ARBITRAGE AND EQUILIBRIUM IN ECONOMIES WITH EXTERNALITIES

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No 588

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

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Arbitrage and Equilibrium in Economies with Externalities*

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June 2000
Current Version, April, 2001
Keywords: arbitrage, externalities, recession cones, general equilibrium

*This paper is to appear in Journal of Global Optimization. The authors, in various combinations, are indebted to several institutions for hospitality and support. These include Paris 1 and the University of Alabama. Page and Wooders are also indebted to CentER (Tilberg University) and the Autonomous University of Barcelona for hospitality and support during the period when this research on arbitrage with externalities was initiated. Wooders is grateful to the Social Sciences and Humanities Research Council of Canada and the Direccio General d'Universitats of Catalonia for support.

Abstract

We introduce consumption externalities into a general equilibrium model with arbitrary consumption sets. To treat the problem of existence of equilibrium, a condition of no unbounded arbitrage, extending the condition of Page (1987) and Page and Wooders (1993,1996) is defined. It is proven that this condition is sufficient for the existence of an equilibrium and both necessary and sufficient for compactness of the set of rational allocations.

1. Introduction

A subject of ongoing interest in economic theory has been conditions ensuring existence of economic equilibrium in models allowing unbounded short sales. When unbounded short sales are allowed, in contrast to Arrow-Debreu-McKenzie general equilibrium models, consumption sets are unbounded below. To illustrate the problem this creates for existence of economic equilibrium, suppose that two agents have diametrically opposed preferences. For example, one agent may want to buy arbitrarily large amounts of one commodity and sell another commodity short while the other agent may prefer to do the opposite. In such a situation, there are unbounded arbitrage opportunities and no equilibrium exists. To ensure existence of equilibrium arbitrage opportunities must be limited. Arbitrage conditions sufficient to guarantee existence of equilibria in general equilibrium models of unbounded exchange economies (e.g., asset exchange economies allowing short sales) have been studied by Werner (1987), Nielsen (1989), Page and Wooders (1993, 1996),[1] and most recently by Dana, Le Van, and Magnien (1999), Page, Wooders, and Monteiro (1999) and Allouch (1999). None of these models allow consumption externalities, that is, an agent's evaluation of a trade is not allowed to depend on the trades engaged in by other members of the economy. With the speculative behavior that sometimes appears to dominate financial markets in mind, this would seem to be a significant limitation of the existing models.

In this paper we extend the general equilibrium models noted above to allow consumption externalities and closed, convex, and possibly unbounded consumption sets. We also extend the condition of no unbounded arbitrage of Page (1987) for asset market models, applied to general equilibrium models in Page

[1] These two papers are essentially the same; the earlier version does not restrict to preferences representable by concave functions, which enables the later version to be shorter.

and Wooders (1993,1996), to treat situations with externalities. It is shown that our generalized condition of no unbounded arbitrage is sufficient for the existence of equilibrium and necessary and sufficient for compactness of the set of rational allocations.

Our generalized condition of no unbounded arbitrage in the presence of externalities has the same intuition that applies in situations without externalities. The condition of no unbounded arbitrage in Page (1987) and Page and Wooders (1993,1996) is an assumption on the relationships of preferences of agents in the economy, ruling out the possibility that for large trades, no two agents' preferences become diametrically opposed. The assumption is stated in terms of the recession cones of the sets of trades that each agent prefers to his endowment. Our generalized condition of no unbounded arbitrage is also essentially a similarity assumption on preferences. The assumption, however, to treat economies with externalities, is stated in terms of sequences of allocations.

The importance of externalities is widely recognized in the economics literature and a number of papers have studied equilibria in abstract economies which allow externalities. These include, for example, the classic papers of Shafer and Sonnenschein (1975) and Borglin and Keiding (1976). Since in their models consumption sets are compact, arbitrage considerations do not play a critical role. Motivation for the introduction of externalities into a general equilibrium framework with unbounded short sales comes primarily from the theory of financial markets. Externalities permit us to model the fact that the possibilities for profitable arbitrage perceived by individual agents may be affected by the trading activities of others. This seems particularly natural in asset markets. The observed demands of others for assets may well be taken as indicators of the desirability of assets, and the trading activities of others may convey information about expected asset returns. In fact, in their models of asset markets, Hart (1974), Hammond (1983), and Page (1987) all allow price dependent preferences, a form of externalities. The general equilibrium models with unbounded short sales noted above, however, do not permit this aspect of asset market models.

2. An Economy with Externalities

Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ denote an unbounded exchange **economy**. Each agent j has **choice set** $X_j \subset R^L$ and **endowment** $\omega_j \in X_j$. The j^{th} agent's preferences, defined over $\prod_{j=1}^n X_j$, are specified via a **utility function** $u_j(\cdot) : \prod_{j=1}^n X_j \to R$. Define

$X := \prod_{j=1}^{n} X_j$ and $X_{-j} := \prod_{i \neq j} X_i$, with typical element denoted by x_{-j}.

The set of **rational allocations** is given by

$$A = \{(x_1, ..., x_n) \in X : \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \omega_j \\ \text{and for each } j,\ u_j(x_j, x_{-j}) \geq u_j(\omega_j, x_{-j})\}. \tag{2.1}$$

For each $(x_j, x_{-j}) \in \prod_{j=1}^{n} X_j$, the **preferred set** is given by

$$P_j(x_j, x_{-j}) := \{x \in X_j : u_j(x, x_{-j}) > u_j(x_j, x_{-j})\}, \tag{2.2}$$

while the weakly preferred set is given by

$$\widehat{P}_j(x_j, x_{-j}) := \{x \in X_j : u_j(x, x_{-j}) \geq u_j(x_j, x_{-j})\}. \tag{2.3}$$

We will maintain the following assumptions on the economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^{n}$ throughout the remainder of the paper. For each $j = 1, ..., n$,

[A-1] $\left\{ \begin{array}{l} X_j \text{ is closed and convex, and } \omega_j \in int X_j, \\ \text{where "}int\text{" denotes "interior".} \end{array} \right.$

[A-2] $\left\{ \begin{array}{l} \text{For each } (x_j, x_{-j}) \in X,\ u_j(\cdot, x_{-j}) \text{ is quasi-concave on } X_j, \\ \text{and } u_j(\cdot,\cdot) \text{ is continuous on } X_j \times X_{-j}. \end{array} \right.$

[A-3] $\left\{ \begin{array}{l} \text{For each } (x_j, x_{-j}) \in A,\ P_j(x_j, x_{-j}) \neq \emptyset, \\ \text{and } clP_j(x_j, x_{-j}) = \widehat{P}_j(x_j, x_{-j}). \end{array} \right.$

Note that in [A-1] we do not assume that consumption sets are bounded. Also, note that given [A-2], for all $(x_j, x_{-j}) \in X$ the preferred set $P_j(x_j, x_{-j})$ is nonempty and convex, while the weakly preferred set $\widehat{P}_j(x_j, x_{-j})$ is nonempty, closed and convex. Finally, note that [A-3] implies that there is **local nonsatiation at rational allocations.**

Given commodity prices $p \in R^L$, the cost of a consumption vector $x = (x_1, ..., x_L)$ is $\langle p, x \rangle = \sum_{\ell=1}^{L} p_\ell \cdot x_\ell$. The **budget set** is given by[2]

$$B_j(p, \omega_j) = \{x \in X_j : \langle p, x \rangle \leq \langle p, \omega_j \rangle\}. \tag{2.4}$$

[2] The restriction of the budget set to be a subset of the consumption set entails no losss of substance or generality.

Without loss of generality we can assume that commodity prices are contained in the unit ball

$$\mathcal{B} := \{p \in R^L : \|p\| \leq 1\}.$$

An **equilibrium** for the economy $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ is an $(n+1)$-tuple of vectors $(\overline{x}_1, ..., \overline{x}_n, \overline{p})$ such that

(i) $(\overline{x}_1, ..., \overline{x}_n) \in A$ (the allocation is feasible);

(ii) $\overline{p} \in \mathcal{B} \backslash \{0\}$ (prices are in the unit ball and not all prices are zero); and

(iii) for each j, $\overline{x}_j \in B_j(\overline{p}, \omega_j)$ and $P_j(\overline{x}_j, \overline{x}_{-j}) \cap B_j(\overline{p}, \omega_j) = \emptyset$ (i.e., $\overline{x}_j$ maximizes $u_j(x_j, \overline{x}_{-j})$ over $B_j(\overline{p}, \omega_j)$).[3]

3. Arbitrage and Compactness

We begin by recalling a few basic facts about recession cones (see Section 8 in Rockafellar (1970)). Let X be a convex set in R^L. The **recession cone** $0^+(X)$ corresponding to X is given by

$$0^+(X) = \{y \in R^L : x + \lambda y \in X \text{ for all } \lambda \geq 0 \text{ and } x \in X\}. \tag{3.1}$$

If X is also closed, then the set $0^+(X)$ is a closed convex cone containing the origin. Moreover, if X is closed, then $x + \lambda y \in X$ for **some** $x \in X$ and all $\lambda \geq 0$ implies that $x' + \lambda y \in X$ for **all** $x' \in X$ and all $\lambda \geq 0$. Thus, if X is closed, then we can conclude that $y \in 0^+(X)$ if for some $x \in X$ and all $\lambda \geq 0$, $x + \lambda y \in X$. Perhaps the most useful fact is the following:

if X is closed, then X is compact if and only if $0^+(X) = \{0\}$.

Now, on to arbitrage.

We say that an n-tuple of net trade vectors, $(y_1, \ldots, y_n)$, is mutually compatible and utility nondecreasing if

$$\sum_{j=1}^n y_j = 0$$

and for all j,

$$\omega_j + y_j \in X_j, \text{ and}$$

$$u_j(\omega_j + y_j, \omega_{-j} + y_{-j}) \geq u_j(\omega_j, \omega_{-j} + y_{-j})$$

[3]Under assumptions [A-1]-[A-3], it follows that in equilibrium budget constraints are satisfied with equality, that is, $\langle \overline{p}, \overline{x}_j \rangle = \langle \overline{p}, \omega_j \rangle$.

The following **generalized no unbounded arbitrage condition** (GNUA) guarantees that there are no unbounded sequences of mutually compatible and utility nondecreasing net trades:

$$\left.\begin{array}{c} GNUA: \\ \text{if the } n\text{-tuple } (y_1,\ldots,y_n) \text{ of net trades is such that} \\ (y_1,\ldots,y_n) = \lim_k (t^k x_1^k,\ldots,t^k x_n^k), \\ \text{where} \\ \{t^k\}_k \text{ is a sequence of positive real numbers such that } t^k \downarrow 0 \\ \text{and} \\ \{(x_1^k,\ldots,x_n^k)\}_k \text{ is a sequence of rational allocations,} \\ \text{then} \\ y_j = 0 \text{ for all } j. \end{array}\right\} \quad (3.2)$$

Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ be an economy with externalities satisfying assumptions [A-1]-[A-3]. To see that GNUA guarantees an absence of unbounded sequences of mutually compatible and utility nondecreasing net trades consider the following. Suppose $\{(y_1^k,\ldots,y_n^k)\}_k$ is an unbounded sequence of mutually compatible and utility nondecreasing net trades. For all j and k let $x_j^k = \omega_j + y_j^k$. Thus,

$$\begin{array}{c} \{(x_1^k,\ldots,x_n^k)\}_k \subset A \\ \text{with} \\ \sum_j \|x_j^k\| \to \infty \text{ as } k \to \infty. \end{array}$$

Letting, $t^k := \frac{1}{\sum_j \|x_j^k\|}$, we have $t^k \downarrow 0$ and for some subsequence $\{(x_1^{k'},\ldots,x_n^{k'})\}_{k'}$, we also have

$$(t^{k'} x_1^{k'},\ldots,t^{k'} x_n^{k'}) \to (y_1,\ldots,y_n).$$

But $\sum_j \|y_j\| = 1$, and thus, we have a contradiction of GNUA.

If the economy satisfies GNUA, we can say much more.

Theorem 3.1. *(GNUA is equivalent to the compactness of A)*

Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ be an economy with externalities satisfying assumptions (A-1)-(A-3). Then the following statements are equivalent:

1. *$(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ satisfies GNUA.*
2. *The set of rational allocations is compact.*
3. *$0^+(\overline{co}A) = \{0\}$.*[4]

[4]Here $\overline{co}A$ denotes the closed, convex hull of A.

4. *There are no unbounded sequences of mutually compatible and utility nondecreasing net trades.*

Proof. (1)$\Rightarrow$ (2) (GNUA $\Rightarrow$ compactness). Since A is closed, we have just to prove that A is bounded. Suppose not. Then there is a sequence $\left\{(x_1^k, \ldots, x_n^k)\right\}_k \subset A$ such that $\sum_j \left\|x_j^k\right\| \to \infty$ as $k \to \infty$. Letting $t^k := \frac{1}{\sum_j \left\|x_j^k\right\|}$, and repeating the argument immediately above, we have for some subsequence $\left\{(x_1^{k'}, \ldots, x_n^{k'})\right\}_{k'}$,

$$(t^{k'} x_1^{k'}, \ldots, t^{k'} x_n^{k'}) \to (y_1, \ldots, y_n)$$
with
$$\textstyle\sum_j \|y_j\| = 1.$$

Thus, we have a contradiction of GNUA.

(2)$\Leftrightarrow$(3) (compactness $\Leftrightarrow 0^+(\overline{co}A) = \{0\}$). First, A compact $\Rightarrow$ coA compact $\Rightarrow \overline{co}A$ compact $\Rightarrow 0^+(\overline{co}A) = \{0\}$. Second, $0^+(\overline{co}A) = \{0\} \Rightarrow \overline{co}A$ compact. Since $A \subset R^L$ is closed, $\overline{co}A$ compact and $A \subset \overline{co}A$ implies that A is also bounded and hence compact.

(2)$\Rightarrow$ (1) (compactness $\Rightarrow$ GNUA). Let $(t^k x_1^k, \ldots, t^k x_n^k) \to (y_1, \ldots, y_n)$ where $\left\{(x_1^k, \ldots, x_n^k)\right\}_k \subset A$ and $t^k \downarrow 0$. Since $A \subseteq \overline{co}A$, $(y_1, \ldots, y_n) \in 0^+(\overline{co}A) = \{0\}$. Thus, GNUA holds.

(4) $\Rightarrow$ (1) (GNUA $\Rightarrow$ no unbounded sequences). This implication follows directly from the equivalence of GNUA and compactness of the set of rational allocations.

(1) $\Rightarrow$ (4) (no unbounded sequences $\Rightarrow$ GNUA). Consider the rational allocations $\left\{(x_1^k, .., x_n^k)\right\}_{k=0,..,\infty}$. For all i and for all k, let $y_j^k = x_j^k - \omega_j$. Each component of the sequence $\left\{(y_1^k, .., y_n^k)\right\}_k$ describes a vector of mutually compatible and utility nondecreasing net trades. By assumption, the sequence is bounded; that is, there is a constant K such that for each component of the sequence and each term in the sequence y_j^k it holds that $\left\|y_j^k\right\| < (K, ..., K)$. Let $\left\{t^k\right\}$ a positive sequence converging to 0. We have $\lim_k \left\{(t^k x_1^k, .., t^k x_n^k)\right\} = 0$. Thus, GNUA holds. ■

Before moving on to the existence question, several observations are in order.

1. Define the set of utility possibilities, $U(A)$, as follows:

$$U(A) := \{(u_1, \ldots, u_n) \in \mathsf{R}^n : \exists (x_1, ..., x_n) \in A \text{ such that } u_j(\omega_j, x_{-j}) \leq u_j \leq u_j(x_j, x_{-j}) \forall j\}. \tag{3.3}$$

   An immediate consequence of Theorem (3.1) is that $U(A)$ is compact.

2. In economies without externalities, GNUA coincides with the condition of no unbounded arbitrage introduced in Page (1987). Indeed, without externalities the set of rational allocations is given by

$$A = \{(x_1, ..., x_n) \in X : \textstyle\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \omega_j$$
$$\text{and for each } j,\ u_j(x_j) \geq u_j(\omega_j)\},$$

and no unbounded arbitrage can be stated very compactly as

$$0^+ A = \{0\}. \tag{3.4}$$

Since without externalities A is closed and convex, $0^+ A = 0^+ (coA)$.

3. In an economic model similar to the model presented here, but without externalities, Dana, Le Van, and Magnien (1999) have shown that compactness of the set of utility possibilities, $U(A)$, is sufficient for the existence of equilibrium. However, in the presence of externalities compactness of $U(A)$, as a condition limiting arbitrage opportunities, seems not to be sufficient. Here we only sketch the difficulty: For simplicity, assume that utility functions are strictly quasi-concave and that $P_j(x_j^*, x_{-j}^*) \neq \emptyset$ for all $(x_j^*, x_{-j}^*) \in A$ (these two simplifying assumptions imply local nonsatiation at rational allocations). Consider a sequence of truncated economies. For each of these economies there exists a quasi-equilibrium (x^k, p^k). Since $U(A)$ is compact we can assume without loss of generality that

$$\big(u_1(x_1^k, x_{-1}^k), \ldots, u_j(x_j^k, x_{-j}^k), \ldots, u_n(x_n^k, x_{-n}^k)\big) \to (z_1, \ldots, z_j, \ldots, z_n) \in U(A).$$

We can also assume without loss of generality that $p^k \to p^* \neq 0$. By the definition of the set of utility possibilities there exists $(x_1^*, ..., x_n^*) \in A$ such that or all j, $u_j(x_j^*, x_{-j}^*) \geq z_j$. Moreover, since $P_j(x_j^*, x_{-j}^*) \neq \emptyset$, there exists x_j such that

$$z_j \leq u_j(x_j^*, x_{-j}^*) < u_j(x_j, x_{-j}^*),$$

and by strict quasi-concavity,

$$u_j(\lambda x_j + (1-\lambda)x_j^*, x_{-j}^*) > u_j(x_j^*, x_{-j}^*).$$

Finally, by continuity of utility functions we have for k large enough,

$$u_j(\lambda x_j + (1-\lambda)x_j^*, x_{-j}^*) > u_j(x_j^k, x_{-j}^k).$$

Unfortunately, we cannot write

$$u_j(\lambda x_j + (1-\lambda)x_j^*, x_{-j}^k) > u_j(x_j^k, x_{-j}^k).$$

However, if A is compact, then we can assume without loss of generality that $x^k \to x^*$, and therefore, we can conclude via continuity of utility functions that for large k,

$$u_j(\lambda x_j + (1-\lambda)x_j^*, x_{-j}^k) > u_j(x_j^k, x_{-j}^k).$$

Moreover, since for all k,

$$\lambda \left\langle p^k, x_j \right\rangle + (1-\lambda) \left\langle p^k, x_j^* \right\rangle \geq \left\langle p^k, \omega_j \right\rangle,$$

we can conclude that

$$\lambda \left\langle p^*, x_j \right\rangle + (1-\lambda) \left\langle p^*, x_j^* \right\rangle \geq \left\langle p^*, \omega_j \right\rangle.$$

Letting $\lambda \to 0$, we obtain $\left\langle p^*, x_j^* \right\rangle \geq \left\langle p^*, \omega_j \right\rangle, \forall\ j$ and then $\left\langle p^*, x_j^* \right\rangle = \left\langle p^*, \omega_j \right\rangle, \forall\ j$. It is easy to check that (x^*, p^*) is a quasi-equilibrium.

4. In an economic model without externalities, Allouch (1999) introduces a condition limiting arbitrage weaker than the conditions limiting arbitrage found in both Page and Wooders (1996) and Page, Wooders, and Monteiro (2000). With some mild assumptions on the economic model, Allouch shows that his condition is equivalent to compactness of the set of utility possibilities $U(A)$.

4. Existence of Equilibrium

4.1. Existence for Bounded Economies with Externalities

We begin by defining a k-bounded economy,

$$(X_{kj}, \omega_j, u_j(\cdot))_{j=1}^n, \tag{4.1}$$

In the k-bounded economy, the j^{th} agent's consumption set is

$$X_{kj} := X_j \cap B_k(\omega_j), \tag{4.2}$$

where $B_k(\omega_j)$ is a closed ball of radius k centered at the agent's endowment, ω_j.

Define

$$X_k := \prod_{j=1}^{n} X_{kj}.$$

The set of k-bounded rational allocations is given by

$$\begin{aligned} A_k = \{(x_1, ..., x_n) \in X_k : \textstyle\sum_{j=1}^n x_j = \sum_{j=1}^n \omega_j \\ \text{and for each } j,\ u_j(x_j, x_{-j}) \geq u_j(\omega_j, x_{-j})\}. \end{aligned} \tag{4.3}$$

An **equilibrium** for the k-bounded economy, $(X_{kj}, \omega_j, u_j(\cdot))_{j=1}^n$, is an $(n+1)$-tuple of vectors $(x_1^k, \ldots, x_n^k, p^k)$ such that

(*i*) $(x_1^k, \ldots, x_n^k) \in A_k$, (the allocation is feasible);

(*ii*) $p^k \in \mathcal{B} \backslash \{0\}$ (prices are in the unit ball and not all prices are zero); and

(*iii*) for each j, $x_j^k \in B_{kj}(p^k, \omega_j)$ and $P_{kj}(x_j^k, x_{-j}^k) \cap B_{kj}(p^k, \omega_j) = \emptyset$ (i.e., x_j^k maximizes $u_j(x_j, x_{-j}^k)$ over $B_{kj}(p^k, \omega_j)$).[5]

Here,

$$P_{kj}(x_j^k, x_{-j}^k) := P_j(x_j^k, x_{-j}^k) \cap X_{kj},$$
and
$$B_{kj}(p^k, \omega_j) := B_j(p^k, \omega_j) \cap X_{kj}.$$

We now have our main existence result for bounded economies.

Theorem 4.1. *(Existence of equilibria for k-bounded economies)*

Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ be an economy with externalities satisfying assumptions [A-1]-[A-3]. Then, for all k sufficiently large the k-bounded economy,

$$(X_{jk}, \omega_j, u_j(\cdot))_{j=1}^n,$$

has an equilibrium, $(x_1^k, \ldots, x_n^k, p^k)$, with

$$p^k \in \mathcal{B}_u := \left\{p \in R^L : \|p\| = 1\right\}.$$

In particular, $(X_{jk}, \omega_j, u_j(\cdot))_{j=1}^n$, has an equilibrium for all k larger than *the k^*, where k^* is such that $A_k = A$ for all $k \geq k^*$.*

[5]Under assumptions [A-1]-[A-3], it follows that in equilibrium budget constraints are satisfied with equality, that is, $\langle p^k, x_j^k \rangle = \langle p^k, \omega_j \rangle$.

Proof: One can refer to Florenzano (1981,Proposition 2, p. 96) where preferences depend on consumptions of the other agents and also on prices. Because in our paper preferences do not depend on prices, a simpler proof will suffice; this is provided in the appendix. Our proof is based on work on abstract games. So we show that this "tool" can be used in economies with or without externalities.

4.2. Existence for Unbounded Economies with Externalities

Our main existence result for unbounded economies with externalities is the following:

Theorem 4.2. *(Existence for unbounded economies with externalities)*

Let $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ be an economy with externalities satisfying assumptions [A-1]-[A-3]. If the economy satisfies the generalized condition of no unbounded arbitrage (GNUA), then $(X_j, \omega_j, u_j(\cdot))_{j=1}^n$ has an equilibrium, $(\overline{x}_1, \ldots, \overline{x}_n, \overline{p})$, with

$$\overline{p} \in \mathcal{B}_u := \left\{ p \in R^L : \|p\| = 1 \right\}.$$

.

Proof. For each k sufficiently large the k-bounded economy $(X_{jk}, \omega_j, u_j(\cdot))_{j=1}^n$ has an equilibrium

$$(x_1^k, \ldots, x_n^k, p^k) = (x^k, p^k) \in A_k \times \mathcal{B}_u \subseteq A \times \mathcal{B}_u.$$

Since $A \times \mathcal{S}_u$ is compact, we can assume without loss of generality that

$$(x_1^k, \ldots, x_n^k, p^k) \to (\overline{x}_1, \ldots, \overline{x}_n, \overline{p}) \in A \times \mathcal{B}_u.$$

Moreover, since for all j and k, $\left\langle p^k, x_j^k \right\rangle = \left\langle p^k, \omega_j \right\rangle$, we have for all j, $\langle \overline{p}, \overline{x}_j \rangle = \langle \overline{p}, \omega_j \rangle$.

Let $u_j(x_j, \overline{x}_{-j}) > u_j(\overline{x}_j, \overline{x}_{-j})$. Then, for k sufficiently large, $x_j \in X_{jk}$ and $u_j(x_j, x_{-j}^k) > u_j(x_j^k, x_{-j}^k)$ which implies that $\left\langle p^k, x_j \right\rangle > \left\langle p^k, \omega_j \right\rangle$. Thus, in the limit $\langle \overline{p}, x_j \rangle \geq \langle \overline{p}, \omega_j \rangle$. Hence, $(\overline{x}_1, \ldots, \overline{x}_n, \overline{p})$ is a quasi-equilibrium. Since for all j, $\omega_j \in intX_j$ (see [A-1]), and since utility functions are continuous (see [A-2]), in fact, $(\overline{x}_1, \ldots, \overline{x}_n, \overline{p})$ is an equilibrium. ■

5. Appendix

Proof of Theorem 4.1. For each k, we have corresponding to the economy $(X_{kj}, \omega_j, u_j(\cdot))_{j=1}^n$ the abstract game,

$$G_k := \{(X_{kj}, H_{kj}(\omega, \cdot), v_j(\cdot, \cdot))_{j=1}^{n+1}\}_k,$$

with

$$\text{constraint mappings } p \to H_{kj}(\omega, p),$$

$$\text{payoff functions } (x, p) \to v_j(x, p), \text{ and}$$

$$\text{where } X_{kn+1} := \mathcal{B} \text{ and } (x, p) = (x_1, ..., x_n, p) \in X_{k1} \times ... \times X_{kn} \times X_{kn+1}.$$

For $(x_1, ..., x_n, p) \in X_{k1} \times ... \times X_{kn} \times X_{kn+1}$, and agents $j = 1, 2, ..., n$, define

$$H_{kj}(\omega, p) := \{x_j \in X_{kj} : \langle x_j, p\rangle \leq \langle \omega_j, p\rangle + 1 - \|p\|\},$$

$$v_j(x, p) = v_j(x_j, x_{-j}, p) := u_j(x_j, x_{-j});$$

and for $(x_1, ..., x_n, p) \in X_{k1} \times ... \times X_{kn} \times X_{kn+1}$ and agent $j = n+1$ (the market), define

$$H_{kn+1}(\omega, p) := \mathcal{B},$$

$$v_{n+1}(x, p) := \left\langle \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} \omega_j, p \right\rangle.$$

For $j = 1, 2, ..., n, n+1$ and all k we have,

1. for each p, $H_{kj}(\omega, p)$ is nonempty, convex, and compact;
2. the mapping $p \to H_{kj}(\omega, p)$ is continuous (see Hildenbrand (1974), p. 33, Lemma 1);
3. for $j = 1, 2, ..., n$, $v_j(\cdot, x_{-j}, p)$ is quasi-concave and $v_j(\cdot, \cdot, \cdot)$ is continuous;
4. for $j = n+1$, $v_j(x_j, x_{-j}, \cdot)$ is quasi-concave (in fact linear) and $v_j(\cdot, \cdot, \cdot)$ is continuous.

Given observations 1 - 3 above, it follows from the Theorem 2 in Tian and Zhou (1992) that for each k, the abstract game G_k has an equilibrium. Thus for each k, there exists

$$(x_1^k, ..., x_n^k, p^k) \in X_{k1} \times ... \times X_{kn} \times X_{kn+1}$$

such that for $j = 1, 2, ..., n$

$$\left.\begin{array}{c} x_j^k \in H_{kj}(\omega, p) \text{ and } x_j^k \text{ maximizes } v_j(x_j, x_{-j}^k, p^k) \text{ over } H_{kj}(\omega, p), \\ \text{or equivalently} \\ x_j^k \in H_{kj}(\omega, p) \text{ and } P_{kj}(x_j^k, x_{-j}^k) \cap H_{kj}(\omega, p) = \emptyset. \end{array}\right\} \tag{5.1}$$

and for $j = n + 1$

$$\left.\begin{array}{c} p^k \in \mathcal{B} \text{ and } p^k \text{ maximizes } v_{n+1}(x_j^k, x_{-j}^k, p) \text{ over } \mathcal{B}, \\ \text{or equivalently} \\ p^k \in \mathcal{B} \text{ and } P_{kn+1}(x^k, p^k) \cap \mathcal{B} = \emptyset, \end{array}\right\} \tag{5.2}$$

where

$$P_{kn+1}(x^k, p^k) := \left\{q \in \mathcal{B} : v_{n+1}(x_j^k, x_{-j}^k, q) > v_{n+1}(x_j^k, x_{-j}^k, p^k)\right\}$$

$$= \left\{q \in \mathcal{B} : \left\langle \textstyle\sum_{j=1}^n x_j - \sum_{j=1}^n \omega_j, q \right\rangle > \left\langle \textstyle\sum_{j=1}^n x_j - \sum_{j=1}^n \omega_j, p \right\rangle \right\}$$

Note that for all k, $\sum_{j=1}^n x_j^k = \sum_{j=1}^n \omega_j$. Otherwise,

$$P_{kn+1}(x^k, p^k) \cap \mathcal{B} = \emptyset$$

would imply that

$$\left\langle \sum_{j=1}^n x_j^k - \sum_{j=1}^n \omega_j, p^k \right\rangle > 0 \text{ and } \|p^k\| = 1.$$

But since for all k and j,

$$x_j^k \in \{x_j \in X_{kj} : \langle x_j, p^k \rangle \leq \langle \omega_j, p^k \rangle + 1 - \|p^k\|\},$$

the latter would imply that for all k and j, $\langle x_j, p^k \rangle \leq \langle \omega_j, p^k \rangle$. Thus,

$$\left\langle \sum_{j=1}^n x_j^k - \sum_{j=1}^n \omega_j, p^k \right\rangle \leq 0,$$

a contradiction. Finally note that $x_j^k \in \widehat{P}_{kj}(\omega_j, x_{-j}^k)$. Otherwise, $u_j(\omega_j, x_{-j}^k) > u_j(x_j^k, x_{-j}^k)$, or equivalently $\omega_j \in P_{kj}(x_j^k, x_{-j}^k)$, contradicting (5.1). Thus, for all k

$$(x_1^k, ..., x_n^k) \in A_k.$$

For $j = 1, 2, \ldots, n$ and for k larger than k^*, $P_{kj}(x_j^k, x_{-j}^k)$ is nonempty and x_j^k is on the boundary of $P_{kj}(x_j^k, x_{-j}^k)$. Thus, $\langle x_j^k, p^k \rangle < \langle \omega_j, p^k \rangle + 1 - \|p^k\|$ would imply that

$$P_{kj}(x_j^k, x_{-j}^k) \cap H_{kj}(\omega, p^k) \neq \emptyset,$$

contradicting (5.1). We must conclude, therefore, that $\langle x_j^k, p^k \rangle = \langle \omega_j, p^k \rangle + 1 - \|p^k\|$. Summing over j yields $\|p^k\| = 1$. Thus, the equilibrium, $(x_1^k, ..., x_n^k, p^k)$, for the abstract game G_k is such that

(i) $(x_1^k, \ldots, x_n^k) \in A_k$;

(ii) $\|p^k\| = 1$; and

(iii) for each j, $x_j^k \in B_{kj}(p^k, \omega_j)$ and $P_{kj}(x_j^k, x_{-j}^k) \cap B_{kj}(p^k, \omega_j) = \emptyset$ (x_j^k maximizes $u_j(x_j, x_{-j}^k)$ over $B_{kj}(p^k, \omega_j)$ and $\langle p^k, x_j^k \rangle = \langle p^k, \omega_j \rangle$).

Therefore, $(x_1^k, ..., x_n^k, p^k)$ is an equilibrium for the k -bounded economy. ■

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