



Nonlinear Inequalities with Double Riesz Potentials

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Abstract

We investigate the nonnegative solutions to the nonlinear integral inequality $u \geq I_\alpha * ((I_\beta * u^p)u^q)$ a.e. in \mathbb{R}^N , where $\alpha, \beta \in (0, N)$, $p, q > 0$ and I_α, I_β denote the Riesz potentials of order α and β respectively. Our approach relies on a nonlocal positivity principle which allows us to derive optimal ranges for the parameters α, β, p and q to describe the existence and the nonexistence of a solution. The optimal decay at infinity for such solutions is also discussed.

Keywords Nonlinear integral inequalities · Riesz potentials · Nonlocal positivity principle · Liouville theorems

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1 Introduction

We study nonnegative solutions of the following integral inequality with double Riesz potentials

$$u \geq I_\alpha * ((I_\beta * u^p)u^q) \quad \text{a.e. in } \mathbb{R}^N, \tag{1}$$

in the range $N \geq 2$, $p, q > 0$, $\alpha, \beta \in (0, N)$ and $I_\gamma(x) := A_\gamma|x|^{-(N-\gamma)}$ is the Riesz potential of order $\gamma \in (0, N)$ where $*$ denotes the standard convolution in \mathbb{R}^N . The choice of the normalisation constant $A_\gamma := \frac{\Gamma((N-\gamma)/2)}{\pi^{N/2}2^\gamma\Gamma(\gamma/2)}$ ensures that $I_\gamma(x)$ could be interpreted as the Green function of $(-\Delta)^{\gamma/2}$ in \mathbb{R}^N , and that the semigroup property $I_{\alpha+\beta} = I_\alpha * I_\beta$ holds for all $\alpha, \beta \in (0, N)$ such that $\alpha + \beta < N$, see for example [9, p. 45].

By a nonnegative *solution* of Eq. 1 we understand a function $u \in L^1_{loc}(\mathbb{R}^N)$, $u \geq 0$, such that the right hand side of Eq. 1 is well-defined, i.e.

$$I_\alpha * ((I_\beta * u^p)u^q) < +\infty \quad \text{a.e. in } \mathbb{R}^N \tag{2}$$

and the inequality Eq. 1 holds a.e. in \mathbb{R}^N . Condition Eq. 2 above is equivalent (see Lemma 2.1 below) to

$$(I_\beta * u^p)u^q \in L^1((1 + |x|)^{-(N-\alpha)}dx, \mathbb{R}^N). \tag{3}$$

Integral inequalities and equations featuring a single Riesz potential have been extensively investigated in the past two decades. The prototype model

$$u = I_\alpha * u^p \quad \text{in } \mathbb{R}^N,$$

has been largely investigated starting with the seminal works in [3, 4]. More recently, various techniques have been devised to deal with systems of equations or inequalities that incorporate anisotropic or more general potentials [2, 8, 11, 12, 19, 20].

Our aim in this paper is to provide an optimal description for the existence and nonexistence of positive solutions to the integral inequality (1).

For any $\alpha > 0$, the fractional Laplacian $(-\Delta)^{\alpha/2}$ is defined by means of the Fourier transform

$$\widehat{(-\Delta)^{\alpha/2}u}(\xi) := |\xi|^\alpha \hat{u}(\xi)$$

for all $u \in \mathcal{S}'$ such that $|\xi|^\alpha u \in \mathcal{S}'$, here \mathcal{S}' stands for the space of tempered distributions on \mathbb{R}^N which is the dual of the Schwartz space \mathcal{S} .

Since for $\alpha \in (0, N)$ the Riesz potential I_α can be interpreted as the inverse of $(-\Delta)^{\alpha/2}$ (cf. [18]*Sect.5.1 or [2, Section 2.1]), under some extra integrability conditions on $u \geq 0$, inequality (1) is equivalent to the elliptic inequality

$$(-\Delta)^{\alpha/2}u \geq (I_\beta * u^p)u^q \quad \text{a.e. in } \mathbb{R}^N, \tag{4}$$

provided that both Eqs. 1 and 4 are well-defined. This is the case, for instance, if Eq. 3 holds and u belongs to the homogeneous Sobolev space $\dot{H}^{\alpha/2}(\mathbb{R}^N)$, so that Eq. 4 is understood in the weak sense. Pointwise interpretations of the inequality Eq. 4 for non-integer $\alpha/2$ are also possible, cf. [2, Theorem 2.13]. For a comparison of different definitions of the higher order fractional Laplacian $(-\Delta)^{\alpha/2}$ see [1].

Inequality (4) is a Choquard type inequality. Equations and inequalities of such structure originate from mathematical physics and have attracted considerable interest of mathematicians in the past decades. Precisely, the equation

$$-\Delta u + u = (I_2 * u^2)u \quad \text{in } \mathbb{R}^3 \tag{5}$$

was introduced by Fröhlich [5, 6] and Pekar [17] as a model for polaron, where free electrons in an ionic lattice interact with phonons associated to deformations of the lattice or

with the polarisation that it creates on the medium. The same equation was proposed by Choquard [13] in the modelling of a one-component plasma. Closely related to the Eq. 5 is the Hartree equation

$$i \partial_t u = \Delta u + (I_2 * u^2)u \quad (x, t) \in \mathbb{R}^3 \times (0, \infty). \tag{6}$$

Indeed, standing waves of Eq. 6 are solutions to Eq. 5. Further, steady state solutions to the Schrödinger equation

$$i \partial_t u = \Delta u + (I_\beta * u^p)u^q \quad (x, t) \in \mathbb{R}^N \times (0, \infty),$$

satisfy Eq. 4 with $\alpha = 2$. More recently, the Choquard Eq. 5 appears with the name of Schrödinger–Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. For a mathematical account on this topic we refer the reader to the survey work [16]. In the second order elliptic case $\alpha = 2$ optimal regimes for the existence and nonexistence of positive solutions to inequality Eq. 4 were fully investigated in [15]. The higher–order polyharmonic case $\alpha/2 = m \in \mathbb{N}$ was recently studied in [7], where (amongst other results) optimal existence and nonexistence regimes for the Eq. 4 were obtained for the exponents $p \geq 1$ and $q > 1$, see [7, Theorem 1.4].

In this work we extend earlier results in [15] and [7] to the full admissible range $\alpha \in (0, N)$ and exponents $p, q > 0$. Our approach is different from the techniques in [7], which were based on the poly–superharmonic properties of $(-\Delta)^m$ in the elliptic framework of Eq. 4. Instead, we work entirely with the double–nonlocal inequality Eq. 1. Our analysis of Eq. 1 employs only direct Riesz kernel estimates, and a new version of the nonlocal positivity principle in Lemma 3.1, inspired by [15, Proposition 3.2]. This has the advantage of incorporating the fractional case of noninteger $\alpha/2$ in a seemingly effortless way, and does not rely on comparison principles or Harnack type inequalities, which are commonly used for similar Liouville type results in the elliptic framework, but which are generally speaking not available in the case of the higher–order fractional Laplacians $(-\Delta)^{\alpha/2}$ with $\alpha > 2$.

The main result of this work related to the existence of positive solutions to Eq. 1 reads as follows.

Theorem 1.1 *Let $p, q > 0$. Then, inequality (1) has a nontrivial nonnegative solution $u \in L^1_{loc}(\mathbb{R}^N)$ which satisfies (2) if and only if*

$$\left\{ \begin{array}{ll} p > \frac{\beta}{N - \alpha} \\ p + q > \frac{N + \beta}{N - \alpha} \\ q > \frac{\beta}{N - \alpha} & \text{if } \beta > N - \alpha, \\ q \geq 1 & \text{if } \beta = N - \alpha, \\ q > 1 - \frac{N - \alpha - \beta}{N} p & \text{if } \beta < N - \alpha. \end{array} \right.$$

The necessary part of the proof follows directly from Propositions 2.3, 4.1–4.5 below. The sufficiency follows from Propositions 5.4–5.7, where we construct explicitly smooth positive radial solutions to Eq. 1. In the case $\alpha = 2$ our results are fully consistent with the results established in [15, Theorem 1] for the 2nd order elliptic inequality (4). The

nonexistence of positive solutions to double–nonlocal inequality Eq. 1 with $p > 1, q > 0$ and $p + q \leq \frac{N+\beta}{N-\alpha}$ was established by different methods in [10, Theorem 1].

Remark 1.2 In Section 5 we also discuss the optimal decay of solutions to Eq. 1 in terms of the parameters α, β, p and q . Clearly (see Eq. 11), if $u \geq 0$ is a nontrivial solution of Eq. 1 then $\liminf_{|x| \rightarrow \infty} u(x)|x|^{N-\alpha} > 0$. In particular, for $R \gg 1$ we have

$$\int_{B_{2R} \setminus B_R} u \geq cR^\alpha. \tag{7}$$

In Proposition 5.1 we establish an integral lower bound

$$\int_{B_{2R} \setminus B_R} u \geq cR^{\frac{\alpha+\beta-Nq}{1-q}}, \tag{8}$$

which is stronger than Eq. 7 when $\alpha + \beta < N$ and $q < \frac{\beta}{N-\alpha} < 1$. In Propositions 5.4 and 5.1 we construct positive radial solutions u to Eq. 1 that confirm the optimality of Eq. 7 when $q > \frac{\beta}{N-\alpha}$ and of Eq. 8 when $q < \frac{\beta}{N-\alpha}$. When $q = \frac{\beta}{N-\alpha}$ the bounds in Eqs. 7 and 8 coincide. In that case in Propositions 5.6 and 5.7 we construct positive radial solutions to Eq. 1 that satisfy Eq. 7 up to a log ($q < 1$) or arbitrary small polynomial ($q = 1$) corrections. In the case $\alpha = 2$ such corrections are necessary, see [15, Proposition 4.12, 4.13].

2 Preliminaries

In this section we collect some useful facts for our approach.

Lemma 2.1 *Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be a nonnegative measurable function. Then, the Riesz potential $I_\alpha * f$ of order $\alpha \in (0, N)$ is well defined, in the sense that*

$$I_\alpha * f < +\infty \quad \text{a.e. in } \mathbb{R}^N, \tag{9}$$

if and only if

$$f \in L^1((1 + |x|)^{-(N-\alpha)} dx, \mathbb{R}^N). \tag{10}$$

Moreover, if Eq. 9 fails then $I_\alpha * f = +\infty$ everywhere in \mathbb{R}^N , see [9]*p.61-62. We present the proof of the lemma for completeness.

Proof Assume first that Eq. 9 holds. Then, for any $x, y \in \mathbb{R}^N, x \neq 0$ we have

$$|x-y|^{N-\alpha} \leq c(|x|^{N-\alpha} + |y|^{N-\alpha}) \leq c \max\{1, |x|^{N-\alpha}\}(1+|y|)^{N-\alpha} \leq c \max\{1, |x|^{N-\alpha}\}(1+|y|)^{N-\alpha}.$$

Thus, for any $x \in \mathbb{R}^N \setminus \{0\}$ such that Eq. 9 holds, we have

$$\infty > (I_\alpha * f)(x) \geq \frac{1}{c \max\{1, |x|^{N-\alpha}\}} \int_{\mathbb{R}^N} \frac{f(y)dy}{(1 + |y|)^{N-\alpha}},$$

which yields Eq. 10.

Conversely, assume now that Eq. 10 holds. For any $R > 0$ we have

$$\int_{B_R} f(y)dy \leq (1 + R)^{N-\alpha} \int_{B_R} \frac{f(y)dy}{(1 + |y|)^{N-\alpha}} \leq (1 + R)^{N-\alpha} \int_{\mathbb{R}^N} \frac{f(y)dy}{(1 + |y|)^{N-\alpha}} < \infty.$$

This shows that $f \in L^1_{loc}(\mathbb{R}^N)$. If $x \in \mathbb{R}^N \setminus \{0\}$, then

$$\begin{aligned} (I_\alpha * f)(x) &= \int_{|y| \leq 2|x|} \frac{f(y)dy}{|x - y|^{N-\alpha}} + \int_{|y| > 2|x|} \frac{f(y)dy}{|x - y|^{N-\alpha}} \\ &\leq \int_{|y| \leq 2|x|} \frac{f(y)dy}{|x - y|^{N-\alpha}} + 2^{N-\alpha} \int_{|y| > 2|x|} \frac{f(y)dy}{|y|^{N-\alpha}} \\ &< \infty. \end{aligned}$$

□

In the same spirit to the above proof, if $f \geq 0$ and Eq. 9 (or, equivalently Eq. 10) holds, then

$$I_\alpha * f(x) \geq \frac{c}{|x|^{N-\alpha}} \int_{B_{|x|}(0)} f(y) dy. \tag{11}$$

One of the elementary yet important for our approach consequences of Eq. 11 is the following estimate, which we will be using frequently, and which to some extent is the counterpart of the Harnack inequalities on the annuli in the study of Eq. 4.

Lemma 2.2 *Let $\alpha \in (0, N)$, $\theta > 0$ and $0 \leq f \in L^1((1 + |x|)^{-(N-\alpha)} dx, \mathbb{R}^N)$. Then for all $R > 0$ we have*

$$\int_{B_{2R} \setminus B_R} (I_\alpha * f)^\theta \geq C R^{N-(N-\alpha)\theta} \left(\int_{B_R} f \right)^\theta. \tag{12}$$

Proof Follows from Eq. 11 by integration. □

An obvious implication of Eq. 11 is that $I_\alpha * f$ can not decay faster than I_α at infinity, even if the function f is compactly supported. Recall also that if $f \geq 0$ then an elementary estimate shows that for every $x \in \mathbb{R}^N$,

$$I_\alpha * f(x) \geq \frac{A_\alpha}{R^{N-\alpha}} \int_{B_R(x)} f(y) dy. \tag{13}$$

As a consequence, if $u \geq 0$ is a solution of Eq. 1 that is positive on a set of positive measure, then u is everywhere strictly positive on \mathbb{R}^N and the following lower bounds must hold:

$$u(x) \geq c(1 + |x|)^{-(N-\alpha)}, \tag{14}$$

$$I_\beta * u^p(x) \geq c(1 + |x|)^{-(N-\beta)}. \tag{15}$$

On the other hand, Eq. 2 requires

$$u^p \in L^1((1 + |x|)^{-(N-\beta)} dx, \mathbb{R}^N), \tag{16}$$

$$(I_\beta * u^p)u^q \in L^1((1 + |x|)^{-(N-\alpha)} dx, \mathbb{R}^N). \tag{17}$$

Combining the competing upper and lower bounds immediately leads to the following nonexistence result.

Proposition 2.3 *Let $p, q > 0$ and assume that either $p \leq \frac{\beta}{N-\alpha}$, or $\alpha + \beta > N$ and $q \leq \frac{\beta}{N-\alpha} - 1$. If $u \geq 0$ is a solution of Eq. 1 then $u \equiv 0$.*

Proof First we note that Eqs. 14 and 16 are incompatible when $p \leq \frac{\beta}{N-\alpha}$. Then we observe that Eqs. 14, 15 and 17 are incompatible when $0 < q \leq \frac{\beta}{N-\alpha} - 1$. □

Remark 2.4 We will see in Proposition 4.3 below that $q \leq \frac{\beta}{N-\alpha} - 1$ is suboptimal for the nonexistence and could be refined.

3 Nonlocal Positivity Principle

The nonexistence result in Proposition 2.3 “decouples” the values of p and q . In order to deduce an estimate which involves the quantity $p + q$ which appears in Theorem 1.1, we need the following lemma, inspired by [14, Proposition 2.1] and [15]*Section 3.

Lemma 3.1 (Nonlocal positivity principle) *Let $\alpha \in (0, N)$ and $V : \mathbb{R}^N \rightarrow [0, \infty)$ be a measurable function. Assume that there exists $u \in L^1_{loc}(\mathbb{R}^N)$ such that $u > 0$ a.e. in \mathbb{R}^N , $Vu \in L^1((1 + |x|)^{-(N-\alpha)} dx, \mathbb{R}^N)$ and*

$$u \geq I_\alpha * (Vu) \quad \text{a.e. in } \mathbb{R}^N. \tag{18}$$

Then for every $R > 0$ and $0 \leq \varphi \in C_c^\infty(B_R)$,

$$\int_{B_R} \varphi^2 \geq CR^{\alpha-N} \left(\int_{B_R} \sqrt{V}\varphi \right)^2. \tag{19}$$

Proof Take $\psi := \frac{\varphi^2}{u}$ as a test function in Eq. 18. Then

$$\begin{aligned} \int_{B_R} \varphi^2 &\geq \iint_{B_R \times B_R} I_\alpha(x-y)V(y)u(y)\frac{\varphi^2(x)}{u(x)} dy dx \\ &= \frac{1}{2} \iint_{B_R \times B_R} I_\alpha(x-y) \left(V(x)u(x)\frac{\varphi^2(y)}{u(y)} + V(y)u(y)\frac{\varphi^2(x)}{u(x)} \right) dx dy \\ &= \iint_{B_R \times B_R} I_\alpha(x-y)\sqrt{V(x)V(y)}\varphi(y)\varphi(x) dx dy \\ &\quad + \frac{1}{2} \iint_{B_R \times B_R} I_\alpha(x-y)u(x)u(y) \left(\sqrt{V(x)}\frac{\varphi(y)}{u(y)} - \sqrt{V(y)}\frac{\varphi(x)}{u(x)} \right)^2 dx dy \\ &\geq \frac{A_\alpha}{2^{N-\alpha}R^{N-\alpha}} \left(\int_{B_R} \sqrt{V(x)}\varphi(x) dx \right)^2, \end{aligned}$$

since

$$I_\alpha(x-y) \geq \frac{A_\alpha}{2^{N-\alpha}R^{N-\alpha}} \quad (x, y \in B_R), \tag{20}$$

which completes the proof. □

Remark 3.2 Nonlocal inequality (18) can be interpreted as the “inversion” of the fractional Schrödinger inequality

$$(-\Delta)^{\alpha/2}u - Vu \geq 0 \quad \text{in } \mathbb{R}^N.$$

In this context Lemma 3.1 can be seen as a higher-order version of the fractional Agmon–Allegretto–Piepenbrink’s positivity principle: if Eq. 18 has a positive supersolution then a certain variational inequality which involves the potential V must hold. We will see that

Lemma 3.1 alongside with the standard integral estimate Eq. 12 of the Riesz potentials are sufficient for the complete analysis of the existence and nonexistence of positive solutions of the nonlinear inequality Eq. 1.

Using Lemma 3.1 we establish the following estimate.

Proposition 3.3 *Let $p, q > 0$ and $u > 0$ be a solution of Eq. 1. Then, for every $R > 0$ and every $\varphi \in C_c^\infty(B_R)$,*

$$\int_{B_R} \varphi^2 \geq CR^{\alpha+\beta-2N} \left(\int_{B_R} u^p \right) \left(\int_{B_R} u^{\frac{q-1}{2}} \varphi \right)^2. \tag{21}$$

Proof For every $\varphi \in C_c^\infty(B_R)$, by Lemma 3.1 with $V = (I_\beta * u^p)u^{q-1}$, and using a similar inequality to Eq. 20 for $I_\beta * u^p$, we obtain

$$\begin{aligned} \int_{B_R} \varphi^2 &\geq CR^{\alpha-N} \left(\int_{B_R} ((I_\beta * u^p)u^{q-1})^{1/2} \varphi \right)^2 \\ &\geq C'R^{\alpha+\beta-2N} \left(\int_{B_R} u^p \right) \left(\int_{B_R} u^{\frac{q-1}{2}} \varphi \right)^2. \end{aligned}$$

□

One of the principal tools in the subsequent analysis is the following decay estimate on the solutions of Eq. 1, which is an adaptation of Eq. 21. Note that for $q < 1$ our estimate contains a lower bound on the solution, since the 2nd integral involves a negative power of u .

Corollary 3.4 *Let $p, q > 0$ and $u > 0$ be a solution of Eq. 1. Then for every $R > 0$,*

$$\left(\int_{B_{2R}} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^2 \leq CR^{3N-\alpha-\beta}. \tag{22}$$

Proof Take $\varphi_R(x) = \varphi(x/R)$, where $\varphi \in C_c^\infty(\mathbb{R}^N)$ is such that $\text{supp}(\varphi) \subset \bar{B}_4 \setminus B_{1/2}$, $\varphi \equiv 1$ on $\bar{B}_2 \setminus B_1$ and $0 \leq \varphi \leq 1$. Then, by Eq. 21 we find

$$\begin{aligned} cR^N = \int_{B_{4R}} \varphi_R^2 &\geq C'R^{\alpha+\beta-2N} \left(\int_{B_{4R}} u^p \right) \left(\int_{B_{4R}} u^{\frac{q-1}{2}} \varphi_R \right)^2 \\ &\geq C'R^{\alpha+\beta-2N} \left(\int_{B_{2R}} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^2. \quad \square \end{aligned}$$

4 Nonexistence

In this section we derive several nonexistence results for Eq. 1. Our approach is inspired by [15] which studied the inequality Eq. 4 in the semilinear the case $\alpha = 2$, yet with substantial modifications. In particular, in this work we completely avoid the use of the comparison principle and Harnack’s inequalities, which are not applicable in the framework of the double–nonlocal inequality Eq. 1. It turns out that Harnack inequality estimates in the context of Eq. 1 can be replaced by the estimate Eq. 12.

Proposition 4.1 *Let $p, q > 0$ and assume that $p + q < 1$. If $u \geq 0$ is a solution of Eq. 1 then $u \equiv 0$.*

Proof Since $u^p \in L^1((1 + |x|)^{-(N-\beta)}dx)$ and $\beta < N$, by Lebesgue’s dominated convergence theorem

$$\int_{B_{2R} \setminus B_R} u^p = o(R^{N-\beta}) = o(R^N) \quad \text{as } R \rightarrow \infty.$$

Since $p + q < 1$, we may apply Hölder’s inequality and then the estimate Eq. 22 to obtain

$$\begin{aligned} cR^N &= \int_{B_{2R} \setminus B_R} 1 = \int_{B_{2R} \setminus B_R} u^{\frac{p(1-q)}{2p+1-q}} u^{\frac{(q-1)p}{2p+1-q}} \\ &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q}{2p+1-q}} \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^{\frac{2p}{2p+1-q}} \\ &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q-p}{2p+1-q}} \left[\left(\int_{B_{2R} \setminus B_R} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^2 \right]^{\frac{p}{2p+1-q}} \\ &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q-p}{2p+1-q}} \left(CR^{3N-\alpha-\beta} \right)^{\frac{p}{2p+1-q}} \\ &\leq o(1)R^{N\frac{1-q-p}{2p+1-q}} (R^{3N})^{\frac{p}{2p+1-q}} \\ &= o(1)R^N, \end{aligned}$$

which raises a contradiction. □

Proposition 4.2 *Let $p, q > 0$ and assume that $1 \leq p + q \leq \frac{N+\beta}{N-\alpha}$. If $u \geq 0$ is a solution of Eq. 1 then $u \equiv 0$.*

Proof Assume first $p + q < \frac{N+\beta}{N-\alpha}$. By the Cauchy–Schwarz inequality,

$$\int_{B_{2R}} u^p \geq cR^{-N} \left(\int_{B_{2R}} u^{\frac{p}{2}} \right)^2,$$

and so Eq. 22 implies

$$cR^{4N-\alpha-\beta} \geq \left(\int_{B_{2R}} u^{\frac{p}{2}} \right)^2 \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^2. \tag{23}$$

Using Eq. 23 and the Cauchy–Schwarz again together with $u \geq c|x|^{-(N-\alpha)}$ in $\mathbb{R}^N \setminus B_1$ (that follows from Eq. 14), we obtain

$$\begin{aligned} cR^{4N-\alpha-\beta} &\geq \left(\int_{B_{2R}} u^{\frac{p}{2}} \right)^2 \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^2 \\ &\geq \left(\int_{B_{2R} \setminus B_R} u^{\frac{p+q-1}{4}} \right)^4 \\ &\geq cR^{4N-(N-\alpha)(p+q-1)}, \end{aligned} \tag{24}$$

which is a contradiction since $0 \leq p + q - 1 < \frac{\alpha+\beta}{N-\alpha}$.

Assume now $p + q = \frac{N+\beta}{N-\alpha}$. By Hölder’s inequality we find

$$\begin{aligned} \left(\int_{\mathbb{R}^N} (I_\beta * u^p)u^q\right)^2 &= \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\beta(x-y)u^p(y)u^q(x)\right)\left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} I_\beta(x-y)u^p(x)u^q(y)\right) \\ &\geq \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x)^{\frac{p+q}{2}} I_\beta(x-y)u(y)^{\frac{p+q}{2}} dx dy\right)^2, \end{aligned}$$

so that

$$\int_{\mathbb{R}^N} (I_\beta * u^p)u^q \geq \iint_{\mathbb{R}^N \times \mathbb{R}^N} u(x)^{\frac{p+q}{2}} I_\beta(x-y)u(y)^{\frac{p+q}{2}}.$$

Using the lower bound Eq. 14 and the fact that $\frac{p+q}{2} = \frac{N+\beta}{2(N-\alpha)} > 0$ we deduce

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\beta * u^p)u^q &\geq \int_{\mathbb{R}^N \setminus B_1} \int_{\mathbb{R}^N \setminus B_1} u(x)^{\frac{p+q}{2}} I_\beta(x-y)u(y)^{\frac{p+q}{2}} dx dy \\ &\geq c \int_{\mathbb{R}^N \setminus B_1} \int_{\mathbb{R}^N \setminus B_1} \frac{1}{|x|^{\frac{N+\beta}{2}}} I_\beta(x-y) \frac{1}{|y|^{\frac{N+\beta}{2}}} dx dy = \infty. \end{aligned}$$

Hence,

$$\lim_{R \rightarrow \infty} \int_{B_R} (I_\beta * u^p)u^q = \infty. \tag{25}$$

Since u satisfies Eq. 1, from Lemma 2.2 with $\theta = \frac{p+q-1}{4} = \frac{\alpha+\beta}{4(N-\alpha)}$ and $f = (I_\beta * u^p)u^q$ we find

$$\int_{B_{2R} \setminus B_R} u^{\frac{p+q-1}{4}} \geq \int_{B_{2R} \setminus B_R} (I_\alpha * f)^{\frac{p+q-1}{4}} \geq CR^{N-\frac{\alpha+\beta}{4}} \left(\int_{B_R} (I_\beta * u^p)u^q\right)^{\frac{p+q-1}{4}}.$$

From the above estimate and Eq. 25 we deduce

$$\lim_{R \rightarrow \infty} \frac{1}{R^{N-\frac{\alpha+\beta}{4}}} \int_{B_{2R} \setminus B_R} u^{\frac{p+q-1}{4}} = \infty,$$

which contradicts the upper bound in Eq. 24. □

If $\alpha + \beta \geq N$ we give precise lower bounds on $\int_{B_{2R} \setminus B_R} u^{q-1}$ to obtain a further nonexistence result.

Proposition 4.3 *Let $p, q > 0$ and assume that $\alpha + \beta > N$ and $1 < q \leq \frac{\beta}{N-\alpha}$. If $u \geq 0$ is a solution of Eq. 1 then $u \equiv 0$.*

Proof Assume that $u > 0$ on a set of positive measure and let $R > 1$ be large. From Eq. 14 we obtain

$$\left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}}\right)^2 \geq cR^{2N-(N-\alpha)(q-1)}. \tag{26}$$

On the other hand, by Corollary 3.4,

$$\left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}}\right)^2 \leq \frac{CR^{3N-\beta-\alpha}}{\int_{B_{2R}} u^p} \leq C'R^{3N-\beta-\alpha}. \tag{27}$$

Now Eqs. 26 and 27 yield a contradiction if $q < \frac{\beta}{N-\alpha}$.

In the critical case $q = \frac{\beta}{N-\alpha}$ we use Eq. 11 to find

$$(I_\beta * u^p)(x) \geq \frac{c}{|x|^{N-\beta}} \int_{B_1} u^p \quad \text{for all } x \in B_R \setminus B_1.$$

Combining this estimate with Eq. 14 we obtain

$$\begin{aligned} \int_{B_R} (I_\beta * u^p)u^q &\geq c \left(\int_{B_1} u^p \right) \int_{B_R \setminus B_1} \frac{u(x)^q}{|x|^{N-\beta}} dx \\ &\geq c' \int_{B_R \setminus B_1} \frac{1}{|x|^{N-\beta+(N-\alpha)q}} dx = c'' \log(R), \end{aligned}$$

since $(N - \alpha)q = \beta$. By Lemma 2.2 with $\theta = \frac{q-1}{2} > 0$,

$$\begin{aligned} \int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} &\geq \int_{B_{2R} \setminus B_R} \left(I_\alpha * ((I_\beta * u^p)u^q) \right)^{\frac{q-1}{2}} \\ &\geq cR^{N-(N-\alpha)\frac{q-1}{2}} \left(\int_{B_R} (I_\beta * u^p)u^q \right)^{\frac{q-1}{2}} \tag{28} \\ &\geq c'R^{\frac{2N-(N-\alpha)(q-1)}{2}} \log^{\frac{q-1}{2}}(R) \\ &= c'R^{\frac{3N-\alpha-\beta}{2}} \log^{\frac{q-1}{2}}(R), \end{aligned}$$

which contradicts Eq. 27. □

The transitional locally linear case $q = 1$ requires a separate consideration.

Proposition 4.4 *Let $p > 0$ and assume that $\alpha + \beta > N$ and $q = 1$. If $u \geq 0$ is a solution of Eq. 1 then $u \equiv 0$.*

Proof Using Corollary 3.4, for any $R > 0$ we have

$$R^{3N-\alpha-\beta} \geq c \left(\int_{B_R} u^p \right) \left(\int_{B_{2R} \setminus B_R} 1 \right)^2 = cR^{2N} \left(\int_{B_R} u^p \right).$$

Since $\alpha + \beta > N$, it follows that $u \equiv 0$. □

In the sublinear case $q < 1$ we deduce an additional restriction on the admissible range of the exponent q .

Proposition 4.5 *Let $p, q > 0$ and assume that $p + q \geq 1$, $q < 1$ and*

$$q \leq 1 - \frac{N - \alpha - \beta}{N} p.$$

If $u \geq 0$ is a solution of Eq. 1 then $u \equiv 0$.

Proof Since $q < 1$, by Hölder’s inequality we deduce

$$\begin{aligned}
 cR^N &= \int_{B_{2R} \setminus B_R} 1 = \int_{B_{2R} \setminus B_R} u^p \frac{1-q}{2p+1-q} u^{\frac{q-1}{2}} \frac{2p}{2p+1-q} \\
 &\leq \left(\int_{B_{2R} \setminus B_R} u^p \right)^{\frac{1-q}{2p+1-q}} \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^{\frac{2p}{2p+1-q}} \\
 &= \left[\left(\int_{B_{2R} \setminus B_R} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^2 \right]^{\frac{1-q}{2p+1-q}} \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^{2 \frac{p+q-1}{2p+1-q}}.
 \end{aligned}
 \tag{29}$$

By Corollary 3.4 we have

$$\left(\int_{B_{2R}} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^2 \leq CR^{3N-\alpha-\beta},$$

which yields

$$\left(\int_{B_{2R} \setminus B_R} u^p \right) \left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^2 \leq CR^{3N-\alpha-\beta},
 \tag{30}$$

and on the other hand

$$\left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^2 \leq C \frac{R^{3N-\beta-\alpha}}{\int_{B_{2R}} u^p} \leq C' R^{3N-\alpha-\beta}.
 \tag{31}$$

If $q < 1 - \frac{N-\beta-\alpha}{N} p$, we use Eqs. 30–31 in 29 to raise a contradiction since $p + q \geq 1$.

If $q = 1 - \frac{N-\beta-\alpha}{N} p$, we use Eq. 31 and Hölder’s inequality to further estimate

$$\int_{B_{2R} \setminus B_R} u^p \geq \frac{\left(\int_{B_{2R} \setminus B_R} 1 \right)^{1+\frac{2p}{1-q}}}{\left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^{\frac{2p}{1-q}}} \geq cR^{N-\frac{p}{1-q}(N-\alpha-\beta)} = c.$$

This shows that $u^p \notin L^1(\mathbb{R}^N)$ and

$$\lim_{R \rightarrow \infty} \int_{B_{2R}} u^p = \infty.$$

Using this fact in Eq. 31 we deduce

$$\left(\int_{B_{2R} \setminus B_R} u^{\frac{q-1}{2}} \right)^2 = o(R^{3N-\alpha-\beta}) \quad \text{as } R \rightarrow \infty.$$

We now use this last estimate and Eqs. 30 in 29 to conclude. □

5 Optimal Decay and Existence

If $u \geq 0$ is a solution of Eq. 1, then either $u \equiv 0$ or u must obey the “natural” lower bound Eq. 14, which implies in particular, the integral lower bound

$$\int_{B_{2R} \setminus B_R} u \geq cR^\alpha.
 \tag{32}$$

In the region $q < 1$ the estimate Eq. 22 of Corollary 3.4 leads to an integral lower bound which improves upon Eq. 32 when $\alpha + \beta < N$ and $q < \frac{\beta}{N-\alpha}$.

Proposition 5.1 *Let $p, q > 0$ and assume that $\alpha + \beta < N$ and $q < \frac{\beta}{N-\alpha} < 1$. If $u \geq 0$ is a solution of Eq. 1, then either $u \equiv 0$ or*

$$\int_{B_{2R} \setminus B_R} u \geq cR^{\frac{\alpha+\beta-Nq}{1-q}}. \tag{33}$$

As pointed out in Remark 1.2, since $\alpha + \beta < N$ and $q < \frac{\beta}{N-\alpha} < 1$, the exponent of R in Eq. 33 is greater than α .

Proof From Corollary 3.4 we have

$$\left(\int_{B_{2R} \setminus B_R} u^{-\frac{1-q}{2}} \right)^2 \leq c \frac{R^{3N-\alpha-\beta}}{\int_{B_{2R}} u^p} \leq c' R^{3N-\alpha-\beta}.$$

Further, by Hölder’s inequality (since $0 < q < 1$) we have

$$\left(\int_{B_{2R} \setminus B_R} u^{-\frac{1-q}{2}} \right)^2 \geq \left(\int_{B_{2R} \setminus B_R} u \right)^{-(1-q)} \left(\int_{B_{2R} \setminus B_R} 1 \right)^{3-q}.$$

Now, the above estimates yield

$$\left(\int_{B_{2R} \setminus B_R} u \right)^{-(1-q)} \leq R^{Nq-\alpha-\beta},$$

which leads to Eq. 33. □

Our next step is to construct explicit solutions with the decay which match or near-match the lower bounds in Eq. 32 and Eq. 33. Before we do this, we recall the following simple estimates, cf. [15, Lemma A.1 and A.2] which are frequently used in the proofs below.

Lemma 5.2 *Let $v \in L^1_{loc}(\mathbb{R}^N)$, $\gamma \in (0, N)$ and $s > \gamma$. If*

$$\limsup_{|x| \rightarrow \infty} v(x)|x|^s < \infty,$$

then

$$\begin{aligned} \limsup_{|x| \rightarrow \infty} (I_\gamma * v)(x)|x|^{s-\gamma} &< \infty \text{ if } \gamma < s < N, \\ \limsup_{|x| \rightarrow \infty} (I_\gamma * v)(x) \frac{|x|^{N-\gamma}}{\log |x|} &< \infty \text{ if } s = N, \\ \limsup_{|x| \rightarrow \infty} (I_\gamma * v)(x)|x|^{N-\gamma} &< \infty \text{ if } s > N. \end{aligned}$$

Lemma 5.3 *Let $v \in L^1_{loc}(\mathbb{R}^N)$, $\gamma \in (0, N)$ and $\sigma \in \mathbb{R}$. If*

$$\limsup_{|x| \rightarrow \infty} v(x) \frac{|x|^N}{(\log |x|)^\sigma} < \infty,$$

then

$$\begin{aligned} \limsup_{|x| \rightarrow \infty} (I_\gamma * v)(x) |x|^{N-\gamma} &< \infty \text{ if } \sigma < -1, \\ \limsup_{|x| \rightarrow \infty} (I_\gamma * v)(x) \frac{|x|^{N-\gamma}}{(\log(\log|x|))} &< \infty \text{ if } \sigma = -1, \\ \limsup_{|x| \rightarrow \infty} (I_\gamma * v)(x) \frac{|x|^{N-\gamma}}{(\log|x|)^{\sigma+1}} &< \infty \text{ if } \sigma > -1. \end{aligned}$$

Proposition 5.4 Assume that

$$p > \frac{\beta}{N-\alpha}, \quad p+q > \frac{N+\beta}{N-\alpha} \quad \text{and} \quad q > \frac{\beta}{N-\alpha}. \tag{34}$$

Then, Eq. 1 admits a positive radial solution $u \in C(\mathbb{R}^N)$ which satisfies

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{N-\alpha} < \infty. \tag{35}$$

Proof Let $0 < \varepsilon < q(N-\alpha) - \beta$ and take $u(x) = (1+|x|)^{-(N-\alpha)}$. Since $p(N-\alpha) > \beta$ we can apply the estimates in Lemma 5.2 to deduce

$$(I_\beta * u^p)(x) \leq c_1 \begin{cases} (1+|x|)^{\beta-p(N-\alpha)} & \text{if } p(N-\alpha) < N \\ (1+|x|)^{\beta-N} & \text{if } p(N-\alpha) > N \\ (1+|x|)^{\beta-N} \log(|x|+e) & \text{if } p(N-\alpha) = N \end{cases} \quad \text{in } \mathbb{R}^N,$$

for some constant $c_1 > 0$. Thus,

$$\left[(I_\beta * u^p) u^q \right](x) \leq c_1 \begin{cases} (1+|x|)^{\beta-(p+q)(N-\alpha)} & \text{if } p(N-\alpha) < N \\ (1+|x|)^{\beta-N-q(N-\alpha)} & \text{if } p(N-\alpha) > N \\ (1+|x|)^{\beta-N-q(N-\alpha)} \log(|x|+e) & \text{if } p(N-\alpha) = N \end{cases} \quad \text{in } \mathbb{R}^N.$$

In particular, one may further estimate as

$$\left[(I_\beta * u^p) u^q \right](x) \leq c_1 \begin{cases} (1+|x|)^{\beta-(p+q)(N-\alpha)} & \text{if } p(N-\alpha) < N \\ (1+|x|)^{\beta-N-q(N-\alpha)+\varepsilon} & \text{if } p(N-\alpha) \geq N \end{cases} \quad \text{in } \mathbb{R}^N.$$

Since $(p+q)(N-\alpha) - \beta > N$ and $N - \beta + q(N-\alpha) - \varepsilon > N$ it follows from the third estimate in Lemma 5.2 that

$$I_\alpha * \left[(I_\beta * u^p) u^q \right](x) \leq c_2 (1+|x|)^{\alpha-N} = c_2 u(x) \quad \text{in } \mathbb{R}^N,$$

where $c_2 > 0$ is a constant. Thus, the continuous function $U(x) = c_2^{-1/(p+q-1)} u(x)$ is a solution of Eq. 1 which satisfies Eq. 35. □

Proposition 5.5 Assume that

$$1 - \frac{N-\alpha-\beta}{N} p < q < \frac{\beta}{N-\alpha} < 1.$$

Then, Eq. 1 admits a positive radial solution $u \in C(\mathbb{R}^N)$ which satisfies

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{\frac{N-\alpha-\beta}{1-q}} < \infty.$$

Proof Let $u(x) = (1 + |x|)^{-k}$ where $k = (N - \alpha - \beta)/(1 - q)$. Since $1 - (N - \alpha - \beta)p/N < q$, we have $pk > N$, and hence by the third estimate of Lemma 5.2 we have

$$I_\beta * u^p \leq c_1(1 + |x|)^{\beta - N} \quad \text{in } \mathbb{R}^N,$$

for some constant $c_1 > 0$. Since $\beta - N - kq = -\frac{N - \beta - \alpha q}{1 - q}$, we have

$$(I_\beta * u^p)u^q \leq c_2(1 + |x|)^{-\frac{N - \beta - \alpha q}{1 - q}} \quad \text{in } \mathbb{R}^N.$$

Since $q < \frac{\beta}{N - \alpha} < 1$, we have $\alpha < \frac{N - \beta - \alpha q}{1 - q} < N$. Hence, by the first estimate of Lemma 5.2 we have

$$I_\alpha * \left[(I_\beta * u^p)u^q \right] \leq c_2(1 + |x|)^{-k} \quad \text{in } \mathbb{R}^N,$$

where $c_2 > 0$ is a constant. Thus, $U(x) = c_2^{-1/(p+q-1)}(1 + |x|)^{-k}$ is a continuous solution of Eq. 1. Moreover,

$$\limsup_{|x| \rightarrow \infty} U(x)|x|^k < \infty.$$

□

Proposition 5.6 *Assume that*

$$\alpha + \beta = N, \quad p > \frac{N}{N - \alpha} \quad \text{and} \quad q = 1.$$

Then, for every $m > 0$ inequality Eq. 1 admits a positive radial solution $u \in C(\mathbb{R}^N)$ which satisfies

$$\limsup_{|x| \rightarrow \infty} u(x)|x|^{N - \alpha - m} < \infty.$$

Proof Let $m > 0$. Since $p > \frac{N}{N - \alpha} = \frac{N}{\beta}$, we see that $\beta - \frac{N}{p} > 0$. Set

$$\delta = \begin{cases} m & \text{if } 0 < m < \beta - \frac{N}{p}, \\ \frac{1}{2}(\beta - \frac{N}{p}) & \text{if } m \geq \beta - \frac{N}{p}, \end{cases} \quad \text{and} \quad k = \beta - \delta.$$

Then,

$$kp = \begin{cases} (\beta - m)p > N & \text{if } 0 < m < \beta - \frac{N}{p}, \\ \frac{\beta p}{2} + \frac{N}{2} > N & \text{if } m \geq \beta - \frac{N}{p}. \end{cases}$$

Let $u(x) = (1 + |x|)^{-k}$. By the third estimate of Lemma 5.2 we see that $I_\beta * u^p \leq c_1(1 + |x|)^{\beta - N}$ in \mathbb{R}^N for some constant $c_1 > 0$, and hence

$$(I_\beta * u^p)u^q \leq c_1(1 + |x|)^{\beta - N - kq} \quad \text{in } \mathbb{R}^N.$$

Since $\alpha < -\beta + N + kq < N$, by the first estimate of Lemma 5.2 we see that

$$I_\alpha * \left[(I_\beta * u^p)u^q \right] \leq c_2(1 + |x|)^{\alpha + \beta - N - kq} \quad \text{in } \mathbb{R}^N,$$

for some constant $c_2 > 0$. Since $\alpha + \beta - N - kq = -k$, we have

$$I_\alpha * \left[(I_\beta * u^p)u^q \right] \leq c_2u \quad \text{in } \mathbb{R}^N.$$

Thus, $U(x) = c_2^{-1/(p+q-1)}(1 + |x|)^{-k}$ is a continuous solution of Eq. 1. Moreover,

$$\limsup_{|x| \rightarrow \infty} U(x)|x|^{N - \alpha - m} \leq \limsup_{|x| \rightarrow \infty} U(x)|x|^k < \infty.$$

□

Proposition 5.7 *Assume that*

$$p > \frac{N}{N - \alpha} \quad \text{and} \quad q = \frac{\beta}{N - \alpha} < 1.$$

Then, for $m \geq \frac{N - \alpha}{N - \alpha - \beta}$ inequality Eq. 1 admits a positive radial solution $u \in C(\mathbb{R}^N)$, which satisfies

$$\limsup_{|x| \rightarrow \infty} u(x) |x|^{N - \alpha} (\log |x|)^{-m} < \infty.$$

Proof Take

$$u(x) = (1 + |x|)^{-(N - \alpha)} (\log(e + |x|))^m.$$

Since $-(N - \alpha)p < -N$, by the third estimate of Lemma 5.2 we see that $I_\beta * u^p \leq c_1(1 + |x|)^{\beta - N}$ for some constant $c_1 > 0$, and hence

$$(I_\beta * u^p)u^q \leq c_1(1 + |x|)^{\beta - N - (N - \alpha)q} (\log(e + |x|))^{mq} = c_1(1 + |x|)^{-N} (\log(e + |x|))^{mq}.$$

By the third estimate of Lemma 5.3 we see that

$$I_\alpha * \left[(I_\beta * u^p)u^q \right] \leq c_2(1 + |x|)^{-(N - \alpha)} (\log(e + |x|))^{mq+1} \quad \text{in } \mathbb{R}^N,$$

for some constant $c_2 > 0$. Since $m \geq \frac{N - \alpha}{N - \alpha - \beta}$, we have

$$I_\alpha * \left[(I_\beta * u^p)u^q \right] \leq c_2 u.$$

Thus, $U(x) = c_2^{-1/(p+q-1)} u$ is a continuous solution of Eq. 1. □

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