Journal of Physics: Conference Series

PAPER • OPEN ACCESS

The method of PID controllers synthesis for sixth-order systems

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A P Prokopev, Zh I Nabizhanov, V I Ivanchura and R T Emelyanov

Siberian Federal University, 79, Svobodny ave., Krasnoyarsk, 660041, Russia

E-mail: prok1@yandex.ru

Abstract. The methodical of proportional-integral-derivative (PID) controllers designing for systems the sixth order are suggested. At the first stage of implementation of the method algorithm, the transfer function of the corrected closed control system is determined. Then you can specify the number and type of roots of this polynomial. The roots of the polynomial are set taking into account the root quality indicators. The next step of the algorithm determines the values of the parameters desired polynomial of the corrected closed system. These parameters will depend on the imaginary parts of complex-conjugate roots. Parameters of the polynomial must be positive It follows from the stability condition of the system. For a sixth-order model, the four higher-order parameters of this polynomial do not depend on the controller parameters and their values cannot be changed, since they are determined by calculations. Next, the coefficients of the PID controller model are calculated. You can set the condition for positive values of the regulator parameters. Checking the results of calculating the parameters of the controller is performed by the step response of the system. The efficiency of the proposed method of the designing PID controller is performed on the example of a system with a sixth order object.

1. Introduction

A proportional-integral-derivative controller are globally used throughout systems of automated control of technical processes and machines [1]. Difficulties emerges in the design of control systems of real objects.

We can find the best-known design methods of PID controllers from scientific sources [2-6]. There are 1731 synthesis methods of PID controllers proposed in the paper [6].

For example, usually mathematical models of work processes of road-building machines as control objects are described by differential equations and transfer function (TF) of second or higher order [7-9]. Designing methods of PID controller's synthesis for higher order objects is the vital task.

2. Initial mathematical formulation of system and methodologies

A linear automatic control system with a transfer function of a sixth-order object is investigated. The transfer function of the high-order control object $W_p(s)$ and the PID controller $W_c(s)$ have the form:

$$W_p(s) = \frac{b_m \cdot s^m + b_{m-1} \cdot s^{m-1} + \ldots + b_0}{a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + \ldots + a_0}, \tag{1}$$

$$W_c(s) = K_p + \frac{K_i}{s} + K_d \cdot s. \tag{2}$$

where s – Laplace operator; K_p, K_i, K_d – coefficients of, respectively, proportionality, integration and differentiation.

We considered the problem of PID controller parameters designing methods K_p, K_i, K_d of closed loop control that provides the specified quality indicators when managing a control object with a sixth order transfer function $W_p(s)$ by modal method for setting the values of the poles roots of the characteristic equation of a closed control system: the dominant real pole $s_1 = -\eta_1$ and complex poles

$$s_2 = -\eta_2 + j \cdot \beta_2, s_3 = -\eta_2 - j \cdot \beta_2, \ldots, \ldots,$$
$$s_n = -\eta_n + j \cdot \beta_n, s_n = -\eta_n - j \cdot \beta_n.$$

Design method (approach) of PID controller model includes the following steps (see below).

Step 1: Represent the transfer function of the control object $W_p(s)$ (1) to the following form (in the MathCAD)

$$W_p(s) := \frac{K_0}{\begin{pmatrix} a_0 \cdot s^6 + a_1 \cdot s^5 + a_2 \cdot s^4 \\ +a_3 \cdot s^3 + a_4 \cdot s^2 + a_5 \cdot s + a_6 \end{pmatrix}}, \tag{3}$$

given $a_0 = 1$.

Step 2: Determine the transfer function of a closed corrected system using the formulas (2) and (3):

$$K_p(s) = \frac{K_0 \cdot K_d \cdot s^2 + K_0 \cdot K_p \cdot s + K_0 \cdot K_i}{\begin{pmatrix} a_0 \cdot s^7 + a_1 \cdot s^6 + a_2 \cdot s^5 \\ +a_3 \cdot s^4 + a_4 \cdot s^3 + (a_5 + K_0 \cdot K_d) \cdot s^2 \\ +(a_6 + K_0 \cdot K_p) \cdot s + K_0 \cdot K_i \end{pmatrix}}. \tag{4}$$

From the transfer function (4) of the corrected closed system, it can be seen that the coefficients of the PID controller affect the last three terms of the characteristic polynomial of this system. The remaining coefficients of this polynomial do not depend of the controller parameters.

Step 3: Set the number and type of roots of this polynomial.

Options for specifying the number and type of the characteristic polynomial roots of the corrected closed system can be different. This determines the research of the proposed method for designing the PID controller of the objects the sixth order.

Polynomial roots are set taking into account the root quality indicators.

For research study, the following task option is accepted:

one real root and three pairs of complex-conjugate roots of the characteristic polynomial of the corrected closed system

$$s_1 = -\alpha_1,\ s_2 = -\alpha_2 + j \cdot \beta_1,\ s_3 = -\alpha_2 - j \cdot \beta_1,$$
$$s_4 = -\alpha_3 + j \cdot \beta_2,\ s_5 = -\alpha_3 - j \cdot \beta_2,$$
$$s_6 = -\alpha_4 + j \cdot \beta_3,\ s_7 = -\alpha_4 - j \cdot \beta_3.$$

Studies have shown that the coefficient a_1 of the transfer function (4) will be determined only by the real parts of these roots. For a sixth-order object model (n = 6), the characteristic polynomial of the corrected closed system will have an order by one unit more, i.e. $n_p = n + 1$. For the considered variant of setting the roots, we get the expression

$$a_1 = (\alpha_1 + 2\cdot\alpha_2 + 2\cdot\alpha_3 + 2\cdot\alpha_4). \tag{5}$$

Formula (5) will determine the choice of values for these real parts. For example, set α_1, α_2, α_3 and α_4 is defined from the expression (5), i.e.

$$\alpha_4 := \frac{a_1 - (\alpha_1 + 2\cdot\alpha_2 + 2\cdot\alpha_3)}{2}.$$

Step 4: Determine the values of the desired polynomial coefficients of the corrected closed system when the values of the characteristic polynomial roots of this system are found. These coefficients will depend on the imaginary parts of complex-conjugate roots, where i = 1, 2, 3.

From the structure of the characteristic polynomial of the corrected closed system, it follows that the last three coefficients depend of the PID controller parameters, i.e.

$$D(s) := a_0\cdot s^7 + a_1\cdot s^6 + a_2\cdot s^5 + a_3\cdot s^4 + a_4\cdot s^3 + (a_5 + K_0\cdot K_d)\cdot s^2 + (a_6 + K_0\cdot K_p)\cdot s + K_0\cdot K_i. \tag{6}$$

We obtain an expression for the desired characteristic polynomial $D_p(s)$ of the corrected system

$$D_p(s) = a_{0p}\cdot s^7 + a_{1p}\cdot s^6 + a_{2p}\cdot s^5 + a_{3p}\cdot s^4 + a_{4p}\cdot s^3 + a_{5p}\cdot s^2 + a_{6p}\cdot s + a_{7p}. \tag{7}$$

In polynomial (7), the coefficients a_{5p}, a_{6p}, a_{7p} are defined by the following expressions

$$\begin{gathered} a_{5p} := a_5 + K_0\cdot K_d,\ a_{6p} := a_6 + K_0\cdot K_p, \\ a_{7p} := K_0\cdot K_i. \end{gathered} \tag{8}$$

Moreover, the coefficients of the desired characteristic polynomial of the corrected system a_{1p}, a_{2p}, a_{3p}, a_{4p} don’t depend on the regulator parameters and their values can’t be changed, because they are determined by calculations.

Construction of the characteristic polynomial corrected closed system expressed in terms of real and complex roots of the characteristic polynomial.

Using the *collect* function for reducing such terms (in the MathCAD), we construct a characteristic polynomial of the corrected closed system, expressed in terms of the real and complex poles

$$\begin{gathered}\left[s^3 + (\alpha_1 + 2\cdot\alpha_2)\cdot s^2 + (\alpha_2^2 + 2\cdot\alpha_1\cdot\alpha_2 + \beta_1^2)\cdot s + \alpha_1\cdot(\alpha_2^2 + \beta_1^2)\right]\cdot(s^2 + 2\cdot\alpha_3\cdot s + \alpha_3^2 + \beta_2^2) \\ \times(s^2 + 2\cdot\alpha_4\cdot s + \alpha_4^2 + \beta_3^2)\ \text{collect}, s \rightarrow \ldots \end{gathered} \tag{9}$$

Characteristic polynomial coefficients of the corrected system

$$a_1 := \alpha_1 + 2\cdot\alpha_2 + 2\cdot\alpha_3 + 2\cdot\alpha_4,$$

$$a_2 := \alpha_2^2 + \beta_1^2 + \alpha_3^2 + \beta_2^2 + \alpha_4^2 + \beta_3^2 + 2\cdot\alpha_1\cdot\alpha_2 + 4\cdot\alpha_3\cdot\alpha_4 + \left(\alpha_1 + 2\cdot\alpha_2\right) + \left(2\cdot\alpha_3 + 2\cdot\alpha_4\right),$$

$$\begin{aligned} a_3 := {} & \left(\alpha_1 + 2\cdot\alpha_2\right)\cdot\left(\alpha_3^2 + 4\cdot\alpha_3\cdot\alpha_4 + \beta_2^2 + \alpha_4^2 + \beta_3^2\right) + \alpha_1\cdot\left(\alpha_2^2 + \beta_1^2\right) + 2\cdot\alpha_4\cdot\left(\alpha_3^2 + \beta_2^2\right) \\ & + 2\cdot\alpha_3\cdot\left(\alpha_4^2 + \beta_3^2\right) + \left(2\cdot\alpha_3 + 2\cdot\alpha_4\right)\cdot\left(\alpha_2^2 + 2\cdot\alpha_1\cdot\alpha_2 + \beta_1^2\right), \end{aligned}$$

$$\begin{aligned} a_4 := {} & \left(\alpha_2^2 + 2\cdot\alpha_1\cdot\alpha_2 + \beta_1^2\right)\cdot\left(\alpha_3^2 + 4\cdot\alpha_3\cdot\alpha_4 + \beta_2^2 + \alpha_4^2 + \beta_3^2\right) + \left(\alpha_3^2 + \beta_2^2\right)\cdot\left(\alpha_4^2 + \beta_3^2\right) \\ & + \left(\alpha_1 + 2\cdot\alpha_2\right)\cdot\left[2\cdot\alpha_4\cdot\left(\alpha_3^2 + \beta_2^2\right) + 2\cdot\alpha_3\cdot\left(\alpha_4^2 + \beta_3^2\right)\right] \\ & + \alpha_1\cdot\left(2\cdot\alpha_3 + 2\cdot\alpha_4\right)\cdot\left(\alpha_2^2 + \beta_1^2\right), \end{aligned}$$

$$\begin{aligned} a_5 := {} & \left[2\cdot\alpha_4\cdot\left(\alpha_3^2 + \beta_2^2\right) + 2\cdot\alpha_3\cdot\left(\alpha_4^2 + \beta_3^2\right)\right]\cdot\left(\alpha_2^2 + 2\cdot\alpha_1\cdot\alpha_2 + \beta_1^2\right) + \alpha_1\cdot\left(\alpha_2^2 + \beta_1^2\right) \\ & \times\left(\alpha_3^2 + 4\cdot\alpha_3\cdot\alpha_4 + \beta_2^2 + \alpha_4^2 + \beta_3^2\right) + \left(\alpha_1 + 2\cdot\alpha_2\right)\cdot\left(\alpha_3^2 + \beta_2^2\right)\cdot\left(\alpha_4^2 + \beta_3^2\right), \end{aligned}$$

$$\begin{aligned} a_6 := {} & \left(\alpha_3^2 + \beta_2^2\right)\cdot\left(\alpha_4^2 + \beta_3^2\right)\cdot\left(\alpha_2^2 + 2\cdot\alpha_1\cdot\alpha_2 + \beta_1^2\right) \\ & + \alpha_1\cdot\left[2\cdot\alpha_4\cdot\left(\alpha_3^2 + \beta_2^2\right) + 2\cdot\alpha_3\cdot\left(\alpha_4^2 + \beta_3^2\right)\right]\cdot\left(\alpha_2^2 + \beta_1^2\right), \end{aligned}$$

$$a_7 := \alpha_1\cdot\left(\alpha_2^2 + \beta_1^2\right)\cdot\left(\alpha_3^2 + \beta_2^2\right)\cdot\left(\alpha_4^2 + \beta_3^2\right).$$

We introduce notations taking into account the structure of characteristic polynomials (6) – (9), expressed in terms of real and imaginary parts of the poles.

It follows from the stability condition of the system that the coefficients of the characteristic polynomial (7) must be positive. Moreover, the coefficients $a_{1p}, a_{2p}, a_{3p}, a_{4p}$ of this polynomial do not depend on the regulator parameters and their values can't be changed, because they are determined by calculations.

Conditions must be fulfilled

$$a_{1p}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = a_1,\ a_{2p}(\beta_1, \beta_2, \beta_3) = a_2,\ a_{3p}(\beta_1, \beta_2, \beta_3) = a_3,\ a_{4p}(\beta_1, \beta_2, \beta_3) = a_4.$$

In addition, the conditions determined by the requirement of positive parameters of the PID controller must be fulfilled

$$a_{5p} > a_5 + K_0 \cdot K_d\,, \tag{10}$$

$$a_{6p} > a_6 + K_0 \cdot K_p\,. \tag{11}$$

Conditions (10) and (11) allow us to determine the imaginary parts of complex roots $\beta_1, \beta_2, \beta_3$ and the desired coefficients of the characteristic polynomial a_{5p}, a_{6p}, a_{7p}.

Step 5: The parameters calculate of the PID controller model.

PID controller parameters are determined from the expressions (8):

$$K_d := \frac{a_{5p} - a_5}{K_0}, K_p := \frac{a_{6p} - a_6}{K_0}, K_i := \frac{a_{7p}}{K_0}. \tag{12}$$

Step 6: Check of results of calculating the parameters of the PID controller model on the step response.

The found values of the characteristic polynomial roots allow us to determine the numerical values of their coefficients.

3. Example of parameters synthesis of a system PID controller model with a sixth-order object

As an example, of the PID controller designing for the sixth-order object is considered. Formulas are presented using the MathCAD.

A sixth order control object is set $W_p(s)$ in form (1), where a_1:=29, a_2:=328, a_3:=2025, a_4:=8225, a_5:=19000, a_6:=12500, K_0:=42500.

Calculations were performed using the proposed method.

The simulation model of the corrected control system in the MATLAB&Simulink environment for design the transient characteristic is shown in figure 1.

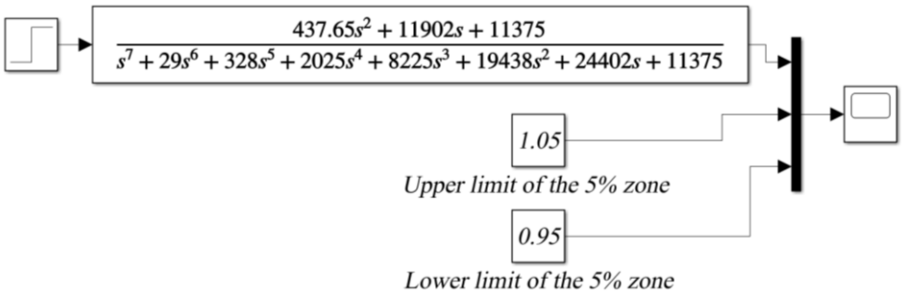

Figure 1. Control system model in MATLAB/Simulink.

The result of designing the transient characteristic $h(t)$ in the MATLAB/Simulink environment is shown in figure 2. Dotted lines in figure 2 shows the values of the five-percent border of the zone – 0.95; 1.05.

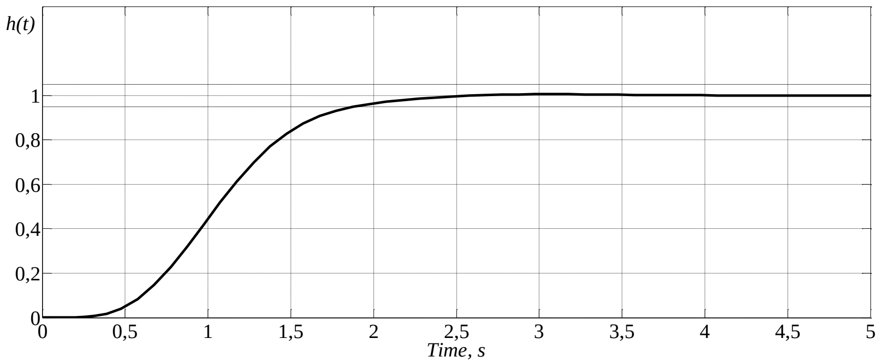

Figure 2. Step response of a feedback control system.

Expected response time is determined by the known [10, 11, 12] approximate formula

$$t_p \approx 3/\eta,$$

where η – stability index, the degree of stability, defined as the minimum value of the real part of the root $\eta = |\alpha|_{min}$.

For this example, the expected response time is $t_{p0} = 3/\min(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = 3s$. The response time was $t_p = 1.89\ s$, overshoot was $\sigma\% < 1\%$. There is a delay in the transient characteristic $\tau = 0.7\ s$.

4. Conclusion

Proposed a method for designing the parameters of the PID controller, which provides quality indicators for system in terms of response time, overshoot, and the type of transfer process at a given location of the poles, with a sixth-order object. The advantage of the proposed algorithm over methods based on restrictions on stability reserves is that the poles of the system are directly related to direct estimates of the transfer a quality, such as time response and overshoot.

The following conditions must be taken into account when parametric designing of the PID controllers using the proposed method:

real values must be set $\alpha_1,\alpha_2,\alpha_3$ and calculate α_4 the poles of the corrected system using the coefficient a_1 of the corrected system $a_1 = (\alpha_1 + 2\cdot\alpha_2 + 2\cdot\alpha_3 + 2\cdot\alpha_4)$ and the restrictions determined by the coefficients a_5, a_6: $a_{5p} > a_5 + K_0 \cdot K_d$; $a_{6p} > a_6 + K_0 \cdot K_p$ that affect the values of the imaginary parts β_1, β_2, β_3 of poles;

there is no arbitrary value assignment of coefficients a_5, a_6, which is possible only if the order of the corrected system is $n = 2$;

the minimum response time is limited.

Acknowledgment

This project was supported by the Russian Foundation for Basic Research (grant no. 19-37-90052).

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