Financial Network Games

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We study financial systems from a game-theoretic standpoint. A financial system is represented by a network, where nodes correspond to firms, and directed labeled edges correspond to debt contracts between them. The existence of cycles in the network indicates that a payment of a firm to one of its lenders might result to some incoming payment. So, if a firm cannot fully repay its debt, then the exact (partial) payments it makes to each of its creditors can affect the cash inflow back to itself. We naturally assume that the firms are interested in their financial well-being (*utility*) which is aligned with the amount of incoming payments they receive from the network. This defines a game among the firms, that can be seen as utility-maximizing agents who can strategize over their payments.

We are the first to study *financial network games* that arise under a natural set of payment strategies called priority-proportional payments. We investigate both the existence and the (in)efficiency of equilibrium strategies, under different assumptions on how the firms' utility is defined, on the types of debt contracts allowed between the firms, and on the presence of other financial features that commonly arise in practice. Surprisingly, even if all firms' strategies are fixed, the existence of a unique payment profile is not guaranteed. So, we also investigate the existence and computation of valid payment profiles for fixed payment strategies.

Additional Key Words and Phrases: financial networks, equilibria, price of anarchy, price of stability

1 INTRODUCTION

A financial system comprises a set of institutions, such as banks and firms, that engage in financial transactions. The interconnections showing the *liabilities* (financial obligations or debts) among the firms are represented by a network and can be highly complex. Hence, the wealth and financial well-being of a firm depends on the entire network and not just on the well-being of its immediate borrowers/*debtors*. For example, a possible bankruptcy of a firm and the corresponding damage to its immediate lenders/*creditors* resulting from the firm's failure to repay (known as credit risk), can be propagated through the financial network by causing the creditors' (and other firms', in sequence) inability to repay their debts, thus having a global effect.

In this work, we examine the global effect of payment decisions of individual firms. We assume that each firm has a fixed amount of *external assets* (not affected by the network) which are measured in the same currency as the liabilities. A firm's *total assets* comprise its external assets and its incoming payments, and can be used for (outgoing) payments to its creditors. A firm's payment decision, for example, can specify the priority to be given to each of its debts or creditors. An authority (such as the government or the court) is assumed to monitor the decisions of the firms in order to guarantee that they comply with general regulatory principles, such as the absolute priority and the limited liability ones (see, e.g., [8]). According to these principles, a firm can leave a liability unpaid only if its assets are not enough to fully repay its liabilities. In particular, the *absolute priority principle* requires that all creditors must be paid off before a firm's stakeholders can split assets, and the *limited liability principle* implies that a firm that does not have enough assets to pay its liabilities in full, has to spend all its remaining assets to pay its creditors. Surprisingly, not all decisions of the firms lead to valid solutions (or payment profiles)[1], or there can be ambiguity in the overall solution despite the individual fixed payment decisions of the firms[2] [8]. This gives

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[1]An example of such a network appears in [20] and assumes the presence of default costs and Credit Default Swaps (definitions appear in Section 2).
[2]Consider, for example, four firms A, B, C, and D, with zero external assets each, and the following liabilities: A owes ℓ coins to B and another ℓ coins to C; B owes ℓ coins to A and another ℓ coins to D. Assume that the payment strategy of firm

rise to the *clearing problem* aiming to determine the final payment profile of the network for fixed individual payment decisions. Such payments are called *clearing payments* and are compatible with the given individual payment decisions. Computing clearing payments is a necessary step before performing a game-theoretic analysis.

We investigate the consequences of individual firm decisions in clearing from a game-theoretic standpoint [15]. Following the recent work of Bertschinger et al. [4], we diverge from the common assumption of proportionality in payments, that dictates that a firm pays its creditors proportionally to their respective liabilities. We assume that firms strategize about their payments by appropriately deciding the priority of their payment actions, consistently with a predefined payment scheme (and the current regulation); the chosen priorities may have an effect on the amount of incoming payments of a given firm. We consider a natural payment scheme that allows fixed allocations of creditors to priority-classes independently of the available assets. We refer to this payment scheme as priority-proportional scheme, and the strategies that conform with it as *priority-proportional strategies*.

Our paper is the first to perform a game-theoretic analysis of priority-proportional payments in financial networks. Payments with priorities are simple to express as well as quite common and very well-motivated in the financial world. Indeed, bankruptcy codes allow for assigning priorities to the payout to different creditors, in case an entity is not able to repay all its obligations. Such a distribution of payments can be part of a reorganization plan ordered by the court [3]. Priority classes in bankruptcy law have also been considered in [10, 12], among others. Moving beyond regulated financial contexts, similar behavior is common in everyday transactions between individuals with pairwise debt relations. This is the first time that the impact of priority-proportional payments is assessed in a strategic setting, as are other elements of our analysis, even though such payments have been considered in the past (most recently by Papp and Wattenhofer [16]). Our work demonstrates that, despite their simplicity, priority-proportional strategies exhibit certain desired properties with respect e.g., to equilibria existence and quality. They are also not very restrictive, which can, however, lead to unattractive instances. Our work can be seen as an attempt to examine whether such priority-based payment plans are indeed stable equilibria; this can be useful from a mechanism design perspective.

Our game-theoretic analysis considers two different definitions of utility motivated by the financial literature, namely *total assets*, computed as the sum of external assets and incoming payments, or *equity*, respectively. Traditionally, the financial health of a firm has been measured by its equity, which is equal to the amount of remaining assets after payments (total assets minus liabilities) if this is positive, and 0 otherwise. This means that all firms that have more debt than assets have equity 0, so equities fail to capture the potentially different available assets these firms might have. Total assets can be seen as a refinement of the equities: indeed, for any financially healthy firm (that can repay all its debt) its total assets equal its equity plus its liabilities (a fixed term), while for other firms, total assets allow them to distinguish among different states that are indistinguishable in terms of equity. Note that, since by the absolute priority principle a firm is obliged by law to fully repay all its creditors if it has enough assets, it is only the payment decisions of firms whose debt is more than their assets that can have an effect on the network. For this reason, computing the firms' utility as their total assets is a suitable approach in a game-theoretic context. Overall, both definitions are aligned with the individual financial wealth and welfare of a firm.

A is to prioritize payments towards B over C, and that the payment strategy of firm B is to prioritize payments towards A over D (firms C and D are not strategic since they do not have multiple creditors—or any for that matter). Then there exist infinitely many valid solutions that are consistent with this strategy profile: Indeed, A paying B the amount of λ coins and B paying A the same amount (λ coins), is a valid solution for any positive value $\lambda \leq \ell$.

Our model is based on the seminal and widely adopted work of Eisenberg and Noe [8] who provide a basic financial network model, i.e., proportional payments, non-negative external assets, and debt-only liabilities. In addition to considering a different payment scheme, we enhance the basic model by also considering financial features commonly arising in practice, such as default costs [17], Credit Default Swap (CDS) contracts [20], and negative external assets [7] (definitions appear in Section 2). We analyze the efficiency of the states arising from various clearing payments of financial networks, with a focus on ones consistent with equilibrium strategies. Note that even though, as stated above, both utility definitions are aligned with the individual financial wealth and welfare of a firm, it turns out that these notions of individual utility are not always aligned with the welfare of the whole financial system.

Our contribution. We quantify the extent to which strategic behavior of the firms affects the welfare of the society, by analyzing the *financial network games* that are defined by a particular utility function, and possibly allow the presence of other financial instruments. In particular, we consider financial network games under priority-proportional strategies, defined for different utility functions, such as total assets or equities, and which potentially allow CDS contracts, default costs, or negative external assets. We derive structural results that have to do with the existence, the computation, and the properties of clearing payments for fixed payment decisions in a non-strategic setting, and/or the existence and quality of equilibrium strategies. In particular, in Section 3 we prove the existence of maximal clearing payments under priority-proportional strategies, even in the presence of default costs, and provide an algorithm that computes them efficiently. We are then able to prove existence of equilibria when the utility is defined as the equity, but show that equilibria are not guaranteed to exist when the utility is captured by the total assets. We then turn our attention to the efficiency of equilibria and provide an almost complete picture of the price of anarchy [13] and the price of stability [2, 21]. Our results for total assets appear in Section 4, while the case of equities is treated in Section 5.

Related work. Financial networks and their related properties have been analyzed in various works that follow the standard (non-strategic) model developed by Eisenberg and Noe [8]. They introduce a financial network model allowing debt-only contracts among firms with non-negative external assets that make proportional payments. Among other results, they prove that there always exist maximum and minimum clearing payments and they identify sufficient conditions so that uniqueness is guaranteed; they also present a simple iterative algorithm that computes them in polynomial time.

Following the model introduced by Eisenberg and Noe [8], a series of papers extend and enrich the model by adding default costs [17], cross-ownership [9, 22], liabilities with various maturities [1, 14] and CDS contracts [16, 20]. In [17], Rogers and Veraart prove the existence of maximal clearing payments in the presence of default costs and provide an algorithm that computes them. Schuldenzucker et al. [20] extend the previous models by introducing CDS contracts and show that, in general, there can be zero or many clearing payments. Furthermore, they provide sufficient conditions for the existence of unique clearing payments. Demange [7] proposes a model that allows for firms to be indebted to entities outside the network, and captures this by allowing negative external assets.

The strategic aspects of financial networks have been considered very recently. Most relevant to our setting is the paper by Bertschinger et al. [4] who also study the inefficiency of equilibria in financial networks. They follow the standard model of [8] and focus mainly on two payment schemes, namely coin-ranking and edge-ranking strategies, while also considering the total assets (as opposed to equity) as a measure of the individual utility of each firm. Apart from defining the graph-theoretic version of the clearing problem, they present a large range of results on the

existence and quality of equilibria. Our work extends this line of research by considering a different payment scheme (which extends edge-ranking strategies by allowing ties in the ranking) and the possibility of additional common financial features (default costs, CDS contracts, and negative external assets). In a similar spirit, Papp and Wattenhofer [16] consider the impact of individual firm actions, such as removing an incoming debt, donating extra funds to another firm, or investing more external assets, when CDS contracts are also allowed. They mostly focus on the case where firms have predefined priorities over their creditors and they remark that even by redefining such priorities, a firm cannot affect its equity.

Schuldenzucker and Seuken [18] consider the problem of portfolio compression, where a set of liabilities forming a directed cycle in the financial network may be simultaneously removed, if all participating firms approve it. They consider questions related to the firms' incentives to participate in such a compression, while Schuldenzucker et al. [19] show that finding clearing payments when CDS contracts are allowed is PPAD-complete. Additional strategic considerations, albeit less related to our setting, are the focus of Allouch and Jalloul [1] who consider liabilities with two different maturity dates and study how firms may strategically deposit some amount from their first-period endowment in order to increase their assets in the second period, while Csóka and Herings [6] consider liability games and study how to distribute the assets of a firm in default, among its creditors and the firm itself.

2 PRELIMINARIES

In the following, we denote by $[n]$ the set of integers $\{1, \ldots, n\}$. The critical notions of this section and their graphical representation are presented in an example (Figure 1), at the end of this Section.

Financial networks. Consider a set $\mathcal{N} = \{v_1, \ldots, v_n\}$ of n firms, where each firm v_i initially has some *external assets* e_i corresponding to income received from entities outside the financial system; note that e_i may also be negative.

Firms have payment obligations, i.e., *liabilities*, among themselves. These are in the form of a simple debt contract or of a Credit Default Swap. A *debt contract* creates a liability l_{ij}^0 of firm v_i (the debtor) to firm v_j (the creditor); we assume that $l_{ij}^0 \geq 0$ and $l_{ii}^0 = 0$. Note that both $l_{ij}^0 > 0$ and $l_{ji}^0 > 0$ may hold. Firms with sufficient funds to pay their obligations in full are called *solvent* firms, while ones that cannot are *in default*. The *recovery rate*, r_k, of a firm v_k that is in default, is defined as the fraction of its total liabilities that it can fulfill. A *Credit Default Swap* (CDS) is a conditional liability of $(1 - r_k)l_{ij}^k$ of the debtor v_i to the creditor v_j, subject to the default of v_k, called the *reference entity*. Overall, the total liability of firm v_i to firm v_j is $l_{ij} = l_{ij}^0 + \sum_{k \in [n]} (1 - r_k)l_{ij}^k$. Let $L_i = \sum_j l_{ij}$ be the total liabilities of firm v_i and set $\mathbf{l} = (L_1, \ldots, L_n)$.

Let p_{ij} denote the payment from v_i to v_j; we assume that $p_{ii} = 0$. These payments define a payment matrix $\mathbf{P} = (p_{ij})$ with $i, j \in [n]$. Then, $p_i = \sum_{j \in [n]} p_{ij}$ represents the total outgoing payments of firm v_i, while $\mathbf{p} = (p_1, \ldots, p_n)$ is the payment vector; this should not be confused with the breakdown of individual payments of firm v_i that is denoted by $\mathbf{p}_i = (p_{i1}, \ldots, p_{in})$. A firm in default may need to liquidate its external assets or make payments to entities outside the financial system (e.g., to pay wages). This is modeled using *default costs* defined by values $\alpha, \beta \in [0, 1]$. A firm in default can only use an α fraction of its external assets (when this is positive) and a β fraction of its incoming payments. The absolute priority and limited liability regulatory principles, discussed in the introduction, imply that a solvent firm must repay all its obligations to all its creditors, while a firm in default must repay as much of its debt as possible, taking default costs also into account.

Summarizing, it must hold that $p_{ij} \leq l_{ij}$ and, furthermore, $\mathbf{P} = \Phi(\mathbf{P})$, where

$$\Phi(\mathbf{x})_i = \begin{cases} L_i, & \text{if } L_i \leq e_i + \sum_{j=1}^n x_{ji} \\ \alpha e_i + \beta \sum_{j=1}^n x_{ji}, & \text{otherwise.} \end{cases} \tag{1}$$

Payments $\mathbf{P}$ that satisfy these constraints are called *clearing payments*[3]. We define the notion of *proper* clearing payments, which are clearing payments where all the money circulating in the financial network have originated from some firm with positive external assets. In the following, we only consider proper clearing payments.

Financial network games. These games arise naturally when we view the firms as strategic agents. We denote the strategy of firm v_i by $s_i(\cdot)$; this dictates how v_i allocates its existing funds, for any possible value these might have. Similarly, we can define the strategy profile $\mathbf{s} = (s_1(\cdot), \ldots, s_n(\cdot))$.

Given clearing payments $\mathbf{P}$, we define firm v_i's utility using either of the following two notions. The *total assets* $a_i(\mathbf{P})$ (see also [4, 5]) are

$$a_i(\mathbf{P}) = e_i + \sum_{j \in [n]} p_{ji},$$

while the *equity* $E_i(\mathbf{P})$ (see also [16]) is

$$E_i(\mathbf{P}) = \max\{0, a_i(\mathbf{P}) - L_i\}.$$

Proportional payments have been frequently studied in the financial literature (e.g., in [7, 8, 17]). Given clearing payments $\mathbf{P}$ (see Equation (1)), each p_{ij} must also satisfy $p_{ij} = \min\{l_{ij}, (e_i + \sum_{k \in [n]} p_{ki}) \frac{l_{ij}}{L_i}\}$. Note that, when constrained to use proportional payments, there is no strategic decision making involved.

We focus on *priority-proportional payments*, where a firm's strategy is independent of its total assets and consists of a complete ordering of its creditors allowing for ties. Creditors of higher priority must be fully repaid before any payments are made towards creditors of lower priority, while creditors of equal priority are treated as in proportional payments. For example, a firm v_i having firms $a, b, c,$ and d as creditors may select strategy $s_i = (a, b|c|d)$ that has firms a, b in the top priority class, followed by c and, finally, with d in the lowest priority class.

Let $L_i^{(m)}$ denote the total liability of firm v_i to firms in its m-th priority class. We use parameters k_{ij} to imply that firm v_j is in the k_{ij}-th priority class of firm v_i and denote by π'_{ij} the relative liability of v_i towards v_j in the corresponding priority class, i.e., $\pi'_{ij} = \frac{l_{ij}}{L_i^{(k_{ij})}}$ if $L_i^{(k_{ij})} > 0$, and $\pi'_{ij} = 0$ if $L_i^{(k_{ij})} = 0$. For given priority-proportional strategies for all firms, the clearing payments $\mathbf{P}$ (see Equation (1)) must also satisfy

$$p_{ij} = \min\left\{ \max\left\{ \left(p_i - \sum_{m=1}^{k_{ij}-1} L_i^{(m)} \right) \cdot \pi'_{ij}, 0 \right\}, l_{ij} \right\}. \tag{2}$$

That is, the payment of v_i to a creditor in priority class $L_i^{(k_{ij})}$ occurs only after all payments to creditors of higher priority have been guaranteed. Then, payments to creditors in $L_i^{(k_{ij})}$ are made proportionally to their claims in that priority class. Finally, we also have $0 \leq p_{ij} \leq l_{ij}$.

We will now define the notion of *Nash equilibrium* in a financial network game. First, let us stress that a strategy profile has consistent clearing payments which are not necessarily unique. It is standard practice (see, e.g. [8, 17]) to focus attention to maximal clearing payments (such payments point-wise maximize all corresponding payments) to avoid this ambiguity. So, we say

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[3]Clearing payments are not necessarily unique.

that a strategy profile **s** is a *Nash equilibrium* if no firm can increase her utility by deviating to another payment strategy. We only consider pure Nash equilibria and clarify that the utility of a firm for a given strategy profile is computed based on the assumption that the maximal clearing payments will be realized every time. In Section 3, we show how we can compute such payments efficiently. Extending strategy deviations to coalitions and joint deviations, we are interested in *strong equilibria* where no coalition can cooperatively deviate so that all coalition members obtain strictly greater utility.

Social welfare. Given clearing payments **P**, the social welfare $SW(\mathbf{P})$ is the sum of the firm utilities; the particular utility notion (total assets or equities) will be clear from the context. The optimal social welfare is denoted by OPT. Let $\mathbf{P_{eq}}$ be the set of clearing payments consistent with (pure) Nash equilibrium strategy profiles. The price of anarchy (PoA) of a particular instance is defined as the worst-case ratio of the optimal social welfare over the social welfare achieved at any equilibrium at the instance, $\text{PoA} = \max_{\mathbf{P} \in \mathbf{P_{eq}}} \frac{OPT}{SW(\mathbf{P})}$. In contrast, the price of stability (PoS) of a given instance of a game measures how far the highest social welfare that can be achieved at equilibrium is from the optimal social welfare, i.e., $\text{PoS} = \min_{\mathbf{P} \in \mathbf{P_{eq}}} \frac{OPT}{SW(\mathbf{P})}$. The *Price of Anarchy (Price of Stability, respectively)* of a game is the maximum PoA (PoS, respectively) of any instance of the given game.

Example 2.1. We represent a financial network by a graph as follows. Nodes correspond to firms and black edges correspond to debt-liabilities; a directed edge from node v_i to node v_j with label l_{ij}^0 implies that firm v_i owes firm v_j an amount of money equal to l_{ij}^0. Nodes are also labeled, their label appears in a rectangle and denotes their external assets; we omit these labels for firms with external assets equal to 0. A pair of red edges (one solid and one dotted) represents a CDS contract: a solid directed edge from node v_i to node v_j with label l_{ij}^k and a dotted undirected edge connecting this edge with a third node v_k implies that firm v_i owes v_j an amount of money equal to $(1-r_k)l_{ij}^k$.

Figure 1 depicts a financial network with five firms having external assets $e_1 = e_4 = 1$ and $e_2 = e_3 = e_5 = 0$. There exist four debt contracts, i.e., firm v_1 owes v_2 and v_3 two coins and one coin, respectively; v_2 owes v_1 one coin, and v_3 owes v_5 one coin. There is also a CDS contract between v_4, v_5, and v_3, with nominal liability $l_{45}^3 = 1$, with v_4 being the debtor, v_5 the creditor, and v_3 the reference entity. v_4 will only need to pay v_5 if v_3 is in default; the amount owed would be equal to $1(1-r_3)$.
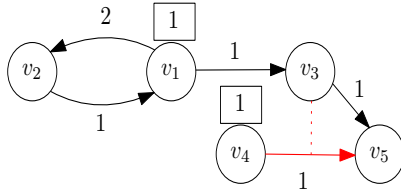

Fig. 1. An example of a financial network.

In this network, only v_1 can strategize about its payments, hence we focus just on v_1's strategy:

- Let v_1 select the priority-proportional strategy $s_1 = (v_2|v_3)$. Then, the payment vector would be $\mathbf{p} = (2,1,0,1,0)$ with $\mathbf{p}_1 = (0,2,0,0,0)$. Note that this is valid since $r_3 = \frac{p_3}{L_3} = \frac{p_{35}}{l_{35}} = \frac{0}{1} = 0$ and $p_{45} = (1-r_3)l_{45}^3 = 1$. The total assets of the firms are $a_1(\mathbf{P}) = 2$, $a_2(\mathbf{P}) = 2$, $a_3(\mathbf{P}) = 0$, $a_4(\mathbf{P}) = 1$ and $a_5(\mathbf{P}) = 1$, and the social welfare is $SW(\mathbf{P}) = 6$.

- If v_1's strategy is $s_1' = (v_3|v_2)$, the only consistent payment vector would be $\mathbf{p}' = (1, 0, 1, 0, 0)$ with $\mathbf{p}_1' = (0, 0, 1, 0, 0)$. Then, $a_1(\mathbf{P}') = 1$, $a_2(\mathbf{P}') = 0$, $a_3(\mathbf{P}') = 1$, $a_4(\mathbf{P}') = 1$ and $a_5(\mathbf{P}') = 1$, resulting in $SW(\mathbf{p}') = 4$.
- If v_1 decides to pay proportionally, that is $s_1'' = (v_2, v_3)$, the payment vector would be $\mathbf{p}'' = (2, 1, 2/3, 1/3, 0)$ with $\mathbf{p}_1'' = (0, 4/3, 2/3, 0, 0)$; note that $p_{45}'' = l_{45}^3(1 - r_3) = 1(1 - \frac{p_{35}}{l_{35}}) = 1/3$. Then, $a_1(\mathbf{P}'') = 2$, $a_2(\mathbf{P}'') = 4/3$, $a_3(\mathbf{P}'') = 2/3$, $a_4(\mathbf{P}'') = 1$ and $a_5(\mathbf{P}'') = 1$, resulting in $SW(\mathbf{P}'') = 6$.

Comparing the total assets under the different strategies discussed above, v_1 would select either strategy $(v_2|v_3)$ or (v_2, v_3), returning the maximum possible total assets (i.e., 2). Therefore, any strategy profile $\mathbf{s}$ where either $s_1 = (v_2|v_3)$ or $s_1 = (v_2, v_3)$ is a Nash equilibrium.

3 EXISTENCE AND COMPUTATION OF CLEARING PAYMENTS

This section contains our results relating to the existence and the properties of clearing payments in priority-proportional games. We begin by arguing that, given a strategy profile, proper clearing payments always exist even in presence of default costs. Furthermore, in case there are multiple clearing payments, there exist maximal payments, i.e., ones that point-wise maximize all corresponding payments, and we provide a polynomial-time algorithm that computes them. Note that this result is in a non-strategic context, but it is necessary in order to perform our game-theoretic analysis as it allows us to argue about well-defined deviations by considering clearing payments consistently among different strategy profiles.

LEMMA 3.1. *In priority-proportional games with default costs, there always exist maximal clearing payments under a given strategy profile.*

We now show how such maximal clearing payments can be computed. Given a strategy profile in a priority-proportional game, Algorithm 1 below, that extends related algorithms in [8, 17], computes the maximal (proper) clearing payments in polynomial time. In particular, and since the strategy profile is fixed, we will argue about the payment vector consisting of the total outgoing payments for each firm; the detailed payments then follow by the strategy profile.

LEMMA 3.2. *The payment vectors computed in each round of Algorithm 1 are pointwise non-increasing, i.e., $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ for any round $\mu \geq 0$.*

PROOF. We prove the lemma by induction. The base of our induction is $\mathbf{p}^{(0)} \leq \mathbf{p}^{(-1)} = \mathbf{l}$. It holds that $p_i^{(0)} = L_i$ if $v_i \in \mathcal{N} \setminus \mathcal{D}_0$, so it suffices to compute $p_i^{(0)}$ and show that $p_i^{(0)} \leq L_i$ for $v_i \in \mathcal{D}_0$.

We wish to find the solution $\mathbf{x}$ to the following system of equations

$$x_i = \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_0} x_{ji} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_0} l_{ji} \right), \quad \forall v_i \in \mathcal{D}_0$$
$$x_i = L_i, \quad \forall v_i \in \mathcal{N} \setminus \mathcal{D}_0 \quad (3)$$

We compute $\mathbf{x}$ using a recursive method and starting from $\mathbf{x}^{(0)} = \mathbf{p}^{(-1)} = \mathbf{l}$. We define $x^{(k)}$, $k \geq 1$, recursively by

$$x_i^{(k+1)} = \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_0} x_{ji}^{(k)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_0} l_{ji} \right).$$

Now for $v_i \in \mathcal{D}_0$, we have

$$x_i^{(1)} = \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_0} x_{ji}^{(0)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_0} l_{ji} \right) < L_i = x_i^{(0)}$$

Algorithm 1: MCP

```
/* The algorithm assumes given strategies (priority classes). For ease of
   exposition, we denote by p_i^(κ) the total outgoing payments of firm v_i at
   round κ and by p^(κ) the vector of total outgoing payments at round κ.   */
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1 Set $\mu = 0, \mathbf{p}^{(-1)} = \mathbf{l}$ and $\mathcal{D}_{-1} = \emptyset$;

2 Compute $E_i^{(\mu)} := e_i + \sum_{j \in [n]} p_{ji}^{(\mu-1)} - L_i$, for $i = 1, \dots, n$;

3 $\mathcal{D}_\mu = \left\{ v_i : E_i^{(\mu)} < 0 \right\}$;

4 if $\mathcal{D}_\mu \neq \mathcal{D}_{\mu-1}$ **then**

5 $\quad$ Compute $\mathbf{p}^{(\mu)}$ that is consistent with Equation (2) and satisfies

$$
p_i^{(\mu)} = \begin{cases} \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_\mu} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_\mu} l_{ji} \right), & \forall v_i \in \mathcal{D}_\mu \\ L_i, & \forall v_i \in \mathcal{N} \setminus \mathcal{D}_\mu. \end{cases} \quad ;
$$

6 $\quad$ Set $\mu = \mu + 1$;

7 $\quad$ **go to** Line 2;

8 else

9 $\quad$ Run PROPER($\mathbf{p}^{(\mu-1)}$)

Algorithm 2: PROPER($\mathbf{x}$)

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/* The algorithm takes in input payments x and returns proper payments.   */
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1 Set MARKED $= \{v_i : e_i > 0\}$ and CHECKED $= \emptyset$;

2 while *MARKED* $\neq \emptyset$ **do**

3 $\quad$ Pick $v_i \in$ MARKED;

4 $\quad$ **for** $v_j \notin$ *MARKED* $\cup$ *CHECKED with* $x_{ij} > 0$ **do**

5 $\quad\quad$ MARKED $=$ MARKED $\cup \{v_j\}$;

6 $\quad$ MARKED $=$ MARKED $\setminus \{v_i\}$ and CHECKED $=$ CHECKED $\cup \{v_i\}$;

7 for $v_i \notin$ *CHECKED* **do**

8 $\quad$ Set all outgoing payments from v_i in $\mathbf{x}$ to 0;

9 Return $\mathbf{x}$

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where the inequality holds because $\alpha, \beta \leq 1$ and by our assumption that $v_i \in \mathcal{D}_0$. Hence, sequence $\mathbf{x}^{(k)}$ is decreasing. Since the solution to Equation (3) is non-negative, $\mathbf{x}$ can be computed as $\mathbf{x} = \lim_{k \to \infty} \mathbf{x}^{(k)}$, which completes the base of our induction.

Now assume that $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ for some $\mu \geq 0$. We will prove that $\mathbf{p}^{(\mu+1)} \leq \mathbf{p}^{(\mu)}$. Similarly to before, $p_i^{(\mu+1)} = p_i^{(\mu)} = L_i$ if $v_i \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}$, so it suffices to compute $p_i^{(\mu+1)}$ and show that $p_i^{(\mu+1)} \leq p_i^{(\mu)}$ for $v_i \in \mathcal{D}_{\mu+1}$.

The desired $p_i^{(\mu+1)}$ is the solution $\mathbf{x}$ to the following system of equations

$$x_i = \begin{cases} \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_{\mu+1}} x_{ji} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right), & \forall i \in \mathcal{D}_{\mu+1} \\ L_i, & \forall v_i \in \mathcal{N} \setminus \mathcal{D}_{\mu+1} \end{cases} \tag{4}$$

We compute $\mathbf{x}$ recursively starting with $\mathbf{x}^{(0)} = \mathbf{p}^{(\mu)}$. We define $x^{(k)}$, $k \geq 1$, recursively by

$$x_i^{(k+1)} = \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_{\mu+1}} x_{ji}^{(k)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right) \tag{5}$$

For $v_i \in \mathcal{D}_{\mu+1}$, we have

$$\begin{aligned} x_i^{(1)} &= \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_{\mu+1}} x_{ji}^{(0)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right) \\ &= \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_{\mu+1}} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right) \\ &= \alpha e_i + \beta \left(\left(\sum_{j \in \mathcal{D}_\mu} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{D}_{\mu+1} \setminus \mathcal{D}_\mu} p_{ji}^{(\mu)} \right) + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right) \\ &= \alpha e_i + \beta \left(\left(\sum_{j \in \mathcal{D}_\mu} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{D}_{\mu+1} \setminus \mathcal{D}_\mu} l_{ji} \right) + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_{\mu+1}} l_{ji} \right) \\ &= \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_\mu} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_\mu} l_{ji} \right), \end{aligned}$$

where we note that our assumption $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ implies that $\mathcal{D}_{\mu+1} \supseteq \mathcal{D}_\mu$. Now we can split the set $\mathcal{D}_{\mu+1}$ into $\mathcal{D}_\mu$ and $\mathcal{D}_{\mu+1} \setminus \mathcal{D}_\mu$. For $v_i \in \mathcal{D}_\mu$ we have $x_i^{(1)} = \mathbf{p}^{(\mu)} = x_i^{(0)}$. For $v_i \in \mathcal{D}_{\mu+1} \setminus \mathcal{D}_\mu$ we have

$$\begin{aligned} x_i^{(1)} &= \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_\mu} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_\mu} l_{ji} \right) \\ &\leq e_i + \sum_{j \in \mathcal{D}_\mu} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_\mu} l_{ji} \\ &= e_i + \sum_{j \in [n]} p_{ji}^{(\mu)} \\ &< L_i = p_i^{(\mu)} = x_i^{(0)} \end{aligned}$$

which implies that the sequence $x^{(k)}$ is decreasing. Since the solution to Equation (4) is non-negative, $\mathbf{x} = \mathbf{p}^{\mu+1}$ can be computed as $\mathbf{x} = \lim_{k \to \infty} \mathbf{x}^{(k)}$, which completes our claim that $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ for any round $\mu \geq 0$. □

THEOREM 3.3. *Algorithm 1 computes the maximal clearing payments under priority-proportional strategies in polynomial time.*

Proof. The algorithm proceeds in rounds. In each round μ, tentative vectors of payments, $\mathbf{p}^{(\mu)} = (p_1^{(\mu)}, \ldots, p_n^{(\mu)})$, and effective equities, $E^{(\mu)}$, are computed. At the beginning of round 0, all firms are marked as tentatively solvent, which we denote by $\mathcal{D}_{-1} = \emptyset$; $\mathcal{D}_\mu$ is used to denote the firms in default after the μ-th round of the algorithm. The algorithm works so that once a firm is in default in some round, then it remains in default until the termination of the algorithm. Indeed, by Lemma 3.2 the vectors of payments are non-increasing between rounds and the strategies are fixed. Algorithm Proper is called when $\mathcal{D}_\mu = \mathcal{D}_{\mu-1}$, which requires at most n rounds; clearly, each round requires polynomial time. The running time of Proper is also polynomial. Indeed, note that each firm can enter set Marked at most once and will leave Marked to join set Checked after each other firm is examined at most once.

Regarding correctness, we start by proving by induction that the payment vector provided as input to Algorithm 2 is at least equal (pointwise) to the maximal clearing vector $\mathbf{p}^*$. As a base of our induction, it is easy to see that $\mathbf{p}^{(-1)} = \mathbf{l} \geq \mathbf{p}^*$. Now assume that $\mathbf{p}^{(\mu-1)} \geq \mathbf{p}^*$ for some $\mu \geq 0$; we will prove that $\mathbf{p}^{(\mu)} \geq \mathbf{p}^*$. We denote by $\mathcal{D}_*$ the firms in default under the maximal clearing vector $\mathbf{p}^*$, i.e., $\mathcal{D}_* = \left\{ v_i : e_i + \sum_{j \in [n]} p_{ji}^* < L_i \right\}$. Our inductive hypothesis $\mathbf{p}^{(\mu-1)} \geq \mathbf{p}^*$ implies $\mathcal{D}_\mu \subseteq \mathcal{D}_*$. Hence, for firms $v_i \in \mathcal{N} \setminus \mathcal{D}_\mu$, we have $\mathbf{p}_i^{(\mu)} = L_i \geq p_i^*$. For $v_i \in \mathcal{D}_\mu$ we refer to the proof of Lemma 3.2 above and consider Equation (5) again while starting the recursive solution with $x_i^{(0)} = p_i^{(\mu)}$. For $v_i \in \mathcal{D}_\mu$, we observe

$$
\begin{aligned}
x_i^{(1)} &= \alpha e_i + \beta \left(\sum_{j \in \mathcal{D}_\mu} p_{ji}^{(\mu)} + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_\mu} l_{ji} \right) \\
&\geq \alpha \cdot e_i + \beta \cdot \left(\sum_{j \in \mathcal{D}_\mu} p_{ji}^* + \sum_{j \in \mathcal{N} \setminus \mathcal{D}_\mu} l_{ji} \right) \\
&\geq \alpha e_i + \beta \left(\sum_{j \in [n]} p_{ji}^* \right) \\
&= L_i^{(0)}.
\end{aligned}
$$

Recursion (5) then implies that $x^{(k)} \geq \mathbf{p}^*$ for all k and hence $\mathbf{p}^{(\mu+1)} = \mathbf{x} = \lim_{k \to \infty} \mathbf{x}^k \geq \mathbf{p}^*$.

We have now proved that the input to Algorithm 2 is at least equal (pointwise) to the desired maximal clearing vector $\mathbf{p}^*$. However, by Lemma 3.2 we also know that $\mathbf{p}^{(\mu)} \leq \mathbf{p}^{(\mu-1)}$ for all $\mu \geq 0$. It holds by design that the input of Algorithm 2 is a clearing vector, so $\mathbf{p}^*$ is the only possible such input.

Finally, it remains to show that Algorithm 2 with input the maximal clearing payments computes the maximal proper clearing payments. Indeed, consider some maximal clearing payments $\mathbf{P}$ and the resulting proper payments $\mathbf{P}'$ obtained by $\mathbf{P}$ via Algorithm 2. We note that $\mathbf{P}'$ are clearing payments since the payments that are deleted, in the first place only reached firms whose outgoing payments are decreased to zero as well. Clearly, if $\mathbf{P}'$ are not maximal proper clearing payments, then there exist proper clearing payments $\tilde{\mathbf{P}}$ with $p_{ij}' < \tilde{p}_{ij}$ for some firms v_i, v_j. If $p_{ij}' = p_{ij}$ we obtain a contradiction to the maximality of $\mathbf{P}$, otherwise if $p_{ij}' < p_{ij}$, then $\tilde{\mathbf{P}}$ cannot be proper. The claim follows. □

4 MAXIMIZING THE TOTAL ASSETS

We now turn our attention to financial network games under priority-proportional strategies when the utility is defined as the total assets. We note that in this case, the maximal clearing payments, computed in Section 3 are weakly preferred by all firms among all clearing payments of the given strategy profile; indeed, the utility is computed as the sum between a fixed term (external assets) and the incoming payments which are by definition maximized. So, in case of various clearing payments, it is reasonable to limit our attention to the (unique) maximal clearing payments computed in Section 3.

We begin with a negative result on regarding the existence of Nash equilibria.

THEOREM 4.1. *Nash equilibria are not guaranteed to exist when firms aim to maximize their total assets.*

Next, we aim to quantify the social welfare loss in Nash equilibria when each firm aims to maximize its total assets. While the focus is on financial network games under priority-proportional payments, we warm-up by considering the well-studied case of proportional payments, where we show that these may lead to outcomes where the social welfare can be far from optimal.

THEOREM 4.2. *Proportional payments can lead to arbitrarily bad social welfare loss with respect to total assets. In acyclic financial networks, the social welfare loss is at most a factor of $n/2$ and this is almost tight.*

We remark that, given clearing payments with proportional payments, a firm may wish to deviate.

REMARK 4.2.1. *Proportional payments may not form a Nash equilibrium when firms aim to maximize their total assets.*

We now turn our attention to priority-proportional strategies. To avoid text repetitions, we omit referring to priority-proportional games in our statements. We start with a positive result on the quality of equilibria when allowing for default costs in the (extreme) case $\alpha = \beta = 0$.

THEOREM 4.3. *The price of stability is 1 if default costs $\alpha = \beta = 0$ apply and firms aim to maximize their total assets.*

In the more general case, however, the price of stability may be unbounded, as the following result suggests. Recall that the setting without default costs corresponds to $\alpha = \beta = 1$.

THEOREM 4.4. *The price of stability is unbounded if default costs $\alpha > 0$ or $\beta > 0$ apply when firms aim to maximize their total assets.*

PROOF. We begin with the case where $\beta > 0$ and consider the financial network shown in Figure 2(a) where M is arbitrarily large. Firm v_1 is the only firm that can strategize about its payments. Its strategy set comprises $(v_2|v_3)$, $(v_3|v_2)$, and (v_2, v_3), which result in utility $1 + \beta$, 1, and $1 + \frac{2\beta^3}{M+2\beta-2\beta^3}$, respectively; note that, unless v_1 selects strategy $(v_2|v_3)$, v_2 is also in default as M is arbitrarily large. For sufficiently large M, we observe that any Nash equilibrium must have v_1 choosing strategy $(v_2|v_3)$, leading to clearing payments $\mathbf{p}_1 = (0, \beta^2 + \beta, 0, 0, 0)$, $\mathbf{p}_2 = (\beta, 0, 0, 0, 0)$, $\mathbf{p}_3 = \mathbf{p}_4 = (0, 0, 0, 0, 0)$, and $\mathbf{p}_5 = (1, 0, 0, 0, 0)$ with $SW(\mathbf{P}) = 2 + \beta^2 + 2\beta$. Now, when v_1 chooses strategy $(v_3|v_2)$, we obtain clearing payments $\mathbf{p}'_1 = (0, 0, \beta, 0, 0)$, $\mathbf{p}'_2 = (0, 0, 0, 0, 0)$, $\mathbf{p}'_3 = (0, 0, 0, M, 0)$, $\mathbf{p}'_4 = (0, 0, M, 0, 0)$, and $\mathbf{p}'_5 = (1, 0, 0, 0, 0)$ with $SW(\mathbf{P}') = 2M + 2 + \beta$. The claim follows since $OPT \geq SW(\mathbf{P}')$.

Now, let us assume that $\alpha > 0$ and $\beta = 0$ and consider the financial network shown in Figure 2(b). Again, firm v_1 is the only firm that can strategize about its payments. v_1's total assets when
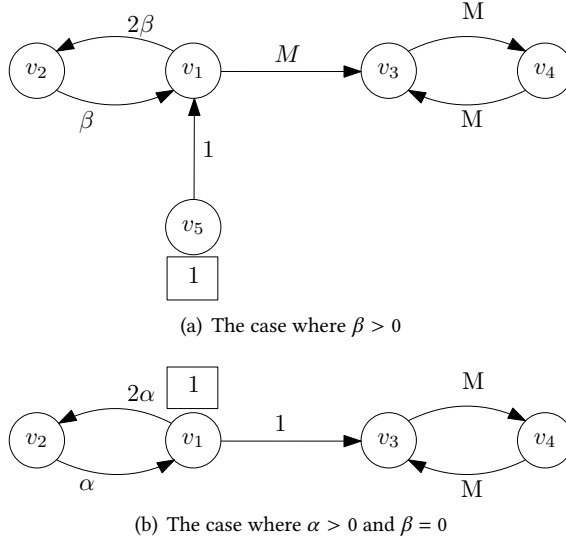
(a) The case where $\beta > 0$

(b) The case where $\alpha > 0$ and $\beta = 0$

Fig. 2. The instances used in the proof of the unbounded price of stability.

choosing strategies $(v_2|v_3)$, $(v_3|v_2)$, and (v_2, v_3) are $1 + \alpha$, 1, and 1, respectively; note that when v_1 chooses strategy (v_2, v_3), firm v_2 is also in default as it receives a payment of $\frac{2\alpha^2}{1+2\alpha}$ which is strictly less than α for any $\alpha > 0$. Hence, in any Nash equilibrium v_1 chooses strategy $(v_2|v_3)$, resulting in clearing payments $\mathbf{p}_1 = (0, \alpha, 0, 0), \mathbf{p}_2 = (\alpha, 0, 0, 0), \mathbf{p}_3 = \mathbf{p}_4 = (0, 0, 0, 0)$ with social welfare $SW(\mathbf{P}) = 1 + 2\alpha$. The optimal social welfare, however, is achieved when v_1 chooses strategy $(v_3|v_2)$, resulting in clearing payments $\mathbf{p}'_1 = (0, 0, \alpha, 0), \mathbf{p}'_2 = (0, 0, 0, 0), \mathbf{p}'_3 = (0, 0, 0, M)$, and $\mathbf{p}'_4 = (0, 0, M, 0)$ with $OPT = 2M + 1 + \alpha$. □

We now show that the price of stability may also be unbounded in the absence of default costs, if negative external assets are allowed.

THEOREM 4.5. *The price of stability is unbounded if negative external assets are allowed and firms aim to maximize their total assets.*

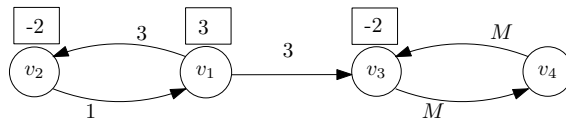PROOF. Consider the financial network shown in Figure 3. Only firm v_1 can strategize about its

Fig. 3. A financial network with unbounded price of stability.

payments. Clearly, v_1's total assets equal 3, unless v_2 pays its debt (even partially). This is only possible when v_1 chooses strategy $(v_2|v_3)$ and prioritizes the payment of v_2 (note v_2's negative external assets will "absorb" any payment that is at most 2). Therefore, the resulting Nash equilibrium leads to the clearing payments $\mathbf{p}_1 = (0, 3, 1, 0), \mathbf{p}_2 = (1, 0, 0, 0), \mathbf{p}_3 = (0, 0, 0, 0), \mathbf{p}_4 = (0, 0, 0, 0)$ with social welfare $SW(\mathbf{P}) = 4$. However, when v_1 chooses strategy $(v_3|v_2)$ we obtain the clearing payments $\mathbf{p}'_1 = (0, 0, 3, 0), \mathbf{p}'_2 = (0, 0, 0, 0), \mathbf{p}'_3 = (0, 0, 0, M), \mathbf{p}'_4 = (0, 0, M, 0)$ with social welfare $SW(\mathbf{P}') = 2M + 2$. Hence, since $OPT \geq SW(\mathbf{P}')$ we obtain the theorem. □

The proof of Theorem 4.5 in fact holds for any type of strategies as v_1 always prefers to pay in full its liability to v_2. This includes the case of a very general payment strategy scheme, namely coin-ranking strategies [4] that are known to have price of stability 1 with non-negative external assets.

Next, we show that the price of anarchy can be unbounded even without default costs, CDS contracts, and negative externals. Bertschinger et al. [4] have shown a similar result for coin-ranking strategies, albeit for a network that has no external assets; our result extends to the case of coin-ranking strategies and strengthens the result of [4] to capture the case of proper clearing payments.

THEOREM 4.6. *The price of anarchy is unbounded when firms aim to maximize their total assets.*

5 MAXIMIZING THE EQUITY

In this section we consider the case of equities. Similarly, the social welfare is defined as the sum of equities. We present interesting properties of clearing payments and observe that Nash equilibria always exist in such games, contrary to the case of total assets.

5.1 Existence and properties of equilibria

We warm up with a known statement in absence of default costs. In particular, each firm obtains the same equity under all clearing payments, so it does not have a preference; this provides additional justification to our choice to focus on maximal clearing payments computed in Section 3, in case of various clearing payments.

LEMMA 5.1 ([11]). *Each firm obtains the same equity under different clearing payments, given a strategy profile. That is, given the firms' strategies, for any two different (not necessarily maximal) clearing payments $\mathbf{P}$ and $\mathbf{P}'$, it holds $E_i(\mathbf{P}) = E_i(\mathbf{P}')$ for each firm v_i.*

Lemma 5.1 also indicates that, for any given strategy profile, any firm is always either solvent or in default in all resulting clearing payments. We exploit this property to obtain the following result; this extends a result by Papp and Wattenhofer (Theorem 7 in [16]) which holds for the maximal clearing payments.

THEOREM 5.2. *Even with CDS contracts, any strategy profile is a Nash equilibrium, when firms aim to maximize their equity. This holds even if the clearing payments that are realized are not maximal.*

Note that Lemma 5.1 no longer holds once default costs are introduced; see e.g., Example 3.3 in [17] where both firms may be in default or solvent depending on the clearing payments. The next result extends Theorem 7 in [16] to the setting with default costs, and guarantees the existence of Nash equilibria (and actually strong ones) when firms wish to maximize their equity.

THEOREM 5.3. *Even with default costs and negative external assets, any strategy profile is a strong equilibrium when firms aim to maximize their equity.*

5.2 (In)efficiency of equilibria

We start by noting that Lemma 5.1 together with Theorem 5.2 imply the following, which we note holds for any payment scheme.

COROLLARY 5.4. *The price of anarchy in financial network games with CDS contracts is* 1 *when firms aim to maximize their equity.*

The above positive result, however, no longer holds when default costs or negative external assets exist. For these cases we derive the following results.

THEOREM 5.5. *The price of anarchy with default costs (α, β) when firms aim to maximize their equity, is*

(a) 1 when $\alpha = \beta = 0$,
(b) unbounded when: i) $\beta \in (0, 1)$, ii) $\beta = 0$ and $\alpha \in (0, 1]$, or iii) $\beta = 1$ and $\alpha = 0$,
(c) at least $1/\alpha - \varepsilon$, if $\beta = 1$ and $\alpha \in (0, 1)$ for any $\varepsilon > 0$.

PROOF. We begin with the case $\alpha = \beta = 0$. We claim that all strategy profiles correspond to the same clearing payments hence admit the same social welfare. It suffices to observe that neither the strategy of a solvent firm, nor the strategy of a firm in default, affect the set of firms in default and consequently the clearing payments. Consider two strategy profiles $\mathbf{s}$ and $\mathbf{t}$. A firm i that is solvent under $\mathbf{s}$, will be solvent and continue to make payments $p_i = L_i$ under any strategy (given the strategies of everyone else), i.e., will be solvent at the clearing payments consistent with strategy vector $(t_i, \mathbf{s}_{-i})$, derived by $\mathbf{s}$ if firm i alone changes her strategy from s_i to t_i. On the other hand, a firm i that is in default under $\mathbf{s}$, will similarly remain in default under $(t_i, \mathbf{s}_{-i})$ and continue to make 0 payments since $\alpha = \beta = 0$, thus not affecting the set of firms in default. The claim follows by considering the individual deviations from s_i to t_i of all firms i sequentially and observing that the set of firms in default and, hence, the clearing payments are unaffected at each step.
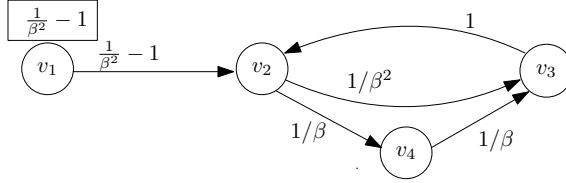

Fig. 4. The instance in the proof of Theorem 5.5, where we assume $\beta \in (0, 1)$.

Regarding the case $\beta \in (0, 1)$, consider the financial network in Figure 4. Clearly, only firm v_2 can strategize and observe that it is always in default irrespective of its strategy. When v_2 selects $s_2 = (v_4|v_3)$, we obtain the clearing payments $\mathbf{p}_1 = (0, \frac{1}{\beta^2} - 1, 0, 0), \mathbf{p}_2 = (0, 0, 0, \frac{1+\beta}{\beta(\beta^2+\beta+1)}), \mathbf{p}_3 = (0, \frac{\beta(1+\beta)}{\beta^2+\beta+1}, 0, 0)$, and $\mathbf{p}_4 = (0, 0, \frac{1+\beta}{\beta^2+\beta+1}, 0)$. Since v_2, v_3 and v_4 are in default, we have $SW(\mathbf{P}) = 0$.

When, however, v_2 selects strategy $s_2' = (v_3|v_4)$, we obtain the clearing payments $\mathbf{p}_1' = (0, \frac{1}{\beta^2} - 1, 0, 0), \mathbf{p}_2' = (0, 0, \frac{1}{\beta}, 0), \mathbf{p}_3' = (0, 1, 0, 0)$, and $\mathbf{p}_4' = (0, 0, 0, 0)$. Now, v_2 and v_4 are in default and we have $SW(\mathbf{P}') = 1/\beta - 1$. Since $OPT \geq SW(\mathbf{P}')$, the claim follows.

Regarding the case $\beta = 0$ and $\alpha \in (0, 1]$, consider the financial network in Figure 5 and note that v_2 is always in default irrespective of its strategy. If v_2 selects strategy $s_2 = (v_6|v_3)$, then v_6 ends up having equity α. However, strategy $(v_3|v_6)$ is also an equilibrium strategy for v_2, but it results in each firm having equity 0.
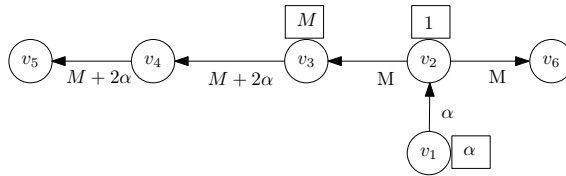

Fig. 5. The instance in the proof of Theorem 5.5, when $\beta \in \{0, 1\}$ and $\alpha \neq \beta$.

Regarding the remaining case, i.e., $\beta = 1$ and $\alpha \in [0, 1)$, consider again the financial network in Figure 5. As before, v_2 is always in default. If v_2 selects strategy $(v_3|v_6)$, then v_5 ends up having equity $M + 2\alpha$. However, $(v_6|v_3)$ is also an equilibrium strategy for v_2, but it results in equity 2α for v_6, equity αM for v_5 and equity 0 for the remaining firms; the theorem follows.   □

We complement this result with a tight upper bound when $\beta = 1$.

THEOREM 5.6. *The price of anarchy with default costs $(\alpha, 1)$ when firms aim to maximize their equity is at most $1/\alpha$. This holds even if the clearing payments that are realized are not maximal.*

PROOF. Consider any clearing payments $\mathbf{P}$ and let $S(\mathbf{P})$ and $D(\mathbf{P})$ be the set of solvent and in default firms under $\mathbf{P}$. Recall that for any firm v_i, we have $E_i(\mathbf{P}) = \max\{0, a_i(\mathbf{P}) - L_i\}$. For any firm $v_i \in S(\mathbf{P})$, it holds that $E_i(\mathbf{P}) = e_i + \sum_{j \in [n]} p_{ji} - \sum_{j \in [n]} p_{ij}$, as $\sum_{j \in [n]} p_{ij} = \min\{e_i + \sum_{j \in [n]} p_{ji}, L_i\}$. Similarly, for any $v_i \in D(\mathbf{P})$, we have $E_i(\mathbf{P}) = \alpha e_i + \sum_{j \in [n]} p_{ji} - \sum_{j \in [n]} p_{ij}$, since $\beta = 1$. By summing over all firms, we get

$$E(\mathbf{P}) = \sum_{i:v_i \in S(\mathbf{P})} E_i(\mathbf{P}) + \sum_{i:v_i \in D(\mathbf{P})} E_i(\mathbf{P})$$

$$= \sum_{i \in [n]} \left(e_i + \sum_{j \in [n]} p_{ji} - \sum_{j \in [n]} p_{ij} \right) - (1 - \alpha) \sum_{i:v_i \in D(\mathbf{P})} e_i$$

$$= \sum_{i \in [n]} e_i - (1 - \alpha) \sum_{i:v_i \in D(\mathbf{P})} e_i.$$

Clearly, the social welfare is maximized when $D(\mathbf{P}) = \emptyset$, i.e., $OPT \leq \sum_{i \in [n]} e_i$, while it is minimized when $D(\mathbf{P})$ includes all firms, i.e., $E(\mathbf{P}) \geq \alpha \sum_{i \in [n]} e_i$; this completes the proof.   □

Extending the setting to allow for negative external assets again may lead to unbounded social welfare loss.

THEOREM 5.7. *The price of anarchy is unbounded when negative external assets are allowed and firms aim to maximize their equity.*

Still, in the presence of default costs or negative external assets, Theorem 5.3 leads to the next positive result.

COROLLARY 5.8. *The strong price of stability is 1 even with default costs and negative external assets.*

If we relax the stability notion and consider the price of stability in super-strong equilibria, where a coalition of firms deviates if at least one strictly improves its utility and no firm suffers a decrease in utility, then we obtain a negative result.

THEOREM 5.9. *The price of stability in super-strong equilibria is unbounded when negative external assets are allowed and firms aim to maximize their equity.*

6 CONCLUSIONS AND DISCUSSION

We have studied strategic payment games in financial networks with priority-proportional payments and we have presented an almost full picture both with respect to structural properties of clearing payments as well as the quality of Nash equilibria.

Our work reveals several interesting open questions. In particular, an interesting restriction on the strategy space is to always prioritize strategies that could lead to incoming payments, i.e., when each firm *i* always ranks higher a payment that might create a cash flow through a directed

cycle (in the liabilities graph) back to i, than a payment that does not. While some of our negative results would still hold, e.g., Theorem 4.1, others, such as Theorem 4.4, crucially rely on the reverse ranking. Furthermore, one could consider addressing computational complexity questions like deciding whether a Nash equilibrium exists or computing a Nash equilibrium when it is guaranteed to exist, for the case of total assets.

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