is rather close to that of the analogous result of Hochschild and Mostow, mentioned above.

Proof of theorem:

(1) Definition of $s(W)$, and of d.

Let $W \in \text{Mod-}\underline{\underline{h}}$. Let

$$0 = W_0 < W_1 < \ldots < W_n = W$$

be a composition series for W. We write

$$S(W) = \bigoplus_{i=1}^{n} (W_i/W_{i-1})$$

and call $S(W)$ the semisimple $\underline{\underline{h}}$-module corresponding to W.

By the Jordan-Holder theorem (see, for example, Curtis and Reiner [2] (13.7), p.79) $S(W)$ is determined up to isomorphism. A subalgebra of $\underline{\underline{h}}$ acts nilpotently on W if and only if it acts trivially on $S(W)$.

Note that a composition series for W can have length (n) at most $\dim_k W$. Write $d = \dim_k W$.

(2) The formulas (A) and (B) and the core of the proof.

$$(\mathrm{Ann}_{U\underline{\underline{h}}}(S(W)))^d \subseteq \mathrm{Ann}_{U\underline{\underline{h}}}(W) \qquad \text{- (A)}$$

$$\subseteq \mathrm{Ann}_{U\underline{\underline{h}}}(S(W)) \qquad \text{- (B)}$$

Since, by hypothesis, $[\underline{\underline{h}},\underline{\underline{s}}] \subseteq \mathrm{Ann}_{U\underline{\underline{h}}}(S(W))$, it follows that $\mathrm{Ann}_{U\underline{\underline{h}}}(S(W))$ is a $\underline{\underline{g}}$-submodule of $U\underline{\underline{h}}$. Hence $(\mathrm{Ann}_{U\underline{\underline{h}}}(S(W)))^d$ is a $\underline{\underline{g}}$-submodule of $U\underline{\underline{h}}$.

If $f \in \mathrm{Hom}_k(U\underline{\underline{h}},W)$, and

$$f(\mathrm{Ann}_{U\underline{\underline{h}}}(W)) = (0),$$

then for all $x \in U\underline{\underline{g}}$

$$\begin{aligned} f^x((\mathrm{Ann}_{U\underline{\underline{h}}}(S(W)))^d) &\subseteq f(x(\mathrm{Ann}_{U\underline{\underline{h}}}(S(W)))^d) \\ &\subseteq f(\mathrm{Ann}_{U\underline{\underline{h}}}(W)) \qquad \text{by (A) above} \\ &= 0. \end{aligned}$$

Now im j_W annihilates $\mathrm{Ann}_{U\underline{\underline{h}}}(W)$, so $IW = (\mathrm{im}\ j_W).U\underline{\underline{g}}$ annihilates $(\mathrm{Ann}_{U\underline{\underline{h}}}(S(W)))^d$. Let us write $J = (\mathrm{Ann}_{U\underline{\underline{h}}}(S(W)))^d$. It is easy to see that IW is embedded in $\mathrm{Hom}_k(U\underline{\underline{h}}/J,W)$ (cf proposition (7.17) of this thesis.)

Since W is finite-dimensional, $\mathrm{Ann}_{U\underline{\underline{h}}}(W)$ is of finite codimension in $U\underline{\underline{h}}$. Hence, by inequality (B), above, $\mathrm{Ann}_{U\underline{\underline{h}}}(S(W))$ is of finite codimension in $U\underline{\underline{h}}$. Now we need a lemma of Zassenhaus.[1]

<u>Lemma</u>: If X and Y are ideals of $U\underline{\underline{h}}$ of finite dimension, then so is XY.

We deduce from this lemma that J is of finite codimension in $U\underline{\underline{h}}$. Thus

$$\dim_k \mathrm{Hom}_k(U\underline{\underline{h}}/J,W) < \infty$$

and so, since $IW \subseteq \mathrm{Hom}_k(U\underline{\underline{h}}/J,W)$,

$$\dim_k(IW) < \infty. \qquad \square$$

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[1] Cf [18] page 263

Chapter 9 - Lie Ideal Subrings and Clifford's Theorem

(9.1) Introduction. The reader should note that the theorems, propositions and lemmas of this chapter are labelled in a different way from those of chapters 1 to 8.

In [14], M.A. Rieffel remarks that there is "one very important part of the theory of induced representations of groups which [he does] not a present see how to generalize to rings, namely Clifford's theory of induced representations of group extensions".

He then notes that the difficulty lies in finding a satisfactory concept of "normal" subring of a ring.

In this chapter, we present a possible candidate for the role of "normal" subring. It is shown that, with this concept of normal subring, the analogues of at least two of the main results of Clifford's theory of induced representations of group extensions hold, with some restrictions on the rings and modules involved.

Throughout this chapter, all rings considered will be assumed to have identity elements. The identity element of a ring R will be denoted by 1_R. By a subring S of a ring R, we shall mean a subset of R closed under subtraction and multiplication, and containing 1_R. All modules will be assumed to be unitary. All modules will be right modules or bimodules. The symbols $[s,r]$ will be used to denote the commutator $sr - rs$ of two elements s and r of a ring. $C_R(S)$ will denote the subring $\{r \in R : \forall\ s \in S \ \ rs = sr\}$ and $Z(R)$ the centre of a ring $R \supseteq S$. If M is an R-module, and X is a subset of M, then $Ann_R(X)$ will denote $\{r \in R : X.r = \{0\}\}$. By an R-component of an R-module M, we shall mean a quotient module V/W, where $W \subseteq V$ and W and V are R-submodules of M.

Definition: A Lie ideal subring S of a ring R is a subring of R which is also a Lie ideal of R, that is, a subring with the following property:

$$\forall s \in S \ \forall r \in R \quad [s,r] \in S.$$

The Lie ideal subring is our candidate for the role of "normal" subring.

Theorem A: Let R be a right Artinian ring and let S be a right Artinian Lie ideal subring of R. Let M be an irreducible R-module which is finitely generated as an S-module. Suppose that $2r \in \mathrm{Ann}_R(M)$ implies $r \in \mathrm{Ann}_R(M)$ for all $r \in R$. Then all irreducible S-components of M are S-isomorphic, and M is completely reducible as an S-module.

This result is a ring-theoretic analogue of (49.2) of Curtis and Reiner [2]. It is also closely analogous to a result of Barnes and Newell [1], page 185. Part of Theorem A is true under much weaker hypotheses: see Proposition 1.

Theorem B: Let R be a ring and let S be a Lie ideal subring of R. Let M be an irreducible R-module and let L be an irreducible S-submodule of M. Set $S^* = \{ r \in R : L.r \subseteq L\}$. Then $M \simeq L \otimes_{S^*} R$.

(9.2) Examples of Lie Ideal Subrings.

(i) If I is an ideal of a ring R, then the subset $\{ i + n.1_R : i \in I,\ n \text{ an integer}\}$ is the smallest Lie ideal subring of R containing I. This Lie ideal subring will henceforth be referred to as $I + \mathbb{Z}1_R$.

(ii) If I is an ideal of R, then I + Z(R) is a Lie ideal subring of R. In fact, if S is any subring of R such that

$$I + \mathbb{Z}1_R \subseteq S \subseteq I + Z(R),$$

then S is a Lie ideal subring of R. A partial converse of this will be proved in (9.4) in the case where R is a right Artinian semisimple ring.

(iii) If S is a Lie ideal subring of R, then so is $C_R(S)$.

(9.3) Equivalence of Irreducible Components.

The result proved in this section is similar to a result of Zassenhaus on Lie algebras, and is proved similarly: see Zassenhaus [18] page 253. It implies one half of Theorem A, but is proved under weaker hypotheses.

Proposition 1: If R is a ring, and S is a Lie ideal subring of R, and if M is an irreducible R-module which contains an irreducible S-submodule, then all the irreducible S-components of M are S-isomorphic.

Proof: Let L be an irreducible S-submodule of M. Consider the set of S-submodules of M which contain L and have all their irreducible S-components isomorphic to L. By Zorn's lemma, this set contains a maximal element, K, say.

We shall show that K = M, by showing that K is an R-submodule of the irreducible R-module M. Suppose $r \in R$. We must show that $K.r \subseteq K$. For all $s \in S$ and all $k \in K$,

$$(kr)s = k.[r,s] + (ks)r \in K + Kr$$

since $[r,s] \in S$, hence K + Kr is an S-submodule of M.

We claim that $\rho_r : K \to (K + Kr)/K$, defined by

$$k \mapsto kr + K \qquad \text{for } k \in K$$

is an S-epimorphism of K onto (K + Kr)/K. The surjectivity part is obvious. If $s \in S$ and $k \in K$, then

$$\begin{aligned}\rho_r(ks) &= ksr + K \\ &= k.[s,r] + krs + K \\ &= krs + K \quad \text{since} \quad [s,r] \in S \\ &= (kr + K).s \\ &= \rho_r(k).s\end{aligned}$$

so ρ_r is an S-homomorphism as claimed. Thus $(K + Kr)/K$ is S-isomorphic to a quotient module of K. It is now easy to see that every irreducible S-component of $K + Kr$ is isomorphic to L.

Hence, by the maximality of K, $K = K + Kr$. That is, $Kr \subseteq K$. Thus K is an R-submodule of M. □

(9.4) Lie Ideal Subrings of Right Artinian Semisimple Rings.

We need Lemma 1.3 of Herstein [4]; we restate this Lemma here and in a convenient form, for ease of reference:

Lemma 2: Let R be a ring with no nonzero nilpotent ideals, in which $2x = 0$ implies $x = 0$. Suppose U is a Lie ideal subring of R. Then either $U \subseteq Z(R)$, or U contains a nonzero ideal of R. □

The next result is the promised partial converse of example (ii) of (9.2).

Corollary 3: Let R be a semisimple right Artinian ring in which $2x = 0$ implies $x = 0$, and let S be a Lie ideal subring of R. Then there exists an ideal I of R, such that

$$I + \mathbb{Z}.1_R \subseteq S \subseteq I + Z(R).$$

Proof: Let I be the unique largest ideal of R contained in S. Recall that every ideal of a semisimple right Artinian ring is a direct summand. Thus there is an ideal J of R such that $R = I \oplus J$. Let ρ be the projection of R onto J. It is easily checked that $S = \rho(S) \oplus I$, and that $\rho(S)$ is a Lie ideal subring of J. Since J is an ideal of R, $2x = 0$ implies $x = 0$ in J, and furthermore, J has no nonzero nilpotent ideals, since J is semisimple (see Lemma 1.2.2 of Herstein [5]).

Hence, by Lemma 2, $\rho(S) \subseteq Z(J)$.

It follows that $\rho(S) \subseteq Z(R)$.

Therefore, $S = I \oplus \rho(S) \subseteq I + Z(R)$. Finally, since $I \subseteq S$ and $1_R \in S$, it follows that $I + \mathbb{Z}.1_R \subseteq S$. □

(9.5) Proof of Theorem A.

For the proof of Theorem A, we need some extra notation and a lemma.

Let R be a ring and M an R-module, and let T be a subring or ideal of R. Noting that $\mathrm{Ann}_R(M)$ is a two-sided ideal of R, we write

$$\bar{T} = (T + \mathrm{Ann}_R(M))/\mathrm{Ann}_R(M).$$

Lemma 4: Let R be a ring and let S be a Lie ideal subring of R. Let M be an R-module and let L be an irreducible S-submodule of M. Suppose that $c \in R$ satisfies $c + \mathrm{Ann}_R(M) \in C_{\bar{R}}(\bar{S})$ and $c \notin \mathrm{Ann}_R(L)$. Then the map $\rho : L \to Lc$ defined by $\ell \mapsto \ell c$ is an isomorphism of S-modules.

Proof: Let $c \in R$ satisfy $c + \mathrm{Ann}_R(M) \in C_{\bar{R}}(\bar{S})$ and $c \notin \mathrm{Ann}_R(L)$. For any $\ell \in L$ and any $s \in S$, $(\ell c)s = (\ell s)c + \ell.[c,s]$. Now $[(c + \mathrm{Ann}_R(M)), (s + \mathrm{Ann}_R(M))] = 0$ by choice of c, hence $[c,s] \in \mathrm{Ann}_R(M)$, and $\ell[c,s] = 0$. That is

$$(\ell c)s = (\ell s)c. \qquad (*)$$

Clearly ρ is additive, and the equation (*) above shows that ρ is an S-homomorphism. Since L is irreducible and $c \notin \mathrm{Ann}_R(L)$, ρ is injective, and clearly ρ is surjective. Thus ρ is an isomorphism. $\square$

Proof of Theorem A: We shall work modulo $\mathrm{Ann}_R(M)$. Clearly M is a faithful irreducible $\bar{R}$-module, hence $\bar{R}$ is a primitive ring. R is right Artinian, so $\bar{R}$ is right Artinian too. Thus by Theorem 2.1.4 of Herstein [5] (page 40), $\bar{R}$ is a complete matrix ring over a skewfield, so is simple. Since $2r \in \mathrm{Ann}_R(M)$ implies $r \in \mathrm{Ann}_R(M)$ for $r \in R$, it follows that $2\bar{r} = 0$ implies $\bar{r} = 0$ for $\bar{r} \in \bar{R}$. Also, it is easy to check that $\bar{S}$ is a Lie ideal subring of $\bar{R}$.

Hence, by Lemma 2, $\bar{S} \subseteq Z(\bar{R})$.

Since M is finitely generated as an S-module and S is right Artinian, M has an irreducible S-submodule, L say, which is also an irreducible $\bar{S}$-module.

Using the fact that M is finitely generated as an S-module again, we can find a finite irredundant list $c_1, c_2, \ldots, c_t$ of elements of R, such that

$$\sum_{i=1}^{t} Lc_i = M.$$

By the irredundancy of $c_1, c_2, \ldots, c_t$ it follows that $c_i \notin \mathrm{Ann}_R(L)$ for $i = 1, 2, \ldots, t$. Since $\bar{S} \subseteq Z(\bar{R})$, $c_i + \mathrm{Ann}_R(M) \in C_{\bar{R}}(\bar{S})$, for $i = 1, 2, \ldots, t$. So, by Lemma 4, Lc_i is an S-module isomorphic to L, for $i = 1, 2, \ldots, t$.

For any $j \in \{1, 2, \ldots, t\}$, $Lc_j \simeq L$ is irreducible, so $Lc_j \cap \sum_{i \neq j} Lc_i = \{0\}$ or Lc_j. The latter possibility would contradict the irredundancy of $c_1, c_2, \ldots, c_t$. Therefore, $Lc_j \cap \sum_{i \neq j} Lc_i = \{0\}$. That is, $M = \bigoplus_{i=1}^{t} Lc_i$. $\square$

(9.6) The Proof of Theorem B.

We need a Lemma.

Lemma 5. Let R be a ring and let S be a Lie ideal subring of R. Let M be an irreducible R-module and let L be an irreducible S-submodule of M. Set $S^* = \{r \in R : Lr \subseteq L\}$. If $\ell_o \in L$, then $\mathrm{Ann}_R(\ell_o) \subseteq S^*$.

Proof of Lemma 5: If $\ell \in L$, then we can write $\ell = \ell_o . s$ for some $s \in S$, because L is irreducible as an S-module. Now for any $a \in \mathrm{Ann}_R(\ell_o)$,

$$\begin{aligned} \ell . a = \ell_o sa &= \ell_o as + \ell_o[s,a] \\ &= 0 + \ell_o[s,a] \in L, \end{aligned}$$

since $[s,a] \in S$, because S is a Lie ideal subring.

That is, $La \subseteq L$, or $a \in S^*$. □

Proof of Theorem B: We shall use the characterization of a tensor product given in Curtis and Reiner [2], sections (12.1) - (12.6).

We shall consider L as a right S^*-module and R as a left S^*-module and right R-module. To prove that M is isomoprhic as an Abelian group to $L \otimes_{S^*} R$, we must show that if

$$\phi : L \times R \to A$$

is any S^*-balanced bilinear map from the Cartesian product $L \times R$ into an arbitrary Abelian group A, then ϕ factors through M by a bilinear balanced map $\mu : L \times R \to M$ which is independent of ϕ.[1] That is, we must show that there exists $\phi' : M \to A$ such that the following diagram commutes:

$$\begin{tikzcd} L \times R \arrow[rr, "\mu"] \arrow[dr, "\phi"'] & & M \arrow[dl, "\phi'"] \\ & A & \end{tikzcd}$$

We choose for μ the restriction to $L \times R$ of the structure map $M \times R \to M$ of the R-module M. That is,

$$\mu(\ell,r) = \ell . r \qquad \text{for } \ell \in L \text{ and } r \in R.$$

---

[1] We must also remark that the elements $\mu(\ell,r)$ generate M as an Abelian group.

It is easy to check that μ is bilinear and S^*-balanced. We construct ϕ' as follows: pick any nonzero $\ell_0 \in L \subseteq M$. M is an irreducible R-module, so for any $m \in M$ there exists an $r \in R$ such that $m = \ell_0.r$. We define $\phi'(m) = \phi(\ell_0,r)$. We must show that this map is well-defined - it certainly has the desired commutativity property. Suppose that $m = \ell_0.r = \ell_0.\bar{r}$ for some $\bar{r} \in R$. Then $r - \bar{r} \in \mathrm{Ann}_R(\ell_0) \subseteq S^*$ by Lemma 5. We must show that $\phi(\ell_0,r) = \phi(\ell_0,\bar{r})$.

We know that

$$\begin{aligned}\phi(\ell_0,r) - \phi(\ell_0,\bar{r}) &= \phi(\ell_0,r - \bar{r}) \\ &= \phi(\ell_0(r-\bar{r}),1_R) \quad \text{since } r-\bar{r} \in S^* \\ &= \phi(0,1_R) \quad \text{since } r-\bar{r} \in \mathrm{Ann}_R(\ell_0) \\ &= 0.\end{aligned}$$

That is,

$$\phi(\ell_0,r) = \phi(\ell_0,\bar{r})$$

and ϕ' is well-defined. Hence $L \otimes_{S^*} R \simeq M$ as Abelian groups, and the isomorphism, ι say, is given by $\iota(\ell \otimes r) = \ell.r$ for $\ell \in L$ and $r \in R$. Finally, we must show that this is an R-homomorphism. Suppose that $\ell \in L$ and $r_1,r_2 \in R$. Then

$$\begin{aligned}\iota((\ell \otimes r_1).r_2) &= \iota(\ell \otimes (r_1r_2)) \\ &= \ell.(r_1r_2) \\ &= (\ell.r_1).r_2 \\ &= \iota(\ell \otimes r_1).r_2 .\end{aligned}$$

Thus ι is an R-homomorphism, and hence an R-isomorphism. $\square$

Appendix - Weak Double Adjoint Functors

(A.0) This appendix contains work done since chapters 7 and 8 were written. It uses terminology which differs from that developed in chapters 4-8, but which is more convenient for the concepts we shall be dealing with.

The principal object of this chapter is to reformulate the statements of (5.8), (5.9), (5.16) and (5.17) in the case where the left and right injectivity axioms of chapter 5 and the splitting axiom of chapter 7 are all satisfied. The result so obtained is illustrated with a new example.

(A.1) Definitions of Types of Weak Adjoint Functor

Let $\underline{\underline{H}}$, $\underline{\underline{G}}$, be categories, and let $R : \underline{\underline{G}} \to \underline{\underline{H}}$, $I : \underline{\underline{H}} \to \underline{\underline{G}}$ be functors. We say that I is an <u>injective weak left adjoint to R</u> if:

$$\left.\begin{array}{l}\text{for all } W \in \underline{\underline{H}},\ V \in \underline{\underline{G}}, \text{ there exists an injection} \\ \qquad\qquad \theta_{WV} : \underline{\underline{G}}(IW,V) \to \underline{\underline{H}}(W,RV) \\ \text{which is natural in W and V.}\end{array}\right\} \qquad (1)$$

(In chapters 5, 7 and 8 this concept was expressed by saying that I and R "satisfied the left injectivity axiom".)

We say that I is an <u>injective weak right adjoint to R</u> if

$$\left.\begin{array}{l}\text{for all } W \in \underline{\underline{H}},\ V \in \underline{\underline{G}}, \text{ there exists an injection} \\ \qquad\qquad \eta_{VW} : \underline{\underline{G}}(V,IW) \to \underline{\underline{H}}(RV,W) \\ \text{which is natural in V and W.}\end{array}\right\} \qquad (2)$$

(In chapters 5, 7 and 8 this concept was expressed by saying that I and R "satisfied the right injectivity axiom".)

Similarly one could define <u>surjective weak left and right adjoints</u> to R.

Notice that for $W \in \underline{\underline{H}}$,

$$\theta_{W,IW}(1_{IW}) \in \underline{\underline{H}}(W,RIW)$$

and $$\eta_{IW,W}(1_{IW}) \in \underline{\underline{H}}(RIW,W).$$

As in chapter 5, we shall denote these two morphisms by j_W and k_W respectively. Both are natural in W, by sections (5.4) and (5.12).

We shall say that I is an <u>injective weak double adjoint to R</u> if (1) and (2) above are satisfied, and also the following condition:

for all $W \in \underline{\underline{H}}$, $k_W \circ j_W = 1_W$. (3)

Similarly, one could define surjective weak double adjoints. Various other combinations are possible.

<u>Double Adjoint Situations.</u> Let $\underline{\underline{H}}$ and $\underline{\underline{G}}$ be categories

and let	$R : \underline{\underline{G}} \to \underline{\underline{H}}$	be a functor.
Suppose that	$L : \underline{\underline{H}} \to \underline{\underline{G}}$	is a left adjoint to R,
and that	$F : \underline{\underline{H}} \to \underline{\underline{G}}$	is a right adjoint to R.
Let	$i : 1_{\underline{\underline{H}}} \dot{\to} RL$	denote the unit of L
and let	$e : RF \dot{\to} 1_{\underline{\underline{H}}}$	denote the counit of F,

in the terminology of MacLane [12], page 81. Then we shall say that the 7-tuple $(\underline{\underline{H}},\underline{\underline{G}},R,L,i,F,e)$ is a <u>double adjoint situation.</u>

Recall that if a category is <u>exact</u>, in the sense of Mitchell [13], page 18, then every morphism has a kernel and a cokernel, an epimorphism is the cokernel of its kernel, and every morphism factors as an epi followed by a monic.

The following lemma is used in the proof of theorem (A.3).

<u>(A.2) Lemma:</u> Let $\alpha : A \to B$ be an epimorphism, let $\beta : C \to D$ be a monomorphism, and let the figure at left be a commutative diagram in an exact category. Then there is a unique map $\varepsilon : B \to C$ such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, two heads, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\delta"] \\ C \arrow[r, tail, "\beta"'] & D \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[r, two heads, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[dl, "\varepsilon"] \\ C & \end{tikzcd} \qquad\qquad \begin{tikzcd} & B \arrow[d, "\delta"] \arrow[dl, "\varepsilon"'] \\ C \arrow[r, tail, "\beta"'] & D \end{tikzcd}$$

<u>Proof:</u> Let $\ker \alpha \xrightarrow{k} A$ be the kernel of α, and consider the diagram

$$\begin{tikzcd} \ker \alpha \arrow[r, "k"] & A \arrow[r, two heads, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[d, "\delta"] \\ & C \arrow[r, tail, "\beta"'] & D \end{tikzcd}$$

$$\begin{aligned} \beta \circ \gamma \circ k &= \delta \circ \alpha \circ k && \text{by commutativity,} \\ &= \delta \circ 0 && \text{since } k \text{ is the kernel of } \alpha \\ &= 0 = \beta \circ 0. \end{aligned}$$

Therefore $\gamma \circ k = 0$, since β is monic.

Hence, since α is the cokernel of k, there exists a unique $\varepsilon : B \to C$ such that $\varepsilon \circ \alpha = \gamma$. Together with the commutativity of the original square, this implies that $\delta \circ \alpha = \beta \circ \gamma = \beta \circ \varepsilon \circ \alpha$. But α is epi, so $\delta = \beta \circ \varepsilon$. The equations $\varepsilon \circ \alpha = \gamma$ and $\beta \circ \varepsilon = \delta$ tell us that the diagrams

$$\begin{tikzcd} A \arrow[r, two heads, "\alpha"] \arrow[d, "\gamma"'] & B \arrow[dl, "\varepsilon"] \\ C & \end{tikzcd} \qquad\qquad \begin{tikzcd} & B \arrow[d, "\delta"] \arrow[dl, "\varepsilon"'] \\ C \arrow[r, tail, "\beta"'] & D \end{tikzcd}$$

commute. □

<u>(A.3) Theorem:</u> Let $(\underline{\underline{H}}, \underline{\underline{G}}, R, L, i, F, e)$ be a double adjoint situation, and assume $\underline{\underline{G}}$ is an exact category. If $\phi : L \overset{\cdot}{\to} F$ is a natural transformation, such that

$$\text{for all } W \in \underline{\underline{H}},\ e_W \circ R\phi_W \circ i_W = 1_W, \tag{4}$$

then ϕ determines an injective weak double adjoint to R.

Conversely, an injective weak double adjoint to R determines a natural transformation $\phi : L \overset{\cdot}{\to} F$ satisfying (4).

<u>Proof:</u> Let $(\underline{\underline{H}}, \underline{\underline{G}}, R, L, i, F, e)$ be a double adjoint situation, and suppose that the category $\underline{\underline{G}}$ is exact.

First, let us suppose that $\phi : L \overset{\cdot}{\to} F$ is a natural transformation satisfying condition (4). Let $W \in \underline{\underline{H}}$. Since $\underline{\underline{G}}$ is exact, ϕ_W factors as shown below:

$$\begin{tikzcd} LW \arrow[rr, "\phi_W"] \arrow[dr, two heads, "\mu_W"'] & & FW \\ & \operatorname{im} \phi_W \arrow[ur, tail, "\nu_W"'] & \end{tikzcd}$$

Define an object function $I : \underline{\underline{H}} \to \underline{\underline{G}}$ by $IW = \operatorname{im} \phi_W$. Suppose $W_1, W_2 \in \underline{\underline{H}}$ and $\psi \in \underline{\underline{H}}(W_1, W_2)$. Consider

$$\begin{tikzcd} LW_1 \arrow[r, two heads, "\mu_{W_1}"] \arrow[d, "L\psi"'] & IW_1 \arrow[r, tail, "\nu_{W_1}"] & FW_1 \arrow[d, "F\psi"] \\ LW_2 \arrow[r, two heads, "\mu_{W_2}"'] & IW_2 \arrow[r, tail, "\nu_{W_2}"'] & FW_2 \end{tikzcd}$$

This diagram can be redrawn as

$$\begin{tikzcd} LW_1 \arrow[r, two heads, "\mu_{W_1}"] \arrow[d, "\mu_{W_2} \circ L\psi"'] & IW_1 \arrow[d, "F\psi \circ \nu_{W_1}"] \\ IW_2 \arrow[r, tail, "\nu_{W_2}"'] & FW_2 \end{tikzcd}$$

In this form, it can be seen that Lemma (A.2) applies, so that there exists a unique morphism, which we denote by $I\psi : IW_1 \to IW_2$, satisfying

$$\mu_{W_2} \circ L\psi = I\psi \circ \mu_{W_1} \qquad \text{and}$$
$$\nu_{W_2} \circ I\psi = F\psi \circ \nu_{W_1}.$$

The uniqueness property of $I\psi$ makes it easy to verify that $I : \underline{\underline{H}} \to \underline{\underline{G}}$ is a functor, and then clearly $\mu : L \stackrel{\cdot}{\to} I$ and $\nu : I \stackrel{\cdot}{\to} R$ are natural transformations.

Define $j = R\mu \circ i : 1_{\underline{\underline{H}}} \stackrel{\cdot}{\to} RI$

and $k = e \circ R\nu : RI \stackrel{\cdot}{\to} 1_{\underline{\underline{H}}}$,

then for $W \in \underline{\underline{H}}$ we have the following commutative diagram:

$$\begin{tikzcd}
RLW \arrow[rr, "R\phi_W"] \arrow[dr, "R\mu_W"'] & & RFW \arrow[dd, "e_W"] \\
& RIW \arrow[ur, "R\nu_W"'] \arrow[dr, "k_W"] & \\
W \arrow[uu, "i_W"] \arrow[ur, "j_W"'] \arrow[rr, phantom, "="] & & W
\end{tikzcd} \tag{5}$$

Next, for $W \in \underline{\underline{H}}$, $V \in \underline{\underline{G}}$, define

$$\eta_{VW} : \underline{\underline{G}}(V,IW) \to \underline{\underline{H}}(RV,W)$$
$$\text{and} \quad \theta_{WV} : \underline{\underline{G}}(IW,V) \to \underline{\underline{H}}(W,RV)$$

by

$$\eta_{VW}(\alpha) = k_W \circ R\alpha$$
$$\theta_{WV}(\beta) = R\beta \circ j_W$$

where $\alpha \in \underline{\underline{G}}(V,IW)$ and $\beta \in \underline{\underline{G}}(IW,V)$.

We claim that η_{VW} and θ_{WV} are injective, and natural in V and W. The naturality follows from that of j_W and k_W, and is proved in a manner similar to the appropriate part of the proofs of theorems (5.8) and (5.16).

Suppose that $\alpha_1, \alpha_2 \in \underline{\underline{G}}(V, IW)$ and

$$\eta_{VW}(\alpha_1) = \eta_{VW}(\alpha_2).$$

That is, $$k_W \circ R\alpha_1 = k_W \circ R\alpha_2.$$

But, by definition, $$k_W = e_W \circ R\nu_W,$$

so $$e_W \circ R\nu_W \circ R\alpha_1 = e_W \circ R\nu_W \circ R\alpha_2.$$

Since, according to MacLane [12], page 80, theorem 1, part (ii), the adjunction $\underline{\underline{G}}(V, FW) \to \underline{\underline{H}}(RV, W)$ is given by $\chi \mapsto e_W \circ R\chi$, the last equation implies that $$\nu_W \circ \alpha_1 = \nu_W \circ \alpha_2.$$
But, by its definition, ν_W is monic. So it follows that

$$\alpha_1 = \alpha_2.$$

Thus η_{VW} is injective. A similar argument shows that θ_{WV} is injective. Finally, for $W \in \underline{\underline{H}}$,

$$\eta_{IW,W}(1_{IW}) = k_W$$

$$\text{and} \quad \theta_{W,IW}(1_{IW}) = j_W .$$

Inspection of the commutative diagram (5) shows that for $W \in \underline{\underline{H}}$,

$$k_W \circ j_W = 1_W,$$

that is, condition (3) is satisfied, so I is an injective weak double adjoint to R.

Now we prove the converse. Let $(\underline{\underline{H}}, \underline{\underline{G}}, R, L, i, F, e)$ be a double adjoint situation and suppose that we are given natural injections

$$\eta_{VW} : \underline{\underline{G}}(V, IW) \to \underline{\underline{H}}(RV, W)$$
$$\text{and} \quad \theta_{WV} : \underline{\underline{G}}(IW, V) \to \underline{\underline{H}}(W, RV)$$

for all $W \in \underline{\underline{H}}$ and $V \in \underline{\underline{G}}$. Set $k_W = \eta_{IW,W}(1_{IW})$ and $j_W = \theta_{W,IW}(1_{IW})$, and suppose further that for all $W \in \underline{\underline{H}}$

$$k_W \circ j_W = 1_W.$$

That is, we are supposing that I is an injective weak double adjoint to R.

By theorems (5.9) and (5.17), there exist natural transformations $\mu_W : LW \to IW$ and $\nu_W : IW \to FW$ with components which are respectively epi and monic. Using Corollaries (5.9a) and (5.17a) and MacLane [12], page 80, Theorem 1, we find that

$$\left.\begin{aligned} j_W &= R\mu_W \circ i_W \\ \text{and}\quad k_W &= e_W \circ R\nu_W \end{aligned}\right\} \tag{6}$$

$\nu \circ \mu$ is a natural transformation from L to F, and, for all $W \in \underline{\underline{H}}$,

$$\begin{aligned} e_W \circ R(\nu \circ \mu)_W \circ i_W &= (e_W \circ R\nu_W) \circ (R\mu_W \circ i_W) \\ &= k_W \circ j_W \\ &= 1_W. \end{aligned}$$

Thus condition (4) is satisfied (with $\phi = \nu \circ \mu$), and the proof of the theorem is complete. □

<u>(A.4) Example</u>: Let $\mathcal{G}$ be a ring with identity element 1, and let $\mathcal{H}$ be a subring such that $1 \in \mathcal{H}$. Let $e : \mathcal{H} \to \mathcal{G}$ denote the inclusion map, and suppose that there is an $(\mathcal{H},\mathcal{H})$-bimodule epimorphism $\gamma : \mathcal{G} \to \mathcal{H}$ such that $\gamma \circ e = 1_{\mathcal{H}}$.

Let $R : \text{Mod-}\mathcal{G} \to \text{Mod-}\mathcal{H}$ denote the change-of-rings functor. R has a left adjoint $L = - \otimes_{\mathcal{H}} \mathcal{G}$ and a right adjoint $F = \text{Hom}_{\mathcal{H}}(\mathcal{G},-)$. Let W be a right $\mathcal{H}$-module. Define a map

$$\hat{j}_W : W \to \text{Hom}_{\mathcal{H}}(\mathcal{G},W)$$

by $$\hat{j}_W(w)(r) = w.\gamma(r) \qquad \text{for } w \in W,\ r \in \mathcal{G}.$$

Set $\bar{I}W = (\text{im}\ \hat{j}_W).\mathcal{G}$. $\bar{I}W$ is a $\mathcal{G}$-module. Let $W_1, W_2 \in \text{Mod-}\mathcal{H}$ and let

$\psi \in \mathrm{Hom}(W_1, W_2)$. Define, for $f \in \bar{I}W_1 \subseteq \mathrm{Hom}_{\mathcal{H}}(\mathcal{G}, W_1)$

$$(\bar{I}\psi)(f) = \psi \circ f .$$

f may be written as $f = \sum_{i=1}^{n} \hat{j}_{W_1}(w_i).r_i$ for suitable $w_i \in W$ and $r_i \in \mathcal{G}$, $i = 1,\ldots,n$, and calculation shows that

$$(\bar{I}\psi)(f) = \sum_{i=1}^{n} \hat{j}_{W_2}(\psi(w_i)).r_i \in \bar{I}W_2$$

and that $\bar{I}\psi$ is a $\mathcal{G}$-homomorphism. Thus

$$\bar{I}\psi \in \mathrm{Hom}_{\mathcal{G}}(\bar{I}W_1, \bar{I}W_2)$$

and $\bar{I}$: Mod-$\mathcal{H} \to$ Mod-$\mathcal{G}$ is a functor.

Define j_W to be $\hat{j}_W$ with codomain restricted to be $R\bar{I}W$, and define $k_W : R\bar{I}W \to W$ by

$$k_W(f) = f(1)$$

for $f \in R\bar{I}W \subseteq \mathrm{Hom}_{\mathcal{H}}(\mathcal{G}, W)$.

It is routine to verify that j_W and k_W are natural in W, and that $k_W \circ j_W = 1_W$, so that j_W is monic and k_W is epi.

Define a map $\bar{\phi}_W : W \times \mathcal{G} \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{G}, W)$ by

$$\bar{\phi}_W(w,g)(s) = w.\gamma(gs) \quad \text{for } w \in W, \text{ and } g,s \in \mathcal{G}.$$

It is easy to check that $\bar{\phi}_W$ is bilinear and that for $h \in \mathcal{H}$

$$\bar{\phi}_W(w.h,g) = \bar{\phi}_W(w,hg)$$

so $\bar{\phi}_W$ induces a map

$$\phi_W : W \otimes_{\mathcal{H}} \mathcal{G} \to \mathrm{Hom}_{\mathcal{H}}(\mathcal{G}, W)$$

given by

$$\phi_W(w \otimes g)(s) = w.\gamma(gs).$$

It is easy to check that ϕ_W is a $\mathcal{G}$-homomorphism and natural in W. We claim that the natural transformation ϕ satisfies condition (4) with respect to the double adjoint situation

$$(\text{Mod-}\mathcal{H}, \text{Mod-}\mathcal{G}, R, -\otimes_{\mathcal{H}}\mathcal{G}, i, \text{Hom}_{\mathcal{H}}(\mathcal{G},-), e)$$

(where i and e are defined below), and so induces an injective weak double adjoint to R, by theorem (A.3).

After verifying this claim, we will show that the injective weak double adjoint I induced by ϕ is, in fact, the functor $\bar{I}$ defined above.

The unit $i_W : W \to W \otimes_{\mathcal{H}}\mathcal{G}$ is given by

$$i_W(w) = w \otimes 1 \quad \text{for} \quad w \in W$$

and the counit $e_W : \text{Hom}_{\mathcal{H}}(\mathcal{G},W) \to W$ is given by

$$e_W(f) = f(1) \qquad \text{for } f \in \text{Hom}\,(\mathcal{G},W).$$

Thus, for $w \in W$,

$$\begin{aligned}(e_W \circ R\phi_W \circ i_W)(w) &= (e_W \circ R\phi_W)(w \otimes 1)\\ &= e_W(R\phi_W(w \otimes 1))\\ &= (R\phi_W(w \otimes 1))(1)\\ &= w.\gamma(1)\\ &= w.\gamma(e(1))\\ &= w.1\\ &= w\ .\end{aligned}$$

So condition (4) is satisfied.

By the proof of theorem (A.3), $IW = \text{im}\,\phi_W$ for $W \in \underline{\underline{H}}$. But, by inspection of the definitions of ϕ_W and $\hat{j}_W$,

$$\text{im}\,\phi_W = \text{im}\,\hat{j}_W.\mathcal{G}$$

$$\text{so} \qquad IW = \operatorname{im} \phi_W = \operatorname{im} \hat{j}_W . \mathcal{G} = \bar{I}W.$$

Thus the object functions of I and $\bar{I}$ coincide.

By lemma (A.2) and the proof of theorem (A.3), the morphism function of I is uniquely determined by the fact that if $W_1, W_2 \in$ Mod-$\mathcal{H}$ and $\psi \in \operatorname{Hom}_{\mathcal{H}}(W_1, W_2)$, then $I\psi$ is the unique $\mathcal{G}$-homomorphism which makes the diagram below commute:

$$\begin{tikzcd}
W_1 \otimes_{\mathcal{H}} \mathcal{G} \arrow[r, "\mu_{W_1}"] \arrow[d, "\psi \otimes \mathcal{G}"'] & IW_1 \arrow[r, "\nu_{W_1}"] \arrow[d, "I\psi"'] & \operatorname{Hom}_{\mathcal{H}}(\mathcal{G}, W_1) \arrow[d, "\operatorname{Hom}(\mathcal{G}, \psi)"] \\
W_2 \otimes_{\mathcal{H}} \mathcal{G} \arrow[r, "\mu_{W_2}"'] & IW_2 \arrow[r, "\nu_{W_2}"'] & \operatorname{Hom}_{\mathcal{H}}(\mathcal{G}, W_2)
\end{tikzcd}$$

where μ_{W_i} is the natural projection onto $\operatorname{im} \phi_{W_i} = IW_i$ and ν_{W_i} is the natural inclusion of $\operatorname{im} \phi_{W_i}$ into the codomain of ϕ_{W_i}, $i = 1,2$.

It is trivial to check that the diagram above actually does commute with $\bar{I}\psi$ in place of $I\psi$, hence $I\psi = \bar{I}\psi$. Thus $I = \bar{I}$.

Thus, by theorem (A.3), I is an injective weak double adjoint to R.

It is possible to calculate that j_W and k_W, as defined in this present section, coincide with the maps j_W and k_W which arise from ϕ_W in the proof of theorem (A.3).

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