

# A Logical Reasoning Based Decision Making Method for Handling Qualitative Knowledge

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## Abstract

Successful decision-making analysis needs to take both advantages of human analysts and computers, and human knowledge is usually expressed in a qualitative way. Computer based approaches are good at handling quantitative data, while it is still challenging on how to well structure qualitative knowledge and incorporate them as part of decision analytics. This paper develops a logical reasoning based decision-making framework for handling qualitative human knowledge. In this framework, an algebraic structure is adopted for modelling qualitative human knowledge in a systematic way, and a logic based approximate reasoning method is then proposed for inferring the final decision based on the structured qualitative knowledge. By taking a non-classical logic as its formal foundation, the proposed logical reasoning based decision making method is able to model and infer with qualitative human knowledge directly without numerical approximation in a strict way.

**Keywords:** Decision making, qualitative knowledge, non-classical logic, algebraic structure, approximate reasoning

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## 1. Introduction

Decision making is the process for choosing the most appropriate one among a set of alternatives under given criteria or preferences, which is the crucial step in many real applications such as financial planning, risk analysis, products evaluation, and so on. During the past decade, the booming development of Artificial Intelligence (AI) has made society-changing in different areas, including decision making. The state of the art AI method is able to learn from a big volume of data in real time to quickly identify newly emerging unknown patterns or make the decision, given good computing resources. However, one of the problems is that most of the current approaches work like a black box where the explanation is missing, and therefore the output is arguable or out of control to certain extent.

As a canonical branch of AI, symbolism, is regaining more attention these years. There is increasing recognition for combining symbolic methods with data-driven ones, in order to make AI behavior controllable and explainable. Logic, as a typical symbolism method, can provide not only a formal way to represent real problems, but also a solid theoretical foundation to guarantee the credibility of reasoning and prediction result [39]. It is therefore of good necessity and importance to explore how to incorporate logic based symbolic methods with data-driven methods effectively.

On the other hand, appropriate decision-making analysis should take advantages of both human analysts and computer computation. Prior knowledge of human being is very helpful on enhancing data analytic for decision making, while the

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challenge is how to well structure relevant and reliable knowledge and incorporate them as part of decision analytics. Prior knowledge of human being is usually expressed in qualitative form, or natural language. Take a simple case as an example, we have the common knowledge that “if someone has a fever and cough, then it is with high possibility that he / she has caught a cold”. Unlike computer, human being is able to handle uncertain, imprecise and incomplete perceptions naturally without explicit employment of any measurements or computations. It is therefore important to investigate how to manipulate qualitative information for solving decision making problems under uncertain environment.

For modeling qualitative information, Zadeh introduced linguistic variable by associating the linguistic terms with fuzzy subsets that are modelled by membership functions [51]. This has inspired the conventional Computing with Words (CW) methodology for dealing with qualitative information that exists widely in human decision making process [52]. Contrasting to the data-driven methods, CW aims at manipulating perceptions or propositions expressed in natural language directly, rather than manipulating numerical values, and enhancing the ability of computers on handling qualitative and imprecise information besides precise data [2, 16, 33, 38, 42-44].

On the other hand, fuzzy set based CW methods usually need to define appropriate membership functions for representing the involved qualitative information, and then the generated result in the form of fuzzy set, usually based on aggregation operator, needs to be converted back to qualitative proposition [35, 36]. It is usually not a plain task to work out the appropriate membership functions for modelling the involved qualitative information which requires lots of investigations due to the subjective nature of human knowledge, and the process of transforming the qualitative information into fuzzy sets and then fuzzy sets back to qualitative proposition is time-consuming, and may cause information loss [19]. It is therefore naturally expected to be able to handle qualitative information in its original form without numerical approximation, and symbolic methods provide such a possibility [47].

Symbolic computational models represent linguistic information mainly based on ordinal scales and aggregate the information using max-min operators or convex combinations, which do not require linguistic approximation needed by the conventional fuzzy set based methods [1, 12, 20, 33]. These methods, typically fuzzy ordinal linguistic approach, usually adopt indexed linguistic labels to model the involved qualitative information, linguistic terms, which is often in ordered structure [35]. For example, as a representative method, 2-tuple linguistic computational model defines a pair  $(L_i, \alpha_i)$ , called as linguistic 2-tuple, for representing the qualitative information, where  $L_i \in S$  is a linguistic label and the numerical value  $\alpha_i \in [-0.5, 0.5)$  is used to support the ‘distance’ of the linguistic label to the closest one in the pre-defined term set, and called as symbolic translation [18, 19, 26, 32].

The advantage of fuzzy ordinal linguistic approaches relies on their ability to handle qualitative information, expressed as indexed linguistic labels, directly without information loss and the membership function definition [19, 26, 32]. On the other hand, one of the shortages is that they require the involved qualitative information being totally ordered, while the fact is that partially ordered information, especially for qualitative information due to its ambiguity, is pervasive in reality, especially when there are multiple criteria or attributes need to be considered [31]. Therefore, it is worth of studying decision making methods for handling both totally ordered and partially ordered qualitative information.

Considering the rich representation ability of lattice as a typical partially ordered algebraic structure in the areas as decision making, cybernetics and so on [6, 14, 15, 25], Ho et al. [21-24] presented Hedge algebra to represent qualitative terms. This algebraic structure is generated by applying the so-called linguistic hedges to some prime terms, where the prime terms are some words like “*true* and *false*,” or “*good* and *bad*”, and the linguistic hedges are applied on the prime terms as linguistic modifiers to strengthen or weaken their meanings. Hedge algebra is generally a partially ordered algebraic structure

as shown in Fig. 1.1, which is able to reflect the rich partially ordered semantic ordering relation among the qualitative information. On the other hand, it provides only an algebraic representation structure, while has not provided the corresponding logical reasoning related methods for further dealing with the qualitative information under consideration [9].

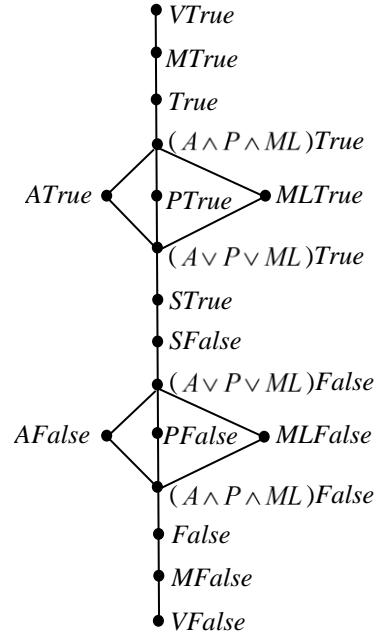


Fig. 1.1<sup>[24]</sup> A sample Hedge algebra  
 $V$ =very,  $M$ =more,  $A$ =approximately,  $P$ =possibly,  $ML$ =more or less,  $S$ =less.

From our point of view, decision making, which is to draw a collective conclusion based on given information, is a mental process that is essentially a reasoning process often with qualitative information rather than numerical calculation process [7], and decision making under qualitative and uncertain environment can be viewed as an approximate reasoning process [10]. Furthermore, the rationality of the corresponding approximate reasoning method is rooted in some non-classical logic [7, 40]. It is therefore worth of investigating the rational approximate reasoning based decision making approach with qualitative and uncertain information in the framework of certain non-classical logic [9, 47].

There are already some attempts on applying logic system and logical reasoning to decision making problems. Generally, logical propositional symbols, such as  $p_1, p_2$ , are used to represent decision making statements, with logical connectives, such as  $\wedge$  (and),  $\vee$  (or) and  $\rightarrow$  (if-then), for generating more complex propositions [3, 4, 5]. Logical reasoning methods are then applied to work on these logical propositions to infer the final comprehensive decision [11].

In order to provide a logic foundation for intelligent information processing, Xu et al. [49] proposed a series of non-classical logic systems, lattice-valued logic systems, which take a kind of logical algebraic structure, lattice implication algebra (LIA), as their truth-value fields. LIA, by combining lattice and implication algebra, serves as not only a class of efficient algebraic structure for representing uncertain information, but also an appropriate link with the non-classical logic systems. In order to make the general LIA based lattice-valued logic more specific for decision making problems under qualitative and uncertain environment, a series of linguistic truth-valued LIAs (L-LIAs) [28, 45, 48] were constructed for representing qualitative information, and the corresponding linguistic truth-valued logic system [29, 46], approximate reasoning approaches [10, 48, 50] and related decision making methods [30, 53, 54] were then proposed. Liu et al. [28] summarized some ideas on lattice ordered linguistic decision making, and presented a systematic framework for lattice ordered linguistic decision making problems from the viewpoint of lattice structure representation and logical reasoning.

On the other hand, these lattice based decision making methods are mainly adopting lattice for representing the qualitative information involved in decision making problems, while the decision result is still obtained by applying certain aggregation operators, instead of logical reasoning, to combine the transformed qualitative information. In order to bridge the gap between algebraic representation and logical reasoning for decision making, we have proposed a linguistic decision making approach based on approximate reasoning, where a linguistic valued algebraic structure is constructed to represent the linguistic information under consideration, and then the logic operators in a lattice-valued logic system, named LP(X) [49], are applied on the linguistic information to infer the final decision directly without numerical approximation. This approach is able to deal with both totally ordered and partially ordered linguistic information directly without numerical approximation, and applies logic based approximate reasoning to infer the final decision result. On the other hand, the logic based approximate reasoning approach proposed in [10] mainly applies some logical operators, similar as aggregation operators, to combine the qualitative decision information, while the justification of the rationality of the decision result from the viewpoint of logic is missing. We think that the rationality of logic based decision making methods comes from its consistency between semantic and syntax parts. In other words, the decision making process and result should not only take the logical representation form, but also semantically sound and syntactically provable in logic, which reflects the essential advantage, strictness, of logic based method. Therefore, we propose in this paper a new logical reasoning based decision making approach base on a gradational lattice-valued logic  $L_{vpl}$ , where the decision result is not only logically interpretable, but also semantically sound and syntactically provable in logic.

The structure of the remainder part is as follows. Section 2 provides the related preliminary knowledge about the logical algebraic structure LIA and the corresponding lattice-valued logical system  $L_{vpl}$ . An algebraic structure is then introduced in Section 3 for representing the qualitative information in a structured way, which also links with the consequent logical approximate reasoning based decision making process that is detailed in Section 4. Section 5 gives an illustrative example to show the feasibility of the proposed approach. Conclusions and discussions are provided in Section 6.

## 2. Preliminary knowledge about LIA and $L_{vpl}$

This section reviews some related concepts and notations about the logical algebraic structure LIA and the corresponding lattice-valued logical system  $L_{vpl}$ . The preliminary knowledge of LIA and L-LIA is reviewed firstly, and the readers may refer to [8, 29, 48, 49] for more details about LIA and  $L_{vpl}$ .

### 2.1 Preliminary concepts of LIA

LIA [49] is proposed by combining lattice and implication algebra, which can then take both the advantages of them, i.e., the efficient representation ability of lattice, and more logical operations, especially “*implication*,” so as to link with logic systems.

**Definition 2.1**<sup>[49]</sup> Suppose that  $(L, \vee, \wedge, O, I)$  is a bounded lattice, where  $O$  and  $I$  are the smallest element and greatest element of  $L$  respectively. We define two operations, an order-reversing involution “ $'$ ” and an implication  $\rightarrow$ :  $L \times L \rightarrow L$  on  $L$ , then  $(L, \vee, \wedge, ', \rightarrow)$  is called a *lattice implication algebra* (LIA) if, for any  $x, y, z \in L$ , it satisfies the following conditions:

- (I<sub>1</sub>)  $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$ ;
- (I<sub>2</sub>)  $x \rightarrow x = I$ ;
- (I<sub>3</sub>)  $x \rightarrow y = y' \rightarrow x'$ ;
- (I<sub>4</sub>)  $x \rightarrow y = y \rightarrow x = I$  implies  $x = y$ ;
- (I<sub>5</sub>)  $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$ ;

$$(I_1) (x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z);$$

$$(I_2) (x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z).$$

**Example 2.1**<sup>[49]</sup> (**Łukasiewicz implication algebra on a finite chain**). Suppose that  $L$  is a finite chain, i.e. its elements are totally ordered, denoted as  $L = \{x_i | 1 \leq i \leq n\}$  and  $O = x_1 < x_2 < \dots < x_n = I$ . For any  $x_i, x_j \in L$ , the corresponding operations are defined as follows.

$$x_i \vee x_j = x_{\max(i,j)}, x_i \wedge x_j = x_{\min(i,j)}, (x_i)' = x_{n-i+1}, x_i \rightarrow x_j = x_{\min(n-i+j, n)}.$$

The above defined  $(L, \vee, \wedge, ', \rightarrow)$  is an LIA, called as Łukasiewicz chain and denoted by  $L_n$ .

Given different numbers of elements, Łukasiewicz chain can be used for modelling different sets of totally ordered qualitative terms in decision making problems. For example, the prime terms of “*dissatisfied*” and “*satisfied*” for describing the evaluations about certain products can be modelled by a Łukasiewicz chain with two elements, while the linguistic modifiers “*absolutely, highly, very, quite, exactly*” can be modelled by a Łukasiewicz chain with five elements.

The following proposition gives some properties of the operations on LIA that will be used in the proposed logical reasoning based decision making method, and interesting readers may refer to [48, 49] for more information.

**Proposition 2.1** Let  $(L, \vee, \wedge, ', \rightarrow)$  be an LIA. For any  $x, y, z \in L$ ,

- (1)  $x \vee y = (x \rightarrow y) \rightarrow y, x \wedge y = (x' \vee y')'$ ;
- (2)  $x \rightarrow O = x', O \rightarrow x = I, x \rightarrow I = I, I \rightarrow x = x$ ;
- (3)  $x \rightarrow y = O$  iff  $x = I$  and  $y = O$ ;
- (4)  $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$ ;
- (5)  $x \rightarrow y \geq x' \vee y$ ;
- (6) If  $x \leq y$ , then  $x \rightarrow z \geq y \rightarrow z$ ;
- (7) If  $y \leq z$ , then  $x \rightarrow y \leq x \rightarrow z$ ;
- (8)  $x \leq y$  if and only if  $x \rightarrow y = I$ .

## 2.2 Related concepts in a lattice-valued logic based on LIA

Some related concepts of the lattice-valued logic  $L_{vpl}$  that takes LIA as its truth-value field are reviewed in this subsection. The logical reasoning based decision making approach will be described using this logic system, and then formalized by the approximate reasoning process in  $L_{vpl}$ . In the following, we always assume that  $(L, \vee, \wedge, ', \rightarrow, O, I)$  is an LIA, in short  $L$ .

**Definition 2.2**<sup>[49]</sup> Let  $X$  be a set of propositional variables,  $X = \{p, q, r, \dots\}$ ,  $T = L \cup \{\vee, \wedge, ', \rightarrow\}$  be a type with  $ar(\vee) = 2$ ,  $ar(\wedge) = 2$ ,  $ar(') = 1$ ,  $ar(\rightarrow) = 2$  and  $ar(a) = 0$  for any  $a \in L$ . The propositional algebra of the lattice-valued propositional calculus on the set  $X$  of propositional variables is a free  $T$  algebra on  $X$  and is denoted by  $L_{vpl}$ .

We have some additional operations defined as follows.

$$p \leftrightarrow q \triangleq (p \rightarrow q) \wedge (q \rightarrow p) \quad (2.1)$$

$$p \otimes q \triangleq (p \rightarrow q')' \quad (2.2)$$

The propositional variables defined in Definition 2.2 are used to represent statements in decision making problems, and the operations, also called logical connectives, is to link the simpler statements to generate compound ones. These operations take the intuitive meanings as in real situations, where  $\wedge$  means *and*,  $\vee$  means *or*,  $'$  means *negation*,  $\rightarrow$  means *if-then* and  $\otimes$  takes the similar meaning as  $\wedge$ .

Denote the set of all formulae of  $L_{vpl}$  as  $\mathcal{F}_p$ , and it can be seen that  $(\mathcal{F}_p, \vee, \wedge, ', \rightarrow)$  is an algebra with the same type as  $(L, \vee, \wedge, ', \rightarrow)$ . Let  $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$ , where  $\mathcal{F}_L(\mathcal{F}_p)$  is the set of all  $L$ -type fuzzy set of  $\mathcal{F}_p$ , and denote  $\mathcal{T}_h \subseteq \{T | T: \mathcal{F}_p \rightarrow L \text{ is a homomorphic mapping}\} \triangleq \mathcal{T}_H$ .

For any  $T \in \mathcal{T}_H$ , it can be seen as an valuation of  $L_{vpl}$  that maps a formula to its truth-value in  $L$ . In real decision making problems under qualitative environments, propositional logical formula  $p$  is used to express an evaluation statement, such as “Dell is *good*,” and  $\alpha \in L$  means the truth-value level of this statement, e.g., “*very true*”. It means that “Dell is *very good*” when the statement is associated with its truth-value.

**Definition 2.3**<sup>[49]</sup> For  $D_n \subseteq \mathcal{F}_p^n$ , the mapping  $r_n$  defined as follows is called as an  $n$ -ary partial operation on  $\mathcal{F}_p$ .

$$r_n: D_n \rightarrow \mathcal{F}_p \quad (2.3)$$

$D_n$  is called the domain of  $r_n$  and denoted as  $D_n(r_n)$ .

**Definition 2.4**<sup>[49]</sup> A mapping

$$t_n: L^n \rightarrow L \quad (2.4)$$

is said to be an  $n$ -ary truth-valued operation on  $L$ , if

- (1)  $\alpha \rightarrow t_n(\alpha_1, \dots, \alpha_n) \geq t_n(\alpha \rightarrow \alpha_1, \dots, \alpha \rightarrow \alpha_n)$  holds for any  $\alpha \in L$  and  $(\alpha_1, \dots, \alpha_n) \in L^n$ , and
- (2)  $t_n$  is isotone in each argument.

Denote

$$R_n \subseteq \{r_n | r_n \text{ is an } n\text{-ary partial operation on } \mathcal{F}_p\}, T_n \subseteq \{t_n | t_n \text{ is an } n\text{-ary truth-valued operation on } L\},$$

$$\mathcal{R}_n \subseteq R_n \times T_n, \mathcal{R} \subseteq \bigcup_{n=0}^{+\infty} \mathcal{R}_n. \quad (2.5)$$

If  $(r, t) \in \mathcal{R}_n$ , then  $(r, t)$  is called as an  $n$ -ary rule of inference in  $L_{vpl}$ .

**Example 2.2** we can express the MP rule in the form of rule of inference in  $L_{vpl}$  as follows.

$$r_2^0: D_2^0 \rightarrow \mathcal{F}_p, (p, p \rightarrow q) \mapsto q,$$

where  $D_2^0 = \{(p, p \rightarrow q) | p, q \in \mathcal{F}_p\}$ .

**Remark 2.1** It is noted that there are two parts in the rules of inference in  $L_{vpl}$ , where  $r$  is used for the formal syntactic inference as in classical logic, while  $t$  is to transfer the truth-value along with the formal inference process. The truth-value operation part is not included in classical logic, because in which the inference is always kept as true, or with truth-value degree always being 1. Under qualitative decision making environments,  $r$  is for describing the formal process from the evaluations of each candidate to the final decision, while  $t$  describes the transmission of truth degrees along with the decision process under uncertain environment.

**Definition 2.5**<sup>[49]</sup> Suppose that  $X \in \mathcal{F}_L(\mathcal{F}_p)$ ,  $(r, t) \in \mathcal{R}_n$ ,  $\alpha \in L$ .  $X$  is said to be  $\alpha$  – I type closed w.r.t.  $(r, t)$  in  $D_n(r)$ , if

$$X \circ r \supseteq \alpha \otimes (t \circ \prod_n X), \quad (2.6)$$

and  $X$  is said to be  $\alpha$  – II type closed w.r.t.  $(r, t)$ , if

$$X \circ r \supseteq t \circ \prod_n (\alpha \otimes X). \quad (2.7)$$

If  $X$  is  $\alpha - I$  ( $\alpha - II$ ) type closed w.r.t.  $(r, t)$  for any  $(r, t) \in \mathcal{R}$ , then  $X$  is called as  $\alpha - I$  ( $\alpha - II$ ) type closed w.r.t.  $\mathcal{R}$ . Note that  $X$  is  $I - I$  type closed w.r.t.  $(r, t)$  if and only if  $X$  is  $I - II$  type closed w.r.t.  $(r, t)$ .

**Remark 2.2** Definition 2.5 and the following Definitions 2.6 and 2.7 are used to show the degreed consistency between valuations and inference rules, which are actually to guarantee the validity of the logical reasoning process. The parameter  $\alpha$  can be seen as the level of consistency between semantics and syntax of the inference rules, which can be interpreted as the belief degree of each decision rule in the rule base for qualitative decision making problems. The following two theorems show that this consistency still holds with the same degree after certain operations.

**Theorem 2.1**<sup>[49]</sup> Let  $X \in \mathcal{F}_L(\mathcal{F}_p)$ . If  $X$  is  $\alpha - i$  type closed w.r.t.  $\mathcal{R}$ , then  $Y = \beta \rightarrow X$  is  $\alpha - i$  type closed w.r.t.  $\mathcal{R}$  for any  $\beta \in L$  and  $i=I, II$ .

**Theorem 2.2**<sup>[49]</sup> Let  $\alpha \in L$ . If

$$\emptyset \neq \mathcal{U} \subseteq \{X | X \text{ is } \alpha - i \text{ type closed w.r.t. } \mathcal{R}\}, \quad (2.8)$$

then  $\bigcap_{X \in \mathcal{U}} X$  is  $\alpha - i$  type closed w.r.t.  $\mathcal{R}$  for  $i=I, II$ .

For  $p, q, g \in \mathcal{F}_p$ ,  $\alpha, \beta, \gamma, \theta, \theta_0 \in L$ , we define three specific and one type of inference rules in  $L_{vpl}$  that will be used in the following logical reasoning based decision making process.

$$\mathcal{R}^* = \{(r_2^0, t_2^*), (r_2^*, t_2^*), (r_2^\Delta, t_2^*)\} \cup \{(r_1^{\theta_0}, t_1^{\theta_0}) | \theta_0 \in L\} \subseteq \mathcal{R}, \quad (2.9)$$

where  $r_2^0$  is defined as in Example 2.2, and

$$t_2^*(\theta, \beta) = \theta \wedge \beta, r_2^*(p \rightarrow g, p \rightarrow q) = p \rightarrow (g \wedge q), r_2^\Delta(p \rightarrow g, g \rightarrow q) = p \rightarrow q$$

$$r_1^{\theta_0}(p) = \theta_0 \rightarrow q, t_1^{\theta_0}(\theta) = \theta_0 \rightarrow \theta.$$

It can be seen that the inference rules defined above are typical in decision making problems. For example, the truth-value operator  $t_2^*$  is to take the minimum truth values of both initial evaluations as that of the compound evaluation. It is also noted that for any  $T \in \mathcal{F}_H$ ,  $T$  is  $\alpha - i$  type closed w.r.t.  $(r_2^0, t_2^*), (r_2^*, t_2^*), (r_2^\Delta, t_2^*)$  and  $(r_1^{\theta_0}, t_1^{\theta_0})$  respectively for  $i=I, II$ .

There are usually two parts in a logic system, i.e., semantics and syntax, where semantics is about the truth value transition along the logical reasoning process and syntax is about the formal deduction in logic system. The following definitions are about the semantics of  $L_{vpl}$ .

**Definition 2.6**<sup>[49]</sup> Let  $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$ . Two kinds of mappings are defined as follows.

$$(1) \quad C_{\mathcal{T}}: \mathcal{F}_L(\mathcal{F}_p) \rightarrow \mathcal{F}_L(\mathcal{F}_p),$$

$$X \mapsto C_{\mathcal{T}}^X,$$

where  $C_{\mathcal{T}}^X(p) = \bigwedge_{T \in \mathcal{T}} [\pi(X \subseteq T) \rightarrow T(p)]$ , and  $\pi(X \subseteq T) = \bigwedge_{q \in \mathcal{F}_p} [X(q) \rightarrow T(q)]$ .

$$(2) \quad C_{(C_{\mathcal{T}}^\emptyset, \mathcal{R}(\alpha-i))}^\beta: \mathcal{F}_L(\mathcal{F}_p) \rightarrow \mathcal{F}_L(\mathcal{F}_p),$$

$$X \mapsto C_{(C_{\mathcal{T}}^\emptyset, \mathcal{R}(\alpha-i))}^{\beta, X},$$

where  $C_{(C_{\mathcal{T}, \mathcal{R}(\alpha-i)}^{\emptyset})}^{\beta, X}(p) = \bigwedge \{Y(p) | Y \supseteq \beta \otimes (C_{\mathcal{T}}^{\emptyset} \cup X), Y \text{ is } \alpha - i \text{ type closed w.r.t. } \mathcal{R}\}, i=I, II, \text{ and } \alpha, \beta \in L.$

**Definition 2.7**<sup>[49]</sup> Let  $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$ ,  $\alpha \in L$ ,  $\mathcal{R}$  is said to be  $\alpha - i$  type sound w.r.t.  $\mathcal{T}$  if  $\mathcal{T}$  is  $\alpha - i$  type closed w.r.t.  $\mathcal{R}$  for any  $T \in \mathcal{T}$ ,  $i=I, II$ .

**Theorem 2.3**<sup>[49]</sup>  $C_{\mathcal{T}}$  is a closure operation, i.e.,  $X \subseteq C_{\mathcal{T}}^X$ ,  $C_{\mathcal{T}}^X \subseteq C_{\mathcal{T}}^Y$  if  $X \subseteq Y$ , and  $C_{\mathcal{T}}^X = C_{\mathcal{T}}^{C_{\mathcal{T}}^X}$ .

**Remark 2.3** It can be seen from Definition 2.6 that  $C_{\mathcal{T}}^X(p)$  is the degree or level of  $p$  being semantically deduced from  $X$  under the valuation  $\mathcal{T}$ . Therefore,  $C_{\mathcal{T}}$  can be seen as a gradational semantic closure operation. Under the decision making circumstance,  $C_{\mathcal{T}}^X$  means the level at which a specific evaluation  $X$  of a candidate can be inferred from the general evaluations  $T$  based on human knowledge, where the inference is reflected by the implication operator  $\rightarrow$  in the definition.

**Definition 2.8**<sup>[49]</sup> Let  $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$ ,  $X \in \mathcal{F}_L(\mathcal{F}_p)$ ,  $\alpha, \beta, \tau \in L$ . If

$$\bigvee \left\{ C_{(C_{\mathcal{T}, \mathcal{R}(\alpha-i)}^{\emptyset})}^{\beta, X}(p) \otimes C_{(C_{\mathcal{T}, \mathcal{R}(\alpha-i)}^{\emptyset})}^{\beta, X}(p') | p \in \mathcal{F}_p \right\} \leq \tau,$$

then  $X$  is said to be  $\tau' - i$  type consistent with respect to  $(\alpha, \beta)$ .

**Remark 2.4** It can be seen from Definition 2.8 that the consistency degree of  $X$  means that there is no contradiction in this logical proposition to certain degree.

The following definitions are about the syntax of  $L_{vpl}$ .

**Definition 2.9**<sup>[49]</sup> Let  $X \in \mathcal{F}_L(\mathcal{F}_p)$ ,  $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$ ,  $p \in \mathcal{F}_p$ ,  $\alpha, \beta, \theta \in L$ . The mapping  $P^i$  is defined as

$$\begin{aligned} P^i: (n) &\rightarrow \mathcal{F}_p \times L \quad ((n) = \{1, 2, \dots, n\}), \\ i &\mapsto (p_i, \theta_i). \end{aligned}$$

$(P^i, (n), X, (p, \theta) - (\alpha, \beta))$  is said to be an  $(\alpha, \beta) - i$  type proof with the truth degree  $\theta$  from  $X$  to  $p$  (shortly,  $\theta - (\alpha, \beta) - i$  type proof from  $X$  to  $p$ ), where  $n$  is said to be the length of  $\theta - (\alpha, \beta) - i$  type proof from  $X$  to  $p$  under  $P^i$ , and denoted as  $l(P^i)$ , if the following conditions hold.

- (1)  $(p_n, \theta_n) = (p, \theta)$  and
- (2)  $\theta_i = \beta \otimes C_{\mathcal{T}}^{\emptyset}(p_i)$ , or
- (3)  $\theta_i = \beta \otimes X(p_i)$ , or
- (4) there exist  $i_1, i_2, \dots, i_k \leq i$ , and  $(r, t) \in \mathcal{R}_k$  such that

$$(p_i, \theta_i) = (r(p_{i_1}, \dots, p_{i_k}), \alpha \otimes t(\theta_{i_1}, \dots, \theta_{i_k})), i=I, \text{ or } (p_i, \theta_i) = (r(p_{i_1}, \dots, p_{i_k}), t(\alpha \otimes \theta_{i_1}, \dots, \alpha \otimes \theta_{i_k})), i=II.$$

It can be seen from Definition 2.9 that the proof sequence in  $L_{vpl}$  is degreed by several parameters, where  $\alpha$  can be seen as the truth value of the knowledge rule in decision making problem,  $\beta$  as the belief degree associated with each inference step, and  $\theta$  as the truth degree of the syntactical proof [8].

The following theorems show that the syntax and semantics of  $L_{vpl}$  are consistent with each other, which will be used to guarantee that the decision result, generated by the proposed logical reasoning based decision making approach, is not only

semantically interpretable as made by usual decision making methods, but also logically sound in a formal way.

**Theorem 2.4**<sup>[49]</sup> Let  $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$ ,  $X \in \mathcal{F}_L(\mathcal{F}_p)$ ,  $\alpha, \beta \in L$ , and the truth-valued operations in  $\mathcal{R}$  satisfy finite semi-continuity, then for any  $p \in \mathcal{F}_p$ ,  $i=I, II$ ,

$$C_{(C_{\mathcal{T}, \mathcal{R}(\alpha-i)}^\emptyset)}^{\beta, X}(p) = \bigvee \{ \theta \mid \text{there exists } (P^i, (n), X, (p, \theta) - (\alpha, \beta)) \}. \quad (2.10)$$

**Theorem 2.5**<sup>[49]</sup> Let  $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$ ,  $X \in \mathcal{F}_L(\mathcal{F}_p)$ ,  $\alpha, \beta \in L$ .  $\mathcal{R}$  is  $\alpha$ - $i$  type sound w.r.t.  $\mathcal{T}$ , and  $C_{(C_{\mathcal{T}, \mathcal{R}(\alpha-i)}^\emptyset)}^{\beta, X} \in \mathcal{T}$ , then for  $i=I, II$ ,

$$C_{(C_{\mathcal{T}, \mathcal{R}(\alpha-i)}^\emptyset)}^{\beta, X} = C_{\mathcal{T}}^{\beta \otimes X}. \quad (2.11)$$

### 3. Algebraic Structure for Representing Qualitative Information

Herrera and Herrera-Viedma [17] introduced a linguistic decision making scheme for decision analysis under qualitative environment by developing the common one, which is generally made of three steps: (1) The choice of the linguistic term set with its semantics. (2) The choice of the aggregation operator of linguistic information. (3) The choice of the best alternatives, which is realized by the two phases of the common decision resolution scheme, i.e., aggregation phase and exploration phase. This has then been widely adopted as a common scheme for decision making under qualitative and uncertain environment [32-34].

We follow but extend this scheme for logical reasoning based decision making as follows: (1) The choice of the appropriate structure for representing qualitative information. (2) The choice of the logic system and reasoning operations of qualitative information. (3) The choice of the best alternatives realized by logical approximate reasoning process. This section focuses on the first step about choosing the appropriate structure for representing the qualitative information under consideration, and an algebraic structure, a L-LIA, is introduced for representing the qualitative information involved in decision making problems.

As mentioned in Section 1, in order to make this general algebraic structure, LIA, more specific for modelling the qualitative information in decision making under uncertain environment, linguistic truth-valued LIAs, a special class of LIA, were constructed, and denoted as L-LIAs [28, 29, 45]. Actually, the construction of L-LIA follows the similar way as that of Hedge algebra, i.e., applying a set of linguistic modifiers to the prime terms for generating an algebraic representation structure. For the prime terms, we choose “*dissatisfied* and *satisfied*” as the proposed approach is mainly for decision making problems with qualitative information, similarly as in our previous paper [10]. For the linguistic modifiers, we adopt the similar ones as in [10], i.e.,  $H = H^+ \cup H^-$ , where  $H^+$  consists of modifiers strengthening the prime terms with  $H^+ = \{absolutely, highly, very, quite, exactly\}$ , and the modifiers in  $H^-$  are those that weaken the prime terms with  $H^- = \{almost, rather, somewhat, slightly\}$ .

We denote the ordering relation between two modifiers  $a$  and  $b$  as  $a \leq b$  if and only if  $a(satisfied) \leq b(satisfied)$  as the same as in natural language.

The generated algebraic structure is a LIA by applying the linguistic modifiers in  $H$  to the prime terms, with the further defined operations  $\vee, \wedge, ', \rightarrow$  as shown in the following Proposition 3.1.

**Proposition 3.1**<sup>[10]</sup> Let  $L_s, L_t$  be two LIAs and  $L_s = \{m_1, \dots, m_s | m_1 \leq \dots \leq m_s\}$ ,  $L_t = \{p_1, \dots, p_t | p_1 \leq \dots \leq p_t\}$ ,  $m_i \rightarrow m_j = m_{s \wedge (s-i+j)}$ ,  $m'_i = m_i \rightarrow m_1$ ,  $p_k \rightarrow p_l = p_{t \wedge (t-k+l)}$ ,  $p'_k = p_k \rightarrow p_1$ . Define the product of  $L_s$  and  $L_t$  as  $L_s \times L_t = \{(m, p) | m \in L_s, p \in L_t\}$ , and the operations on  $L_s \times L_t$  as follows:

$$(m_i, p_j)' = (m'_i, p'_j), (m_i, p_j) \rightarrow (m_k, p_l) = (m_i \rightarrow m_k, p_j \rightarrow p_l). \quad (3.1)$$

Then  $(L_s \times L_t, \vee, \wedge, ', \rightarrow)$  is a LIA with  $(m_1, p_1)$  and  $(m_s, p_t)$  being its smallest and greatest elements, denoted by  $L_{s \times t}$ .

According to Proposition 3.1, the L-LIA generated by the product of two Łukasiewicz chains  $L_{s \times 2} = L_s \times L_2$  is a LIA, where  $L_s = \{m_1, m_2, \dots, m_s\}$  ( $m_1 < m_2 < \dots < m_s$ ) is the set of the linguistic modifiers, and  $L_2 = \{p_1, p_2\}$  ( $p_1 < p_2$ ) is the set of the prime terms, e.g.,  $\{p_1, p_2\} = \{\text{Dissatisfied (Ds for short), Satisfied (Sa)}\}$  with  $Ds < Sa$ . For the linguistic modifiers, we adopt 9 modifiers as this is able to meet the general requirements of decision making problems with qualitative information [10, 45]. These modifiers are denoted as  $L_9 = \{\text{Slightly (Sl), Somewhat (So), Rather (Ra), Almost (Al), Exactly (Ex), Quite (Qu), Very (Ve), Highly (Hi), Absolutely (Ab)}\}$  whose natural semantic ordering relations are  $Sl < So < Ra < Al < Ex < Qu < Ve < Hi < Ab$ . It is worth mentioning that the above L-LIA provides a general framework for constructing the algebraic representation structure, and it is of course possible to select a L-LIA consisting other number of, e.g., 3 or 5, linguistic modifiers for representing evaluation terms given different decision making requirements.

It is noted that the natural meaning of applying linguistic modifiers on the prime term *dissatisfied* is reversed to the ordering. For example, *quite dissatisfied* < *exactly dissatisfied* in natural language, while *exactly dissatisfied* < *quite dissatisfied* in the  $L_{9 \times 2}$  as defined in Proposition 3.1. Therefore, we define a mapping  $f: L\text{-LIA} \rightarrow L_{9 \times 2}$  as follows such that the ordering in the constructed L-LIA keeps the same ordering as its natural meaning.

$$\begin{aligned} f(Ab, Sa) &= (m_9, p_2), f(Hi, Sa) = (m_8, p_2), f(Ve, Sa) = (m_7, p_2), f(Qu, Sa) = (m_6, p_2), f(Ex, Sa) = (m_5, p_2), f(Al, Sa) = (m_4, p_2), f(Ra, Sa) = (m_3, p_2), \\ f(So, Sa) &= (m_2, p_2), f(Sl, Sa) = (m_1, p_2), f(Sl, Ds) = (m_9, p_1), f(So, Ds) = (m_8, p_1), f(Ra, Ds) = (m_7, p_1), f(Al, Ds) = (m_6, p_1), \\ f(Ex, Ds) &= (m_5, p_1), f(Qu, Ds) = (m_4, p_1), f(Ve, Ds) = (m_3, p_1), f(Hi, Ds) = (m_2, p_1), f(Ab, Ds) = (m_1, p_1). \end{aligned}$$

It can be seen that the mapping  $f$  reverses the *dissatisfied* part, while keep the *satisfied* part unchanged, and it is a bijection with its inverse mapping denoted as  $f^{-1}$ . In addition, it is easy to show that  $(L\text{-LIA}, \vee, \wedge, ', \rightarrow)$  is a LIA if for any  $x, y \in L$ ,

$$\begin{aligned} x \vee y &= f^{-1}(f(x) \vee f(y)), x \wedge y = f^{-1}(f(x) \wedge f(y)), \\ x \rightarrow y &= f^{-1}(f(x) \rightarrow f(y)), x' = f^{-1}((f(x))'). \end{aligned}$$

Given  $f$  being a bijection, it can be easily shown that L-LIA is isomorphic to  $L_{9 \times 2}$ , and so we use  $L_{9 \times 2}$  still for denoting the L-LIA with 9 linguistic modifiers in the following.

Fig. 3.1 shows the Hasse diagram of L-LIA  $L_{9 \times 2}$  with  $I=(m_9, p_2)$ ,  $A=(m_8, p_2)$ ,  $B=(m_7, p_2)$ ,  $C=(m_6, p_2)$ ,  $D=(m_5, p_2)$ ,  $E=(m_4, p_2)$ ,  $F=(m_3, p_2)$ ,  $G=(m_2, p_2)$ ,  $H=(m_1, p_2)$ ,  $R=(m_9, p_1)$ ,  $J=(m_8, p_1)$ ,  $K=(m_7, p_1)$ ,  $S=(m_6, p_1)$ ,  $M=(m_5, p_1)$ ,  $N=(m_4, p_1)$ ,  $P=(m_3, p_1)$ ,  $Q=(m_2, p_1)$ ,  $O=(m_1, p_1)$ , where the ordering relations between the linguistic terms are shown clearly.

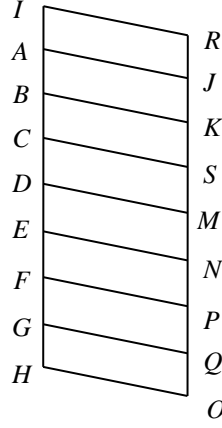


Fig. 3.1<sup>[10]</sup>. Hasse diagram of L-LIA  $L_{9 \times 2}$ .

Although it shares the similar idea with Hedge algebra on constructing the algebraic representation structure, the presented L-LIA takes the advantages of having close relationship with logical systems, giving the introduction of implication operation. As a consequence, the logical system with L-LIA being its truth-value field is then established and the corresponding approximate reasoning approach is introduced for decision making problems under qualitative and uncertain environment, which makes the rational decision making possible.

We introduce two additional operations,  $\otimes$  and  $\oplus$ , in L-LIA as in Definition 3.1 and some of their properties in Proposition 3.2. These two operations will be used in the logical reasoning based decision making approach as presented in Section 4.

**Definition 3.1**<sup>[49]</sup> For any  $m_i, m_k \in L_s, p_j, p_l \in L_t$ ,

$$(m_i, p_j) \otimes (m_k, p_l) = ((m_i, p_j) \rightarrow (m_k, p_l))' \quad (3.2)$$

$$(m_i, p_j) \oplus (m_k, p_l) = (m_i, p_j)' \rightarrow (m_k, p_l) \quad (3.3)$$

Actually, operation  $\otimes$  works similarly like the algebraic operation  $\wedge$  and arithmetic operation  $\times$ , and  $\oplus$  takes the similar role as  $\vee$  and  $+$  respectively.

**Proposition 3.2**<sup>[49]</sup> For any  $(m_i, p_j), (m_k, p_l), (m_s, p_t) \in \text{L-LIA}$ , and  $(m_1, p_1), (m_s, p_t)$  be the smallest and greatest element of L-LIA respectively, the following statements hold.

- (1)  $(m_i, p_j) \otimes (m_k, p_l) = (m_i \otimes m_k, p_j \otimes p_l), \quad (m_i, p_j) \oplus (m_k, p_l) = (m_i \oplus m_k, p_j \oplus p_l);$
- (2)  $(m_i, p_j) \otimes (m_k, p_l) = (m_k, p_l) \otimes (m_i, p_j), \quad (m_i, p_j) \oplus (m_k, p_l) = (m_k, p_l) \oplus (m_i, p_j);$
- (3)  $((m_i, p_j) \otimes (m_k, p_l)) \otimes (m_s, p_t) = (m_i, p_j) \otimes ((m_k, p_l) \otimes (m_s, p_t));$
- (4)  $((m_i, p_j) \oplus (m_k, p_l)) \oplus (m_s, p_t) = (m_i, p_j) \oplus ((m_k, p_l) \oplus (m_s, p_t));$
- (5)  $((m_i, p_j) \otimes (m_k, p_l))' = (m_i, p_j)' \oplus (m_k, p_l)', \quad ((m_i, p_j) \oplus (m_k, p_l))' = (m_i, p_j)' \otimes (m_k, p_l)';$
- (6)  $(m_i, p_j) \otimes (m_k, p_l) \leq (m_i, p_j) \wedge (m_k, p_l), \quad (m_i, p_j) \oplus (m_k, p_l) \geq (m_i, p_j) \vee (m_k, p_l);$
- (7)  $(m_1, p_1) \otimes (m_i, p_j) = (m_1, p_1), \quad (m_s, p_t) \otimes (m_i, p_j) = (m_i, p_j), \quad (m_i, p_j) \otimes (m_i, p_j)' = (m_1, p_1);$
- (8)  $(m_1, p_1) \oplus (m_i, p_j) = (m_i, p_j), \quad (m_s, p_t) \oplus (m_i, p_j) = (m_s, p_t), \quad (m_i, p_j) \oplus (m_i, p_j)' = (m_s, p_t);$
- (9)  $(m_i, p_j) \rightarrow ((m_i, p_j) \otimes (m_k, p_l)) = (m_i, p_j)' \vee (m_k, p_l) = ((m_i, p_j) \oplus (m_k, p_l)) \rightarrow (m_k, p_l);$
- (10)  $((m_i, p_j) \rightarrow (m_k, p_l)) \otimes (m_i, p_j) = (m_i, p_j) \wedge (m_k, p_l).$

With this linguistic algebraic structure,  $L_{9 \times 2}$ , being constructed, the decision making problem with qualitative information under uncertain environment can then be outlined as follows.

Given a set of alternatives  $A = \{a_1, \dots, a_n\}$  under evaluation, and some decision making criteria or attributes that are denoted as  $C = \{c_1, \dots, c_m\}$ , each alternative is then evaluated under these criteria. The experts may evaluate the alternatives using qualitative expressions that are modelled by the L-LIA  $E = L_{9 \times 2}$  presented in this section, where the provided evaluations are denoted as  $R = \{r_1, \dots, r_m\}$  with  $r_i = (\alpha_i, \beta_i) \in E = L_{9 \times 2}$ . The criteria are usually assigned with the corresponding weights  $W = \{\omega_1, \dots, \omega_m\}$  that are also in qualitative form modelled by the modifiers in  $L_9$ , i.e.,  $\omega_i \in L_9$ . The weights are actually the contribution degrees of the criteria to the overall evaluation, just expressed in qualitative form. The multi-criteria decision making task is then to reach a comprehensive decision based on the provided expert evaluations. It is noted that the primary evaluations and criterion weights are usually determined by the experts subjectively, which reflect the domain knowledge from the experts. There are of course some methods on utilizing the quantitative weights or objective ways to determine evaluations and weights, which will be discussed in another paper.

For example, when evaluating several types of laptops, e.g., Dell, Apple, Lenovo and HP, suppose that we have four criteria (attributes): processing speed ( $c_1$ ), price ( $c_2$ ), user friendship ( $c_3$ ) and battery life ( $c_4$ ). We may have certain knowledge from experts or the public about one type of laptop as “Its processing speed is *highly satisfied*, price is *somewhat dissatisfied*, user friendship is *very satisfied*, and battery life is *quite satisfied*”, which can be denoted as  $r_1=(m_8, p_2)$ ,  $r_2=(m_2, p_1)$ ,  $r_3=(m_7, p_2)$ ,  $r_4=(m_6, p_2)$ . It is assumed based on common knowledge that the weights corresponding to the four criteria are *highly*, *quite*, *very* and *somewhat*, which can be seen as the contribution degrees of the four criteria to the compound evaluation. The logical reasoning based decision making under qualitative environment is then proposed in Section 4 to apply an appropriate reasoning mechanism for inferring a comprehensive decision based on the provided qualitative knowledge.

#### 4. Logical Reasoning Decision Making with Qualitative Knowledge

According to the logical reasoning based decision making scheme presented at the beginning of Section 3, the next step after algebraic representation structure construction is to choose the appropriate logical reasoning mechanism and then apply it to the qualitative information represented by the L-LIA  $L_{9 \times 2}$  for inferring the comprehensive decision [10, 17]. It is different to the conventional decision making approaches under qualitative environments, where the qualitative information under consideration is usually transformed into fuzzy membership functions or symbolic models but with numerical indices, and the final decision result is obtained by applying chosen aggregation operators to aggregate the transformed qualitative information. Actually, decision analysis of human being is a mental process that is essentially a reasoning process often with qualitative information rather than numerical calculation process [7]. Therefore, it is more natural for decision making methods to handle the qualitative information directly without numerical approximation which may cause information loss sometimes. Furthermore, the conventional qualitative decision making methods are mainly based on fuzzy set theory, which needs to define membership functions for qualitative information modelling [51]. However, it is usually not a plain task that requires lots of investigations due to the subjective nature of human knowledge.

A logical reasoning based decision making approach is proposed in this section for this purpose, which is capable of working with the qualitative information directly, and takes a lattice-valued logic system as the strict theoretical foundation to guarantee the validity of the proposed approach.

#### 4.1 The logical reasoning based qualitative decision making process

The proposed logical reasoning based approach for decision making problem with qualitative information is essentially a rule-based approach, where the rules are established based on human knowledge. The rules take the form ‘If  $X$ , then  $Y$ ’, with  $X, Y \in \mathcal{F}_L(\mathcal{F}_p)$  being the evaluations with respect to criteria from the given criteria set  $C = \{c_1, \dots, c_m\}$ . As discussed at the end of Section 3, different qualitative weights are assigned to the criteria to denote the contribution degrees of the criteria to the overall evaluation, which are taking into consideration by applying the approximate reasoning operator  $\otimes$ . Take the computer evaluation example as an instance, the rule may take the form like “If a laptop is *highly fast*, *rather cheap*, *very use-friendly* and with *quite satisfied* power consumption, then it is *highly satisfied*”. If the evaluation about candidate  $x$  corresponding to criterion  $c_i$  is  $r_i = (\alpha_i, \beta_i)$ , expressed as  $X(x(c_i)) = r_i = (\alpha_i, \beta_i)$ , then the corresponding evaluation taking into account the weight  $\omega_i$  about criterion  $c_i$  is  $\tilde{X}(x(c_i)) = (\omega_i \otimes \alpha_i, \beta_i)$ , where  $r_i = (\alpha_i, \beta_i) \in L_{9 \times 2}$ ,  $\omega_i \in L_9$ ,  $i = 1, \dots, m$ . The consequence of the rule is the overall evaluation  $Y(x) = (m^*, p^*)$  on candidate  $x$ , given the evaluation  $X$  about each criterion  $c_i$ , where  $(m^*, p^*) \in L_{9 \times 2}$ . Suppose that we have a new evaluation  $Z(x(c_j)) = (\alpha_j, \beta_j)$  on candidate  $x$  corresponding to criterion  $c_j$ , and the evaluation taking weights into consideration is  $\tilde{Z}(x(c_j)) = (\omega_j \otimes \alpha_j, \beta_j)$ , where  $(m_j, p_j) \in L_{9 \times 2}$ ,  $\omega_j \in L_9$ ,  $j = 1, \dots, m$ , then the composite evaluation result, denoted as  $D$ , is the logical consequence by applying the rule to new evaluation as defined in Eq. (4.1).

Formally, we can express the logical reasoning based decision making model interpreted from the obtained knowledge base (rule base) as in Eq. (4.1). The decision making process can be described in a general way as that, given the knowledge (rule) and new evaluations provided by the experts, we can generate the compound decision result based on the logical reasoning process that is defined in in Eq. (4.2).

$$\begin{array}{c} \text{Rule: If evaluation about each criterion is } X, \text{ then the overall evaluation is } Y \\ \text{New evaluation : } Z \\ \hline \text{The composite evaluation result: } D. \end{array} \quad (4.1)$$

In the above model, the evaluations on candidates are qualitative terms taken from  $L_{9 \times 2}$ .

$$X(x(c)) = \begin{cases} (\alpha_i, \beta_i), & c = c_i \in C, \\ (m_1, p_1), & c \notin C; \end{cases}$$

$$Z(x(c)) = \begin{cases} (\alpha_j, \beta_j), & c = c_i \in C, \\ (m_1, p_1), & c \notin C; \end{cases}$$

$$Y(x) = (m^*, p^*).$$

The weighted evaluations, by applying the logical operation  $\otimes$ , are then expressed as:

$$\tilde{X}(x(c)) = \begin{cases} (\omega_i \otimes \alpha_i, \beta_i), & c = c_i \in C, \\ (m_1, p_1), & c \notin C; \end{cases}$$

$$\tilde{Z}(x(c)) = \begin{cases} (\omega_j \otimes \alpha_j, \beta_j), & c = c_i \in C, \\ (m_1, p_1), & c \notin C; \end{cases}$$

where  $(\alpha_i, \beta_i) \in L_{9 \times 2}$ ,  $(m^*, p^*) \in L_{9 \times 2}$ ,  $\omega_i \in L_9$ ,  $i = 1, \dots, m$ .

Given evaluation  $Z(x(c_j)) = (m_j, p_j)$  on a new candidate, with the corresponding weighted evaluation being computed as  $\tilde{Z}(x(c_j)) = (\omega_j \otimes \alpha_j, \beta_j)$ , the composite evaluation result  $D$  is then obtained as

$$D = C_{\mathcal{T}_i}^{\beta \otimes \tilde{Z}}, \quad (4.2)$$

where the operator  $C_{\mathcal{T}_i}$  is defined in Definition 2.6 with  $i=I, II$ .

Let  $\mathcal{R}^* = \{(r_2^0, t_2^*), (r_2^*, t_2^*), (r_2^A, t_2^*)\} \cup \{(r_1^{\theta_0}, t_1^{\theta_0}) | \theta_0 \in L\} \subseteq \mathcal{R}$  be the rules of inference as provided in Section 2.2, and  $\mathcal{T}_i = \{T | T \in \mathcal{F}_L(\mathcal{F}_p), T \text{ is } \alpha\text{-}i \text{ type closed w.r.t. } \mathcal{R}^*\}$ ,  $i=I, II$ , and  $\mathcal{T}_H \triangleq \{T | T: \mathcal{F}_p \rightarrow L \text{ is a homomorphic mapping}\}$  be the set of valuations. It is noted that there is degreed consistency between rules of inference and the corresponding valuations, i.e., the syntax part and the semantics part, which is to guarantee that the decision result is not only semantically interpretable, but also logically sound. On the other hand, there are of course many other rules of inference as defined in Refs. [8, 49], but only suitable rules for decision making problems are applied here.

Sum up above, the process of the logical reasoning based decision making approach with qualitative information is described as follows.

**Step 1.** The algebraic structure L-LIA  $L_{9 \times 2}$  is adopted for modelling the qualitative information involved in the decision making problem, which is a product algebra of the modifier set  $L_9$  and the prime term set  $L_2$  as shown in Section 3.

**Step 2.** Determine the weights corresponding to the criteria  $C = \{c_1, \dots, c_m\}$  and generate the rule base ‘If  $X$ , then  $Y$ ’ as shown in model (4.1) based on the domain expert or public knowledge, where the weights are also qualitative and denoted as  $\omega_i \in L_9$ .

**Step 3.** Introduce alternatives and ask the experts to provide evaluations that are denoted as  $Z(x(c))$ , and then the weights are taken into account to get the weighted evaluations, denoted as  $\tilde{Z}(x(c))$ .

**Step 4.** Obtain the overall evaluations by applying the logical reasoning process (4.2) to the rule-based model (4.1) and new evaluations, and then explore the overall evaluations to give a rank of the alternatives.

#### 4.2 Rationality of the logical reasoning based decision making approach

As discussed previously, the proposed logical reasoning based approach for decision making with qualitative information takes both the advantages of handling the qualitative information directly without numerical approximation and having the strict logical foundation. The first advantage of working with the qualitative information directly has been illustrated in the previous sections, and its logical properties are discussed in this subsection based on the constructed L-LIA and the corresponding logic system [8, 29, 49]. These logical properties are mainly to guarantee that the comprehensive decision is not only semantically interpretable, but also logically rational.

**Definition 4.1** We call that the rule in model (4.1) as  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type representable in  $L_{vpl}$ , if  $X, Y$  in the model are  $\tau'$ - $i$  type consistent with respect to  $(\alpha, \beta, \mathcal{T})$  and  $\tau' \otimes Y \subseteq C_{\mathcal{T}}^X$ , for given  $\alpha, \beta, \tau \in L$  and  $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$ .

It is noted that the requirement of the model being representable in  $L_{vpl}$  comes from two aspects. The first one is that  $X, Y$  should be consistent to certain degree, i.e., there is no contradictory evaluations in  $X, Y$  at certain level. The second one tells us that  $Y$  can be inferred from  $X$ , not only as stated in model (4.1) that  $Y$  is the conclusion part of the ‘If-then’ rule, but also that  $Y$  can be inferred from  $X$  based on formal logical inference.

**Definition 4.2** The logical reasoning based decision making model (4.1) is called as  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type regular model in  $L_{vpl}$ , if the rule in model (4.1) is  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type representable in  $L_{vpl}$ , and the new evaluation  $Z$  is  $\tau'$ - $i$  type consistent with respect to  $(\alpha, \beta, \mathcal{T})$ , for given  $\alpha, \beta, \tau \in L$  and  $\mathcal{T} \subseteq \mathcal{F}_L(\mathcal{F}_p)$ .

The regularity of the logical reasoning based decision making defined in Definition 4.2 requires not only the rule is representable as shown in Definition 4.1, but also the evaluations about new alternative is also consistent, i.e., not contradictory to certain degree. The requirements are natural.

We shall then show some conditions for the logical reasoning based decision making model (4.1) to be regular so that we can apply the logical reasoning process to infer the comprehensive evaluation without question.

**Theorem 4.1** The rule in the logical reasoning based decision making model (4.1) is  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type representable in  $L_{vpl}$  for any  $\alpha, \beta, \tau \in L$  and  $\mathcal{T} = \{T | X \subseteq T, Y \subseteq T\}$ , if there exists  $T_0 \in \mathcal{T}_H, X, Y \subseteq T_0$ .

**Proof.** Given  $\alpha, \beta, \tau \in L$ ,

$$C_{\mathcal{T}}^{\beta \otimes X} \subseteq C_{\mathcal{T}}^{\beta \otimes \cap_{T \in \mathcal{T}} T} \subseteq C_{\mathcal{T}}^{\cap_{T \in \mathcal{T}} (\beta \otimes T)} \subseteq \bigcap_{T \in \mathcal{T}} C_{\mathcal{T}}^{\beta \otimes T} \subseteq \bigcap_{T \in \mathcal{T}} C_{\mathcal{T}}^T = \bigcap_{T \in \mathcal{T}} T \subseteq T_0.$$

So,  $X$  is  $\tau'$ - $i$  type consistent with respect to  $(\alpha, \beta, \mathcal{T})$ , and it can be shown similarly that  $Y$  is also  $\tau'$ - $i$  type consistent with respect to  $(\alpha, \beta, \mathcal{T})$ .

On the other hand,  $\tau' \otimes Y \subseteq Y \subseteq \cap_{T \in \mathcal{T}} T = C_{\mathcal{T}}^X$ .

Therefore, the rule in model (4.1) is  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type representable in  $L_{vpl}$ . This completes the proof.

The following proposition can be obtained from Theorem 4.1 easily.

**Corollary 4.1** If  $\mathcal{T} = \mathcal{T}_H \cap \{T | X \subseteq T, Y \subseteq T\}$  in the logical reasoning based decision making model (4.1) is not empty, then the rule in model (4.1) is  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type representable in  $L_{vpl}$  for any  $\alpha, \beta, \tau \in L$ .

**Theorem 4.2** In the logical reasoning based decision making model (4.1), if  $\mathcal{T} = \{T | \tau' \otimes Y \subseteq T\}$  and  $X, Y$  are  $\tau'$ - $i$  type consistent with respect to  $(\alpha, \beta, \mathcal{T})$ , then the rule in model (4.1) is  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type representable in  $L_{vpl}$ .

**Proof.** Given  $X, Y$  are already  $\tau'$ - $i$  type consistent with respect to  $(\alpha, \beta, \mathcal{T})$ , We need only to prove that  $\tau' \otimes Y \subseteq C_{\mathcal{T}}^X$ .

For any formula  $p \in \mathcal{F}_p$ , according to Definition 2.6,

$$\begin{aligned} C_{\mathcal{T}}^X(p) &= \bigwedge_{T \in \mathcal{T}} [\pi(X \subseteq T) \rightarrow T(p)] \\ &\geq \bigwedge_{T \in \mathcal{T}} [(X(p) \rightarrow T(p)) \rightarrow T(p)] \geq \bigwedge_{T \in \mathcal{T}} T(p) = \tau' \otimes Y(p). \end{aligned}$$

Therefore, the rule in model (4.1) is  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type representable in  $L_{vpl}$ . This completes the proof.

As a straightforward corollary, we have the following proposition.

**Corollary 4.2** Let  $\mathcal{T} = \{T | X \rightarrow (\tau' \otimes Y \otimes T) \subseteq T\}$  in the logical reasoning based decision making model (4.1), and  $X, Y$  be  $\tau'$ - $i$  type consistent with respect to  $(\alpha, \beta, \mathcal{T})$ , then the rule in model (4.1) is  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type representable in  $L_{vpl}$ .

The above theorems are about the representability of the rule in the logical reasoning based decision making model (4.1). The following theorems are about the regularity of model (4.1) given new evaluation  $Z$ , which can be obtained easily based on Definition 4.2 and Theorems 4.1 and 4.2, and so the proofs are not provided.

**Theorem 4.3** In the logical reasoning based decision making model (4.1), if there exists  $T_0 \in \mathcal{T}_H$ ,  $X, Y, Z \subseteq T_0$ , then model (4.1) is  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type regular in  $L_{vpl}$  for any  $\alpha, \beta, \tau \in L$  and  $\mathcal{T} = \{T | X \subseteq T, Y \subseteq T, Z \subseteq T\}$ .

**Theorem 4.4** In the logical reasoning based decision making model (4.1), if  $\mathcal{T} = \mathcal{T}_H \cap \{T | X \subseteq T, Y \subseteq T, Z \subseteq T\}$  is not empty, then model (4.1) is  $(\alpha, \beta, \tau, \mathcal{T})$ - $i$  type regular in  $L_{vpl}$  for any  $\alpha, \beta, \tau \in L$ .

There are also surely some other properties of the proposed logical reasoning based decision making approach, which are not discussed here due to the reason that they are not directly related to the decision making process.

## 5. Illustrative Example

In this section, an example about laptop evaluation is provided to illustrate the feasibility of the proposed logical reasoning based approach for decision making with qualitative information.

**Step 1.** Construct the representation structure.

Assume that there are four types of laptops, Dell ( $x_1$ ), Apple ( $x_2$ ), Lenovo ( $x_3$ ) and HP ( $x_4$ ), which are evaluated with respect to four criteria or attributes: processing speed ( $c_1$ ), price ( $c_2$ ), user friendship ( $c_3$ ) and battery ( $c_4$ ). The qualitative evaluations provided by the experts take values from the L-LIA  $L_{9 \times 2}$  as constructed in Section 3, and the weights related to the four criteria are from the lattice  $L_9$ , whose values are determined by the experts as  $\omega_1 = m_8 = \text{highly}$ ,  $\omega_2 = m_7 = \text{very}$ ,  $\omega_3 = m_5 = \text{exactly}$ , and  $\omega_4 = m_6 = \text{quite}$ . This is the first step of the logical reasoning based decision making approach as described in Section 4.1.

**Step 2.** Establish the evaluation rule base.

The rule comes from the common knowledge of people: “If the laptop’s processing speed is *absolutely satisfied*, price is *exactly satisfied*, user friendship is *quite satisfied*, and battery life is *very satisfied*, then it is *highly satisfied*”, which can be expressed as:

$$\mathbf{R}: \text{If } X, \text{ then } Y, \quad (5.1)$$

where,

$$X(x(c)) = \begin{cases} (m_9, p_2), & c = c_1, \\ (m_5, p_2), & c = c_2, \\ (m_6, p_2), & c = c_3, \\ (m_7, p_2), & c = c_4, \\ (m_1, p_1), & c \notin C, \end{cases} \quad (5.2)$$

$$Y(x) = (m_8, p_2).$$

The weighted evaluations are

$$\tilde{X}(x(c)) = \begin{cases} (\omega_1 \otimes m_9, p_2), & c = c_1, \\ (\omega_2 \otimes m_5, p_2), & c = c_2, \\ (\omega_3 \otimes m_6, p_2), & c = c_3, \\ (\omega_4 \otimes m_7, p_2), & c = c_4, \\ (m_1, p_1), & c \notin C, \end{cases} = \begin{cases} (m_8, p_2), & c = c_1, \\ (m_5, p_2), & c = c_2, \\ (m_5, p_2), & c = c_3, \\ (m_6, p_2), & c = c_4, \\ (m_1, p_1), & c \notin C. \end{cases} \quad (5.3)$$

**Step 3.** Introduce new evaluations.

The evaluations of each kind of laptop against these four criteria are provided in Table 5.1, which is essentially in the form of decision matrix as widely used in decision making problems. The value of each entry  $x_i(c_j)$  of this decision matrix means the valuation of experts about alternative  $x_i$  against criterion  $c_j$ . For example,  $x_2(c_1) = (m_8, p_1)$  expresses the evaluation “the processing speed of Apple is *highly satisfied*”, whose meaning in natural language is actually “the speed of Apple is highly fast”.

Table 5.1. Evaluations on the four laptops

	$c_1$	$c_2$	$c_3$	$c_4$
$x_1$	$(m_7, p_2)$	$(m_4, p_2)$	$(m_8, p_2)$	$(m_7, p_2)$
$x_2$	$(m_8, p_2)$	$(m_5, p_1)$	$(m_8, p_2)$	$(m_7, p_2)$
$x_3$	$(m_7, p_2)$	$(m_3, p_2)$	$(m_7, p_2)$	$(m_6, p_2)$
$x_4$	$(m_6, p_2)$	$(m_6, p_2)$	$(m_6, p_2)$	$(m_3, p_1)$

These evaluations are then expressed in logical form for the following reasoning process. We take that about alternative  $x_1$  as an example, and the others are just in the similar form.

$$Z(x_1(c)) = \begin{cases} (m_7, p_2), & c = c_1, \\ (m_4, p_2), & c = c_2, \\ (m_8, p_2), & c = c_3, \\ (m_7, p_2), & c = c_4, \\ (m_1, p_1), & c \notin C. \end{cases} \quad (5.4)$$

The weighted evaluations can then be obtained, with that about alternative  $x_1$  as follows, after taking the weights into consideration.

$$\tilde{Z}(x_1(c)) = \begin{cases} (\omega_1 \otimes m_7, p_2), & c = c_1, \\ (\omega_2 \otimes m_4, p_2), & c = c_2, \\ (\omega_3 \otimes m_8, p_2), & c = c_3, \\ (\omega_4 \otimes m_7, p_2), & c = c_4, \\ (m_1, p_1), & c \notin C, \end{cases} = \begin{cases} (m_7, p_2), & c = c_1, \\ (m_4, p_2), & c = c_2, \\ (m_5, p_2), & c = c_3, \\ (m_6, p_2), & c = c_4, \\ (m_1, p_1), & c \notin C. \end{cases} \quad (5.5)$$

When providing initial evaluations about the candidate laptops, the experts need only to select suitable evaluation terms from the algebraic structure L-LIA, which are actually in natural language. The experts may need just to spend a bit time on getting familiar with the orderings on the linguistic modifiers, e.g., *Quite* < *Very*, and applying linguistic modifiers on the prime term *dissatisfied* is reversed to the ordering in accordance to the natural meaning, e.g., *Very dissatisfied* < *Quite dissatisfied*.

**Step 4.** Apply the reasoning mechanism to the new evaluations and explore the results.

We apply the logical reasoning based aggregator  $C_{T_i}$  as shown in Eq. (4.2) to the rule base (5.1) and new evaluations as shown in Table 5.1, with the weighted evaluations expressed as in Eq. (5.5), and the overall evaluations about each alternative can be obtained as shown in Eq. (5.6). It can be translated into natural language as “Dell is *quite satisfied*, Apple is *exactly satisfied*, Lenovo is *exactly satisfied*, and HP is *rather satisfied*”.

$$D(x) = \begin{cases} (m_6, p_2), & x = x_1, \\ (m_5, p_2), & x = x_2, \\ (m_5, p_2), & x = x_3, \\ (m_3, p_2), & x = x_4, \end{cases} \quad (5.6)$$

We can then explore the overall evaluations to give a rank of the alternatives, and we can see that Dell is preferable in this illustrative example, which is compatible to our common sense. On the other hand, because the overall evaluations are

obtained based on a formal logic reasoning approach, rather than from certain aggregation operator usually based on mathematical operations, the evaluation results are then logically rational and more reliable.

As a side effect of putting the decision making problem into a formal framework, the logic based approximate reasoning process may seem complicated to the decision makers. Fortunately, the decision makers do not need to be very familiar with each step of the concrete approximate reasoning process, because this process can be realized automatically. They need only to provide the initial evaluations and then obtain the output composite evaluations at the simplest case.

## 6. Conclusions

Human knowledge is helpful on enhancing data-driven method for decision making, and human knowledge is often expressed in qualitative form. Aims at enhancing decision making with qualitative knowledge in a formal way, the present work provides a logical reasoning based decision making framework by modelling the qualitative information with an algebraic structure, linguistic truth-valued lattice implication algebra, and reaching the decision result based on approximate reasoning, i.e., reasoning with words. This method is able to handle qualitative knowledge, no matter totally ordered or partially ordered, in a direct way without transforming it into numerical values, and takes a non-classical logic, lattice-valued logic as its theoretical foundation. It is guaranteed that the comprehensive decision is not only semantically interpretable, but also logically rational, which reflects the essential advantage of logic based methods and leads the direction to rational decision making. An illustrative example about laptop evaluation is provided to show the process of the proposed method.

It helps to point out that, although seems complicated, the logical approximate reasoning process can be realized automatically. The decision makers do not need to learn well about the concrete approximate reasoning process. They just need to provide initial evaluations about the candidates by selecting suitable evaluation terms, which are actually in natural language, from the linguistic term set modelled by the algebraic structure L-LIA. It is also noted that the algebraic structure L-LIA for modeling the qualitative information is a general framework, and different numbers of, say, 3, 5 or 7, linguistic modifiers can be applied for constructing the L-LIA given different decision making problems. It will be helpful to outline how to choose feasible number of linguistic modifiers for real decision making problems as a next step work. We shall also try to combine the proposed logical reasoning based decision making method with data-driven methods in the coming future so as to take advantages of both sides to make it more powerful.

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