Gossip-Based Models for the Opinion Dynamics of Social Networks

Submitted to
Graduate School of Information Science and Technology
Osaka Univesity

January 2021

Emerico Habacon AGUILAR

List of Publications

Journal Articles

1. Emerico Aguilar and Yasumasa Fujisaki. Reaching Consensus via Coordinated Groups. *SICE Journal of Control, Measurement, and System Integration*, 2021. (Accepted)

International Conference Proceedings

1. Emerico Aguilar and Yasumasa Fujisaki. Inter-Clique Influence Networks. *Proceedings of the 52nd ISCIE International Symposium on Stochastic Systems Theory and Its Applications.* Osaka, Japan, 2021. (Accepted)

2. Emerico Aguilar and Yasumasa Fujisaki. Opinion Dynamics of Social Networks with Stubborn Agents via Group Gossiping with Random Participants. *Preprints of the 21st IFAC World Congress*, Berlin, Germany, pp. 11203-11208, 2020.

3. Emerico Aguilar and Yasumasa Fujisaki. Gossip-Based Model for Opinion Dynamics with Probabilistic Group Interactions. *Proceedings of the 51st ISCIE International Symposium on Stochastic Systems Theory and Its Applications*, Fukushima, Japan, pp. 60-64, 2020.
https://doi.org/10.5687/sss.2020.60

4. Emerico Aguilar and Yasumasa Fujisaki. Opinion Dynamics via a Gossip Algorithm with Asynchronous Group Interactions. *Proceedings of the 50th ISCIE International Symposium on Stochastic Systems Theory and Its Applications*, Kyoto, Japan, pp. 99-102, 2019.
https://doi.org/10.5687/sss.2019.99

Conference Presentations

1. Emerico Aguilar and Yasumasa Fujisaki. Clustering of Opinions in Social Networks via Random Group Gossiping with Bounded Confidence. *SICE International Symposium on Control Systems 2020*, 1E2-5, 2020.

2. Emerico Aguilar and Yasumasa Fujisaki. Opinion Shift: An Index for Describing Gossip Algorithms for Opinion Dynamics. *SICE International Symposium on Control Systems 2019*, Kumamoto, Japan, 2I2-4, 2019.

Abstract

Through technology, humans have become more interconnected, especially in the way we are able to communicate our thoughts and views. This means that our ability to influence one another is also increasing, which shapes us not only as individuals but also as a society. Such effects can be better understood through social network analysis, which includes several methods that can be used for investigating the impact of our relationships and interactions on our collective behaviors. Studies on opinion dynamics, in particular, describe the effects of interpersonal influences on the opinion formation in social networks.

Several models exist for describing the opinion dynamics of social networks. In control theory, this is done using agent-based models. These models have mathematical properties that can help in better understanding how social networks may reach consensus, disagreement, and other patterns resulting from collective behaviors. However, these models have rigid specifications that restrict the behavior of agents which leads to an unrealistic representations of real-world scenarios.

More recent works have incorporated time-varying and randomized dynamics to capture the fluidity of human interactions. These include gossip-based models which uses random pairwise interactions to emulate interpersonal communications. The scenarios they capture remain limited, however, and additional schemes are needed to cope with the evolving modes of communications brought about by technological advancements.

Motivated by the growing need for understanding how opinions propagate in this age of social media and smartphones, this dissertation proposes five gossip-based models for the opinion dynamics of social networks. First is a model that extends pairwise gossiping to group interactions. The second model randomizes the participants in group gossiping. In the third model, random group gossiping is applied to a social network with stubborn agents. The fourth model combines random group gossiping with bounded confidence, which is an approach used in other opinion dynamics models. The final model describes how consensus can be reached by interconnected groups. Each proposed model is given detailed analysis, including its convergence properties. Additionally, simulations are provided to demonstrate the behaviors of the models under varying conditions.

Acknowledgements

To my supervisor, Dr. Yasumasa Fujisaki thank you for your guidance and for pushing me to do my best. Through your supervision, I was able to perform my research and deliver the necessary requirements. Pursuing a doctoral degree was a challenging experience for me, especially during the last few months, but I've gained a lot of knowledge and wisdom during my time as your advisee.

To my dissertation committee, namely Dr. Yasumasa Fujisaki, Dr. Hiroshi Morita, Dr. Masayuki Numao, and Dr. Takayuki Wada, thank you for taking the time to review my dissertation and for providing constructive feedback.

To family, thank you for your constant support. My motivation for pushing myself to the best of my abilities is to be able to give back to all of you, regardless if I accomplish my goals or not.

To the relatives who have supported me along the way, I will never forget what you have done for me. I also look forward to the day I will be able to give back in some capacity.

To my friends here in Osaka, thank you for the companionship and the good times we have shared together. Seeing all of you push yourselves have also inspired me to do my best. I hope all of you will achieve success in your respective endeavors.

To my previous colleagues at DLSU, especially Dr. Rafael Cabredo and Dr. Arnulfo Azcarraga, thank you for putting me in the position to pursue this dream of mine. The current times are unpredictable, but I still hope to be part of the DLSU community in some capacity.

To JICA Kansai, you have made my stay in Japan a pleasant experience. I really enjoyed your activities and I am very grateful for the support you have given us scholars. Thank you for all of your hard work.

ありがとうございました

Contents

List of Figures

Chapter 1

Introduction

1.1 Background

A social network is a collection of social actors and the relationships between them. As an abstract concept representing interconnected persons, it is a tool for investigating how humans influence one another. Different types of interactions may occur within a social network, and the aggregate behaviors of its members give rise to various social phenomena. The area of social network analysis utilize mathematical methods to examine the effects of these interactions on individuals, groups and the network as a whole [1, 2]. Recently, there is an emerging field that combines social network analysis with systems theory for describing the processes over social networks [3].

One of the important research topics in social network analysis is understanding the opinion formation process within a group of interconnected individuals. Studies describing the spread of opinions within groups of individuals began decades before the existence of massive communication platforms such as social media applications [4–6]. However, due to the increasing capacity of humans to connect and influence one another, it is becoming more relevant to examine how opinions propagate and shape our views. From a control theory perspective, this can be approached using agent-based models where the agents and their states correspond to social actors and their opinions [1, 7]. Such models describe the evolution of individual opinions over time based on a given set of rules that define the way agents interact and update their states at every time step. Depending on the design considerations implemented by researchers, patterns that resemble collective social behaviors may emerge, such as consensus, clustering, and fragmentation of opinions [8].

Some of the earlier works on opinion formation are the theory of social power by French [4], the consensus model by DeGroot [5], and the social influence network theory by Friedkin and Johnsen [9]. These models have laid the foundation for succeeding researches on opinion dynamics and are still used as a basis by current works on the subject. However, in these classical models, interactions remain static throughout the duration of their processes and opinions are updated simultaneously during each turn. Thus, these models can be described as deterministic and synchronous, which are characteristics that are not reflective of how interactions take place in real-world settings.

More recent models, however, are shifting toward time-varying and randomized processes [2] which are much closer approximation of human behavior. The use of randomized methods for modeling social networks is unsurprising as it has been established as an important tool for analyzing uncertain systems [10] and distributed computation over networks [11]. Of particular interest are models that implement a gossip-based approach for defining how agents can interact within a network. In distributed systems, gossiping refers to random pairwise interactions [12]. This method has been used in various applications such as in communication networks and multi-agent systems. Recently, it has also been incorporated to classical opinion dynamics models. Gossiping addresses two main concerns in earlier models. First, since the selected agents are the result of a random process, it removes the deterministic behavior of social actors. Second, by limiting interactions to one pair at a time, communications become asynchronous which is a more appropriate representation of how humans interact.

While recent innovations on opinion dynamics model have addressed some of the rigid portrayal of social networks [2, 13, 14], the way agents can interact remain restricted and the scenarios that they capture are still limited. Several recent models capture only some important aspects of the dynamics of actual social groups [3]. This observation is especially true when we consider the way we communicate using online and mobile applications which allows us to broadcast our sentiments to multiple recipients and receive information from multiple sources simultaneously. We also have to consider the fact that agents may already belong in social groups, as opposed to groups that are the result of opinion exchange, which is something straightforward gossiping does not consider. These factors should be given serious consideration when designing opinion dynamics models since humans are becoming more and more interconnected via technological advancements which greatly enhances our capabilities to influence one another [15]. To this end, the mechanism provided in gossip-based models for opinion dynamics should be expanded in order to make them more well-suited for representing how opinions spread in social networks.

This research introduces a new set of models that expand the behavior of gossiping to allow a more flexible representation of the interactions within a social network. The new schemes are combined with other existing models in order to take into consideration various scenarios that are addressed by opinion dynamics models. The following models are proposed in this dissertation:

- group gossiping [16]
- random group gossiping [17]
- random group gossiping with stubborn agents [18]
- random group gossiping with bounded confidence [19]
- intergroup opinion dynamics via gossiping [20]

Each proposed model has its corresponding chapter which contains detailed explanation of its process and analyses of its properties, including convergence and expected dynamics. Simulations are also provided for demonstrating the behavior of each model and for highlighting the results of the analyses.

1.2 Purpose and Contribution

The purpose of this dissertation is to develop novel schemes based on gossiping for representing various types of interactions that may occur in the opinion formation process in social networks. The resulting models and the scenarios represent are as follows:

The first model extends pairwise gossiping so agents interact with a group instead. This model corresponds to real-world scenarios where individuals are more likely to communicate with the same group of people regularly, like in families or in workplace settings. Additionally, this model can be used for analyzing the effects of group sizes in the evolution of opinion.

The second is a consensus model that involves a generalized version of group gossiping which allows agents to interact with a random subset of its neighbors. The flexible scheme employed by this model can capture a wide range of scenarios pertaining to human communication, such as in random daily encounters and the interactions that take place in online platforms. This also introduces a new way of reaching consensus based on random group gossiping.

The third model applies random group gossiping in social networks with stubborn agents. Aside from allowing the Friedkin-Johnsen model to be applied to the same scenarios in the previous model, this also presents a different perspective on how stubborn individuals can prevent social networks from attaining consensus.

The fourth model combines the concept of bounded confidence with random group gossiping. From a real-world perspective, this corresponds to humans that are only influenced by like-minded individuals within their social circle. The outcome is a new approach for clustering members of a social network based on their shared beliefs.

Finally, the fifth model presents an intergroup opinion dynamics model based on gossiping. This model introduces a social network based on interconnected groups, which can represent organizations or online groups, and describes a way for them to reach consensus.

1.3 Organization

This dissertation is organized as follows. Chapter 2 contains the preliminaries and the social network representation used in this. Related works and some important results related to this study are discussed in Chapter 3. The succeeding chapters discuss the proposed models for opinion dynamic. In Chapter 4, an extension of pairwise gossiping to group gossiping is presented. Chapter 5 introduces the concept of random group gossiping. This approach is applied to social networks with stubborn agents in Chapter 6. In Chapter 7, random group-gossiping is combined with bounded confidence for clustering. Chapter 8 presents a novel method for modeling intergroup opinion dynamics using a gossip-based approach. Finally, the conclusion of this dissertation is stated in Chapter 9.

1.4 Preliminaries

A directed graph is denoted as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $V = \{1, 2, \ldots, n\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. $\mathcal{G}$ is bidirectional if for any edge (i, j), the edge (j, i) also exists. The neighbors of agent i is given by $\mathcal{N}_i = \{j | (i, j) \in \mathcal{E}\}$. A path is a sequence of nodes where each consecutive pair of nodes is connected by an edge. A cycle is a path that starts and ends on the same node. A node i is globally reachable if all the other nodes has a path to it. A directed graph is strongly connected if there is a path between every pair of nodes. It is weakly connected if making its edges bidirectional results to a strongly connected graph.

A graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ is a subgraph of $\mathcal{G}$ if and only if $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$. A strongly connected component of $\mathcal{G}$ is a subgraph that is strongly connected and no additional nodes or edges from $\mathcal{G}$ can be added to produce another strongly connected subgraph. A strongly connected subgraph is periodic if the length of its cycles is divisible by $c \in \mathbb{Z}^+$, with $c > 1$. Otherwise, it is aperiodic. A graph induced by a square matrix A, given by $\mathcal{G}[A]$, contains nodes for each row in A and its edges correspond to all $a_{ij} > 0$

The vector of ones is given by $\mathbf{1}$. A standard basis vector is denoted by $e_i \in \mathbb{R}^n$, where the ith element is 1 while the rest are zeros. Given a vector $v \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n \times n}$, $diag(v)$ returns a diagonal matrix that has v as its main diagonal while $diag(A)$ returns a diagonal matrix that contains the main diagonal entries of A. The Hadamard product of two matrices A and B, denoted by $A \circ B$, results to a matrix whose each element is given by $a_{ij}b_{ij}$.

The ith eigenvalue of matrix A is denoted by $\lambda_i(A)$, and the eigenvalues of A are ordered as $\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A)$. If $|\lambda_i(A)| > |\lambda_j(A)|$ for all $j \neq i$, then $\lambda_i(A)$ is the dominant eigenvalue and its corresponding eigenvector is the dominant eigenvector. Matrix A is Schur stable if and only if all of its eigenvalues are inside the unit circle.

Chapter 2

Social Network, Opinion and Related Models

2.1 Social Network

A social network is a sociological concept that refers to a set of social units or actors and the relationships between them [21, 22]. The social units commonly denote individual persons, but they can also signify collective entities or objects [21]. Relationships may represent communication, influence, and other kinds of interactions that may occur between social actors [21].

Mathematically, social networks are usually represented using a directed or undirected graph where the nodes and edges respectively symbolize the actors and their relationships [22–24]. Edges may also have associated values that indicate the strength of relationships or the frequency of interactions [22]. Undirected graphs are used when the relationships are mutual, such as friendships [23]. However, there are cases when the relationships between units are asymmetric, like the citation of authors, which requires a directed graph [23]. Graphs are convenient for visualizing social networks, but they are also easily converted to matrices when computations must be performed.

In this dissertation, a social network is represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where the set of nodes $\mathcal{V}$ correspond to social actors or agents and the set of edges $\mathcal{E}$ signify the relationships between agents. Given a pair of agents i and j, the edge (i, j) means that i can be influenced by j. In general, an edge in $\mathcal{G}$ is not necessarily a symmetric relationship. However, in cases where a model deals with agents that mutually exchange information, $\mathcal{G}$ is represented by a bidirectional graph instead. For clarity, the specific type of graph used is indicated in every succeeding chapter that discusses a proposed model. In all models described in this dissertation, $\mathcal{G}$ is assumed to be at least weakly connected. Otherwise, it is treated as multiple social networks.

2.2 Opinion

In psychology, opinion is defined as an attitude, belief, or judgment [25]. A broader definition is adapted for opinion dynamics models, which considers opinion as an cognitive orientation of an individual towards objects, such as persons, issues or events [26, 27]. Such an interpretation permits the representation of opinions as qualitative attributes that express favor or disdain for particular objects. However, when the concern is describing the evolution of opinions, quantitative variables are more appropriate representations that can convey the magnitude of shift in attitudes [28] or the change in certainties when it comes to beliefs [29]. Agent-based models, in particular, require opinions to be discrete or continuous numeric variables in order to allow the modeling of the opinion formation process, although discrete values also allow mapping to qualitative features [7]. Discussion of the different representations of opinions used in opinion dynamics can be found in [1, 7].

For this research, each agent i maintains an opinion $x_i(k) \in \mathbb{R}$ which corresponds to the strength or certainty of its belief for a particular object at time $k \in \mathbb{Z}_{\geq 0}$. The opinions of all agents in a social network at time k is denoted by vector $x(k) \in \mathbb{R}^n$. All models discussed in this dissertation starts with an initial set of opinions $x(0)$. In some models, the opinions of all agents are explicitly defined to be within the interval $[0, 1]$. The same restriction is not applied here, although setting the initial opinions within the interval $[0, 1]$ will guarantee that the opinions at any succeeding time k are also within the same interval because the models proposed in this dissertation use convex combinations for updating opinions.

An important and recurring concept in opinion dynamics is the notion of consensus or agreement. Given $x(0)$ of a social network, reaching consensus means $\lim_{k\to\infty} x(k) = \omega\mathbf{1}$ where $\omega \in \mathbb{R}$ is the consensus value. That is, all agents agree to have the same final opinion. When ω is equal to the average of the initial opinions, that is called average consensus. Consensus is reached in probability if $\lim_{k\to\infty} \mathbb{P}[\max_{i,j\in\mathcal{V}} |x_i(k) - x_j(k)| > \epsilon] = 0$ for all $\epsilon > 0$.

2.3 Consensus Models

A common theme among several studies on opinion dynamics is finding out if a social network can attain consensus and the conditions that make it possible. In fact, some of the earliest works on opinion formation deal with this topic. French [4] described a consensus model where agents in a social network iteratively communicate with their neighbors and then use the average of the opinions they received to update their own opinions. This was later generalized by the work of DeGroot [5], which updates opinions using a weighted average of other opinions.

The DeGroot consensus model can be formally described as follows. Given n agents in a social network $\mathcal{G}$, let $W \in \mathbb{R}^{n\times n}$ be a nonnegative weight matrix, where $w_{ij} > 0$ if and only if $(i,j) \in \mathcal{E}$. The weight w_{ij} reflects how much agent i gives importance to the opinion of agent j. As such, $w_{ij} = 0$ means j has no influence on i i.e $(i,j) \notin \mathcal{E}$. The weights are normalized such that $\sum_{j\in\mathcal{N}_i} w_{ij} = 1$, which makes W a row-stochastic matrix. Starting with the initial opinions $x(0)$, the opinions at each succeeding time k are updated as

$$x(k+1) = Wx(k) \tag{2.1}$$

where the opinion of each agent i is updated as

$$x_i(k+1) = \sum_{j\in\mathcal{N}_i} w_{ij}x_j(k). \tag{2.2}$$

The model (2.1) describes a deterministic process that updates the opinions of all agents synchronously every iteration using a convex combination of a fixed set of opinions given by the neighbors of each agent and whose weights are defined by W. In general, (2.1) corresponds to the averaging dynamics used in different types of networks and multi-agent systems [30, 31].

An interesting property of the DeGroot model is that convergence and consensus are not dependent on the initial opinions, but rather on the structure of the social network [5, 32]. The opinions at time k is given by

$$x(k) = W^k x(0) \tag{2.3}$$

which implies that the model (2.1) is convergent if and only if $\lim_{k\to\infty} W^k$ exists.

Lemma 2.1. *The limit* $\lim_{k\to\infty} W^k$ *exists if and only if all strongly connected components in* $\mathcal{G}$ *with no outgoing edges are aperiodic. If there is only one strongly connected component with no outgoing edges and all the other strongly components has a path* *to it, then* $\lim_{k\to\infty} W^k = \mathbf{1}\pi^T$*, where* π *is the dominant left eigenvector of* W *that is normalized such that* $\sum_i \pi_i = 1$.

The second part of the lemma above provides the consensus condition for the DeGroot model. Applying Lemma 2.1 to (2.3) results to

$$\lim_{k\to\infty} x(k) = \mathbf{1}\pi^T x(0)$$

where $\pi^T x(0)$ is the consensus value. Thus, π characterizes the contribution of each agent on their consensus.

Another important property of the model (2.1) is the rate by which opinions converge to the consensus. For cases that involve symmetric W matrices, this is determined by the second largest eigenvalue of W, denoted by $\lambda_{n-1}(W)$. A $\lambda_{n-1}(W)$ that is close to 1 means opinions approach the consensus at a slower pace, while lower values correspond to faster convergence. In general, the convergence rate of (2.1) is determined by the magnitude of the eigenvalues of W [30].

While the topic of achieving consensus has garnered significant interest in opinion dynamics, its relevance goes beyond social networks and it is also studied in other applications, such as decentralized computers, sensor networks, and distributed systems [33, 34].

2.4 Stubborn Agents

While the previous section describes a process for reaching an agreement, understanding the conditions that fail to achieve consensus is a separate but still an important challenge in the analysis of social networks [27]. In fact, a limitation of the DeGroot model (2.1) is that consensus may still be attained regardless of agents' resistance to interpersonal influence, which is characterized by the cases when $w_{ii} < 1$ [27].

This limitation is addressed by the Friedkin-Johnsen model [9], which expands the work of DeGroot by taking into consideration agents who are attached to their previous opinions. These so-called stubborn agents have varying degree of openness to external opinions based on their prejudices. As indicated in Lemma 2.1, the ability to reach an agreement in the DeGroot model is dependent on the topology of the given social network. By taking into account resistance to influence, convergence to a disagreement may occur in the Friedkin-Johnsen model even under the same situations that lead to consensus in the DeGroot model.

Similar to the DeGroot model, the Friedkin-Johnsen model uses a weight matrix W to denote the interpersonal influences among agents. The matrix $\Gamma = I - diag(W)$ is added into the model to represent the susceptibility of agents to external influence. Given an agent i, $\gamma_i < 1$ means the agent is stubborn and has attachment to its initial opinion, while $\gamma_i = 1$ means it is completely open to the opinions of others. Prejudices or preconceived opinions are given by the vector $u = x(0)$. The model can then be expressed as

$$x(k+1) = \Gamma W x(k) + (I - \Gamma)u. \tag{2.4}$$

Note that $\Gamma = I$ reduces (2.4) to the DeGroot model (2.1). Based on equation (2.4), the opinions at time k is given by

$$x(k) = (\Gamma W)^k u + \sum_{q=0}^{k-1} (\Gamma W)^q (I - \Gamma)u. \tag{2.5}$$

Lemma 2.2. *When all agents are either stubborn or has a path to a stubborn agent in $\mathcal{G}$, ΓW becomes Schur stable. In this scenario, the model (2.4) converges to the limit*

$$\lim_{k\to\infty} x(k) = (I - \Gamma W)^{-1}(I - \Gamma)u. \tag{2.6}$$

As such, disagreements can occur in the Friedkin-Johnsen model even in aperiodic strongly connected graphs, which is not permissible in the DeGroot model.

A time-varying version of the Friedkin-Johnsen model is described in [35], while a gossip-based model is presented in [36]. The gossip-based model is briefly discussed in the next section.

2.5 Opinion Dynamics via Gossiping

In communication networks, gossiping refers to asynchronous pairwise interactions of agents [37]. Motivated by the simultaneous interactions employed by previous opinion dynamics models, which is deemed unrealistic for depicting personal communications, the work in [38] utilized a gossip algorithm [12] to model the opinion formation process in settings involving one-on-one discussions. This model can be described as follows. Consider a social network $\mathcal{G}$ where the edges are bidirectional since the model is concerned with exchange of opinions between pairs of agents i.e. an edge $(i, j) \in \mathcal{E}$ means agent i can receive the opinion of agent j and vice versa. At each time $k \geq 0$:

1. An agent i is selected with uniform probability from $\mathcal{V}$.

2. An agent $j \in \mathcal{N}_i$ is then randomly selected with probability p_{ij}.

3. The opinions of i and j are updated as

$$x_i(k+1) = x_j(k+1) = \frac{x_i(k) + x_j(k)}{2}. \tag{2.7}$$

while other opinions remain the same.

The update scheme in (2.7) can be generalized [33] as:

$$x_i(k+1) = (1-a)x_i(k) + ax_j(k) \tag{2.8a}$$

$$x_j(k+1) = ax_i(k) + (1-a)x_j(k) \tag{2.8b}$$

$$x_l(k+1) = x_l(k) \qquad \forall l \in \mathcal{V} \setminus \{i, j\}, \tag{2.8c}$$

where $a \in (0, 1)$ is the weight given by an agent to the opinion it receives at each round of interaction. The same behavior in (2.7) can be achieved by setting $a = 0.5$. All succeeding mentions of gossiping for opinion dynamics refers to the process described above that uses (2.8) for updating opinions.

Alternatively, the previous process can be expressed as a time-varying version of (2.1). Let $W^{ij} \in \mathbb{R}^{n \times n}$ be defined as

$$W^{ij} = I - a(e_i - e_j)(e_i - e_j)^T, \tag{2.9}$$

which can be considered as a weight matrix based on the interacting agents i and j. Using (2.9), the pairwise gossip-based model for opinion dynamics can also be written as

$$x(k+1) = W(k)x(k), \tag{2.10}$$

where $W(k) = W^{ij}$. Previous results [33, 38] have established that this model achieves average consensus, as stated in the following lemma.

Lemma 2.3. *The gossip model (2.10) converges almost surely to the limit*

$$\lim_{k \to \infty} x(k) = \frac{1}{n}\mathbf{1}\mathbf{1}^T x(0). \tag{2.11}$$

This means that all agents will have the same final opinions which is the average of the initial opinions.

Another useful method in describing the properties of stochastic systems is the computation of its expected dynamics. For (2.10), this can be obtained by letting $P = [p_{ij}]$.

Lemma 2.4. *Let $\bar{x}(k) = \mathbb{E}[x(k)]$ and $\bar{W}^{ij} = \mathbb{E}[W^{ij}]$ given by*

$$\bar{W}^{ij} = I - \frac{a}{n}(diag((P + P^T)\mathbf{1}) - (P + P^T)).$$

Then, the expected dynamics of (2.10) is

$$\bar{x}(k+1) = \bar{W}^{ij}\bar{x}(k).$$

Proof. Since $\mathcal{G}$ is bidirectional and (2.8) updates the opinions of i and j symmetrically, $\mathbb{P}[W(k) = W^{ij}] = \mathbb{P}[W(k) = W^{ji}] = (p_{ij}+p_{ji})/n$. Additionally, note that $\sum_{j\in\mathcal{N}_i} p_{ij} = 1$. Hence,

$$\begin{aligned}
\bar{W}^{ij} &= \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\frac{p_{ij}}{n}W^{ij} \\
&= \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i}\frac{p_{ij}}{n}(I - a(e_i - e_j)(e_i - e_j)^T) \\
&= I - \frac{a}{n}\sum_{(i,j)\in\mathcal{E}}(p_{ij} + p_{ji})(e_i - e_j)(e_i - e_j)^T \\
&= I - \frac{a}{n}(diag((P + P^T)\mathbf{1}) - (P + P^T)).
\end{aligned}$$

□

Gossiping was also applied to the Friedkin-Johnsen model [36]. The dynamics of this model can be described by using the same steps defined for (2.10) and replacing (2.8) with

$$x_i(k+1) = \gamma_i((1 - w_{ij})x_i(k) + w_{ij}x_j(k)) + (1 - \gamma_i)u_i \tag{2.12a}$$

$$x_l(k+1) = x_l(k) \quad \forall l \in \mathcal{V}\setminus\{i\} \tag{2.12b}$$

where the matrices W and Γ and the vector u are the same as in (2.4). Notice that only the opinion of agent i is recomputed, while the others remain the same. This is an example of an asymmetric update, which can be used to represent situations where an individual can retrieve an information from another without necessarily engaging in a discourse. The properties of this model are beyond the scope of this section; however, some of the analysis performed in [36] are also applied in later parts of this dissertation.

2.6 Bounded Confidence

Bounded confidence models describe the opinion formation process in social networks where only like-minded individuals influence one another. Such models are used for analyzing how clusters are formed within communities with diverse opinions.

One of the earliest works that popularized the concept of bounded confidence is the model by Hegselmann and Krause [39]. In this model, all individuals simultaneously update their opinions based on the average of all opinions that are within a given range. Let δ denote the range of acceptable opinions or the confidence threshold. The model can then be described by using a time-varying social network, where the neighbors of each agent i at time k is given by $\mathcal{N}_i(k) = \{j : |x_i(k) - x_j(k)| \leq \delta\}$. The opinion of each agent is iteratively updated as

$$x_i(k+1) = \frac{1}{|\mathcal{N}_i(k)|} \sum_{j \in \mathcal{N}_i(k)} x_j(k). \tag{2.13}$$

The model converges either to a consensus or to a clustering of opinions depending on the initial opinions and the specified range. Aside from updating opinions synchronously, the dynamics of (2.13) is deterministic based on the initial opinions.

Another well-known bounded confidence model is the work by Deffuant and Weisbuch [40]. Their model follows a similar scheme as the gossip-based model for opinion dynamics, but it also incorporates a confidence threshold δ. Let $a \in (0, 1)$ be the weight of another agent's opinion, like in (2.8). At each time $k \geq 0$:

1. A pair of agents i and j are randomly chosen from $\mathcal{V}$.

2. If $|x_i(k) - x_j(k)| \leq \delta$, the opinions of i and j are updated as

$$x_i(k+1) = (1-a)x_i(k) + ax_j(k) \tag{2.14a}$$

$$x_j(k+1) = ax_i(k) + (1-a)x_j(k) \tag{2.14b}$$

   while other opinions remain the same.

Similar to (2.13), the Deffuant-Weisbuch model is convergent despite involving a random process. However, the resulting clusters may vary each time the model is executed.

Chapter 3

Opinion Dynamics via Gossiping with Group Interactions

3.1 Overview

As mentioned in the previous chapter, gossiping updates opinions based on random pairwise interactions. This approach addressed the synchronism and determinism of earlier models. However, while this is a more realistic method, interactions are strictly restricted to one pair at a time.

This chapter describes an opinion dynamics model based on an extension of pairwise gossiping where agents interact with a group instead. Here, a group refers to the neighbors of an agent in the given social network. From a real-world perspective, this model represents scenarios where individuals are more likely to communicate with multiple persons that they regularly encounter. It can also be seen as an intermediate model between the DeGroot consensus model, where all agents interact with their neighbors simultaneously, and pairwise gossiping.

3.2 Group Gossiping

Consider a social network $\mathcal{G}$ composed of n agents. Since the model proposed in this chapter is concerned with agents exchanging views, $\mathcal{G}$ is bidirectional. Let $b_{ij} \in [0, 1]$ be a scalar variable such that $\sum_{j \in \mathcal{N}_i} b_{ij} = 1$ and $b_{ij} > 0$ if and only if $j \in \mathcal{N}_i$. This variable represents the amount of information or opinion that is transmitted when an agent i contacts its neighbors. As such, it serves a similar purpose as a in the gossip model (2.10), but each neighbor is given its own corresponding weight. To take into account personal opinions, which are also included in pairwise gossiping, it is assumed $b_{ii} > 0$ for all agent i. This means there is an implicit self-loop for each agent, although this does not affect the behavior of the model.

The group gossiping model for opinion dynamics can then be described as follows. At each time $k \geq 0$:

1. An agent i is selected with uniform probability from $\mathcal{V}$.
2. The opinions of i and all agent $j \in \mathcal{N}_i$ are updated as

$$x_i(k+1) = (1 - \sum_{j \in \mathcal{N}_i} b_{ij})x_i(k) + \sum_{j \in \mathcal{N}_i} b_{ij}x_j(k) \tag{3.1a}$$

$$x_j(k+1) = b_{ij}x_i(k) + (1 - b_{ij})x_j(k). \tag{3.1b}$$

while other opinions remain the same.

The combination of (3.1a) and (3.1b) signifies the scenario when i expresses its opinion to all its neighbors who, in turn, also give their opinions to i. Regardless of whether i is included in $\mathcal{N}_i$ or not, the computations in step 2 remain the same since (3.1a) will become

$$\begin{aligned} x_i(k+1) &= (1 - \sum_{j \in \mathcal{N}_i} b_{ij})x_i(k) + \sum_{j \in \mathcal{N}_i} b_{ij}x_j(k) \\ &= (1 - \sum_{j \in \mathcal{N}_i, j \neq i} b_{ij})x_i(k) + \sum_{j \in \mathcal{N}_i, j \neq i} b_{ij}x_j(k) \end{aligned}$$

and (3.1b) will become

$$x_j(k+1) = x_i(k+1) = b_{ii}x_i(k) + (1 - b_{ii})x_i(k) = x_i(k)$$

when the edge (i, i) is ignored.

Similar to (2.10), the model can also be alternatively represented as a time-varying discrete-time system using a random matrix $W^i \in \mathbb{R}^{n \times n}$ whose value depends on the selected agent i at time k and defined as

$$W^i = I - \sum_{j \in \mathcal{N}_i} b_{ij}(e_i - e_j)(e_i - e_j)^T. \tag{3.2}$$

Thus, the model can also be expressed as

$$x(k+1) = W(k)x(k). \tag{3.3}$$

where $W(k) = W^i$.

Let $P \in \mathbb{R}^{n\times n}$ be a matrix where $p_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$. Each p_{ij} denotes the probability that agent i contacts agent $j \in \mathcal{N}_i$ at time k when only one neighbor can be chosen, such as in pairwise gossiping, which implies $\sum_{j\in\mathcal{N}_i} p_{ij} = 1$. This means P is defined similarly as in the previous chapter and can be used to show the relationship between the group gossip model and the pairwise gossip model, as stated by the following lemma.

Lemma 3.1. *Let $b_{ij} = ap_{ij}$. Then $\mathbb{E}[W^i] = \mathbb{E}[W^{ij}]$.*

Proof.

$$\begin{aligned}
\bar{W^i} &= \sum_{i\in\mathcal{V}} \frac{1}{n} W^i \\
&= \sum_{i\in\mathcal{V}} \frac{1}{n} (I - \sum_{j\in\mathcal{N}_i} b_{ij}(e_i - e_j)(e_i - e_j)^T) \\
&= I - \frac{1}{n} \sum_{(i,j)\in\mathcal{E}} (b_{ij} + b_{ji})(e_i - e_j)(e_i - e_j)^T \\
&= I - \frac{a}{n} (diag((P + P^T)\mathbf{1}) - (P + P^T)).
\end{aligned}$$

□

Moreover, both models actually reach the same consensus even if their weights are defined independently of one another. Similar to (2.10), the dynamics (3.3) also achieves average consensus.

Theorem 3.2. *The model (3.3) converges almost surely to the limit*

$$\lim_{k\to\infty} x(k) = \frac{1}{n}\mathbf{1}\mathbf{1}^T x(0). \tag{3.4}$$

Proof. Since $\mathcal{G}$ is at least weakly connected and it is also bidirectional because of the symmetric exchange of opinions in the model (3.3), $\mathcal{G}$ is strongly connected. Note that the sequence $\{W(k)\}_{k\geq 0}$ is i.i.d, where each $W(k)$ is doubly stochastic and has a positive diagonal based on (3.2). Because $\mathcal{G}$ is strongly connected and $\mathbb{P}[W(k)_{ij} > 0]$ for any $(i, j) \in \mathcal{E}$, then $\mathbb{E}[W(k)]$ is irreducible and $\lambda_{n-1}(\mathbb{E}[W(k)]) < 1$ via the Perron-Frobenius theorem. Given the properties of $W(k)$ and $\mathbb{E}[W(k)]$, the model (3.3) converges almost surely to the consensus using Theorem 3 in [41]. Additionally, since $\mathbb{E}[W(k)]$ is also doubly stochastic, the consensus is given by the limit (3.4). □

The results of Lemma 3.1 and Theorem 3.2 enable additional comparisons to be performed on the group gossip model and the pairwise gossip model. This is done using simulations in the next section.

3.3 Numerical Examples

Since the models (3.3) and (2.10) converge to the same limit, their behaviors can be compared using the same initial opinions. Also, by making $\mathbb{E}[W^i] = \mathbb{E}[W^{ij}]$, the dynamics of both models can also be compared to the DeGroot model by using either $\mathbb{E}[W^i]$ or $\mathbb{E}[W^{ij}]$ as the weight matrix in (2.1) which results to the expected dynamics of (3.3) and (2.10).

Consider the social network in Figure 3.1. Let $a = 0.3$, $p_{ij} = 1/\mathcal{N}_i$ and $b_{ij} = ap_{ij}$. Set the initial opinions as

$$x(0) = \begin{bmatrix} 0.4664 & 0.5070 & 0.8022 & 0.8805 & 0.2055 \\ 0.0202 & 0.9304 & 0.7237 & 0.8057 & 0.5268 \end{bmatrix}.$$

Figure 3.2b show the results of the three models for the given values. It can be seen that, while all three models converged to the same consensus, there is significant difference on how their opinions evolve from one time step to another. The pairwise gossiping produces drastic changes in opinions, while the consensus model has smooth transitions. The behavior of the group gossiping has more gradual changes compared to pairwise gossiping and can be described as an intermediate between the other two models.

While the consensus model is deterministic, the interactions of the pairwise gossip model and the group gossip model are based on random processes. To further explore the distinction between both gossip-based models, their dynamics are compared with the DeGroot consensus model that uses $\mathbb{E}[W^i]$ in Lemma 3.1 as its weight matrix. The comparisons are performed on three randomly generated graphs with $n = 10$, $n = 50$, and $n = 100$. For each graph, the Euclidean distance between the opinions of the gossip-based models and the consensus model are computed each time step. This is performed in one hundred trials for each graph. The initial opinions are randomly generated at the start of each trial and the same initial opinions are used for all the models. Figure 3.3 shows the results. In all graphs, the average distance between the opinions generated by the group gossip model and the consensus model is significantly less than the distance between the opinions of the pairwise gossip model and the consensus model during the earlier time steps. However, as the value of k increases, the differences become less pronounced since the opinions converge to the consensus value.

The observations regarding Figure 3.3 are consistent with how opinions are updated in the models under consideration. For these models, assuming that the conditions for reaching consensus are met, the updated opinions at each iteration satisfy the inequality

$$\min_i x_i(k) \le x_i(k+1) \le \max_i x_i(k). \tag{3.5}$$

For the consensus model, this implies that all opinions gradually approach the consensus value since they are all updated at each time k. The group gossip model updates an average of $\frac{1}{n}\sum_i |\mathcal{N}_i| + 1$ opinions per turn, where $|\mathcal{N}_i| \ge 1$ since the social networks under consideration are assumed to be at least weakly connected. The pairwise gossip model, on the other hand, updates only two opinions each time step. Thus, based on the inequality (3.5), there is less gap between the opinions of the group gossip model and the DeGroot model when the consensus is not yet reached.

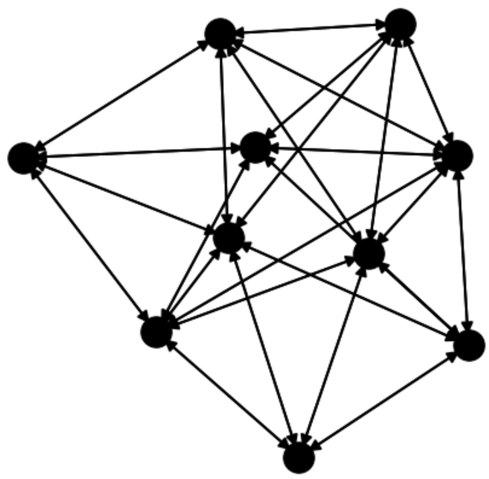
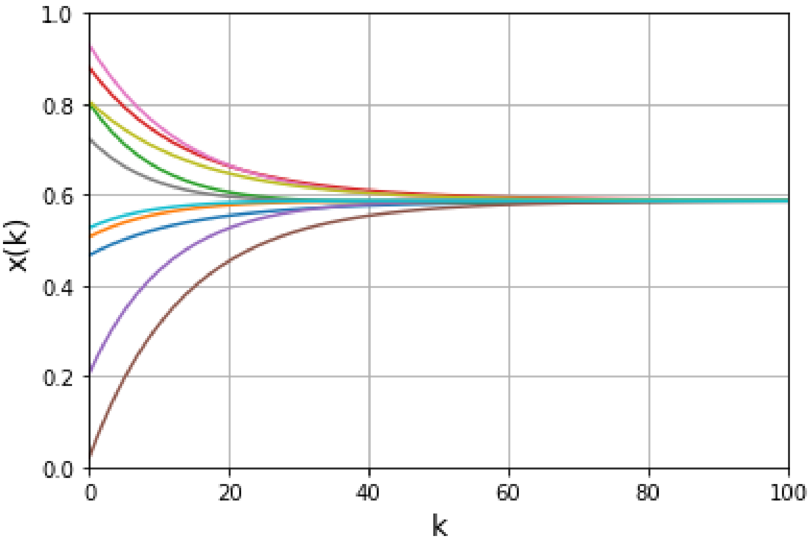


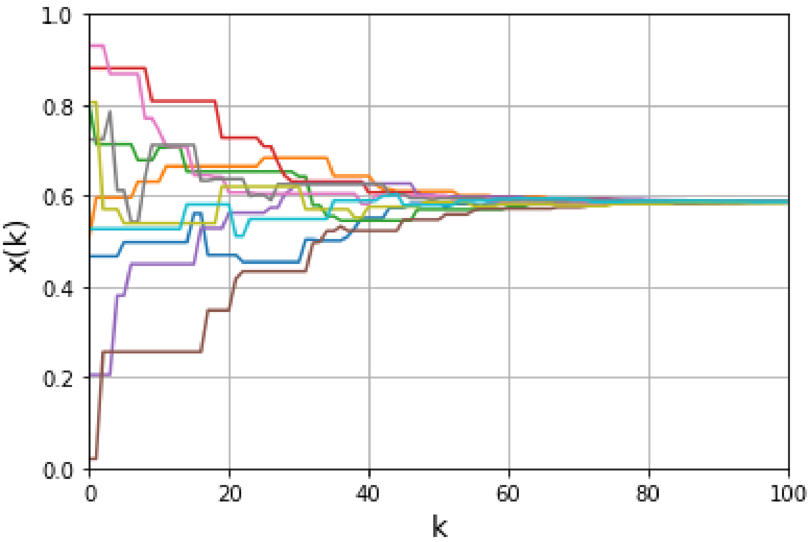
Figure 3.1: Social network

3.4 Summary

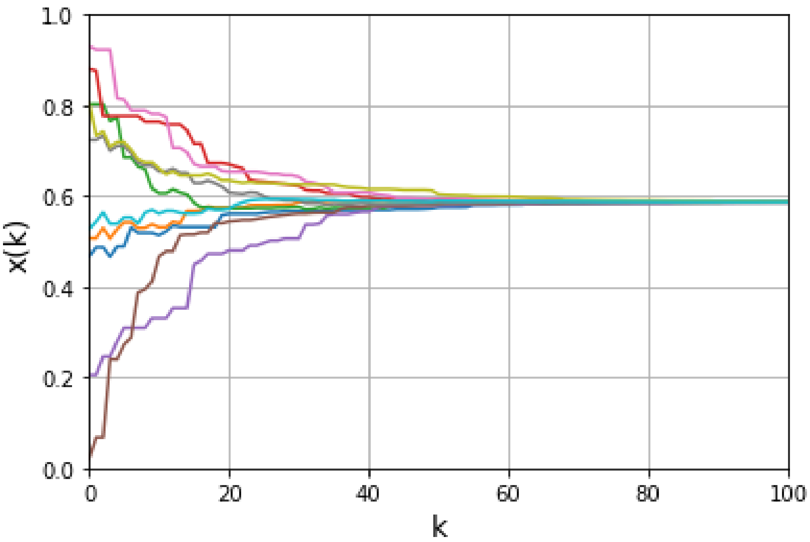
In this chapter, the gossip model for opinion dynamics is extended to incorporate group interactions. Both the pairwise gossip model and group gossip model approach the same consensus. Moreover, their expected dynamics can be the same by modifying the weights used in the group gossip model. However, the evolution of the opinions in group gossiping is less drastic compared to pairwise gossiping, especially during the early stages of their opinion formation process.



(A) DeGroot Model

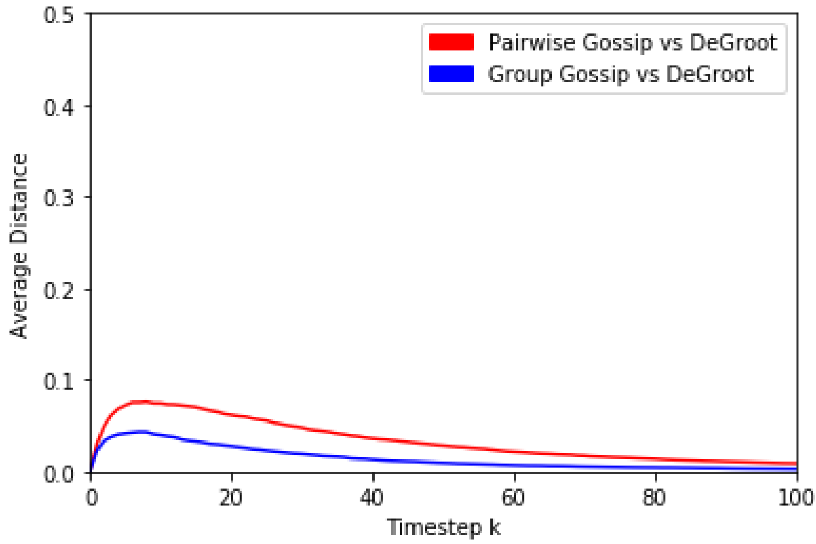


(B) Pairwise Gossiping

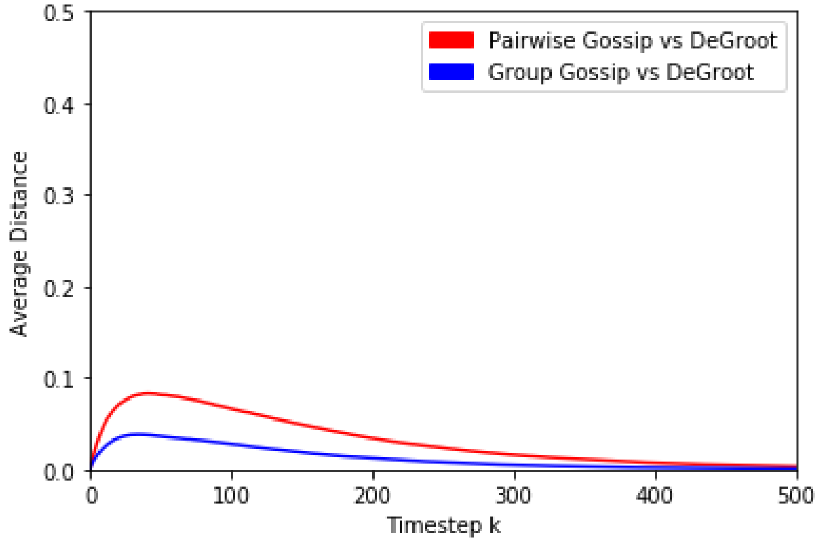


(C) Group Gossiping

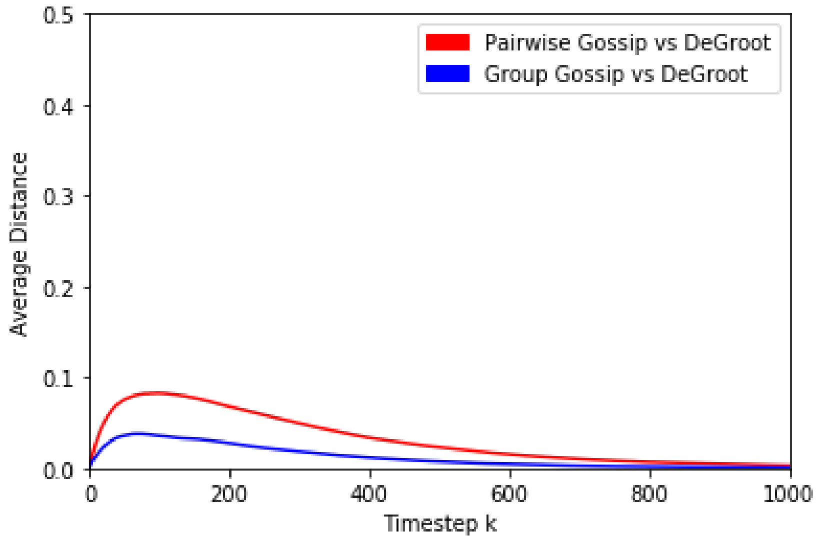
FIGURE 3.2: Resulting dynamics



(A) $n = 10$



(B) $n = 50$



(C) $n = 100$

FIGURE 3.3: Distance between opinions

Chapter 4

Gossip-Based Model with Probabilistic Group Interactions

4.1 Overview

The previous chapter introduced group gossiping, which not only extends the behavior of pairwise gossiping but also demonstrates the effects of group sizes on the evolution of opinions. This chapter proposes a generalization of group gossiping by allowing interactions with randomly selected participants. This novel approach to representing interactions in social networks can be combined with other models in order to depict a wider range of scenarios. When applied to the DeGroot model, consensus can still be attained although the final opinions are no longer deterministic. Additionally, the frequency of interactions can affect the rate by which agents converge to consensus.

4.2 Random Group Gossiping

The DeGroot model (2.1) is an intuitive depiction of how agents in a social network can come to an agreement. Gossiping (2.10) makes interactions random and asynchronous, while group gossiping (3.3) expands the coverage of interactions. In this chapter, the core features of these models are combined then modified in order to attain a more flexible behavior. The result is a gossip-based extension of the DeGroot model with probabilistic group interactions.

Consider a directed graph $\mathcal{G}$ as the given social network. Let $R \in \mathbb{R}^{n\times n}$ be a nonnegative matrix, where $r_{ij} \in [0, 1]$ and $r_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$. In this matrix, each r_{ij} corresponds to the independent probability that agent i receives the opinion of agent

j at time k. This is different from the matrix P defined in the previous chapters, where each row gives the probabilities of mutually exclusive events. So, unlike P, R is not row stochastic. Let $S_i(k) \subseteq \mathcal{N}_i$ be a random subset of the neighbors of i where each member is determined by an independent Bernoulli trial, denoted by the random variable $\phi_{ij}(k)$ that is defined as

$$\phi_{ij}(k) = \begin{cases} 1 & j \in S_i(k) \\ 0 & j \notin S_i(k) \end{cases}$$

where

$$\mathbb{P}[\phi_{ij}(k) = 1] = r_{ij}.$$

In this model, the term "group" refers to the randomly selected members of $S_i(k)$.

Let $W \in \mathbb{R}^{n \times n}$ be a weight matrix that has the same properties and usage as the W matrix in the model (2.1). To be consistent with the objective of generalizing group gossiping and other related models, the requirement of a self-weight w_{ii} is excluded from this model. The consensus model via random group gossiping can be described as follows. At each time $k \geq 0$:

1. An agent i is selected with uniform probability from $\mathcal{V}$.
2. The members of $S_i(k)$ are chosen by performing an independent Bernoulli trial $\phi_{ij}(k)$ on each $j \in \mathcal{N}_i$.
3. The opinion of agent i is then updated as

$$x_i(k+1) = \left(1 - \sum_{j \in S_i(k)} w_{ij}\right) x_i(k) + \sum_{j \in S_i(k)} w_{ij} x_j(k) \tag{4.1}$$

   while the opinions of the other agents remain unchanged.

The process described above conveys the idea that at any given time, individuals may receive information or sentiment from different persons they know and this usually happens unpredictably.

The equation (4.1) is an asymmetric update, like (2.12), which only updates the opinion of one agent, as opposed to symmetric updates where opinions are updated in a pairwise manner. The choice of an asymmetric update gives this model flexibility in representing various scenarios; however, this can be easily extended to a symmetric update while the analyses in this chapter can still be readily applied. The update performed in (4.1) can also be accomplished using a weight matrix $A(k)$ given by

$$A = I - \sum_{j \in S_i(k)} w_{ij} e_i e_i^T + \sum_{j \in S_i(k)} w_{ij} e_i e_j^T. \tag{4.2}$$

Thus, the consensus model via random group gossiping can be compactly expressed as

$$x(k+1) = A(k)x(k). \tag{4.3}$$

Note that if $S_i(k) = \mathcal{N}_i$, which can occur if all neighbors of i are selected, the opinion of i is updated using (3.1a), where $b_{ij} = w_{ij}$. This is also the same as using (2.2), since

$$\begin{aligned} x_i(k+1) &= \left(1 - \sum_{j \in \mathcal{N}_i} w_{ij}\right) x_i(k) + \sum_{j \in \mathcal{N}_i} w_{ij} x_j(k)) \\ &= \sum_{j \in \mathcal{N}_i} w_{ij} x_j(k). \end{aligned}$$

On the other hand, if $|S_i(k)| = 1$ because only one neighbor $j \in \mathcal{N}_i$ is selected at the current turn, then the opinion of i is updated using (2.8a), where $a = w_{ij}$. These show how the proposed model in this chapter relates to the DeGroot model, the pairwise gossip model, and the group gossip model.

Even if the opinions received by an agent at each time step results from a random process, the dynamics (4.3) still achieves probabilistic consensus under certain conditions.

Theorem 4.1. *If $\mathcal{G}$ contains a globally reachable node and every node has a self-loop, that is $(i,i) \in \mathcal{E}$ for all $i \in \mathcal{V}$, then consensus is reached in (4.3).*

Proof. Let $Q(k,r) = A(k+r)A(k+r-1)\dots A(k)$. Theorem 3.1 in [33] states that a random network achieves probabilistic consensus if and only if

$$\mathbb{P}[\{\exists k, \exists h | Q_{ih}(0,k) Q_{jh}(0,k) > 0\}] = 1 \tag{4.4}$$

for all $i, j \in \mathcal{V}$. If every node in $\mathcal{G}$ has a self-loop, then $A_{ii}(k) \geq w_{ii}$ for any i selected at time k. Thus, $A_{ij}(k+1) > 0$ implies $Q_{ij}(k,1) > 0$. If i has a path to a node h, then

$$\mathbb{P}[\{\exists k, \exists r | Q_{ih}(k,r) > 0\}] > 0.$$

Let

$$\kappa_{ij}(k,t) = \{\exists k, \exists t | Q_{ih}(k,t) Q_{jh}(k,t) > 0\} \tag{4.5}$$

for any $i, j \in V$. If h is a globally reachable node, then $\mathbb{P}[\kappa_{ij}(k,t)] > 0$. As $k \to \infty$, then $\mathbb{P}[\kappa_{ij}(k,t)] \to 1$. Notice that (4.5) implies that $Q_{ih}(k,t) > 0$ for all i. Since $A(k)$ is row-stochastic with a positive diagonal, then $Q_{ih}(0, k+t) > 0$ for all i, which satisfies (4.4). □

The consensus value that the proposed model (4.3) attains varies depending on the members of $S_i(k)$ at each time step. In order to have a better understanding of its behavior, its expected dynamics is analyzed.

Lemma 4.2. *Let $\bar{A} = \mathbb{E}[A(k)]$. The expected dynamics of the model (4.3) is*

$$\mathbb{E}[x(k+1)] = \bar{A}\mathbb{E}[x(k)] \tag{4.6}$$

where

$$\bar{A} = I - \frac{1}{n}(diag((R \circ W)\mathbf{1}) - R \circ W). \tag{4.7}$$

Proof. At any given time k, $\mathbb{P}[j \in S_i(k)] = r_{ij}$. Since the selection of $j \in S_i(k)$ is an independent event, then

$$\bar{A} = \frac{1}{n}\sum_{i \in \mathcal{V}} \left(I - \left(\sum_{j \in \mathcal{N}_i} r_{ij} w_{ij} \right) e_i e_i^T + \sum_{j \in \mathcal{N}_i} r_{ij} w_{ij} e_i e_j^T \right).$$

Note that

$$\sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} w_{ij} e_i e_i^T = diag(W\mathbf{1})$$
$$\sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{N}_i} w_{ij} e_i e_j^T = W.$$

Thus

$$\bar{A} = I - \frac{1}{n}(diag((R \circ W)\mathbf{1}) - R \circ W).$$

□

Theorem 4.3. *If $\mathcal{G}$ contains a globally reachable node, then the expected dynamics (4.6) reaches consensus and the consensus value is given by $\mathbf{1}\pi^T x(0)$ where π is the dominant left eigenvector of $\bar{A}$.*

Proof. Notice that (4.6) is the same as the model (2.1), which means Lemma 2.1 can be applied. Let $\mathcal{G}[\bar{A}]$ be the graph induced by matrix $\bar{A}$. It can be directly observed that any $(i, j) \in \mathcal{E}$ means that $\bar{A}_{ij} > 0$. Hence, the existence of a globally reachable node in $\mathcal{G}$ implies the same for $\mathcal{G}[\bar{A}]$. Since $\bar{A}$ has a positive diagonal, then all strongly connected components in $\mathcal{G}[\bar{A}]$ are aperiodic. Also, any globally reachable node in $\mathcal{G}[\bar{A}]$ is part of a strongly connected component with no outgoing edges since any outgoing edge attached to it will include a new node to that strongly connected component. This is the only strongly connected component in $\mathcal{G}[\bar{A}]$ with no outgoing edges because all other strongly connected components, if they exists, have a path to it. If there are multiple globally reachable nodes in $\mathcal{G}[\bar{A}]$, they belong to the same strongly connected component. Therefore, by Lemma 2.1, the expected dynamics (4.6) reaches the consensus value stated in Theorem 4.3. □

The convergence behavior and consensus value of (4.6) and (2.1) are dependent on the properties of the matrices $\bar{A}$ and W, respectively. An important distinction of the proposed model, however, is the presence of the R matrix which allows the modification of its dynamics without having to alter the structure of the given social network. This idea is demonstrated via the succeeding corollary, which involves a special case of the model (4.3).

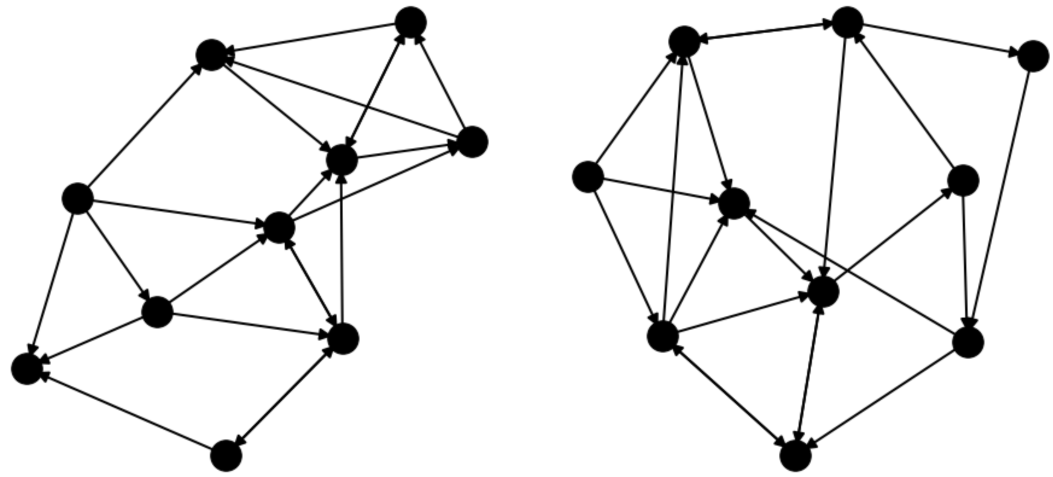
Corollary 4.4. *Let $r_{ij} = \hat{r}$ for all $i, j \in \mathcal{V}$, which means the probability for selecting a neighbor is uniform for all agents (i.e. $R \circ W = \hat{r}W$). Let $A_0 = I - diag(W\mathbf{1}) + W$ which is a row-stochastic matrix. Then any eigenvalue $\lambda_i(\bar{A}) = 1 - \hat{r}/n + (\hat{r}/n)\lambda_i(A_0)$.*

Proof. Based on the assumptions in Corollary (4.4), $\bar{A}$ can be computed as

$$\begin{aligned}\bar{A} &= I - \frac{\hat{r}}{n}(diag(W\mathbf{1}) - W)\\ &= (1 - \frac{\hat{r}}{n})I + \frac{\hat{r}}{n}(I - diag(W\mathbf{1}) + W)\\ &= (1 - \frac{\hat{r}}{n})I + \frac{\hat{r}}{n}A_0,\end{aligned}$$

from which we can characterize its eigenvalues. □

The corollary above does not guarantee that the magnitude of the eigenvalues of $\bar{A}$ is inversely proportional to $\hat{r}$. However, it can be generalized that an $\hat{r}$ value closer to 1 means lower eigenvalue magnitudes while an $\hat{r}$ value closer to 0 means higher eigenvalue magnitudes. Note that the convergence rate of (4.6) is dependent on the magnitude of the eigenvalues of $\bar{A}$. Thus, Corollary (4.4) implies that higher interaction frequencies can lead to a faster convergence to consensus.



(A) Does not contain a globally reachable node (B) Globally reachable nodes are present

FIGURE 4.1: Two social networks with different structures

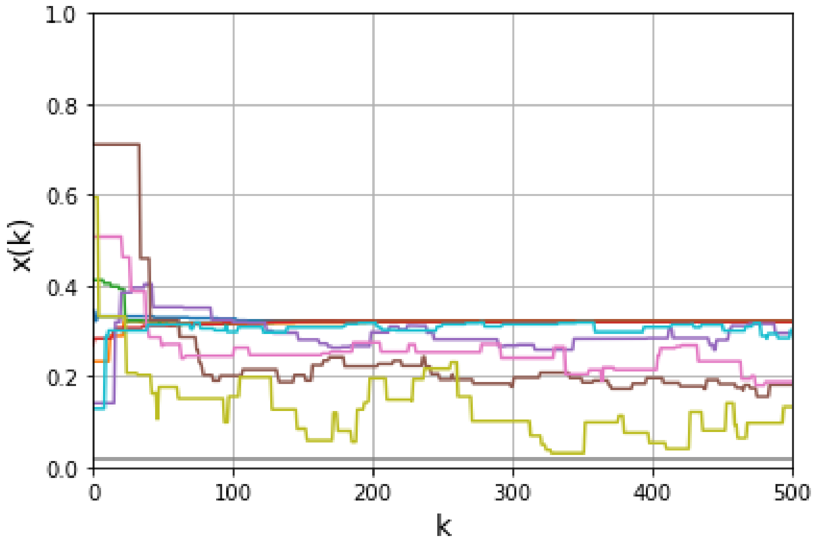
4.3 Numerical Examples

Consider the social networks in Figure 4.1. The first network is a weakly connected graph that does not contain a globally reachable node, while the second is also weakly connected but contains some globally reachable nodes. The nodes in both graphs are given self-loops that are not shown in the figures. The W and R matrices for both networks are randomly generated such that the properties of both matrices are satisfied. Let the initial opinions for both network be

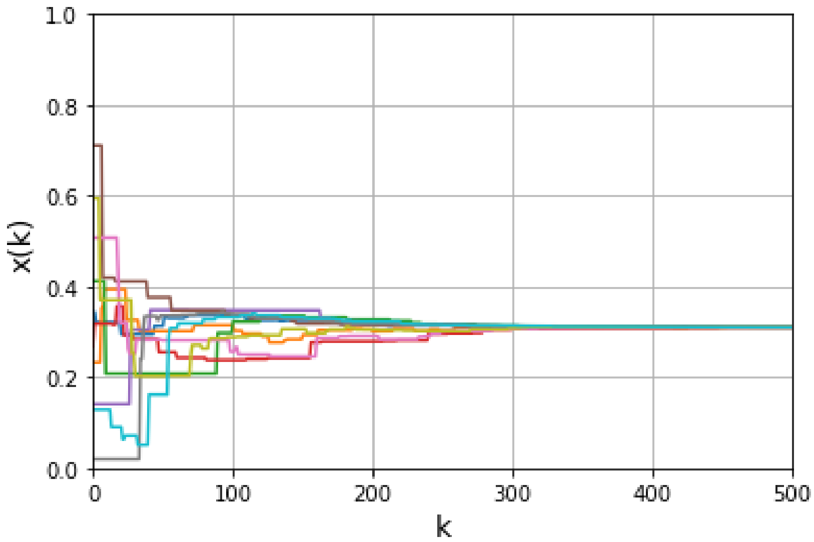
$$x(0) = \begin{bmatrix} 0.3436 & 0.2327 & 0.4117 & 0.2685 & 0.1403 \\ 0.7108 & 0.5072 & 0.0197 & 0.5942 & 0.1286 \end{bmatrix}.$$

Figure 4.2 shows the result of the model (4.3) on both networks. The opinions in Figure 4.2a do not converge, while the dynamics in Figure 4.2b converged to a consensus which demonstrates Theorem 4.1.

To demonstrate the effects of having low and high interaction frequencies, the expected dynamics (4.6) is applied on the social network 4.1b using R matrices with $\hat{r} = 0.15$ and $\hat{r} = 0.85$. The expected dynamics in Figure 4.3b reached consensus significantly faster than the expected dynamics in Figure 4.3a. This highlights the implication of Corollary 4.4.



(A) First social network

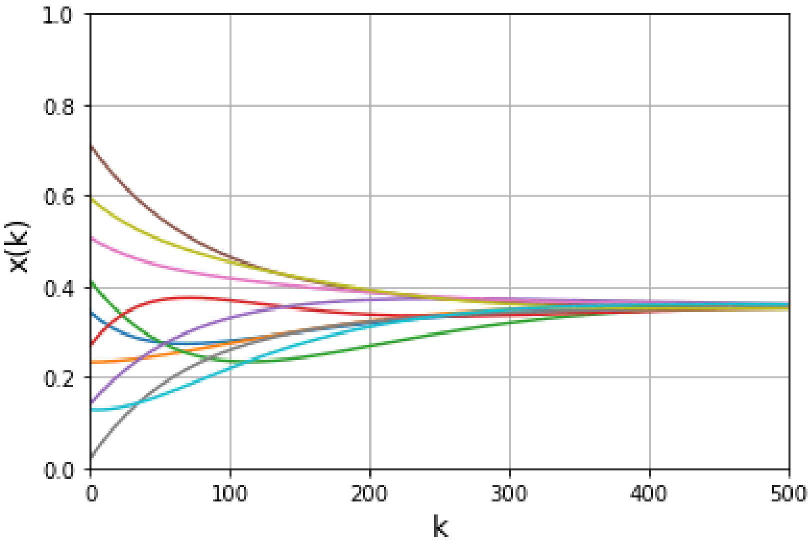


(B) Second social network

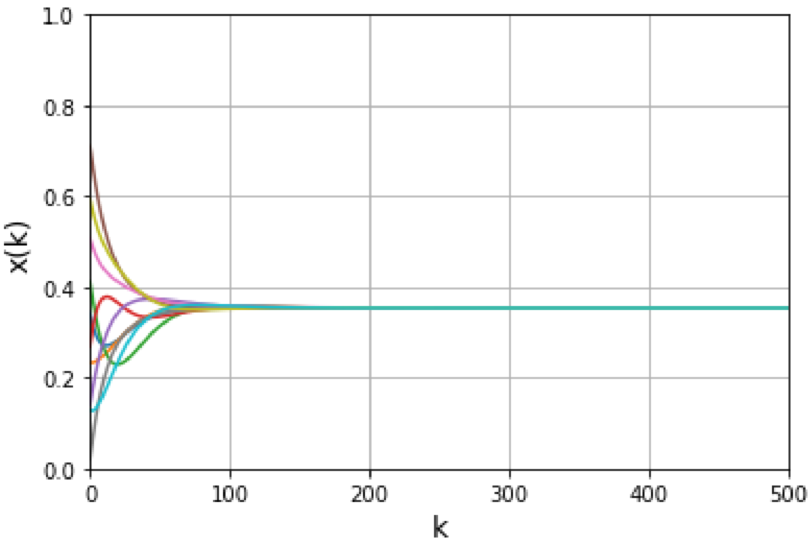
FIGURE 4.2: Resulting dynamics

4.4 Summary

This chapter proposed a gossip-based model for opinion dynamics that involves group interactions with random participants. Compared to traditional models and other gossip-based models, the proposed model is a closer representation of real-world interactions. The results in this chapter have established that, under suitable conditions, the model achieves probabilistic consensus. Furthermore, interaction frequencies may slow down or speed up the rate of convergence.



(A) $\hat{r} = 0.15$



(B) $\hat{r} = 0.85$

FIGURE 4.3: Expected dynamics

Chapter 5

Random Group Gossiping with Stubborn Agents

5.1 Overview

The work in the previous chapter introduced the concept of random group gossiping. It incorporates features from the DeGroot model and gossip-based models, particularly group gossiping, and then added a more generalized form of interaction. In this chapter, random group gossiping is applied to social networks with stubborn agents, which is achieved by extending the Friedkin-Johnsen model. While the opinions tend to oscillate in the resulting model, its expected dynamics and time-averaged opinions coincide over time.

5.2 Random Group Gossiping with Stubborn Agents

The model described in this chapter uses the same setup in the previous chapter. Additionally, let $\Gamma = I - diag(W)$ be a diagonal matrix denoting susceptibility to external influence and let $u = x(0)$ be the prejudices of the agents. Both Γ and u are adapted from the Friedkin-Johnsen model (2.4). The stubborn agents are all i such that $\gamma_i < 1$. Random group gossiping in networks with stubborn agents follows the same steps defined in the previous chapter, but with a modified update formula. At each time $k \geq 0$:

1. An agent i is selected with uniform probability from $\mathcal{V}$.
2. The members of $S_i(k)$ are chosen by performing an independent Bernoulli trial $\phi_{ij}(k)$ on each $j \in \mathcal{N}_i$.

3. The opinion of agent i is then updated as

$$x_i(k+1) = \gamma_i \left(\left(1 - \sum_{j \in S_i(k)} w_{ij} \right) x_i(k) + \sum_{j \in S_i(k)} w_{ij} x_j(k) \right) + (1-\gamma_i) u_i \quad (5.1)$$

while the opinions of the other agents remain unchanged.

The model can also be written as

$$x(k+1) = A(k)x(k) + B(k)u \quad (5.2)$$

where

$$A(k) = (I - e_i e_i^T (I - \Gamma)) \left(I - \left(\sum_{j \in S_i(k)} w_{ij} \right) e_i e_i^T + \sum_{j \in S_i(k)} w_{ij} e_i e_j^T \right)$$

and

$$B(k) = e_i e_i^T (I - \Gamma).$$

Hence, (5.2) is a time-varying version of (2.4) that is dependent on the participating i and $S_i(k)$ at time k. This model also extends the gossip-based version of the Friedkin-Johnsen model (2.12) that was proposed in [36]. Setting $r_{ij} = 1$ for all $(i, j) \in \mathcal{E}$ converts this model to the classical Friedkin-Johnsen model (2.4), while making $\Gamma = I$ transforms this to the model (4.3) in the previous chapter.

While its possible for this model to achieve consensus by manipulating the values in W, R, and Γ, it does not usually converge to stable opinions. Thus, the analysis on this chapter is focused on its expected behavior.

Lemma 5.1. *Let $\bar{A} = \mathbb{E}[A(k)]$ and $\bar{B} = \mathbb{E}[(B(k)]$. The expected dynamics of the model (5.2) is*

$$\mathbb{E}[x(k+1)] = \bar{A}\mathbb{E}[x(k)] + \bar{B}u \quad (5.3)$$

where

$$\begin{aligned} \bar{A} &= I - \frac{1}{n}(I - \Gamma - \Gamma(R \circ W - diag(R \circ W)\mathbf{1}))) \\ \bar{B} &= \frac{1}{n}(I - \Gamma). \end{aligned} \quad (5.4)$$

Proof. Considering that the selection of $j \in S_i(k)$ is an independent event, then

$$\begin{aligned}
\bar{A} =& \frac{1}{n}\sum_{i\in\mathcal{V}}((I - e_ie_i^T(I-\Gamma))(I - (\sum_{j\in\mathcal{N}_i} r_{ij}w_{ij})e_ie_i^T + \sum_{j\in\mathcal{N}_i} r_{ij}e_ie_j^T)) \\
=& \frac{1}{n}\sum_{i\in\mathcal{V}}(I - (\sum_{j\in\mathcal{N}_i} r_{ij}w_{ij})e_ie_i^T + \sum_{j\in\mathcal{N}_i} r_{ij}e_ie_j^T - e_ie_i^T(I-\Gamma) + \\
& e_ie_i^T(I-\Gamma)(\sum_{j\in\mathcal{N}_i} r_{ij}w_{ij})e_ie_i^T - e_ie_i^T(I-\Gamma)\sum_{j\in\mathcal{N}_i} r_{ij}e_ie_j^T) \\
=& \frac{1}{n}(nI - diag((R\circ W)\mathbf{1}) + R\circ W - (I-\Gamma) + (I-\Gamma)diag((R\circ W)\mathbf{1}) - \\
& (I-\Gamma)(R\circ W)) \\
=& I - \frac{1}{n}(I - \Gamma - \Gamma(R\circ W - diag((R\circ W)\mathbf{1})))
\end{aligned}$$

and

$$\begin{aligned}
\bar{B} &= \frac{1}{n}\sum_{i\in\mathcal{V}} e_ie_i^T(I-\Gamma) \\
&= \frac{1}{n}(I-\Gamma).
\end{aligned}$$

□

Note that Γ, $\bar{A}$, and $\bar{B}$ can be rearranged as

$$\Gamma = \begin{bmatrix}\Gamma^{11} & 0 \\ 0 & I\end{bmatrix} \qquad \bar{A} = \begin{bmatrix}\bar{A}^{11} & \bar{A}^{12} \\ 0 & \bar{A}^{22}\end{bmatrix} \qquad \bar{B} = \begin{bmatrix}\frac{I-\Gamma^{11}}{n} & 0 \\ 0 & 0\end{bmatrix} \tag{5.5}$$

where the submatrices Γ^{11} and $[\bar{A}^{11} \quad \bar{A}^{12}]$ correspond to all stubborn agents i, i.e. $\gamma_i < 1$, or has a path to a stubborn agent j in $\mathcal{G}$, and $\bar{A}^{22}$ corresponds to all the remaining agents.

Theorem 5.2. *Let*

$$x_* = \lim_{k\to\infty} \mathbb{E}[x(k)]. \tag{5.6}$$

The expected dynamics of (5.2) converges to

$$x_* = \begin{bmatrix}\frac{(I-\bar{A}^{11})^{-1}(I-\Gamma^{11})}{n} & (1-\bar{A}^{11})^{-1}\bar{A}^{12}\bar{A}^{22}_* \\ 0 & \bar{A}^{22}_*\end{bmatrix} u \tag{5.7}$$

where

$$\bar{A}^{22}_* = \lim_{k\to\infty}(\bar{A}^{22})^k.$$

Proof. The proof below follows the arguments used in [26]. From (5.3), the expected opinions at time k is

$$\mathbb{E}[x(k)] = \bar{A}^k u + \sum_{q=0}^{k-1} \bar{A}^q \bar{B} u. \tag{5.8}$$

Applying (5.5) on (5.8) results to

$$\mathbb{E}[x(k)] = \begin{bmatrix} (\bar{A}^{11})^k & \sum_{q=0}^{k-1} (\bar{A}^{11})^{k-q-1} \bar{A}^{12} (\bar{A}^{22})^k \\ 0 & (\bar{A}^{22})^k \end{bmatrix} u + \begin{bmatrix} \frac{1}{n} \sum_{q=0}^{k-1} \bar{A}^q (I - \Gamma_{11}) & 0 \\ 0 & 0 \end{bmatrix} u. \tag{5.9}$$

Note that $\sum_{j \in \mathcal{N}_i} \bar{A}_{ij} = 1 - \frac{1-\gamma_i}{n}$, which implies that in $\bar{A}^{11}$, $\sum_{j \in \mathcal{N}_i} \bar{A}^{11}_{ij} < 1$ for all i. Thus, $\bar{A}^{11}$ is Schur stable and $\lim_{k \to \infty} (\bar{A}^{11})^k = 0$. Also, $\bar{A}^{22}$ is stochastic and its diagonal entries are positive, which means that the strongly connected components in $\mathcal{G}[\bar{A}^{22}]$ are aperiodic. Therefore, $\lim_{k \to \infty} (\bar{A}^{22})^k$ exists based on Lemma 2.1. Using the previous statements on (5.9) gives

$$x_* = \begin{bmatrix} 0 & (I - \bar{A}^{11})^{-1} \bar{A}^{12} \bar{A}^{22}_* \\ 0 & \bar{A}^{22}_* \end{bmatrix} u + \begin{bmatrix} \frac{(I - \bar{A}^{11})^{-1} (I - \Gamma^{11})}{n} & 0 \\ 0 & 0 \end{bmatrix} u,$$

from which (5.7) can be obtained. □

Corollary 5.3. *If $\gamma_i < 1$ for all agents, then $\bar{A}$ is Schur stable and the limit x_* is*

$$x_* = (I - \Gamma - \Gamma(R \circ M - diag((R \circ M)\mathbf{1})))^{-1} (I - \Gamma) u. \tag{5.10}$$

Proof. Applying the condition in Corollary 5.3 to (5.5) implies that

$$\Gamma = \Gamma^{11} \qquad \bar{A} = \bar{A}^{11} \qquad \bar{B} = \frac{I - \Gamma^{11}}{n}$$

which converts the limit (5.7) to

$$x_* = \frac{(I - \bar{A})^{-1} (I - \Gamma)}{n} u \tag{5.11}$$

by reapplying Theorem 5.2. The limit (5.10) can be obtained by applying (5.4) on (5.11). □

Theorem 5.2 describes the resulting opinions of the expected dynamics of the model (5.3). The presence of stubborn agents can turn the limit (5.7) into a disagreement of opinions, which is also the case with the Friedkin-Johnsen model. In fact, when $r_{ij} = 1$ for all i and j, the limit (5.11) in Corollary 5.3 will be the same as (2.6).

Another interesting aspect of random group gossiping with stubborn agents is that its time-averaged opinions approaches its expected opinions at time k (5.8).

Theorem 5.4. *Let*

$$\bar{x}(k) = \frac{1}{k+1} \sum_{q=0}^{k} x(q).$$

Then, the dynamics (5.2) is mean-square ergodic, such that

$$\lim_{k \to \infty} \mathbb{E}[\|\bar{x}(k) - x_*\|^2] = 0.$$

Proof. Frasca et al. [36] provided their analysis in order to prove the mean-square ergodicity of their gossip-based implementation of the Friedkin-Johnsen model (2.12). This method was generalized in the work of Ravazzi et al. [42]. Their method is used here for proving the same property applies to the model (5.2).

From (5.1)

$$\min_j x_j(0) \leq x_i(k) \leq \max_j x_j(0). \tag{5.12}$$

Let $e(k) = x(k) - x_*$. Then,

$$\begin{aligned} \bar{x}(k) - x_* &= \frac{1}{k+1} \sum_{q=0}^{k} x(q) - x_* \\ &= \frac{1}{k+1} \sum_{q=0}^{k} e(q). \end{aligned} \tag{5.13}$$

The expected squared Euclidean norm of (5.13) is

$$\mathbb{E}[\|\bar{x}(k) - x_*\|^2] = \mathbb{E}\left[\left\| \frac{1}{k+1} \sum_{q=0}^{k} e(q) \right\|^2\right]. \tag{5.14}$$

Note that

$$\left(\sum_{q=0}^{k} e(q)\right)^2 = \sum_{q=0}^{k} e(q)^T e(q) + 2\sum_{q=0}^{k-1}\sum_{t=1}^{k-q} e(q)^T e(q+t). \tag{5.15}$$

Combining (5.14) and (5.15) results to

$$\mathbb{E}[\|\bar{x}(k) - x_*\|^2] = \frac{1}{(k+1)^2}\mathbb{E}\left[\sum_{q=0}^{k} e(q)^T e(q) + 2\sum_{q=0}^{k-1}\sum_{t=1}^{k-q} e(q)^T e(q+t)\right]. \tag{5.16}$$

Based on (5.12), there is a constant upper bound for $(x(k) - x_*)^T(x(k) - x_*)$ for all k. Let the upper bound be η. Then, from (5.16)

$$\begin{aligned}\mathbb{E}\left[\sum_{q=0}^{k} e(q)^T e(q)\right] &\leq \sum_{q=0}^{k} \eta \\ &\leq \eta(k+1).\end{aligned} \tag{5.17}$$

Also, from (5.16)

$$\begin{aligned}\mathbb{E}[e(q)^T e(q+t)] &= \mathbb{E}[\mathbb{E}[e(q)^T e(q+t)|x(q)]] \\ &= \mathbb{E}[e(q)^T \mathbb{E}[e(q+t)|x(q)]] \\ &= \mathbb{E}[e(q)^T \mathbb{E}[x(q+t) - x_*|x(q)]] \\ &= \mathbb{E}[e(q)^T (\mathbb{E}[x(q+t)|x(q)] - x_*)].\end{aligned} \tag{5.18}$$

By recursively applying (5.3) on $\mathbb{E}[x(q+t)|x(q)]$ until $x(q)$ is reached, the following is obtained

$$\mathbb{E}[x(q+t)|x(q)] = \bar{A}^t x(q) + \sum_{s=0}^{t-1} \bar{A}^s \bar{B} u. \tag{5.19}$$

Using the same principle on x_* yields

$$x_* = \bar{A}^t x_* + \sum_{s=0}^{t-1} \bar{A}^s \bar{B} u. \tag{5.20}$$

Then applying (5.19) and (5.20) on (5.18) produces

$$\begin{aligned}
\mathbb{E}[e(q)^T e(q+t)] &= \mathbb{E}[e(q)^T(\bar{A}^t x(q) - \bar{A}^t x_*)] \\
&= \mathbb{E}[e(q)^T \bar{A}^t (x(q) - x_*)] \\
&= \mathbb{E}[e(q)^T \bar{A}^t e(q)] \\
&\leq \mathbb{E}[e(q)^T \rho^t e(q)] \\
&\leq \eta \rho^t
\end{aligned} \tag{5.21}$$

where ρ is a constant such that $v^T \bar{A} v \leq v^T \rho v$ for any vector v.

Applying (5.21) on (5.16) yields

$$\begin{aligned}
\mathbb{E}[\|\bar{x}(k) - x_*\|^2] &\leq \frac{1}{(k+1)^2}\left(\eta(k+1) + 2\sum_{q=0}^{k-1}\sum_{t=1}^{k-q}\eta\rho^t\right) \\
&\leq \frac{\eta}{(k+1)^2}\left(k+1+2\sum_{q=0}^{k-1}\sum_{t=1}^{k-q}\rho^t\right) \\
&\leq \frac{\eta}{(k+1)^2}\left(1+2\sum_{t=1}^{k}\rho^t\right) \\
&\leq \frac{\eta}{(k+1)^2}\left(1+\frac{2}{1-\rho}\right).
\end{aligned}$$

This completes the proof. □

5.3 Numerical Examples

Consider the social network in Figure 5.1 with 10 agents. Let the values in W be randomly generated such that there are agents i with $w_{ii} > 0$ so that $\gamma_i < 1$, which makes them stubborn agents. R is also randomly generated. Let the initial opinions be

$$x(0) = \begin{bmatrix} 0.6227 & 0.3108 & 0.8258 & 0.6248 & 0.4064 \\ 0.0401 & 0.5294 & 0.2153 & 0.9236 & 0.5654 \end{bmatrix},$$

which are also the prejudices of the agents.

Figure 5.2a shows the behavior of random group gossiping when the prejudices of stubborn agents are taken into consideration. Instead of converging to fixed values, the opinions are continuously fluctuating. The expected dynamics of the model (Figure 5.2b), on the other hand, converged to a disagreement, which is the presumption when there are stubborn agents in a social network. More importantly, the time-averaged opinions (Figure 5.2c) follow the same trajectory as the expected dynamics, which validates the ergodic property of the model stated in Theorem 5.4.

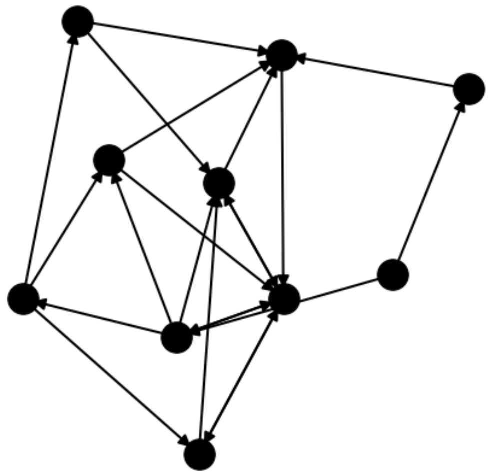
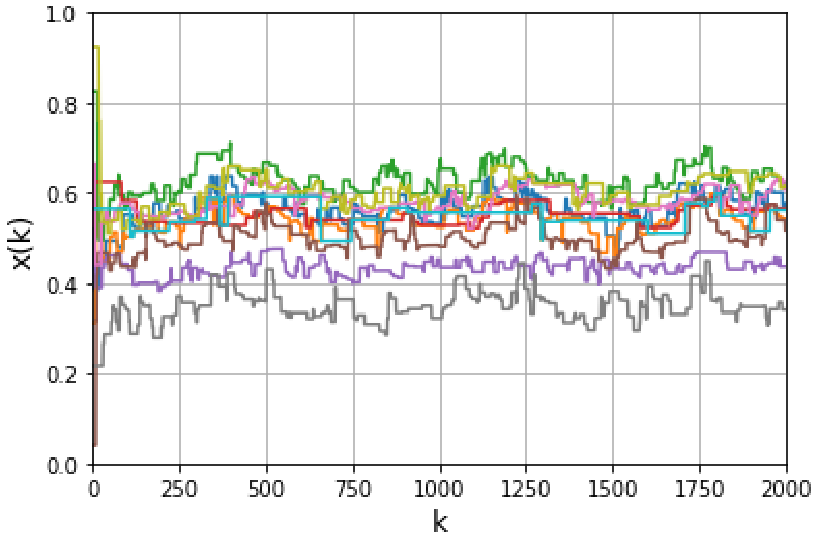


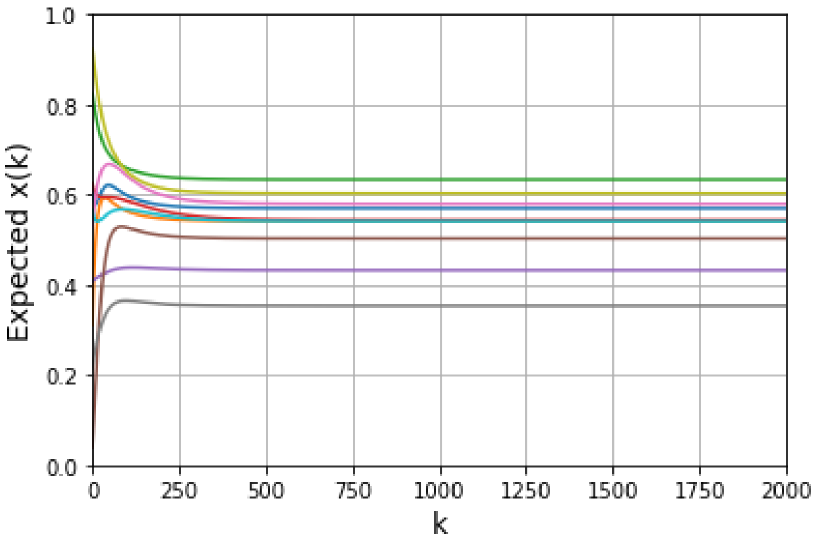
FIGURE 5.1: Social network

5.4 Summary

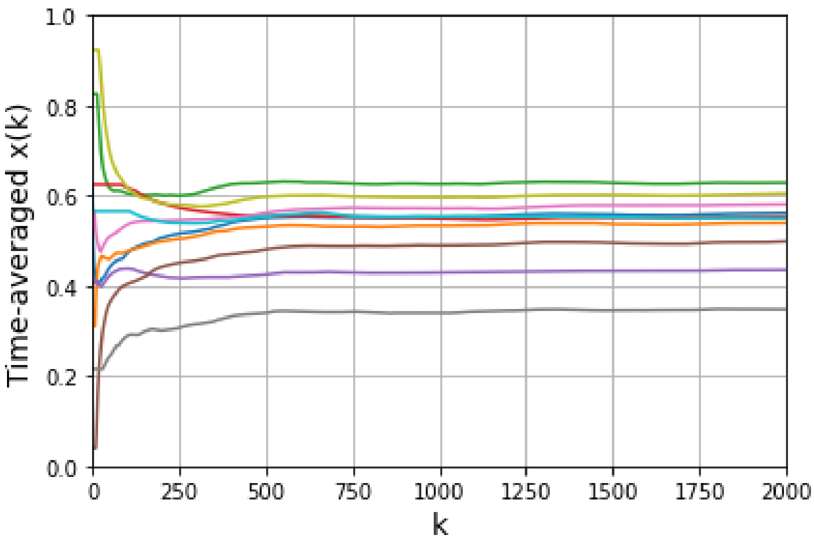
In this chapter, random group gossiping is used for modeling the opinion formation process in social networks with stubborn agents. The resulting model is a time-varying version of the Friedkin-Johnsen model with non-deterministic interactions, which is more appropriate for representing real-world scenarios. While the model is not guaranteed to converge to stable opinions, the analysis in this chapter has shown that its expected dynamics is convergent regardless of the network topology. Furthermore, its expected opinions and its time-averaged opinions approach the same values as the number of iterations increases. This was proven through analysis of its mathematical properties and demonstrated via simulations.



(A) Dynamics



(B) Expected dynamics



(C) Time-averaged opinions

FIGURE 5.2: Resulting dynamics

Chapter 6

Clustering of Opinions via Random Group Gossiping with Bounded Confidence

6.1 Overview

Bounded confidence models involve time-varying dynamics that are determined by the interactions of agents with similar opinions. Two of the most influence works on bounded confidence are the models by Hegselmann and Krause [39], and Deffuant and Weisbuch [40]; both of which have served as the foundation of other similar models.

However, these models disregard relationships in social networks, allowing any pair of individuals to interact as long as they have similar views. In real-world social networks, meaningful interactions are more likely to occur within the same social circles. This idea is addressed by the work of Parasnis et al. [43] which modifies the Hegselmann-Krause model by limiting the interactions to neighbors in the social network graph. The bounded confidence model by Nguyen et al. [44] also restricted the interactions based on relationships while employing a gossip-based approach similar to the Deffuant-Weisbuch model.

In this chapter, random group gossiping is combined with bounded confidence. Similar to the models in [43] and [44], the this model only permits interactions to occur based on existing connections. However, the pairwise gossiping in [40] and [44] is extended by allowing interactions with groups of varying sizes. The proposed model is demonstrated using an actual social network graph data, which resulted to multiple clusters with each cluster representing a set of agents coming to an agreement.

6.2 Random Group Gossiping with Bounded Confidence

Similar to Chapter 6, the model proposed in this chapter uses the same setup in Chapter 5. It also follows a similar process as the previous random group gossiping models. The rules for updating the opinions, however, are modified to incorporate bounded confidence. Let δ denote the confidence threshold for all agents which determines the opinions that they are willing to accept. That is, given an agent i and $j \in \mathcal{N}_i$, the opinion of j can only influence i if and only if $|x_i(k) - x_j(k)| \leq \delta$. Random group gossiping with bounded confidence can thus be described as follows. For each time $k \geq 0$:

1. An agent i is selected with uniform probability from $\mathcal{V}$.
2. The members of $S_i(k)$ are chosen by performing an independent Bernoulli trial $\phi_{ij}(k)$ on each $j \in \mathcal{N}_i$.
3. The opinion of agent i is then updated as

$$x_i(k+1) = \left(1 - \sum_{j \in S_i(k)} \mathbb{1}_{\{|x_i(k)-x_j(k)| \leq \delta\}} w_{ij}\right) x_i(k) + \sum_{j \in S_i(k)} \mathbb{1}_{\{|x_i(k)-x_j(k)| \leq \delta\}} w_{ij} x_j(k) \tag{6.1}$$

   where $\mathbb{1}$ is the indicator function such that $\mathbb{1}_{\{\theta\}} = 1$ if θ is true, otherwise $\mathbb{1}_{\{\theta\}} = 0$. The opinions of the other agents remain unchanged.

Theorem 6.1. *The model above always converges to a limit*

$$\lim_{k \to \infty} x(k) = x^*, \tag{6.2}$$

where either $x_i^* = x_j^*$ or $|x_i^* - x_j^*| > \delta$ for any i and $j \in \mathcal{N}_i$.

Proof. At each $k+1$, either

$$|x_i(k+1) - x_j(k)| \leq |x_i(k) - x_j(k)| \tag{6.3}$$

or

$$|x_i(k+1) - x_j(k)| > |x_i(k) - x_j(k)| \tag{6.4}$$

for any pair of i and $j \in \mathcal{N}_i$. Since the probability of (6.3) and (6.4) occurring are independent based on (6.1), as $k \to \infty$, there will be a sequence involving combinations of (6.3) and (6.4) such that, at a time k_0, either

$$\begin{aligned}|x_i(k_0) - x_j(k_0)| &\leq \delta_0 \leq \delta \\ |x_i(k_0) - x_l(k_0)| &> \delta \\ |x_j(k_0) - x_l(k_0)| &> \delta \\ \forall l &\neq i, j\end{aligned} \tag{6.5}$$

or

$$|x_i(k_0) - x_j(k_0)| > \delta \tag{6.6}$$

for all pairs of i and $j \in \mathcal{N}_i$. Once this scenario is reached, only the neighbors that satisfy (6.5) can influence one another. From hereon,

$$\lim_{k_0 \to \infty} |x_i(k_0) - x_j(k_0)| = 0 \tag{6.7}$$

for all pairs of i and $j \in \mathcal{N}_i$ that satisfy (6.5) since all succeeding recomputations of their opinions will result to (6.3). This completes the proof. □

Despite the randomness of interactions, Theorem 6.1 guarantees that (6.1) will always converge to stable opinions regardless of how the agents are connected in the given social network. The resulting opinion profile corresponds to a consensus or to a clustering of agents based on their final opinions.

Within the context of real-world scenarios, the dynamics (6.1) represents the situations where individuals interact with some people they know and then only consider the opinions that are similar to their views. This kind of behavior is consistent with the concept of homophily which asserts that like-minded individuals tend to associate with one another. Because of this tendency, clusters can be formed within communities, where each cluster represents a shared belief system.

Since opinions in (6.1) can only be given to neighbors in $\mathcal{G}$, there may be pairs of i and $j \notin \mathcal{N}_i$ such that $|x_i^* - x_j^*| \leq \delta$. In reality, such situations do occur since there are other people who may have similar views as us but we have no opportunity to interact with them to further align our views.

6.3 Numerical Examples

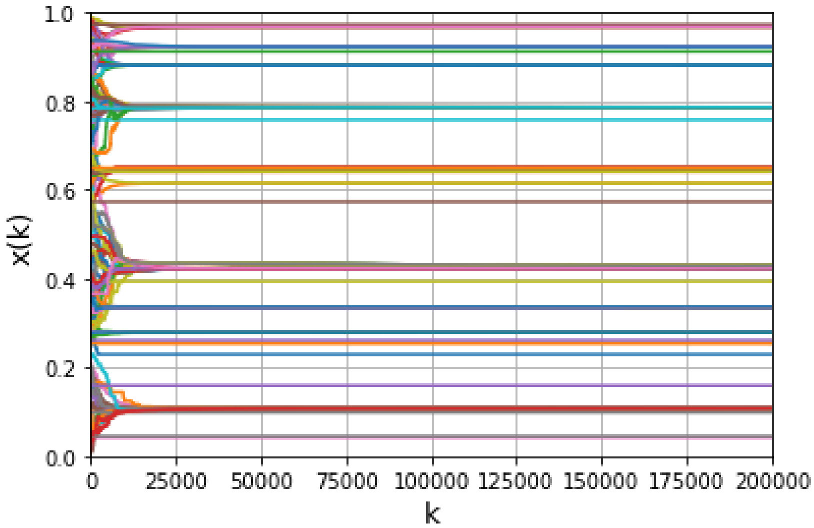
To demonstrate the behavior of the proposed model and to show clustering within social networks based on similarity of views, this section includes numerical examples using an actual social network graph data taken from the Facebook ego dataset of the Stanford Network Analysis Project [45]. This particular network contains 148 nodes that are connected via 1692 edges. The average number of neighbors per node is 22.86. Since the "friend" connection in Facebook is symmetric, the edges in the graph are bidirectional. The dataset does not include information about interpersonal influences and frequency of interactions. Weights and probabilities are randomly assigned to the edges, which correspond to the values in the W and R matrices, respectively. The initial opinions are also randomly generated such that each $x_i(0) \in [0, 1]$. To explore the effects of varying confidence thresholds, three simulations are performed using $\delta = 0.1$, $\delta = 0.2$, and $\delta = 0.5$. The same W, R, and $x(0)$ are used in all simulations.

Figure 6.1 shows the convergence of opinions based on the various confidence thresholds used. The dynamics in Figure 6.1a resulted to the most number of clusters because of its low confidence threshold which heavily restricts the possible interactions of all agents. Figure 6.1b also resulted to multiple clusters reaching a disagreement, but there are significantly less clusters compared to the previous example. The use of a higher confidence threshold in Figure 6.1c led to consensus since it enabled agents to interact with most of their neighbors.

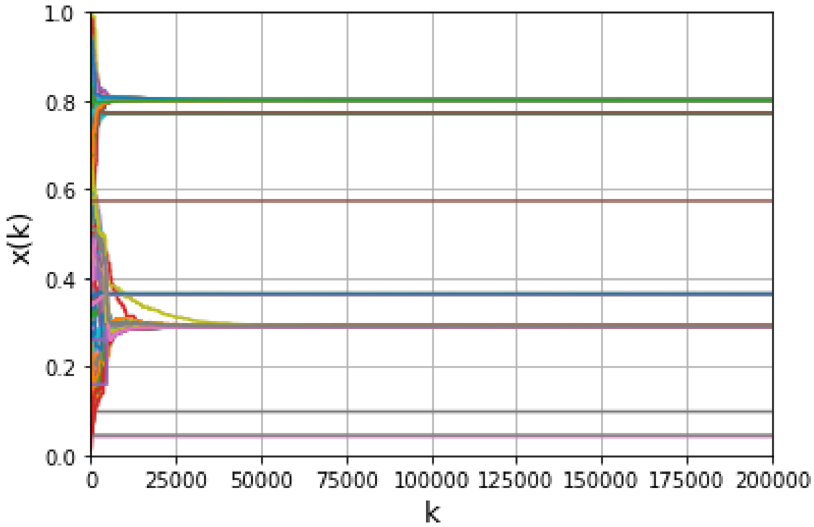
Figure 6.2 shows the resulting clusters in the given social network. Note that the clusters are not based on the location of agents in the network, but on the proximity of their opinions. Figure 6.2 exhibits how views can spread out within a social network even among agents that are not directly connected with one another. Different trials may yield different clusters because the model is non-deterministic. But in general, smaller confidence thresholds produce more clusters, while larger confidence thresholds result to less clusters or may even result to a consensus.

6.4 Summary

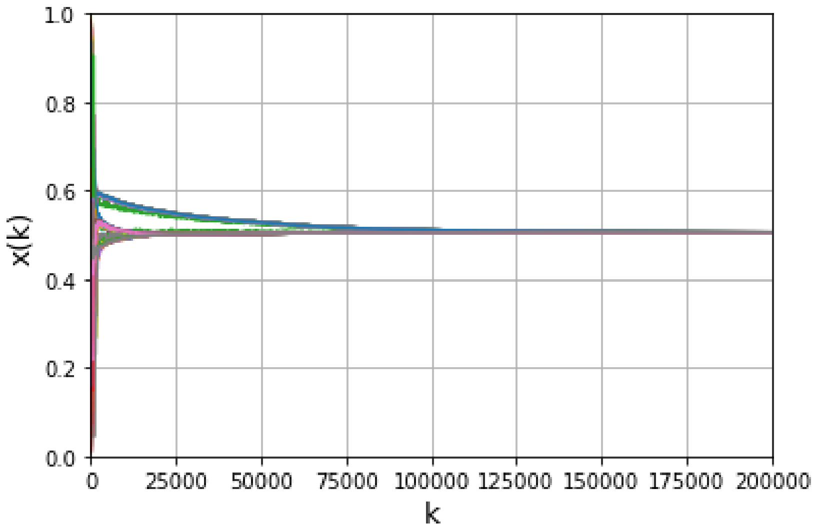
This chapter presented an alternative approach to bounded confidence models for opinion dynamics by using random group gossiping. Unlike the popular bounded confidence models and their variants, the model (6.1) limits interactions only to neighboring agents. Also, compared to other gossip-based bounded confidence models, the proposed model allows interactions with groups of varying sizes. The model is applied to an actual social network graph data, which demonstrated how opinions can propagate within the network and showed how clustering of opinions develop by adjusting the range of opinions that agents are willing to accept.



(A) $\delta = 0.1$

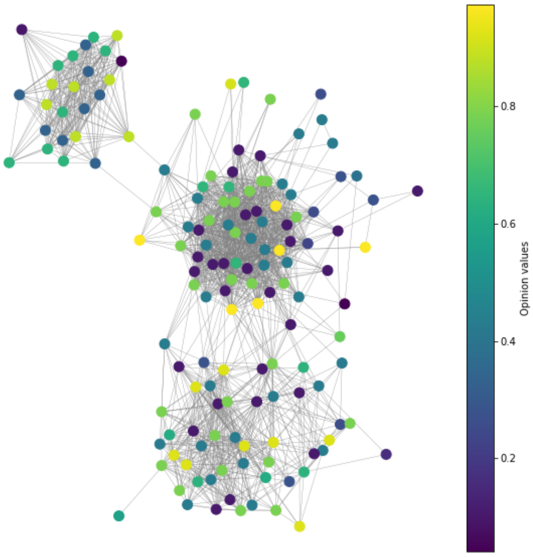


(B) $\delta = 0.2$

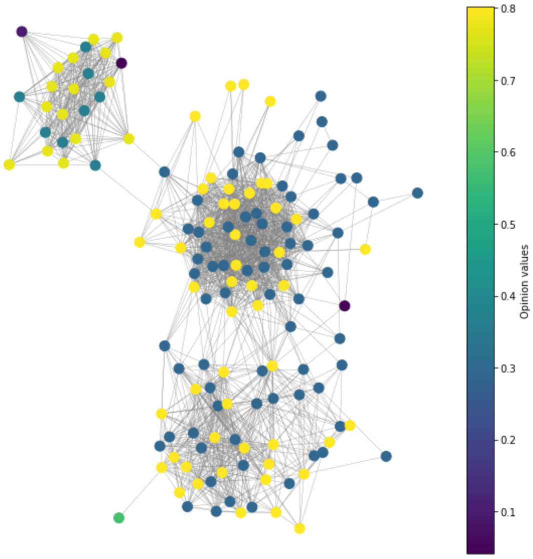


(C) $\delta = 0.5$

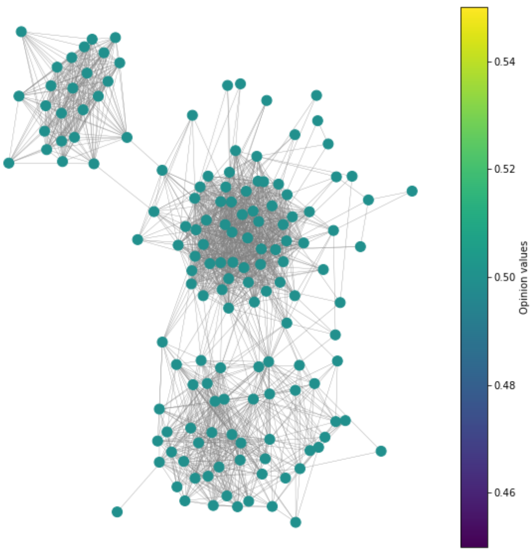
FIGURE 6.1: Convergence of opinions leading to clusters



(A) $\delta = 0.1$



(B) $\delta = 0.2$



(C) $\delta = 0.5$

FIGURE 6.2: Clusters formed in the social network graph

Chapter 7

Reaching Consensus via Coordinated Groups

7.1 Overview

Opinion dynamics models are often concerned with describing the behavior of an entire network based on the characteristics of individual agents and the interactions that may take place. However, important patterns may also be exhibited by smaller groups within a social network. Exploring group dynamics is especially relevant during this age when it is easier to connect with other people, leading to the spread of communities with shared ideologies that shape different aspects of our societies.

One way to explore group behaviors is through a bottom up approach where there are no predefined groupings, but they may later surface based on the dynamics of the model employed. Examples that follow this principle are bounded confidence models, such as the model proposed in the previous chapter. The clusters formed by these models depend on the range of opinions that agents are willing to accept. An alternative to this approach is to design a model with an assumption that there are already explicit groups in place before observing the resulting dynamics [46, 47]. This enables the analysis of how intergroup dynamics affect an entire social network.

In this chapter, a novel model is proposed for the opinion dynamics of social networks with intragroup and intergroup interactions. The model, which also employs a gossiping scheme, enables multiple groups to reach consensus via the presence of coordinators that act as links between groups. Analysis of its expected dynamics shows that the number of groups and the frequency of interactions may affect the convergence behavior of the groups in the network.

7.2 Gossip-Based Intergroup Model

The social network $\mathcal{G}$ considered for this model is bidirectional since it is concerned with changes in opinions brought about by exchanges of views. Every agent in the network belongs to a group $\mathcal{G}^s = (\mathcal{V}^s, \mathcal{E}^s)$, where $\mathcal{V}^s \subseteq \mathcal{V}$ and $\mathcal{E}^s \subseteq \mathcal{E}$. Each $\mathcal{G}^s$ should be a strongly connected subgraph, otherwise it is decomposed into multiple groups. However, groupings may be selected arbitrarily. The groups are labeled as $s = 1, 2, \ldots, m$, where $m \geq 2$ to avoid trivialities. While in reality, individuals may belong in multiple groups, the scope of this chapter is restricted to cases when agents belong in exactly one group only. Fig. 7.1 shows a social network composed of interconnected groups.

In this model, members of a group are classified as either a coordinator or a follower. A coordinator is an agent $i \in \mathcal{V}^C \subseteq \mathcal{V}$ that is connected by an edge to a member of another group. All the remaining agents are followers. It is assumed that each group contains at least one coordinator and there is a path between any pair of coordinators, thus the graph induced by the set of coordinators, $\mathcal{G}^C = (\mathcal{V}^C, \mathcal{E}^C)$, is also strongly connected. Additionally, every coordinator should have at least one follower neighbor, that is $|\mathcal{N}_i^s \setminus \mathcal{V}^C| > 0$. The total number of coordinators is given by $c = |\mathcal{V}^C|$. The set $\mathcal{N}_i^s \subseteq \mathcal{N}_i$ denote the neighbors of i on the same group and $\mathcal{N}_i^C \subseteq \mathcal{N}_i$ are the neighbors of i that are coordinators.

The intergroup opinion dynamics model can be described as follows. Let $q \in (0, 1)$ be the weight given to the opinion of other agents, which is similar to the variable a in 2.10. Let $\rho \in (0, 1)$ be the frequency which agents interact with members from other groups. At each time $k \geq 0$:

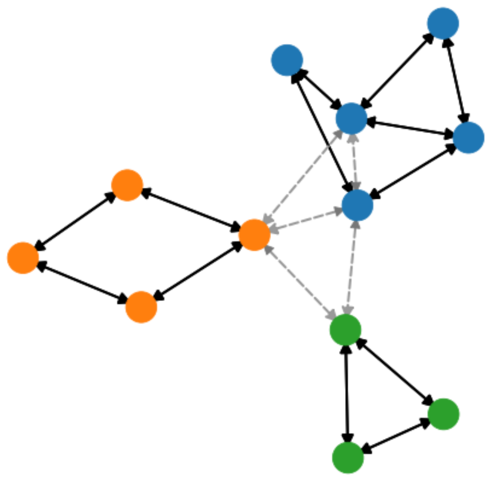


FIGURE 7.1: Social network with three groups containing five, four, and three members. Dashed edges connect the coordinators.

1. Agent i is chosen with uniform probability from $\mathcal{V}$. The group of i is $\mathcal{G}^s$, where $i \in \mathcal{V}^s$.

2. If $i \notin \mathcal{V}^C$, agent $j \in \mathcal{N}_i^s$ is chosen with probability $1/|\mathcal{N}_i^s|$. If $i \in \mathcal{V}^C$, agent $j \in \mathcal{N}_i^s$ is chosen with probability $(1-\rho)/|\mathcal{N}_i^s|$, while agent $j \in \mathcal{N}_i^C$ is chosen with probability $\rho/|\mathcal{N}_i^C|$.

3. The opinion of i and j are updated as

$$x_i(k+1) = (1-q)x_i(k) + qx_j(k) \tag{7.1a}$$

$$x_j(k+1) = qx_i(k) + (1-q)x_j(k), \tag{7.1b}$$

   while others remain the same.

The process above represents a random sequence of exchanges between a pair of coordinators, a pair of followers, or a coordinator-follower pair. The exchange of opinions in 7.1 implements the same scheme from 2.8. Coordinators make it possible for different groups to influence one another. The frequency of interactions between different groups may increase or decrease depending on the value of ρ.

Similar to the other models introduced in this dissertation, the model for intergroup interactions can also be compactly expressed as linear discrete-time system by introducing a time-varying matrix. Let $M^{ij} \in \mathbb{R}^{n\times n}$ be the random matrix based on the selected pair of agents i and j at time k, and defined as

$$M^{ij} = I - q(e_i - e_j)(e_i - e_j)^T.$$

The proposed model can then be written as

$$x(k+1) = M(k)x(k) \tag{7.2}$$

where $M(k) = M^{ij}$. Since the communication between agents occur independently based on a fixed probability distribution, then the sequence $\{M(k)\}_{k\geq 0}$ is i.i.d. Additionally, $M^{ij}\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^T M^{ij} = \mathbf{1}^T$ for any $i, j \in \mathcal{V}$, hence M^{ij} is always doubly stochastic. From here, the first result can be stated.

Theorem 7.1. *The model (7.2) converges almost surely to the limit*

$$\lim_{k\to\infty} x(k) = x^*.$$

where

$$x^* = \frac{1}{n}\mathbf{1}\mathbf{1}^T x(0). \tag{7.3}$$

Proof. The sequence $\{M(k)\}_{k\geq 0}$ is i.i.d, where each $M(k)$ is a doubly stochastic matrix with a positive diagonal. Since $0 < \rho < 1$, for any pair $i, j \in \mathcal{V}$, $\mathbb{P}[M(k) = M^{ij}] > 0$, which also implies $\mathbb{E}[M_{ij}(k)] > 0$. Note that $\mathcal{G}$ is strongly connected, thus $\mathbb{E}[M(k)]$ is irreducible. Additionally, $M(k)$ is doubly stochastic for any k, therefore $\mathbb{E}[M(k)]$ is also doubly stochastic. By the Perron-Frobenius theorem, $\lambda_{n-1}(\mathbb{E}[M(k)]) < \lambda_n(\mathbb{E}[M(k)]) = 1$. Given the previous statements, the limit (7.3) can be reached based on the Theorem 3 in [41]. □

Theorem 7.1 shows that the model (7.2) can be interpreted as multiple groups coming to an agreement through the presence of coordinators. In order to further understand the effects of the frequency of interactions among groups, the expected dynamics of the model (7.2) must be analyzed. This first done through the following lemma.

Lemma 7.2. *Let $\bar{x}(k) = \mathbb{E}[x(k)]$ and $\bar{M} = \mathbb{E}[M(k)]$ which is given by*

$$\bar{M} = (1-\alpha)W + \alpha H,$$

where

$$\begin{aligned} W &= \frac{1}{n-\rho c}\left(\sum_{i\in\mathcal{V}}\frac{1}{|\mathcal{N}_i^s|}\sum_{j\in\mathcal{N}_i^s}M^{ij} - \rho\sum_{i\in\mathcal{V}^C}\frac{1}{|\mathcal{N}_i^s|}\sum_{j\in\mathcal{N}_i^s}M^{ij}\right) \\ H &= \frac{1}{c}\sum_{i\in\mathcal{V}^C}\frac{1}{|\mathcal{N}_i^C|}\sum_{j\in\mathcal{N}_i^C}M^{ij} \\ \alpha &= \frac{\rho c}{n}, \end{aligned}$$

where W and H are doubly stochastic matrices. Then, the expected dynamics of the model (7.2) is

$$\bar{x}(k+1) = \bar{M}\bar{x}(k). \tag{7.4}$$

Proof. The proof of this lemma can be obtained by direct computation

$$\begin{aligned}
\mathbb{E}[M(k)] &= \frac{1}{n} \sum_{i \in \mathcal{V},\ i \notin \mathcal{V}^C} \frac{1}{|\mathcal{N}_i^s|} \sum_{j \in \mathcal{N}_i^s} M^{ij} \\
&\quad + \frac{1}{n} \sum_{i \in \mathcal{V}^C} \left(\frac{1-\rho}{|\mathcal{N}_i^s|} \sum_{j \in \mathcal{N}_i^s} M^{ij} + \frac{\rho}{|\mathcal{N}_i^C|} \sum_{j \in \mathcal{N}_i^C} M^{ij} \right) \\
&= \frac{1}{n} \sum_{i \in \mathcal{V},\ i \notin \mathcal{V}^C} \frac{1}{|\mathcal{N}_i^s|} \sum_{j \in \mathcal{N}_i^s} M^{ij} + \frac{1}{n} \sum_{i \in \mathcal{V}^C} \frac{1}{|\mathcal{N}_i^s|} \sum_{j \in \mathcal{N}_i^s} M^{ij} \\
&\quad - \frac{\rho}{n} \sum_{i \in \mathcal{V}^C} \frac{1}{|\mathcal{N}_i^s|} \sum_{j \in \mathcal{N}_i^s} M^{ij} + \frac{\rho}{n} \sum_{i \in \mathcal{V}^C} \frac{1}{|\mathcal{N}_i^C|} \sum_{j \in \mathcal{N}_i^C} M^{ij} \\
&= \frac{1}{n} \sum_{i \in \mathcal{V}} \frac{1}{|\mathcal{N}_i^s|} \sum_{j \in \mathcal{N}_i^s} M^{ij} - \frac{\rho}{n} \sum_{i \in \mathcal{V}^C} \frac{1}{|\mathcal{N}_i^s|} \sum_{j \in \mathcal{N}_i^s} M^{ij} \\
&\quad + \frac{\rho}{n} \sum_{i \in \mathcal{V}^C} \frac{1}{|\mathcal{N}_i^C|} \sum_{j \in \mathcal{N}_i^C} M^{ij}.
\end{aligned}$$

If α is defined as in the lemma, the third term becomes αH, where the non-negative matrix H satisfies $H\mathbf{1} = \mathbf{1}$ and $\mathbf{1}^T H = \mathbf{1}^T$, which means H is doubly stochastic. The first term

$$\frac{1}{n} \sum_{i \in \mathcal{V}} \frac{1}{|\mathcal{N}_i^s|} \sum_{j \in \mathcal{N}_i^s} M^{ij} = W_s$$

is doubly stochastic, and the second term can be rewritten as

$$\frac{\rho}{n} \sum_{i \in \mathcal{V}^C} \frac{1}{|\mathcal{N}_i^s|} \sum_{j \in \mathcal{N}_i^s} M^{ij} = \alpha \left(\frac{1}{c} \sum_{i \in \mathcal{V}^C} \frac{1}{|\mathcal{N}_i^s|} \sum_{j \in \mathcal{N}_i^s} M^{ij} \right) = \alpha W_C,$$

where W_C is also doubly stochastic. This means that $(W_s - \alpha W_C)\mathbf{1} = (1-\alpha)\mathbf{1}$ and $\mathbf{1}^T(W_s - \alpha W_C) = (1-\alpha)\mathbf{1}^T$ hold true. Therefore, $W_s - \alpha W_C$ can be represented as $(1-\alpha)W$, where W should be a doubly stochastic matrix. This completes the proof. □

The dynamics (7.4) can be alternatively be viewed as a deterministic version of the model (7.2) which describes the evolution of the opinions in a social network based on an alternating sequence of intragroup and intergroup interactions. The matrix W defines the interpersonal influence within members of the same group, which acts similarly as the W matrix in the previous models but on a group level. The matrix H, on the other hand, specifies how much the coordinators influence each other and enables opinions to spread across groups.

The parameter α can be seen as a weight that determines the amount of outside influence that a group is willing to accept per iteration. It can also be viewed as the frequency of interactions among coordinators, while $1-\alpha$ is the frequency of intragroup discussions. Since α is proportional to c, the number of coordinators is a significant factor that affects the impact of groups on one another during each round of interactions.

Corollary 7.3. *The expected dynamics (7.4) converges to the limit*

$$\lim_{k\to\infty} \bar{x}(k) = \bar{x}_*$$

where

$$\bar{x}_* = \frac{1}{n}\mathbf{1}\mathbf{1}^T x(0). \tag{7.5}$$

Since the same limit (7.3) is always achieved in Theorem 7.1, Corollary 7.3 can be directly obtained from (7.4). While (7.4) have the same limit as (7.2), it can be used to analyze the eigenvalues of $\bar{M}$. This can give a better understanding of how the various groups in $\mathcal{G}$ and the frequency of interactions among coordinators can affect the way groups converge to the consensus.

Theorem 7.4. *Suppose that the number of groups and the number of coordinators satisfy the inequality $m \le n-c+1$. Then*

$$\max(1-\alpha,\alpha) \le \lambda_i(\bar{M}) \le 1 \qquad\qquad i = n-m+1,\ldots,n.$$

Proof. Since W, H, and $\bar{M}$, are stochastic, their dominant eigenvalues are equal to 1. Note that W can be rearranged as

$$W = \begin{bmatrix} W^1 & 0 & \cdots & 0 \\ 0 & W^2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & W^m \end{bmatrix}$$

where each W^s corresponds to agents of the same group, and H can be rearranged as

$$H = \begin{bmatrix} H^C & 0 \\ 0 & I \end{bmatrix}$$

where H^C contains only the agents in $\mathcal{V}^C$. It can be directly inferred that W and H have m and $n-c+1$ dominant eigenvalues, respectively. Based on the eigenvalue inequalities of the sum of two Hermitian matrices stated in Theorem 4.3.1 of [48],

$$\lambda_{i-j+1}((1-\alpha)W) + \lambda_j(\alpha H) \le \lambda_i(\bar{M}) \le 1 \qquad\qquad j = 1\ldots i$$

for $i = 1 \ldots n$. From here, Theorem 7.4 can be obtained. $\square$

The theorem above implies that, when α is sufficiently small, the expected dynamics (7.4) has m eigenvalues that are close or equal to 1. This means that the number of groups is an important indicator for describing how the opinions of the interconnected groups converge to the consensus.

A special case of the model (7.2) is discussed in [49] where the groups are cliques, meaning all the members are close to one another so each group is given by a complete subgraph.

7.3 Numerical Examples

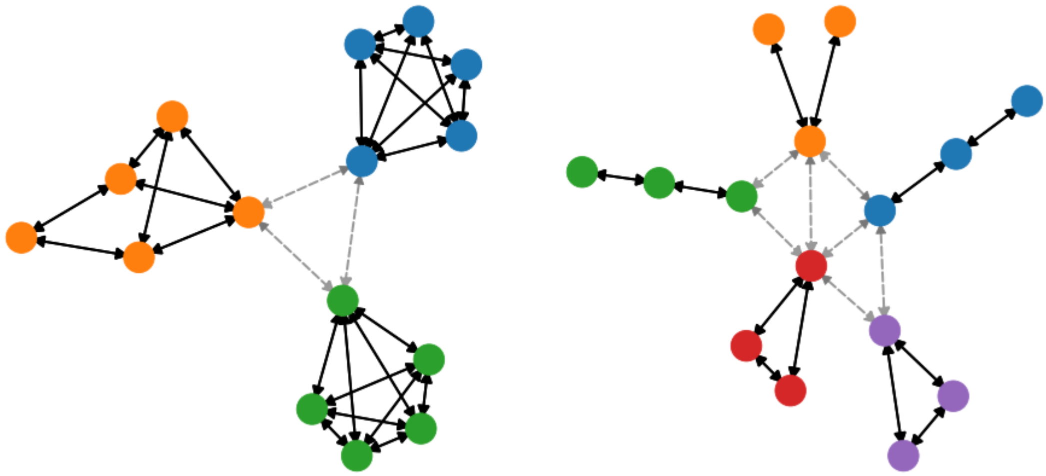
Consider a collection of 15 agents, whose initial opinions are given by

$$x(0) = \begin{bmatrix} 0.1937 & 0.4754 & 0.7152 & 0.2560 & 0.6766 \\ 0.7251 & 0.7382 & 0.1089 & 0.1012 & 0.8487 \\ 0.2705 & 0.6122 & 0.1337 & 0.3151 & 0.5430 \end{bmatrix}.$$

The agents are organized into two different social networks, one with three groups containing five members each (Figure 7.2a) and the other with five groups containing three members each (Figure 7.2b). All groups in both networks contain one coordinator. For the opinion updates, $q = 0.6$.

Figure 7.3 shows the resulting dynamics of the model on both networks when $\rho = 0.30$. In both examples, the opinions converge towards to the limit (7.3) that is the average of the initial opinions, which confirms Theorem (7.1).

To demonstrate the effect of interaction frequencies with other groups, the expected dynamics (7.4) is applied on the two given social networks using $\rho = 0.15$, $rho = 0.30$, and $rho = 0.45$. These correspond to α values of 0.03, 0.06, and 0.09 for the social network with three groups, and 0.05, 0.10, and 0.15 for the social network with five groups. All of the results, shown in Figure 7.4 and 7.5, converge to the same consensus but with varying rates and patterns. In the cases where α is small, it is easier to see the evolution of opinions for each group since these opinions already approach one another as they slowly reach the consensus. This illustrates the implication of Theorem 7.4, where the m largest eigenvalues can signify the convergence behavior of each group's



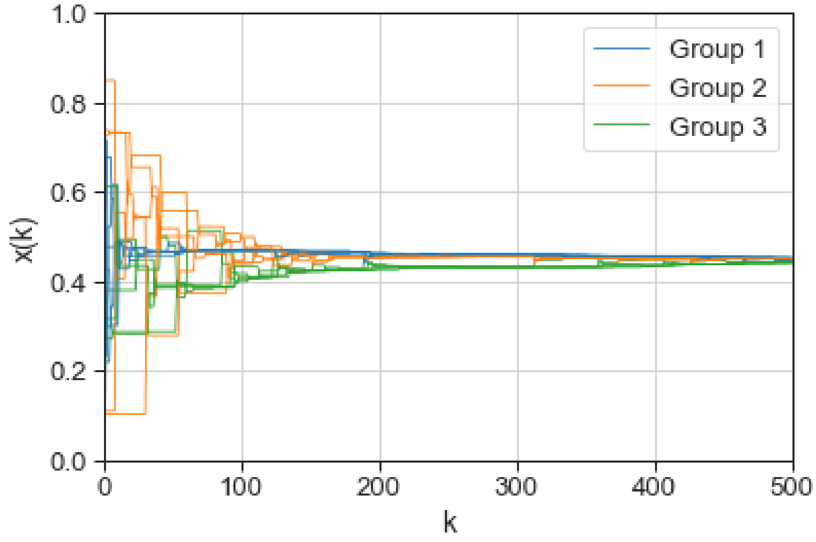
(A) $m = 3$ (B) $m = 5$

FIGURE 7.2: Social networks with interconnected groups

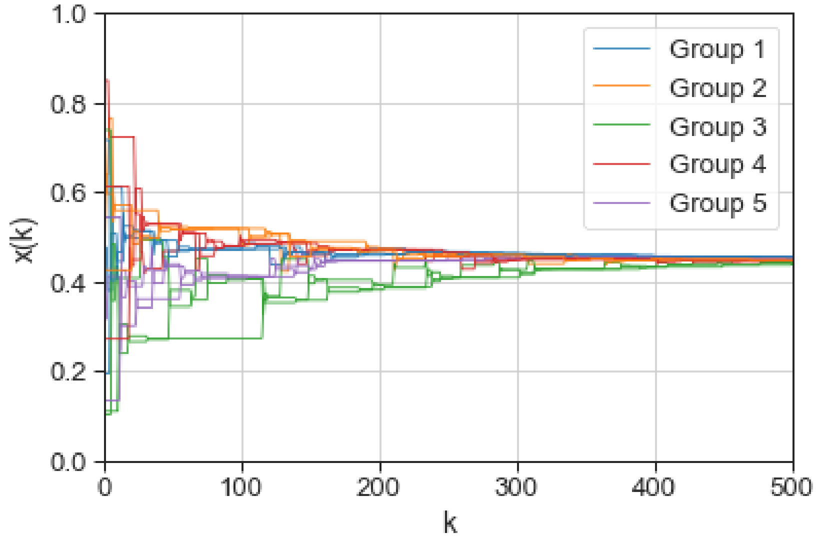
opinions when α is significantly small. The corresponding eigenvalues of the given social networks are shown in Figure 7.6, which emphasizes the characterization of the m largest eigenvalues described in Theorem 7.4.

7.4 Summary

This chapter proposed a gossip-based model for achieving consensus in a social network composed of interconnected groups. The members of each group are classified as either coordinators or followers. The presence of coordinators enables groups to exchange opinions, making it possible for the entire network to reach a consensus. The model can be seen as a combination of intergroup and intragroup opinion dynamics. Aside from allow groups to reach a consensus, analysis have shown that the number of groups and the frequency of interactions between their members can influence the behavior by which the group's opinions converge to the consensus.

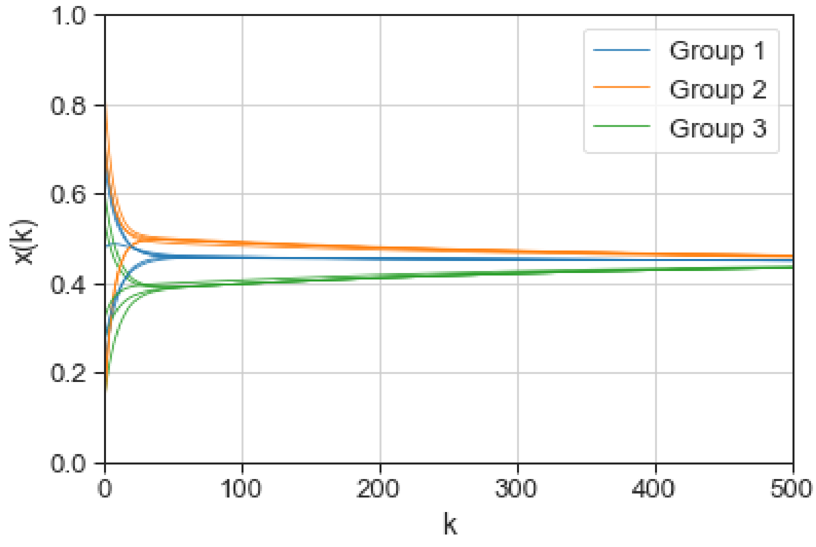


(A) $m = 3$, $\rho = 0.30$

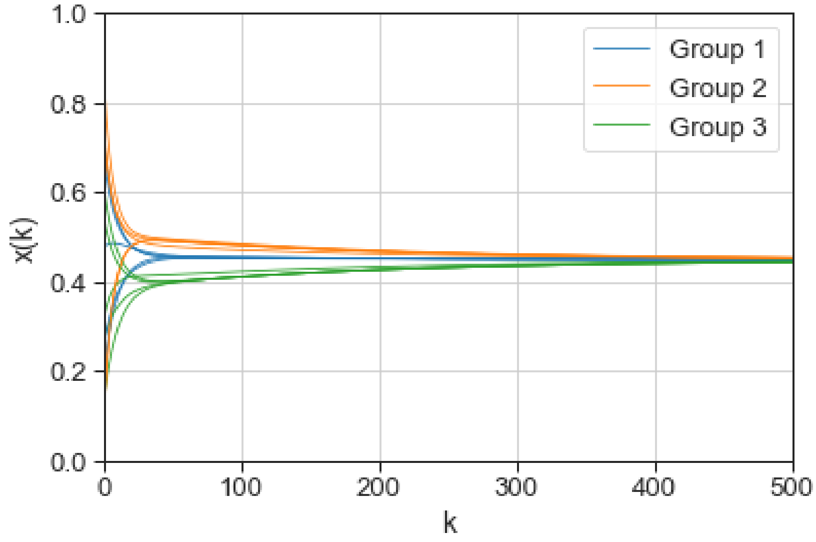


(B) $m = 5$, $\rho = 0.30$

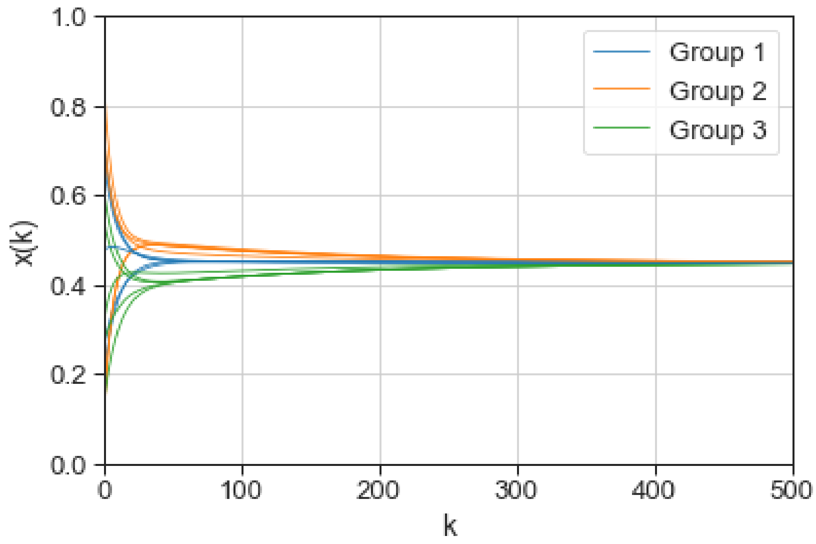
FIGURE 7.3: Resulting dynamics



(A) $\alpha = 0.03$

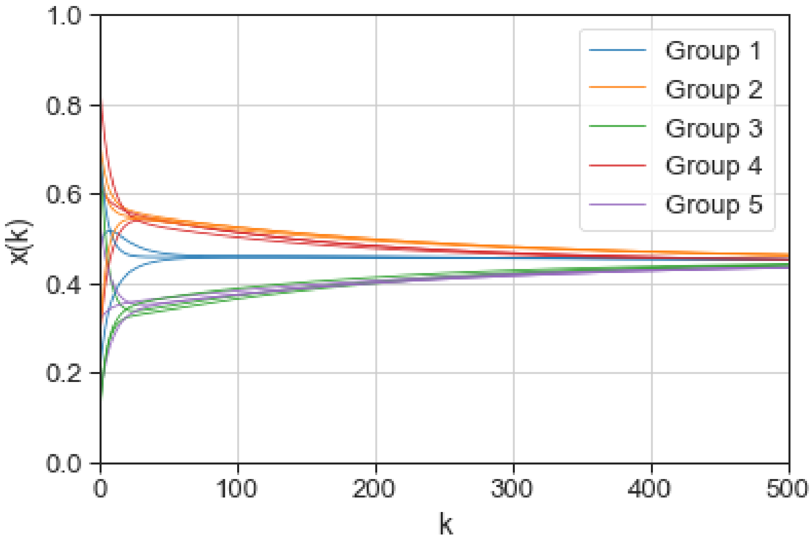


(B) $\alpha = 0.06$

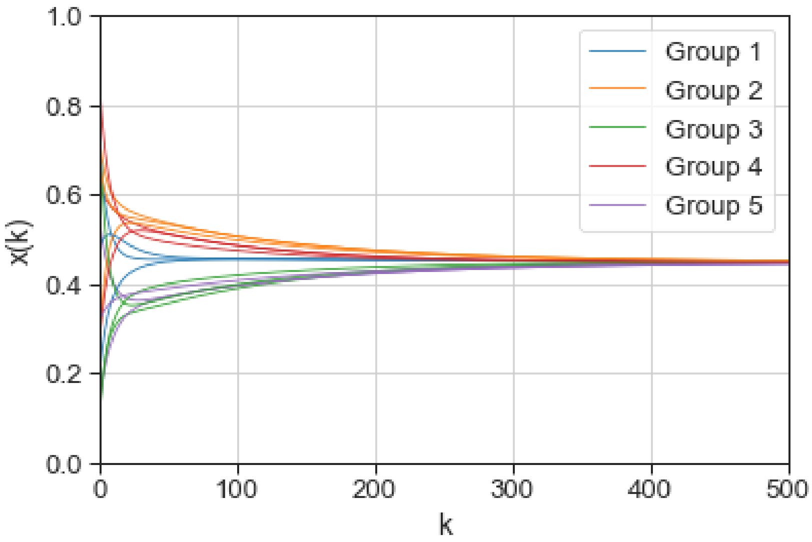


(C) $\alpha = 0.09$

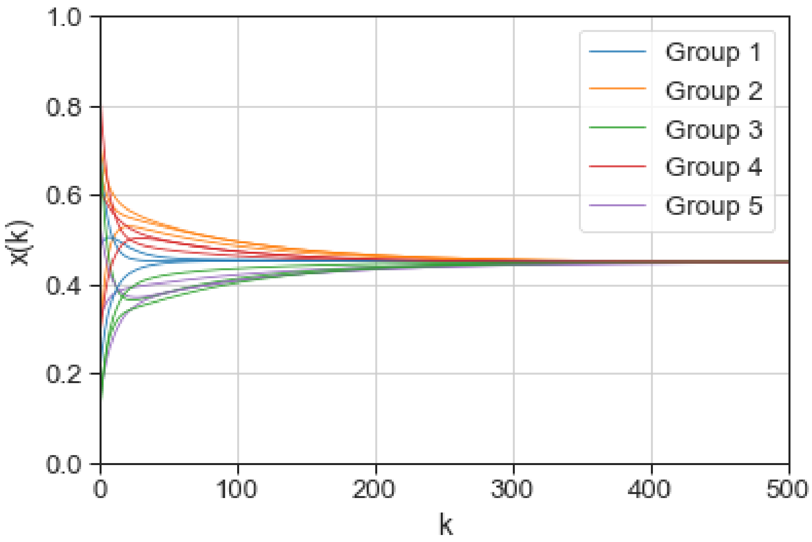
FIGURE 7.4: Expected dynamics for the social network with $m = 3$



(A) $\alpha = 0.05$



(B) $\alpha = 0.10$



(C) $\alpha = 0.15$

FIGURE 7.5: Expected dynamics for the social network with $m = 5$

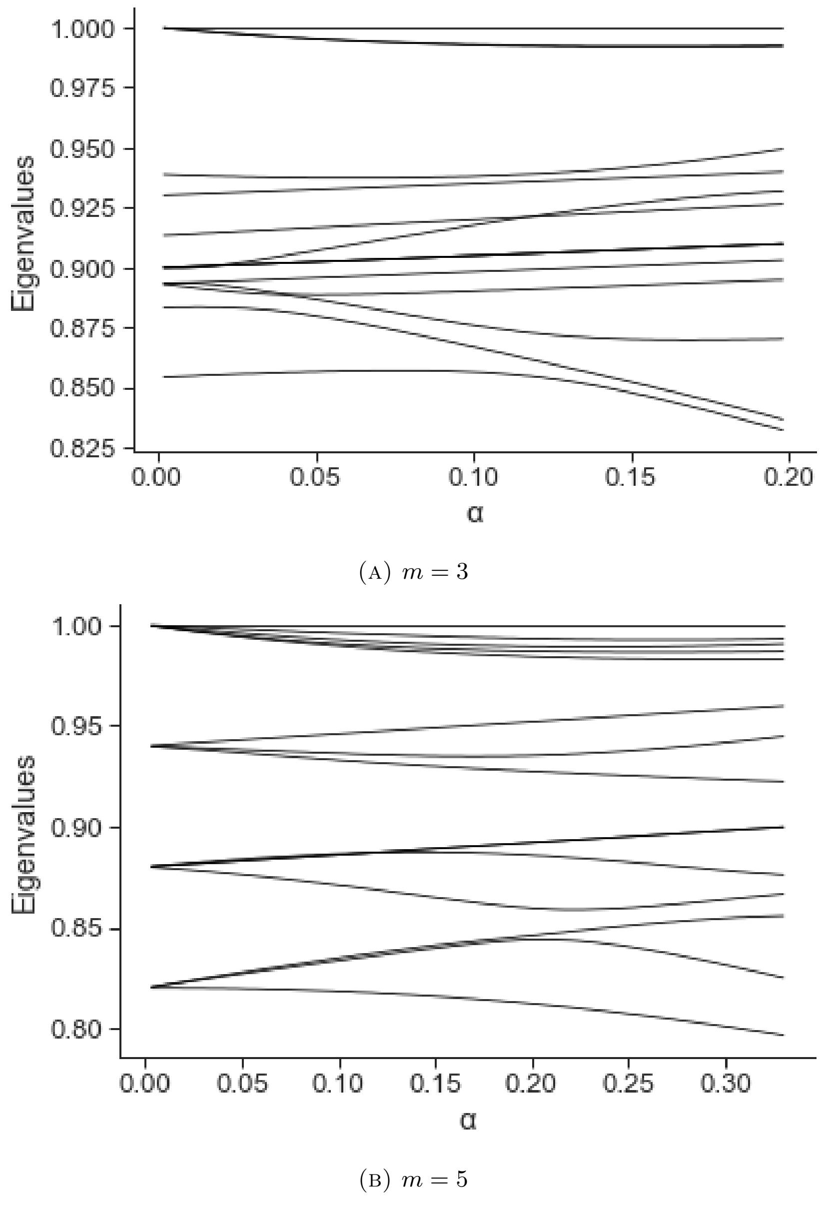


FIGURE 7.6: Eigenvalues of the given social networks

Chapter 8

Conclusion

8.1 Summary

Modeling the opinion formation of social networks is a unique challenge in complex systems. Compared to engineered systems, human behavior is hard to predict. Unlike other problems in natural systems, whose properties can be accurately measured, opinions are difficult to quantify. Despite these challenges, research on opinion dynamics is an important endeavor since it can give us a better understanding of how our interactions shape us individually and our society as a whole. And as we expand our capacity to communicate and connect with others, the need to recognize how these affects us becomes increasingly relevant.

Previous works on opinion dynamics enabled us to see how opinions propagate in social networks from a broader perspective. While the representation of social actors and their relationships are quite simple, such as in the case of agent-based models, this approach makes it possible to test different scenarios that are difficult, if not impossible, to duplicate in real-world settings. Now, there exist various models that demonstrate how agreement can be reached or how individuals can have polarized views by using processes that follow a similar intuition as how we exchange opinions. However, given the tricky nature of capturing human behavior, there is always room for improvement in these models.

In these dissertation, some of the limitations in earlier opinion dynamics models are addressed. In particular, the objective is to alter the way interactions are represented in these models in order to have a closer depiction of how information and sentiments are conveyed in real-world settings. This was achieved by adapting and expanding the gossip approach used by other models. The result is a set of models that can be used for depicting various scenarios.

Group gossiping portrays interactions in settings we normally encounter the same group of people. While less flexible compared random group gossiping, it has shown that having multiple sources of opinions affects the way social networks converge to a consensus. Specifically, the evolution of opinions in group gossiping is more gradual compared to pairwise gossiping.

Random group gossiping employs a more generalized scheme which permits varying opinion sources every time an agent updates its opinion. While the steps involved in pairwise gossiping are simpler and faster to execute, this approach depicts how humans interact in day-to-day situations i.e. we talk to different people, whether individuals or groups, at different times. When applied to the DeGroot model, consensus can be achieved in the probabilistic sense depending on the structure of the network. Additionally, increasing the frequency of interactions between agents may help in attaining consensus at a faster rate.

In the case of social networks with stubborn agents, random group gossiping results to opinions that tend to oscillate. However, its expected dynamics and its time-averaged opinions approach the same limit in the mean-square sense. The observation of this result as a product of random group gossiping provides a new insight on the effects of stubborn agents, especially on how they can cause disagreements.

The combination of bounded confidence and random group gossiping converges to an opinion profile that reflects the clustering of individuals based on the similarity of their opinions. Although this behavior is similar with other bounded confidence models, the proposed model achieved this using a more realistic interaction scheme.

The gossip-based integroup model introduced in the previous chapter allows analysis of opinion dynamics at a group level. Aside from describing a process for the groups to reach consensus, it has been shown that number of groups and the frequency of interactions between members of different groups can affect the way their opinions converge to the consensus. Unlike the other models proposed in this study, this model uses pairwise gossiping. However, it can be extended by applying the schemes used in the other proposed models.

While there are several opinion formation processes not covered by the five models proposed in this dissertation, they addressed some of the most important themes in opinions dynamics, namely: consensus, disagreement, stubborn agents, bounded confidence, and randomized dynamics. Additionally, this dissertation has shown that the convergence behavior of previous models can still be attained by applying the novel interaction schemes described in this study.

8.2 Future Works

The proposed models can still be modified to produce more realistic interactions. Random group gossiping, in particular, can be extended such that there are multiple random agents whose opinions will be updated each turn. For the intergroup model, the effects of using more complex gossiping schemes can be explored. Additional analyses can also be performed to further characterize the convergence properties of the proposed models.

Bibliography

[1] A. V. Proskurnikov and R. Tempo. A tutorial on modeling and analysis of dynamic social networks. part i. *Annual Reviews in Control*, 43:65–79, 2017.

[2] A. V. Proskurnikov and R. Tempo. A tutorial on modeling and analysis of dynamic social networks. part ii. *Annual Reviews in Control*, 45:166–190, 2018.

[3] A.V. Proskurnikov, C. Ravazzi, and F. Dabbene. Dynamics and structure of social networks from a systems and control viewpoint: A survey of roberto tempo's contributions. *Online Social Networks and Media*, 7:45–59, 2018.

[4] J.R.P. French Jr. A formal theory of social power. *Psychological Review*, 63(3): 181–194, 1956.

[5] M. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69(345):118–121, 1974.

[6] K. Lehrer. Social consensus and rational agnoiology. *Synthese*, 31:141–160, 1975.

[7] L. Mastroeni, P. Vellucci, and M. Naldi. Agent-based models for opinion formation: A bibliographic survey. *IEEE Access*, 7(3):58836–58848, 2019.

[8] A. Sîrbu, V. Loreto, V.D. Servedio, and F. Tria. Opinion dynamics: models, extensions and external effects. *Participatory sensing, opinions and collective awareness*, pages 363–401, 2017.

[9] N.E. Friedkin and E.C. Johnsen. Social influence networks and opinion change. *Advances in Group Processes*, 16:1–29, 1999.

[10] R. Tempo, G. Calafiore, and F. Dabbene. *Randomized Algorithms for Analysis and Control of Uncertain Systems*. Springer-Verlag, 2005.

[11] P. Frasca, H. Ishii, C. Ravazzi, and R. Tempo. Distributed randomized algorithms for opinion formation, centrality computation and power systems estimation: A tutorial overview. *European Journal of Control*, 24:2–13, 2015.

[12] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah. Randomized gossip algorithms. *IEEE Transactions on Information*, 52(6):2508–2530, 2006.

[13] B. Anderson and M. Ye. Recent advances in the modelling and analysis of opinion dynamics on influence networks. *International Journal of Automation and Computing*, 16(2):129–149, 2019.

[14] H. Xia, H. Wang, and Z. Xuan. Opinion dynamics: a multidisciplinary review and perspective on future research. *International Journal of Knowledge and Systems Science*, 2(4):72–91, 2011.

[15] M. Conti and A. Passarella. The internet of people: A human and data-centric paradigm for the next generation internet. *Computer Communications*, 131:51–65, 2018.

[16] E. Aguilar and Y. Fujisaki. Opinion dynamics via a gossip algorithm with asynchronous group interactions. *Proceedings of the 50th ISCIE International Symposium on Stochastic Systems Theory and Its Applications*, pages 99–102, 2019.

[17] E. Aguilar and Y. Fujisaki. Gossip-based model for opinion dynamics with probabilistic group interactions. *Proceedings of the 51st ISCIE International Symposium on Stochastic Systems Theory and Its Applications*, pages 60–64, 2020.

[18] E. Aguilar and Y. Fujisaki. Opinion dynamics of social networks with stubborn agents via group gossiping with random participants. *Preprints of the 21st IFAC World Congress*, pages 11203–11208, 2020.

[19] E. Aguilar and Y. Fujisaki. Clustering of opinions in social networks via random group gossiping with bounded confidence. *SICE International Symposium on Control Systems 2020*, 1E2-5, 2020.

[20] E. Aguilar and Y. Fujisaki. Reaching consensus via coordinated groups. *SICE Journal of Control, Measurement, and System Integration*, 2021 (Accepted).

[21] A. O'Malley and P. Marsden. The analysis of social networks. *Health Services and Outcomes Research Methodology*, 8(4):222–269, 2008.

[22] S. Wasserman and K. Faust. *Social Network Analysis: Methods and Applications*. Cambridge University Press, 1994.

[23] M. Jackson. *Social and Economic Networks*. Princeton University Press, 20084.

[24] M. Mesbahi and M. Egerstedt. *Graph Theoretic Methods in Multiagent Networks*. Princeton University Press, 2010.

[25] American psychological association dictionary. URL `https://dictionary.apa.org/opinion`.

[26] S. Parsegov, A. Proskurnikov, R. Tempo, and N. Friedkin. Novel multidimensional models of opinion dynamics in social networks. *IEEE Transactions on Automatic Control*, 62(5):2270–2285, 2017.

[27] N.E. Friedkin. The problem of social control and coordination of complex systems in sociology: A look at the community cleavage problem. *IEEE Control Systems Magazine*, 35(3):40–51, 2015.

[28] J. Hunter, J. Danes, and S. Cohen. *Mathematical Models of Attitude Change*, volume 1. Academic Press, Inc., 1984.

[29] J. Halpern. The relationship between knowledge, belief, and certainty. *Annals of Mathematics and Artificial Intelligence*, 4(3):301–322, 1991.

[30] F. Fagnani and P. Frasca. *Introduction to Averaging Dynamics Over Networks.* Springer-Verlag, 2017.

[31] F. Bullo. *Lectures on Network Systems.* CreateSpace, 2018.

[32] R. Berger. A necessary and sufficient condition for reaching a consensus using degroot's method. *Journal of the American Statistical Association*, 76(374): 415–418, 1981.

[33] F. Fagnani and S. Zampieri. Randomized consensus algorithms over large scale networks. *IEEE Journal on Selected Areas in Communications*, 26(4):634–649, 2008.

[34] S. Kia, B.V. Scoy, J. Cortes, R. Freeman, K. Lynch, and S. Martinez. Tutorial on dynamic average consensus: The problem, its applications, and the algorithms. *IEEE Control Systems Magazine*, 39(3):40–72, 2019.

[35] A.V. Proskurnikov, R. Tempo, M. Cao, and N.E. Friedkin. Opinion evolution in time-varying social influence networks with prejudiced agents. *IFAC-PapersOnLine*, 50(1):11896–11901, 2017.

[36] P. Frasca, C. Ravazzi, R. Tempo, and H. Ishii. Gossips and prejudices: Ergodic randomized dynamics in social networks. *4th IFAC Workshop on Distributed Estimation and Control in Networked Systems*, 46(27):212–219, 2013.

[37] S. Hedetniemi, S. Hedetniemi, and A. Liestman. A survey of gossiping and broadcasting in communication networks. *Networks*, 18(4):319–349, 1988.

[38] L. T. H. Nguyen, T. Wada, I. Masubuchi, T. Asai, and Y. Fujisaki. Opinion formation over signed gossip networks. *SICE Journal of Control, Measurement, and System Integration*, 10(3):266–273, 2017.

[39] R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence models, analysis, and simulation. *Journal of Artificial Societies and Social Simulation*, 5(3):1–33, 2002.

[40] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch. Mixing beliefs among interacting agents. *Advances in Complex Systems*, 3(1):87–98, 2001.

[41] A. Tahbaz-Salehi and A. Jadbabaie. Consensus over ergodic stationary graph processes. *IEEE Transactions on Automatic Control*, 55(1):225–230, 2010.

[42] C. Ravazzi, P. Frasca, R. Tempo, and H. Ishii. Ergodic randomized algorithms and dynamics over networks. *IEEE Transactions on Control of Network Systems*, 2(1):78–87, 2015.

[43] R. Parasnis, M. Franceschetti, and B. Touri. Hegselmann-krause dynamics with limited connectivity. *2018 IEEE Conference on Decision and Control*, pages 5364–5369, 2018.

[44] L. T. H. Nguyen, T. Wada, I. Masubuchi, T. Asai, and Y. Fujisaki. Bounded confidence gossip algorithms for opinion formation and data clustering. *IEEE Transactions on Automatic Control*, 64(3):1150–1155, 2019.

[45] J. McAuley and J. Leskovec. Learning to discover social circles in ego networks. *Advances in Neural Information Processing Systems*, 1:539–547, 2012.

[46] M. Moussaïd, J.E. Kämmer, P.P. Analytis, and H. Neth. Social influence and the collective dynamics of opinion formation. *PLoS ONE*, 8(11):e78433, 2013.

[47] F. Gargiulo and S. Huet. Opinion dynamics in a group-based society. *Europhysics Letters Association*, 91(5):58004, 2010.

[48] R.A. Horn and C.R. Johnson. *Matrix Analysis, 2nd Edition*. Cambridge University Press, 2012.

[49] E. Aguilar and Y. Fujisaki. Inter-clique influence networks. *52nd ISCIE International Symposium on Stochastic Systems Theory and Its Applications*, 2020.

[50] E. Aguilar and Y. Fujisaki. Opinion shift: An index for describing gossip algorithms for opinion dynamics. *SICE International Symposium on Control Systems 2019*, 2I2-4, 2019.