THE CLASSIFICATION OF 2-CONNECTED 7-MANIFOLDS

Abstract. We present a comprehensive classification of closed smooth 2-connected manifolds of dimension 7. This builds on the almost-smooth classification from the first author's thesis. The main new ingredient is a generalisation of the Eells–Kuiper invariant that is defined for any closed spin 7-manifold M, regardless of whether the spin characteristic class $p_M \in H^4(M)$ is torsion.

We also determine the inertia group of 2-connected M—equivalently the number of oriented smooth structures on the underlying topological manifold—in terms of p_M and the torsion linking form.

1. Introduction

Throughout this paper M will be a closed smooth spin 7-manifold and all homeomorphisms and diffeomorphisms are assumed to preserve spin structures, unless stated otherwise.

1.1. **Background.** Wall classified $(s-1)$-connected $(2s+1)$-manifolds up to connected sum with homotopy spheres except when $s = 1, 2, 3$ or 7 [50, Theorem 7]. In this paper, we leave connected 3-manifolds aside, recall that Barden classified 1-connected 5-manifolds [3] and focus on 2-connected 7-manifolds (leaving dimension 15 to Remark 1.15 below).

The topologically simplest 7-manifolds are homotopy 7-spheres, whose spin (equivalently oriented) diffeomorphism classes form the group Θ_7. Kervaire and Milnor [29] computed that $\Theta_7 \cong \mathbb{Z}/28$. Eells and Kuiper [16] defined an invariant $\mu(M)$ of certain spin 7-manifolds M with rationally trivial first Pontrjagin class, which distinguishes all homotopy 7-spheres.

At this point, it made sense to study 7-manifolds up to almost diffeomorphism; *i.e.*, up to the action of Θ_7 via connected sum. Wilkens did this in his PhD [52], using the triple of invariants (see Section 2.1)

$$(H^4(M), b_M, p_M),$$

where $H^4(M)$ is the integral cohomology group, $b_M \colon TH^4(M) \times TH^4(M) \to \mathbb{Q}/\mathbb{Z}$ is the torsion linking form and $p_M \in 2H^4(M)$ is the spin characteristic class of M. We call this triple the *base* of M. Modulo a finite ambiguity if $|TH^4(M)|$ is even, Wilkens proved that the base classifies 2-connected M up to almost diffeomorphism. When M is the total space of an S^3-bundle over S^4, this ambiguity was resolved by the first author and Escher [9].

The first author completed the almost diffeomorphism classification of 2-connected M in his PhD [7] by defining a quadratic refinement q_M of the torsion linking form b_M when $H^4(M)$ is torsion, and a family of such refinements in general. When $H^4(M) = TH^4(M)$ is torsion, [7] also proves that the triple $(TH^4(M), q_M, \mu(M))$ gives a complete diffeomorphism invariant. This left the smooth classification open when $H^4(M)$ is infinite: the difficulty being that the classical Eells-Kuiper invariant is not defined when $p_M \in H^4(M)$ has infinite order.

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In this paper we present a comprehensive smooth classification of closed 2-connected 7-manifolds by defining a generalisation of the Eells-Kuiper invariant for all spin 7-manifolds. The uniqueness part of this classification has also been proven by Kreck [32, Theorem 1]. Our main classification results are stated in Sections 1.2 and 1.3 and Section 1.4 describes the definition of the Generalised Eells-Kuiper invariant. We give applications of the classification results to the inertia groups and mapping class groups of 7-manifolds in Section 1.5 and we continue the discussion of the background in Sections 1.6 and 1.7.

An important motivation for this paper is the study of Riemannian manifolds with holonomy the exceptional Lie group G_2: such manifolds always have p_M of infinite order (see Joyce [26, Proposition 10.2.7]). In [12] we use the generalized Eells-Kuiper invariant to distinguish pairs of closed G_2-manifolds which are homeomorphic but not diffeomorphic.

1.2. **The classification.** To any closed smooth spin 7-manifold M we shall associate the following algebraic invariants:

- The integral cohomology group $H^4(M)$, which is a finitely generated abelian group;
- The torsion linking form $b_M : TH^4(M) \times TH^4(M) \to \mathbb{Q}/\mathbb{Z}$, which is a torsion form on the torsion subgroup $TH^4(M) \subseteq H^4(M)$, by this we mean that b_M is symmetric, bilinear and nonsingular (see (8) and Lemma 2.22);
- The spin characteristic class p_M, which is an even element of $H^4(M)$ (see Lemma 2.2(i)). It is a homeomorphism invariant by Remark 2.1, and it is related to the first Pontrjagin class by $2p_M = p_1(M)$;
- The *quadratic linking family* q_M° (see Definition 2.23), which is a family of quadratic refinements of the base $(H^4(M), b_M, p_M)$;
- The *generalised Eells-Kuiper invariant* μ_M (see Definition 1.8), which is a mod 28 Gauss refinement of the triple $(H^4(M), q_M^\circ, p_M)$.

Let us describe q_M° and then indicate the type of μ_M (leaving a more detailed introduction of μ_M to §1.4).

A quadratic refinement of b_M is a function $q\colon TH^4(M) \to \mathbb{Q}/\mathbb{Z}$ which satisfies the equation $q(x+y) = q(x) + q(y) + b_M(x,y)$ and we denote the set of such q by $\mathcal{Q}(b_M)$. We note that for $t \in TH^4(M)$, the function $q_t(x) := q(x) + b(x,t)$ also belongs to $\mathcal{Q}(b_M)$. The homogeneity defect of $q \in \mathcal{Q}(b_M)$ is the unique element $\beta \in 2TH^4(M)$ such that $q(x) - q(-x) = b_M(x,\beta)$. Let

$$S_2 := \{h \in H^4(M) : p_M - 2h \text{ is torsion}\}.$$

That q_M° is a *family of quadratic refinements* of $(H^4(M), b_M, p_M)$ means that it is a function

$$q_M^\circ\colon S_2 \to \mathcal{Q}(b_M), \quad h \mapsto q_M^h,$$

such that $q_M^{h+t} = (q_M^h)_{-t}$ for all $t \in TH^4(M)$ and q_M^h has homogeneity defect $\beta_h := p_M - 2h$. The family of quadratic refinements q_M° is defined in Definition 2.23.

Let d_π be the greatest integer dividing p_M modulo torsion (or $d_\pi := 0$ if p_M is torsion), $\tilde{d}_\pi := \operatorname{lcm}(4, d_\pi)$ and $\widehat{d}_\pi := \gcd\big(\frac{\tilde{d}_\pi}{4}, 28\big)$. If $d_\pi > 0$ we set

$$S_{d_\pi} := \{k \in H^4(M) : p_M - d_\pi k \text{ is torsion}\},$$

and if $d_\pi = 0$ set $S_{d_\pi} := TH^4(M)$. We set $\beta_k := p_M - d_\pi k$ for each $k \in S_{d_\pi}$ and note that for $e_\pi := d_\pi/2$ we have $e_\pi k \in S_2$. By saying that the generalised Eells-Kuiper invariant of M is a *mod 28 Gauss refinement* of $(H^4(M), q_M^\circ, p_M)$ we mean (see Definition 2.34) that it is a function

$$\mu_M \colon S_{d_\pi} \to \mathbb{Q}/\widehat{d}_\pi\mathbb{Z},$$

such that $\mu_M(k) = A(q_M^{e_\pi k}) \bmod \mathbb{Z}$ (where A is the Arf invariant of a quadratic refinement, computed in terms of a Gauss sum in (10)), and such that the following transformation rule

$$\mu_M(k+t) - \mu_M(k) \;=\; e_\pi q_M^{e_\pi k}(t) - \tbinom{e_\pi+1}{2} b_M(t,t) \mod \widehat{d}_\pi \tag{1}$$

holds for all $k \in S_{d_\pi}$ and $t \in TH^4(M)$ (note that both terms on the RHS have coefficient divisible by $\frac{\widehat{d}_\pi}{4}$, so are in particular well-defined in $\mathbb{Q}/\widehat{d}_\pi\mathbb{Z}$). The Generalised Eells-Kuiper invariant of M is defined in Definition 1.8.

Two of the main consequences of (1) are that a Gauss refinement is defined by its value at a single element in S_{d_π}, and that the difference between two Gauss refinements of $(H^4(M), q_M^\circ, p_M)$ is constant. The constraint in terms of the Arf invariant then forces this constant to take values in $\mathbb{Z}/\widehat{d}_\pi\mathbb{Z}$.

Remark 1.1. If p_M is a torsion element then $d_\pi = 0$ and $\widehat{d}_\pi = 28$, while $S_{d_\pi} = TH^4(M)$ contains the distinguished element 0. The value $\frac{1}{28}\mu_M(0) \in \mathbb{Q}/\mathbb{Z}$ recovers the original Eells-Kuiper invariant. See Remark 2.38 for related statements even when p_M is not torsion.

If G is a finitely generated abelian group, $p \in 2G$ and b is a torsion form on $T \subseteq G$, the torsion subgroup of G, then we call (G, b, p) a *base*. If q° is a family of quadratic refinements of (G, b, p) then we call (G, q°, p) a *refinement*; we suppress b since it can be recovered from q^h for any h and hence from q°. If μ is a mod 28 Gauss refinement of (G, q°, p), then we call the quadruple (G, q°, μ, p) a *mod 28 distillation*. If $F : G' \to G$ is a group isomorphism then we can define another mod 28 distillation $(G', F^\#q, F^\#\mu, F^\#p)$ by pulling back: $F^\#(p) := F^{-1}(p)$, $(F^\#q)^h(x) := q^{F(h)}(F(x))$, and $F^\#\mu := \mu \circ F$.

The mod 28 distillation $(H^4(M), q_M^\circ, \mu_M, p_M)$ of M is an invariant of diffeomorphisms: if $f \colon M \to M'$ is a diffeomorphism then $f^* \colon H^4(M') \to H^4(M)$ is an isomorphism and $(q_{M'}^\circ, \mu_{M'}, p_{M'}) = ((f^*)^\# q_M^\circ, (f^*)^\#\mu_M, (f^*)^\# p_M)$. In fact, only μ_M depends on the smooth structure and the refinement $(H^4(M), q_M^\circ, p_M)$ is also invariant under spin homeomorphisms.

An almost diffeomorphism $f \colon M_0 \approxeq M_1$ is a homeomorphism which is smooth except perhaps at a finite number of points. It follows from results of the first author's thesis, see Lemma 3.1, that *2-connected* 7-manifolds are classified up to almost diffeomorphism and homeomorphism by their refinements.

Theorem 1.2 (Almost diffeomorphism and homeomorphism classification)**.** *Every refinement (G, q°, p) is isomorphic to $(H^4(M), q_M^\circ, p_M)$ for some 2-connected 7-manifold M. Moreover, if M_0 and M_1 are 2-connected, then an isomorphism $F \colon H^4(M_1) \to H^4(M_0)$ is realised as f^* for some almost diffeomorphism $f \colon M_0 \approxeq M_1$ if and only if $(q_{M_1}^\circ, p_{M_1}) = F^\#(q_{M_0}^\circ, p_{M_0})$.*

The same statement holds with "almost diffeomorphism" replaced by "homeomorphism".

The central result of this paper is that the generalised Eells-Kuiper invariant is precisely what needs to be added to Theorem 1.2 to obtain a smooth classification of 2-connected

7-manifolds. Consequently, 2-connected 7-manifolds are classified up to diffeomorphism by their mod 28 distillations.

Theorem 1.3 (Smooth classification)**.** *Every mod 28 distillation (G, q°, μ, p) is isomorphic to $(H^4(M), q^\circ_M, \mu_M, p_M)$ for some 2-connected 7-manifold M. Moreover, if M_0 and M_1 are 2-connected, then an isomorphism $F\colon H^4(M_1) \to H^4(M_0)$ is realised as f^* for some diffeomorphism $f\colon M_0 \cong M_1$ if and only if $(q^\circ_{M_1}, \mu_{M_1}, p_{M_1}) = F^\#(q^\circ_{M_0}, \mu_{M_0}, p_{M_0})$.*

1.3. **Elaboration of the classification.** Theorem 1.3 is a "polarised" classification result in the sense that it identifies whether a given isomorphism of the cohomology is realised by some diffeomorphism. If we are simply interested in whether M_0 and M_1 are diffeomorphic (without specifying how the diffeomorphism acts on cohomology) then we can consider a coarser invariant than the generalised Eells-Kuiper invariant. Let $\mathrm{Aut}(b_M)$ be the group of automorphisms of the linking form b_M and recall that for each $k \in S_{d_\pi}$ we have $\beta_k = p_M - d_\pi k \in TH^4(M)$, which is the homogeneity defect of the quadratic refinement $q_M^{e_\pi k}$. We define the *smooth splitting set* of M to be the set

$$\bar{\mathcal{Q}}(M) := \{\big([\beta_k], \mu_M(k)\big) : k \in S_{d_\pi}\} \subset \big(2TH^4(M)/\mathrm{Aut}(b)\big) \times \mathbb{Q}/\widehat{d}_\pi\mathbb{Z}.$$

An isomorphism $F\colon (H^4(M_1), b_{M_1}, p_{M_1}) \cong (H^4(M_0), b_{M_0}, p_{M_0})$ induces the map

$$F^\#\colon \big(2TH^4(M_0)/\mathrm{Aut}(b_0)\big) \to \big(2TH^4(M_1)/\mathrm{Aut}(b_1)\big), \quad [\beta] \mapsto [F^{-1}(\beta)].$$

The following theorem generalises [9, Theorem 1.5] from the case when M is the total space of a smooth S^3-bundle over S^4 to all 2-connected M.

Theorem 1.4 (Unpolarised smooth classification)**.** *Let M_0 and M_1 be 2-connected, and let $F\colon (H^4(M_1), b_{M_1}, p_{M_1}) \to (H^4(M_0), b_{M_0}, p_{M_0})$ be an isomorphism. Then the following are equivalent:*

(i) *M_0 is diffeomorphic to M_1;*
(ii) *$(F^\# \times \mathrm{Id})\big(\bar{\mathcal{Q}}(M_0)\big) = \bar{\mathcal{Q}}(M_1)$;*
(iii) *$(F^\# \times \mathrm{Id})\big(\bar{\mathcal{Q}}(M_0)\big) \cap \bar{\mathcal{Q}}(M_1) \neq \emptyset$.*

The corresponding result for almost diffeomorphisms is given in Corollary 3.5.

We now formulate the polarised classification of Theorem 1.3 in categorical language, giving more information about the monoidal structure of 2-connected 7-manifolds under connected sum. Let $\mathfrak{D}$ denote the category of mod 28 distillations (G, q°, μ, p) with morphisms isomorphisms:

$$\mathrm{Ob}(\mathfrak{D}) = \{(G, q^\circ, \mu, p)\}$$

Let $\mathcal{M}^{\mathrm{Spin}}_{7,2}$ denote the category of 2-connected spin 7-manifolds with morphisms diffeomorphisms:

$$\mathrm{Ob}(\mathcal{M}^{\mathrm{Spin}}_{7,2}) = \{M : \pi_1(M) = 0 = \pi_2(M)\}$$

Given a diffeomorphism $f\colon M_0 \cong M_1$, write $f^*\colon H^4(M_1) \cong H^4(M_0)$ for the induced action on cohomology. Hence we obtain the contravariant functor

$$\mathcal{D}\colon \mathcal{M}^{\mathrm{Spin}}_{7,2} \to \mathfrak{D}, \quad \begin{cases} M & \mapsto & (H^4(M), q^\circ_M, \mu_M, p_M), \\ f\colon M_0 \cong M_1 & \mapsto & f^*. \end{cases}$$

The operations of connected sum and reversing orientation in $\mathcal{M}^{\mathrm{Spin}}_{7,2}$ are mirrored by corresponding operations in $\mathfrak{D}$. For $i = 0, 1$ the orthogonal sum of two distillations $(G_i, q_i^\circ, \mu_i, p_i)$ is defined as follows. Noting that $d_{\pi_i} = c_i d_{\pi_0 \oplus \pi_1}$ for some integer c_i, in which case $c_0 S_{d_{\pi_0}} \times c_1 S_{d_{\pi_1}} \subseteq S_{d_{\pi_0 \oplus \pi_1}}$, we define the orthogonal sum $q_0 \oplus q_1$ at $c_0 k_0 + c_1 k_1$ by

$$(q_0 \oplus q_1)^{c_0 k_0 + c_1 k_1} := q_0^{k_0} \oplus q_1^{k_1},$$

and

$$(\mu_0 \oplus \mu_1)(c_0 k_0 + c_1 k_1) := \mu_0(k_0) + \mu_1(k_1) \mod \gcd\left(28, \tfrac{\tilde{d}_{\pi_0 \oplus \pi_1}}{4}\right).$$

Since $q_0^\circ \oplus q_1^\circ$ and $\mu_0 \oplus \mu_1$ are determined by their values on a single $k \in S_{d_{\pi_0 \oplus \pi_1}}$, this suffices to define the sum of distillations and the transformations laws for refinements and distillations ensure that the orthogonal sum is well-defined. We define the negative of a distillation by

$$-(G, q^\circ, \mu, p) := (G, -q^\circ, -\mu, p).$$

Theorem 1.5 (Categorical version of smooth classification)**.** *The functor $\mathcal{D}\colon \mathcal{M}^{\mathrm{Spin}}_{7,2} \to \mathfrak{D}$ is surjective and faithful. Moreover*

(i) $\mathcal{D}(M_0 \sharp M_1) = \mathcal{D}(M_0) \oplus \mathcal{D}(M_1)$ *and*
(ii) $\mathcal{D}(-M) = -\mathcal{D}(M)$.

We next present an oriented homotopy classification for 2-connected M. Such a classification was given in [7, Theorem 6.11] and we re-formulate that classification in the setting of this paper. An important feature of the homotopy classification is that $p_M \in H^4(M)$ is not a homotopy invariant but $\rho_{24}(p_M) \in H^4(M; \mathbb{Z}/24)$, the mod 24-reduction of p_M, is a homotopy invariant, [37, Theorem 1]. As a consequence, there is a precise sense in which the homotopy classification is the "mod 24 reduction" of the homeomorphism classification.

For a linking form (b, T) define $J\mathcal{Q}(b)$ to be the quotient of $\mathcal{Q}(b)$ where we identify two refinements q_0 and q_1 if $q_1 = (q_0)_{12t}$ for some $t \in T$ and write $\rho_{12}\colon \mathcal{Q}(b) \to J\mathcal{Q}(b)$ for the quotient map. A *J-quadratic refinement* of a base (G, b, p) is a triple $(G, Jq^\circ, \rho_{24}(p))$ where $Jq^\circ\colon S_2 \to J\mathcal{Q}(b)$ is a map such that $Jq^{h+t} = (Jq^h)_{-t}$ and $\rho_{24}(\beta_h) = \rho_{24}(p - 2h) \in T \otimes \mathbb{Z}/24$. The pull-back of J-refinements is defined analogously to the pull-back of refinements and the J-refinement of M is defined to be the triple $(H^4(M), \rho_{12} \circ q_M^\circ, \rho_{24}(p_M))$.

Theorem 1.6 (Homotopy classification)**.** *Every J-refinement $(G, Jq^\circ, \rho_{24}(p))$ is isomorphic to $(H^4(M), \rho_{12} \circ q_M^\circ, \rho_{24}(p_M))$ for some smooth 2-connected 7-manifold M. Moreover, if M_0 and M_1 are 2-connected, then an isomorphism $F\colon H^4(M_1) \to H^4(M_0)$ is realised as f^* for some orientation preserving homotopy equivalence $f\colon M_0 \simeq M_1$ if and only if $(\rho_{12} \circ q_{M_1}^\circ, \rho_{24}(p_{M_1})) = F^\#(\rho_{12} \circ q_{M_0}^\circ, \rho_{24}(p_{M_0}))$.*

1.4. **The generalised Eells-Kuiper invariant.** As explained in Section 1.1, the main novelty of Theorem 1.3 lies in the smooth classification when $H^4(M)$ is infinite. The key ingredient is the generalisation of the Eells-Kuiper invariant.

Let X be a closed spin 8-manifold. By the index theorem [2, Theorem 5.3] $\widehat{A}(X)$, the $\widehat{A}$-genus of X, is equal to the index of the Dirac operator on X, and so is an integer. The

classical Eells-Kuiper invariant is derived from the relation

$$p_X^2 - \sigma(X) = 224\widehat{A}(X), \tag{2}$$

where X has signature $\sigma(X)$ and spin characteristic class p_X: the latter is defined in Section 2.1. If M is a closed 7-manifold such that p_M is a *torsion* class (so rationally trivial) and W is a spin coboundary of M, then p_W^2 has a well-defined integral over W (it might in general take values in $\mathbb{Q}$ and not just $\mathbb{Z}$), and (2) implies that

$$\mu(M) := \frac{p_W^2 - \sigma(W)}{8} \in \mathbb{Q}/28\mathbb{Z} \tag{3}$$

is independent of the choice of coboundary W. This defines the classical Eells-Kuiper invariant, modulo normalisation by a factor of 28.

To define an analogue when p_M is not a torsion class we have to let it take values not modulo 28 but modulo the integer $\widehat{d}_\pi = \gcd(\frac{\tilde{d}_\pi}{4}, 28)$, depending on the divisibility of p_M modulo torsion as above. Moreover, the generalisation is not simply a constant in $\mathbb{Q}/\widehat{d}_\pi\mathbb{Z}$ but a function.

To define the generalised Eells-Kuiper invariant μ_M, suppose that W is a spin coboundary of M and that there exists $n \in H^4(W)$ such that the image of $p_W - d_\pi n$ under the restriction map $j\colon H^4(W) \to H^4(M)$ is a torsion class; equivalently $j(n) \in S_{d_\pi}$. If W is 3-connected then such n exist and any spin M has 3-connected coboundaries: see the start of Section 2.2. Since $j(p_W - d_\pi n)$ is torsion we can define (*cf.* (23))

$$g_W\big(j(n)\big) := \frac{(p_W - d_\pi n)^2 - \sigma(W)}{8} \in \mathbb{Q}/\tfrac{\tilde{d}_\pi}{4}\mathbb{Z} \tag{4}$$

and then extend g_W to a function $S_{d_\pi} \to \mathbb{Q}/\frac{\tilde{d}_\pi}{4}\mathbb{Z}$ by the transformation rule (1). Then g_W is independent of the choices of n. The following lemma (*cf.* (24)) implies that the residue $\mu_M := g_W \mod \widehat{d}_\pi$ is independent of the choice of W, and functorial.

Lemma 1.7. *Let $f : \partial W_0 \to \partial W_1$ be a diffeomorphism and $X := (-W_0) \cup_f W_1$. Then*

$$g_{W_1} - (f^*)^{\#} g_{W_0} = 28\widehat{A}(X) \mod \tfrac{\tilde{d}_\pi}{4}.$$

Definition 1.8. The *generalised Eells-Kuiper invariant* of M is defined to be the function $\mu_M\colon S_{d_\pi} \to \mathbb{Q}/\widehat{d}_\pi\mathbb{Z}$.

The idea of the definition is that the simplest way to change (3) to something that is well-defined when the restriction of p_W to the boundary is rationally non-trivial is to compensate by subtracting from p_W a class that is divisible by d_π and has the same rational image in $H^4(M)$. The essentially different ways of doing that are parametrised by S_{d_π} and that is why the generalised Eells-Kuiper invariant is a function defined on S_{d_π}.

The definition of the s_1 invariant by Kreck and Stolz [33] provides as a byproduct a way to compute the classical Eells-Kuiper invariant in terms of coboundaries that are not spin, but merely spin^c. Proposition 2.43 gives a similar way to compute the generalised Eells-Kuiper invariant via spin^c coboundaries. We use this method in [12] to compute the

generalised Eells-Kuiper invariants of certain closed 7-manifolds with holonomy G_2 that are homeomorphic but not diffeomorphic.

1.5. **Inertia and reactivity.** Let $\Theta_7 = \{\Sigma : \Sigma \simeq S^7\}$ be the group of spin diffeomorphism classes of homotopy 7-spheres Σ. This is equivalent to the standard definition of Θ_7 in [29], since homotopy spheres are simply connected. By [29], Θ_7 is an abelian group under connected sum and $\Theta_7 \cong \mathbb{Z}/28$. The *inertia group* of M is defined to be the following subgroup of Θ_7:

$$I(M) := \{\Sigma : M\sharp\Sigma \cong M\}$$

Remark 1.9. Let M_+ denote the oriented manifold underlying M. If M is simply connected then $I(M) = I(M_+)$, where $I(M_+)$ is the usual inertia group of M_+, which is defined using orientation preserving diffeomorphisms $f_+ \colon M_+\sharp\Sigma_+ \cong M_+$.

It turns out that even with Theorem 1.3 in hand, the determination of $I(M)$ can be a delicate problem. The reason is that μ_M is not a constant but rather a function and so it is possible for almost diffeomorphisms of M to act non-trivially on μ_M. Equivalently, the automorphism group of a refinement (G, q°, p) can act non-trivially on the set of mod 28 Gauss refinements.

The inertia group is closely related to what we (therefore) call the *reactivity* of M. Let $\mathrm{ADiff}(M)$ denote the group of spin almost diffeomorphisms of M. Given $f \in \mathrm{ADiff}(M)$, the mapping torus T_f of f has as well-defined spin characteristic class $p_{T_f} \in H^4(T_f)$ and we define the integer $p^2(f) := \langle p_{T_f}^2, [T_f]\rangle$. This defines a homomorphism

$$p^2 \colon \mathrm{ADiff}(M) \to \mathbb{Z}, \quad f \mapsto p^2(f),$$

and the reactivity of M is the non-negative integer $R(M)$ defined by

$$p^2(\mathrm{ADiff}(M)) = R(M)\mathbb{Z}. \tag{5}$$

Clearly $R(M)$ is an almost diffeomorphism invariant of M. Since T_f has zero signature and p_{T_f} is characteristic for the intersection form of T_f we have $R(M) \in 8\mathbb{Z}$. It is well understood that $f \in \mathrm{ADiff}(M)$ is pseudo-isotopic to a diffeomorphism if and only if $p^2(f)$ is divisible by 224 (see Lemma 3.7) and consequently

$$I(M) = \frac{R(M)}{8}\Theta_7. \tag{6}$$

To determine $R(M)$ when M is 2-connected, we first determine the values of $p^2(f)$ which are realised when $H^*(f) = \mathrm{Id}$ (see Proposition 3.10). This reduces the determination of $R(M)$ to understanding the action of the automorphism group $\mathrm{Aut}_{q^\circ}(H^4(M))$ of $(H^4(M), q^\circ_M, p_M)$ on mod 28 Gauss refinements. That can in turn be reduced to understanding the automorphism group $\mathrm{Aut}_b(H^4(M))$ of $(H^4(M), b_M, p_M)$, which is much easier to deal with in practice. In fact, $R(M)$ is almost completely determined just using the following 'intermediate' notion of the divisibility of p_M, whose significance was pointed out by Wilkens [54, Conjecture p. 548]:

$$d_o := \begin{cases} 0 & \text{if } p_M \text{ is torsion,} \\ \mathrm{Max}\{s : s, m \in \mathbb{Z},\ sm^2 \text{ divides } mp_M\} & \text{otherwise.} \end{cases} \tag{7}$$

Corollary 4.17 and (6) give the next theorem, where for a fraction $\frac{a}{b}$ written in lowest terms $\mathrm{Num}\left(\frac{a}{b}\right) = a$.

Theorem 1.10. *Let M be 2-connected and let $d_o = d_o(M)$. There is an integer $r \in \{0, 1, 2\}$ depending only on the base $(H^4(M), b_M, p_M)$, such that*

$$R(M) = \mathrm{lcm}(8, 2^r d_o).$$

In particular, by (6)*,*

$$I(M) = \mathrm{Num}\left(\frac{2^r d_o}{8}\right)\Theta_7.$$

If $TH^4(M)$ has odd order then $r = 1$.

If $H^4(M)$ does have some 2-torsion then in general one needs to look at the torsion linking form in detail to determine r. Wilkens' conjecture [54, Conjecture p. 548] for the inertia group is equivalent to supposing that $r = 1$ always, which is not true. The invariant $r = r(G, b, p)$ is defined in Definition 4.5 and while we do not have a closed formula for r, it is feasible to compute r for any given example. For examples where $r = 0, 1$ or 2, see Example 5.2.

We next discuss some consequences of Theorem 1.10 and its proof. If N is a closed smooth manifold, let $n_+(N)$ denote the number of oriented diffeomorphism classes of smooth structures on the topological manifold underlying N. From Theorems 1.2 and 1.10 and Remark 1.9 we deduce

Corollary 1.11. *If M is 2-connected then $n_+(M) = \gcd\left(\mathrm{Num}(2^{r-3}d_o), 28\right)$.*

We call a homotopy equivalence $f\colon N_0 \to N_1$ of smooth manifolds *tangential* if there is a bundle isomorphism $f^*TN_1 \cong TN_0$, where TN_i is the tangent bundle of N_i, $i = 0, 1$. In Lemma 5.5 we show that a homotopy equivalence $f\colon M_0 \simeq M_1$ of 2-connected 7-manifolds with $f^*p_{M_1} = p_{M_0}$ is tangential. Together with Theorem 1.10 this entails

Corollary 1.12. *Let M_0 and M_1 be 2-connected and let $f\colon M_0 \simeq M_1$ be a tangential homotopy equivalence. Then $I(M_0) = I(M_1)$.*

One may wonder if Corollary 1.12 is true because tangentially homotopy equivalent 2-connected 7-manifolds are almost diffeomorphic (equivalently homeomorphic by Theorem 1.2). However this was shown not to be the case in [7, p. 114], contradicting statements of Madsen, Taylor and Williams [35, Theorem C and Theorem 5.10]: see Proposition 5.6 and Remark 5.7.

The computation of the reactivity of M also has applications in G_2-topology. We define the *smooth reactivity of M*, $R^{\mathrm{Diff}}(M)$, using the equation $p^2(\mathrm{Diff}(M)) = R^{\mathrm{Diff}}(M)\mathbb{Z}$, and in [11, Section 6] we show that $R^{\mathrm{Diff}}(M)$ determines the number of G_2-structures on M modulo homotopies and diffeomorphisms. By Corollary 4.17,

$$R^{\mathrm{Diff}}(M) = \mathrm{lcm}(2^r d_o, 224)$$

for 2-connected M, and this allows us to generalise Theorem 1.3 to give a classification of 2-connected 7-manifolds equipped with a G_2-structure, up to diffeomorphisms and homotopies of G_2-structures [11, Theorem 6.9].

The proof of Theorem 1.10 gives subtle information about the mapping class group of M. Let $I_H(M) \subseteq I(M)$ be the subgroup of the inertia group of M consisting of homotopy spheres Σ such that there is a diffeomorphism $f\colon M\sharp\Sigma \cong M$ where $H^*(f) = \mathrm{Id}$, considering $M\sharp\Sigma$ and M as the same topological space. Using the delicate algebra in Section 4.3 we construct a surjective homomorphism

$$\widehat{P}\colon \mathrm{Aut}_{q^\circ}(H^4(M)) \to I(M)/I_H(M),$$

such that $F \in \mathrm{Aut}_{q^\circ}(H^4(M))$ is realised by a diffeomorphism of M if and only if $\widehat{P}(F) = 0$. Now by Theorem 1.2, every $F \in \mathrm{Aut}_{q^\circ}(H^4(M))$ is realised by an almost diffeomorphism of M and in Proposition 6.4 we prove that every nested pair of subgroups $I_1 \subseteq I_2 \subseteq \Theta_7$ can be realised as $I(M)_H \subseteq I(M)$ for some 2-connected M. As consequence we have

Theorem 1.13. *There exist 2-connected M with automorphisms $F \in \mathrm{Aut}_{q^\circ}(H^4(M))$ which are not realised by any diffeomorphism of M. Necessarily every such F is realised by an almost diffeomorphism of M.*

Remark 1.14. Let us call an homeomorphism $f\colon M \to M$ exotic if it is not topologically isotopic to a diffeomorphism. Applying Theorem 1.13 gives examples of exotic homeomorphisms $f\colon M \to M$ whose exoticness is detected by their action on integral cohomology; see Example 6.5. To the best of our knowledge, these are the first examples of exotic homeomorphisms of this kind.

1.6. **An overview of the proof of Theorem 1.3 and some remarks on surgery.** Every 2-connected M bounds a 3-connected 8-manifold W and we define the *characteristic form* of W to be the triple $(H^4(W,\partial W), \lambda_W, p_W)$, where

$$\lambda_W\colon H^4(W,\partial W) \times H^4(W,\partial W) \to \mathbb{Z},$$

is the intersection form of W, and $p_W \in H^4(W)$ is the spin characteristic class of W (see Section 2.1). A key feature of dimension 8 is that p_W is *characteristic for* λ_W, which means that $\lambda_W(x,x) \equiv x \cup p_W \bmod 2$ for all $x \in H^4(W,\partial W)$ (see Lemma 2.2(iii)). In [48] Wall classified the manifolds W by proving that every isomorphism of characteristic forms is realised by a diffeomorphism. He also proved that every abstract triple (H,λ,α), where $\lambda\colon H \times H$ is a symmetric bi-linear form and $\alpha\colon H \to \mathbb{Z}$ is characteristic for λ in the sense above, is realised as the characteristic form of some W.

Following Wall, Wilkens [52, Theorem 3.2] proved that any diffeomorphism $f\colon M_0 \to M_1$ between 2-connected M extends to a diffeomorphism $F\colon W_0 \to W_1$ for some 3-connected coboundaries W_i of M_i, $i = 0, 1$. The results of Wall and Wilkens' reduced the classification of 2-connected 7-manifolds to the classification of characteristic forms up to isometry and orthogonal sum with *spherical* forms, where we call a characteristic form spherical if the boundary of the corresponding handlebody is diffeomorphic to S^7; *cf.* [50, §14]. This algebraic problem boils down to finding the correct notion of the *algebraic boundary* of a characteristic form. Wilkens' base (G,b,p) and the first author's refinement (G,q°,p) were partial solutions to this problem and the mod 28 distillation (G,q°,μ,p) of this paper gives a complete solution.

The fundamental input to Wall and Wilkens' theorems is Smale's h-cobordism theorem [46]. In addition, both proofs make use of handlebody theory. Hence the topological inputs to our proofs are relatively elementary from a modern perspective. The reader may ask whether developments in manifold theory, *e.g.* the classical surgery theory of Browder-Novikov-Sullivan-Wall or the modified surgery of Kreck, give more powerful tools to classify 2-connected 7-manifolds?

In the case of classical surgery, the answer to the above question is simply "no". The homotopy classification of 2-connected 7-manifolds via the study of CW-decompositions and attaching maps, is surely harder than the smooth classification of these manifolds. The reader may consult [45] as a starting point. Even if the homotopy classification is known, the computation of the surgery structure set via the surgery exact sequence and then the action of the self-equivalences on the structure set is a delicate problem. Here the reader may consult [8, Theorem 2.2] for the case $M = S^3 \times S^4$.

The situation is different with modified surgery, which provides a powerful tool for classifying 1-connected 7-manifolds. Until recently the relevant results from modified surgery came from working over the normal 2-type and rested on the general classification theorem [31, Theorem 6], which in the 2-connected case makes the very restrictive hypothesis that $H^4(M)$ is generated by p_M. In [32] Kreck defines an enhanced normal 2-type which applies to all 2-connected M and which he uses to give an alternative proof of the uniqueness part of Theorem 1.3. The enhanced normal 2-type encodes what Kreck calls a *d-structure* which, in the notation of this paper, is a pair (M, k), where $k \in S_{d_\pi}$.

1.7. **Dimensions 7 and 15.** Dimension 7 and 15 were exceptional for Wall's methods in [50] because the tangent bundles of S^3 and S^7 are trivial and this prevented Wall from defining a quadratic refinement of the linking form. We discuss this further in Section 2.8, where we show how Wall's methods can be extended to 2-connected 7-manifolds by adding additional tangential structure. In this way we are able to give an *intrinsic* definition of the quadratic refinement of the linking form.

Remark 1.15. To discuss dimension 15, let $String := O\langle 6\rangle$ denote the 6-connected cover of the stable orthogonal group. This is a well-defined homotopy type with models which are topological groups (see [47, Theorem 5.1]). In particular there is a well-defined notion of a stable string structure on a manifold and hence a well-defined notion of a stable string manifold. For the almost diffeomorphism classification, a 15-dimensional analogue of Theorem 1.2 was proven in [7, Theorem B]. For the smooth classification, we have $\Theta_{15} \cong \mathbb{Z}/8{,}128 \oplus \mathbb{Z}/2$, where the $\mathbb{Z}/8{,}128$ summand is the subgroup of homotopy 15-spheres which bound string manifolds and the $\mathbb{Z}/2$ summand maps isomorphically to $\Omega_{15}^{\mathrm{String}}$, [29, 17]. The 15-dimensional analogue of Theorem 1.3 requires the following modifications. Firstly the universe of 6-connected 15-dimensional manifolds has two disjoint classes: those manifolds which bound string manifolds and those which do not. Secondly, within each of these classes a version of Theorem 1.3 holds, where mod 28 Gauss refinements are replaced by mod 8,128 Gauss refinements, in the sense of Definition 2.34.

1.8. **Organisation.** The rest of this paper is organised as follows. In Section 2 we define the invariants used in Theorems 1.2 and 1.3. In particular, families of quadratic refinements, Gauss refinements and the generalised Eells-Kuiper invariant are defined in Sections 2.4, 2.5 and 2.6 respectively. In Section 3 we prove our main classification results and we discuss the connected sum splitting of 2-connected 7-manifolds in Theorems 3.4 and 3.14. Section 4 is an algebraic section in which we analyse the automorphisms of refinements and bases and the action of these automorphisms on Gauss refinements. This section contains the proof of Theorem 1.10, which follows from the computation of the reactivity of M in Corollary 4.17.

In Section 5 we illustrate the classification of 2-connected M with examples and we also we present a refinement of Wilkens' identification of the set of indecomposable generators for the monoid of almost diffeomorphism classes of 2-connected 7-manifolds under the operation of connected sum; see Theorem 5.8. In Section 6 we investigate the relationship between the inertia groups of M and the mapping class groups of M and prove Theorem 1.13.

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2. Invariants

In this section we define the invariants needed to classify 2-connected spin 7-manifolds M. In Section 2.1 we introduce the linking form b_M of M and the spin characteristic class $p_M \in 2H^4(M)$. In Section 2.2 we recall the characteristic form

$$(H^4(W, \partial W), \lambda_W, p_W)$$

of a spin coboundary W for M and identify it as the salient algebraic model for W. In sections Sections 2.3, 2.4 and 2.5 we progressively build algebraic "boundary invariants" of characteristic forms. Section 2.3 recalls the theory of refinements of torsion forms and Section 2.4 shows how a characteristic form defines a family of refinements on its boundary. In Section 2.5 we define the generalised Eells-Kuiper invariant μ_M of M using Hirzebruch's characteristic class formulae for the $\widehat{A}$-genus and the L-genus. The generalised Eells-Kuiper invariant is a reduced defect invariant of the $\widehat{A}$-genus. In Section 2.7 we show how μ_M can be computed via a coboundary W which is spin^c rather than spin. Finally, in Section 2.8 we give an intrinsic definition of the quadratic refinements defined via coboundaries in Section 2.4.

2.1. **Basic invariants.** To any closed spin 7-manifold M we associate its integral cohomology group $H^4(M)$, torsion linking form b_M and spin characteristic class $p_M \in 2H^4(M)$. We call the triple $(H^4(M), b_M, p_M)$ the base of M. More generally, a base is a triple (G, b, p) consisting a finite abelian group G, a torsion form b on the torsion subgroup of G and an element $p \in 2G$. For later use we introduce the category $\mathfrak{B}$ consisting of bases with morphisms isomorphisms

$$\mathrm{Ob}(\mathfrak{B}) = \{(G, b, p)\}.$$

We now define in the invariants b_M and p_M in turn.

The linking form b_M. Recall that the linking form of a closed oriented $(4k-1)$-manifold N is a nonsingular symmetric bilinear pairing

$$b_N : TH^{2k}(N) \times TH^{2k}(N) \to \mathbb{Q}/\mathbb{Z}$$

defined on the torsion subgroup of $H^{2k}(N)$. Given $x, y \in TH^{2k}(N)$ and $\widehat{x} \in H^{2k-1}(N; \mathbb{Q}/\mathbb{Z})$, a lift of x along the Bockstein $\beta\colon H^{2k-1}(N; \mathbb{Q}/\mathbb{Z}) \to TH^{2k}(N)$ associated to the coefficient sequence $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$, the linking form b_N of N is defined by the equation

$$b_N(x, y) := \langle \widehat{x} y, [N] \rangle \in \mathbb{Q}/\mathbb{Z}.$$

If N is the boundary of an oriented $4k$-manifold Y, then the linking form of N and the intersection form of Y are related, as explained in [1, II]. Let $i\colon H^{2k}(Y, N; \mathbb{Q}) \to H^{2k}(Y; \mathbb{Q})$ be the natural map and define a rational-valued intersection form on the image of i by

$$\bar{\lambda}_Y\colon \mathrm{Im}(i) \times \mathrm{Im}(i) \to \mathbb{Q}, \quad \bar{\lambda}_Y(i(w), i(z)) := \langle w \cup i(z), [Y] \rangle.$$

Let $j\colon H^{2k}(Y) \to H^{2k}(N)$ be the restriction map. If $x \in TH^{2k}(N)$ and $\bar{x} \in H^{2k}(Y)$ is a preimage, $j(\bar{x}) = x$, then the image of $\bar{x}$ in $H^{2k}(Y; \mathbb{Q})$ is in the kernel of the restriction $H^{2k}(Y; \mathbb{Q}) \to H^{2k}(N; \mathbb{Q})$. Thus the image of $j^{-1}(TH^{2k}(N)) \subset H^{2k}(Y)$ in $H^{2k}(Y; \mathbb{Q})$ equals $\mathrm{Im}(i)$. The linking form of N satisfies

$$b_N(x, y) = -\bar{\lambda}_Y(\bar{x}, \bar{y}) \bmod \mathbb{Z}, \tag{8}$$

whenever $\bar{x}, \bar{y} \in H^{2k}(Y)$ are lifts of x and y respectively. Note that if the image of j contains $TH^{2k}(N)$, then (8) describes b_M completely. The appearance of the minus sign in (8) is explained in [1, Proof of Theorem 2.1] and also in [21, §3].

The spin characteristic class p_M. The classifying space $B\mathrm{Spin}$ is 3-connected and has $\pi_4(B\mathrm{Spin}) \cong \mathbb{Z}$. It follows that $H^4(B\mathrm{Spin}) \cong \mathbb{Z}$ is infinite cyclic. A generator is denoted $\pm\frac{p_1}{2}$ and the notation is justified since for the canonical map $\pi\colon B\mathrm{Spin} \to BSO$ we have $\pi^* p_1 = 2\frac{p_1}{2}$ where $p_1 \in H^4(BSO)$ is the first Pontrjagin class, see *e.g.* [36, Lemma 2.2].

One way to explain the claims in the previous paragraph is to note that the canonical homomorphism $SU \to \mathrm{Spin}$, which maps the stable special unitary group to the stable spin group, induces an isomorphism $H^4(B\mathrm{Spin}) \to H^4(BSU)$. Since $H^4(BSU)$ is cyclic with a generator (namely the universal second Chern class c_2) whose image in $H^4(BSO)$ is $2p_1$, the same is true for $H^4(B\mathrm{Spin})$ (see also Lemma 2.39 below).

Given a spin manifold N we write

$$p_N := \frac{p_1}{2}(N) \in H^4(N).$$

Remark 2.1. In order to prove the topological invariance of invariants we define in the later subsections, we consider p_Y for general topological spin manifolds Y. We let $BTop$ denote the classifying space for stable topological microbundles, see [30, Essay IV, Proposition 8.1], and $BTop\langle 4\rangle$ its 3-connected cover. Equivalently, $BTop\langle 4\rangle$ is the classifying space for stable spin topological microbundles. By [24, (3)] there is a split short exact sequence

$$0 \to \pi_4(B\mathrm{Spin}) \to \pi_4(BTop\langle 4\rangle) \to \mathbb{Z}/2 \to 0.$$

It follows that the canonical homomorphism $H^4(BTop\langle 4\rangle) \to H^4(B\mathrm{Spin})$ is an isomorphism and so $p_N \in H^4(N)$ is a homeomorphism invariant of topological spin manifolds.

By [33, Lemma 6.5], the mod 2 reduction of p_M is the $4th$ Stiefel-Whitney class w_4. This has the following consequences for the parity of the characteristic class of a closed spin manifold.

Lemma 2.2.

(i) *Let M be a closed spin 7-manifold. Then $p_M \in 2H^4(M)$.*
(ii) *Let X be a closed spin 8-manifold. For all $x \in H^4(X;\mathbb{Z}/2)$*

$$x^2 = x \cup p_X \in H^8(X;\mathbb{Z}/2).$$

(iii) *Let W be a compact spin 8-manifold with boundary M. For all $x \in H^4(W, M;\mathbb{Z}/2)$*

$$x^2 = x \cup p_W \in H^8(W, M;\mathbb{Z}/2).$$

Proof. By Wu's formula, see *e.g.* [42, Theorem 11.14], $w_4 = v_4$ for any closed spin manifold since the first three Wu classes of a spin manifold vanish.

(i) Now $v_4(M) = 0$ since M is 7-dimensional, the Wu class satisfies $v_4(M) \cup x = Sq^4(x)$ for all $x \in H^3(M;\mathbb{Z}_2)$ by definition, and Sq^4 vanishes on classes of degree three.
(ii) $x^2 = Sq^4(x) = x \cup v_4(X) = x \cup p_X$.
(iii) Let $X := W \cup_{\mathrm{Id}_M} (-W)$. The push-forward $i_*\colon H^*(W, M) \to H^*(X)$ of the inclusion $i\colon W \hookrightarrow X$ is dual under the Poincaré pairing to the restriction $i^*\colon H^4(X) \to H^4(W)$. Since $i^*p_X = p_W$, (ii) gives

$$x^2 = (i_*x)^2 = i_*x \cup p_X = x \cup p_Y,$$

where the equalities take place in $\mathbb{Z}/2 \cong H^8(X;\mathbb{Z}/2) \cong H^8(W, M;\mathbb{Z}/2)$. □

Remark 2.3. The characteristic class p_M is the primary and final obstruction to the triviality TM, the tangent bundle of M; *i.e.* TM is trivial if and only if $p_M = 0$. This is because of Bott periodicity, which states that $\pi_5(BSO(7)) = \pi_6(BSO(7)) = 0$ and we have the exceptional fact that $\pi_7(BSO(7)) = 0$ by [28, p. 162]. Hence all obstructions to the triviality of the tangent bundle of M after p_M vanish. Indeed, any rank 7 (or higher) spin vector bundle E over a CW-complex X is trivial over the 7-skeleton of X if and only if $p(E) = 0$.

2.2. **Algebraic models of coboundaries.** Let M be a closed spin 7-manifold. Since the bordism group Ω_7^{Spin} vanishes by [40], there is a compact spin 8-manifold W such that $\partial W = M$. Applying surgery below the middle dimension to W [39, Theorem 3], we can assume that W is 3-connected. We define $FH^4(W, \partial W) := H^4(W, \partial W)/TH^4(W, \partial W)$ to be the torsion-free quotient of $H^4(W, \partial W)$. Since W is 3-connected, $H^4(W)$ is torsion-free and so the relative cohomology sequence of (W, M) gives exactness of

$$FH^4(W, \partial W) \to H^4(W) \to H^4(M) \to 0. \tag{9}$$

For an abelian group H, let $H^* := \mathrm{Hom}(H, \mathbb{Z})$ be the dual of H. Since $H^4(W)$ is torsion-free, the composition $H^4(W) \to H_4(W, \partial W) \to H^4(W, \partial W)^*$ of the Kronecker homomorphism with the Poincaré-Lefschetz duality isomorphism is an isomorphism. Hence the first homomorphism in (9) can be thought of as the adjoint homomorphism,

$$\widehat{\lambda}_W \colon FH^4(W, \partial W) \to FH^4(W, \partial W)^*,$$

of the intersection pairing λ_W on $FH^4(W, \partial W)$. The principle we follow is to regard the pair $(FH^4(W, \partial W), \lambda_W)$ as a "model" for a coboundary W.

Let us set up some terminology to deal with these models. We say that (H, λ) is an integral form if H is a finitely generated free abelian group and $\lambda \colon H \times H \to \mathbb{Z}$ is symmetric and bilinear. Let $\widehat{\lambda}$ denote the adjoint homomorphism $H \to H^*$. The "boundary" of (H, λ) is $G := \mathrm{coker}(\widehat{\lambda})$. We say that an element $\alpha \in H^*$ is *characteristic* for λ if $\lambda(x, x) = \alpha(x) \mod 2$ for all $x \in H$. We then call (H, λ, α) a *characteristic form*.

If W is a 3-connected coboundary of M then the pair $(FH^4(W, \partial W), \lambda_W)$ is an integral form with boundary $H^4(M)$. By Lemma 2.2(iii), $(FH^4(W, \partial W), \lambda_W, p_W)$ is characteristic. (If M is 2-connected then Wall's classification of 3-connected 8-manifolds [48] ensures that the characteristic form of W is a complete invariant of W under diffeomorphisms; *i.e.* every isomorphism of characteristic forms is realised by a diffeomorphism, see [7, Corollary 2.5].) In the next subsections we study the structures that an integral or characteristic form induces on its boundary. By applying this to the algebraic model of a 3-connected coboundary of M we obtain the desired algebraic invariants of M. To prove that they are independent of the choice of W we will combine a splitting result for the algebraic constructions with the following lemma whose proof is a simple application of the Mayer-Vietoris theorem.

Lemma 2.4. *Let W_i be compact 3-connected spin 8-manifolds with 2-connected boundaries, $f \colon \partial W_0 \to \partial W_1$ a homeomorphism, and $X := (-W_0) \cup_f W_1$ (a closed spin topological manifold). Then for $i = 0, 1$ we have injections $H^4(W_i, \partial W_i) \hookrightarrow H^4(X)$ whose images are orthogonal to each other with respect to the intersection form λ_X of X. Further, the restriction map $H^4(X) \to H^4(M)$ is surjective, with kernel $H^4(W_0, \partial W_0) + H^4(W_1, \partial W_1)$.* □

Remark 2.5. We note that by Lemma 2.2(ii), the triple $(H^4(X), \lambda_X, p_X)$ of the manifold X in Lemma 2.4 is a (nonsingular) characteristic form. Moreover, the image of p_X under the restriction map $H^4(X) \to H^4(W_i)$ is of course p_{W_i}.

2.3. **Torsion forms and quadratic refinements on finite groups.** Throughout this paper T is a *finite* abelian group. We say that $b \colon T \times T \to \mathbb{Q}/\mathbb{Z}$ is a *torsion form* on T if it is

symmetric, bilinear, and nonsingular in the sense that the induced map $T \to \mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z})$ is an isomorphism. We call a function $q\colon T \to \mathbb{Q}/\mathbb{Z}$ a *quadratic refinement* of b if

$$q(x+y) = q(x) + q(y) + b(x,y),\ \forall\, x, y \in T.$$

The *homogeneity defect* of q is the unique element $\beta = \beta(q) \in 2T$ such that for all $x \in G$ $q(x) - q(-x) = b(x, \beta)$. If $\beta = 0$ then $q(x) = q(-x)$ and q is called *homogeneous*. We define

$$\mathcal{Q}(b) := \{q : q \text{ is quadratic refinement of } b\}$$

and we let $\mathcal{Q}^0(b) \subseteq \mathcal{Q}(b)$ be the set of homogeneous quadratic refinements of b. In this subsection we consider the problem of classifying the quadratic refinements in $\mathcal{Q}(b)$ up to isomorphism. For $\mathcal{Q}^0(b)$ this problem was solved by Nikulin [43] and the general solution was given independently by Deloup and Massuyeau [14] and the first author [7].

The first basic results [7, Lemmas 2.30 & 2.31] are that $\mathcal{Q}(b)$ and $\mathcal{Q}^0(b)$ are both non-empty and that T acts freely and transitively on $\mathcal{Q}(b)$ via the action

$$\mathcal{Q}(b) \times T \to \mathcal{Q}(b), \quad (q,t) \mapsto q_t,$$

where we recall from the introduction that for all $t \in T$,

$$q_t(x) = q(x) + b(x,t) = q(x+t) - q(t).$$

It is clear that the homogeneity defects of q and q_t are related by $\beta(q_t) = \beta(q) + 2t$.

Example 2.6. If $T \cong \mathbb{Z}/r\mathbb{Z}$ is cyclic then all torsion forms and refinements on T are given by the following examples. Given $\theta \in \mathbb{Z}/r$ coprime to r, let $\left\langle \frac{\theta}{r} \right\rangle$ denote $\mathbb{Z}/r$ equipped with the torsion form

$$b(x,y) := \frac{\theta x y}{r} \in \mathbb{Q}/\mathbb{Z}.$$

Given $\theta \in \mathbb{Z}/2r$ coprime to r and $\gamma \in \mathbb{Z}/r$ (so that $2\gamma \in \mathbb{Z}/2r$), we define a quadratic refinement $\left\langle\!\left\langle \frac{\theta}{2r} \right\rangle\!\right\rangle_\gamma$ of $\left\langle \frac{\theta}{r} \right\rangle$ by

$$q(x) := \theta \left(\frac{x^2 + 2\gamma x}{2r} \right) \in \mathbb{Q}/\mathbb{Z}.$$

Beyond the homogeneity defect, we introduce two further equivalent invariants of q. The first of these is the *Gauss sum of* q which is the complex number

$$GS(q) := \sum_{x \in T} e^{2\pi i q(x)} \in \mathbb{C},$$

where $i = \sqrt{-1}$ and e is Euler's number. From the fact that $q_t(x) = q(x+t) - q(t)$ one easily obtains the following useful

Lemma 2.7 ([14, (4.1)])**.** $GS(q_t) = e^{-2\pi i q(t)} GS(q)$. □

It is a theorem of Milgram [41, Theorem, p. 127, Appendix 4] that if q is homogeneous, then $GS(q)$ is a non-zero complex number with modulus $\sqrt{|T|}$: by Lemma 2.7, this holds

for all $q \in \mathcal{Q}(b)$. We define the *Arf invariant* of q to be the number $A(q) \in \mathbb{Q}/\mathbb{Z}$ which is the argument of $GS(q)$ divided by 2π. That is

$$GS(q) = \sqrt{|T|}e^{2\pi i A(q)} \in \mathbb{C}. \tag{10}$$

Then Lemma 2.7 is equivalent to

$$A(q_t) = A(q) - q(t). \tag{11}$$

Before giving the classification theorems for $\mathcal{Q}(b)$, we review how elements of $\mathcal{Q}(b)$ can be presented as the boundaries of nondegenerate characteristic forms (H, λ, α) and how $A(q)$ is determined by (H, λ, α) in this situation. If λ is nondegenerate then the boundary $T := \operatorname{coker}(\widehat{\lambda})$ of (H, λ) fits into the short exact sequence

$$0 \to H \xrightarrow{\widehat{\lambda}} H^* \xrightarrow{j} T \to 0. \tag{12}$$

We write $\lambda_{\mathbb{Q}} \colon (H \otimes \mathbb{Q}) \times (H \otimes \mathbb{Q}) \to \mathbb{Q}$ for the rational form induced by λ. Its adjoint $\widehat{\lambda}_{\mathbb{Q}} \colon H \otimes \mathbb{Q} \to H^* \otimes \mathbb{Q}$ is an isomorphism, and we use the inverse $(\widehat{\lambda}_{\mathbb{Q}})^{-1} \colon H^* \otimes \mathbb{Q} \to H \times \mathbb{Q}$ to pull back the form $\lambda_{\mathbb{Q}}$ on $H \otimes \mathbb{Q}$. We obtain a rational symmetric bilinear form on $H^* \otimes \mathbb{Q}$, and restricting to $H^* \subset H^* \otimes \mathbb{Q}$ gives the rational-valued bilinear form

$$\bar{\lambda} := (\widehat{\lambda}_{\mathbb{Q}}^{-1})^*(\lambda_{\mathbb{Q}})|_{H^* \times H^*} \colon H^* \times H^* \to \mathbb{Q}.$$

Explicitly, if $y, z \in H^*$ and if $y = k\widehat{\lambda}(\widetilde{y})$ and $z = l\widehat{\lambda}(\widetilde{z})$ for some integers k and l then,

$$\bar{\lambda}(y, z) = \lambda_{\mathbb{Q}}(\widehat{\lambda}_{\mathbb{Q}}^{-1}(y), \widehat{\lambda}_{\mathbb{Q}}^{-1}(z)) = \frac{\lambda(\widetilde{y}, \widetilde{z})}{kl} = \langle \widehat{\lambda}_{\mathbb{Q}}^{-1}(y), z \rangle.$$

Remark 2.8. In [7] the form $\bar{\lambda} \colon H^* \times H^* \to \mathbb{Q}$ is denoted λ^{-1}.

Remark 2.9. When the sequence (12) is the sequence $H^4(W, \partial W) \longrightarrow H^4(W) \xrightarrow{j} H^4(M)$ of a 3-connected coboundary W as in Section 2.2 with $H^4(M) = TH^4(M)$, then the form $\bar{\lambda} \colon H^* \times H^* \to \mathbb{Q}$ is precisely the restriction of the rational-valued intersection form of W, $\lambda_W \colon H^4(W; \mathbb{Q}) \times H^4(W; \mathbb{Q}) \to \mathbb{Q}$, to $j^{-1}(TH^4(M)) = H^4(W) \subset H^4(W; \mathbb{Q})$.

Given a nondegenerate characteristic form (H, λ, α) and $x, y \in T$, let $\bar{x}, \bar{y} \in H^*$ be such that $j(\bar{x}) = x$ and $j(\bar{y}) = y$. We define the torsion form b_λ

$$b_\lambda \colon T \times T \to \mathbb{Q}/\mathbb{Z}, \quad (x, y) \mapsto -\bar{\lambda}(\bar{x}, \bar{y}) \bmod \mathbb{Z},$$

and the quadratic refinement of b_λ

$$q_{\lambda,\alpha} \colon T \to \mathbb{Q}/\mathbb{Z}, \quad x \mapsto \frac{-\bar{\lambda}(\bar{x}, \bar{x}) - \bar{\lambda}(\bar{x}, \alpha)}{2} \bmod \mathbb{Z}. \tag{13}$$

We regard $(T, q_{\lambda,\alpha}, j(\alpha))$ as the *boundary* of (H, λ, α) and note that the homogeneity defect of $q_{\lambda,\alpha}$ is exactly $j(\alpha)$.

Remark 2.10. The minus signs in (13) are introduced to correspond to the sign in (8). The sign differs from [7, Definition 2.32]. As a consequence, the definition of the linking form and quadratic linking family in [7, Definition 2.50] have the wrong signs.

Example 2.11. Let us discuss the calculation of (T, b) from (H, λ) in more detail. If H has basis $\{v_1, \dots, v_n\}$ and λ is represented with respect to this basis by the symmetric integer matrix B then B is invertible over $\mathbb{Q}$ and the rational symmetric matrix B^{-1} expresses $\lambda\colon H^* \times H^* \to \mathbb{Q}$ with respect to the dual basis $\{v_1^*, \dots, v_n^*\}$ of H^*. It follows that the mod $\mathbb{Z}$ values of B^{-1} express the linking form b with respect to the generating set $\{j(v_1^*), \dots, j(v_n^*)\}$ of T.

For example, suppose that $H = \mathbb{Z}^2$ with basis $\{v_1, v_2\}$, and λ and α are given by

$$B = \begin{pmatrix} 0 & 2^i \\ 2^i & 0 \end{pmatrix}, \quad \alpha(v_1) = 2a_1, \quad \alpha(v_2) = 2a_2,$$

where $a_1, a_2 \in \mathbb{Z}$. Then $T = \mathbb{Z}/2^i \oplus \mathbb{Z}/2^i$ with generating set $\{(j(v_1^*), (j(v_2^*))\}$, b has linking matrix

$$\begin{pmatrix} 0 & -2^{-i} \\ -2^{-i} & 0 \end{pmatrix}$$

with respect to $\{j(v_1^*), j(v_2^*)\}$ and $q_{\lambda,\alpha}$ is given by the formula

$$q_{\lambda,\alpha}\big(kj(v_1^*) + lj(v_2^*)\big) = \frac{-kl - a_2 l - a_1 k}{2^i}.$$

The following fundamental theorem of Wall states that every linking form and quadratic refinement are realised as the boundary of some even nondegenerate form.

Theorem 2.12 ([49, Theorem 6])**.** *For all torsion forms b and for every $q \in \mathcal{Q}^0(b)$, there is an even nondegenerate form (H, λ) and an isomorphism $q \cong q_{\lambda,0}$.*

We now state Milgram's theorem on the Gauss sums of homogeneous quadratic torsion forms.

Theorem 2.13 (Milgram [41, Theorem, p. 127, Appendix 4])**.** *Let $q \in \mathcal{Q}^0(b)$ and (H, λ) be an even nondegenerate integral form with signature $\sigma(\lambda)$. Then*

(i) $8A(q) \in \mathbb{Z}$*;*
(ii) $8A(q_{\lambda,0}) \equiv -\sigma(\lambda)$ *mod* 8.

Following Milgram's theorem, we can restate Nikulin's classification of homogeneous quadratic refinements of b as follows.

Theorem 2.14 ([43, Theorem 1.11.3])**.** *If $q_0, q_1 \in \mathcal{Q}^0(b)$ then q_0 is isomorphic to q_1 if and only if $A(q_0) = A(q_1)$.*

For general quadratic refinements of b we have the following results.

Proposition 2.15 ([7, Proposition 5.19])**.** *Any nondegenerate characteristic form (H, λ, α) has*

$$A(q_{\lambda,\alpha}) = \frac{\bar{\lambda}(\alpha, \alpha) - \sigma(\lambda)}{8} \in \mathbb{Q}/\mathbb{Z}.$$

Theorem 2.16 ([7, Theorem 5.22], [14, Theorem 4.1])**.** *Let $q_0, q_1 \in \mathcal{Q}(b)$ be quadratic refinements with homogeneity defects β_0 and β_1 respectively. Then q_0 and q_1 are isomorphic if and only if the following hold:*

(i) *There is an automorphism* $f\colon T \cong T$ *of* b *such that* $f(\beta_0) = \beta_1$*;*
(ii) $A(q_0) = A(q_1)$.

Remark 2.17. The proof of Theorem 2.14 in [43] and the proof of Theorem 2.16 in [7] both apply classification results for torsion forms and case by case checking. In contrast, the proof of Theorem 2.16 in [14] is short and general, with one elegant argument covering all cases.

2.4. **Families of quadratic refinements.** Let G be a finitely generated abelian group, p an element of $2G$, and b a torsion form on the torsion subgroup T; *i.e.* (G, b, p) is a base and so is an object in the category $\mathfrak{B}$. Define

$$S_2 := \{h \in G : p - 2h \in T\},$$

and for $h \in S_2$ write $\beta_h := p_M - 2h$. Note that T acts simply transitively on S_2 by addition.

Definition 2.18. A *family of quadratic refinements* of a base (G, b, p) is defined to be a function $q^\circ\colon S_2 \to \mathcal{Q}(b),\ h \mapsto q^h$, such that:

(i) the homogeneity defect of q^h is β_h;
(ii) $q^{h+t} = q^h_{-t}$ for any $t \in T$.

The triple (G, q°, p) is called a *refinement* of (G, b, p).

An isomorphism $F\colon G \to G'$ obviously maps $S_2 \to S'_2$, and F pulls back a family of quadratic refinements q'° on G' to one on G by setting

$$(F^\# q')^h := q'^{F(h)} \circ F_{|T}.$$

In this case q° and q'° are isomorphic via F, and so are (G, q°, p) and $(G', q'^\circ, F^{-1}(p))$.

The orthogonal sum of two refinements (G_0, q_0°, p_0) and (G_1, q_1°, p_1) is the refinement $(G_0 \oplus G_1, q_0^\circ \oplus q_1^\circ, p_0 \oplus p_1)$ as defined in Section 1.3. The negative of a refinement (G, q°, p) of (G, b, p) is the refinement $(G, -q^\circ, p)$ of $(Q, -b, p)$ defined by $(-q)^h = -q^h$. For later use we introduce the category $\mathfrak{R}$ consisting of refinements with morphisms isomorphisms

$$\mathrm{Ob}(\mathfrak{R}) = \{(G, q^\circ, p)\}.$$

Refinements as in Definition 2.18 are defined naturally on the boundaries of characteristic forms (H, λ, α) when λ is allowed to be degenerate. First we define the base (G, b, p). Let $G := \mathrm{coker}(\widehat{\lambda})$, let $K := \ker(\widehat{\lambda}) \subset H$ be the radical of λ and let $R \subseteq H^*$ the annihilator of K. Then $R \cong (H/K)^*$, and the form λ descends to a nondegenerate form $\lambda_{/K}\colon H/K \times H/K \to \mathbb{Z}$ with $R/\operatorname{Im}(\widehat{\lambda}_{/K}) \cong T$. Hence we obtain a torsion form $b = b_\lambda$ on T as in Section 2.3. To define p we let $j\colon H^* \to G$ be the projection and set $p := j(\alpha)$. Regardless of whether λ is degenerate or not, the classification of $\mathbb{Z}_2$-valued bilinear forms implies that there is always an $x \in H$ such that $\lambda(x, y) = \lambda(y, y) \bmod 2$ for any $y \in H$. Then $\alpha - \widehat{\lambda}(x) \in H^*$ is even, so $p = j(\alpha) \in G$ is even as required.

Definition 2.19. The *boundary base* of a characteristic form (H, λ, α) is defined to be the triple $(G, b, p) := \big(\mathrm{coker}(\widehat{\lambda}), b_\lambda, j(\alpha)\big)$.

Next we define the induced family of quadratic refinements. For any $h \in S_2$, pick $m \in H^*$ such that $j(m) = h$ and set $\alpha_m = \alpha - 2m$. Then $j(\alpha_m)$ is a torsion element and so $\alpha_m \in R$ which is characteristic for $(H/K, \lambda_{/K})$ and we let

$$q^h_{(H,\lambda,\alpha)} := q_{\lambda_{/K},\alpha_m}$$

be the quadratic refinement of b defined in (13) in the previous subsection, *i.e.* if $\bar{x} \in R$ and $j(\bar{x}) = x$ then

$$q^h(x)_{(H,\lambda,\alpha)} = \frac{\bar{\lambda}(\bar{x},\bar{x}) + \bar{\lambda}(\alpha_m,\bar{x})}{2} = \frac{\langle \widehat{\lambda}^{-1}_{\mathbb{Q}}(\bar{x}), \bar{x} + \alpha_m\rangle}{2} \in \mathbb{Q}/\mathbb{Z}. \tag{14}$$

This is independent of the choice of m, since if $m' = m + \widehat{\lambda}(r)$ then

$$\bar{\lambda}(2m',\bar{x}) - \bar{\lambda}(2m,\bar{x}) = 2\langle r,\bar{x}\rangle \in 2\mathbb{Z}.$$

That (i) of Definition 2.18 is satisfied is immediate from $\beta_h = j(\alpha_m)$. Meanwhile, if $h' = h+t$ for some $t \in T$ then $j(m'-m) = t$, so

$$q^{h'}_{(H,\lambda,\alpha)}(x) - q^h_{(H,\lambda,\alpha)}(x) = \langle -\widehat{\lambda}^{-1}_{\mathbb{Q}}(\bar{x}), m - m'\rangle = -b_\lambda(x,t),$$

which shows that (ii) of Definition 2.18 holds.

Definition 2.20. The *boundary* of a characteristic form (H,λ,α) is the triple

$$\partial(H,\lambda,\alpha) := (\mathrm{coker}(\widehat{\lambda}), q^\circ_{(H,\lambda,\alpha)}, j(\alpha)).$$

It is a refinement of the boundary base $(\mathrm{coker}(\widehat{\lambda}), b_\lambda, j(\alpha))$ of Definition 2.19.

It is clear that an isomorphism of characteristic forms $E\colon (H_0,\lambda_0,\alpha_0) \cong (H_1,\lambda_1,\alpha_1)$ induces an isomorphism $\partial E\colon \partial(H_0,\lambda_0,\alpha_0) \cong \partial(H_1,\lambda_1,\alpha_1)$ of the boundary refinements. It is also clear that the boundary of an orthogonal sum of characteristic forms is the orthogonal sum of the boundaries and that $\partial(H,-\lambda,\alpha) = -\partial(H,\lambda,\alpha)$.

We call a characteristic form (H,λ,α) *nonsingular* if λ is, *i.e.* if the adjoint $\widehat{\lambda}\colon H \to H^*$ is an isomorphism. Suppose that (H,λ,α) is a nonsingular characteristic form, H_0 is some primitive subgroup of H and H_1 is the λ-orthogonal subspace to H_0. Let $\alpha_i \in H_i^*$ be the restrictions of α to H_i. Let λ_1 be the restriction of λ to H_1, and λ_0 the restriction of $-\lambda$ to H_0. In this case we say that (H_0,λ_0,α_0) and (H_1,λ_1,α_1) are *orthogonal in* (H,λ,α). For the groups $G_i = \mathrm{coker}(\widehat{\lambda}_i)$ of the boundaries of (H_i,λ_i,α_i), the restriction maps $H^* \to H_i^*$ and the isomorphism $\widehat{\lambda}\colon H \cong H^*$ give rise to homomorphisms $H \to H^* \to H_i^* \to G_i$ which induce isomorphisms

$$\Pi_i\colon H/(H_0 \oplus H_1) \cong G_i.$$

Given (H_0,λ_0,α_0) orthogonal to (H_1,λ_1,α_1) in (H,λ,α), we thus have the canonical isomorphism

$$F_\lambda := \Pi_1 \circ \Pi_0^{-1}\colon G_0 \cong G_1. \tag{15}$$

(There is a slight asymmetry in the definition of orthogonal forms: If (H_0,λ_0,α_0) is orthogonal to (H_1,λ_1,α_1) in (H,λ,α), then (H_1,λ_1,α_1) is orthogonal to (H_0,λ_0,α_0) in $(H,-\lambda,\alpha)$. However, the isomorphisms $F_\lambda : G_0 \to G_1$ and $F_{-\lambda} : G_1 \to G_0$ are precisely inverse to each

other.) The following lemma is a routine calculation using (14) and the fact that (H, λ, α) is nonsingular: see [7, Lemma 3.10] for the case where $(H_i, \lambda_i, \alpha_i)$ are nondegenerate.

Lemma 2.21. *Let $(H_0, \lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$ be orthogonal characteristic forms in the nonsingular characteristic form (H, λ, α). If for $i = 0, 1$, (G_i, q_i°, p_i) denotes the boundary of $(H_i, \lambda_i, \alpha_i)$, then the canonical isomorphism F_λ of (15) induces an isomorphism of the boundaries:*

$$F_\lambda^\#(q_1^\circ, p_1) = (q_0^\circ, p_0). \qquad \square$$

As discussed in Section 2.2, if W is a 3-connected coboundary of a closed spin 7-manifold M then $(FH^4(W, \partial W), \lambda_W, p_W)$ is a characteristic form. Note that the associated boundary in $\mathfrak{B}$ in the sense of Definition 2.19 is precisely the base of M, $(H^4(M), b_M, p_M)$, where b_M the torsion linking form of M as described in (8). Hence we have the following

Lemma 2.22. *The base of a spin 7-manifold M is the boundary base of the characteristic form of any 3-connected coboundary W of M;* i.e.

$$(H^4(M), b_M, p_M) = \partial(H^4(W, \partial W), \lambda_W, p_W). \qquad \square$$

Definition 2.23. The *quadratic linking family* q_M° of M is the family of quadratic refinements of $(H^4(M), b_M, p_M)$ defined by the characteristic form $(FH^4(W, \partial W), \lambda_W, p_W)$, where W is any 3-connected coboundary W of M. Explicitly, applying (14) we obtain for all $h \in S_2$ that

$$q_M^h(x) = \frac{-\lambda_W(\bar{x}, \bar{x}) - \lambda_W(\alpha_m, \bar{x})}{2},$$

where $\bar{x} \in H^4(W)$ is a lift of x, $\alpha_m = p_W - 2m$ and $m \in H^4(W)$ is a lift of h.

Moreover, if $d_\pi = 0$—in particular if M is a rational homology sphere—then we have the preferred element $0 \in S_2 = TH^4(M)$ and $q_M := q_M^0$ is *the quadratic refinement of M*.

If W_0 and W_1 are 3-connected coboundaries of M_0 and M_1 respectively and $f : M_0 \to M_1$ is a homeomorphism, let X be the closed topological spin manifold $(-W_0) \cup_f W_1$. Then Lemma 2.4 and Remark 2.5 imply that the characteristic forms $(FH^4(W_0, M_0), \lambda_{W_0}, p_{W_0})$ and $(FH^4(W_1, M_1), \lambda_{W_1}, p_{W_1})$ are orthogonal in $(FH^4(X), \lambda_X, p_X)$ and also that the induced isomorphism $F_{\lambda_X} : H^4(M_0) \to H^4(M_1)$ is precisely $(f^*)^{-1}$. Together with Lemma 2.21, this implies that q_M° is independent of the choice of W and natural under homeomorphisms (in the sense that $(f^*)^\# q_{M_0}^\circ = q_{M_1}^\circ$ for any homeomorphism $f \colon M_0 \to M_1$).

Remark 2.24. If $d_\pi = 0$ then by [10, Definition 1.4 and Theorem 2.4], the function q_M can be defined analytically using the eta invariant of a Dirac operator on M, twisted by appropriate quaternionic line bundles. This definition is intrinsic to M, in the sense that no co-boundary is required. For an alternative intrinsic definition of q_M° in the case of 2-connected M, see Section 2.8 below.

Remark 2.25. The proof following Definition 2.23 that q_M° is a homeomorphism invariant relies on Remark 2.1 and Lemma 2.2. It is simpler than the proof given in [7, Theorem 6.1] which used the full apparatus of smoothing theory.

Notice, however, that smoothing theory and Theorem 1.2 imply that every 2-connected M with $H^2(M;\mathbb{Z}/2) \neq 0$ admits exotic self-homeomorphisms; by which we mean homeomorphisms which are not isotopic to piecewise linear homeomorphisms. Self-homotopy equivalences which are homotopic to exotic self-homeomorphisms were defined on certain rational homotopy spheres in [10, §2.b], see [10, Lemma 2.17].

Remark 2.26. Given a section $\sigma\colon G/T \to G$ of the projection $\pi\colon G \to G/T$, the image of σ is isomorphic to the free part of G, and there is a unique $k(\sigma) \in S_{d_\pi} \cap \mathrm{Im}(\sigma)$. We can therefore define the family of quadratic refinements as a function on the set of sections $\mathrm{Sec}(\pi)$ of π so that $q^\bullet\colon \mathrm{Sec}(\pi) \to \mathcal{Q}(b)$, $q^\sigma := q^{e_\pi k(\sigma)}$. This presentation is relevant for considering connected-sum splittings of M and is discussed further in Section 3.2.

2.5. **Gauss refinements.** We can associate a further boundary invariant to a characteristic form which we refer to as a Gauss refinement of the family of quadratic refinements. Let $(G,b,p) \in \mathfrak{B}$, *i.e.* G is a finitely generated abelian group, $p \in 2G$ and $b\colon T \times T \to \mathbb{Q}/\mathbb{Z}$ is a torsion form. Let $\pi\colon G \to G/T$ be the projection and define d_π to be the greatest integer dividing $\pi(p)$ if $\pi(p) \neq 0$ and set $d_\pi := 0$ if $\pi(p) = 0$. If $d_\pi \neq 0$, we define

$$S_{d_\pi} := \{k \in G : p - d_\pi k \in T\}$$

and if $d_\pi = 0$ set $S_{d_\pi} := T$. Given $k \in S_{d_\pi}$ write $\beta_k := p - d_\pi k$ and note that T acts simply transitively on S_{d_π} by addition. As in the introduction, we abbreviate $d_\pi/2$ as e_π.

Given $(G,q^\circ,p) \in \mathfrak{R}$, *i.e.* if q° is a family of quadratic refinements of (G,b,p), let us define $\widetilde{\Delta}(k,t) \in \mathbb{Q}/2\tilde{d}_\pi\mathbb{Z}$ by

$$\widetilde{\Delta}(k,t) = 4d_\pi q^{e_\pi k}(t) - d_\pi(d_\pi{+}2)\, b(t,t) \tag{16}$$

(note that if $k \in S_{d_\pi}$ then $e_\pi k \in S_2$, so $q^{e_\pi k}$ is a well-defined quadratic refinement of b).

Definition 2.27. Given $(G,q^\circ,p) \in \mathfrak{R}$, we call a function $g\colon S_{d_\pi} \to \mathbb{Q}/\frac{\tilde{d}_\pi}{4}\mathbb{Z}$ a *Gauss refinement* of q° if

$$g(k) = A(q^{e_\pi k}) \mod \mathbb{Z} \tag{17a}$$

for all $k \in S_{d_\pi}$, and the transformation rule

$$g(k+t) - g(k) \;=\; \frac{\widetilde{\Delta}(k,t)}{8} \tag{17b}$$

holds for all $k \in S_{d_\pi}$ and $t \in T$.

A Gauss refinement is completely determined by its value at any single $k \in S_{d_\pi}$, using (17b). The difference between two Gauss refinements of the same family of quadratic refinements is a constant, and by (17a) the constant takes values in $\mathbb{Z}/\frac{\tilde{d}_\pi}{4}\mathbb{Z}$.

Now suppose that (H,λ,α) is a characteristic form. Given $k \in S_{d_\pi}$, pick $n \in H^*$ such that $j(n) = k$, and set $\alpha_n := \alpha - d_\pi n$. Note that $j(\alpha_n) = \beta_k$, and that $\alpha_n \in R \cong (H/K)^*$ is a characteristic element for the intersection form on H/K. Let

$$g_H(k) := \frac{\bar{\lambda}(\alpha_n,\alpha_n) - \sigma(\lambda)}{8} \in \mathbb{Q}/\tfrac{\tilde{d}_\pi}{4}\mathbb{Z}. \tag{18}$$

Lemma 2.28. *g_H is well-defined, independent of the choices of n.*

Proof. Replacing n by $n' := n + \widehat{\lambda}(r)$ for some $r \in H^*$, so $\alpha_{n'} = \alpha_n - d_\pi \widehat{\lambda}(r)$, changes the value of $g_H(k)$ by

$$\frac{-2d_\pi \bar{\lambda}\big(\alpha_n, \widehat{\lambda}(r)\big) + d_\pi^2 \bar{\lambda}\big(\widehat{\lambda}(r), \widehat{\lambda}(r)\big)}{8} = \frac{-d_\pi}{4}\left(\langle r, \alpha_n\rangle - \frac{d_\pi}{2}\lambda(r,r)\right).$$

The last factor is an integer, and it is even when d_π is not divisible by 4 (*i.e.* when $\tilde{d}_\pi = 2d_\pi$) because α_n is characteristic for λ. □

Lemma 2.29. *g_H is a Gauss refinement of $q^\circ_{(H,\lambda,\alpha)}$.*

Proof. First we check the condition (17a). The α_n used in the definition of $g_H(k)$ co-incides with the α_m used in the definition of $q^{e_\pi k}$ in (14). Since α_n is characteristic for λ, Proposition 2.15 immediately gives (17a).

Next we check the transformation law (17b). Given $k \in S_{d_\pi}$ and $t \in T$, pick n' such that $j(n') = k + t$. Then $\alpha_{n'} - \alpha_n = -d_\pi(n' - n)$, and $j(n'-n) = t$, so

$$\begin{aligned}\bar{\lambda}(\alpha_{n'}, \alpha_{n'}) - \bar{\lambda}(\alpha_n, \alpha_n) &= -2d_\pi \bar{\lambda}(\alpha_n, n'-n) + d_\pi^2 \bar{\lambda}(n'-n, n'-n)\\ &= -4d_\pi \frac{\bar{\lambda}(n'-n, n'-n) + \bar{\lambda}(\alpha_n, n'-n)}{2} + d_\pi(d_\pi+2)\bar{\lambda}(n'-n, n'-n)\\ &= \widetilde{\Delta}(k,t) \mod 2\tilde{d}_\pi.\end{aligned}$$

□

An isomorphism $F\colon G' \to G$ with $F(p') = p$ maps $S'_{d_\pi} \to S_{d_\pi}$. If $F^\# q^\circ = q'^\circ$ then we have $\Delta(F(k), F(t)) = \Delta'(k,t)$ for all $k \in S'_{d_\pi}$ and $t \in T'$, so if $g\colon S_{d_\pi} \to \mathbb{Q}/\frac{\tilde{d}_\pi}{4}\mathbb{Z}$ is a Gauss refinement of q° then

$$F^\# g := g \circ F$$

is a Gauss refinement of q'°.

Gauss refinements of orthogonal characteristic forms. We recall that if (H, λ, α) is nonsingular and $H_0 \subset H$ is primitive with orthogonal complement H_1, then for the characteristic forms $(H_i, \lambda_i, \alpha_i)$ defined by restriction from $(\pm\lambda, \alpha)$, there is a canonical isomorphism $F_\lambda\colon \partial(H_0, \lambda_0, \alpha_0) \cong \partial(H_1, \lambda_1, \alpha_1)$ of the associated refinements.

Lemma 2.30. *Let $(H_0, \lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$ be orthogonal characteristic forms in the nonsingular characteristic form (H, λ, α). The canonical isomorphism F_λ of (15) pulls back g_{H_1} to a Gauss refinement of the linking family q_0° of $(H_0, \lambda_0, \alpha_0)$, and*

$$F_\lambda^\# g_{H_1} - g_{H_0} = \frac{\bar{\lambda}(\alpha,\alpha) - \sigma(\lambda)}{8} \mod \tfrac{\tilde{d}_\pi}{4}.$$

Proof. Note that since (H, λ) is nonsingular, $(\widehat{\lambda})^{-1}\colon H^* \cong H$ is an isomorphism from $\bar{\lambda}$ to λ. Also, we have homomorphisms $H^* \to H_i^* \to G_i$ where we recall that $G_i = \operatorname{coker}(\widehat{\lambda}_i)$. Pick a $k \in S_{d_\pi}(G_0)$, and then pick $n \in H^*$ whose image in G_0 equals k (the set-up means that the image of n in G_1 is $F_\lambda(k)$). Let n_i be the image of n in H_i^*, and set $\alpha_{n_i} := \alpha_i - d_\pi n_i \in R_i$ as in the definition of g_{H_i}. Since $\sigma(\lambda) = \sigma(\lambda_1) - \sigma(\lambda_0)$, it suffices to show that

$$\bar{\lambda}(\alpha,\alpha) = \bar{\lambda}_1(\alpha_{n_1}, \alpha_{n_1}) - \bar{\lambda}_0(\alpha_{n_0}, \alpha_{n_0}) \mod 2\tilde{d}_\pi. \tag{19}$$

The image of $\alpha - d_\pi n$ in $H^* \otimes \mathbb{Q}$ can be written as a sum $\widehat{\lambda}_0(\gamma_0) + \widehat{\lambda}_1(\gamma_1)$ where $\gamma_i \in H_i \otimes \mathbb{Q}$ and $\widehat{\lambda}_i(\gamma_i) = \alpha_{n_i}$. Thus, since the hypothesis involves λ restricting to λ_1 on H_1 and $-\lambda_0$ on H_0,

$$\begin{aligned}\bar{\lambda}(\alpha,\alpha) = \lambda_1(\gamma_1,\gamma_1) - \lambda_0(\gamma_0,\gamma_0) + 2d_\pi\bar{\lambda}(n,\alpha - d_\pi n) + d_\pi^2\bar{\lambda}(n,n) \\ = \bar{\lambda}_1(\alpha_{n_1},\alpha_{n_1}) - \bar{\lambda}_0(\alpha_{n_0},\alpha_{n_0}) \mod 2\tilde{d}_\pi;\end{aligned}$$

that equality holds mod $4d_\pi$ when d_π is not divisible by 4 follows from $\alpha - d_\pi n$ being a characteristic element for λ. □

Remark 2.31. Let us call a characteristic form (H, λ, α) *neutral* if it is nonsingular and $\lambda(\alpha,\alpha) = \sigma(\lambda)$ and say that two characteristic forms are *neutrally isomorphic* if they become isomorphic after addition of neutral forms (so this is a sharper condition than stable isomorphism). Lemma 2.30 implies that Gauss refinements are invariant under neutral isomorphism. The gluing and splitting arguments for characteristic forms reviewed in Section 3.2, in particular Theorem 3.2, can be used to show that characteristic forms are classified up to neutral isomorphism by their boundary distillations (G, q°, g, p).

Linked functions. There is a certain redundancy in the definition of a Gauss refinement g, in that the constraint (17a) on g mod $\mathbb{Z}$ forces the transformation rule (17b) to hold mod $\mathbb{Z}$. In the analysis of the action of automorphisms on Gauss refinements in §4, it will prove convenient to replace (17b) with a condition that can be expressed purely in terms of the base (G, b, p) rather than the refinement (G, q°, p), but nevertheless implies (17b) when (17a) is assumed.

We call a function $g\colon S_{d_\pi} \to \mathbb{Q}/\frac{d_\pi}{4}\mathbb{Z}$ (b,p)-*linked* if for all $k \in S_{d_\pi}$ and $t \in T$

$$g(k+t) = g(k) + \frac{\Delta(k,t)}{8}, \tag{20}$$

where

$$\Delta(k,t) := -d_\pi^2 b(t,t) + 2d_\pi b(\beta_k,\, t) \in \mathbb{Q}/2d_\pi\mathbb{Z}. \tag{21}$$

Lemma 2.32. *$g : S_{d_\pi} \to \mathbb{Q}/2\tilde{d}_\pi\mathbb{Z}$ is a Gauss refinement of (G, q°, p) if and only if* (17a) *holds for some $k \in S_{d_\pi}$, and the mod $2d_\pi$ reduction of g is (b,p)-linked.*

Proof. For $q^{e_\pi k}$ to be a refinement of b with inhomogeneity β_k implies from the definitions that

$$2q^{e_\pi k}(t) = b(t,t) + b(\beta_k,\, t) \in \mathbb{Q}/\mathbb{Z},$$

which in turn gives

$$\widetilde{\Delta}(k,t) = \Delta(k,t) \mod 2d_\pi.$$

Similarly, combining

$$q^{e_\pi k}(2t) = 2b(t,t) + b(\beta_k,\, t) \in \mathbb{Q}/\mathbb{Z}$$

and

$$q^{e_\pi k}(-e_\pi t) = -e_\pi q^{e_\pi k}(t) + \tbinom{e_\pi+1}{2} b(t,t) \in \mathbb{Q}/\mathbb{Z}$$

gives that

$$\widetilde{\Delta}(k,t) = -8q^{e_\pi k}(-e_\pi t) \mod 8.$$

By the Chinese remainder theorem, these two constraints completely characterise $\widetilde{\Delta}(k,t)$ as an element of $\mathbb{Q}/2\tilde{d}_\pi\mathbb{Z}$. Now observe that

$$A(q^{e_\pi(k+t)}) \;=\; A(q^{e_\pi k}_{-e_\pi t}) \;=\; A(q^{e_\pi k}) - q^{e_\pi k}(-e_\pi t) \in \mathbb{Q}/\mathbb{Z}$$

by 2.18(ii) and (11). Thus (17b) is equivalent to requiring that $g(k) - A(q^{e_\pi k}) \bmod \mathbb{Z}$ is constant and that (20) holds. □

Remark 2.33. We could make an analogy with factors of automorphy of automorphic forms and think of Δ as a "term of automorphy". For any linked functions to exist is clearly equivalent to the cocycle condition

$$\Delta(k, s+t) = \Delta(k+s, t) + \Delta(k, s), \tag{22}$$

which can be checked directly from the definition in (20). The difference of two functions with the same term of automorphy is invariant under the T action; since T acts transitively on S_{d_π} that simply means that the difference between two (p,b)-linked functions is a constant in $\mathbb{Q}/2d_\pi\mathbb{Z}$.

2.6. **The generalised Eells–Kuiper invariant.** Let M be a spin 7-manifold and W a 3-connected coboundary of M. Let $g_W : S_{d_\pi} \to \mathbb{Q}/\frac{\tilde{d}_\pi}{4}\mathbb{Z}$ be the Gauss refinement of $(H^4(M), q^\circ_M, p_M)$ defined by the characteristic form $(FH^4(W, \partial W), \lambda_W, p_W)$. Applying (18), this means that for $n \in H^4(W)$ such that $j(n) \in S_{d_\pi} \subseteq H^4(M)$,

$$g_W(j(n)) = \frac{\bar{\lambda}_W(\alpha_n, \alpha_n) - \sigma(\lambda_W)}{8} = \frac{(p_W - d_\pi n)^2 - \sigma(W)}{8}, \tag{23}$$

as defined in (4) in the introduction. We pointed out before that if $f\colon M_0 \to M_1$ is a spin homeomorphism then $X := (-W_0) \cup_f W_1$ is a closed topological spin 8-manifold, Lemma 2.4 means that $(FH^4(W_0, M_0), \lambda_{W_0}, p_{W_0})$ and $(FH^4(W_1, M_1), \lambda_{W_1}, p_{W_1})$ are orthogonal in the nonsingular form $(FH^4(X), \lambda_X, p_X)$, and the induced isomorphism $F_{\lambda_X} : H^4(M_0) \to H^4(M_1)$ is precisely $(f^*)^{-1}$. Hence Lemma 2.30 implies

$$g_{W_1} - (f^*)^\# g_{W_0} = \frac{p_X^2 - \sigma(X)}{8} \mod \tfrac{\tilde{d}_\pi}{4}\mathbb{Z}. \tag{24}$$

If f is a diffeomorphism then X is smooth, the RHS of (24) equals $28\widehat{A}(X)$, and $\widehat{A}(X)$ is an integer; this proves Lemma 1.7. Letting

$$\widehat{d}_\pi := \gcd\left(\tfrac{\tilde{d}_\pi}{4}, 28\right) \tag{25}$$

as in the introduction, it follows that

$$\begin{aligned} &\mu_M\colon S_{d_\pi} \to \mathbb{Q}/\widehat{d}_\pi\mathbb{Z}, \\ &\mu_M := g_W \mod \widehat{d}_\pi \end{aligned} \tag{26}$$

is independent of the choice of W and natural under diffeomorphisms: If $f\colon M_0 \to M_1$ is a diffeomorphism then $(f^*)^\#\mu_{M_0} = \mu_{M_1}$ by (24). Now μ_M satisfies a transformation rule that is a mod 28 reduction of (17b), and we say that this makes μ_M a mod 28 Gauss refinement of q°_M.

Definition 2.34. Given $(G, q^\circ, p) \in \mathfrak{R}$ and a positive integer N, we call a function $\mu\colon S_{d_\pi} \to \mathbb{Q}/\gcd(\frac{\tilde{d}_\pi}{4}, N)\mathbb{Z}$ a *mod N Gauss refinement* of q° if

$$\mu(k) = A(q^{e_\pi k}) \mod \mathbb{Z}$$

for all $k \in S_{d_\pi}$, and the transformation rule

$$\mu(k+t) - \mu(k) \;=\; \frac{\widetilde{\Delta}(k,t)}{8} \mod \gcd(\tfrac{\tilde{d}_\pi}{4}, N)$$

holds for all $k \in S_{d_\pi}$ and $t \in T$.

For $N = 28$ this transformation rule is equivalent to (1) stated in the introduction.

If W is not 3-connected, then (4) defines g_W only on the subset $S_{d_\pi} \cap j(H^4(W))$ of S_{d_π}. However, as long as that set is non-empty, this completely determines μ_M by the transformation rule, so the description of μ_M from the introduction is valid. This point can be seen as a special case of Proposition 2.43 below.

Remark 2.35. Analogously to Remark 2.26, we can define Gauss refinements (and μ_M) as functions of sections $\sigma\colon G/T \to G$ rather than on S_{d_π}, $g_W(\sigma) := g_W(k(\sigma))$. Then $g_W(\sigma) = A(q^\sigma) \bmod \mathbb{Z}$, and the transformation rule (20) can also be rewritten in these terms.

Remark 2.36. Recall Remark 1.1 saying that if p_M is torsion then $S_{d_\pi} = T$ contains the distinguished element 0 and $\frac{1}{28}\mu_M(0) \in \mathbb{Q}/\mathbb{Z}$ recovers the original Eells-Kuiper invariant $\mu(M)$.

Although defined extrinsically using spin co-boundaries, the original Eells-Kuiper invariant $\mu(M)$ was shown by Donnelly [15, Theorem 4.2] to have an intrinsic definition in terms of the eta invariant of the Dirac operator of M. It would be interesting to find an intrinsic definition of the generalised Eells-Kuiper invariant when $p_M \neq 0 \in H^4(M;\mathbb{Q})$.

For further information about the role of eta invariants in the classification of 7-manifolds, we refer the reader to [18, §4].

Remark 2.37. In [7, §4.4], a pair of characteristic forms (H, λ, α) are called *smoothly equivalent* if they become isomorphic after addition of nonsingular characteristic forms with $\lambda(\alpha,\alpha) \equiv \sigma(\lambda) \bmod 224$ (so this is a weakening of the notion of neutral equivalence from Remark 2.31). In algebraic terms, the definition of the generalised Eells-Kuiper invariant can be used to show that the mod 28 distillation of M, $(H^4(M), q^\circ_M, \mu_M, p_M)$, is a complete invariant of the smooth equivalence class of the characteristic form $(FH^4(W, \partial W), \lambda_W, p_W)$ of a 3-connected coboundary for M. Hence Theorem 1.3 is a development of the dimension 7 case of [7, Theorem 4.9], which classifies 2-connected 7-manifolds up to diffeomorphism by the smooth equivalence class of the characteristic form of a 3-connected coboundary.

Remark 2.38. Let us conclude this subsection by considering how the information captured by the function $\mu_M : S_{d_\pi} \to \mathbb{Q}/\frac{\tilde{d}_\pi}{4}\mathbb{Z}$ can in some special cases be presented more simply. If p_M is torsion or if the greatest divisor of p_M is the same as d_π (the greatest divisor modulo torsion), then S_{d_π} contains the distinguished element 0, and the function μ_M can be naturally identified by the value $\mu_M(0) \in \mathbb{Q}/\frac{\tilde{d}_\pi}{4}\mathbb{Z}$.

More generally, for any divisor c of d_π we can relate μ_M to functions defined on $S_c = \{k \in G : p_M - ck \in T\}$. Let us focus on the case when c is even—because that is more subtle than when c is odd—and let $\tilde{c} = \operatorname{lcm}(4, c)$. We can then define a function $\bar{g}_W : S_c \to \mathbb{Q}/\frac{\tilde{c}}{4}\mathbb{Z}$ analogously to g_W. If $d_\pi = rc$, then rS_{d_π} is a non-empty subset of S_c. For any $k \in S_{d_\pi}$, the mod $\frac{\tilde{c}}{4}$ reduction of $g_W(k)$ equals $\bar{g}_W(rk)$.

Thus the mod $\frac{\tilde{c}}{4}$ reduction of g_W is completely determined by $\bar{g}_W$. In particular, if we take $c = \gcd(28, d_\pi)$, then $\bar{g}_W$ determines μ_M. Meanwhile $\bar{g}_W$ can sometimes be easier to describe.

In particular, if c divides p_M, then S_c contains 0, and the function $\bar{g}_W$ can be naturally identified with its value at 0. In fact more is true: $\bar{g}_W$ must be constant, except when d_o is an odd multiple of c *and* the parameter r from Theorem 1.10 is 0. That c divides p_M means that the image of p_W in $H^4(W; \mathbb{Z}_c)$ is contained in the image of $H^4(W, M; \mathbb{Z}_c)$, and thus has a well-defined square in $H^8(W, M; \mathbb{Z}_c) \cong \mathbb{Z}_c$. The mod $\frac{\tilde{c}}{8}$ reduction of $\bar{g}_W$ is always constant, determined by

$$\bar{g}_W = \frac{p_W^2 - \sigma(W)}{8} \mod \tfrac{c}{8}.$$

One can attempt to compute $\bar{g}_W$ itself in a similar way using the Pontrjagin square $\wp(\bar{p}) \in H^8(W, M; \mathbb{Z}_{2c})$ of a pre-image $\bar{p} \in H^4(W, M; \mathbb{Z}_c)$ of p_W. This is independent of the choice of $\bar{p}$ if and only if $\wp(\partial x) = 0$ for all $x \in H^3(M; \mathbb{Z}_c)$. Because the suspension of the Pontrjagin square is the Postnikov square, that is equivalent to requiring that, for $j = \operatorname{ord}_2 c$, there are no 2^j-torsion classes $y \in H^4(M)$ with $2^j b(y, y)$ odd. Thus—in the terminology of §4.1—if there are no split 2^j-torsion elements in $H^4(M)$ then there is a well-defined Pontrjagin square $\wp(p_W) \in H^8(W, M; \mathbb{Z}_{2c}) \cong \mathbb{Z}_{2c}$, and $\bar{g}_W$ is determined by

$$\bar{g}_W = \frac{\wp(p_W) - \sigma(W)}{8} \mod \tfrac{c}{4}.$$

(This is compatible with the claim above that $\bar{g}_W$ could be non-constant if $r = 0$, because Lemma 4.3 means that if $H^4(M)$ lacks certain split summands then $r \neq 0$.)

On the other hand, for any divisor c of d_π, $\bar{g}_W$ is determined by its value at a single element of S_c, so $\bar{g}_W$ is completely determined by g_W. Regardless of the possible convenience in some special settings of considering divisors c other than d_π, using d_π captures the maximal possible amount of information. For the purposes of studying the general classification theory there is thus no advantage to considering anything other than g_W and μ_M as functions of S_{d_π}, and that is therefore what we do in the rest of the paper.

2.7. **The computation of μ_M via spin^c coboundaries.** Inspired by calculations of Kreck and Stolz for their s_1 invariant [33], we derive an expression for μ_M in terms of coboundaries that are not necessarily spin (never mind 3-connected) but just spin^c.

For a principal spin^c bundle we use the canonical homomorphisms $\mathrm{Spin}^c(n) \to SO(n)$ and $\mathrm{Spin}^c(n) \to U(1)$ to define an associated real vector bundle E together with a complex

line bundle L such that $c_1(L) = w_2(E) \bmod 2$. We can then define the characteristic classes

$$\begin{aligned} z &:= c_1(L), \\ \widehat{p} &:= p(E \oplus L), \\ \breve{p} &:= \widehat{p} - z^2. \end{aligned}$$

So $2\widehat{p} = p_1(E \oplus L) = p_1(E) + z^2$ and $2\breve{p} = p_1(E) - z^2$. Recall that any U-bundle has a natural spin^c structure, defined as follows: if $i : U(n) \to SO(2n)$ is the natural inclusion then the homomorphism $i \times \det : U(n) \to SO(2n) \times U(1)$ has a lift under the double cover $\mathrm{Spin}^c(2n) \to SO(2n) \times U(1)$. If E is a complex vector bundle then the fundamental line bundle L of the corresponding spin^c bundle is $L := \det E$.

Lemma 2.39.

(i) *$\breve{p}$ and z^2 form a basis for $H^4(B\mathrm{Spin}^c)$.*
(ii) *$\breve{p}(E) = -c_2(E)$ for any complex bundle E.*
(iii) *$\breve{p}(E) = w_4(E) \bmod 2$ for any spin^c bundle E.*

Proof. Observe that $\mathrm{Spin}^c/U \cong \mathrm{Spin}/SU$ is 5-connected, since $\mathrm{Spin}(6)/SU(3) \cong S^7$ and $\mathrm{Spin}(6)$ and $SU(3)$ have the same homotopy groups as Spin and SU in degree ≤ 5. Letting $\pi : BU \to B\mathrm{Spin}^c$ denote the classifying map for EU considered as a spin^c-bundle, the maps $\pi^* \colon H^4(B\mathrm{Spin}^c) \to H^4(BU)$ and $\pi^* \colon H^2(B\mathrm{Spin}^c) \to H^2(BU)$ are therefore isomorphisms (with both $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients). Patently $\pi^* z = c_1$.

We know that $H^4(BU)$ has basis $\{c_2, c_1^2\}$. Because there is no 2-torsion, the equation $2\pi^*\breve{p} = p_1 - (\pi^* z)^2 = (-2c_2 + c_1^2) - c_1^2$ implies $\pi^*\breve{p} = -c_2$, proving (i) and (ii).

The isomorphism on $H^4(-; \mathbb{Z}_2)$ implies that it suffices to check that (iii) holds when E is complex. But that follows from (ii). □

Corollary 2.40. *If X is a compact spin^c 8-manifold then $\widehat{p}_X$ is characteristic for the intersection form λ_X of X.*

Proof. Lemma 2.39 gives

$$\widehat{p} = w_4 + w_2^2 \mod 2.$$

Wu's formula implies that for any closed orientable manifold X the fourth Wu class is $v_4(X) = w_4(X) + w_2(X)^2$, and by definition v_n is characteristic for the intersection form of a closed $2n$-manifold. The compact case follows from the closed case, as in the proof of Lemma 2.2(iii). □

Lemma 2.41. *If X is a closed spin^c 8-manifold then the Dirac operator of the fundamental complex spinor bundle has*

$$28 \operatorname{ind} \not{D}^+ \;=\; \frac{\widehat{p}_X^2 - \sigma(X)}{8} - \frac{5z^2\widehat{p}_X}{12} + \frac{z^4}{4}. \tag{27}$$

Proof. [34, Theorem D.15] expresses $\operatorname{ind} \not{D}^+$ as the integral of $\exp\left(\frac{z}{2}\right) \widehat{A}(X)$, whose degree 8 part expands to

$$\frac{-4p_2 + 7p_1^2}{2^7.45} - \frac{z^2 p_1}{24.8} + \frac{z^4}{24.16} \;=\; \frac{p_1^2}{2^7.7} - \frac{L}{2^5.7} - \frac{z^2 p_1}{2^6.3} + \frac{z^4}{2^7.3}.$$

Then substitute $p_1 = 2\widehat{p} - z^2$ to obtain (27). □

Now suppose M is a spin 7-manifold and W a spin^c coboundary, such that the restriction of $z \in H^2(W)$ to M is trivial. Then z has a pre-image $\bar{z} \in H^2(W, M)$, and $\bar{z}^2 \in H^4(W, M)$ is independent of the choice of $\bar{z}$.

Definition 2.42. Given $k \in S_{d_\pi}$, suppose there is $n \in H^4(W)$ such that $j(n) = k$. Then let $\widehat{\alpha}_n := \widehat{p}_W - d_\pi n$, and

$$g^c_W(k) := \frac{\bar{\lambda}_W(\widehat{\alpha}_n, \widehat{\alpha}_n) - \sigma(W)}{8} - \frac{5\bar{z}^2\widehat{p}_W}{12} + \frac{z^4}{4} \in \mathbb{Q}/\tfrac{\tilde{d}_\pi}{4}\mathbb{Z}.$$

If $z = 0$ then of course $g^c_W = g_W$. The proof that $g^c_W(k)$ does not depend on the choice of n is analogous to Lemma 2.28, using that $\widehat{\alpha}$ is characteristic for intersection form λ_W.

Proposition 2.43. *Let (W_1, z_1) be a spin^c coboundary of M and $j_1 \colon H^4(W_1) \to H^4(M)$ the natural homomorphism. Then*

(i) $g^c_{W_1}(k) = \mu_M(k) \bmod 28$ *for all $k \in S_{d_\pi} \cap j_1(H^4(W_1))$;*
(ii) *The defined values of $g^c_{W_1}$ satisfy the transformation rule* (17b), i.e.

$$g^c_{W_1}(k') = g^c_{W_1}(k) + \frac{\widetilde{\Delta}(k, k'-k)}{8}$$

whenever $k, k' \in S_{d_\pi} \cap j_1(H^4(W_1))$, where $\widetilde{\Delta}$ in (16) *is defined in terms of q°_M and b_M.*

Proof. For part (i), let W_0 be a 3-connected coboundary for M, and $X := (-W_0) \cup_{\mathrm{Id}_M} W_1$. Then X is a smooth spin^c manifold, possibly with more than one choice of $z \in H^2(X)$ restricting to z_1 on W_1 and 0 on W_0. While we do not trouble ourselves with separating the algebra from the topology in this case, we essentially adapt the proof of Lemma 2.30 to show

$$g^c_{W_1} - g_{W_0} = \frac{\widehat{p}^2_X - \sigma(X)}{8} - \frac{5z^2\widehat{p}_X}{12} + \frac{z^4}{4} \mod \tfrac{\tilde{d}_\pi}{4}\mathbb{Z}. \tag{28}$$

Since the RHS equals $28 \operatorname{ind} \not{D}^+$ by Lemma 2.41, while $g_W = \mu_M \bmod \gcd(28, \frac{\tilde{d}_\pi}{4})$ by definition, the result then follows.

Pick some $n_1 \in H^4(W_1)$ such that $j_1(n_1) \in S_{d_\pi}$ as in Definition 2.42. As W_0 is 3-connected, there is some $n \in H^4(X)$ whose restriction to W_1 equals n_1. Then $\widehat{p}_X - d_\pi n$ is a sum of push-forwards of $\gamma_i \in H^4(W_i, M; \mathbb{Q})$ and $\widehat{\lambda}_{W_i}(\gamma_i) = \widehat{\alpha}_i$. Meanwhile, note that regardless of the choice of z, $z^2 \in H^4(X)$ is the push-forward of $\bar{z}_1^2 \in H^4(W_1, M)$. Hence

$$\begin{aligned} \widehat{p}^2_X - \frac{10z^2\widehat{p}_X}{3} + 2z^4 &= \gamma_1^2 + \gamma_0^2 + 2d_\pi n(\widehat{p}_X - d_\pi n) + d_\pi^2 n^2 - \frac{10}{3}\bar{z}_1^2\widehat{p}_{W_1} + 2z_1^4 \\ &= \bar{\lambda}_{W_1}(\widehat{\alpha}_1, \widehat{\alpha}_1) - \frac{10}{3}\bar{z}_1^2\widehat{p}_{W_1} + 2z_1^4 - \bar{\lambda}_{W_0}(\widehat{\alpha}_0, \widehat{\alpha}_0) \mod 2\tilde{d}_\pi. \end{aligned}$$

The fact that the equality holds $\bmod 4d_\pi$ when d_π is not divisible by 4 is due to $\widehat{p}_X - d_\pi n$ being a characteristic element for the intersection form on X.

Part (ii) follows from (28) since g_{W_0} satisfies (17b) and the RHS of (28) is constant. □

As a consequence of Proposition 2.43 (ii), we can extend g^c_W to a well-defined Gauss refinement so long as $S_{d_\pi} \cap j(H^4(W))$ is non-empty. Hence the generalised Eells-Kuiper

invariant μ_M can be computed in terms of any spin^c coboundary W of M where the intersection $S_{d_\pi} \cap j(H^4(W))$ is non-empty.

2.8. **An intrinsic definition of q°_M.** In this subsection we define $\mathrm{Spin}\langle 4,2\rangle$-structures on spin manifolds and use them to give an intrinsic definition of the linking family q°_M for 2-connected M.

Recall from the proof of Lemma 2.2 that the mod 2 reduction of the universal spin class $\rho_2(p) \in H^4(B\mathrm{Spin}; \mathbb{Z}/2)$ is identified with the 4th Wu class of the universal bundle over $B\mathrm{Spin}$. We regard v_4 as a map

$$v_4 \colon B\mathrm{Spin} \to K(\mathbb{Z}/2, 4)$$

and define $B\mathrm{Spin}\langle 4,2\rangle$ to be the homotopy fibre of v_4. By construction there is a map $\gamma^{\langle 4,2\rangle} \colon B\mathrm{Spin}\langle 4,2\rangle \to B\mathrm{Spin}$ and a sequence of maps

$$K(\mathbb{Z}/2,3) \longrightarrow B\mathrm{Spin}\langle 4,2\rangle \xrightarrow{\gamma^{\langle 4,2\rangle}} B\mathrm{Spin} \xrightarrow{v_4} K(\mathbb{Z}/2,4),$$

where both successive pairs of maps defines a fibration sequence. Let N be a spin manifold and let $\nu_N \colon N \to B\mathrm{Spin}$ the classifying map for the stable normal bundle of N. A $\mathrm{Spin}\langle 4,2\rangle$-*structure* on N is an vertical homotopy class of lift $\bar\nu_N \colon N \to B\mathrm{Spin}$. In particular, there is a commutative diagram

$$\begin{array}{ccc} & & B\mathrm{Spin}\langle 4,2\rangle \\ & \overset{\bar\nu}{\nearrow} & \big\downarrow {\scriptstyle \gamma^{\langle 4,2\rangle}} \\ N & \xrightarrow{\nu_N} & B\mathrm{Spin}. \end{array}$$

The diagram above ensures that $\bar\nu \colon N \to B\mathrm{Spin}\langle 4,2\rangle$ is canonically covered by a map of stable vector bundles from the normal bundle of N to the pull-back of the universal bundle over $B\mathrm{Spin}$ along $\gamma^{\langle 4,2\rangle}$.

Lemma 2.44.

(i) *Every spin 7-manifold M admits a* $\mathrm{Spin}\langle 4,2\rangle$*-structure.*
(ii) *The set of equivalence classes of* $\mathrm{Spin}\langle 4,2\rangle$*-structures on M is a torsor for $H^3(M;\mathbb{Z}/2)$.*
(iii) *The induced map $\gamma^{\langle 4,2\rangle *} \colon H^4(B\mathrm{Spin}) \to H^4(B\mathrm{Spin}\langle 4,2\rangle)$ is isomorphic to $\times 2 \colon \mathbb{Z} \to \mathbb{Z}$.*

Proof. By Lemma 2.2(i) we have $\rho_2(p_M) = 0$ and so Part (i) follows from the right-hand fibration in the sequence of maps defining $B\mathrm{Spin}\langle 4,2\rangle$ above. Part (ii) and (iii) follow from the left-hand fibration in the sequence of maps defining $B\mathrm{Spin}\langle 4,2\rangle$ above. □

For later use, we point out that Lemma 2.44(iii) shows that $B\mathrm{Spin}\langle 4,2\rangle$-manifolds $(X, \bar\nu)$ have a naturally defined characteristic class $p^{\bar\nu}_X \in H^4(X)$ such that $2p^{\bar\nu}_X = p_X$. For spin 7-manifolds M we set

$$\bar S_2 := \{h \in G : p_M = 2h\} \subset S_2$$

and then Lemma 2.44(ii) shows that every $h \in \bar S_2$ arises as $p^{\bar\nu}$ for some $\mathrm{Spin}\langle 4,2\rangle$-structure $\bar\nu$ on M. Of course, $\bar S_2$ is a torsor for ${}_2TH^4(M)$, the subgroup of 2-torsion elements of $TH^4(M)$.

In the usual way, we define the bordism groups of closed n-manifolds with $\mathrm{Spin}\langle 4,2\rangle$-structure,

$$\Omega_n^{\mathrm{Spin}\langle 4,2\rangle} = \{[N,\bar{\nu}]\},$$

where $[N,\bar{\nu}]$ denotes the $B\mathrm{Spin}\langle 4,2\rangle$-bordism class of $(N,\bar{\nu})$.

Lemma 2.45. $\Omega_7^{\mathrm{Spin}\langle 4,2\rangle} = 0$*;* i.e. *every closed spin 7-manifold has a spin coboundary W with p_W even.*

Proof. Consider a $B\mathrm{Spin}\langle 4,2\rangle$-manifold $\bar{\nu}\colon M \to B\mathrm{Spin}\langle 4,2\rangle$. Since the space $B\mathrm{Spin}\langle 4,2\rangle$ is 3-connected, surgery below the middle dimension as in by [31, Proposition] ensures that we may replace $(M,\bar{\nu})$ in its $B\mathrm{Spin}\langle 4,2\rangle$-bordism class by a homotopy sphere with $\mathrm{Spin}\langle 4,2\rangle$-structure $(\Sigma,\bar{\nu}_\Sigma)$. By Lemma 2.44(ii), Σ has a unique $\mathrm{Spin}\langle 4,2\rangle$-structure. By [29, Theorem 3.1] Σ is stably parallelisable and and so its $\mathrm{Spin}\langle 4,2\rangle$-structure is induced from a stably framing. By [29] Σ bounds a parallelisable manifold. Hence $(\Sigma,\bar{\nu}_\Sigma)$ bounds a $B\mathrm{Spin}\langle 4,2\rangle$-manifold and so $\Omega_7^{\mathrm{Spin}\langle 4,2\rangle} = 0$. □

Fix a $\mathrm{Spin}\langle 4,2\rangle$-structure $\bar{\nu}$ on M and recall the characteristic class $p_M^{\bar{\nu}} \in H^4(M)$. For 2-connected M we show how to define a homogeneous quadratic form

$$q^{\bar{\nu}}\colon TH^4(M) \to \mathbb{Q}/\mathbb{Z}$$

using just $(M,\bar{\nu})$ and in particular no coboundary. Moreover this definition recovers the quadratic form obtained by evaluating the quadratic linking family q_M° at $h = p_M^{\bar{\nu}} \in \bar{S}_2$; *i.e.* $q^{\bar{\nu}} = q^{p_M^{\bar{\nu}}}$. The idea is to repeat Wall's definition of the quadratic refinement of the linking form for $(s{-}1)$-connected $(2s{+}1)$-manifolds for $s \neq 3,7$ from [50, §12A]. We assume the reader is familiar with this definition, recalling only its essential features.

Following Wall we work with the dual group $TH_3(M)$, the torsion subgroup of $H_3(M)$. For brevity, we write $\widehat{x} \in TH_3(M)$ for the Poincaré dual of $x \in TH^4(M)$. Since M is 2-connected every element $\widehat{x} \in TH_3(M)$ is represented by an embedding $S^3 \hookrightarrow M$ and since every linear bundle over S^3 is trivial, this extends to an embedding $f_{\widehat{x}}\colon D^4 \times S^3 \hookrightarrow M$. To compute the self-linking number $b_M(x,x)$ we need to push $f_{\widehat{x}}(\{0\}\times S^3)$ off itself and this can be achieved by taking a section $s\colon S^3 \to S^3 \times S^3$ of the unit normal bundle $S^3 \times S^3 \to S^3$. Following Wall, we set $X := M \setminus \mathrm{Int}(f_{\widehat{x}}(D^4 \times S^3))$, note that M is obtained from X by attaching a 4-handle and a 7-handle and let $y_1 := [s] \in H_3(X)$ and $y_2 \in H_3(X)$ be the homology class of the meridian $S^3 \times \{*\}$. For $i\colon X \to M$ the inclusion, y_2 generates the kernel of $i_*\colon H_3(X) \to H_3(M)$ and $i_*(y_1) = x$ has order r for some positive integer r. Hence $i_*(ry_1) = 0$ and so $ry_1 = \lambda(s)y_2$ for $\lambda(s) \in \mathbb{Z}$. The homological definition of the linking form gives

$$b_M(x,x) = \frac{\lambda(s)}{r}.$$

Wall defined a refinement of b_M by restricting the choice of section s, and hence the possible integers $\lambda(s)$ appearing in the description of b_M above. To achieve a similar restriction on the choice of sections in dimension 7 we use the $\mathrm{Spin}\langle 4,2\rangle$-structure $\bar{\nu}$ on M. The codimension-0 submanifold $f_{\widehat{x}}(D^4 \times S^3) \subset M$ inherits a $\mathrm{Spin}\langle 4,2\rangle$-structure from $(M,\bar{\nu})$ and this induces a $\mathrm{Spin}\langle 4,2\rangle$-structure on $S^3 \times S^3$, which we also denote by $\bar{\nu}$. By

construction the universal bundle on $B\mathrm{Spin}\langle 4,2\rangle$ in Wu 4-oriented in the sense of Brown [4, Definition 1.10]. Now for any closed 6-manifold Y with a Wu 4-orientation $\bar{\nu}_Y$, Brown [4, Corollary 1.11] defines a quadratic refinement, $\phi^{\bar{\nu}_Y}\colon H^3(Y;\mathbb{Z}/2)\to\mathbb{Z}/2$, of the mod 2-intersection form of Y. In particular, we have the quadratic refinement

$$\phi^{\bar{\nu}}\colon H^3(S^3\times S^3;\mathbb{Z}/2)\to\mathbb{Z}/2.$$

We then define $q^{\bar{\nu}}\colon TH^4(M)\to\mathbb{Q}/\mathbb{Z}$ by the equation

$$q^{\bar{\nu}}(x):=\frac{\lambda(s)}{2r}\in\mathbb{Q}/\mathbb{Z},$$

where we restrict to sections $s\colon S^3\to S^3\times S^3$ such that $\phi^{\bar{\nu}}(s^*(u))=0$ for $u\in H^3(S^3;\mathbb{Z}/2)$ the generator.

Lemma 2.46. *$q^{\bar{\nu}}\colon TH^4(M)\to\mathbb{Q}/\mathbb{Z}$ is well-defined and refines b_M. Moreover $q^{\bar{\nu}}=q^{p^{\bar{\nu}}_M}$.*

Proof. That $q^{\bar{\nu}}$ is a well-defined refinement of b_M follows the proof of [50, Lemma 26], using the fact that Brown's form is a refinement of the mod 2 intersection for a 6-manifold.

To see $q^{\bar{\nu}}=q^{p^{\bar{\nu}}_M}$, we let $(W,\bar{\nu}_W)$ be a $B\mathrm{Spin}\langle 4,2\rangle$-coboundary for $(M,\bar{\nu})$, which exists by Lemma 2.45. As for spin coboundaries, we may assume that W is 3-connected and consider the characteristic form $(H^4(W,\partial W),\lambda_W,p_W)$ of W. In the definition of $q^{p^{\bar{\nu}}_M}$ in (14), we may take $m=p^{\bar{\nu}_W}$ so that $\alpha_m=0$. Then for $x\in TH^4(M)$ and $y\in H^4(W)$ with $j(y)=x$, we have

$$q^{p^{\bar{\nu}}_M}(x)=\frac{-\bar{\lambda}_W(y,y)}{2},\tag{29}$$

where we note that λ_W is even since $\rho_2(p_W)=0$. But in the proof of [50, Theorem 8] Wall identifies his topologically defined refinement with the algebraically defined refinement appearing in (29). It follows that Wall's arguments in the proof of [50, Theorem 8] can be repeated to show that $q^{\bar{\nu}}=q^{p^{\bar{\nu}}_M}$. □

Remark 2.47. Using Lemma 2.46 we can define q^h intrinsically on 2-connected M for every $h\in\bar{S}_2\subset S_2$ and then use the transformation rule of Definition 2.18(ii) to determine q°_M. For example, if $H^4(M)$ is torsion then for each $\mathrm{Spin}\langle 4,2\rangle$-structure $\bar{\nu}$ on M we have

$$q_M=q^0_M=q^{\bar{\nu}}_{p^{\bar{\nu}}}.$$

3. The classification of 2-connected 7-manifolds

In this section we classify closed smooth spin 2-connected 7-manifolds M up to diffeomorphism. Recall that a homotopy 7-sphere Σ is a spin manifold which is homotopy equivalent to S^7. In Section 3.1 we recall that an almost diffeomorphism $f\colon M_0\cong M_1$ defines a diffeomorphism $f\colon M_0\sharp\Sigma\cong M_1$, for some Σ. In Section 3.2 we relate the algebra of Section 2 to the algebra used in [7] and so give the almost diffeomorphism classification of 2-connected M in terms of their refinements $(H^4(M),q^\circ_M,p_M)$.

With the almost diffeomorphism classification in hand, we consider the inertia group of M, which is the group of homotopy spheres Σ such that $M\sharp\Sigma\cong M$. In Section 3.4 we establish basic facts relating the inertia group of M, the reactivity of M and certain mapping class

groups of M. We also construct an important family of almost diffeomorphisms $f\colon M \cong M$ in Proposition 3.10. The almost diffeomorphisms of Proposition 3.10 allow us to show that the generalised Eells-Kuiper invariant of M, μ_M, precisely measures the gap between the almost diffeomorphism classification and the diffeomorphism classification. In Section 3.5 we prove that the mod 28 distillation of M, $(H^4(M), q^\circ_M, \mu_M, p_M)$, is a complete invariant of diffeomorphisms.

3.1. **Almost diffeomorphisms.** In this subsection we briefly review the almost smooth spin category in dimension 7. An *almost diffeomorphism* $f\colon M_0 \cong M_1$ is a homeomorphism which is smooth except perhaps at a finite set of singular points $\{m_0, \dots, m_a\} \subset M_0$. Notice that *we do not require f to be non-smooth* at m_i, but we rather *allow* it. The composition of almost diffeomorphisms is again an almost diffeomorphism and so almost diffeomorphism defines an equivalence relation on smooth spin 7-manifolds.

Let $f\colon M_0 \cong M_1$ be an almost diffeomorphism with singular set $\{m_0, \dots, m_a\}$. We shall associate a homotopy 7-sphere to each singular point m_i. For $i = 0, \dots, a$, let $D^7_i \ni m_i$ be a small disc containing m_i and disjoint from D^7_j if $i \neq j$. The manifold $f(D^7_i) \subset M_1$ is a co-dimension zero submanifold of M_1 and so inherits a smooth structure from M_1 such that

$$\widehat{f_i} := f|_{D^7_i - \{m_i\}}\colon D^7_i - \{m_i\} \cong f(D^7_i - \{m_i\})$$

is a diffeomorphism. We can therefore define the smooth homotopy 7-sphere

$$\Sigma_i := D^7_i \cup_{\widehat{f_i}} (-f(D^7_i))$$

by gluing D^7_i and $-f(D^7_i)$ together along $\widehat{f_i}$

We set $\Sigma_f := \Sigma_0 \sharp \Sigma_1 \sharp \dots \sharp \Sigma_a$. If $D^7 \subset M_0$ contains the singular points of f in its interior, then by [7, Proposition 2.1] there is a diffeomorphism $f'\colon M_0 \sharp \Sigma_f \to M_1$ such that $f'|_{M_0 - D^7} = f|_{M_0 - D^7}$. It follows that M_0 is almost diffeomorphic to M_1 if and only if there is a homotopy sphere Σ and a diffeomorphism $M_0 \sharp \Sigma \cong M_1$.

Before defining pseudo-isotopy for almost diffeomorphisms with one singular point we recall the definition for diffeomorphisms. Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms of M. A pseudo-isotopy between $f_0, f_1 \in \mathrm{Diff}(M)$ is a diffeomorphism $F\colon M \times I \cong M \times I$ which restricts to f_i on $M \times \{i\}$. We define

$$\widetilde{\pi}_0\mathrm{Diff}(M) := \{[f\colon M \cong M]\},$$

the group of the pseudo-isotopy classes of diffeomorphisms of M.

For $m_0 \in M$, let $\mathrm{ADiff}(M, m_0)$ be the group of almost diffeomorphisms of M with singular point m_0. A pseudo-isotopy between $f_0, f_1 \in \mathrm{ADiff}(M, m_0)$ is a homeomorphism $F\colon M \times I \to M \times I$ with $F|_{M_0 \times \{i\}} = f_i$ and which is smooth, except possibly along $\{m_0\} \times I$. We define

$$\widetilde{\pi}_0\mathrm{ADiff}(M, m_0) := \{[f\colon M \cong M]\},$$

the group of pseudo-isotopy classes of almost diffeomorphisms of M with singular point m_0.

3.2. **The almost diffeomorphism classification.** In this subsection we show how Theorem 1.2 follows from the classification results of [7]. The almost diffeomorphism classification given in [7] used a different but closely related definition of a quadratic linking family. We begin by explaining the relationship between the two definitions of linking family and showing that Theorem 1.2 is equivalent to [7, Theorem B]. We then describe the main ideas of the proof of [7, Theorem B] and interpret linking families in terms of connected sum splittings. Throughout this subsection M is 2-connected and we have the global notation $G = H^4(M)$ with torsion subgroup $T \subseteq G$ and free quotient $F = G/T$.

Let us start with some elementary algebra for the group G. Let $\iota\colon T \to G$ be the inclusion and $\pi\colon G \to F$ be the canonical projection. We let $\mathrm{Sec}(\pi) := \{\sigma\colon F \to G\}$ be the set of sections of $\pi\colon G \to F$ and we let $\mathrm{Proj}(\iota) := \{\tau\colon G \to T\}$ be the set of projections over ι; *i.e.* $\tau \circ \iota = \mathrm{Id}_T$. The sets $\mathrm{Sec}(\pi)$ and $\mathrm{Proj}(\iota)$ are in bijection by mapping $\sigma \mapsto \tau_\sigma$, where $\mathrm{Im}(\sigma) = \ker(\tau_\sigma)$. Both sets admit simple transitive actions of $\hom(F,T)$ via addition of functions. For $\phi \in \hom(F,T)$, $f \in F$ and $g \in G$ we have

$$(\sigma + \phi)(f) = \sigma(f) + \phi(f) \quad \text{and} \quad (\tau + \phi)(g) = \tau(g) + \phi(\pi(g)).$$

Notice that $\tau_{\sigma+\phi} = \tau_\sigma - \phi$.

Remark. The action of $\hom(F,T)$ on $\mathrm{Sec}(\pi)$ used above differs by a sign from the corresponding action in [7, p. 39].

Let (G, b, p) be a base so that b is a torsion form on T and $p \in 2G$. Recall that $\mathcal{Q}(b)$ is the set of refinements of b and given $q \in \mathcal{Q}(b)$, let us write $\beta(q)$ for the homogeneity defect of q: see Section 2.3. In [7, Definition 2.39] a quadratic linking family on a base (G, b, p) was defined as a function

$$q^\bullet\colon \mathrm{Sec}(\pi) \to \mathcal{Q}(b)$$

such that for all $\sigma \in \mathrm{Sec}(\pi)$ and for all $\phi \in \hom(G,T)$,

$$q^{\sigma+\phi} = q^\sigma_{-\phi(\pi(p)/2)} \quad \text{and} \quad \beta(q^\sigma) = \tau_\sigma(p).$$

We explain the topological significance of these conditions below, focussing for now on the algebra.

In this paper we work with linking families which are functions on S_2 and we now explain how to pass between linking families defined on S_2 and linking families defined on $\mathrm{Sec}(\pi)$. Given a section $\sigma \in \mathrm{Sec}(\pi)$ there is a unique element $k(\sigma) := \sigma(\pi(p)/d_\pi)$ of S_{d_π} which lies in $\mathrm{Im}(\sigma)$ and so we obtain the function

$$\mathrm{Sec}(\pi) \to S_{d_\pi}, \quad \sigma \mapsto k(\sigma) \in \mathrm{Im}(\sigma) \cap S_{d_\pi}.$$

Now multiplication by $e_\pi = \frac{d_\pi}{2}$ gives a map $S_{d_\pi} \to S_2$ and we set $\widehat{S}_2 := e_\pi S_{d_\pi} \subset S_2$. Given a refinement $q^\circ\colon S_2 \to \mathcal{Q}(b)$ we define

$$q^\bullet\colon \mathrm{Sec}(\pi) \to \mathcal{Q}(b), \quad q^\sigma := q^{e_\pi k(\sigma)}. \tag{30a}$$

Conversely, given $q^\bullet\colon \mathrm{Sec}(\pi) \to \mathcal{Q}(b)$ we define

$$q^\circ\colon \widehat{S}_2 \to \mathcal{Q}(b), \quad q^{e_\pi k(\sigma)} := q^\sigma \tag{30b}$$

and extend q° to all of S_2 by the transformation rule of Definition 2.18 (ii). The transformation rules for $q^\bullet$ and q° ensure that they are determined by their value on a single section or element of S_2. Moreover, these transformation rules are compatible since $k(\sigma+\phi) = k(\sigma) + \phi(\pi(p)/d_\pi)$, $q^{\sigma+\phi} = q^\sigma_{-\phi(\pi(p)/2)}$ and

$$q^{e_\pi k(\sigma+\phi)} = q^{e_\pi(k(\sigma)+\phi(\pi(p)/d_\pi))} = q^{e_\pi k(\sigma)+\phi(\pi(p)/2)} = q^{e_\pi k(\sigma)}_{-\phi(\pi(p)/2)}.$$

Hence we have

Lemma 3.1. *The mappings $q^\circ \mapsto q^\bullet$ and $q^\bullet \mapsto q^\circ$ of* (30a) *and* (30b) *define inverse equivalences of categories between linking families defined on S_2 and linking families defined on* $\mathrm{Sec}(\pi)$. □

Proof of Theorem 1.2. Let M be 2-connected, q°_M the linking family of M as defined in Definition 2.23 and $-q^\bullet_M$ the linking family of M as defined in [7, Definition 2.39] (we have introduced the sign to correct the mistake in [7, Definition 2.50]: see Remark 2.10.) Comparing these definitions, we see that for each $\sigma \in \mathrm{Sec}(\pi)$

$$q^{e_\pi k(\sigma)}_M = q^\sigma_M. \tag{31}$$

Now [7, Theorem B] states that all linking families defined on $\mathrm{Sec}(\pi)$ arise as the quadratic linking families of 2-connected M and that any isomorphism of linking families defined on $\mathrm{Sec}(\pi)$ is realised by an almost diffeomorphism. Hence Theorem 1.2 follows by combining [7, Theorem B], (31) and Lemma 3.1. □

We now explain the proof of [7, Theorem B]. Recall that every 2-connected M is the boundary of a 3-connected W and that the characteristic form of W, $(H^4(W,\partial W), \lambda_W, \alpha_W)$, is a complete invariant of 3-connected W under diffeomorphisms by [48]; see [7, Corollary 2.5]. Let $\natural$ denote the boundary connected sum of manifolds with boundary. A foundational theorem of Wilkens [52, Theorem 3.2] (see also [7, Theorem 2.24]) states that for any diffeomorphism $f\colon \partial W_0 \cong \partial W_1$, there are W_2 and W_3 with $\partial W_2 = \partial W_3 = S^7$ and a diffeomorphism

$$g\colon W_0 \natural W_2 \cong W_1 \natural W_3$$

extending f. The boundary of W is a homotopy sphere, if and only if $(H^4(W,\partial W), \lambda_W, \alpha_W)$ is nonsingular. Hence we say that two characteristic forms are *stably isomorphic* if they become isomorphic after addition of nonsingular characteristic forms. The above discussion shows that classifying 2-connected 7-manifolds up to almost diffeomorphism is equivalent to classifying characteristic forms up to stable isomorphism. This was achieved in [7, Theorem 3.4] by extending ideas of Wall [51, Theorem p. 156] from the setting of even forms to the setting of characteristic forms. The point is that an isomorphism $F\colon \partial(H_0,\lambda_0,\alpha_0) \cong \partial(H_1,\lambda_1,\alpha_1)$ of the boundaries of characteristic forms can be used to glue them together to obtain a nonsingular characteristic form

$$(H_0, -\lambda_0, \alpha_0) \cup_F (H_1, \lambda_1, \alpha_1).$$

It is then possible to explicitly write down an isomorphism of characteristic forms

$$E\colon (H_0,\lambda_0,\alpha_0)\oplus\big((H_0,-\lambda_0,\alpha_0)\cup_F(H_1,\lambda_1,\alpha_1)\big)\to \\ (H_1,\lambda_1,\alpha_1)\oplus\big((H_0,-\lambda_0,\alpha_0)\cup_{\mathrm{Id}}(H_0,\lambda_0,\alpha_0)\big),$$

such that $\partial E = F$. Combined with Lemma 2.21, these methods give the following theorem, which is a refinement of a special case of [7, Theorem 3.4].

Theorem 3.2 (*cf.* [7, Theorem 3.4])**.** *For $i=0,1$, let (H_i,λ_i,α_i) be two characteristic forms. The following are equivalent:*

(i) *There is an isomorphism of refinements $F\colon \partial(H_0,\lambda_0,\alpha_0)\cong\partial(H_1,\lambda_1,\alpha_1)$;*
(ii) *There are nonsingular characteristic forms (H_j,λ_j,α_j), $j=2,3$, and an isomorphism*

$$E\colon (H_0,\lambda_0,\alpha_0)\oplus(H_2,\lambda_2,\alpha_2)\cong(H_1,\lambda_1,\alpha_1)\oplus(H_3,\lambda_3,\alpha_3)$$

such that $\partial E = F$;
(iii) *There is a nonsingular characteristic form (H,λ,α) containing $(H_0,-\lambda_0,\alpha_0)$ and (H_1,λ_1,α_1) as orthogonal summands.*

In addition, there is a canonical isomorphism $F_\lambda = F\colon \partial(H_0,\lambda_0,\alpha_0)\cong\partial(H_1,\lambda_1,\alpha_1)$ in case (iii).

Remark. By Lemma 3.1 the statement of Theorem 3.2 and the discussion before it applies equally well to linking families defined over $\mathrm{Sec}(\pi)$ and linking families defined over S_2.

Proof of Theorem 3.2. This follows from [7, Lemma 3.12] and the proof of [7, Theorem 3.4]. □

We now explain how the linking family q°_M of M parametrises connected sum decompositions of M into a summand with torsion-free homology and a summand which is a rational homotopy sphere. By Theorem 1.2, 2-connected rational homotopy spheres M with torsion linking form (T,b) are classified up to almost diffeomorphism by their quadratic refinements $q^0_M\in\mathcal{Q}(b)$. We shall write $M(q)$ for any rational homotopy sphere with linking form isomorphic to q. The simplest examples of 2-connected M with $H^4(M)$ torsion-free are given in the following

Definition 3.3. Let $F\cong\mathbb{Z}^b$ be a free abelian group of rank b and d_π be an even integer. We define the 2-connected 7-manifold

$$M(F,d_\pi) := \sharp_b(S^3\tilde{\times}_{d_\pi}S^4),$$

where $S^3\tilde{\times}_{d_\pi}S^4$ is the total space of the S^3-bundle over S^4 with trivial Euler class and first Pontrjagin class equal to $2d_\pi$ times the the preferred generator of $H^4(S^4)$ and $\sharp_b$ denotes the b-fold connected sum of a manifold with itself. The base of $M(F,d_\pi)$ is identified with $(\mathbb{Z}^b,0,(d_\pi,\dots,d_\pi))$.

We define an *almost splitting* of M to be an almost diffeomorphism with a singular point $m_0\in M$

$$f\colon M\approxeq M(q^f)\sharp M(F,d_\pi),$$

where q^f is some quadratic refinement of b_M and $f_0(m_0) \in M(q^f)$. Two almost splittings f_0 and f_1 are called H^*-*equivalent* if there is an almost diffeomorphism $g \in \mathrm{ADiff}(M, m_0)$ with $g(m_0) = m_0$ and $H^*(g) = \mathrm{Id}$, an almost diffeomorphism $g_T\colon M(q^{f_0}) \approxeq M(q^{f_1})$ with singular point $f_0(m_0)$ and a diffeomorphism $g_F\colon M(F, d_\pi) \cong M(F, d_\pi)$ such that $g_T(f_0(m_0)) = f_1(m_0)$ and the following diagram commutes up to pseudo-isotopy:

$$\begin{array}{ccc} M & \xrightarrow{f_0} & M(q^{f_0})\sharp M(F, d_\pi) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_T\sharp g_F} \\ M & \xrightarrow{f_1} & M(q^{f_1})\sharp M(F, d_\pi) \end{array}$$

We define $\mathrm{ASplit}(M) := \{[f] : f \text{ an almost splitting of } M\}$ to be the set of H^*-equivalence classes of almost splittings of M and note that there is a well-defined map

$$\mathrm{ASplit}(M) \mapsto \mathrm{Sec}(\pi), \quad [f] \mapsto \sigma(f),$$

where $\mathrm{Im}(\sigma(f)) = f^*\big(H^4(M(F, d_\pi))\big)$. The following theorem is implicit in [7, Definition 2.50].

Theorem 3.4. *Let M have linking family $q_M^\bullet\colon \mathrm{Sec}(\pi) \to \mathcal{Q}(b)$. For each $\sigma \in \mathrm{Sec}(\pi)$ there is a unique H^*-equivalence class of almost splitting*

$$f_\sigma\colon M \approxeq M(q_M^\sigma)\sharp M(F, d_\pi).$$

Consequently the map $\mathrm{ASplit}(M) \to \mathrm{Sec}(\pi)$ is a bijection.

Proof. Let W be a 3-connected coboundary of M with characteristic form $(H, \lambda, \alpha) = (H^4(W, \partial W), \lambda_W, \alpha_W)$. We recall from the definition of the linking family defined by W in (14), that there are orthogonal splittings of (H, λ, α)

$$\psi\colon (H, \lambda, \alpha) \cong (R, \lambda_R, \alpha_\psi) \oplus (F, 0, \alpha|_F),$$

where $(R, \lambda_R, \alpha_\psi)$ is nondegenerate. For every such splitting ψ, the classification of 3-connected coboundaries (see [7, Corollary 2.5]) implies that there is a corresponding boundary connected sum splitting $g_\psi\colon W \cong W_\psi \natural W_F$. In addition, there is a corresponding section $\sigma = \sigma(\psi) \in \mathrm{Sec}(\pi)$ where $\mathrm{Im}(\sigma) = j(H^4(W_f))$, for j the natural homomorphism $H^4(W, M) \to H^4(M)$. By definition, (*cf.* [7, Definition 2.50]),

$$q_M^\sigma = \partial(R, \lambda_R, \alpha_\psi),$$

and we define $f_\sigma\colon M \cong M(q_M^\sigma)\sharp M(F, d_\pi)$ to be the diffeomorphism on the boundary induced by the the splitting g_ψ. This shows that $\mathrm{ASplit}(M) \to \mathrm{Sec}(\pi)$ is onto.

Suppose that f_0 and f_1 are two splittings of M defining the same section σ. Then the H^*-equivalence class of f_i is determined by the almost diffeomorphism type of $M(q^{f_i})$. Now the Poincaré dual of $\mathrm{Im}(\sigma)$ is a finitely generate free abelian group $\widehat{F} \subset H_3(M)$. We choose a basis $\{x_1, \dots, x_b\}$ for $\widehat{F}$ and this is represented by a set of disjoint embeddings $\phi\colon \sqcup_{i=1}^b\colon D^4 \times S^3 \subset M$. We let M_ϕ be the outcome of surgery on ϕ. Clearly there are choices ϕ_0 and ϕ_1 for ϕ so that $M_{\phi_i} \approxeq M(q^{f_i})$. We claim that the almost diffeomorphism type of M_ϕ is independent of the choice of ϕ and this implies that $\sigma\colon \mathrm{ASplit}(M) \to \mathrm{Sec}(\pi)$ in injective.

To prove the claim, let W_ϕ be the trace of surgeries on ϕ and let $W_1 := W \cup_M W_\phi$ be the union of W_ϕ and our original 3-connected coboundary. By construction, we see that there

is a fixed $\alpha_\sigma \in R^*$ such that the characteristic form of W_1 is isomorphic to the orthogonal sum $(R, \lambda_R, \alpha_\sigma) \oplus (H_1, \lambda_1, \alpha_1)$ where $(H_1, \lambda_1, \alpha_1)$ is nonsingular. It follows that the almost diffeomorphism type of M_ϕ is well-defined. □

We conclude this subsection by identifying a simpler complete almost diffeomorphism invariant of 2-connected M. Recall that the quadratic refinements $q \in \mathcal{Q}(b)$ of a torsion form (b, T) are classified by their homogeneity defect $\beta \in 2T$ and Arf invariant $A(q) \in \mathbb{Q}/\mathbb{Z}$. For a refinement (G, q°, p) with torsion form (b, T), $\beta_h = p - 2h$ is the homogeneity defect of q^h and $\mathrm{Aut}(b)$, the group of automorphisms of b, acts on $\mathcal{Q}(b)$, the set of refinements of b. We define the *almost smooth splitting set* of M to be the set

$$\bar{\mathcal{Q}}^{\mathrm{as}}(M) := \{\big([\beta_h], A(q^h)\big) : h \in S_2\} \subset \big(2TH^4(M)/\mathrm{Aut}(b)\big) \times \mathbb{Q}/\mathbb{Z}.$$

The following classification theorem is a direct corollary of Theorem 3.4 and (31).

Corollary 3.5. *Let $F\colon (H^4(M_1), b_{M_1}, p_{M_1}) \to (H^4(M_0), b_{M_0}, p_{M_0})$ be an isomorphism of the bases of M_0 and M_1. The following are equivalent:*

(i) *M_0 is almost diffeomorphic to M_1;*
(ii) *$(F^\# \times \mathrm{Id})(\bar{\mathcal{Q}}^{\mathrm{as}}(M_0)) = \bar{\mathcal{Q}}^{\mathrm{as}}(M_1)$;*
(iii) *$(F^\# \times \mathrm{Id})(\bar{\mathcal{Q}}^{\mathrm{as}}(M_0)) \cap \bar{\mathcal{Q}}^{\mathrm{as}}(M_1) \neq \emptyset$.* □

3.3. **The homotopy classification.** In [7, §6] 2-connected M were classified up to homotopy equivalence using *J-quadratic linking families*, as we now review. For a torsion form (b, T), $\mathcal{Q}_J(b) \subset 2^{\mathcal{Q}(b)}$ was defined to be the set of subsets of $\mathcal{Q}(b)$ of the form

$$S(q) := \{q_{12t} : t \in T\}.$$

Notice that for $q_0, q_1 \in S(q)$, $\beta(q_0) - \beta(q_1) \in 24T$ and so $S(q)$ has a well-defined homogeneity defect $\beta(S(q)) \in T \otimes \mathbb{Z}/24$. For a group G, recall that $T \subseteq G$ is the torsion subgroup, $\pi\colon G \to F = G/T$ is the map to the torsion-free quotient of G, $\sigma\colon F \to G$ denotes a section of π and $\tau_\sigma\colon G \to T$ denotes the projection defined by σ. A *J-quadratic linking family* was defined to be a triple $(G, q_J^\bullet, \rho_{24}(p))$, where $\rho_{24}(p) \in G \otimes \mathbb{Z}/24$ is an even element and $q_J^\circ\colon \mathrm{Sec}(\pi) \to \mathcal{Q}_J(b)$ is a function such that for all $\phi \in \mathrm{Hom}(F, T)$ we have $q_J^{\sigma+\phi} = (q_J^\sigma)_{-\phi(\pi(p))}$ and $\beta(q_J^\sigma) = (\tau_\sigma \otimes \mathrm{Id})(\rho_{24}(p))$. The J-quadratic linking family of M, written $(H^4(M), q_{J,M}^\bullet, \rho_{24}(p_M))$ is induced from its quadratic linking family in the obvious way.

We now up-date the notion of a J-quadratic linking family to that of a J-refinement. Recall from Section 1.3 that $\rho_{12}\colon \mathcal{Q} \to J\mathcal{Q}(b)$ is the quotient map which identifies quadratic refinements $q \simeq q_{12t}$ and that a J-refinement of a base (G, b, p) is a triple $(G, Jq^\circ, \rho_{24}(p))$ where $Jq^\circ\colon S_2 \to J\mathcal{Q}(b)$ is a function satisfying $Jq^{h+t} = (Jq^j)_{-t}$ and $\rho_{24}(\beta_h) = \rho_{24}(p - 2h)$. The J-refinement of M is the triple $(H^4(M), \rho_{12} \circ q_M^\circ, \rho_{24}(p_M))$.

Given a J-refinement $(G, Jq^\circ, \rho_{24}(p))$ we define the corresponding J-quadratic linking family by setting

$$q^\bullet\colon \mathrm{Sec}(\pi) \to \mathcal{Q}_J(b), \quad q_J^\sigma := Jq^{e_\pi k(\sigma)} \tag{32a}$$

and given the function $q^\bullet\colon \mathrm{Sec}(\pi) \to \mathcal{Q}_J(b)$ of a J-quadratic linking family we define the corresponding J-refinement by setting

$$q^\circ\colon \widehat{S}_2 \to \mathcal{Q}(b), \quad q^{e_\pi k(\sigma)} := q^\sigma \tag{32b}$$

and we then extend the definition of Jq° to all of S_2 using the transformation rule for J-refinements. The correspondence between quadratic linking functions define on $\mathrm{Sec}(\pi)$ and on S_2 identified in Lemma 3.1 is easily modified to give

Lemma 3.6. *The mappings $Jq^\circ \mapsto q^\bullet_J$ and $q^\bullet_J \mapsto Jq^\circ$ of* (32a) *and* (32b) *define inverse equivalences of categories between J-refinements defined on S_2 and J-quadratic linking families defined on* $\mathrm{Sec}(\pi)$. □

Proof of Theorem 1.6. Let $(G, Jq^\circ, \rho_{24}(p))$ be a J-refinement. The transformation rule for J-refinements ensures that a J-refinement is determined by $[q^h]$ for any $h \in S_2$. Since $\mathcal{Q}(b) \to \mathcal{Q}_{12}(b)$ is onto, it follows that every J-refinement $(G, [q^\circ], \rho_{24}(p))$ is the mod 24 reduction of a refinement (G, q°, p). Theorem 1.2 then entails that every J-refinement is realised as $(H^4(M), \rho_{12} \circ q^\circ_M, \rho_{24}(p_M))$, for a 2-connected M.

If $F\colon (H^4(M_1), b_{M_1}, p_{M_1}) \to (H^4(M_0), b_{M_0}, p_{M_0})$ is an isomorphism of bases where M_0 and M_1 are 2-connected, then by [7, Theorem 6.11], $F = f^*$ for a homotopy equivalence $f\colon M_0 \to M_1$ if and only if $(q^\bullet_{J,M_0}, \rho_{24}(p_{M_0})) = F^\sharp(q^\bullet_{J,M_1}, \rho_{24}(p_{M_1}))$ and Lemma 3.6, this is happens if and only if $(Jq^\circ_{M_0}, \rho_{24}(p_{M_0})) = F^\sharp(Jq^\circ_{M_1}, \rho_{24}(p_{M_1}))$. □

3.4. **Inertia and reactivity in more detail.** Recall that $I(M)$, the inertia group of M, is the subgroup of the group of homotopy spheres $\Sigma \in \Theta_7$ such that $M\sharp\Sigma \cong M$, and that

$$I_H(M) \subseteq I(M),$$

is the subgroup of homotopy spheres Σ for which there is a diffeomorphism $f\colon M\sharp\Sigma \cong M$ such that $H^*(f) = \mathrm{Id}$, where we regard $M\sharp\Sigma$ and M as the same topological space.

One might expect that a complete understanding of $I(M)$ is needed to pass from the almost diffeomorphism classification of 2-connected 7-manifolds to the diffeomorphism classification, but it turns that that a lower bound on the order of $I_H(M)$ suffices. The main result of this subsection, Proposition 3.10, establishes this required lower bound on $I_H(M)$ for 2-connected M: see Remark 3.11. In general, computing $I(M)$ is a delicate problem which we take up for 2-connected M in Section 4. We begin this section by relating the groups $I_H(M)$ and $I(M)$ to certain mapping class groups of M.

Given an almost diffeomorphism $f \in \mathrm{ADiff}(M, m_0)$, we consider the problem of deciding whether f is pseudo-isotopic to a diffeomorphism. From Section 3.1 we recall the homotopy sphere Σ_0 which measures the singularity of f at m_0. From the definition of pseudo-isotopy in Section 3.1 we see that the diffeomorphism class of the homotopy sphere

$$\Sigma_f := \Sigma_0$$

is invariant under pseudo-isotopies. Moreover, it is clear that f defines a diffeomorphism

$$f\colon M\sharp\Sigma_f \cong M, \tag{33}$$

and that $\Sigma_{f\circ g} = \Sigma_f \sharp \Sigma_g$. Further, an application of the Alexander trick—see Rourke and Sanderson [44, Proposition 3.22]—shows that f is pseudo-isotopic to a diffeomorphism if and only if $\Sigma_f \cong S^7$. It follows that there is a *singularity homomorphism*,

$$\partial\colon \widetilde{\pi}_0\mathrm{ADiff}(M, m_0) \to \Theta_7, \quad [f] \mapsto \Sigma_f,$$

with kernel isomorphic to the image of $\widetilde{\pi}_0\mathrm{Diff}(M)$ in $\widetilde{\pi}_0\mathrm{ADiff}(M, m_0)$. We define the subgroup $\widetilde{\pi}_0\mathrm{ADiff}_H(M, m_0) \subseteq \widetilde{\pi}_0\mathrm{ADiff}(M, m_0)$ of pseudo-isotopy classes inducing the identity on $H^*(M)$ and define the singularity homomorphism $\partial_H\colon \widetilde{\pi}_0\mathrm{ADiff}_H(M, m_0) \to \Theta_7$ to be the restriction of ∂. From (33) we see that

$$I_H(M) = \mathrm{Im}(\partial_H) \quad \text{and} \quad I(M) = \mathrm{Im}(\partial). \tag{34}$$

Given $f \in \mathrm{ADiff}(M, m_0)$ we now show how to determine $\Sigma_f \in \Theta_7$ using the mapping torus of f, T_f, which is the almost smooth manifold constructed from the cylinder $M \times I$ by using f to identify points at either end:

$$T_f := (M \times [0,1])/(m, 0) \sim (f(m), 1)$$

Since f is an almost diffeomorphism, the closed 8-manifold T_f admits a smooth structure except perhaps at the point $\overline{m}_0 = [m_0, 0]$ corresponding to the singular point of f. Indeed if $B_0^8 \ni \overline{m}_i$ is a small open ball containing $\overline{m}_0$, then

$$W_f := T_f - B_0^8 \tag{35}$$

is a compact smooth manifold with boundary

$$\partial W_f \cong \Sigma_f.$$

We choose a spin structure on T_f and denote the corresponding 8-dimensional almost smooth spin manifold by T_f also: no confusion shall arise since we are interested only in the characteristic number

$$p^2(f) := \langle p_{T_f}^2, [T_f] \rangle \in \mathbb{Z},$$

which depends only on the oriented almost diffeomorphism type of T_f since $2p_{T_f} = p_1(T_f)$ and $H^8(T_f) \cong \mathbb{Z}$ (in fact p_{T_f} is independent of the choice of spin structure by [5, p. 170]). It follows that $p^2(f)$ is an invariant of the pseudo-isotopy class of f. For the statement of the next lemma, we recall the renormalised Eells-Kuiper invariant of a homotopy sphere Σ, $\mu(\Sigma) \in \mathbb{Z}/28$, defined in (3). By [16, (13)], $\mu(\Sigma_1) = \mu(\Sigma_2)$ if and only if $\Sigma_1 \cong \Sigma_2$.

Lemma 3.7. *For every almost diffeomorphism $f \in \mathrm{ADiff}(M, m_0)$ the following hold:*

(i) $p^2(f) \in 8\mathbb{Z}$,
(ii) $\mu(\Sigma_f) = \frac{p^2(f)}{8} \in \mathbb{Z}/28$,
(iii) *f is pseudo-isotopic to a diffeomorphism if and only if $p^2(f) \in 224\mathbb{Z}$.*

Proof. (i) This follows since by Lemma 2.2(iii), p_{T_f} is characteristic for the intersection form of T_f. Hence by [41, Lemma 5.2, §5], $p^2(f) \equiv \sigma(T_f) \bmod 8$. But by Novikov additivity, the signature of T_f is zero.
(ii) This follows since W_f defined in (35) above is a smooth spin coboundary for Σ_f and so can be used to compute $\mu(\Sigma_f)$. Since $\sigma(W_f) = \sigma(T_f) = 0$, applying (3) gives the result.

(iii) The almost diffeomorphism f is pseudo-isotopic to a diffeomorphism if and only if $\Sigma_f \cong S^7$. Hence (iii) follows directly from (ii). □

In the light of Lemma 3.7, we define the function

$$p^2 \colon \widetilde{\pi}_0\mathrm{ADiff}(M, m_0) \to \mathbb{Z}, \quad [f] \mapsto p^2(f).$$

Since the image of p^2 plays a key role, we define non-negative integers called the *reactivity of* M, $R(M)$, and the *(co)homologically fixed reactivity* of M, $R_H(M)$, by the following equations

$$p^2(\widetilde{\pi}_0\mathrm{ADiff}(M, m_0)) = R(M)\mathbb{Z} \quad \text{and} \quad p^2(\widetilde{\pi}_0\mathrm{ADiff}_H(M, m_0)) = R_H(M)\mathbb{Z}.$$

By Lemma 3.7 (i), $R(M)$ and $R_H(M)$ are both divisible by 8. By Lemma 3.7 (ii) and the definition of reactivity we have

Proposition 3.8.

(i) $I(M) = \frac{R(M)}{8}\Theta_7$,
(ii) $I_H(M) = \frac{R_H(M)}{8}\Theta_7$. □

For other problems, for example counting the number of deformation equivalence classes of G_2-structures on M as in [11], it important to know the value of p^2 for diffeomorphisms. Hence we defined the *smooth reactivity of* M, $R^{\mathrm{Diff}}(M)$, and the *smooth (co)homologically fixed reactivity of* M, $R_H^{\mathrm{Diff}}(M)$ by the equations

$$p^2(\widetilde{\pi}_0\mathrm{Diff}(M)) = R^{\mathrm{Diff}}(M)\mathbb{Z} \quad \text{and} \quad p^2(\widetilde{\pi}_0\mathrm{Diff}_H(M)) = R_H^{\mathrm{Diff}}(M)\mathbb{Z},$$

where $\widetilde{\pi}_0\mathrm{Diff}_H(M) \subseteq \widetilde{\pi}_0\mathrm{Diff}(M)$ is the subgroup of pseudo-isotopy classes acting trivially on $H^*(M)$. By Lemma 3.7 (iii) we have

Lemma 3.9.

(i) $R^{\mathrm{Diff}}(M) = \mathrm{lcm}(R(M), 224)$,
(ii) $R_H^{\mathrm{Diff}}(M) = \mathrm{lcm}(R_H(M), 224)$. □

We next construct almost diffeomorphisms $f \in \mathrm{ADiff}(M, m_0)$ on 2-connected M with $p^2(f) \neq 0$. Recall that d_π is the divisibility of $\pi(p_M) \in H^4(M)/TH^4(M)$ and $\tilde{d}_\pi = \mathrm{lcm}(4, d_\pi)$.

Proposition 3.10. *If M is 2-connected then $R_H(M)|2\tilde{d}_\pi$;* i.e. $2\tilde{d}_\pi\mathbb{Z} \subseteq p^2(\widetilde{\pi}_0\mathrm{ADiff}_H(M, m_0))$.

Remark 3.11. If M is 2-connected, Propositions 3.8 and 3.10 together give $\frac{\tilde{d}_\pi}{4}\Theta_7 \subseteq I_H(M)$. In Corollary 4.17 (ii) below we will show that $R_H(M) = 2\tilde{d}_\pi$ and hence $\frac{\tilde{d}_\pi}{4}\Theta_7 = I_H(M)$.

For the proof of Proposition 3.10 it will be useful to compute the characteristic number $p^2(T_f)$ using a co-bounding spin 8-manifold W. We define the closed almost smooth 8-manifold

$$X_f := (-W) \cup_f W.$$

Lemma 3.12. *With the notation above, $p^2(f) = \langle p^2_{X_f}, [X_f]\rangle$.*

Proof. Since $p^2(f) = \langle p^2_{T_f}, [T_f]\rangle$ and $\langle p^2_{X_f}, [X_f]\rangle$ are characteristic numbers, if suffices to prove that T_f is oriented bordant to X_f. Consider the manifolds $M \times I$ and $W \sqcup -W$. Both have boundary $-M \sqcup M$, T_f is formed from $M \times I$ by gluing $-M$ to M via f and X_f if formed from $-W \sqcup W$ by gluing $-M$ to M via f. It therefore suffices to prove that $-W \sqcup W$ is bordant relative to the boundary to $M \times I$. But the manifold $W \times I$ is a rel. boundary bordism from $-W \sqcup W$ to $M \times I$, and we are done. □

Proof of Proposition 3.10. We assume $d_\pi \neq 0$, since otherwise there is nothing to prove. By Theorem 3.4 or by [53, Theorem 1], we may decompose M as a connected sum of spin manifolds

$$M \cong M_0 \sharp M_1$$

where $M_0 = M(\mathbb{Z}, d_\pi) = S^3 \tilde{\times}_{d_\pi} S^4$ is the total space of a 3-sphere bundle over S^4 from Definition 3.3. We shall produce the required almost diffeomorphisms on the manifold M_0 and then extend by the identity to M. Let

$$M_0^\bullet := M_0 - \mathrm{Int}(D^7)$$

be M_0 minus a small open disc. Since M_0 is the total space of an S^3-bundle over S^4, there is a diffeomorphism

$$M_0^\bullet \cong (D^3 \tilde{\times}_{d_\pi} S^4) \cup_{S^2 \times D^4} (D^3 \times D^4),$$

where $D^3 \tilde{\times}_{d_\pi} S^4$ is a tubular neighbourhood of a section of $M_0 \to S^4$ and $D^3 \times D^4$ is a 3-handle added to $D^3 \tilde{\times}_{d_\pi} S^4$ along the tubular neighbourhood of a fibre 2-sphere, $S^2 \times D^4 \subset S^2 \tilde{\times}_{d_\pi} S^4$.

By [48, p. 171 (2)] we may identify $\pi_3(SO(4))$ as the group of pairs of integers (n, p) where $n \equiv p \bmod 2$, so that the corresponding S^3-bundle over S^4 has Euler class $n \in H^4(S^4) = \mathbb{Z}$ and first Pontrjagin class $2p$. Let $\gamma_{n,p} \colon (D^3, S^2) \to (SO(4), \mathrm{Id})$ be a smooth function representing (n, p). We define a diffeomorphism

$$f^\bullet_{n,p} \colon M_0^\bullet \cong M_0^\bullet$$

where $f^\bullet_{n,p}|_{D^3 \tilde{\times}_\alpha S^4}$ is the identity and on the 3-handle we use the D^3 co-ordinate to twist the D^4-coordinate using $\gamma_{n,p}$. To be explicit:

$$f^\bullet_{n,p}|_{D^3 \times D^4}(u, v) = (u, \gamma_{n,p}(u)(v)).$$

We observe that there is a subspace $S^3 \vee S^4 \subset M_1^\bullet$ such that the restriction $f^\bullet_{n,p}|_{S^3 \vee S^4}$ is the identity and $M_1^\bullet$ deformation retracts to $S^3 \vee S^4$. It follows that $f^\bullet_{n,p}$ acts trivially on cohomology.

Let $m_0 \subset D^7 \subset M_0$ be the centre of the disc removed to make $M_0^\bullet$. By coning the restriction of $f^\bullet_{n,p}$ to the boundary of $M_0^\bullet$, we extend $f^\bullet_{n,p}$ to an almost diffeomorphism $f_{n,p}$ of M_0 with a single singular point m_0. Since $f^\bullet_{n,p}$ acts trivially on cohomology, so does $f_{n,p}$. Since M_0 admits a unique spin structure for each orientation and since $f_{n,p}$ is orientation preserving, $f_{n,p}$ is a spin almost diffeomorphism. By construction, $f_{n,p}$ is the identity on any 7-disc contained in $D^3 \tilde{\times}_{d_\pi} S^4$ and hence we may we extend $f_{n,p}$ to M by taking the connected sum with the identity on M_1. Thus we define the spin almost diffeomorphism

$$g_{n,p} := f_{n,p} \sharp \mathrm{Id}_{M_1} \colon M \cong M$$

with single singularity at m_0 and which acts trivially on cohomology. We claim that

$$p^2(g_{n,p}) = p^2(f_{n,p}) = d_\pi(2p - nd_\pi). \tag{36}$$

The manifold $M_0 \cong S^3 \tilde{\times}_{d_\pi} S^4$ bounds the 8-dimensional D^4-bundle $W_0 := D^4 \tilde{\times}_{d_\pi} S^4$, and we let W_1 be any spin coboundary for M_1. We form the twisted doubles $X_{f_{n,p}} := (-W_0) \cup_{f_{n,p}} W_0$ and

$$X_{g_{n,p}} := (-W_0 \natural - W_1) \cup_{g_{n,p}} (W_0 \natural W_1) \cong X_{f_{n,p}} \sharp \big((-W_1) \cup_{\mathrm{id}} W_1\big). \tag{37}$$

Applying Lemma 3.12 we have,

$$p^2(g_{n,p}) = \langle p^2(X_{g_{n,p}}), [X_{g_{n,p}}] \rangle = \langle p^2(X_{f_{n,p}}), [X_{f_{n,p}}] \rangle = p^2(f_{n,p}),$$

where the second equality holds by (37) since the characteristic number p^2 is a bordism invariant, which is additive for connected sums and $(-W_1) \cup W_1 = \partial(W_1 \times I)$. Writing $X_{n,p} := X_{f_{n,p}}$, it therefore remains to compute $\langle p^2(X_{n,p}), [X_{n,p}] \rangle$. From the construction of $X_{n,p}$ we see that $H_4(X_{n,p}) \cong \mathbb{Z}(x) \oplus \mathbb{Z}(y)$ where x is represented by the zero section of W_1 and $y = [D^4 \cup D^4]$ is represented by an embedded 4-sphere obtained by gluing two fibres of the D^4-bundle W_1 together, one from each copy of W_0. By construction, the normal bundle of the 4-sphere $D^4 \cup D^4$ has characteristic function $\gamma_{n,p}$ and hence Euler number n. It follows that the intersection form of $X_{n,p}$ with respect to the basis $\{x, y\}$ is given by the following matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$$

Moreover since x is represented by an embedded 4-sphere with tubular neighbourhood diffeomorphic to $D^4 \tilde{\times}_{d_\pi} S^4$ and since y is represented by an embedded 4-sphere with normal bundle $\gamma_{n,p}$, we have $p_{X_{n,p}}(x) = d_\pi$ and $p_{X_{n,p}}(y) = p$. We conclude that the Poincaré dual of $p_{X_{n,p}}$ is given by

$$PD(p_{X_{n,p}}) = (p - nd_\pi)x + d_\pi y.$$

It follows that $\langle p^2_{X_{n,p}}, [X_{n,p}] \rangle = 2d_\pi(p - nd_\pi) + nd_\pi^2 = d_\pi(2p - nd_\pi)$, and the claim (36) is proven.

Finally we need to choose n and p so that $d_\pi(2p - nd_\pi) = 2\tilde{d}_\pi$. Recall that we may choose n and p freely subject to the constraint that $n \equiv p \bmod 2$. If $d_\pi = 4k + 2$, then $2\tilde{d}_\pi = 4d_\pi$ and we choose $(n, p) = (0, 2)$. If $d_\pi = 4k$, then $2\tilde{d}_\pi = 2d_\pi$ and we set $(n, p) = (1, 2k+1)$. □

3.5. **The proof of the main classification theorem.** The mod 28 distillation of M is the quadruple $(H^4(M), q^\circ_M, \mu_M, p_M)$ where q°_M is the quadratic linking family of M as in Definition 2.23 and the generalised Eells-Kuiper invariant

$$\mu_M \colon S_{d_\pi} \to \mathbb{Q}/\widehat{d}_\pi\mathbb{Z}$$

is the mod 28 Gauss refinement of q°_M defined by (26). In this subsection we prove Theorem 1.3 which states, in part, that mod 28 distillations give a complete invariant of diffeomorphisms of 2-connected M. For the remainder of the subsection M is 2-connected.

Recall that the (renormalised) classical Eells-Kuiper invariant, as defined by (3), gives a group isomorphism

$$\Theta_7 \cong \mathbb{Z}/28\mathbb{Z}, \quad \Sigma \mapsto \mu(\Sigma) := \mu_\Sigma(0).$$

The following lemma is obvious from the definitions of q°_M and μ_M.

Lemma 3.13. *For all $\Sigma \in \Theta_7$, $q^\circ_{M\sharp\Sigma} = q^\circ_M$ and $\mu_{M\sharp\Sigma} = \mu_M + [\mu(\Sigma)]$, where $[\mu(\Sigma)]$ is the mod $\widehat{d}_\pi$ reduction of $\mu(\Sigma)$.* □

Proof of Theorem 1.3. The existence of a smooth M with mod 28 distillation isomorphic to a prescribed $(G, q^\circ, \mu, p) \in \mathfrak{D}$ follows from the corresponding existence statement in Theorem 1.2, since Lemma 3.13 lets us freely change the Eells-Kuiper invariant of a manifold with a prescribed refinement $(G, q^\circ, p) \in \mathfrak{R}$.

By Lemma 1.7, which is proven in (24), the generalised Eells-Kuiper invariant is a diffeomorphism invariant. Now we suppose $F^\#(q^\circ_{M_0}, \mu_{M_0}, p_{M_0}) = (q^\circ_{M_1}, \mu_{M_1}, p_{M_1})$. As explained in (33), Theorem 1.2 means that there is a homotopy sphere Σ and a diffeomorphism

$$f\colon M_0\sharp\Sigma \cong M_1$$

such that $H^*(f) = F\colon H^4(M_1) \cong H^4(M_0)$. It remains to show that $\Sigma \in I_H(M_0)$. For if so, there is a diffeomorphism

$$h\colon M_0 \cong M_0\sharp\Sigma$$

with $H^*(h) = \mathrm{Id}$ and then $f \circ h\colon M_0 \cong M_1$ is a diffeomorphism with $H^*(f \circ h) = H^*(f) = F$.

Since f is a diffeomorphism it preserves the mod 28 Gauss refinements. Applying Lemma 3.13 we have

$$\mu_{M_1} = F^\#(\mu_{M_0} + \mu(\Sigma)) = F^\#(\mu_{M_0}) + \mu(\Sigma).$$

On the other hand, our assumption is that $F^\#(\mu_{M_0}) = \mu_{M_1}$. Since $d_{M_0} = d_{M_1}$,

$$\mu(\Sigma) = \mu_{M_1} - F^\#(\mu_{M_0}) = 0 \in \mathbb{Z}/d_{M_1}\mathbb{Z} = \mathbb{Z}/d_{M_0}\mathbb{Z}.$$

By Remark 3.11, $\Sigma \in I_H(M_0)$ and this completes the proof. □

Proof of Theorem 1.5. That the functor $\mathcal{D}\colon \mathcal{M}^{\mathrm{Spin}}_{7,2} \to \mathfrak{D}$ is surjective and faithful is a restatement of Theorem 1.3. To see that $\mathcal{D}$ is additive and compatible with orientation reversal, let $i = 0, 1$, and let $M_i = \partial W_i$ where W_i has characteristic from $(H_i, \lambda_i, \alpha_i)$ with boundary refinement q°_i. The mod 28 Gauss refinement of M_i is $(q^\circ_i, \partial g_{W_i})$ where we set $\partial g_{W_i} := g_{W_i} \bmod \widehat{d}_\pi$ as in (26). Since the characteristic form of $-W_i$ is $(H_i, -\lambda_i, \alpha_i)$ and the characteristic form of the boundary connected sum $W_0\natural W_1$ is $(H_0, \lambda_0, \alpha_0) \oplus (H_1, \lambda_1, \alpha_1)$, it follows that mod 28 Gauss refinements of $-M_i$ and $M_0\sharp M_1$ are $(-q^\circ_i, -\partial g_{W_i})$ and $(q^\circ_1, \partial g_{W_0}) \oplus (q^\circ_1, \partial g_{W_1})$ respectively. □

3.6. **Smooth splitting functions.** In this subsection we consider connected sum splittings of 2-connected M in the smooth category and we prove a smooth analogue of Theorem 3.4. We also prove Theorem 1.4, which is the smooth analogue of Corollary 3.5. Throughout this subsection M is 2-connected.

Let (T, b) be a torsion linking form and d an even integer. We define the set

$$\widehat{\mathcal{Q}}_d(b) := \{(q, s)\} \subset \mathcal{Q}(b) \times \mathbb{Q}/d\mathbb{Z},$$

which consists of pairs of quadratic refinements q of b and rational residues mod d where $A(q) \equiv s \bmod \mathbb{Z}$. By Theorem 1.3, rational homotopy spheres M with torsion linking forms

$(H^4(M), b_M) = (T, b)$ are classified up to diffeomorphism by the pair $(q_M, \mu(M)) \in \widehat{\mathcal{Q}}_{28}(b)$. We denote this rational homotopy sphere by

$$M = M(q, s),$$

where $q = q_M$ and $s = \mu(M)$. Suppose that we are given a base (G, b, p) with $F = G/T \cong \mathbb{Z}^b$ and $\pi(p) \in F$ of divisibility d_π; as in §2.5, π denotes the projection $G \to F$. By Theorem 3.4, if M has base $(H^4(M), p_M, b_M) \cong (G, b, p)$, then for some rational homotopy sphere $M(q, s(f))$ and base-point $m_0 \in M$, there is a connected sum splitting

$$f\colon M \cong M(q, s(f))\sharp M(F, d_\pi),$$

where $f(m_0) \in M(q, s(f))$. Recall that $\sigma(f) \in \mathrm{Sec}(\pi)$ is given by $\mathrm{Im}(\sigma) = f^*(H^4(M(F, d_\pi)))$. In considering the uniqueness of $M(q, s)$ in such a splitting, we note that by Theorem 1.3, $I(M(F, d_\pi)) = \widehat{d}_\pi\Theta_7$ (see also Remark 3.11). As a consequence, for $i = 0, 1$, we see that if $(q, s_i) \in \mathcal{Q}_{28}(b)$ and $s_0 \equiv s_1 \bmod \widehat{d}_\pi\mathbb{Z}$, then there is a diffeomorphism

$$h\colon M(q, s_0)\sharp M(F, d_\pi) \cong M(q, s_1)\sharp M(F, d_\pi) \tag{38}$$

such that $H^*(h)$ preserves the induced splittings of H^4. We define two splittings f_0 and f_1 to be H^*-equivalent if there is an almost diffeomorphism $g \in \mathrm{ADiff}(M, m_0)$ with $g(m_0) = m_0$, $H^*(g) = \mathrm{Id}$ and $\Sigma_g \in \widehat{d}_\pi\Theta_7$, an almost diffeomorphism $g_T\colon M(q, s_0) \cong M(q, s_1)$ with singular point $f_0(m_0)$ and $\Sigma_{g_T} = \Sigma_g$ and a diffeomorphism $g_F\colon M(F, d_\pi) \cong M(F, d_\pi)$ such that $g_T(f_0(m_0)) = f_1(m_0)$ and the following diagram commutes up to pseudo-isotopy:

$$\begin{array}{ccc} M & \xrightarrow{f_0} & M(q^{f_0}, s(f_0))\sharp M(F, d_\pi) \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_T\sharp g_F} \\ M & \xrightarrow{f_1} & M(q^{f_1}, s(f_1))\sharp M(F, d_\pi) \end{array}$$

We define $\mathrm{Split}(M) := \{[f]\}$ to be the set of H^*-equivalence classes of splittings of M. We also define the *smooth splitting function* of M

$$\widehat{q}^{\bullet}_M\colon \mathrm{Sec}(\pi) \to \widehat{\mathcal{Q}}_{\widehat{d}_\pi}(b), \quad \sigma \mapsto \widehat{q}^{\sigma}_M := \big(q_M^{e_\pi k(\sigma)}, \mu_M(k(\sigma))\big),$$

where we recall that $k(\sigma) \in S_{d_\pi}$ is defined by $k(\sigma) \in \mathrm{Im}(\sigma) \cap S_{d_\pi}$. From the diffeomorphism in (38), we see that $M\big(q^{e_\pi k(\sigma)}, \mu(k(\sigma))\big)\sharp M(F, d_\pi)$ gives a well-defined diffeomorphism type for each section σ.

Theorem 3.14. *Let M have smooth splitting function $\widehat{q}^{\bullet}_M\colon \mathrm{Sec}(\pi) \to \widehat{\mathcal{Q}}_{\widehat{d}_\pi}(b_M)$. For each $\sigma \in \mathrm{Sec}(\pi)$ there is a unique H^*-equivalence class of splitting*

$$f_\sigma\colon M \cong M(\widehat{q}^{\sigma}_M)\sharp M(F, d_\pi).$$

Consequently the map $\mathrm{Split}(M) \to \mathrm{Sec}(\pi), [f] \mapsto \sigma(f)$ is a bijection.

Proof. The proof is a refined version of the proof Theorem 3.4 and we adopt the notation of that proof so that M has 3-connected coboundary W. Specifically, the proof of the existence of f_σ is verbally the same, except that now by [7, Definition 2.50] we have

$$\widehat{q}^{\sigma}_M = \big(\partial(R, -\lambda_R, \alpha_\psi), \big[\big(\bar{\lambda}_R(\alpha_\psi, \alpha_\psi) - \sigma(\lambda_R)\big)/8\big]\big)\,.$$

Hence the splitting $W \cong W_\psi \natural W_F$ defines the splitting $f_\sigma \colon M \cong M(\widehat{q}_M^\sigma) \sharp M(F, d_\pi)$.

To show that splittings f_0 and f_1 defining the same section σ are H^*-equivalent, we consider the nonsingular characteristic form $(H_1, \lambda_1, \alpha_1)$. The symmetric form (H_1, λ_1) has a Lagrangian $L \subset H_1$ corresponding to $H^3(M)$ and hence $\alpha_1(L) = d_\pi \mathbb{Z}$. The proof of Proposition 3.10 now shows that $\frac{\tilde{d}_\pi}{4}$ divides $(\bar{\lambda}_1(\alpha_1, \alpha_1) - \sigma(\lambda_1))/8$. Consequently the diffeomorphism type of M_ϕ is determined up to connected sum with $\Sigma \in \widehat{d}_\pi \Theta_7$, and this shows that f_0 and f_1 are H^*-equivalent splittings. □

Proof of Theorem 1.4. For a 2-connected M, define an action of $\mathrm{Aut}(b_M)$ on $\widehat{\mathcal{Q}}_{\widehat{d}_\pi}(b_M)$ by $F \cdot (q, s) = (q \circ F, s)$ and let $[q, s]$ denote the $\mathrm{Aut}(b_M)$ orbit of (q, s). We define the map

$$\beta \colon \widehat{\mathcal{Q}}_{\widehat{d}_\pi}(b_M) \to 2T/\mathrm{Aut}(b_M) \times \mathbb{Q}/\widehat{d}_\pi \mathbb{Z}, \quad (q, s) \mapsto ([\beta_q], s).$$

Since the Gauss sum of each q is given by $A(q) = s \bmod \mathbb{Z}$, we note that Theorem 2.16 ensures that $[q, s] = [q', s']$ if and only if $\beta(q, s) = \beta(q', s')$.

Now for $i = 0, 1$, let M_0 and M_1 have smooth splitting functions $\widehat{q}_i^\bullet \colon \mathrm{Sec}(\pi_i) \to \widehat{\mathcal{Q}}_{\widehat{d}_\pi}(b_{M_i})$, where $\pi_i \colon H^4(M_i) \to H^4(M_i)/TH^4(M_i)$ is the projection and suppose there there is an isomorphism $F \colon (H^4(M_1), b_{M_1}, p_{M_1}) \to (H^4(M_0), b_{M_0}, p_{M_0})$ of their bases. By Theorem 3.14, M_0 and M_1 are diffeomorphic if and only if there are sections σ and σ', a homotopy sphere $\Sigma \in \widehat{d}_\pi \Theta_7$ and a diffeomorphism $M(\widehat{q}_0^\sigma) \cong M(\widehat{q}_1^{\sigma'}) \sharp \Sigma$. But this happens if and only if there is an isomorphism $q_0^\sigma \cong q_1^{\sigma'}$ and $\mu_0(k(\sigma)) = \mu_1(k(\sigma'))$; *i.e.* if and only if $(F^\# \times \mathrm{Id})(\beta(\widehat{q}_0^\sigma)) = \beta(\widehat{q}_1^{\sigma'})$. With the notation above the smooth splitting set of M_i, defined in the introduction just prior to the statement of Theorem 1.4, is the set

$$\bar{\mathcal{Q}}(M_i) = \{\beta(\widehat{q}_i^\sigma) : \sigma \in \mathrm{Sec}(\pi_i)\}.$$

The above shows that M_0 and M_1 are diffeomorphic if and only if the sets $(F^\# \times \mathrm{Id})\big(\bar{\mathcal{Q}}(M_0)\big)$ and $\bar{\mathcal{Q}}(M_1)$ intersect and consequently this happens if and only if these sets coincide. □

4. Automorphisms of $H^4(M)$

The smooth classification Theorem 1.3 implies that the number of different smooth structures on the same 2-connected almost-smooth 7-manifold corresponds to the number of different mod 28 Gauss refinements of the linking family $(H^4(M), q_M^\circ, p_M)$. The first estimate of the number of smooth structures on M, $\widehat{d}_\pi = \gcd(\frac{\tilde{d}_\pi}{4}, 28)$, only counts smooth structures on M modulo almost diffeomorphisms that act trivially on $H^4(M)$. To get the full picture, we need to understand how automorphisms of the quadratic linking family q_M° act on the Gauss refinements. We begin this process in Section 4.2.

Conveniently enough, it turns out that this problem can be reduced to understanding how automorphisms of the base $(H^4(M), b_M, p_M)$ act on linked functions: see Proposition 4.10 in Section 4.3. While we do not have a complete description of this action in general we still have control up to a factor 2^r, where $r \in \{0, 1, 2\}$ is explicitly defined in (42). Moreover it is feasible to understand it for explicit examples: see Examples 4.7, 4.8, 4.12 and 4.13. With Proposition 4.10 in hand, we proceed in Section 4.4 to determine the reactivity of 2-connected M in terms of r and the integer d_o defined in (7) and recalled in Section 4.1.

4.1. **Notation.** We begin by setting up some terminology. Given a finitely generated abelian group G, $p \in 2G$, and $b\colon T \times T \to \mathbb{Q}/\mathbb{Z}$ a torsion form, let Aut_b denote the group of isomorphisms $F\colon G \to G$ preserving p and b. If q° is a family of quadratic refinements of (G, b, p), let Aut_{q° be the subgroup of Aut_b that preserves q° too.

Let $\pi\colon G \to G/T$ be the projection to the free quotient of G. Let $\mathrm{Shr}_p \subseteq \mathrm{Aut}_b$ be the subgroup of "pure shears", *i.e.* F acting trivially on T and G/T. In other words, $F = \mathrm{Id}_G + \rho \circ \pi$ for some homomorphism $\rho\colon G/T \to T$ such that $\rho\left(\frac{\pi(p)}{d_\pi}\right)$ is d_π-torsion (the last condition is equivalent to $F(p) = p$); so actually Shr_p does not depend on b at all. Similarly let $\mathrm{Shr}_{p/2} = \mathrm{Shr}_p \cap \mathrm{Aut}_{q^\circ}$, the subgroup of shears in Aut_{q°. For $h \in S_2$, $(F^\# q)^h = q^{F(h)} = q^{h+\rho\left(\frac{\pi(p)}{2}\right)} = q^h_{-\rho\left(\frac{\pi(p)}{2}\right)}$, so for F to preserve q we need $\rho\left(\frac{\pi(p)}{2}\right) = 0$. Hence $\mathrm{Shr}_{p/2}$ simply corresponds to homomorphisms $\rho\colon G/T \to T$ such that $\rho\left(\frac{\pi(p)}{d_\pi}\right)$ is $\frac{d_\pi}{2}$-torsion. In particular: $\mathrm{Shr}_{p/2}$ actually depends on neither q° nor b but only on p, and if $F \in \mathrm{Shr}_p$ then $F^2 \in \mathrm{Shr}_{p/2}$.

We say that a cyclic subgroup $C \subseteq T$ is a *split summand* if T is a direct sum of C and its b-orthogonal complement. We call $x \in T$ *split* if it generates a split summand; this is equivalent to

$$b(x, x) = \frac{m}{n},$$

where n is the order of x and m is coprime to n.

Given the element $p \in 2G$ we consider the following notions of its divisibility (if p is a torsion element we set all three integers to be 0):

$$\begin{aligned} d &:= \mathrm{Max}\{s \in \mathbb{Z} : s \text{ divides } p \in G\},\\ d_\pi &:= \mathrm{Max}\{s \in \mathbb{Z} : s \text{ divides } \pi(p) \in G/T\},\\ d_o &:= \mathrm{Max}\{s : s, m \in \mathbb{Z},\ sm^2 \text{ divides } mp \in G\}. \end{aligned}$$

We have an obvious chain of divisibilities

$$2 \mid d \mid d_o \mid d_\pi.$$

Further $d = d_\pi$ if and only if $d_o = d_\pi$, since the latter implies that the maximum in the definition of d_o is attained with $m = 1$.

For an integer s, let $\mathrm{ord}_2 s$ denote the exponent of 2 in the prime factorisation of s; *e.g.* $\mathrm{ord}_2 2^j = j$.

Definition 4.1. A non-negative integer e is a 2-*extremal exponent* for (G, p) if for some m such that $d_o m^2$ divides mp, $\mathrm{ord}_2 m = e$.

Example 4.2. Let $p = (2^a, 2^c) \in \mathbb{Z} \times \mathbb{Z}/2^b\mathbb{Z}$ with $a, b \geq c \geq 1$. Then $d_\pi = 2^a$, $d = 2^c$, and $d_o = \max(2^c, 2^{a-b+c})$. The 2-extremal exponents are 0 for $a \leq b$, and $b - c$ for $a \geq b$.

4.2. **The action of** Aut_b **on linked functions.** Given $F \in \mathrm{Aut}_b$ and any $k \in S_{d_\pi}$, set $t := F(k) - k \in T$ (not necessarily d_π-torsion, unless $F_{|T}$ is the identity) and $\beta_k := p - d_\pi k$, and let

$$P(F) := d_\pi^2 b(t, t) - 2d_\pi b(\beta_k, t) \in \mathbb{Q}/2d_\pi\mathbb{Z}. \tag{39}$$

In other words, $P(F) = -\Delta(k,t)$ from (21). Equivalently, we can characterise $P(F)$ by

$$F^{\#}g = g - \frac{P(F)}{8} \mod \frac{d_\pi}{4}\mathbb{Z} \tag{40}$$

for any linked function g (use that $(F^{\#}g)(k) = g(F(k)) = g(k+t) = g(k) + \frac{\Delta(k,t)}{8}$ by the condition (20) for g to be a linked function.) The first characterisation, (39), is independent of g and the second, (40), of k, so in fact P depends on neither. If F preserves a family of quadratic refinements, then taking g to be a Gauss refinement of that family shows that P takes values in $8\mathbb{Z}/2d_\pi\mathbb{Z}$ (in the next subsection we study a corresponding $8\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$-valued function $\widetilde{P}$). Even if F does not preserve a family of quadratic refinements, the fact that the mod $\frac{1}{4}\mathbb{Z}$ reduction of the Arf invariant of a quadratic refinement of (b,p) depends only on (b,p) itself shows that P takes values in $2\mathbb{Z}$. It is also clear from (40), or from (39) together with (22), that P is a homomorphism $\mathrm{Aut}_b \to 2\mathbb{Z}/2d_\pi\mathbb{Z}$.

Let $j_\pi = \mathrm{ord}_2 d_\pi$, and $j_o = \mathrm{ord}_2 d_o$.

Lemma 4.3. *$P(\mathrm{Aut}_b) \subseteq d_o\mathbb{Z}/2d_\pi\mathbb{Z}$. If b lacks a split 2^{2e+j_o} summand for some 2-extremal exponent e, then $P(\mathrm{Aut}_b) \subseteq 2d_o\mathbb{Z}/2d_\pi\mathbb{Z}$.*

Proof. Pick some $y \in G$ such that $m^2 d_o y = mp$. Then $s := Fy - y$ is an $m^2 d_o$-torsion element. It suffices to show that

$$P(F) = m^2 d_o^2 b(s,s) \mod 2d_o, \tag{41}$$

because the RHS is d_o if the 2-primary part of s is split, and 0 otherwise.

Note that $u := \frac{d_\pi}{d_o}$ and $\frac{m}{u}$ are integers. Let $k := \frac{m}{u}y$. Then $k \in S_{d_\pi}$, and $ut = ms$, so (39) implies

$$P(F) = u^2 d_o^2 b(t,t) - 2ud_o b(\beta_k, t) = d_o^2 b(ms, ms) - 2d_o b(\beta_k, ms) \mod 2d_o.$$

Since $\beta_k = p - d_\pi k$ is m-torsion, (41) and the result follows. □

If $F \in \mathrm{Shr}_p \subseteq \mathrm{Aut}_b$, *i.e.* $F = \mathrm{Id}_G + \rho \circ \pi$ for some homomorphism $\rho\colon G/T \to T$, then $t = F(k) - k = \rho\big(\frac{\pi(p)}{d_\pi}\big)$ is independent of the choice of $k \in S_{d_\pi}$. Since $\frac{\pi(p)}{d_\pi} \in G/T$ is a primitive element of a free abelian group, we can prescribe its image under a homomorphism ρ arbitrarily. Determining the image $P(\mathrm{Shr}_p)$ therefore amounts to computing the RHS of (39) for all d_π torsion elements $t \in T$.

Lemma 4.4. *$4d_o\mathbb{Z}/2d_\pi\mathbb{Z} \subseteq P(\mathrm{Shr}_p)$. Moreover, if $j_\pi \neq j_o + 1$ or if b has no split 2^{j_π} summand, then $2d_o\mathbb{Z}/2d_\pi\mathbb{Z} \subseteq P(\mathrm{Shr}_p)$.*

Proof. The key claim is that there exists a d_π-torsion element t such that $b(\beta_k, t) = \frac{1}{u}$, where $d_\pi = ud_o$. By the non-degeneracy of b, this is equivalent to β_k having order at least u, and not being divisible by more than d_o. That any divisor of β_k also divides d_o is obvious, and if $m\beta_k = 0$ then mp is divisible by md_π, which indeed implies $u \mid m$ by the definition of d_o.

Let ρ be any homomorphism $G/T \to T$ mapping $\frac{\pi(p)}{d_\pi} \mapsto t$, and $F := \mathrm{Id}_G + \rho \circ \pi \in \mathrm{Shr}_p$. If the 2-primary part of t does not generate a split 2^{j_π} summand then $d_\pi^2 b(t,t)$ is divisible

by $2d_\pi$, so

$$P(F) = d_\pi^2 b(t,t) - 2d_\pi b(\beta_k, t) = 2d_o \mod 2d_\pi,$$

and we are done. Otherwise $P(F) = d_\pi - 2d_o = (u-2)d_o \bmod 2d_\pi$. The subgroup this generates is precisely $nd_o\mathbb{Z}/2d_\pi\mathbb{Z}$, where $n = \gcd(u-2, 2u) = \gcd(u-2, 4)$. Clearly n is 1 or 2 except when $j_\pi = j_o + 1$, in which case $n = 4$. □

Lemmas 4.3 and 4.4 imply that the following is well-defined.

Definition 4.5. Define $r = r(G,p,b) \in \{0,1,2\}$ by

$$\operatorname{Im} P = 2^r d_o\mathbb{Z}/2d_\pi\mathbb{Z}. \tag{42}$$

Remark 4.6. Lemmas 4.3 and 4.4 provide necessary conditions for $r = 0$ or $r = 2$. In particular, if G has no 2-torsion then $r = 1$. The next examples show that there are bases with $r = 0$ and bases with $r = 2$.

Example 4.7. Let $G = \mathbb{Z} \oplus \mathbb{Z}/2^j$, $b = \langle \frac{1}{2^j} \rangle$ and $p = (2^j, 0)$ (so $d_\pi = d_o = 2^j$). Then the shear $F\colon (x,y) \mapsto (x, x+y)$ has $P(F) = 2^j \bmod 2^{j+1}$, *i.e.* $P(F) = d_o \bmod 2d_\pi$. Thus $r = 0$.

Example 4.8. Let $G = \mathbb{Z} \oplus \mathbb{Z}/2^j$, $b = \langle \frac{1}{2^j} \rangle$ and $p = (2^j, 2^{j-1})$ (so $d_\pi = 2^j$, while $d_o = 2^{j-1}$). Now any $t \in T$ has $d_\pi^2 b(t,t) + 2d_\pi(\beta_k, t) = 0 \bmod 2^{j+1}$, so $r = 2$.

4.3. **The action of Aut_{q° on Gauss refinements.** Now let q° be a family of quadratic refinements of the base (G,b,p), and let Aut_{q° denote its group of automorphisms. For an automorphism $F \in \mathrm{Aut}_{q^\circ}$ we define $\widetilde{P}(F) \in 8\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$ by

$$\widetilde{\Delta}(F) = -4d_\pi q^{e_\pi k}(t) + d_\pi(d_\pi+2)\, b(t,t)$$

for any $k \in S_{d_\pi}$ and $t := F(k) - k$; equivalently, $\widetilde{P}(F) = -\widetilde{\Delta}(k,t)$. Now $\widetilde{P}$ is a homomorphism $\widetilde{P}\colon \mathrm{Aut}_{q^\circ} \to 8\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$, such that

$$F^\# g = g - \frac{\widetilde{P}(F)}{8} \mod \frac{\tilde{d}_\pi}{4}\mathbb{Z} \tag{43}$$

for any Gauss refinement g of q°. Notice that, similarly to the proof of Lemma 2.32, $\widetilde{P}$ can alternatively be characterised by

$$\widetilde{P}(F) = P(F) \mod 2d_\pi,$$
$$\widetilde{P}(F) = 0 \mod 8.$$

We can therefore get some control on the image of the shear subgroup $\mathrm{Shr}_{p/2} \subseteq \mathrm{Aut}_{q^\circ}$ just from the observation that $F^2 \in \mathrm{Shr}_{p/2}$ for any $F \in \mathrm{Shr}_p$.

Lemma 4.9. $\widetilde{P}(\mathrm{Shr}_{p/2}) \supseteq 4d_o\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$.

Proof. The proof of Lemma 4.4 showed that we can achieve $P(F) = 2d_o$ or $P(F) = 2d_o + d_\pi$ for some $F \in \mathrm{Shr}_p$. Then $F^2 \in \mathrm{Shr}_{p/2}$ has $P(F^2) = 4d_o$. □

Conveniently, it turns out that the image of $\widetilde{P}$ can be determined directly from the image of P.

Proposition 4.10. $\operatorname{Im}\widetilde{P} = \{n \in 8\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z} : n \bmod 2d_\pi \in \operatorname{Im} P\} = \operatorname{lcm}(8, 2^r d_o)\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$.

Proof. If d_π is not divisible by 4 then $\operatorname{lcm}(8, 2^r d_o) = 4d_o$, so the result follows from Lemma 4.9.

If $4 \mid d_\pi$ and $F \in \operatorname{Aut}_b$, then that for any $k \in S_{d_\pi}$, $t := F(k) - k$ and $h := \frac{d_\pi}{2}k$ we have

$$q^h\left(-\tfrac{d_\pi}{2}t\right) = \frac{P(F)}{8} \mod \mathbb{Z}.$$

Thus $P(F) = n \in 8\mathbb{Z}/2d_\pi\mathbb{Z}$ implies that $(F^\# q)^h = q^{F(h)} = q^h_{-\frac{d_\pi}{2}t}$ and q^h, which have equal inhomogeneity $\beta_h = p - 2h$ by definition, also have equal Arf invariant by (11). Therefore by Theorem 2.16 there is an automorphism F_T of (T, b) such that $(F^\# q)^h \circ F_T = q^h$ (necessarily F_T fixes β_h).

Now suppose that σ is a section of π, and $k \in \operatorname{Im}\sigma \cap S_{d_\pi}$ (*cf.* Remark 2.26). Then $G \cong \operatorname{Im}\sigma \oplus T$, and we may define $\operatorname{Id}_{\operatorname{Im}\sigma} + F_T \in \operatorname{Aut}_b$. This fixes k and h, so the composition $F' := F \circ (\operatorname{Id}_{\operatorname{Im}\sigma} + F_T)$ has

$$(F'^\# q)^h = \left((\operatorname{Id}_{\operatorname{Im}(\sigma)} + F_T)^\# F^\# q\right)^h = (F^\# q)^h \circ F_T = q^h.$$

Hence $F' \in \operatorname{Aut}_{q^\circ}$, and $F'(k) = F(k)$ implies $P(F') = P(F)$. □

Example 4.11. For the base $(\mathbb{Z} \oplus \mathbb{Z}/2^j, \langle\frac{1}{2^j}\rangle, (2^j, 0))$ of Example 4.7 let q° be the refinement with $q^{(2^{j-1},0)} = \left\langle\!\left\langle \frac{1}{2^{j+1}} \right\rangle\!\right\rangle$. The isomorphism F of the base in Example 4.7 does not preserve q°: if $j > 1$ then F alters the homogeneity defect of $q^{(2^{j-1},0)}$ and if $j = 1$ then F alters the Arf invariant. However, if $j \geq 3$ then $F'\colon (x, y) \mapsto (x,\ x + (2^{j-1}{+}1)y)$ is an isomorphism of q° with $\widetilde{P}(F') = P(F) = 2^m \bmod 2^{j+1}$.

The following examples illustrate that r, and hence $\operatorname{Im}\widetilde{P}$, can depend on b as well as (G, p).

Example 4.12. Let $G = \mathbb{Z} \oplus (\mathbb{Z}/2^j)^2$ and $p = (2^j, 0, 0)$ (so $d_\pi = d_o = 2^j$). Choosing the torsion form $b_0 = \langle\frac{1}{2^j}\rangle \oplus \langle\frac{1}{2^j}\rangle$ on TG, using Example 4.7 shows that $r_0 = 0$. Let b_1 be the hyperbolic torsion form on TG with matrix

$$\begin{pmatrix} 0 & 2^{-j} \\ 2^{-j} & 0 \end{pmatrix}.$$

Since $d_\pi = d_o$, it follows that $r_1 = 0$ or 1. But b_1 contains no split cyclic summands, and so by Lemma 4.4, we conclude that $r_1 = 1$.

The next example shows that r cannot be determined merely from the type of splitting of b (cyclics versus hyperbolics), but can depend on the isomorphism classes of split cyclic summands.

Example 4.13. Let $G = \mathbb{Z} \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/64 \oplus \mathbb{Z}/512$ with torsion form $\left\langle\frac{1}{8}\right\rangle \oplus \left\langle\frac{1}{64}\right\rangle \oplus \left\langle\frac{\epsilon}{512}\right\rangle$ ($\epsilon = \pm 1$), and $p = (64, 0, 8, 0)$. Then $d_\pi = 64$ and $d_o = 8$, so r is 0 or 1. The 2-extremal exponents are 0 and 3. If $F \in \operatorname{Aut}_b$ then by (41)

$$\begin{aligned} P(F) = d_o \bmod 2d_o &\Leftrightarrow (\operatorname{Id} - F)(1, 0, 0, 0) \text{ split 512-torsion} \\ &\Leftrightarrow (\operatorname{Id} - F)(8, 0, 1, 0) \text{ split 8-torsion.} \end{aligned}$$

Thus if $r = 0$ there must be some automorphism f of (T, b) such that $(\mathrm{Id} - f)(0, 1, 0)$ plus a split 8-torsion element is divisible by 8, *i.e.* $f(0, 1, 0) = (a, 8b + 1, 8c)$ with a odd. If $\epsilon = +1$ then the would-be image has norm $\frac{17}{64}$ for any a, b, c, so there can be no such f; hence $r = 1$. On the other hand, if $\epsilon = -1$ we can define such an f by the matrix

$$\begin{pmatrix} 1 & -8 & 0 \\ 1 & 1 & 8 \\ 1 & 1 & 1 \end{pmatrix}.$$

Setting $F = \mathrm{Id}_{\mathbb{Z}} + \rho + f$ with $\rho\colon \mathbb{Z} \to T, n \mapsto (0, 0, -n)$ makes $\mathrm{Id} - F$ map $(8, 0, 1, 0) \in G$ to the split 8-torsion element $(1, 0, 0) \in T$ (and $(1, 0, 0, 0)$ to $(0, 0, -1)$), so $P(F) = d_o \bmod 2d_o$, and $r = 0$.

Corollary 4.14. *Modulo the action of* Aut_{q°*, the number of possible Gauss refinements of* (G, q°, p) *is*

$$\mathrm{Num}\left(\frac{2^r d_o}{8}\right),$$

and the number of possible mod 28 Gauss refinements is

$$\gcd\left(28, \mathrm{Num}\left(\frac{2^r d_o}{8}\right)\right).$$

Remark 4.15. Notice that Corollary 4.14 combined with Theorems 1.2 and 1.3 gives the computation of the inertia group $I(M)$ for 2-connected M from Theorem 1.10.

4.4. **The computation of reactivity.** In this subsection we use Proposition 4.10 to prove lower bounds on the reactivity of every spin 7-manifold M. When M is 2-connected we also prove that this lower bound is sharp and so compute the reactivity of 2-connected M. Recall from Section 3.4 that if f is a self-almost diffeomorphism of M, then the mapping torus T_f is almost smooth, the spin characteristic class $p_{T_f} \in H^4(T_f)$ is well defined and so is the integer

$$p^2(f) = \langle p_{T_f}^2, [T_f] \rangle \in 8\mathbb{Z}.$$

The next lemma provides the bridge between the algebraic arguments of Sections 4.2 and 4.3 and the computation of reactivity of M, as defined in (5). Note that if f is a self-almost diffeomorphism of M, then the induced map $f^* : H^4(M) \to H^4(M)$ preserves q_M°, *i.e.* we have $f^* \in \mathrm{Aut}_{q_M^\circ}$ and so $\widetilde{P}(f^*) \in 8\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$ is defined.

Proposition 4.16. $\widetilde{P}(f^*) = p^2(f) \bmod 2\tilde{d}_\pi$ *for any self-almost diffeomorphism of* M*.*

Proof. Let $f\colon M \cong M$ be a self-almost diffeomorphism and let W be a 3-connected spin coboundary for M. In §2.6 we used g_W to denote the Gauss refinement of q_M° induced by the form $(FH^4(W, M), \lambda_W, p_W)$ (and used that to define μ_M). We can use f to glue two copies of W together along M and form the almost smooth spin manifold $X := (-W) \cup_f W$. Lemma 3.12 gives $p^2(f) = p_X^2$. Applying (43) to $F = f^*$ and combining with the comparison of Gauss refinements in (24) we obtain

$$\widetilde{P}(f^*) \equiv 8(g_W - (f^*)^\# g_W) \equiv p_X^2 - \sigma(X) \equiv p_X^2 \equiv p^2(f) \mod 2\tilde{d}_\pi,$$

where $\sigma(X) = \sigma(W) - \sigma(W) = 0$ by Novikov additivity. □

Corollary 4.17. *For any closed spin 7-manifold M we have that:*

(i) $R_H(M)$ *is divisible by* $2\tilde{d}_\pi$*;*
(ii) $R(M)$ *is divisible by* $\operatorname{lcm}(8, 2^r d_o)$*;*
(iii) $R_H^{\mathrm{Diff}}(M)$ *is divisible by* $\operatorname{lcm}(224, 2\tilde{d}_\pi)$*;*
(iv) $R^{\mathrm{Diff}}(M)$ *is divisible by* $\operatorname{lcm}(224, 2^r d_o)$.

If M is 2-connected then equality holds in each case.

Proof. For part (i), recall that for $[f]$ to belong to $\widetilde{\pi}_0\mathrm{ADiff}_H(M, m_0)$ by definition means that $f^* = \mathrm{Id}$ on $H^4(M)$. Thus $\widetilde{P}(f^*) = 0 \in 8\mathbb{Z}/2\tilde{d}_\pi$, and Proposition 4.16 implies that $2\tilde{d}_\pi \mid R_H(M)$. Meanwhile Proposition 3.10 shows that $R_H(M) \mid 2\tilde{d}_\pi$ if M is 2-connected.

For part (ii), let M have refinement (G, q°, p). Proposition 4.10 computes the image $\operatorname{Im}\widetilde{P} = \operatorname{lcm}(8, 2^r d_o)\mathbb{Z}/2\tilde{d}_\pi$, so Proposition 4.16 gives $\operatorname{lcm}(8, 2^r d_o) \mid R(M)$. On the other hand, if M is 2-connected then Theorem 1.2 states that every automorphism of (G, q°, p) is realised by an almost diffeomorphism $f\colon M \approxeq M$, and so part (i) and Proposition 4.10 imply the equality.

Parts (iii) and (iv) follow from parts (i) and (ii) and Lemma 3.9. □

Proof of Theorem 1.10. The computation of $R(M) = \operatorname{lcm}(8, 2^r d_o)$ is given in Corollary 4.17(ii). Then $I(M) = \mathrm{Num}\big(\frac{2^r d_o}{8}\big)\Theta_7$ by Proposition 3.8(i). By Remark 4.6, $r = 1$ if $TH^4(M)$ is of odd order. □

5. Examples

Ever since Milnor's discovery of exotic 7-spheres [38], 2-connected 7-manifolds have provided interesting examples in topology and geometry. In this section we discuss various examples of 2-connected 7-manifolds. In Section 5.1 we consider the total spaces of 3-sphere bundles over S^4 and their connected sums. In Section 5.2 we mention some examples admitting interesting metrics. In Section 5.3 we give examples which are tangentially homotopy equivalent but not homeomorphic. Finally in Section 5.4 we present a refinement of Wilkens' list [53, Theorem 1] of the indecomposable generators for the monoid of almost diffeomorphism classes of 2-connected 7-manifolds.

5.1. **3-sphere bundles over S^4 and their connected sums.** Following the notation of [10], let (n, p) be integers with same parity and let $M_{n,p} := S(\xi_{n,p})$ denote the total space of the 3-sphere bundle over S^4 which there corresponding vector bundle $\xi_{n,p}$ has Euler class $e(\xi_{n,p}) = n \in H^4(S^4)$ and spin characteristic class $\frac{p_1}{2}(\xi_{n,p}) = p \in H^4(S^4)$. By definition, we have $M_{0,p} = M(\mathbb{Z}, p)$, where $M(\mathbb{Z}, p)$ is as defined in Definition 3.3. Using (14) and (18) and recalling the notation of Example 2.6, we compute for $n \neq 0$ that there is a diffeomorphism

$$M_{n,p} \cong M\left(\left\langle\!\left\langle \frac{-1}{2n} \right\rangle\!\right\rangle_{-p},\ \left[\frac{p^2 - |n|}{8n}\right]\right).$$

Example 5.1. The Milnor sphere, $\Sigma_{\mathrm{Mi}} := M_{1,3}$, is homeomorphic to S^7 but not diffeomorphic to S^7 since $\mu(\Sigma_{\mathrm{Mi}}) = 1 \neq 0 \bmod 28$: see [38] and [16].

In [9] the total spaces of 3-sphere bundles over S^4 were classified up to homotopy homeomorphism and diffeomorphism.

We now give an example which illustrates the subtleties of the inertia group. Building on Examples 4.7 and 4.8, Theorem 1.10 gives to the following

Example 5.2. The connected sums

$$M_0 := M_{-8,0}\sharp M_{0,8}, \quad M_1 := M_{-8,2}\sharp M_{0,8} \quad \text{and} \quad M_2 := M_{-8,4}\sharp M_{0,8},$$

have $r(M_i) = i$. In each case $d_\pi(M_i) = 8$, whereas $d_o(M_0) = 8$, $d_o(M_1) = 2$ and $d_o(M_2) = 4$. From Theorem 1.10 we have $I(M_0) \cong I(M_1) \cong \Theta_7$ and $I(M_2) \cong 2\Theta_7$.

Notice that when $r = 1$ the [54, Conjecture p. 548] correctly predicts $I(M_1) = \Theta_7$. However when $r \neq 1$, [54, Conjecture p. 548] incorrectly predicts that $I(M_0)$ is $2\Theta_7$ and that $I(M_2)$ is Θ_7.

Example 5.3. While [54, Theorem 1] and Theorem 1.10 give $I(M(\mathbb{Z}^b, d)) = \mathrm{Num}\left(\frac{d}{4}\right)\Theta_7$, the classical Eells-Kuiper invariant is not defined for $M(\mathbb{Z}^b, d)$ when $d_\pi = d \neq 0$. Using (18) we compute that

$$\mu\big(M(\mathbb{Z}^b, d)\sharp\Sigma\big) = [\mu(\Sigma)] \in \mathbb{Z}/\widehat{d}_\pi\mathbb{Z}.$$

Hence we have $\mu(M(\mathbb{Z}^b, 8)) = 0$, whereas $\mu(M(\mathbb{Z}^b, 8)\sharp\Sigma_{\mathrm{Mi}}) = 1 \in \mathbb{Z}/2\mathbb{Z}$ and we see that the generalised Eells-Kuiper invariant distinguishes the diffeomorphism types of $M(\mathbb{Z}^b, 8)$ and $M(\mathbb{Z}^b, 8)\sharp\Sigma_{\mathrm{Mi}}$.

We can also deduce from Theorem 1.5, for example, that $M(\mathbb{Z}^b, 8)\sharp\Sigma_{\mathrm{Mi}}$ admits an orientation reversing diffeomorphism, whereas $M(\mathbb{Z}^b, 16)\sharp\Sigma_{\mathrm{Mi}}$ does not.

Example 5.4. Let N be a simply-connected oriented 6-manifold with $\pi_2(N) \cong \mathbb{Z}$ and suppose that $S^1 \to M \to N$ is a principal S^1 bundle with primitive first Chern class. Then M is 2-connected with a preferred orientation and hence spin structure. Conversely, by [25, Lemma 2.1], every free S^1 action on M is equivalent to such a principal bundle action.

In [25, Theorem 1.3] Yi Jiang identifies the homeomorphism and diffeomorphism types of all 2-connected M which admit free circle actions. In particular, by [25, Theorem 1.3] every such M is almost diffeomorphic to a connected sum $M_{bk,b(k+12m)}\sharp_{2r}M_{0,0}$ for $b \in \{1, 2\}$, $r \in \mathbb{Z}^{\geq 0}$ and $m, k \in \mathbb{Z}$.

5.2. **Examples from geometry.** There are a many 2-connected 7-manifolds that admit metrics with interesting geometric properties. Indeed, according to [13, Theorem B], every 2-connected 7-manifold admits a metric with positive Ricci curvature.

The Gromoll-Meyer sphere. Let $Sp(n)$ denote the n-dimensional symplectic group of orthogonal $n \times n$ quaternionic matrices. The Gromoll-Meyer sphere is a certain quotient of $Sp(2) \times Sp(1)$ by $Sp(1) \times Sp(1)$ and the smooth manifold underlying the Gromoll-Meyer sphere, Σ_{GM}, is an exotic 7-sphere admitting a metric of non-negative sectional curvature [22]. By [22, Theorem 1], there are diffeomorphisms $\Sigma_{GM} \cong M_{-1,-5} \cong 3\Sigma_{\mathrm{Mi}}$.

Berger Space. The smooth manifold underlying the Berger space B is a homogeneous space of the form $B = SO(5)/SO(3)$ (where $SO(3) \hookrightarrow SO(5)$ by the adjoint representation) that admits a metric of positive sectional curvature. The Berger space is 2-connected with $H^4(B) \cong \mathbb{Z}/10$ and Goette, Kitchloo and Shankar [20, Corollary 2] proved there is a diffeomorphism

$$B \cong M_{10,8}.$$

The manifold P_2. More recently Grove, Verdiani and Ziller [23, Theorem A] constructed a metric of positive sectional curvature on a 2-connected 7-manifold P_2 with an isomorphism $H^4(P_2) \cong \mathbb{Z}/2$. Applying [7, Theorem A], they deduced that there is an almost diffeomorphism $P_2 \approxeq S(TS^4)$, where $S(TS^4) = M_{2,0}$ is the unit tangent sphere bundle of S^4. In [19, Theorem 0.3, Example 3.12] Goette proved that there are diffeomorphisms

$$P_2 \cong M_{2,2}\sharp(-\Sigma_{\mathrm{Mi}}) \quad \text{and} \quad P_2 \cong -M_{2,4}.$$

Computation shows that P_2 is not orientation preserving diffeomorphic to the total space of any S^3-bundle over S^4.

G_2-manifolds. In [6] Corti, Haskins, the second author and Pacini constructed a very large class of examples of simply connected manifolds with G_2 holonomy metrics. Many of these examples are 2-connected with $H^4(M)$ torsion-free. For instance, [6, Table 3] gives 7 explicit ways to construct holonomy G_2 metrics on $M(\mathbb{Z}^{85}, 2)$. By [54, Theorem 1(ii)], see also Corollary 1.11, the underlying topological manifold admits a unique smooth structure. In [12] we find examples of manifolds with G_2 holonomy where the smooth structure is not unique and calculating the Generalised Eells-Kuiper invariant we find pairs of closed G_2-manifolds that are homeomorphic but not diffeomorphic. For example $(M(\mathbb{Z}^{89}, 8), M(\mathbb{Z}^{89}, 8)\sharp\Sigma_{\mathrm{Mi}})$ is a pair of homeomorphic but not diffeomorphic smooth manifolds both of which admit metrics with G_2 holonomy.

5.3. **Tangentially homotopy equivalent manifolds.** Let N_0 and N_1 be closed smooth manifolds with tangent bundles TN_0 and TN_1. A homotopy equivalence $f\colon N_0 \to N_1$ is called *tangential* if there is a bundle isomorphism $f^*TN_1 \cong TN_0$. It is natural to ask under what conditions tangentially homotopy equivalent manifolds are necessarily homeomorphic, and this question was studied in detail by Madsen, Taylor and Williams in [35].

In [7, p. 144] it was proven the 2-connected manifolds give rise to examples of non-homeomorphic tangentially homotopy equivalent manifolds. We present a simplified version of the proof here, which starts with the following

Lemma 5.5. *Let M_0 and M_1 be 2-connected and let $f\colon M_0 \simeq M_1$ be a homotopy equivalence such that $f^*p_{M_1} = p_{M_0}$. Then f is tangential.*

Proof. The proof is a relative version of Remark 2.3. The bundles TM_0 and f^*TM_1 are classified by maps $M_0 \to BSO(7)$. Since M_0 is 2-connected, the primary obstruction to a null-homotopy between these maps may be identified with $p_{M_0} - f^*p_{M_1}$. The computations of [28] show that $\pi_i(BSO(7)) = 0$ for $i = 5, 6, 7$ and so there are no further obstructions to finding a homotopy between the classifying maps of TM_0 and f^*TM_1. Hence if $p_{M_0} = f^*p_{M_1}$ then $f^*TM_0 \cong TM_1$. □

Proposition 5.6 (*cf.* [7, p. 114])**.** *The manifolds $M_{-8,1}$ and $M_{-8,5}$ are tangentially homotopy equivalent but not homeomorphic.*

Proof. We first show that $M_{-8,2}$ and $M_{8,-10}$ are tangentially homotopy equivalent. By Definition 2.19, both manifolds have base $\left(\mathbb{Z}/8, \left\langle \frac{-1}{8} \right\rangle, \rho_8(2)\right)$ and applying (14), we see that their quadratic refinements are respectively $\left\langle\!\left\langle \frac{-1}{16} \right\rangle\!\right\rangle_{-2}$ and $\left\langle\!\left\langle \frac{-1}{16} \right\rangle\!\right\rangle_{-10}$. Now

$$\left\langle\!\left\langle \frac{-1}{16} \right\rangle\!\right\rangle_{-10} = \left(\left\langle\!\left\langle \frac{-1}{16} \right\rangle\!\right\rangle_{-2}\right)_{\rho_8(4)}$$

and $\rho_8(4) \in 12(\mathbb{Z}/8)$. By Theorem 1.6, it follows that $M_{-8,2}$ and $M_{-8,5}$ are orientation preserving homotopy equivalent via a homotopy equivalence $f\colon M_{-8,2} \to M_{-8,5}$, which is the identity with respect to the above bases. It follows that $f^* p_{M_{-8,5}} = p_{M_{-8,1}}$, and so f is tangential by Lemma 5.5.

Applying Proposition 2.15 we compute that $A(q_{M_0}) = -1/16 \bmod \mathbb{Z}$ but $A(q_{M_1}) = -9/16 \bmod \mathbb{Z}$ and by Theorem 1.2, the quadratic refinement q_M is a homeomorphism invariant and hence $M_{-8,2}$ and $M_{-8,10}$ are not homeomorphic. □

Remark 5.7. Proposition 5.6 contradicts [35, Theorem C and Theorem 5.10] where it is stated, amongst other things, that all tangentially homotopy equivalent 2-connected 7-manifolds are homeomorphic. The source of the mistake in the arguments of [35] can be found in [35, Theorem 3.12] which is not correct. It is claimed that a certain cohomology class

$$f^*\pi^*(l_n) \in H^{4n}(S^2\Omega^2(SG[3,\infty]); \mathbb{Z}_{(2)})$$

vanishes. Here $SG[3,\infty]$ is the 2-connected cover of SG, the space of orientation preserving stable self-homotopy equivalences of the sphere, $S^2\Omega^2$ denotes the double suspension of the double loop space, the coefficient group $\mathbb{Z}_{(2)}$ is the integers localised at 2 and we shall not define the maps f or π or the class l_n. However the argument given for the proof of [35, Theorem 3.12] only shows that $f^*\pi^*(l_n) = 2x$ for some $x \in H^{4n}(S^2\Omega^2(SG[3,\infty]); \mathbb{Z}_{(2)})$ and not that $f^*\pi^*(l_n) = 0$. To the best of our knowledge, this is the only flaw in the arguments of [35].

5.4. **Generators for the monoid of 2-connected 7-manifolds.** The connected sum operation gives the set of spin diffeomorphism classes of 2-connected 7-manifolds the structure of a commutative monoid with unit S^7. Owing to the existence of homotopy 7-spheres, every M has non-trivial connected sum splittings

$$M \cong (M \sharp \Sigma) \sharp (-\Sigma)$$

for each $\Sigma \in \Theta_7$. Hence we call M *topologically decomposable* if there is a diffeomorphism

$$M \cong M_0 \sharp M_1$$

where neither M_0 nor M_1 is a homotopy sphere and *topologically indecomposable* otherwise.

By Theorem 1.5, every connected sum splitting of M gives rise to an orthogonal splitting of the refinement of M, and by Theorem 1.3 every orthogonal splitting of the refinement of M is realised by a connected sum splitting of M. Hence we call a refinement (G, q°, p)

or a base (G, b, p) decomposable if it can be written as a non-trivial orthogonal sum and indecomposable otherwise. It is clear from the definitions that a refinement is indecomposable if and only if its base if indecomposable. Moreover, the indecomposable bases are of the form $(\mathbb{Z}, 0, p)$ and (T, b, p) where b is an indecomposable torsion form; *i.e.* b cannot be written as a non-trivial orthogonal sum. In this case we also call q indecomposable. A list of all isomorphism classes of indecomposable torsion forms was given by Wall [49, Theorem 4] and torsion forms were then classified by Kawauchi and Kojima [27, Theorem 4.1]. We do not go into details but note that if (T, b, p) is indecomposable then $T \cong \mathbb{Z}/r^k$ for a prime r or $T \cong (\mathbb{Z}/2^k)^2$. Summarising the above discussion we have the following refinement of a theorem of Wilkens.

Theorem 5.8 (*cf.* [53, Theorem 1])**.** *Every 2-connected M is diffeomorphic to a connected sum of topologically indecomposable manifolds M_i :*

$$M \cong \sharp_{i=1}^{n} M_i.$$

Moreover, M is topologically indecomposable if and only if it is almost diffeomorphic to a manifold of one of the following forms

$$S^7, \quad M(\mathbb{Z}, d), \quad M(q, s),$$

where in the final case, q is a prime refinement and hence $H^4(M(q,s)) \cong \mathbb{Z}/r^k$ for r a prime or $H^4(M(q,s)) \cong (\mathbb{Z}/2^k)^2$. □

Even up to almost diffeomorphism, the splitting $M \cong \sharp_{i=1}^{n} M_i$ of Theorem 5.8 is in general far from being unique. For example the manifolds $M_{4,0}$ and $M_{4,2}$ have non-isomorphic bases $(\mathbb{Z}/4, \langle \frac{-1}{4} \rangle, 0)$ and $(\mathbb{Z}/4, \langle \frac{-1}{4} \rangle, 2)$ respectively, but $M_{4,0} \sharp M_{0,2}$ and $M_{4,2} \sharp M_{0,2}$ are diffeomorphic. Even when H^4 is torsion, there are many examples of torsion forms where $b_0 \oplus b_2 \cong b_1 \oplus b_3$ but b_0 is isomorphic to neither b_1 nor b_3 and the same holds for b_2: see for example [27, §3]. This leads to non-uniqueness of connected sum splittings for manifolds with torsion linking form isomorphic to $b_0 \oplus b_2$.

6. Mapping class groups and inertia

In this section we point out some implications of our classification results for mapping class groups of 2-connected M. Throughout this section M will be 2-connected.

Recall the mapping class group $\widetilde{\pi}_0 \mathrm{Diff}(M)$ of pseudo-isotopy classes of diffeomorphisms from Section 3.1 and the subgroup $\widetilde{\pi}_0 \mathrm{Diff}_H(M) \subseteq \widetilde{\pi}_0 \mathrm{Diff}(M)$ of classes acting trivially on $H^*(M)$ from Section 3.4. For brevity, let $\mathrm{Aut}_\mu(H^4(M))$ denote the group of automorphisms of the mod 28 distillation $(H^4(M), q_M^\circ, \mu_M, p_M)$ and let $\mathrm{Aut}_{q^\circ}(H^4(M))$ denote the group of automorphisms of the refinement $(H^4(M), q_M^\circ, p_M)$. As an immediate consequence of Theorem 1.3 we obtain

Proposition 6.1. *For each 2-connected M, there is a short exact sequence*

$$0 \to \widetilde{\pi}_0 \mathrm{Diff}_H(M) \to \widetilde{\pi}_0 \mathrm{Diff}(M) \to \mathrm{Aut}_\mu(H^4(M)) \to 0. \quad \square$$

Remark 6.2. The exact sequence of Proposition 6.1 serves as a starting point for studying the mapping class groups $\widetilde{\pi}_0\mathrm{Diff}(M)$. The determination of $\widetilde{\pi}_0\mathrm{Diff}_H(M)$ and the extension in the sequence of Proposition 6.1 lie outside the scope of this paper. For example, we do not currently know if $\widetilde{\pi}_0\mathrm{Diff}_H(M)$ is abelian in general.

Recall the mapping class groups $\widetilde{\pi}_0\mathrm{ADiff}_H(M,m_0) \subseteq \widetilde{\pi}_0\mathrm{ADiff}(M,m_0)$ defined in Sections 3.1 and 3.4 and the homomorphism $\widetilde{P}\colon \mathrm{Aut}_{q^\circ}(H^4(M)) \to \mathrm{lcm}(8, 2^r d_o)\mathbb{Z}/2\tilde{d}_\pi\mathbb{Z}$ from Section 4.3; see (43) and Proposition 4.10. Noting that $\widehat{d}_\pi = \gcd(\frac{\tilde{d}_\pi}{4}, 28)$, we define

$$\widehat{P}\colon \mathrm{Aut}_{q^\circ}(H^4(M)) \to \mathrm{Num}\left(2^{r-3}d_o\right)\mathbb{Z}/\widehat{d}_\pi\mathbb{Z}, \quad F \mapsto \frac{\widetilde{P}(F)}{8} \mod \widehat{d}_\pi,$$

to be the mod $\widehat{d}_\pi$ reduction of $\widetilde{P}$ divided by 8. By Theorem 1.10 and Remark 3.11 we have $I(M)/I_H(M) \cong \mathrm{Num}\left(2^{r-3}d_o\right)\mathbb{Z}/\widehat{d}_\pi\mathbb{Z}$ and so we can equally regard $\widehat{P}$ as a homomorphism $\widehat{P}\colon \mathrm{Aut}_{q^\circ}(H^4(M)) \to I(M)/I_H(M)$.

Theorem 6.3. *For each 2-connected M there is a commutative diagram of group homomorphisms with short exact sequences for rows and with exact columns:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\pi}_0\mathrm{Diff}_H(M) & \longrightarrow & \widetilde{\pi}_0\mathrm{Diff}(M) & \longrightarrow & \mathrm{Aut}_\mu(H^4(M)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{\pi}_0\mathrm{ADiff}_H(M,m_0) & \longrightarrow & \widetilde{\pi}_0\mathrm{ADiff}(M,m_0) & \longrightarrow & \mathrm{Aut}_{q^\circ}(H^4(M)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_H} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \widehat{P}} & & \\
0 & \longrightarrow & I_H(M) & \longrightarrow & I(M) & \longrightarrow & I(M)/I_H(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In particular, an automorphism $F \in \mathrm{Aut}_{q^\circ}(H^4(M))$ is realised by a diffeomorphism of M if and only if $\widehat{P}(F) = 0$.

Proof. The top row is the exact sequence of Proposition 6.1. The exactness of the second row follows from Theorem 1.2. The first two columns are exact by the discussion at the beginning of Section 3.4 and in particular (34) and the third column is exact by the definition of $\widehat{P}$. The only part of the commutativity of the diagram which needs comment is the bottom right hand square, where the commutativity follows from from Lemma 3.7 (ii) and Proposition 4.16. The final statement follows from the exactness of final row and the top column. □

We shall call an almost diffeomorphism *exotic* if it is not pseudo-isotopic to a diffeomorphism. A feature of the diagram in Proposition 6.3 is that when $I(M)/I_H(M) \neq 0$, M admits exotic almost diffeomorphisms which are detected by their action on $H^4(M)$. Specifically, if $f\colon M \cong M$ is an almost diffeomorphism, then $\widehat{P}(f^*)$ is the obstruction to $f^*\colon H^4(M) \cong H^4(M)$ being induced by *any diffeomorphism* of M. Since $\widehat{P}$ is onto, it is enough to find cases where $I(M)/I_H(M)$ is non-zero to show that $\widehat{P}$ is non-zero.

Proposition 6.4. *Any pair of subgroups $I_0 \subseteq I_1 \subseteq \Theta_7$ can arise as the pair of inertia groups $(I_0, I_1) = (I_H(M), I(M))$ for some 2-connected M.*

Proof. There are three pairs of subgroups (T_0, T_1) in $\mathbb{Z}/7$ and six pairs of subgroups (T_0, T_1) in $\mathbb{Z}/4$, leaving 18 cases to realise. By Theorem 1.10 and Remark 3.11, $I(M)$ and $I_H(M)$ depend only on the base (G, b, p). We list manifolds, their bases (G, b, p) and the pairs (I_H, I) of inertia groups they realise in the following table, where it is helpful to note that $112 = 7 \times 16$:

M	G	b	p	I_H	I	I/I_H
S^7	$\{0\}$	0	0	0	0	0
$M_{0,56}$	$\mathbb{Z}$	0	56	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0
$M_{0,28}$	$\mathbb{Z}$	0	28	$\mathbb{Z}/4$	$\mathbb{Z}/4$	0
$M_{0,16}$	$\mathbb{Z}$	0	16	$\mathbb{Z}/7$	$\mathbb{Z}/7$	0
$M_{0,8}$	$\mathbb{Z}$	0	8	$\mathbb{Z}/14$	$\mathbb{Z}/14$	0
$M_{0,4}$	$\mathbb{Z}$	0	4	$\mathbb{Z}/28$	$\mathbb{Z}/28$	0
$M_{-16,0}\sharp M_{0,112}$	$\mathbb{Z}/16 \oplus \mathbb{Z}$	$\langle 1/16 \rangle$	$(0, 112)$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$M_{-8,0}\sharp M_{0,56}$	$\mathbb{Z}/8 \oplus \mathbb{Z}$	$\langle 1/8 \rangle$	$(0, 56)$	$\mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}/2$
$M_{-16,2}\sharp M_{0,112}$	$\mathbb{Z}/16 \oplus \mathbb{Z}$	$\langle 1/16 \rangle$	$(2, 112)$	0	$\mathbb{Z}/4$	$\mathbb{Z}/4$
$M_{-16,0}\sharp M_{0,16}$	$\mathbb{Z}/16 \oplus \mathbb{Z}$	$\langle 1/16 \rangle$	$(0, 16)$	$\mathbb{Z}/7$	$\mathbb{Z}/14$	$\mathbb{Z}/2$
$M_{-8,0}\sharp M_{0,8}$	$\mathbb{Z}/8 \oplus \mathbb{Z}$	$\langle 1/8 \rangle$	$(0, 8)$	$\mathbb{Z}/14$	$\mathbb{Z}/28$	$\mathbb{Z}/2$
$M_{-16,2}\sharp M_{0,16}$	$\mathbb{Z}/16 \oplus \mathbb{Z}$	$\langle 1/16 \rangle$	$(2, 16)$	$\mathbb{Z}/7$	$\mathbb{Z}/28$	$\mathbb{Z}/4$
$M_{-7,1}\sharp M_{0,112}$	$\mathbb{Z}/7 \oplus \mathbb{Z}$	$\langle 1/7 \rangle$	$(1, 112)$	0	$\mathbb{Z}/7$	$\mathbb{Z}/7$
$M_{-7,1}\sharp M_{0,56}$	$\mathbb{Z}/7 \oplus \mathbb{Z}$	$\langle 1/7 \rangle$	$(1, 56)$	$\mathbb{Z}/2$	$\mathbb{Z}/14$	$\mathbb{Z}/7$
$M_{-0,14}\sharp M_{0,14}$	$\mathbb{Z}/7 \oplus \mathbb{Z}$	$\langle 1/7 \rangle$	$(1, 14)$	$\mathbb{Z}/4$	$\mathbb{Z}/28$	$\mathbb{Z}/7$
$M_{-7,1}\sharp M_{112,16}$	$\mathbb{Z}/112 \oplus \mathbb{Z}$	$\langle 1/112 \rangle$	$(16, 112)$	0	$\mathbb{Z}/14$	$\mathbb{Z}/14$
$M_{-56,0}\sharp M_{0,56}$	$\mathbb{Z}/56 \oplus \mathbb{Z}$	$\langle 1/56 \rangle$	$(8, 56)$	$\mathbb{Z}/2$	$\mathbb{Z}/28$	$\mathbb{Z}/14$
$M_{-112,2}\sharp M_{0,112}$	$\mathbb{Z}/112 \oplus \mathbb{Z}$	$\langle 1/112 \rangle$	$(2, 112)$	0	$\mathbb{Z}/28$	$\mathbb{Z}/28$

□

Theorem 1.13 follows immediately from Theorem 6.3 and Proposition 6.4. We conclude with an example drawn from the bottom line of the table above.

Example 6.5. Let $M = M_{-112,2}\sharp M_{0,112}$ so that $H^4(M) = \mathbb{Z}/112 \oplus \mathbb{Z}$ and consider the automorphism of $(H^4(M), q_M^\circ, p_M)$ defined by

$$F = \begin{pmatrix} 1 & [1] \\ 0 & 1 \end{pmatrix} : \mathbb{Z}/112 \oplus \mathbb{Z} \cong \mathbb{Z}/112 \oplus \mathbb{Z}.$$

In this case $\widehat{d}_\pi = 28$ and from the proof of Proposition 6.4 we see that M admits an almost diffeomorphism $f\colon M \cong M$ with $f^* = F$, and $\widehat{P}(f) = 1 \in \mathbb{Z}/28\mathbb{Z}$. By Theorem 6.3, F^n is realised by a diffeomorphism of M if and only if $n \equiv 0 \bmod 28$.

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