SOME FIRST RESULTS FOR NONCOOPERATIVE PREGAMES: SOCIAL CONFORMITY AND EQUILIBRIUM IN PURE STRATEGIES

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Some First Results for Noncooperative Pregames:
Social Conformity and Equilibrium in Pure Strategies[*]

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Abstract: We introduce the framework of noncooperative pregames and demonstrate that for all games with sufficiently many players, there exist approximate (ε) Nash equilibria in pure strategies. In fact, every mixed strategy equilibrium can be used to construct an ε-equilibrium in pure strategies – ours is an 'ε-purification' result. Our main result is that there exists an ε-equilibrium in pure strategies with the property that most players choose the same strategies as all other players with similar attributes. More precisely, there is an integer L, depending on ε but not on the number of players, so that any sufficiently large society can be partitioned into fewer than L groups, or cultures, consisting of similar players, and all players in the same group play the same pure strategy. In ongoing research, we are extending the model to cover a broader class of situations, including incomplete information.

We would be grateful for any comments that might help us improve the paper.

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1 Introduction: Learning from similar individuals.

A society or culture is a group of individuals who have commonalities of language, social and behavioural norms, and customs. Social learning consists, at least in part, in learning the norms and behaviour patterns of the society into which one is born and in those other societies which one may join – our professional associations, our workplace, and our community, for example. Social learning may also include learning a set of skills from others that will enable us to "fit into the society." The society in question may be broad as "Western civilization" or Canada, or as small as the Econometric Society. If most people observe "similar" people and learn by mimicking other individuals, then a stable society depends on the existence of an equilibrium where most individuals who are similar choose the same strategies. If most individuals learn from and mimic similar individuals, then the existence of such equilibria is important; indeed, it is fundamental to the social sciences.

To ask whether equilibria where most players choose the same strategies as similar players exist, we must first have an appropriate model. One of the main contributions of the current paper is the introduction of a noncooperative counterpart to the pregame framework of cooperative game theory.[1] In cooperative game theory this framework has led to a number of results, especially results showing that large games with small effective groups resemble, or in fact *are,* competitive economies. It appears that our framework of noncooperative games may be equally useful. In this paper, we demonstrate, for all games with sufficiently many players, existence of ε-equilibria in pure strategies and, with a more restrictive space of player types, existence of ε-equilibria in pure strategies where almost all players (that is, all except at most some bounded and finite number) choose the same strategy as all sufficiently similar players.

As in the cooperative pregame framework, we take as given a set of attributes of players; here, these attributes index payoff functions. We require two anonymity assumptions. The first is that payoffs do not depend on the identities of other players, only on their attributes and, of course, on the strategies they choose. The second is that when there are many players (but still a finite number), then the actions of a small subset of players do not significantly affect payoffs of members of the complementary player set. We

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[1]See, for example, Wooders (1979,1983,1994) and Wooders and Zame (1984).

require also some continuity conditions and demonstrate two main results:

Theorem 1: Existence. Given $\varepsilon > 0$ there exists an integer $\eta(\varepsilon)$ with the property that every game with at least $\eta(\varepsilon)$ players has an ε-equilibrium in pure strategies.

Theorem 2: Social conformity. Assume that the set of player attributes for any fixed society can be parameterized by an index set satisfying a property of 'convex separation'. Then, given $\varepsilon > 0$, there is an integer $L(\varepsilon)$ such that the ε-equilibrium can be chosen so that for some partition of Ω into fewer than L subsets, $\{\Omega^\ell\}_{\ell=1}^L$, all players with attributes represented by points in the same convex set Ω^ℓ play the same strategy.

The integer $L(\varepsilon)$ is a measure of social conformity; the smaller $L(\varepsilon)$ the greater the extent to which players who are dissimilar in their attributes can conform in their choice of strategy.[2] Thus, it is an important feature of Theorem 2 that, given ε, the integer $L(\varepsilon)$ is fixed, independently of the numbers of players. Roughly, a space is said to satisfy convex separation if there is some ordering on the space with the property that if all points in one set are "less than" all points in another set, then the convex hulls of both sets of points are disjoint. To illustrate the nonrestrictiveness of convex separation, we note here that it is a a property of finite dimensional Euclidean space, as we will prove.

An important aspect of Theorem 2 is indexing of attributes. An attribute is a complete description of the possible characteristics of a player, such as gender, height, IQ, personality, and the payoff function for any possible society in which a player with those characteristics appears. Thus, we must address the question of what classes of attributes satisfy convex separation. It would not be very interesting, for example, if all players had to have payoff functions that differed only by monotonic transformations. In fact, it appears that the indexing problem allows a reasonably rich class of attributes. We demonstrate, for example, that player's attributes may consist of his attributes in some space of characteristics, such as intelligence, educational level, height, metabolic rate, eye color, personality and so on, and payoff functions satisfying the property that the payoff to mixed strategies is the expectation of payoffs to pure strategies.

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[2]We are grateful to Roland Benabou for suggesting we emphasize this aspect of Theorem 2.

To compare our noncooperative pregame framework to the cooperative pregame framework, it is important to note a major and significant difference. In the cooperative framework, the payoff to a coalition is fixed and independent of the society in which that coalition is embedded. Although this is possible within the current noncooperative framework, it is not built into the model and thus may or may not hold. Noncooperative games derived from a (noncooperative) pregame are parameterized by the numbers of players of each type in the player set and may vary considerably depending on the attributes of the players actually represented in the society. For example, there may be little relationship between derived games where all players have the same attribute, male for example, and games where some players have a different attribute, female, for example. Moreover, even in the case where all players are identical, there is no necessary relationship between a game with n players and another with $n + 1$ players. That said, however, it should be noted that asymptotically, only the distribution of players' attributes matters; that is, the games become anonymous. Just to be sure this is clear, in games with many players, the percentages of males and females are still relevant, but whether a male is labelled i or j is irrelevant, and a few males or females, more or less, are both of no great consequence.

One interesting similarity between the two frameworks is that, in the cooperative pregame framework, the condition of *small group effectiveness* plays an important role, cf., Wooders (1994).[3] This condition dictates that all or almost all gains to collective activities can be realized by groups of players bounded in size. An *equivalent* condition is that *small groups are negligible*: in large cooperative games derived from pregames, small groups are effective if and only if small groups cannot have significant effects on aggregate per capita payoff (Wooders 1993). The main substantive condition of the current paper can be interpreted as the negligibility of small groups of players; that is, the effects of the actions of any small set of players on the complementary set of players become negligible in games with many players. For cooperative pregames with side payments, the condition of small group negligibility implies that large games are market games, as defined by Shapley and Shubik (1969). The full implications of the condition for noncooperative pregames have not been fully explored but we expect there to be many.

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[3]A strong form of this condition, a sort of *strict* small group effectiveness was originally introduced in Wooders (1979) and earlier versions of that paper. For our pruposes here, this form of small group effectiveness is not useful. Neither is the 'boundedness of marginal contributions' of Wooders and Zame (1984).

Our first result is actually a "purification result," showing that, for all sufficiently large games, every mixed strategy equilibrium generates a pure strategy ε-equilibrium. Our result differs from a number of purification results in the literature in that prior papers all have a continuum of players (cf. Schmeidler 1973, Mas-Colell 1984, Khan 1989, Pascoa 1993,1998, Khan et al. 1997, Araujo and Pascoa 2000). With a continuum of players, small group negligibility is *built into* the framework and thus does not appear as a separate assumption. Of course for a number of these results, it is easy to see that one could consider a sequence of large finite games with player distribution converging to the distribution of player types in the given continuum, and from the results for the continuum, establish existence of ε-equilibrium in pure strategies for all sufficiently large games in the sequence. Our results differ in that we are not restricted to one limiting distribution of player types; our results hold for all sufficiently large games derived from a non-cooperative pregame. In particular, our results allow for the possibility that there are player types who appear in arbitrarily small percentages in large finite games. An important part of our work is defining the model of non-cooperative pregames and establishing that a set of conditions, especially small group negligibility, that allow us to obtain our purification result. We note that related results have recently been obtained by Kalai (2000); these are discussed further later.

2   The Model. Noncooperative Pregames.

We first introduce the concept of a society, then that of strategies and the set of 'weight functions' derived from the set of strategy vectors. We conclude by introducing the game corresponding to a society.

2.1   Societies.

We assume a compact metric space Ω of player types. Let N be a finite set and let α be a mapping from N to Ω, called an *attribute function*. The pair (N, α) is a *society*.

Let $\mathcal{Z}_+$ denote the set of non-negative integers. The *profile of a society* (N, α) is a function $\rho(N, \alpha) : \Omega \to \mathcal{Z}_+$ given by

$$\rho(N, \alpha)(\omega) = \left| \alpha^{-1}(\omega) \right|$$

Thus, the profile of a player set tells us the number of players with each attribute in the set. Let $support(\rho(N, \alpha))$ denote the support of the function $\rho(N, \alpha)$, that is,

$$support(\rho(N, \alpha)) = \{\omega \in \Omega : \rho(N, \alpha)(\omega) \neq 0\}.$$

Let $P(\Omega)$ denote the set of all functions from Ω to $\mathcal{Z}_+$ with finite support. Note that for each possible society (N, α) the profile of N is in $P(\Omega)$. Note also that the sum of profiles (defined pointwise) is also a profile.

Before introducing the game corresponding to any society, we require some preliminary concepts.

Let $S = \{s_1, ..., s_K\}$ be a finite set of pure strategies. Let $\Delta(S)$ denote the set of mixed strategies. In each game $\Gamma(N, \alpha)$, each player will have the strategy choice set $\Delta(S)$. The support of a mixed strategy σ_i is denoted by $support(\sigma_i)$, where "support" is defined as above. A mixed strategy is called *pure* if it puts unit weight on a single pure strategy.

A *strategy vector* is given by $\sigma = (\sigma_1, ..., \sigma_{|N|}) \in \times_{i \in N} \Delta(S)$ where σ_i denotes the strategy of player i. Let σ_{ik} denote the probability with which player i plays pure strategy s_k. We denote the set of all strategy vectors by Σ. A strategy vector σ is called *degenerate* if for each i, for some k, $\sigma_{ik} = 1$; that is, each player's strategy is a pure strategy.

Given an attribute function α (or a profile $\rho(N, \alpha)$) we define a *weight function* $w(\cdot, \cdot; \alpha)$ (or $w(\cdot, \cdot; \rho(N, \alpha))$) as a mapping from $\Omega \times S$ into $\mathcal{R}_+$ satisfying the conditions that

$$\sum_k w(\omega, s_k; \alpha) = \rho(N, \alpha)(\omega).$$

Thus, given an attribute function α, a weight function is an assignment of a non-negative real number to each attribute-strategy pair (ω, s) so that the sum, over strategies, of the weights attached to the pairs (ω, s) equals the number of players with that attribute. It follows that

$$\sum_k \sum_{\omega \in support(\rho(N, \alpha))} w(\omega, s_k; \alpha) = |N|.$$

It is convenient to also define weight functions relative to strategy vectors. Given an attribute function α and a strategy vector σ, define a *weight function* $w(\cdot, \cdot; \alpha, \sigma)$ *relative to* α *and* σ by:

$$w(\omega, s_k; \alpha, \sigma) = \sum_{i \in N : \alpha(i) = \omega} \sigma_i(s_k)$$

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for each $s_k \in S$ and for each $\omega \in \Omega$. We interpret $w(\omega, s; \alpha, \sigma)$, as showing, for each $\omega \in \Omega$, the 'weight' given to pure strategy s_k in the strategy vector σ by players assigned type ω by α. That is, given the society (N, α) and the strategy vector σ, $\frac{w(\omega, s_k; \alpha, \sigma)}{p(N, \alpha)(\omega)}$ is the expected proportion of times strategy s_k will be played by the average player of type ω. It is immediate that a weight function relative to an attribute function α and a strategy vector σ is a weight function, as defined above. In particular

$$\sum_k w(\omega, s_k; \alpha, \sigma) = p(N, \alpha)(\omega)$$

for each $\omega \in \Omega$. Note that there may be many strategy vectors that generate a given weight function relative to an attribute function. Let W_α denote the set of all possible weight functions for the society (N, α).

For a strategy $p \in \Delta(S)$ and attribute $\omega_0 \in \Omega$ we denote by $\chi(\cdot, \cdot; \omega_0, p)$ the *individual weight function.*

$$\chi(\omega, s_k; \omega_0, p) \quad = \quad p_k \text{ if } \omega = \omega_0 \text{ and}$$

$$\chi(\omega, s_k; \omega_0, m) \quad = \quad 0 \text{ otherwise.}$$

for each $s_k \in S$, where p_k denotes the probability pure strategy $s_k \in S$ is played, given the mixed strategy p.

Given an attribute function α and a strategy vector $\sigma \in \Sigma$ with corresponding weight function $w(\cdot, \cdot; \alpha, \sigma) \in W_\alpha$, let $w_{-i}(\cdot, \cdot; \alpha, \sigma)$ denote the weight function in which player i's contribution is not included. That is,

$$w_{-i}(\omega, s; \alpha, \sigma) = w(\omega, s; \alpha, \sigma) - \chi_i(\omega, s; \omega_0, \sigma_i)$$

for all $\omega \in \Omega$, all $s \in S$ and for all $i \in N$, where $\alpha(i) = \omega_0$ and $\chi_i(\cdot, \cdot; \omega_0, \sigma_i)$ is the individual weight function of player i given attribute function α and strategy vector σ.

Given (N, α) and $i \in N$ let $W_{\alpha - \chi_i}$ denote the set of weight functions for the society $(N \backslash \{i\}, \alpha \big|_{N \backslash \{i\}})$.

2.2 The games $\Gamma(N, \alpha)$.

With the above definitions in place, we can now define the games $\Gamma(N, \alpha)$. For any society (N, α), the game $\Gamma(N, \alpha)$ is given by the pair (S, H) where

S is the finite set of strategies and

$$H = \{h_\omega(\cdot, \cdot; \rho(N, \alpha)),\ \omega \in\ support\rho(N, \alpha)\}$$

where, for each $i \in N$, $h_{\alpha(i)}(\cdot, \cdot; \rho(N, \alpha))$ is a given *payoff function* mapping $\Delta S \times W_{\alpha-\chi_i}$ into $\mathcal{R}_+$.

Given an attribute function α, a strategy vector σ and the corresponding weight function $w(\cdot, \cdot; \alpha, \sigma) \in W_\alpha$, the payoff of player $i \in N$ is given by

$$h_{\alpha(i)}(\sigma_i, w_{-i}(\omega, s_k; \alpha, \sigma), \rho(N, \alpha)) \in \mathcal{R}_+. \tag{1}$$

The interpretation is that $h_\omega(\sigma_i, w_{-i}(\cdot, \cdot; \alpha, \sigma), \rho(N, \alpha))$ is the payoff to a player $i \in N$ with $\alpha(i) = \omega$, in the game $\Gamma(N, \alpha)$, from playing the (possibly) mixed strategy σ_i when the strategy choices of the remaining players are represented by $w_{-i}(\cdot, \cdot; \alpha, \sigma)$. Note that payoff functions are parameterized by the population profile $\rho(N, \alpha)$ since different population profiles correspond to different games. We make the standard assumption that the payoff to a mixed strategy is the expected payoff from pure strategies, that is,

$$h_\omega(p, w_{-i}(\cdot, \cdot; \alpha), \rho(N, \alpha)) = \sum_k p_k h_\omega(s_k, w_{-i}(\cdot, \cdot; \alpha), \rho(N, \alpha)). \tag{2}$$

Note that implicit in the definition of the payoff function there is an anonymity assumption. For example, consider two players $i, j \in N$, where $a(i) = \alpha(j)$, and two alternative scenarios. In the first scenario player i plays pure strategy s_1 and player j plays pure strategy s_2. In the second scenario, roles are reversed so that player i plays s_2 and player j plays s_1. Then, assuming everything else remains the same, the payoff to a third player $i' \in N$ is indifferent to this switch between i and j. This example is a special case of a continuity assumption (continuity 1) below.

The standard definition of a Nash equilibrium applies. A strategy vector σ is a *Nash equilibrium* only if, for each $i \in N$ and for each pure strategy $s_k \in support(\sigma_i)$, it holds that

$$h_{\alpha(i)}(s_k, w_{-i}(\cdot, \cdot; \alpha, \sigma), \rho(N, \alpha)) \geq h_{\alpha(i)}(t, w_{-i}(\cdot, \cdot; \alpha, \sigma), \rho(N, \alpha))\ \text{for all}\ t \in S.$$

2.3 Large anonymous games

We now introduce the following assumptions about growing sequences of games which together constitute a *large anonymous game property*. First,

without loss of generality we can suppose that the furthest distance between any two points in Ω is less than one. For ease in notation, since any function f with finite support from Ω to Z_+ completely describes the profile of some society (N, α), where $f(\omega) = |\alpha^{-1}(\omega)|$, we will refer to such a function as a profile. Define $\|f\| = \sum_{\substack{\omega \in \Omega \\ f(\omega) \neq 0}} f(\omega)$ and note that when $f = \rho(N, \alpha)$, then $\|f\| = |N|$.

Observe that a profile f induces a probability measure $\frac{|1|}{\|f\|} f$ on Ω where each singleton set $\{\omega\}$ is assigned the probability

$$\frac{|f(\omega)|}{\|f\|};$$

let us call this probability measure $\mu(f)$. Similarly, for a given attribute function α, a weight function w induces a probability measure $\frac{|\omega(\omega, s_k; \alpha)|}{\|w\|} = \mu(w(\cdot, \cdot; \alpha))$ on $\Omega \times S$ where probability of a singleton set $\{(\omega, s_k)\}$ is $\frac{|\omega(\omega, s_k; \alpha)|}{\|w\|}$ and $\|w\| = \sum_k \sum_{\substack{\omega \in \Omega \\ w(\omega) \neq 0}} w(\omega, s_k)$.

To define a metric between societies (N, α) and (N', α'), we consider two cases, $|N| = |N'|$ and then $|N| \neq |N'|$. For the first case, label the points in N by $1, ..., |N|$ and those in N' by $1', ..., |N'|$ so that $d((N, \alpha), (N', \alpha')) = \sup dist(\alpha(i), \alpha(i'))$ is minimized, where $dist$ denotes the metric on Ω. In the second case, $|N| \neq |N'|$ define $d((N, \alpha), (N', \alpha')) = |N - N'|$. Then d is a well-defined metric. In fact, when $|N| = |N'|$, d corresponds to the Prohorov metric.[4]

Throughout the following, let $\{N^\nu\}$ be a sequence of player sets with $|N^\nu|$ becoming large as ν becomes large and let $\{\alpha^\nu\}$ be a sequence of attribute functions, $\alpha^\nu \colon N^\nu \to \Omega$. For ease in notation, let $\{f^\nu\}$ be the sequence of profiles where, for each ν, we have $f^\nu = \rho(N^\nu, \alpha^\nu)$.

We give two variants of a continuity property - the second implies the first. These continuity properties are both formulated as Lipschitz conditions on large games and are with respect to changes in attributes of players. Both conditions dictate that if we change attributes of players in large player sets only slightly, then for any given strategy vector, the change in payoffs of players is small. The second continuity condition states that in addition, if we change the attribute of a player only slightly then the change in his own payoff is small.

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[4]See, for example, Kirman in the *Handbook of Mathematical Economics*, pages 197-198.

Continuity with respect to attributes: Given $\varepsilon > 0$ there exists a *similarity parameter $\delta(\varepsilon)$* such that:

for any sequence of attribute functions $\{\overline{\alpha}^\nu\}$, where $\overline{\alpha}^\nu : N^\nu \to \Omega$ and satisfies:

$$dist(\alpha^\nu(i), \overline{\alpha}^\nu(i)) < \delta(\varepsilon) \text{ for all } i \in N^\nu \text{ and for all } \nu,$$

and for any sequence of strategy vectors $\{\sigma^\nu\}$, weight functions $\{w^\nu(\cdot, \cdot; \alpha^\nu, \sigma^\nu)\}$ and $\{w^\nu(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu)\}$, and profiles $\{\overline{f}^\nu\}$ where $\overline{f}^\nu = \rho(N^\nu, \overline{\alpha}^\nu)$.:

Continuity 1 (with respect to the attributes of others):

$$\lim_{\nu \to \infty} \left| h_{\alpha^\nu(i)}(\sigma_i, w_{-i}^\nu(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) - h_{\alpha^\nu(i)}(\sigma_i, w_{-i}^\nu(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu), \overline{f}^\nu) \right| < \varepsilon$$

for all $\sigma_i \in \Delta(S)$ and for all i with $\alpha^\nu(i) = \overline{\alpha}^\nu(i)$.
(Note that since a player's payoff function is parameterized by the profile of the society in which he's a player, it is indeed possible that his payoff changes when the attributes of other players change.[5] But although the payoff functions of the players have been changed – for each player j, $\alpha^\nu(j)$ changes to $\overline{\alpha}^\nu(j)$– the *actions* of the players remain unchanged. This is possible since a strategy vector lists a strategy for each player $i \in N$ and the N remains unchanged – only the payoff functions of the players in N have possibly changed, not the set of players nor their strategies. Thus, if one finds it reasonable that the payoff functions of players are affected only by the *actions* of others and not by their payoff functions, then this form of continuity is very mild.[6])

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[5] In standard models of economies, the preferences of players are taken as independent of the preferences of other players. Thus, this assumption would be immediately satisifed by such models.

[6] There are situations where individuals claim to be affected by the feelings, loyalties or thoughts of others, independent of their actions. In Arthur Miller's celebrated book, *The Crucible*, Rachel has been a pious woman, known for her good deeds and kind works, all through her long life. But the witch hunters of Salem interpreted Rachel's apparent goodness as just a clever disguise to hide her love of the devil. Rachel was put to death as a witch; for witch hunters, the private feelings of others and their thoughts are significant.

Continuity 2 (with respect to all attributes):

$$\lim_{\nu \to \infty} \left| h_{\alpha^\nu(i)}(\sigma_i, w^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(\sigma_i, w^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu), \overline{f}^\nu) \right| < \varepsilon$$

for all $\sigma_i \in \Delta(S)$ and all i.

In continuity 1, we essentially only consider the changes in payoffs, from perturbing the attribute function, to players who keep the same attribute type in both societies (N^ν, α^ν) and $(N^\nu, \overline{\alpha}^\nu)$. In continuity 2, we consider the change in payoffs to players who themselves also have their attribute type slightly perturbed between societies (N^ν, α^ν) and $(N^\nu, \overline{\alpha}^\nu)$ and impose continuity. In a later section, we will discuss attributes more fully.

Societies (N, α) and individual games $\Gamma(N, \alpha)$ derived from noncooperative pregames have an anonymity property as noted previously; the game $\Gamma(N, \alpha)$ and the payoff $h_{\alpha(i)}(\cdot, \cdot, \rho(N, \alpha))$ to an individual player $i \in N$ do not depend on the names of other players, only on the profile of the player set. To obtain our results we require further conditions on payoff functions as the numbers of players in the games becomes large. For our current results, we use the strong anonymity condition below. Note that this condition, along with continuity, can also be viewed as a *small group negligibility* assumption. In brief, strong anonymity ensures that the actions of near-negligible sets of players do not significantly change the payoffs of the complementary set.[7]

Strong anonymity: Assume that for some finite set of points $\{\omega_1, ..., \omega_J\}$ in Ω, $\{\omega_1, ..., \omega_J\} = support(f^\nu)$ for each ν. Also suppose that for each $\omega_j \in \{\omega_1, ..., \omega_J\}$, $\frac{f^\nu(\omega_j)}{\|f^\nu\|}$ converges as $\nu \to \infty$. Let $\{w^\nu(\cdot, \cdot; \alpha^\nu)\}$ and $\{g^\nu(\cdot, \cdot; \alpha^\nu)\}$ be sequences of weight functions where w^ν and, respectively, g^ν are relative to attribute function α^ν.

If, for each pure strategy $s_k \in S$, for some real numbers $\{\theta_{jk} : j = 1, ..., J\}$ it holds that:

$$\lim_{\nu \to \infty} \frac{w^\nu(\omega_j, s_k; \alpha^\nu)}{\|w^\nu\|} = \lim_{\nu \to \infty} \frac{g^\nu(\omega_j, s_k; \alpha^\nu)}{\|g^\nu\|} = \theta_{jk},^8 \qquad (3)$$

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[7] It is interesting that, in cooperative pregames, the condition of small group negligibility is equivalent to the condition of small group effectiveness; see Wooders (2001).

then:

$$\lim_{\nu \to \infty} h_{\alpha^\nu(i)}(p, w_{-i}^\nu, f^\nu) = \lim_{\nu \to \infty} h_{\alpha^\nu(i)}(p, g_{-i}^\nu, f^\nu)$$

for all $p \in \Delta(S)$ and all i.

If preferences satisfy strong anonymity then, for games with many players, the payoff to a player depends only on his own strategy choice and the *proportion*, relative to the total population of players, of each type playing each strategy.

Note that in the definition of strong anonymity, for each ν, it holds that $\{\omega_1, ..., \omega_J\} \subset support(f^\nu)$ but in general it may be that the supports of the functions f^ν become infinite as ν becomes large. The definition of strong anonymity requires, however, while the player sets tend to become infinitely large, 'in the limit' the number of attributes represented in the population tends towards a fixed, finite subset of the attributes represented in the individual games. The actions of players with attributes represented in the populations (N, α^ν) but in vanishingly small proportions do not have significant effects on the payoffs of other players. Although strong anonymity is a restriction only on special sorts of sequences of societies, along with our continuity assumption it allows uniform results – that is, results for all sufficiently large games rather than for sequences of games.

Given $\omega \in \Omega$, let the ball around ω with diameter δ be denoted $B(\omega, \sigma)$ and defined by

$$B(\omega, \delta) = \left\{ \omega^* \in \Omega : d(\omega, \omega^*) < \frac{\delta}{2} \right\}.$$

3   Results.

We first state two useful lemmas. The first lemma applies to any game and concerns approximation of mixed strategy vectors by degenerate strategy vectors. The second lemma concerns limiting approximations for sequences of games. With these two lemmas in hand, in the following subsection, we next prove our purification result and then, in the final subsection, we prove our social conformity result.

3.1 Two lemmas.

We firstly introduce some notation. We say a vector $a = (a_1, ..., a_N) \geq b = (b_1, ..., b_N)$ if and only if $a_i \geq b_i$ for all $i = 1, ..., N$. Let $\mathcal{Z}_+^K$ denote the set of K dimensional vectors for which every element is a non-negative integer.

For any strategy vector $\sigma = (\sigma_1, .., \sigma_N)$ (where $\sigma_i = (\sigma_{i1}, ..., \sigma_{iK}) \in \Delta^K$ for $i = 1, ..., N$), let $\mathcal{M}(\sigma)$ denote the set of vectors $m = (m_1, .., m_N)$ such that, for each $i \in N$:

(a) $m_i = (m_{i1}, ..., m_{iK}) \in \mathcal{Z}_+^K$,

(b) $\|m_i\| = 1$ and,

(c) For all k, $m_{ik} = 1$ implies $\sigma_{ik} > 0$.

Informally, $\mathcal{M}(\sigma)$ denotes the set of strategy vectors with the property that each player i is assigned, as a pure strategy, some strategy in the support of σ_i. If σ were a Nash equilibrium then in the strategy vector m, each player would be assigned a strategy in his best response set for σ.

The following lemma shows that given any choices of mixed strategies, $(\sigma_i, \ i = 1, ..., N)$, one for each player, we can select pure strategies m_i for each player so that each player's pure strategy is a best response to the initially given mixed strategy choices and so that the total number of players assigned strategy s_k is within K of the total weight assigned to s_k by the initially given mixed strategy vector, that is,

$$\left| \sum_i \sigma_{ik} - \sum_{i:m_{ik}=1} m_{ik} \right| \leq K. \tag{4}$$

Actually, the result is somewhat stronger.

Lemma 1: For any strategy vector $\sigma = (\sigma_1, ..., \sigma_{|N|}) \in \Delta^{KN}$ and for any vector $\overline{g} \in \mathcal{Z}_+^K$ such that $\sum_i \sigma_i \geq \overline{g}$, there exists a pure strategy vector $m = (m_1, ..., m_{|N|}) \in \mathcal{M}(\sigma)$ such that:

$$\sum_i m_i \geq \overline{g}.$$

(To relate this to the interpretation above, choose $\overline{g}$ so that for each $k = 1, ..., K$, $\sum_i \sigma_{ik} - \overline{g}_k < 1$. Then, since $\sum_k \sum_i \sigma_{ik} = |N|$ it holds that $|N| -$

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$\sum_k \overline{g}_k < K$. Note also that $\sum_k \sum_i m_{ik} = |N|$. It follows that for each pure strategy s_k, (4) holds.)

Proof[9]**:** Suppose the statement of the lemma is false. Then, there exists a strategy vector $\sigma = (\sigma_1, ..., \sigma_i, ..., \sigma_N)$ and a vector $\overline{g} \in \mathcal{Z}^K$ where $\sum_i \sigma_i \geq \overline{g}$ and such that, for any vector $m = (m_1, ..., m_{|N|}) \in \mathcal{M}(\sigma)$ there must be at least one $\widehat{k}$ for which $\sum_i m_{i\widehat{k}} < \overline{g}_{\widehat{k}}$. For each vector $m \in \mathcal{M}(\sigma)$ let L be defined as follows:

$$L(m) = \sum_{k:\overline{g}_k - \sum_i m_{ik} > 0} \left(\overline{g}_k - \sum_i m_{ik} \right)$$

Select $m^0 \in \mathcal{M}(\sigma)$ for which $L(m)$ attains its minimum value over all $m \in \mathcal{M}(\sigma)$. Intuitively the vector m^0 is 'as close' as we can get to satisfying the lemma. Pick a strategy $\widehat{k}$ such that $\overline{g}_{\widehat{k}} - \sum_i m^0_{i\widehat{k}} > 0$.

For any subset I of N and any vector $m \in M(\sigma)$ let the set $S(I, m) \subset S$ be such that:

$$S(I, m) = \left\{ s_{\widehat{k}} \right\} \cup \{s_k \in S : m_{ik} = 1 \text{ for some } i \in I\}$$

We can now define sets $I^n \left(\widehat{k} \right)$ for $n = 0, 1, ...$ as follows:

$$I^0 \left(\widehat{k} \right) = \left\{ i \in N : m^0_{i\widehat{k}} = 1 \right\},$$

$$I^{n+1} \left(\widehat{k} \right) = I^n \left(\widehat{k} \right) \cup \left\{ \begin{array}{c} j \in N : \sigma_{jk} > 0 \text{ and } m^0_{jk} = 0 \\ \text{for some } k \in S \left(I^n \left(\widehat{k} \right), m^0 \right) \end{array} \right\}$$

Consider a player $i_1 \in I^1 \left(\widehat{k} \right) \backslash I^0 \left(\widehat{k} \right)$. Then, $m^0_{i_1 \widehat{k}} = 0$, $\sigma_{i_1 \widehat{k}} > 0$ and $m^0_{i_1 k_1} = 1$ for some $s_{k_1} \in S$. Thus, there exists an $m^* \in M(\sigma)$ such that $m^*_i = m^0_i$ for all $i \neq i_1$ while $m^*_{i_1 \widehat{k}} = 1$ and $m^*_{i_1 k_1} = 0$. Suppose that $\sum_{i \in N} m^0_{ik_1} > \overline{g}_{k_1}$. This implies, given that $m^0_{ik_1}$ and $\overline{g}_{k_1}$ are integers, that $\sum_{i \in N} m^0_{ik_1} \geq \overline{g}_{k_1} + 1$. Then, it follows, by the definition of $L(m)$ that $L(m^*) =$

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[9] The proof of this result could perhaps also be derived from the Shapley-Folkman Theorem (see, for example, Green and Heller 1991). It appears, however, to be an extension since Lemma 1 implies the Shapley-Folkman Theorem and more.

$L(m^0) - 1$. This contradicts that we chose the vector $m^0 \in M(\sigma)$ with minimum L.

Consider a player $i_n \in I^n\left(\widehat{k}\right) \backslash I^{n-1}\left(\widehat{k}\right)$ where $n \geq 2$ and $m_{i_n k_n} = 1$ for some pure strategy s_{k_n}. By the construction of $I^n\left(\widehat{k}\right)$ if

$$i_n \in I^n\left(\widehat{k}\right) \backslash I^{n-1}\left(\widehat{k}\right)$$

there must exist some strategy

$$s_{k_{n-1}} \in S\left(I^{n-1}\left(\widehat{k}\right), m^0\right) \backslash S\left(I^{n-2}\left(\widehat{k}\right) \backslash I^{n-1}\left(\widehat{k}\right), m^0\right)$$

such that $\sigma_{i_n k_{n-1}} > 0$ and $m^0_{i_n k_{n-1}} = 0$. This implies, by the definition of $S(I, m)$, that there exists a player

$$i_{n-1} \in I^{n-1}\left(\widehat{k}\right) \backslash I^{n-2}\left(\widehat{k}\right)$$

such that $m^0_{i_{n-1} k_{n-1}} = 1$. Continuing this chain as far as necessary, it follows that there exists a set of players $C = \{i_1, ..., i_n\} \subset I^n\left(\widehat{k}\right)$, where for all $i_t \in C$, $i_t \in I^t\left(\widehat{k}\right)$ and $m^0_{i_t k_t} = 1$, and a vector $m^* \in M(\sigma)$ such that:

for i_1 : $m^*_{i_1 k_1} = 0$ and $m^*_{i_1 \widehat{k}} = 1$,
for all $i_t \in C \backslash \{i_1\}$: $m^*_{i_t k_t} = 0$ and $m^*_{i_t k_{t-1}} = 1$, and
for all $i \notin C$ and all $s_k \in S$: $m^*_{ik} = m^0_{ik}$.

Suppose that

$$\sum_{i \in N} m^0_{ik_n} > \overline{g}_{k_n}.$$

Then, again noting this implies that

$$\sum_{i \in N} m^0_{ik_n} \geq \overline{g}_{k_n} + 1,$$

we have that

$$L(m^*) = L(m^0) - 1.$$

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To see this we note that

$$\sum_{i \in N} m^0_{ik_t} = \sum_{i \in N} m^*_{ik_t}$$

for all $1 \leq t \leq n-1$ while

$$\sum_{i \in N} m^0_{ik_n} = \sum_{i \in N} m^*_{ik_n} + 1$$

and

$$\sum_{i \in N} m^0_{i\widehat{k}} + 1 = \sum_{i \in N} m^*_{i\widehat{k}}.$$

Then use the definition of $L(m)$. This contradicts the choice of $m^0 \in M(\sigma)$ to minimize $L(\cdot)$.

Ultimately, for some $n^* \geq 1$ we must have that $I^{n^*+1}\left(\widehat{k}\right) = I^{n^*}\left(\widehat{k}\right)$. This is an immediate consequence of the finiteness of the player set.

The above has shown that if there exists a strategy $s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)$ where:

$$\sum_{i \in N} m^0_{ik} > \overline{g}_k$$

then we have the desired contradiction. This implies, for all $s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)$ that:

$$\sum_{i \in N} m^0_{ik} \leq \overline{g}_k. \tag{5}$$

Using the definition of $I^n\left(\widehat{k}\right)$ and that $I^{n^*+1} = I^{n^*}$, there can exist no player $j \in N \backslash I^{n^*}\left(\widehat{k}\right)$ such that $\sigma_{jk} > 0$ for some $s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)$, unless $m^0_{jk} = 1$. This implies that:

$$\sum_{i \in N \backslash I^{n^*}\left(\widehat{k}\right)} m^0_{ik} \geq \sum_{i \in N \backslash I^{n^*}\left(\widehat{k}\right)} \sigma_{ik} \tag{6}$$

for all $s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)$.

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Using the definition of $S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)$, we have that:

$$\sum_{k:s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)} \sum_{i \in I^{n^*}\left(\widehat{k}\right)} m_{ik}^0 \geq \sum_{k:s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)} \sum_{i \in I^{n^*}\left(\widehat{k}\right)} \sigma_{ik}. \qquad (7)$$

Combining 6 and 7 and using the statement of the lemma, we see that:

$$\sum_{k:s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)} \sum_{i \in N} m_{ik}^0 \geq \sum_{k:s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)} \sum_{i \in N} \sigma_{ik} \geq \sum_{k:s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)} \overline{g}_k$$

However, by assumption:

$$\overline{g}_{\widehat{k}} > \sum_{i \in N} m_{i\widehat{k}}^0$$

and also by assumption, $s_{\widehat{k}} \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)$. Thus, there must exist at least one $s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)$ such that:

$$\overline{g}_k < \sum_{i \in N} m_{ik}^0.$$

This contradicts 5 and completes the proof.$\blacklozenge$

Having completed the proof let us provide an intuitive explanation of the sets $I^n\left(\widehat{k}\right)$ with reference to figure 1 below.

The set $I^0\left(\widehat{k}\right)$ contains those players who are 'assigned' the pure strategy $s_{\widehat{k}}$ by vector m^0. That is, if $i_0 \in I^0(\widehat{k})$ then $m_{i_0\widehat{k}}^0 = 1$. The set $I^1\left(\widehat{k}\right) \setminus I^0\left(\widehat{k}\right)$ consists of those players who could have been assigned the strategy $\widehat{k}$ according to the definition of $M(\sigma)$, but were not. That is, if $i_1 \in I^1\left(\widehat{k}\right) \setminus I^0\left(\widehat{k}\right)$, then $\sigma_{i_1\widehat{k}} > 0$ but $m_{i_1\widehat{k}}^0 = 0$.

Suppose that $m_{i_1k_1}^0 = 1$. That is, player 1 was assigned pure strategy s_{k_1}. In looking at the set $I^2\left(\widehat{k}\right) \setminus I^1\left(\widehat{k}\right)$ pure strategies $k \neq \widehat{k}$ play a role. In particular, if there exists a player i_2 such that $m_{i_2k_2}^0 = 1$ and $\sigma_{i_2k_1} > 0$ then player $i_2 \in I^2\left(\widehat{k}\right) \setminus I^1\left(\widehat{k}\right)$. That is, player i_2 could have been assigned the strategy s_{k_1} but was actually assigned strategy s_{k_2}. Further, there exists a player $i_1 \in I^1\left(\widehat{k}\right)$ using pure strategy s_{k_1}.[10] In adding the next group of

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[10] We also require that $\sigma_{i_2\widehat{k}} = 0$ otherwise player i_2 would be included in the set $I^1\left(\widehat{k}\right)$.

players, $I^3\left(\widehat{k}\right) \setminus I^2\left(\widehat{k}\right)$, we can now take into account the pure strategy s_{k_2}. Thus, we start looking for some player i_3 such that $m^0_{i_3 k_3} = 1$ and $\sigma_{i_3 k_2} > 0$. And so on.

$i_0 \in I^0\left(\widehat{k}\right)$ $i_1 \in I^1\left(\widehat{k}\right) \setminus I^0\left(\widehat{k}\right)$ $i_2 \in I^2\left(\widehat{k}\right) \setminus I^1\left(\widehat{k}\right)$ $i_3 \in I^3\left(\widehat{k}\right) \setminus I^2\left(\widehat{k}\right)$ $\cdots$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$ $\qquad\qquad$ $\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\sigma_{i_0\widehat{k}} > 0$ $\qquad$ $\sigma_{i_1\widehat{k}} > 0$ $\qquad$ $\sigma_{i_2\widehat{k}} = 0$ $\qquad$ $\sigma_{i_3\widehat{k}} = 0$

$m^0_{i_0\widehat{k}} = 1.$ $\qquad$ $m^0_{i_1\widehat{k}} = 0$ $\qquad$ $m^0_{i_2\widehat{k}} = 0.$ $\qquad$ $m^0_{i_3\widehat{k}} = 0.$

For k_1, $\qquad\qquad$ For k_1,

and for some k_1, $\qquad$ $\sigma_{i_2 k'} > 0,$ $\qquad\qquad$ $\sigma_{i_3 k_1} = 0,$

$\sigma_{i_1 k'} > 0$ $\qquad\qquad$ $m^0_{i_2 k_1} = 0$ $\qquad\qquad$ $m^0_{i_3 k_1} = 0.$

$\underline{m^0_{i_1 k_1} = 1}$.

and for some k_2 $\qquad\qquad$ For k_2

$\sigma_{i_2 k_2} > 0$ $\qquad\qquad$ $\sigma_{i_3 k_2} > 0$

$\underline{m^0_{i_2 k_2} = 1}$. $\qquad\qquad$ $m^0_{i_3 k_2} = 0$

and for some k_3

$\sigma_{i_3 k_3} > 0$

$\underline{m^0_{i_3 k_3} = 1.}$

Figure 1

Any of the players in $I^1\left(\widehat{k}\right)\backslash I^0\left(\widehat{k}\right)$, such as i_1, could have been assigned the pure strategy $\widehat{k}$ (since $\sigma_{i_1\widehat{k}} > 0$) but instead i_1 was assigned the pure strategy s_{k_1}. Player i_2 could have been assigned pure strategy i_1 (since $\sigma_{i_2k_1} > 0$) but was assigned pure strategy s_{k_2}. Generalising, any player $i_n \in I^n\left(\widehat{k}\right)\backslash I^{n-1}\left(\widehat{k}\right)$ could have been assigned pure strategy $s_{k_{n-1}}$ but was actually assigned pure strategy s_{k_n}.

Now, suppose that there exists a strategy $s_k \in S\left(I^{n^*}\left(\widehat{k}\right), m^0\right)$ such that:

$$\sum_{i \in N} m_{ik}^0 \geq \overline{g}_k + 1$$

We can do the following reallocation: put $m_{i_nk_n}^0$ equal to zero and set $m_{i_nk_{n-1}}^0 = 1$ for all $n \geq 2$ and set $m_{i_1k_1} = 0$ and $m_{i_1\widehat{k}} = 1$. This leaves the numbers allocated to strategy s_{k_t} for all $1 \leq t \leq n-1$ unchanged. The number playing s_{k_n} reduces by one and the number playing $s_{\widehat{k}}$ increases by one. Essentially, the player i_n has been allocated to a strategy s_{k_n} where 'it is not needed'. Thus, we can take player i_n away from strategy s_{k_n} and allocate it to strategy $s_{k_{n-1}}$. Repeating this chain we finish by putting $m_{i_1\widehat{k}} = 1$. If there was a shortfall in the number of players using strategy $s_{\widehat{k}}$ we thus reduce this shortfall to at the worst one less than we began with. At best, we can, of course, overcome the shortfall completely and repeatedly applying the above procedure will eventually do so.

Roughly, our next Lemma shows that, for any growing sequence of games, if there is only a finite number of types that appear in positive proportions in the limit, then in the limit, strategy vectors can be purified. Suppose, as is standard in papers showing purification of mixed strategy equilibria, we had a continuum of players with a finite number of types where type ω_a appears in the proportion θ_{ak}. Then the following result demonstrates that we can approach the continuum purification in large finite games.[11] But it shows more. The games considered in Lemma 2 could have vanishingly small percentages of players of some types. Our conditions ensure that these players cannot significantly effect payoffs to other players and are, in the continuum limit, negligible.

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[11]In fact, such a result was obtained in Rashid (1983).

Before introducing our next lemma, we introduce an additional term. A weight function $w(\cdot, \cdot; \alpha) \in W_\alpha$ is called *integer-valued* if $w(\omega, s_k; \alpha) \in \mathcal{Z}_+$ for each $\omega \in \Omega$ and each $s_k \in S$. We typically denote an integer-valued weight function by $g(\cdot, \cdot; \alpha)$. If $g(\cdot, \cdot; \alpha)$ is an integer-valued weight function there exists strategy vectors σ such that σ_i is degenerate for all players $i \in N$ and $g(\cdot, \cdot; \alpha, \sigma)(\omega) = g(\cdot, \cdot; \alpha)(\omega)$ for all $\omega \in \Omega$. Moreover, every degenerate strategy vector σ generates an integer-valued weight function. Given the profile $\rho(N, \alpha)$ and a degenerate strategy vector σ, the interpretation is that, for each attribute ω and strategy $s_k \in S$, $g(\omega, s_k, \alpha)$ denotes the number of players i in N with attribute ω whose strategy σ_i places weight 1 on some pure strategy s_k.

Lemma 2: Let $\{N^\nu\}$ be a sequence of player sets with $|N^\nu|$ becoming large as becomes large. Let $\{\omega_1, ..., \omega_J\} \equiv L$ be a finite set of points in Ω. Let $\{\alpha^\nu\}$ be a sequence of attribute functions, $\alpha^\nu: N^\nu \rightarrow \Omega$ and $\{f^\nu\}$ a corresponding seqence of profiles, $f^\nu = \rho(N^\nu, \alpha^\nu)$, such that $support(f^\nu) = \{\omega_1, ..., \omega_J\}$ for all ν. Let $\{\sigma^\nu\}$ be a sequence of strategy vectors and $\{w^\nu\}$ be a sequence of weight functions, where w^ν is relative to strategy vector σ^ν and attribute function α^ν, such that the $\lim_{\nu \rightarrow \infty} \left(\frac{w^v(\omega_j, s_k; \alpha^\nu, \sigma^\nu)}{\|w^v\|} \right) = \theta_{jk}$ exists for all $s_k \in S$ and all $\omega_j \in L$. Then, there exists a sequence $\{s^\nu\}$ of degenerate strategy vectors and a sequence $\{g^\nu\}$ of integer-valued weight functions, where g^ν is relative to strategy vector s^ν and attribute function α^ν, such that:

1. for all $s_k \in S$ and all $\omega_j \in L$:

$$\lim_{\nu \rightarrow \infty} \frac{g^\nu(\omega_j, s_k; \alpha^\nu, s^\nu)}{\|g^\nu\|} = \lim_{\nu \rightarrow \infty} \frac{g^\nu(\omega_j, s_k; \alpha^\nu, s^\nu) - 1}{\|g^\nu\|} = \theta_{jk}$$

2. the strategy vector s^ν is such that $s_i^\nu \in support(\sigma_i^\nu)$ for all $i \in N^\nu$ and for all ν.

Proof: Suppose the statement of the lemma is false. Let $\mathcal{S}$ denote the set of sequences of degenerate strategy vectors such that for any sequence $\{s^\nu\} \in \mathcal{S}$, $s_i^\nu \in support(\sigma_i^\nu)$ for all $i \in N^\nu$ and for all ν. Given any strategy vector σ^ν there always exists a degenerate strategy vector s^ν such that $s_i^\nu \in support(\sigma_i^\nu)$ for all $i \in N^\nu$ and therefore $\mathcal{S}$ is non empty. By assumption there exists no sequence $\{s^\nu\} \in \mathcal{S}$ satisfying the statement of the Lemma. Thus, there

exists an $\varepsilon_0 > 0$ such that for any sequence of strategy vectors $\{s^\nu\} \in \mathcal{S}$, with corresponding sequence of integer-valued weight functions $\{g^\nu\}$, and for any ν there exists a $\nu_0 > \nu$, an $\omega_j \in L$ and a pure strategy $s_k \in S$ such that:

$$\left| \frac{g^{\nu_0}(\omega_j, s_k; \alpha^{\nu_0}, s^{\nu_0})}{\|g^{\nu_0}\|} - \theta_{jk} \right| > \varepsilon_0$$

and/or:

$$\left| \frac{g^{\nu_0}(\omega_j, s_k; \alpha^{\nu_0}, s^{\nu_0}) - 1}{\|g^{\nu_0}\|} - \theta_{jk} \right| > \varepsilon$$

By way of contradiction let us construct a sequence of integer-valued weight functions as follows. For each ν, for each $s_k \in S$ and each $\omega_j \in L$ define the integer $\overline{g}^\nu(\omega_j, s_k)$ as the largest integer less than or equal to $w^\nu(\omega_j, s_k; \alpha^\nu, s^\nu)$. Formally:

$$\overline{g}^\nu(\omega_j, s_k) = \{x \in \mathcal{Z}_+ : x = \min\{(y - w^\nu(\omega_j, s_k; \alpha^\nu, s^\nu)) : \ y \in \mathcal{Z}_+ \text{ and } y \leq w^\nu(\omega_j, s_k; \alpha^\nu, s^\nu)\}$$

The construction of $\overline{g}$ implies that:

$$1 \geq w^\nu(\omega_j, s_k; \alpha^\nu, s^\nu) - \overline{g}^\nu(\omega_j, s_k) \geq 0$$

for all $s_k \in S$, all $\omega_j \in L$ and for all ν.

Now fix ω_j. We next apply Lemma 1 to a stituation where all the players i with attribute ω_j are taken as the total player set; thus, the analogue of σ in the statement of Lemma 1 is $\{\sigma_i^\nu : \alpha^\nu(i) = \omega_j\}$. The analogue of $\overline{g}$ in the statement of Lemma 1 is $\overline{g}(\omega_j, \cdot)$. Lemma 1 implies that we can choose, for each ν, degenerate strategy vectors s^ν (the analogues of m^ν in the statement of Lemma 1) with corresponding integer valued weight functions g^ν such that:

1. $s_i^\nu \in support(\sigma_i^\nu)$ for all $i \in N^\nu$.

2. $\overline{g}^\nu(\omega_j, s_k) + L^\nu \geq g^\nu(\omega_j, s_k) \geq \overline{g}^\nu(\omega_j, s_k)$ for all $k = 1, ..., K$, where $L^\nu \in \mathcal{Z}^+$ is defined for each ν as:

$$L^\nu = f^\nu(\omega_j) - \sum_{k=1}^{K} \overline{g}^\nu(\omega_j, s_k)$$

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We note that $L^\nu \leq K$ for all ν. Thus:

$$|w^\nu(\omega_j, s_k; \alpha^\nu, s^\nu) - g^\nu(\omega_j, s_k)| \leq \max\{1, K-1\}$$
$$|w^\nu(\omega_j, s_k; \alpha^\nu, s^\nu) - g^\nu(\omega_j, s_k) + 1| \leq \max\{2, K-2\}$$

Therefore, given that $\|f^\nu\| \to \infty$ as $\nu \to \infty$, for any ε_1 there exists a ν_1 such that for all $\nu > \nu_1$ we have that:

$$\max\left\{ \frac{|w^\nu(\omega_j, s_k; \alpha^\nu, s^\nu) - g^\nu(\omega_j, s_k)|}{\|f^\nu\|}, \frac{|w^\nu(\omega_j, s_k; \alpha^\nu, s^\nu) - g^\nu(\omega_j, s_k) + 1|}{\|f^\nu\|} \right\} < \varepsilon_1$$

However, if we select $\varepsilon_1 \in (0, \varepsilon)$ then for all $\nu > \nu_1$ and for all $\omega_j \in L$ and $s_k \in S$:

$$\left| \frac{g^\nu(\omega_j, s_k)}{\|g^\nu\|} - \theta_{jk} \right| < \varepsilon$$

and

$$\left| \frac{g^\nu(\omega_j, s_k) - 1}{\|g^\nu\|} - \theta_{jk} \right| < \varepsilon$$

This gives the desired contradiction.♦

3.2   Existence of ε-equilibrium in pure strategies.

In the following Theorem, we demonstrate that, given $\varepsilon > 0$ there is an integer η sufficiently large so that every game $\Gamma(N, \alpha)$ has an ε-equilibrium in pure strategies. To obtain this result, at a point in the proof we arbitrarily select a Nash equilibrium for each game in a sequence and show that if there are sufficiently many players, this Nash equilibrium can be used to construct an ε-equilibrium in pure strategies. Since the selection of the Nash equilibrium was arbitrary, our result can be viewed as a purification theorem – in sufficiently large games, every Nash equilibrium can be purified. The section concludes with a discussion of recent related research in Kalai (2000).

Theorem 1: Given a real number $\varepsilon > 0$ there exists a real number $\eta_0(\varepsilon) > 0$ such that for all societies (N, α), where preferences satisfying continuity 1 and strong anonymity, and where $\|\rho(N, \alpha)\| > \eta(\varepsilon)$, the induced game $\Gamma(N, \alpha)$

has an ε-equilibrium in pure strategies. Moreover, for any mixed strategy equilibrium there exists an ε-purification.

Proof: Suppose that the statement of the Theorem is false. Then there is some $\varepsilon > 0$ such that, for each integer ν there is a society (N^ν, α^ν) and induced game $\Gamma(N^\nu, \alpha^\nu)$ with profile $\rho(N^\nu, \alpha^\nu) > \nu$ for which there does not exist an ε-equilibrium in pure strategies. For ease of notation, denote $\rho(N^\nu, \alpha^\nu)$ by f^ν. That is, for each induced game $\Gamma(N^\nu, \alpha^\nu)$ there does not exist a degenerate strategy vector s^ν, with corresponding integer-valued weight function $g^\nu(\cdot, \cdot; \alpha^\nu, \sigma^\nu)$ such that:

$$h_{\alpha(i)}(s, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) \geq h_{\alpha(i)}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu)$$

for all $t \in S$ and for all $i \in N^\nu$ where $s \in support(\sigma^\nu_i)$.

Observe, however that the game $\Gamma(N^\nu, \alpha^\nu)$ has a mixed strategy Nash equilibrium; this is an immediate application of Nash's well known theorem. Denote a Nash equilibrium (NE) of the game $\Gamma(N^\nu, \alpha^\nu)$ by σ^ν with the appropriate weight function $w^\nu(\cdot, \cdot; \alpha^\nu, \sigma^\nu)$. Since σ^ν is a Nash equilibrium, for each ν and for each $i \in N^\nu$ we have:

$$h_{\alpha^\nu(i)}(\sigma^\nu_i, w^\nu_{-i}, f^\nu) \geq h_{\alpha^\nu(i)}(t, w^\nu_{-i}, f^\nu)$$

for all $s \in S$ and

$$h_{\alpha^\nu(i)}(s, w^\nu_{-i}, f^\nu) \geq h_{\alpha^\nu(i)}(t, w^\nu_{-i}, f^\nu)$$

for all $t \in S$ and for all $s \in support(\sigma^\nu_i)$.

Let $\delta\left(\frac{\varepsilon}{8}\right)$ be the similarity parameter as defined by (Lipschitz) continuity for a required payoff bound of $\frac{\varepsilon}{8}$. Use compactness of Ω to write Ω as the disjoint union of a finite number of non-empty subsets $\Omega_1, ..., \Omega_A$, each of diameter less than δ. For each a, choose and fix a point $\omega_a \in \Omega_a$.

We define the attribute function $\overline{\alpha}^\nu$ as follows, for all ν and for all $i \in N^\nu$:

$$\overline{\alpha}^\nu(i) = \omega_a \text{ if and only if } \alpha(i) \in \Omega_a$$

Given the weight function $w^\nu(\cdot, \cdot; \alpha^\nu, \sigma^\nu)$ relative to society (N^ν, α^ν) and Nash equilibrium strategy vector σ^ν let $w^\nu(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu)$ denote the weight function relative to $\overline{\alpha}^\nu$ and σ^ν.

For each $a = 1, ..., A$ and for each $k = 1, .., K$ define θ^ν_{ak} as follows:

$$\theta^\nu_{ak} = \frac{w^\nu(\omega_a, s_k; \overline{\alpha}^\nu, \sigma^\nu)}{|N^\nu|}$$

By passing to a subsequence if necessary assume that the $\lim_{\nu \to \infty} \theta^\nu_{ak} = \theta_{ak}$ exists for all $a = 1, .., A$ and all $k = 1, ..., K$.

By Lemma 2 there exists a sequence $\{s^\nu\}$ of strategy vectors and a sequence $\{g^\nu(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu)\}$ of integer-valued weight functions relative to attribute function $\overline{\alpha}^\nu$ and the degenerate strategy vector s^ν, such that:

1. for all ν and for all $s_k \in S$ and all $\omega_a \in \Omega$,

$$\lim_{\nu \to \infty} \frac{g^\nu(\omega_a, s_k; \overline{\alpha}^\nu, s^\nu)}{\|g^\nu\|} = \lim_{\nu \to \infty} \frac{g^\nu(\omega_a, s_k; \overline{\alpha}^\nu, s^\nu) - 1}{\|g^\nu\|} = \theta_{ak} \qquad (8)$$

2. for all ν and for all $i \in N^\nu$, $s^\nu_i \in support(\sigma^\nu_i)$.

Given the weight function $g^\nu(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu)$, let $g^\nu(\cdot, \cdot; \alpha^\nu, s^\nu)$ denote the integer valued weight function relative to society (N^ν, α^ν) and strategy vector s^ν.

Consider the payoff to player $i \in N^\nu$ from changing the strategy vector σ^ν to s^ν. We let $\overline{f}^\nu_{-i}$ denote the profile of a society in which the attribute of a player $j \in N^\nu \backslash \{i\}$ is given by $\overline{\alpha}(j)$ and the attribute of player i is given by $\alpha(i)$.

By continuity 1 and the choice of δ we have that there exists a ν_1 such that for all $\nu > \nu_1$:

$$\left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu), \overline{f}^\nu_{-i}) \right|$$
$$< \tfrac{\varepsilon}{8}$$

for any $t \in S$.

Given Lemma 2 and strong anonymity, we have that for any $\varepsilon_2 > 0$ there exists a ν_2 such that for all $\nu > \nu_2$:

$$\left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu), \overline{f}^\nu_{-i}) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu), \overline{f}^\nu_{-i}) \right|$$
$$< \varepsilon_2$$

for any $t \in S$. Set $\varepsilon_2 \in (0, \frac{\varepsilon}{4})$.

Again, using continuity 1 and the choice of δ we have that there exists a ν_3 such that for all $\nu > \nu_3$:

$$\left| h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu), \overline{f}^\nu_{-i}) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) \right|$$
$$< \frac{\varepsilon}{8}$$

for any $t \in S$.

Thus, for any $\nu > \max\{\nu_1, \nu_2, \nu_3\}$ we have that:

$$\left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) \right|$$
$$\leq \left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu), \overline{f}^\nu_{-i}) \right|$$
$$+ \left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu), \overline{f}^\nu_{-i}) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu), \overline{f}^\nu_{-i}) \right|$$
$$+ \left| h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \sigma^\nu), \overline{f}^\nu_{-i}) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) \right|$$
$$< \tfrac{\varepsilon}{8} + \tfrac{\varepsilon}{4} + \tfrac{\varepsilon}{8} = \tfrac{\varepsilon}{2}$$

for any $t \in S$.

However, given that

$$h_{\alpha(i)^\nu}(s, w^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) \geq 0$$

for all $s \in support(\sigma^\nu_i)$, for all $i \in N$, for all $t \in S$ and for all ν, this implies that for $\nu > \max\{\nu_1, \nu_2, \nu_3\}$:

$$h_{\alpha(i)^\nu}(s^\nu_i, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu)$$
$$\geq - \left| h_{\alpha(i)^\nu}(t, w^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) \right| -$$
$$\left| h_{\alpha(i)^\nu}(s^\nu_i, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) - h_{\alpha(i)^\nu}(s^\nu_i, w^\nu_{-i}(\cdot, \cdot; \alpha^\nu, \sigma^\nu), f^\nu) \right|$$
$$\geq -\tfrac{\varepsilon}{2} - \tfrac{\varepsilon}{2} = -\varepsilon$$

which gives the desired contradiction

We note that the equilibrium mixed strategy vector with which we started our proof was arbitrary. Thus, any mixed strategy can be ε-purified. (See, for example, Aumann et. al. 1984).◆

Some recent related results are developed in Kalai (2000). In that paper, Kalai introduces the appealing concept of information proof Bayesian equilibrium, a type of equilibrium immune to changes in the prior probability of

types, in the probabilities of mixed pure actions, and in the order of play, as well as to information leakage and the possibility of revision. Remarkably, he demonstrates that this is a property of Bayesian equilibrium for a broad class of large games. Kalai also uses a sort of pregame consisting of specifications of sets of possible player types $\mathcal{T}$ and possible player actions $\mathcal{A}$, yielding a finite set of possible player compositions $\mathcal{C} = \mathcal{A} \times \mathcal{T}$, and a specification of an equi-continuous family of payoff functions. From these components, large games can be constructed. We understand, from discussions with Ehud Kalai, that for games with complete information his current results for large games imply the existence of ε-Nash equilibria in pure strategies. Similarly, our Theorem 1 can be modified to apply to games with incomplete information and imply existence and ε-purification of Bayesian ε-equilibrium in pure strategies; this is being carried out in work in progress (Cartwright and Wooders 2001).

The frameworks of Kalai (2000) and this paper are sufficiently dissimilar so that there is no immediate comparison between either the models or the results. It does appear that Kalai's results are stronger but his framework appears more restrictive. For example, in Kalai (2000), only the distribution of opponents over compositions is relevant, and not numbers of players. In our framework, exact numbers of players can matter, but the larger the total number of players in the game, the less they matter. In addition to existence results, Kalai (2000) also contains a number of characterization results for information-proof equilibria.

4   Social conformity.

Besides permitting results such as Theorem 1 (and various extensions), our framework has the advantage that it allows us to address and provide new formulations, of different questions than currently in the game-theoretic literature, as exemplified by the following social conformity result. The interpretation of this result also requires us to look more deeply at the attribute space. We proceed with the statement and proof of our social conformity result and postpone discussion of indexing attributes to the next section.

An important aspect of the following result is that the number of distinct cultures required to partition the total player set into connected intervals, with the property that all players in the same interval play the same pure strategy, is bounded by a constant, $J(\varepsilon)K$, which is independent of the size

of the total player set. The smaller the bound $J(\varepsilon)$, the stronger the social conformity, since the smaller $J(\varepsilon)$, the more dissimilar players who are choosing the same strategy may be.

Before stating Theorem 2 we require the following definition.

convex separation. Let Ω be a well ordered set with binary relation $\geq$ such that any two elements $\omega, \omega' \in \Omega$ are equal, written $\omega = \omega'$, if and only if they are identical in the normal sense. Furthermore, suppose that given any two finite set of points $\Omega_J = \{\omega_1, ..., \omega_J\}$ and $\Omega_Q = \{\omega_1', ..., \omega_Q'\}$, where $\omega_1, ..., \omega_J, \omega_1', ..., \omega_Q' \in \Omega$ if: $\omega_1 \leq \omega_2 \leq ... \leq \omega_J < \omega_1' \leq \omega_2' \leq ... \leq \omega_Q'$ then the convex hulls of the sets Ω_J and Ω_Q are distinct.

it appears that convex separation is satisfied by interesting class of metric spaces. After the proof of Theorem 2, we demonstrate that a closed subset of finite dimensional Euclidean space satisfies convex separation.

We can now state and prove our second theorem:

Theorem 2: Assume that Ω satisfies convex separation. Given a real number $\varepsilon > 0$, there exists a real number $\eta_1(\varepsilon) > 0$ and an integer $J(\varepsilon)$ such that for all societies (N, α), where:

1. $|N| > \eta_1(\varepsilon)$.

2. Preferences satisfy continuity 2 and strong anonymity.

3. For some fixed number B, for all $\omega \in \Omega$, $|\alpha(\omega)| \leq B$.

there exists a partition of Ω into $C \leq J(\varepsilon)K$ convex subsets $\{\omega_c\}_{c=1}^C$ such that the induced game $\Gamma(N, \alpha)$ has an ε-equilibrium in pure strategies with the property that, for each $c = 1, ..., C$, all players in Ω_c choose the same pure strategy.

Proof: Suppose not. Then, there is some $\varepsilon_0 > 0$ such that for each integer ν there is a society (N^ν, α^ν) and induced game $\Gamma(N^\nu, \alpha^\nu)$ with profile f^ν, where $\|f^\nu\| > \nu$ and for which no ε-equilibrium satisfies the conditions of the lemma.

We begin by noting that, by Theorem 1, for any ε_0 there exists a number $\eta_0(\frac{8}{18}\varepsilon_0)$ and ν^* such that if $\|f^\nu\| > \|f^{\nu^*}\| \geq \eta(\frac{8}{18}\varepsilon_0)$ the society (N^ν, α^ν) has

an $\frac{8}{18}\varepsilon_0$-equilibrium in pure strategies. Denote an $\frac{8}{18}\varepsilon_0$ equilibrium of society (N^ν, α^ν) by s^ν with corresponding weight function $g^\nu(\cdot, \cdot; \alpha^\nu, s^\nu)$.

Given ε_0, let $\delta\left(\frac{\varepsilon_0}{18}\right)$ be the similarity parameter as defined by (Lipschitz) continuity 2 for a required payoff bound of $\frac{\varepsilon_0}{18}$. Use compactness of Ω to write Ω as the disjoint union of a finite number of *convex* non-empty subsets $\Omega_1, ..., \Omega_{J(\varepsilon_0)}$, each of diameter less than δ.

Given the binary relation $\leq$ assume, without loss of generality, that for all ν and for all $i, j \in N^\nu$, if $\alpha(i) < \alpha(j)$ then $i < j$. For each ν we rearrange the strategy vector s^ν in two stages:

1. Let n_{kl}^ν denote the number of players i such that $\alpha(i) \in \Omega_l$ and $s_{ik} = 1$. That is n_{kj}^ν denotes the number of players with attributes in the set Ω_l playing pure strategy s_k with probability 1. Then for each $j = 1, ..., J(\varepsilon_0)$ starting with the minimum integer $i \in \mathcal{N}$ such that $\alpha^\nu(i) \in \Omega_j$ allocate players in ascending order to strategy 1 until n_{1j}^ν players are allocated to strategy 1. Then move onto strategies $2, ..., K$. This procedure will clearly reallocate the assignment of strategies within the partition Ω_j so that the weight within Ω_j to each pure strategy remains the same.

2. We have still, however, yet to create convex subsets in which all players use the same strategy. For example we may have B players with attribute type ω where the first player is allocated to pure strategy s_k and the next $B - 1$ players to pure strategy s_{k+1}. So, the second part of the reallocation is to allocate all those players with the same type to a unique pure strategy that at least one player previously used. It is relatively easy to see that the total number of people whose pure strategy we may have to change in this second part of the reallocation is less than or equal to $(K - 1)J(\varepsilon_0)(B - 1)$.

The above procedure reassigns the $\frac{8}{18}\varepsilon_0$-equilibrium s^ν to create a new strategy vector $\overline{s}^\nu$. For each ν, let N_{kl}^ν denote the set of players $i \in N^\nu$ such that $\alpha(i) \in \Omega_l$ and $\overline{s}_{ik} = 1$. That is, a player $i \in N_{kl}^\nu$ if the players attribute is in the set Ω_l and the player is allocated, in the new strategy profile, the strategy s_k. Given the set of attributes $\{\alpha^\nu(i) : i \in N_{kl}^\nu\}$ let Ω_{kl}^ν denote the convex hull of these attributes. Given the steps 1 and 2 above and the definition of Ω we have that the sets Ω_{kl}^ν are distinct and convex. Furthermore, every player $i \in N^\nu$ belongs to some subset Ω_{kl}^ν. Thus,

(given some appropriate reallocation of the remaining attribute space) we have created, for all ν, a partition of Ω into convex subsets $\Omega_1^\nu, ..., \Omega_C^\nu$, where $C \leq J(\varepsilon)K$ and such that any two players $i, j \in \Omega_c^\nu$ use the same pure strategy.

We now consider the change in payoffs from this reallocation.

The first part of the reallocation process can be seen as mathematically equivalent to changing the attribute types of players. That is, instead of thinking of swapping, say, the strategies that players i and j use, we can think of it as swapping the attribute types of players i and j while keeping their strategies unchanged. Thus, stage 1 is mathematically equivalent to creating a new society $(N^\nu, \overline{\alpha}^\nu)$ satisfying $dist(\alpha^\nu(i), \overline{\alpha}^\nu(i)) < \delta\left(\frac{\varepsilon_0}{18}\right)$ for all $i \in N^\nu$. The weight function relative to strategy vector s^ν and society $\overline{\alpha}^\nu$ is given by $g^\nu(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu)$. (We note that the profile of societies (N^ν, α^ν) and $(N^\nu, \overline{\alpha}^\nu)$ are equivalent to f^ν.) This interpretation and the choice of δ allows us to make use of continuity 2 by arguing that there exists a ν_1 such that for all $\nu > \nu_1$:

$$\left| h_{\alpha^\nu(i)}(t, g_{-i}^\nu(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g_{-i}^\nu(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu), f^\nu) \right| < \frac{1}{18}\varepsilon_0$$

for all $t \in S$ and for all $i \in N^\nu$.

Consider, now the second part of the reallocation in which at most finite number $(K-1)L(\varepsilon_0)(B-1)$ players change pure strategy. Assume this changes the strategy vector to $\overline{s}^\nu$ and the weight function to $g^\nu(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu)$. Because, only a finite number of players change strategy, it can be shown, using continuity 2 (or 1), strong anonymity and an argument analogous to that in Theorem 1, that there exists a ν_2 such that for all $\nu > \nu_2$:

$$\left| h_{\overline{\alpha}^\nu(i)}(t, g_{-i}^\nu(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu), \overline{f}^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g_{-i}^\nu(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu), \overline{f}^\nu) \right| < \frac{4}{18}\varepsilon_0$$

for all $t \in S$ and all $i \in N^\nu$. The intuition is clear - there are only a bounded and given number of players changing strategies - note that $J(\varepsilon_0)$ can be fixed at say $\frac{1}{\delta} + 1$ - and so for large enough populations the weight function is relatively unchanged by these players changing strategies. Thus

for $\nu > \max\{\nu_1, \nu_2\}$:

$$\left| h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu), \overline{f}^\nu) \right| \leq$$
$$\left| h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu), \overline{f}^\nu) \right|$$
$$+ \left| h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu), \overline{f}^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, s^\nu), \overline{f}^\nu) \right|$$
$$< \tfrac{1}{18}\varepsilon_0 + \tfrac{4}{18}\varepsilon_0 = \tfrac{5}{18}\varepsilon_0$$

for all $t \in S$ and all $i \in N^\nu$.

We began by noting that there exists a finite ν^* such that for all $\nu > \nu^*$ there exits an $\frac{8}{18}\varepsilon_0$-equilibrium in pure strategies s^ν and corresponding weight function $g^\nu(\cdot, \cdot; \alpha^\nu, s^\nu)$, implying that:

$$h_{\alpha^\nu(i)}(s, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) \geq h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - \frac{8}{18}\varepsilon_0$$

for all $t \in S$ and all $s \in support(s^\nu_i)$ and for all $i \in N^\nu$.

This implies, for all $\nu > \max\{\nu_1, \nu_2, \nu^*\}$ that:

$$h_{\overline{\alpha}^\nu(i)}(s, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu), \overline{f}^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu), \overline{f}^\nu) \geq \qquad (9)$$
$$- \left| h_{\alpha^\nu(i)}(s, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(s, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu), \overline{f}^\nu) \right| \qquad (10)$$
$$- \left| h_{\alpha^\nu(i)}(s, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) \right|$$
$$- \left| h_{\alpha^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \alpha^\nu, s^\nu), f^\nu) - h_{\overline{\alpha}^\nu(i)}(t, g^\nu_{-i}(\cdot, \cdot; \overline{\alpha}^\nu, \overline{s}^\nu), \overline{f}^\nu) \right|$$
$$\geq -\frac{5}{18}\varepsilon_0 - \frac{8}{18}\varepsilon_0 - \frac{5}{18}\varepsilon_0 = -\varepsilon_0$$

for all $s \in support(\overline{s}^\nu_i)$ and for all $i \in N^\nu$.

The above expression, however, gives the desired contradiction. To see this we make two observations. Firstly, we repeat the analogy that swapping the strategies of players is 'equivalent' to swapping their attribute types. Thus, if stage 1 of the reallocation swaps the strategy of players i and j the above shows that player j is at an ε_0-equilibrium. Secondly, we have to consider the players who were allocated a new and different strategy in stage 2 of the reallocation. We recall that, say $b \leq B$, players were of the same attribute type and at most $b - 1$ were reallocated a different strategy. However, this implies that at least one player i did retain their original strategy

s_i^ν and so remain at an ε_0-equilibrium. Given the other $b-1$ players have the same attribute type they must also be at an ε_0-equilibrium.◆

In an appendix, we provide several examples illustrating various spaces that satisfy convex separation, including Euclidean space, both finite or infinite dimensional.

5   Attribute indexing.

5.1   An example of attribute indexing

A particular example of attribute indexing may be helpful. Note first that that an attribute can be given by a sequence of real numbers $x = (x_1, x_2, ...)$ where $x_i \in [a, b]$ for some finite $a, b \in \mathcal{R}$. Given two sequences $x = (x_1, x_2, ...)$ and $(y_1, y_2, ...)$, a standard metric is given by:

$$\rho_s(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{1 + |x_i - y_i|} 2^{-i} \tag{11}$$

which makes Ω a metric space furnished with the *product topology*. The metric space (Ω, ρ) is compact and therefore separable. This metric, however, may not appear very appropriate in that traits described 'early in the sequence' receive undue weight when measuring the distance between two sequences. This may not be as severe as it may seem as the ordering of the numbers in the sequence is of 'our choice'.[12] We can, however, improve on the metric above.

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[12]We may prefer the metric in which the distance between two sequences $x = (x_1, x_2, ...)$ and $(y_1, y_2, ...)$ is given by:

$$\rho_{l_\infty}(x, y) = \sup_i |x_i - y_i|$$

The metric space $(\Omega, \rho_{l_\infty})$ is, however, not separable and therefore not compact. To see this think of a set of sequences: $\{(1, 0, 0, ...), (0, 1, 0, 0, ...), (0, 0, 1, 0, 0, ...)\}$. This set is not finite and the distance between any two points in the set is 1. This implies that there cannot be a finite ε-net for $\varepsilon < \frac{1}{2}$. That is, for $\varepsilon < \frac{1}{2}$ there cannot be a finite set of points $\{\omega_1, ..., \omega_G\}$ belonging to Ω such that the collection of balls $\{B(\omega_i, \varepsilon)\}$ covers Ω. Thus, the space $(\Omega, \rho_{l_\infty})$ is not totally bounded and therefore not compact. (To show the space is not separable use the set of all binary sequences.)

We begin by introducing some notation: Given a sequence of real numbers $x = (x_1, x_2, ...)$ and a finite set of positive integers $\mathcal{P} = (1, ..., P)$, a sequence $\overline{x} = (\overline{x}_1, \overline{x}_2, ...)$ is a *permutation of x with respect to $\mathcal{P}$* if:

1. $\overline{x}_i = x_i$ for all $i \notin \mathcal{P}$ and,

2. there exists a one-to-one mapping p from $\mathcal{P}$ to $\mathcal{P}$ such that $\overline{x}_i = x_{p_i}$ for all $i \in \mathcal{P}$.

For any $x \in \Omega$ let $\pi_{\mathcal{P}}(x)$ denote the set of permutations of x with respect to $\mathcal{P}$. Then we can define a metric, given $\mathcal{P}$, by:

$$\rho_{\mathcal{P}}(x, y) = \sup_{z \in \pi_{\mathcal{P}}(x-y)} \sum_{i=1}^{\infty} \frac{|z_i|}{1 + |z_i|} 2^{-i}$$

This is indeed a metric for any finite set $\mathcal{P}$. Furthermore, the metric space $(\Omega, \rho_{\mathcal{P}})$ is compact. Thus, in this metric we essentially pick a finite set of 'representative points' and measure the distance between two attributes so the representative points are the most important.

To see how a sequence of numbers $x = (x_1, x_2, ...)$ can be interpreted as an attribute firstly, assume that a player's endowment or physical characteristics, such as endowment of commodities, profession, or power measured in some way, are given by a point in a closed subset of $\mathcal{R}^C$ for some finite positive integer C. Also assume that payoffs to mixed strategies are the natural convex combinations of the payoffs to pure strategies.[13] We begin by taking a society (N, α) as given. Taking the attributes of players $i \in N$ as fixed and noting that there are only a finite number of strategies K, there are only a finite number of possible integer valued weight functions g for the society (N, α). Furthermore, this implies that there are only a finite number of possible strategy/integer valued weight function pairs (s_k, g) with respect to player i for the society (N, α). Thus, suppose we associate a real number $\{-1\} \cup [0, B]$ to each of these possible strategy/integer valued weight function pairs. We assign the value -1 only to strategy/integer valued pairs that are not feasible - for example player i with attribute $\alpha(i)$ playing strategy s where $g(\alpha(i), s) = 0$. The real number attached to a strategy/integer

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[13]For example, if there are two players and player 1 chooses the pure strategy s_1 while player two chooses the mixed strategy s_1 with probability p_1 and s_2 with probability p_2, then the payoff to player 1 is p_1 times his payoff to both players choosing s_1 plus p_2 times his payoff to his strategy s_1 and strategy s_2 for player two.

valued weight function pair is interpreted as the payoff from a realization of that strategy/integer valued weight function pair. This implies that, for the given society (N, α), the payoff function of player i can be defined by a finite sequence of bounded real numbers.

A simple example may be clarifying. Consider a society of four people indexed $1, 2, 3$ and 4 and two strategies indexed A and B. Suppose further that $\alpha(N, \alpha)(1) = \alpha(N, \alpha)(2) = \omega_1$ and $\alpha(N, \alpha)(3) = \alpha(N, \alpha)(4) = \omega_3$. Let us consider the payoff function of player 1. The payoff function must assign values to the following strategy/integer valued weight function pairs:-

$$(A; (2 \times A; 2 \times A)) \; ; \; (A; (2 \times A; 2 \times B)) \; ; \; (A; (2 \times A; 1 \times B, 1 \times A)) \; ; \; (A; (2 \times B; 2 \times A))$$

etc. where $(2 \times A; 2 \times B)$ is interpreted as the integer valued weight function in which two players with attribute ω_1 play strategy A and two players of attribute ω_3 play strategy B. Note that the value assigned to the final strategy/integer value weight function pair given is -1.

An attribute type and payoff function must, however, specify the payoff from all possible societies. Given that Ω is separable let Ω^* denote a countable, everywhere dense, subset of Ω. Assume, for any society (N, α), that $\alpha(i) \in \Omega^*$ for all $i \in N$. Then, since the union of a countable number of possible societies is countable, there are only a countable number of possible societies (N, α). Since each society (N, α) has a finite number of players $|N|$, there are only a finite number of possible strategy/integer valued weight function pairs for that society. Thus, it is possible to describe the payoff from all strategy/integer valued weight function pairs for all societies (N, α) for which $\alpha(i) \in \Omega^*$ for all $i \in N$ by a countable sequence of real numbers.

Consider finally the payoff to strategy/integer valued weight function pairs for societies (N, α) for which $\alpha(i) \notin \Omega^*$ for some $i \in N$. Fixing the size of the society at $|N|$ and using the Prohorov metric, which we will denote by d_P on the space of possible weight functions for a society of size $|N|$, given any integer valued weight function g and any $\varepsilon > 0$ there exists an integer valued weight function g^* such that the support of g^* is Ω^* and $d_P(g, g^*) < \varepsilon$. This follows from Ω being separable. Thus assume there exists an $\bar{\varepsilon}$ such that the payoff from any strategy/integer valued weight function pair (s, g^*) is within $\bar{\varepsilon}$ of the payoff from any strategy/integer valued weight function pair (s, g^*) where the support of g^* is Ω^* and $d_P(g, g^*) < \bar{\varepsilon}$. The payoffs to all possible strategy/integer valued weight function pairs can thus be given within some

bound of $\overline{\varepsilon}$ by a countable sequence of real numbers. Theorems 1 and 2 then follow with slight adjustment for the $\overline{\varepsilon}$.

6   Conclusions.

The noncooperative framework promises to be fruitful. First, the techniques developed in this paper may be useful in other applications. One potential application, currently in progress, is to games with incomplete information (Cartwright and Wooders 2001). So far there appears to be no major obstacles to obtaining uniform large (but finite) analogues of the sorts of results of Aumann et al. (1983). In particular, note that Lemma 1 applies to *any* game and we conjecture that an extension of our model to incomplete information would be obtained using that Lemma similarly to how it is used in this paper. Indeed, it seems that as long as we restrict to compact metric spaces and to the appropriate Lipschitzian continuity conditions, analogues of Theorem 1 will continue to hold.

Other possible applications concern the so called "Equivalence Principle" of cooperative outcomes of large ("competitive") exchange economies. In exchange economies with many players, the set of equilibrium outcomes, represented by the induced utilities of members of the economy, coincides with the core of the core of the game generated by the economy and the value outcomes; see Debreu and Scarf (1963), Aumann (1963, 1985). We conjecture that when noncooperative games derived from pregames are required to satisfy the conditions of this paper (satisfied, in spirit, for exchange economies for which the Equivalence Principle holds) and, in addition, the condition of self-sufficiency – that what a coalition of players can achieve is independent of the society in which it is embedded – then analogues of the Equivalence Principle can be obtained for large noncooperative games. More precisely, we conjecture that under self sufficiency, (approximate) strong equilibrium outcomes are close to Pareto optimal and also treat similar individuals similarly – that is, strong equilibrium outcomes have the equal treatment property.

Comparing our model with those for cooperative pregames, in spirit the frameworks have significant similarities. The cooperative pregame framework, however, is not totally satisfactory. One shortcoming is that some of the results depend on the framework itself (cf. Wooders 1994, Theorem 4, relating small group effectiveness and boundedness of average or per capita payoff). This, the inability of the pregame framework to treat widespread

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externalities, and a desire to highlight what drives the results, led to the introduction of 'parameterized collections' of games (cf. Kovalenkov and Wooders 1996, 1997). We anticipate that eventually such a framework will be introduced for noncooperative pregames. In particular, one advantage of the framework of Kovalenkov and Wooders (1997) is that an exact bound on the size of a game to ensure nonemptiness of ε-cores can be calculated; we hope to eventually obtain such a bound for 'parameterized collections of noncooperative pregames.'

A different direction of research may lead to more insight into social norms and the difficulties of achieving economic efficiency. When individuals mimic similar individuals, the metric that they place on 'similarity' is crucial. A bright and highly capable young woman, living in a low income area in Liverpool, for example, may aspire to occupations similar to those of the more successful women in her community – nurses, bank clerks, school teachers, for example – rather than occupations similar to those of successful males in her community with similar intellectual ability – doctors, bankers, school principals, for example. It may be that if the 'similarity metrics' that people use are biased to place too much weight on similarities of gender, race, color, or religion rather than on similarities of ability, interests, and so on, there may be (non-Nash) 'stable equilibrium' outcomes that are quite different than Nash outcomes. Some of the motivation for developing the current model is to explore such issues.

Related questions concern concepts of equilibrium based on imitation and learning. Typically, such equilibrium outcomes are not Nash equilibria. This is, in some senses, at odds with our Theorem 2 and its motivation. It may be fruitful to investigate what sorts of learning and imitation dynamics would lead to the sort of ε-equilibrium shown to exist in Theorem 2.

7 Appendix: Euclidean space satisfies convex separation.

7.1 Convex separation.

To provide a simple example of convex separation, we show that a closed subset of finite dimensional Euclidean space satisfies convex separation:

Lemma 3: For any finite integer M the linear space $\times_{m=1}^{M}[0,1]$ satisfies convex separation.

Proof. We define the binary relations $<$ and $=$ as follows: Take any two points $x = (x_1, ..., x_M), x' = (x'_1, ..., x'_M) \in \times_{m=1}^{M}[0,1]$. We say that $x = x'$ if $x_m = x'_m$ for all $m = 1, ..., M$. We say that $x < x'$ if either:

1. $\sum_m x_m < \sum_m x'_m$ or,

2. $\sum_m x_m = \sum_m x'_m$ and for some $m^* \in \{1, ..., M-1\}$ $x_{m^*} < x'_{m^*}$ and $x_m = x'_m$ for all $m < m^*$.

Suppose $\Omega_J = \{x_1, x_2, ..., x_J\}$ and $\Omega_Q = \{x'_1, x'_2, ..., x'_Q\}$ are two sets of points in $\times_{m=1}^{M}[0,1]$. Then, if $x_j < x'_q$ for all $x_j \in \Omega_J$ and all $x_q \in \Omega_Q$ we claim that: the convex hulls of Ω_J and Ω_Q are disjoint. Denote the convex hull of a set of points S by *con S* where

$$con\ S = \left\{ \begin{array}{c} x | x = \sum_{i=1}^{n} \beta_i s_i \text{ for some numbers } \beta_i, 0 \le \beta_i \le 1, \\ \sum_{i=1}^{n} \beta_i = 1, \text{ and for some set of points } s_i \in S \end{array} \right\}.$$

Thus, suppose the claim is false. Then there exists a point x such that $x \in con\ \Omega_J$ and $x \in con\ \Omega_Q$. Thus, for some numbers $\beta_1, ..., \beta_Q$ and $\gamma_1, ..., \gamma_Q$ we have that:

$$x = \sum_{j=1}^{J} \gamma_j x_j \text{ and } x = \sum_{q=1}^{Q} \beta_q x'_q. \tag{12}$$

which implies that:

$$\sum_{j=1}^{J} \gamma_j \sum_{m=1}^{M} x_{jm} = \sum_{q=1}^{Q} \beta_q \sum_{m=1}^{M} x'_{qm}$$

Suppose, for some x_{j*} and x'_{q*} we have that $\sum_m x_{jm} < \sum_m x'_{qm}$. Then, given that $x_j < x'_q$ for all $x_j \in \Omega_J$ and all $x'_q \in \Omega_Q$, we must have that either $\gamma_{j*} = 0$ or $\beta_{q*} = 0$. Let Ω_J^+ denote the set of $x_j \in \Omega_J$ given positive weight $\gamma_j > 0$ and Ω_Q^+ the set of all $x'_q \in \Omega_Q$ given positive weight $\beta_q > 0$. Then $\sum_m x_{jm} = \sum_m x'_{qm}$ for all $x_j \in \Omega_J^+$ and $x'_q \in \Omega_Q^+$. Then for any pair of points $x_j \in \Omega_J^+$ and $x'_q \in \Omega_Q^+$ there must exist some $m^* \in \{1, ..., M-1\}$ for which

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$x_{m^*} < x'_{m^*}$ and $x_m = x'_m$ for all $m < m^*$. Take the minimum of these m^* over all points $x_j \in \Omega_J^+$ and $x'_q \in \Omega_Q^+$. Then by 12 we have that:

$$\sum_{j=1}^{J} \gamma_j x_{jm^*} = \sum_{q=1}^{Q} \beta_q x'_{qm^*}$$

However, by choice of m^* we have that $x_{jm^*} \leq x_{qm^*}$ for all j, q and $x_{jm^*} < x_{qm^*}$ for some j, q which implies that:

$$\sum_{j=1}^{J} \gamma_j > \sum_{q=1}^{Q} \beta_q$$

giving the desired contradiction.◆

The same line of reasoning exemplified by the example above can be extended to other spaces. Here are two examples we subsequently use below. We can use the following binary relation on the space of bounded sequences of real numbers: two sequences $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$ are equal, written $x = y$, if and only if $x_i = y_i$ for all $i \in \mathcal{Z}^+$. We say that the sequence x is strictly less than the sequence y if:

$$\sum_{i=1}^{\infty} \frac{x_i}{1 + |x_i|} 2^{-i} < \sum_{i=1}^{\infty} \frac{y_i}{1 + |y_i|} 2^{-i}$$

The remaining case is one in which the above sum is equal but the two sequences are not equal. This implies that there is some smallest integer $i \in \mathcal{Z}^+$ at which they differ, say $x_i < y_i$, and in this case we say that $x < y$.

Consider now the set of real valued, bounded, measurable functions on the closed unit interval. Two functions f and g are identical, written $f = g$, only if $f(x) = g(x)$ for all $x \in [0, 1]$. If $\int_{[0,1]} f(x)d\mu < \int_{[0,1]} g(x)d\mu$ then we say $f < g$. Once again, the remaining case is when the integrals are equal but the functions are not. This implies some smallest number $x \in [0, 1]$ such that, say $f(x) < g(x)$, and thus we say $f < g$..

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