

A note on the Cops and Robber game on graphs embedded in non-orientable surfaces

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Abstract We consider the two-player, complete information game of Cops and Robber played on undirected, finite, reflexive graphs. A number of cops and one robber are positioned on vertices and take turns in sliding along edges. The cops win if, after a move, a cop and the robber are on the same vertex. The minimum number of cops needed to catch the robber on a graph is called the cop number of that graph.

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Let $c(g)$ be the supremum over all cop numbers of graphs embeddable in a closed orientable surface of genus g , and likewise $\tilde{c}(g)$ for non-orientable surfaces. It is known (Andreae, 1986) that, for a fixed surface, the maximum over all cop numbers of graphs embeddable in this surface is finite. More precisely, Quilliot (1985) showed that $c(g) \leq 2g+3$, and Schröder (2001) sharpened this to $c(g) \leq \frac{3}{2}g+3$. In his paper, Andreae gave the bound $\tilde{c}(g) \in O(g)$ with a weak constant, and posed the question whether a stronger bound can be obtained. Nowakowski & Schröder (1997) obtained $\tilde{c}(g) \leq 2g+1$.

In this short note, we show $\tilde{c}(g) \leq c(g-1)$, for any $g \geq 1$. As a corollary, using Schröder's results, we obtain the following: the maximum cop number of graphs embeddable in the projective plane is 3, the maximum cop number of graphs embeddable in the Klein Bottle is at most 4, $\tilde{c}(3) \leq 5$, and $\tilde{c}(g) \leq \frac{3}{2}g + 3/2$ for all other g .

Keywords Games on graphs · cops and robber game · cop number · graphs on surfaces

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1 Introduction

For an integer $k \geq 1$, the *Cops and Robber game with k cops* is a pursuit game played on a reflexive graph, i.e. a graph with a loop at every vertex. There are two opposing sides, a set of k cops and a single robber. The cops begin the game by each choosing a (not necessarily distinct) vertex to occupy, and then the robber chooses a vertex. The two sides move alternately, where a move is to slide along an edge or along a loop. The latter is equivalent to passing were the game played on a loopless graph. There is perfect information, and the cops win if any of the cops and the robber occupy the same vertex at the same time, after a finite number of moves. Graphs on which one cop suffices to win are called *copwin* graphs. In general, we say that a graph G is *k -copwin* if k cops can win on G . The minimum number of cops that suffice to win on G is the cop number of G , denoted $c(G)$. The game has been considered on infinite graphs but, here, we only consider finite graphs.

Nowakowski and Winkler [9] and Quilliot [10] have characterized the class of copwin graphs. The class of k -copwin graphs, $k > 1$, has been characterized by Clarke and MacGillivray [6]. Families of graphs with unbounded cop number have been constructed [1], even families of d -regular graphs, for each $d \geq 3$ [2].

By a surface, we mean a closed surface, i.e. a compact two-dimensional topological manifold without boundary. For any non-negative integer g , we denote by $c(g)$ the supremum over all $c(G)$, with G ranging over all graphs embeddable in an orientable surface of genus g , and we call this the cop number of the surface. Similarly, we define the cop number $\tilde{c}(g)$ of a non-orientable surface of genus g to be the supremum over all $c(G)$, with G ranging over all graphs embeddable in this surface.

Aigner and Fromme [1] proved that the cop number of the sphere is equal to three; i.e. $c(0) = 3$. Quilliot [12] gave an inductive argument to the effect that the cop number of an orientable surface of genus g is at most $2g + 3$. Schröder [13] was able to sharpen this result to $c(g) \leq \frac{3}{2}g + 3$. He also proved that the cop number of the double torus is at most 5.

Andreae [3] generalized the work of Aigner and Fromme. He proved that, for any graph H satisfying a mild connectivity assumption, the class of graphs which do not contain H as a minor has cop number bounded by a constant depending on H . Using this, and the well known formula for the non-orientable genus of a complete graph, he obtained an upper bound for the cop number of a non-orientable surface of genus g , namely

$$\tilde{c}(g) \leq \binom{\lfloor 7/2 + \sqrt{6g + 1/4} \rfloor}{2}.$$

Nowakowski and Schröder [8] used a series of technically challenging arguments to prove a much stronger bound: $\tilde{c}(g) \leq 2g + 1$.

In this short note, we prove the following.

Theorem 1 *For a non-negative integer g , $c(\lfloor g/2 \rfloor) \leq \tilde{c}(g) \leq c(g - 1)$.*

This immediately improves the best known upper bound for the non-orientable surface of genus g to $\tilde{c}(g) \leq \frac{3}{2}(g - 1) + 3 = \frac{3}{2}(g + 1)$. The following table gives the new and status quo for the concrete upper bounds.

¹ Using Schröder's [13] result that $c(2) \leq 5$.

N/o genus	1	2	3	4	5	6	7
N. & S. [8]	3	5	7	9	11	13	15
Here	3	4	5 ¹	7	9	10	12

Table 1 Comparison of the new and status quo upper bounds for $\tilde{c}(g)$.

We say that a *weak cover* of H by G is a surjective mapping $p: V(G) \rightarrow V(H)$ which maps vertex neighborhoods onto vertex neighborhoods; i.e. for every vertex u of G , we have $p(N(u)) = N(p(u))$. (This terminology lends itself to the classical definition of a “cover” without weak, where the restriction to the vertex neighborhood $p: N(u) \rightarrow N(p(u))$ is required to be a bijection.) Using the same technique as for the inequality “ \leq ” in the proof of Theorem 1, it is possible to show the following:

Lemma 1 *If G is a weak cover of H , then $c(H) \leq c(G)$.*

This is similar in spirit to the seminal result of Berarducci and Intrigila [4], saying that if H is a retract of G , then the same inequality holds. Note, however, that neither of the two notions generalizes the other. We will not prove Lemma 1; the proof is only slightly more technical than the geometric proof of Theorem 1.

2 Proof

Familiarity with the classification of combinatorial surfaces is assumed. See any standard textbook on topology, such as [5]. We will make use of the standard representation of surfaces as quotients of polygonal discs with labelled and directed edges. Each label occurs twice, and the two edges with the same label are identified according to their orientations. Reading the labels of the edges in counterclockwise (i.e. *positive*) order and adding an exponent -1 whenever the orientation of the edges is negative (i.e. clockwise) gives the *word* of the surface.

For a graph G , let $\gamma(G)$ denote the smallest integer g such that G can be embedded in an orientable surface of genus g ; similarly define $\tilde{\gamma}(G)$ as the smallest integer g such that G can be embedded in a non-orientable surface of genus g . For the proof of Theorem 1, we use the following well-known fact. Its proof can be found in [7].

Lemma 2 (Folklore) For any graph G , $\tilde{\gamma}(G) \leq 2\gamma(G) + 1$.

In the proof of the inequality $c(g) \leq c(g - 1)$, we make use of the well-known fact that every manifold X has a 2-sheeted covering $X' \rightarrow X$ by an orientable manifold. If X is a non-orientable surface of genus g , it is easy to see that the standard construction (again, see [5]) yields a surface of genus $g - 1$. This is Lemma 3. The proof is straightforward (consider Figure 1), and is thus omitted.

Lemma 3 A non-orientable surface of genus g has an orientable surface of genus $g - 1$ as a 2-sheeted covering space.

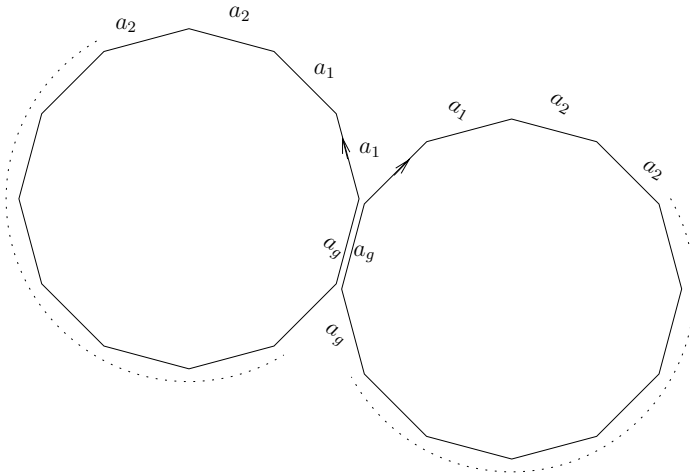


Fig. 1 A figure to accompany Lemma 3.

We are now ready for the proof of our main result.

Proof of Theorem 1. Lemma 3 immediately implies that $c(g) \leq \tilde{c}(2g+1)$, and hence $\tilde{c}(g) \geq c(\lfloor g/2 \rfloor)$.

For the proof of the remaining inequality $\tilde{c}(g) \leq c(g - 1)$, let X be the non-orientable surface of genus g on which a graph G is embedded. We identify the graph G with its embedding; i.e. we think of the vertex set $V(G)$ as a set of points of X and the edge set $E(G)$ as a set of internally disjoint injective curves connecting the respective end vertices of the edges.

By Lemma 3, there exists a covering $p: X' \rightarrow X$ of X by an orientable surface X' with genus $g' := g - 1$. Consider the graph G' whose vertex set is $\{p^{-1}(V(G))\}$ and whose edge set consists of the curves obtained by lifting the edges of G . By construction, G' is embedded in the orientable surface X' of genus g' .

We now give a strategy for $k := c(g')$ cops to win the Cops and Robber game on G , by “simulating” a game on G' and using any winning strategy for k cops on this graph, who chase an “imaginary” robber. In such a strategy, the k cops first choose their starting vertices $u_1, \dots, u_k \in V(G')$. In the strategy for G , we let the starting vertices be $p(u_1), \dots, p(u_k)$. Suppose now that, in the game on G , the robber chooses a starting vertex r . We choose a starting vertex for an imaginary robber on G' arbitrarily in the fibre $p^{-1}(r)$.

Throughout the game, the position of each player in G' will be in the fibre $p^{-1}(x)$ of the position x of the corresponding player in G . Moreover, the movements of the players on G describe curves on X , which can be lifted (uniquely, although this is not essential) to curves on X' forming walks in G' .

Now, whenever it is the cops’ turn in any game on G , the robber is at a certain vertex s of G' and the k cops are on vertices v_1, \dots, v_k . The strategy for the cops on G' now prescribes moves for the cops. The corresponding moves in G are then given as images under p .

Since we have a winning strategy, after a finite number of moves, the imaginary robber on G' will be on the same vertex as a cop in G' . Consequently, the same holds on G , and thus the cops have won the game on G . \square

3 Conclusion

We conclude with a conjecture.

Conjecture 1 *For a non-negative integer g , $\tilde{c}(g) = c(\lfloor g/2 \rfloor)$.*

One might wonder whether it is possible to improve Theorem 1 by taking a different covering, or possibly a branched covering. This is impossible: It is a well-known fact that, whenever $p: X' \rightarrow X$ is a (branched) covering with X' orientable and X

non-orientable, then p lifts to a (branched) covering $\tilde{p}: X' \rightarrow \tilde{X}$, where \tilde{X} is the orientable double cover constructed in Lemma 3.

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