Noname manuscript No.
(will be inserted by the editor)

Hamiltonicity of 3-arc graphs

Guangjun Xu · Sanming Zhou

Received: date / Accepted: date

Abstract An arc of a graph is an oriented edge and a 3-arc is a 4-tuple (v, u, x, y) of vertices such that both (v, u, x) and (u, x, y) are paths of length two. The 3-arc graph of a graph G is defined to have vertices the arcs of G such that two arcs uv, xy are adjacent if and only if (v, u, x, y) is a 3-arc of G. We prove that any connected 3-arc graph is hamiltonian, and all iterative 3-arc graphs of any connected graph of minimum degree at least three are hamiltonian. As a corollary we obtain that any vertex-transitive graph which is isomorphic to the 3-arc graph of a connected arc-transitive graph of degree at least three must be hamiltonian. This confirms the conjecture, for this family of vertex-transitive graphs, that all vertex-transitive graphs with finitely many exceptions are hamiltonian. We also prove that if a graph with at least four vertices is Hamilton-connected, then so are its iterative 3-arc graphs.

G. Xu
Department of Mathematics and Statistics, The University of Melbourne
Parkville, VIC 3010, Australia
E-mail: gjxu11@gmail.com

S. Zhou
Department of Mathematics and Statistics, The University of Melbourne
Parkville, VIC 3010, Australia
E-mail: smzhou@ms.unimelb.edu.au

Keywords 3-Arc graph · Hamilton cycle · Hamiltonian graph · Hamilton-connected graph · Vertex-transitive graph

1 Introduction

A path or cycle which contains every vertex of a graph is called a *Hamilton path* or *Hamilton cycle* of the graph. A graph is *hamiltonian* if it contains a Hamilton cycle, and is *Hamilton-connected* if any two vertices are connected by a Hamilton path. The hamiltonian problem, that of determining when a graph is hamiltonian, is a classical problem in graph theory with a long history. The reader is referred to [3], [4, Chapter 18], [8, Chapter 10] and [10] for results on Hamiltonicity of graphs.

In this paper we present a large family of hamiltonian graphs. Such graphs are defined by means of a graph operator, called the 3-arc graph construction, which bears some similarities with the line graph operator. This construction was first introduced in [17, 24] in studying a family of arc-transitive graphs whose automorphism group contains a subgroup acting imprimitively on the vertex set. (A graph is *arc-transitive* if its automorphism group is transitive on the set of oriented edges.) It was used in classifying or characterizing certain families of arc-transitive graphs [9, 12, 17, 18, 23, 25].

All graphs in this paper are finite and undirected without loops. We use the term *multigraph* when parallel edges are allowed. An *arc* of a graph $G = (V(G), E(G))$ is an ordered pair of adjacent vertices, or equivalently an oriented edge. For adjacent vertices u, v of G, we use uv to denote the arc from u to v, vu ($\neq uv$) the arc from v to u, and $\{u, v\}$ the edge between u and v. A *3-arc* of G is a 4-tuple of vertices (v, u, x, y), possibly with $v = y$, such that both (v, u, x) and (u, x, y) are paths of G.

Notation: We follow [4] for graph-theoretic terminology and notation. The degree of a vertex v in a graph G is denoted by $d(v)$, and the minimum degree of G is denoted by $\delta(G)$. The set of arcs of G with tail v is denoted by $A(v)$, and the set of arcs of G is denoted by $A(G)$.

The general 3-arc construction [17, 24] involves a self-paired subset of the set of 3-arcs of a graph. The following definition is obtained by choosing this subset to be the set of all 3-arcs of the graph.

Definition 1 *Let G be a graph. The 3-arc graph of G, denoted by $X(G)$, is defined to have vertex set $A(G)$ such that two vertices corresponding to two arcs uv and xy are adjacent if and only if (v, u, x, y) is a* 3*-arc of G.*

It is clear that $X(G)$ is an undirected graph with $2\,|E(G)|$ vertices and $\sum_{\{u,v\} \in E(G)}(d(u)-1)(d(v)-1)$ edges. We can obtain $X(G)$ from the line graph $L(G)$ of G by the following operations [14]: split each vertex $\{u, v\}$ of $L(G)$ into two vertices, namely uv and vu; for any two vertices $\{u, v\}, \{x, y\}$ of $L(G)$ that are distance two apart in $L(G)$, say, u and x are adjacent in G, join uv and xy by an edge. On the other hand, the quotient graph of $X(G)$ with respect to the partition $\mathcal{P} = \{\{uv, vu\} : \{u, v\} \in E(G)\}$ of $A(G)$ is isomorphic to the graph obtained from the square of $L(G)$ by deleting the edges of $L(G)$. The reader is referred to [14, 13, 2] respectively for results on the diameter and connectivity, the independence, domination and chromatic numbers, and the edge-connectivity and restricted edge-connectivity of 3-arc graphs.

The following is the first main result in this paper.

Theorem 1 *Let G be a graph without isolated vertices. The 3-arc graph of G is hamiltonian if and only if*

(a) $\delta(G) \geq 2$*;*

(b) *no two degree-two vertices of G are adjacent; and*

(c) *the subgraph obtained from G by deleting all degree-two vertices is connected.*

We remark that Theorem 1 can not be obtained from known results on the hamiltonicity of line graphs, though $X(G)$ and $L(G)$ are closely related as mentioned above. As a matter of fact, even if $L(G)$ is hamiltonian, $X(G)$ is not necessarily hamiltonian, as witnessed by stars $K_{1,t}$ with $t \geq 3$.

We define the *iterative 3-arc graphs* of G by

$$X^1(G) = X(G), \;\; X^{i+1}(G) = X(X^i(G)), \;\; i \geq 1.$$

Theorem 1 together with [14, Theorem 2] implies the following result.

Theorem 2 (a) *A 3-arc graph is hamiltonian if and only if it is connected.*
(b) *If G is a connected graph with $\delta(G) \geq 3$, then $X^i(G)$ is hamiltonian for every integer $i \geq 1$.*

We will prove Theorems 1 and 2 in Section 3. In Section 4 we will prove the following result.

Theorem 3 *Let G be a 2-edge connected graph with $\delta(G) \geq 3$. If G contains a path of odd length between any two distinct vertices, then its 3-arc graph is Hamilton-connected.*

A basic strategy in the proof of Theorems 1 and 3 is to find an Eulerian tour or an open Eulerian trail in a properly defined multigraph that produces the required Hamilton cycle or path. This is similar to the observation [5] that an Eulerian tour of a graph produces a Hamilton cycle of its line graph.

Theorem 3 implies the following result.

Theorem 4 *If a graph G with at least four vertices is Hamilton-connected, then so are its iterative 3-arc graphs $X^i(G)$, $i \geq 1$.*

Given vertex-disjoint graphs G and H, the join $G \vee H$ of them is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\}$. Theorem 3 implies the following result.

Corollary 1 *Let G and H be graphs such that $\max\{\delta(G), \delta(H)\} \geq 2$. Then $X(G \vee H)$ is Hamilton-connected.*

In the case when G has a large order but small maximum degree, $X(G)$ has a large order but relatively small maximum degree. In this case the Hamiltonicity of $X(G)$ may not be derived from known sufficient conditions for Hamilton cycles such as the degree conditions in the classical Dirac's or Ore's Theorem (see [3, 4, 8, 10]).

In spirit, Theorems 1 and 2 are parallel to the well-known conjecture of Thomassen [20] which asserts that every 4-connected line graph is hamiltonian. This conjecture is still open; see [6, 10, 11, 16, 22]. In contrast, Theorem 1 solves the hamiltonian problem for 3-arc graphs completely.

A well-known conjecture due to Lovász, formulated by Thomassen [21], asserts that all connected vertex-transitive graphs, with finitely many exceptions, are hamiltonian. Since the 3-arc graph of an arc-transitive graph is vertex-transitive, Theorem 2 implies the following result, which confirms this conjecture for a large family of vertex-transitive graphs. (The family of arc-transitive graphs is large from a group-theoretic point of view [19].)

Corollary 2 *If a vertex-transitive graph is isomorphic to the 3-arc graph of a connected arc-transitive graph of degree at least three, then it is hamiltonian.*

The Lovász conjecture has been confirmed for several families of vertex-transitive graphs [15], including connected vertex-transitive graphs of order kp, where $k \leq 4$, (except for the Petersen graph and the Coxeter graph) of order p^j, where $j \leq 4$, and of order $2p^2$, where p is prime, and some families of Cayley graphs. Tools from group theory were used in the proof of almost all these results. Corollary 2 has a different flavour and its proof does not rely on group theory.

There has also been considerable interest on Hamilton-connectedness of vertex-transitive graphs. Theorem 4 implies that if a vertex-transitive graph (with at least four vertices) is Hamilton-connected, then so are its iterative 3-arc graphs. For example, it is known that every connected non-bipartite Cayley graph of degree at least three on a finite abelian group [7] or a Hamiltonian group [1] is Hamilton-connected. (A finite non-abelian group in which every subgroup is normal is called a Hamiltonian group.) From this and Theorem 4 we know immediately that all iterative 3-arc graphs of such a Cayley graph are also Hamilton-connected.

2 Preliminaries

Let G^* be a multigraph. A *walk* in G^* of length l is a sequence $v_0, e_1, v_1, \dots, v_{l-1}, e_l, v_l$, whose terms are alternately vertices and edges of G^* (not necessarily distinct), such that v_{i-1} and v_i are the end-vertices of e_i, $1 \leq i \leq l$. A walk is *closed* if its initial and terminal vertices are identical, is a *trail* if all its edges are distinct, and is a *path* if all its vertices are distinct. Often we present a trail by listing its sequence of vertices only, with the understanding that the edges used are distinct. A trail that traverses every edge of G^* is called an *Eulerian trail* of G^*, and a closed Eulerian trail is called an *Eulerian tour*. A multigraph is *Eulerian* if it admits an Eulerian tour. It is well known that a multigraph is Eulerian if and only if all its vertices have even degrees.

A *2-trail* of G^* is a trail of length two (and so is a path or cycle of length two). We call a 2-trail (u, x, v) with mid-vertex x a *visit to* x (if $u = v$, then (u, x, u) is thought as entering and leaving x on parallel edges). When there is no need to make distinction between (u, x, v) and (v, x, u), or the orientation of the visit is unknown, we write $[u, x, v]$. Two visits (u, x, v) and (u', x, v') are called *twin visits* if $\{u, v\} = \{u', v'\}$ and the four edges involved are distinct. In particular, when $u = v$, two twin visits (u, x, u) and (u, x, u) use four parallel edges between u and x.

Denote by $E^*(x)$ the set of edges of G^* incident with $x \in V(G^*)$, and $d^*(x) = |E^*(x)|$ the degree of x in G^*. In the case when $d^*(x)$ is even, a decomposition of $E^*(x)$ into a set of visits to x is called a *visit-decomposition* of $E^*(x)$ (at x). In this definition the orientations of the visits in the decomposition are not important in our subsequent discussion. So we may view each visit (u, x, v) in such a visit-decomposition as a non-oriented path (if $u \neq v$) or cycle (if $u = v$) of length two. As an example, if $E^*(x) = \{\{x, y\}, \{x, y\}, \{x, z\}, \{x, z\}\}$, where $\{x, y\}$ and $\{x, y\}$ are viewed as distinct edges between x and y, then both $\{[y, x, y], [z, x, z]\}$ and $\{[y, x, z], [y, x, z]\}$ are visit-decompositions of $E^*(x)$.

Definition 2 Given a visit-decomposition $J(x)$ of $E^*(x)$, define $H(x)$ to be the bipartite graph with vertex bipartition $\{J(x), A(x)\}$ such that $p \in J(x)$

and $xy \in A(x)$ are adjacent if and only if y is not in p, where $A(x)$ is the set of arcs of the underlying simple graph of G^* with tail x.

We emphasize that $H(x)$ relies on $J(x)$. One can verify the following result by using Hall's marriage theorem.

Lemma 1 *Suppose x is a vertex of G^* such that $d^*(x) \geq 6$ is even and either x is joined to every neighbour of x by exactly two parallel edges, or x is joined to one of its neighbours by exactly three parallel edges, another neighbour by a single edge, and each of the remaining neighbours by exactly two parallel edges. Let $J(x)$ be a visit-decomposition of $E^*(x)$. Then the bipartite graph $H(x)$ with respect to $J(x)$ has no perfect matchings if and only if $d^*(x) = 6$ and $J(x)$ contains two twin visits.*

Proof We have $|J(x)| = |A(x)| = d^*(x)/2$ and $\delta(H(x)) \geq (d^*(x)/2) - 2 \geq 1$. One can show that, if $d^*(x) \geq 8$, then the neighbourhood $N_{H(x)}(S)$ in $H(x)$ of each $S \subseteq J(x)$ has size at least $|S|$. Thus, by Hall's marriage theorem, $H(x)$ has a perfect matching when $d^*(x) \geq 8$.

Suppose $H(x)$ has no perfect matchings, so that $d^*(x) = 6$ and $|J(x)| = |A(x)| = 3$. Then there exists $S \subseteq J(x)$ such that $|N_{H(x)}(S)| < |S|$. This implies $|S| = 2$ and so $|N_{H(x)}(S)| \leq 1$. Denote $S = \{(u, x, v), (y, x, z)\}$, where $u, v, y, z \in N(x)$ (the neighbourhood of x in G^*). Then $N_{H(x)}(S) = (A(x) - \{xu, xv\}) \cup (A(x) - \{xy, xz\}) = A(x) - (\{xu, xv\} \cap \{xy, xz\})$. Since $|N(x)| = 3$ and $|N_{H(x)}(S)| \leq 1$, it follows that $\{u, v\} = \{y, z\}$, and therefore (u, x, v) and (y, x, z) are twin visits.

Conversely, if $d^*(x) = 6$ and $J(x)$ contains twin visits, then $H(x)$ consists of two paths of length two and hence has no perfect matchings. □

Definition 3 Let $C : v_0, e_1, v_1, e_2, v_2, \ldots, v_{l-2}, e_{l-1}, v_{l-1}, e_l, v_l$ be an Eulerian trail of G^*, possibly with $v_l = v_0$. The visit (v_{i-1}, v_i, v_{i+1}) to v_i is said to be induced by C, $1 \leq i \leq l - 1$. In addition, if C is an Eulerian tour, then (v_{l-1}, v_0, v_1) is also a visit to v_0 induced by C.

Denote by $C(x)$ the set of visits to $x \in V(G^*)$ induced by C.

Define $H_C(x)$ to be the bipartite graph at x as defined in Definition 2 with respect to the visit-decomposition $C(x)$ of $E^*(x)$. (We leave $H_C(v_0)$ and $H_C(v_l)$ undefined if C is an open Eulerian trail.)

Note that a vertex may be visited several times by C because the vertices on C may be repeated. Indeed, $C(x)$ is a visit-decomposition of $E^*(x)$ for all vertices x, except v_0 and v_l when $v_0 \neq v_l$.

Definition 4 Let C be an Eulerian tour of G^* and $J(x)$ a visit-decomposition of $E^*(x)$. We say that C is compatible with $J(x)$, written $C(x) \prec J(x)$, if for every $(a, x, b) \in J(x)$, either $(a, x, b) \in C(x)$ or $(b, x, a) \in C(x)$.

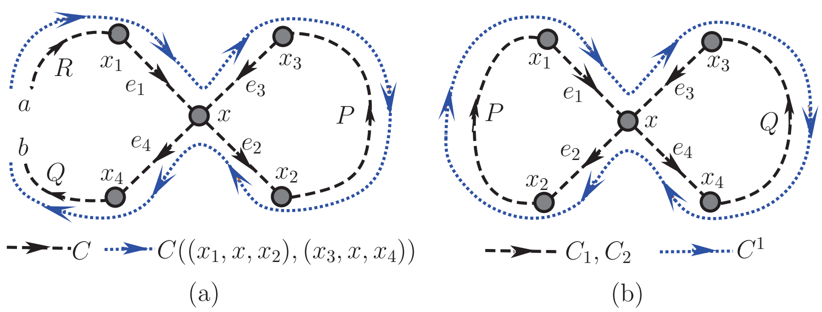

Fig. 1 (a) Bow-tie operation; (b) Concatenation operation.

Definition 5 Let C be a trail of G^* with length at least four. Let $(x_1, x, x_2), (x_3, x, x_4) \in C(x)$ be distinct visits, so that C can be expressed as

$$C : \overbrace{a, \ldots, x_1}^{R}, e_1, x, e_2, \overbrace{x_2, \ldots, x_3}^{P}, e_3, x, e_4, \overbrace{x_4, \ldots, b}^{Q},$$

possibly with $a = b$.

Define

$$C((x_1, x, x_2), (x_3, x, x_4)) : \overbrace{a, \ldots, x_1}^{R}, e_1, x, e_3^{-1}, \overbrace{x_3, \ldots, x_2}^{P^-}, e_2^{-1}, x, e_4, \overbrace{x_4, \ldots, b}^{Q}$$

where P^- is the trail obtained from P by reversing its direction, and e_2^{-1} and e_3^{-1} are the same edges as e_2 and e_3 but with reversed orientations, respectively. (See Figure 1 (a).)

We call $C \to C((x_1, x, x_2), (x_3, x, x_4))$ the bow-tie operation on C with respect to (x_1, x, x_2) and (x_3, x, x_4).

Definition 6 Let

$$C_1 : x_1, e_1, x, e_2, \overbrace{x_2, \ldots, x_1}^{P}; \qquad C_2 : x_3, e_3, x, e_4, \overbrace{x_4, \ldots, x_3}^{Q}.$$

be edge-disjoint closed trails of G^* with x as a common vertex. Define

$$C^1 : x_1, e_1, x, e_3^{-1}, \overbrace{x_3, \ldots, x_4}^{Q^{-1}}, e_4^{-1}, x, e_2, \overbrace{x_2, \ldots, x_1}^{P}.$$

We call $(C_1, C_2) \to C^1$ the concatenation operation with respect to $(C_1, C_2, (x_1, x, x_2), (x_3, x, x_4))$. (See Figure 1 (b).)

Remark 1 Some of x_1, x_2, x_3, x_4 or even all of them in Definitions 5 and 6 are allowed to be the same vertex. Each of P, Q (and R in Definition 5) may visit some of x, x_1, x_2, x_3, x_4 several times, and they may have common vertices.

In each operation above, the visits (x_1, x, x_2), (x_3, x, x_4) are replaced by (x_1, x, x_3), (x_4, x, x_2), respectively. All other visits induced by C (in Definition 5) or $C_1 \cup C_2$ (in Definition 6) are retained or with orientation reversed.

In Definition 6, C^1 is a closed trail which covers every edge covered by C_1 and C_2. In particular, if C_1 and C_2 collectively cover all edges of G^*, then C^1 is an Eulerian tour of G^*.

3 Proof of Theorems 1 and 2

Proof of Theorem 1 Denote by S_i the set of vertices of G with degree i, for $i \geq 1$.

Suppose that G has no isolated vertices and $X(G)$ is hamiltonian. We show that (a), (b) and (c) hold. Note first that if G has a degree-one vertex, then the unique arc emanating from it gives rise to an isolated vertex of $X(G)$. Similarly, if $x, y \in S_2$ are adjacent, say, $N(x) = \{u, y\}, N(y) = \{x, v\}$, then the edge of $X(G)$ between xu and yv is an isolated edge no matter whether $u \neq v$ or not. Since $X(G)$ is assumed to be hamiltonian, it follows that G is connected with $\delta(G) \geq 2$ and S_2 is an independent set of G.

It remains to prove that $G - S_2$ is connected. Suppose otherwise. Then we can choose a minimal subset S of S_2 such that $G - S$ is disconnected. Note that $S \neq \emptyset$ as G is connected. Let H be a component of $G - S$. The minimality of S implies that each vertex of S has exactly one neighbour in $V(H)$, and each vertex of S_2 with both neighours in H (if such a vertex exists) is contained in $V(H)$. Denote by A_1 the set of arcs of G with tails in S and heads outside of $V(H)$. Denote by A_2 the set of arcs of G with tails in $V(H)$ (and heads in $V(H)$ or S). One can verify that the subgraph of $X(G)$ induced by $A_1 \cup A_2$ is a connected component of $X(G)$. Since there are arcs of G not in $A_1 \cup A_2$, it follows that $X(G)$ is disconnected, contradicting our assumption. Hence $G - S_2$ is connected.

Suppose that G satisfies (a), (b) and (c). We aim to prove that $X(G)$ is hamiltonian. Note that G is connected by (c). Let G^* be the multigraph obtained from G by doubling each edge. Then the degree $d^*(v) = 2d(v)$ of each $v \in V(G)$ in G^* is even. Hence G^* is Eulerian. We will prove the existence of an Eulerian tour of G^* such that the corresponding bipartite graph (see Definition 3) at each vertex has a perfect matching. We will then exploit such an Eulerian tour to construct a Hamilton cycle of $X(G)$.

We claim first that there exists an Eulerian tour C of G^* such that

$$\text{if } v \in S_2 \text{ with } N(v) = \{u, w\}, \text{ then } C(v) \prec \{(u, v, u), (w, v, w)\}. \tag{1}$$

To construct such an Eulerian tour, we can start from any vertex and travel as far as possible without repeating any edge such that, whenever the tour reaches a vertex of S_2, it returns to the previous vertex immediately. Since $G - S_2$ is connected, an Eulerian tour C of G^* satisfying (1) can be constructed this way. Note that $G^* - S_2$ is Eulerian because it is connected and all its vertices have even degrees.

For an Eulerian tour C of G^* satisfying (1), let $Z(C)$ denote the set of vertices x such that $H_C(x)$ has no perfect matchings. Since for every $x \in S_2$, $H_C(x) \cong 2K_2$ is a perfect matching, by Lemma 1 we have $Z(C) \subseteq S_3$.

Now we choose an Eulerian tour C of G^* satisfying (1) such that $|Z(C)|$ is minimum. We claim that $Z(C) = \emptyset$. Suppose otherwise. Then by Lemma 1, $C(x)$ contains twin visits for each $x \in Z(C)$. Denote $N(x) = \{x_1, x_2, x_3\}$ for a fixed $x \in Z(C)$, and assume without loss of generality that $C(x) = \{(x_1, x, x_2), (x_1, x, x_2), (x_3, x, x_3)\}$. Denote $C' = C((x_1, x, x_2), (x_3, x, x_3))$. Then C' is an Eulerian tour of G^* and $C'(x) = \{(x_1, x, x_2), (x_1, x, x_3), (x_2, x, x_3)\}$. One can see that $H_{C'}(x)$ is a perfect matching of three edges, and $H_{C'}(y)$ is isomorphic to $H_C(y)$ for each $y \neq x$. Thus $Z(C')$ is a proper subset of $Z(C)$, and moreover (1) is respected by C' at every $v \in S_2$. Since this contradicts the choice of C, we conclude that $Z(C) = \emptyset$; that is, $H_C(v)$ has a perfect matching for each $v \in V(G)$.

Let C be a fixed Eulerian tour of G^* satisfying (1) such that $Z(C) = \emptyset$. Let us fix a perfect matching of $H_C(v)$ for each $v \in V(G)$. Every traverse of C to v corresponds to a visit to v, say, (u, v, w), and in the chosen perfect matching of $H_C(v)$, (u, v, w) is matched to an arc of $A(v)$ other than vu and vw. Denote this arc by $\phi(u, v, w)$. Then for any two consecutive visits $(u, v, w), (v, w, x)$ induced by C (that is, (u, v, w, x) is a segment of C), $\phi(u, v, w)$ and $\phi(v, w, x)$ are adjacent in $X(G)$. Since C is an Eulerian tour of G^* and a perfect matching of each $H_C(v)$ is used, every arc of G is of the form $\phi(u, v, w)$ for some segment (u, v, w) of C. Therefore, if, say, $C = (u, v, w, x, y, \ldots, a, b, c, u)$, then the sequence

$$\phi(u, v, w), \phi(v, w, x), \phi(w, x, y), \ldots, \phi(a, b, c), \phi(b, c, u), \phi(c, u, v), \phi(u, v, w)$$

of arcs of G gives rise to a Hamilton cycle of $X(G)$. □

We illustrate the proof above by the following example.

Example 1 Since the Petersen graph PG (see Figure 2) satisfies the conditions in Theorem 1, its 3-arc graph $X(PG)$ is hamiltonian. Let

$$C: a_1, a_2, a_3, a_4, a_5, a_1, b_1, b_4, b_2, b_5, b_3, b_1, a_1, a_2, b_2, \\ b_5, a_5, a_4, b_4, b_2, a_2, a_3, b_3, b_1, b_4, a_4, a_3, b_3, b_5, a_5, a_1.$$

Then C is an Eulerian tour of the multigraph PG^* obtained from PG by doubling each edge. One can verify that at each a_i or b_i, $H_C(a_i)$ or $H_C(b_i)$ has a perfect matching. In $H_C(a_2)$ the 'vertex' (a_1, a_2, a_3) is matched to the 'vertex' a_2b_2, and in $H_C(a_3)$, (a_2, a_3, a_4) is matched to a_3b_3, and so on. Continuing, one can verify that C gives rise to the following Hamilton cycle of $X(PG)$:

$$a_2b_2, a_3b_3, a_4b_4, a_5b_5, a_1a_2, b_1b_3, b_4a_4, b_2a_2, b_5a_5, b_3a_3, b_1b_4, a_1a_5, a_2a_3, b_2b_4, b_5b_3, \\ a_5a_1, a_4a_3, b_4b_1, b_2b_5, a_2a_1, a_3a_4, b_3b_5, b_1a_1, b_4b_2, a_4a_5, a_3a_2, b_3b_1, b_5b_2, a_5a_4, a_1b_1, a_2b_2.$$

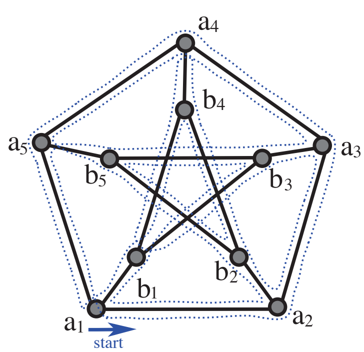

Fig. 2 An Eulerian tour of PG^* which produces a Hamilton cycle of the 3-arc graph of the Petersen graph PG.

Proof of Theorem 2 (a) Let G be a graph. Define $\hat{G}$ to be the graph obtained from G by replacing each degree-two vertex v by a pair of nonadjacent vertices each joining to exactly one neighbour of v in G. In [14, Theorem 2] it is proved that, if $\delta(G) \geq 2$, then $X(G)$ is connected if and only if $\hat{G}$ is connected. One can verify that $\delta(G) \geq 2$ and $\hat{G}$ is connected if and only if (a), (b) and (c) in Theorem 1 hold. Thus, by Theorem 1, if $X(G)$ is connected, then it is hamiltonian. The converse of this statement is obvious.

(b) If G is connected with $\delta(G) \geq 3$, then $\hat{G} = G$ and so $X(G)$ is connected by [14, Theorem 2]. Hence, by (a), $X(G)$ is hamiltonian. Since $\delta(G) \geq 3$, we have $\delta(X(G)) \geq 3$. Thus, by applying (a) to $X(G)$, we see that $X^2(G)$ is hamiltonian. Continuing, by induction we can prove that $X^i(G)$ is hamiltonian for every $i \geq 1$. □

4 Proof of Theorems 3 and 4

Let us first introduce an operation that will be used in the proof of Theorem 3. Let G^* be an Eulerian multigraph and C an Eulerian tour of G^*. Let (z_1, x, z_2) be a visit of C to x. Write

$$C : z_1, e_1, x, e_2, \overbrace{z_2, \ldots, z_1}^{T},$$

where e_1 is the oriented edge from z_1 to x, e_2 the oriented edge from x to z_2, and T the segment of C from z_2 to z_1 covering all edges of G^* except e_1 and e_2. Add two new vertices t, t' to G^* and join them to x by edges $e_t, e_{t'}$, respectively, with orientation towards x. Denote the resultant multigraph by $G^*_C(z_1, x, z_2)$. Set

$$W = W_C(z_1, x, z_2) : t, e_t, x, e_2, \overbrace{z_2, \ldots, z_1}^{T}, e_1, x, e_{t'}^{-1}, t'.$$

Since C is an Eulerian tour of G^*, W is an open Eulerian trail of $G^*_C(z_1, x, z_2)$. Denote by $W(x)$ the set of visits to x induced by W. As the first and last visits induced by W, (t, e_t, x, e_2, z_2) and $(z_1, e_1, x, e_{t'}^{-1}, t')$ are members of $W(x)$. Note that $xt, xt' \notin A(x)$.

Definition 7 Define $K_C(z_1, x, z_2)$ to be the bipartite graph with bipartition $\{W(x), A(x)\}$ such that an arc in $A(x)$ is adjacent to a visit $p \in W(x)$ if and only if its head does not appear in p. Denote by $L_C(z_1, x, z_2)$ the graph obtained from $K_C(z_1, x, z_2)$ by deleting the vertices (t, e_t, x, e_2, z_2), $(z_1, e_1, x, e_{t'}^{-1}, t')$, xz_1 and xz_2.

To prove Theorem 3, we need to prove that, for any two distinct arcs xy, uv of G, there exists a Hamilton path of $X(G)$ between xy and uv. We will prove the existence of such a path by constructing a specific Eulerian trail in a certain auxiliary multigraph G^*. We treat the cases $x = u$ and $x \neq u$ separately in the next two lemmas.

Lemma 2 *Under the condition of Theorem 3, for any distinct arcs $xy, xv \in A(G)$ with the same tail, there exists a Hamilton path of $X(G)$ between xy and xv.*

Proof By our assumption there exists a path in G of odd length connecting y and v. Let

$$P : y = x_0, x_1, x_2, \ldots, x_{l-1}, x_l = v$$

be a path in G between y and v with minimum possible odd length $l \geq 1$. Denote $E_0(P) = \{\{x_j, x_{j+1}\} \mid j = 0, 2, \ldots, l-1\}$ and $E_1(P) = \{\{x_j, x_{j+1}\} \mid j = 1, 3, \ldots, l-2\}$.

Case 1. $x \notin V(P)$. In this case let G^* be obtained from G by doubling each edge of $E(G) - (E(P) \cup \{\{x, y\}, \{x, v\}\})$ and tripling each edge of $E_0(P)$.

Case 2. $x \in V(P)$. In this case we have $l \geq 3$ and $x = x_j$ for some $1 \leq j \leq l-1$. If $2 \leq j \leq l-2$, then since l is odd, one of the two paths $y, x_1, \ldots, x_{j-1}, x, v$ and $y, x, x_{j+1}, \ldots, x_{l-1}, v$ would be a path of odd length connecting y and v that is shorter than P, contradicting the choice of P. Therefore, either $x = x_1$ or $x = x_{l-1}$. Assume without loss of generality that $x = x_1$. Define G^* to be the multigraph obtained from G by doubling each edge of $E(G) - [(E(P) - \{\{x, y\}\}) \cup \{\{x, v\}\}]$ and tripling each edge of $E_0(P) - \{\{x, y\}\}$.

In each case above, $d^*(x) = 2d(x) - 2$ and $d^*(z) = 2d(z)$ for every $z \neq x$, and hence G^* is Eulerian.

Set $a = y$ in Case 1 and $a = x_2$ in Case 2. By extending the 2-path a, x, v to an Eulerian tour, we see that there are Eulerian tours of G^* which pass through (a, x, v). Choose C to be an Eulerian tour of G^* with $(a, x, v) \in C(x)$

such that $|Z(C)|$ is minimum, where $Z(C)$ is the set of vertices $w \neq x$ of G^* such that $H_C(w)$ has no perfect matching.

Claim 1. $Z(C) = \emptyset$; that is, $H_C(w)$ has a perfect matching for every $w \neq x$.

Proof of Claim 1. We prove this by way of contradiction. Suppose $H_C(w)$ has no perfect matching for some $w \neq x$. By Lemma 1, $d^*(w) = 6$ and $C(w)$ contains twin visits. Since $w \neq x$, we have $d(w) = 3$ by the construction of G^*. Denote $N(w) = \{w_1, w_2, w_3\}$. In the case when each of w_1, w_2 and w_3 is joined to w by two parallel edges, we apply the bow-tie operation at w with respect to one of the twin visits and the third visit of $C(w)$. Similar to the proof of Theorem 1, for the resultant Eulerian tour C' of G^*, $H_{C'}(w)$ has a perfect matching, and the visit-decomposition at any other vertex is unchanged. Thus $(a, x, v) \in C'(x)$ and $Z(C')$ is a proper subset of $Z(C)$, contradicting the choice of C.

It remains to consider the case where exactly one vertex of $N(w)$ is joined to w by one, two or three (parallel) edges, respectively. Without loss of generality we may assume that there is one edge between w_3 and w, two parallel edges between w_1 and w, and three parallel edges between w_2 and w. Then $C(w) = \{[w_1, w, w_2], [w_1, w, w_2], [w_3, w, w_2]\}$. Reversing the orientation of C when necessary, we may assume $(w_1, w, w_2) \in C(w)$. Denote by e_1, e_3 the oriented parallel edges from w_1 to w, by e_2, e_4, e_6 the oriented parallel edges from w to w_2, and by e_5 the oriented edge from w to w_3.

Case (a): $C(w) = \{(w_1, w, w_2), (w_1, w, w_2), [w_3, w, w_2]\}$. We may assume

$$C : w_1, e_1, w, e_2, w_2, f, \ldots, g, w_1, e_3, w, e_4, w_2, h, \ldots, k, w_1.$$

Let

$$C' : w_1, e_1, w, e_3^{-1}, w_1, g^{-1}, \ldots, f^{-1}, w_2, e_2^{-1}, w, e_4, w_2, h, \ldots, k, w_1.$$

Then C' is an Eulerian tour of G^* and $C'(w) = \{(w_1, w, w_1), (w_2, w, w_2), [w_3, w, w_2]\}$. Moreover, $H_{C'}(w)$ has a perfect matching which matches (w_1, w, w_1), (w_2, w, w_2), $[w_3, w, w_2]$ to ww_2, ww_3, ww_1 respectively.

Case (b): $C(w) = \{(w_1, w, w_2), (w_2, w, w_1), [w_3, w, w_2]\}$. We may assume

$$C : w_1, e_1, w, e_2, w_2, f, \ldots, g, w_2, e_4^{-1}, w, e_3^{-1}, w_1, h, \ldots, k, w_1.$$

Denote

$$C_1 : w_1, e_1, w, e_3^{-1}, w_1, h, \ldots, k, w_1; \qquad C_2 : w_2, e_4^{-1}, w, e_2, w_2, f, \ldots, g, w_2.$$

Note that each of C_1 and C_2 is a closed trail, and $[w_3, w, w_2]$ is a segment of exactly one of C_1 and C_2.

In the case when $(w_3, w, w_2) \in C(w)$ and it is in C_2, we first rewrite C_2 to highlight the position of (w_3, w, w_2) in C_2:

$$C_2' : w_3, e_5^{-1}, w, e_6, w_2, \ldots, w_3.$$

Applying the concatenation operation to $(C_1, C_2', (w_1, w, w_1), (w_3, w, w_2))$ yields:

$$C' : w_1, e_1, w, e_5, w_3, \ldots, w_2, e_6^{-1}, w, e_3^{-1}, w_1, h, \ldots, k, w_1.$$

We have $C'(w) = \{(w_1, w, w_3), (w_2, w, w_1), [w_2, w, w_2]\}$. Hence $H_{C'}(w)$ has a perfect matching which matches (w_1, w, w_3), (w_2, w, w_1), $[w_2, w, w_2]$ to ww_2, ww_3, ww_1 respectively.

In the case when $(w_3, w, w_2) \in C(w)$ and it is in C_1, we first rewrite C_1 to highlight the position of $[w_3, w, w_2]$ in C_1:

$$C_1' : w_3, e_5^{-1}, w, e_6, w_2, \ldots, w_3.$$

Applying the concatenation operation to $(C_2, C_1', (w_2, w, w_2), (w_3, w, w_2))$ yields:

$$C' : w_2, e_4^{-1}, w, e_5, w_3, \ldots, w_2, e_6^{-1}, w, e_2, w_2, f, \ldots, g, w_2.$$

Since $C'(w) = \{(w_2, w, w_3), (w_2, w, w_2), [w_1, w, w_1]\}$, $H_{C'}(w)$ has a perfect matching which matches (w_2, w, w_3), (w_2, w, w_2), $[w_1, w, w_1]$ to ww_1, ww_3, ww_2 respectively.

The remaining case when $(w_2, w, w_3) \in C(w)$ can be dealt with similarly.

In all possibilities above we obtain a new Eulerian tour C' of G^* such that $H_{C'}(w)$ has a perfect matching whilst the visit-decomposition at any other vertex is unchanged. Thus $(a, x, v) \in C'(x)$ and $Z(C')$ is a proper subset of $Z(C)$, contradicting the choice of C. This completes the proof of Claim 1.

Claim 2. There exists an Eulerian tour C^* of G^* together with a visit $(u_1, x, u_2) \in C^*(x)$ such that (i) $H_{C^*}(z)$ has a perfect matching for every $z \neq x$, and (ii) the bipartite graph $K_{C^*}(u_1, x, u_2)$ (as defined in Definition 7) has a perfect matching under which the first and last visits induced by $W_{C^*}(u_1, x, u_2)$ are matched to xy and xv resepctively.

Note that, for $z \neq x$, $H_{C^*}(z) = H_W(z)$, where $W = W_{C^*}(u_1, x, u_2)$.

Proof of Claim 2. We will prove the existence of C^* and $(u_1, x, u_2) \in C^*(x)$ based on C as in Claim 1.

Case (a): G^* was constructed in Case 1. Then $(a, x, v) = (y, x, v) \in C(x)$ and all edges of G incident with x except $\{x, y\}$ and $\{x, v\}$ were doubled.

In the case when $d(x) = 3$, let z_1 be the neighbour of x in G other than y and v. One can see that $K_C(z_1, x, z_1)$ has a perfect matching which matches (t, x, z_1), (y, x, v), (z_1, x, t') to xy, xz_1, xv, respectively.

In the case when $d(x) = 4$, let z_1 and z_2 be the neighbours of x in G other than y and v. Since $(y, x, v) \in C(x)$, without loss of generality we may assume $C(x) \prec \{(z_1, x, z_1), (z_2, x, z_2), (y, x, v)\}$ or $\{(z_1, x, z_2), [z_1, x, z_2], (y, x, v)\}$. If $C(x) \prec \{(z_1, x, z_1), (z_2, x, z_2), (y, x, v)\}$, then $K_C(y, x, v)$ has a perfect matching which matches (t, x, v), (z_1, x, z_1), (z_2, x, z_2), (y, x, t') to xy, xz_2, xz_1, xv, respectively. In the case when $C(x) \prec \{(z_1, x, z_2), [z_1, x, z_2], (y, x, v)\}$, by applying the bow-tie operation at x with respect to $((z_1, x, z_2), (y, x, v))$ we obtain a new Eulerian tour $C' = C((z_1, x, z_2), (y, x, v))$ for which $C'(x) = \{[z_1, x, z_2], (z_j, x, y), (z_{j'}, x, v)\}$, where $\{j, j'\} = \{1, 2\}$. Without loss of generality we may assume $C'(x) = \{(z_1, x, z_2), (z_j, x, y), (z_{j'}, x, v)\}$. One can see that $K_{C'}(z_1, x, z_2)$ contains a perfect matching which matches (t, x, z_2), (z_j, x, y), $(z_{j'}, x, v)$, (z_1, x, t') to xy, $xz_{j'}$, xz_j, xv, respectively.

Assume $d(x) \geq 5$. If $L_C(y, x, v)$ has a perfect matching, then adding the edges $\{(t, x, v), xy\}$, $\{(y, x, t'), xv\}$ to it yields a perfect matching of $K_C(y, x, v)$ which matches the first and last visits of $W_C(y, x, v)$ to xy, xv, respectively.

Suppose that $L_C(y,x,v)$ has no perfect matchings. Similar to Lemma 1, by using Hall's marriage theorem we can prove that $d(x)=5$ and $C(x)$ contains twin visits, say, $[z_1,x,z_2]$; that is, $C(x) \prec \{[z_1,x,z_2],[z_1,x,z_2],[z_3,x,z_3],(y,x,v)\}$. Without loss of generality we may assume $(z_1,x,z_2) \in C(x)$. It is not hard to see that $K_C(z_1,x,z_2)$ has a perfect matching which matches (t,x,z_2), (z_1,x,z_2), $[z_3,x,z_3]$, (y,x,v), (z_1,x,t') to xy, xz_3, xz_2, xz_1, xv, respectively.

Case (b): G^* was constructed in Case 2. Then $(x_2,x,v) \in C(x)$ and all edges of G incident with x except $\{x,x_2\}$ and $\{x,v\}$ were doubled.

In the case when $d(x)=3$, we have $C(x) \prec \{(x_2,x,v),(y,x,y)\}$ and $K_C(x_2,x,v)$ has a perfect matching which matches $(t,x,v),(y,x,y),(x_2,x,t')$ to xy, xx_2, xv, respectively.

In the case when $d(x)=4$, we have $C(x) \prec \{(x_2,x,v),[z_1,x,y],[z_1,x,y]\}$ or $C(x) \prec \{(x_2,x,v)(z_1,x,z_1),\ (y,x,y)\}$, where z_1 is the neighbour of x other than y,v,x_2. If $C(x) \prec \{(x_2,x,v),[z_1,x,y],[z_1,x,y]\}$, let $(z_1,x,y) \in C(x)$, say. Then $K_C(y,x,z_1)$ has a perfect matching, namely $(t,x,z_1),(x_2,x,v),[z_1,x,y]$, (y,x,t') are matched to xy, xz_1, xx_2, xv, respectively. If $C(x) \prec \{(x_2,x,v)(z_1,x,z_1),(y,x,y)\}$, then $K_C(z_1,x,z_1)$ has a perfect matching which matches (t,x,z_1), $(x_2,x,v),(y,x,y),(z_1,x,t')$ to xy, xz_1, xx_2, xv, respectively.

Assume $d(x) \geq 5$ hereafter. In the case when $L_C(x_2,x,v)$ has a perfect matching, say, M, let xy be matched to (w_1,x,w_2) by M, where $w_1,w_2 \in N(x)-\{x_2,v,y\}$. Deleting $\{(w_1,x,w_2),xy\}$ from M and then adding $\{(w_1,x,w_2),$ $xx_2\},\{(t,x,v),xy\}$ and $\{(x_2,x,t'),xv\}$ yields a perfect matching of $K_C(x_2,x,v)$ satisfying (ii) in Claim 2.

Suppose $L_C(x_2,x,v)$ has no perfect matchings. Similar to Lemma 1, we can prove that $d(x)=5$ and $C(x)$ contains twin visits. Denote by $z_1,z_2 \neq y,v,x_2$ the other two neighbours of x. Let (w_1,x,w_2) be one of the twin visits in $C(x)$, where $w_1,w_2 \in \{y,z_1,z_2\}$ are distinct, and let w_3 denote the unique vertex in $\{y,z_1,z_2\}-\{w_1,w_2\}$. Then $C(x) \prec \{(x_2,x,v),(w_1,x,w_2),[w_1,x,w_2],(w_3,x,w_3)\}$. Since w_1 and w_2 are distinct, one of them, say, w_2, is not equal to y.

Thus $K_C(w_1, x, w_2)$ has a perfect matching which matches (t, x, w_2), (x_2, x, v), $[w_1, x, w_2]$, (w_3, x, w_3), (w_1, x, t') to xy, xw_2, xw_3, xx_2, xv, respectively.

Since $H_C(z)$ has a perfect matching for every $z \neq x$, one can see that in all possibilities above, condition (i) in Claim 2 is satisfied by the underlying Eulerian tour (which is C or C'). This proves Claim 2.

Choose an Eulerian tour $C^* : w_l, x, w_1, w_2, w_3, \ldots, w_l$ of G^* together with a visit $(w_l, x, w_1) \in C^*(x)$ satisfying the conditions of Claim 2. Then $W = W_{C^*}(w_l, x, w_1) : t, x, w_1, w_2, w_3, \ldots, w_{l-1}, w_l, x, t'$. Denote by $\phi(t, x, w_1)$ ($\phi(w_l, x, t')$, respectively) the arc of G with tail x that is matched to (t, x, w_1) ((w_l, x, t'), respectively) by a perfect matching of $K_{C^*}(w_l, x, w_1)$ satisfying (ii) in Claim 2. Let $\phi(x, w_1, w_2)$ denote the arc matched to (x, w_1, w_2) in a perfect matching of $H_{C^*}(w_1)$ $(= H_W(w_1))$, and let $\phi(w_1, w_2, w_3)$, $\ldots$, $\phi(w_{l-1}, w_l, x)$ be interpreted similarly. Conditions (i) and (ii) in Claim 2 ensure that

$$xy = \phi(t, x, w_1), \phi(x, w_1, w_2), \phi(w_1, w_2, w_3), \ldots, \phi(w_{l-1}, w_l, x), \phi(w_l, x, t') = xv$$

is a Hamilton path of $X(G)$ connecting xy and xv. □

Lemma 3 *Under the condition of Theorem 3, for distinct $xy, uv \in A(G)$ with $x \neq u$, there exists a Hamilton path of $X(G)$ between xy and uv.*

Proof We have five possibilities to consider: $x = v$ and $y = u$; x, y, u, v are pairwise distinct; $x = v$ and $y \neq u$; $y = v$ and $x \neq u$; $y = u$ and $x \neq v$. The following treatment covers all of them.

By our assumption there exists a path of odd length connecting x and u in G. Let

$$P : x = x_0, x_1, x_2, \ldots, x_{l-1}, x_l = u \tag{2}$$

be such a path with shortest (odd) length $l \geq 1$. (It may happen that $y = x_1$ and/or $v = x_{l-1}$.) Define G^* to be the multigraph obtained from G by doubling each edge of G outside of P and tripling each edge $\{x_j, x_{j+1}\}$ for $j = 1, 3, \ldots, l-2$. Then $d^*(x) = 2d(x)-1$, $d^*(u) = 2d(u)-1$ and $d^*(z) = 2d(z)$ for $z \neq x, u$.

Let $G^*_{x,u}(t,t')$ be the multigraph obtained from G^* by adding two new vertices t,t' and joining them to x,u respectively by a single edge. Then all vertices of $G^*_{x,u}(t,t')$ except t and t' have even degrees in $G^*_{x,u}(t,t')$. Hence $G^*_{x,u}(t,t')$ has Eulerian trails connecting t and t'.

Since $\delta(G) \geq 3$, we can choose x' to be a neighbour of x other than y and x_1, and u' a neighbour of u other than v and x_{l-1}. In addition, if $d(x) = d(u) = 3$, $y = x_1$ and $v = x_{l-1}$, say, $N(x) = \{y, x', z\}$ and $N(u) = \{v, u', w\}$, then we can choose x' and u' in such a way that the edges $\{x, z\}$ and $\{u, w\}$ do not form an edge cut of G. In fact, if $\{\{x, z\}, \{u, w\}\}$ is an edge cut of G in this case, then since G is assumed to be 2-edge connected, $G - \{\{x, z\}, \{u, w\}\}$ has two connected components, say, G_0 and G_1 with $z, w \in V(G_0)$ and P in G_1. Since x' is in G_1 and removal of $\{x, x'\}$ does not disconnect G, one can see that $\{\{x, x'\}, \{u, w\}\}$ is not an edge-cut of G. Thus interchanging the roles of x' and z produces the desired x' and u'. (In general, at most one of x' and u' lies on P since P is a path between x and u with minimum odd length.)

With x' and u' as above, let

$$W' : t, x, x', \overbrace{x, x_1, x_2, \ldots, x_{l-1}, u}^{P}, u', u, t',$$

where P is the path given in (2). Then W' is a trail of $G^*_{x,u}(t,t')$. Let W be an Eulerian trail of $G^*_{x,u}(t,t')$ obtained by extending W' to cover all edges of $G^*_{x,u}(t,t')$ while maintaining (t, x, x') and (u', u, t') as its first and last visits respectively. Such a trail W exists because removing the four edges in (t, x, x') and (u', u, t') from $G^*_{x,u}(t,t')$ results in a connected multigraph with x' and u' as the only odd-degree vertices. In addition, if $d(x) = 3$ and $y = x_1$, say, $N(x) = \{y, x', z\}$, since $\{\{x, z\}, \{u, w\}\}$ is not an edge cut of G by our choices of x' and u', we can choose W in such a way that (x', x, x_1) is a visit induced by W; similarly, we can choose W such that (u', u, x_{l-1}) is a visit induced by W, if $d(u) = 3$ and $v = x_{l-1}$, say, $N(u) = \{v, u', w\}$. (Such a W can be constructed as follows: extend W' to an Eulerian trail of the multigraph obtained by deleting the parallel edges between x and z and/or that between

u and w, and then insert the visits (z, x, z) and/or (w, u, w) to this trail.) In this way we obtain an Eulerian trail W of $G^*_{x,u}(t, t')$ such that

(A) (t, x, x') and (u', u, t') are its first and last visits, respectively; and

(B) if $d(x) = 3$ and $y = x_1$, say, $N(x) = \{y, x', z\}$, then $(x', x, x_1) \in W(x)$; and, if $d(u) = 3$ and $v = x_{l-1}$, say, $N(u) = \{v, u', w\}$, then $(u', u, x_{l-1}) \in W(x)$.

Similar to Claim 1, one can show that there exists an Eulerian trail of $G^*_{x,u}(t, t')$, denoted by W hereafter, satisfying (A), (B) and

(C) $H_W(z)$ has a perfect matching for every $z \in V(G) - \{x, u\}$.

Note that $|W(z)| = |A(z)| = d(z)$ for every $z \in V(G)$.

Claim 3. There exists an Eulerian trail W^* of $G^*_{x,u}(t, t')$ such that (i) (t, x, x') and (u', u, t') are its first and last visits, respectively; (ii) $H_{W^*}(x)$ has a perfect matching under which (t, x, x') is matched to xy; (iii) $H_{W^*}(u)$ has a perfect matching under which (u', u, t') is matched to uv; and (iv) $H_{W^*}(z)$ has a perfect matching for every $z \in V(G) - \{x, u\}$.

Proof of Claim 3. Let $p = (t, x, x')$ denote the first visit of W, and let $L_W(x) = H_W(x) - \{p, xy\}$ be the subgraph of $H_W(x)$ obtained by deleting vertices p and xy. For $S \subseteq W(x) - \{p\}$, denote by $N_{L_W(x)}(S)$ the neighbourhood of S in $L_W(x)$.

Case (a): $y \neq x_1$. If $d(x) \geq 5$, then $|N_{L_W(x)}(S)| \geq |S|$ for any S, and so $L_W(x)$ contains a perfect matching by Hall's marriage theorem.

Suppose $d(x) = 4$. Then $|N_{L_W(x)}(S)| \geq |S|$ for every S with $|S| = 1$ or 3. Suppose $|S| = 2$ and $S = \{(a, x, b), (a', x, b')\}$. Then $N_{L_W(x)}(S) = [(A(x) - \{xy\}) - \{xa, xb\}] \cup [(A(x) - \{xy\}) - \{xa', xb'\}] = [(A(x) - \{xy\})] - (\{xa', xb'\} \cap \{xa, xb\})$. Thus, if $|\{xa', xb'\} \cap \{xa, xb\}| \leq 1$, then $|N_{L_W(x)}(S)| \geq |S|$. If $|\{xa', xb'\} \cap \{xa, xb\}| = 2$, then $\{a, b\} = \{a', b'\}$ and $\{x', x_1\} \cap \{a, b\} = \emptyset$, which implies $y \in \{a, b\}$ and $|N_{L_W(x)}(S)| = |(A(x) - \{xa, xb\}| = 2$. Hence $L_W(x)$ contains a perfect matching by Hall's theorem.

Suppose $d(x) = 3$. Then $W(x) = \{p, (x', x, y), (y, x, x_1)\}$ or $W(x) = \{p, (x', x, x_1), (y, x, y)\}$. In the former case $L_W(x)$ clearly has a perfect matching.

In the latter case, apply the bow-tie operation to W with respect to (x', x, x_1) and (y, x, y) to obtain a new Eulerian trail W_0 such that $L_{W_0}(x)$ has a perfect matching.

Case (b): $y = x_1$. Similar to Case (a), if $d(x) \geq 5$, then $L_W(x)$ has a perfect matching. If $d(x) = 4$, let $N(x) = \{x', x_1, z_1, z_2\}$. Then $|N_{L_W(x)}(S)| \geq |S|$ unless $S = \{(z_1, x, z_2), [z_1, x, z_2]\}$. In this exceptional case, $W(x) = \{p, (x', x, x_1), (z_1, x, z_2), [z_1, x, z_2]\}$, and we apply the bow-tie operation to W with respect to (x', x, x_1) and (z_1, x, z_2) to obtain a new Eulerian trail W_0. One can show that $L_{W_0}(x)$ has a perfect matching.

If $d(x) = 3$, let $N(x) = \{x', x_1, z\}$. By (B), (x', x, x_1) is a visit to x induced by W. Hence $W(x) = \{p, (x', x, x_1), (z, x, z)\}$ and $L_W(x)$ has a perfect matching.

So far we have proved that there exists an Eulerian trail W_1 of $G^*_{x,u}(t, t')$ (which is either W or W_0) satisfying (A) such that $L_{W_1}(x)$ has a perfect matching. This matching together with the edge between (t, x, x') and xy is a perfect matching of $H_{W_1}(x)$. Moreover, since W satisfies (C), from the proof above one can see that W_1 satisfies (C) as well. If $H_{W_1}(u)$ has a perfect matching which matches (u', u, t') to uv, then set $W^* = W_1$ and we are done. Otherwise, beginning with W_1 and using similar arguments as above, we can construct an Eulerian trail W^* of $G^*_{x,u}(t, t')$ satisfying all requirements in Claim 3. This completes the proof of Claim 3.

Similar to the proof of Lemma 2, we can show that the Eulerian trail W^* in Claim 3 produces a Hamilton path in $X(G)$ connecting xy and uv. □

Proof of Theorem 3 This follows from Lemmas 2 and 3 immediately. □

In the proof of Theorem 4 we will use the following lemma which may be known in the literature. We give its proof since we are unable to allocate a reference.

Lemma 4 *In any Hamilton-connected graph with at least four vertices, there exists a path of odd length connecting any two distinct vertices.*

Proof Let G be such a graph. Then for any distinct $u, v \in V(G)$ there exists a Hamilton path $P: u = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = v$, where $n = |V(G)| - 1$. It suffices to consider the case when n is even. Denote $A = \{x_0, x_2, \ldots, x_n\}$ and $B = \{x_1, x_3, \ldots, x_{n-1}\}$. Since $\{A, B\}$ is a partition of $V(G)$ and any bipartite graph other than K_2 is not Hamilton-connected, there exist adjacent vertices x_i, x_j both in A or B, where $j \geq i+2$. Thus $x_0, x_1, \ldots, x_{i-1}, x_i, x_j, x_{j+1}, \ldots, x_n$ is a path of odd length between u and v. □

Proof of Theorem 4 It can be verified that any Hamilton-connected graph with at least four vertices is 2-edge connected and has minimum degree at least three. Hence Theorem 3 and Lemma 4 together imply that the 3-arc graph of such a graph is Hamilton-connected (with more than four vertices). Applying this iteratively, we obtain Theorem 4. □

Acknowledgements Guangjun Xu was supported by the MIFRS and SFS scholarships of the University of Melbourne. Sanming Zhou was supported by a Future Fellowship (FT110100629) of the Australian Research Council.

References

1. B. Alspach, Y. S. Qin, Hamilton-connected Cayley graphs on Hamiltonian groups, *Europ. J. Combin.* **22** (2001), 777–787.
2. C. Balbuena, L. P. Montejano, P. García-Vázquez, On the connectivity and restricted edge-connectivity of 3-arc graphs, In *Proceedings of the 3rd International Workshop on Optimal Networks Topologies, (IWONT 2010)*, Barcelona, Spain, 9-11 June, 2010, pp. 79–90, Barcelona, Spain, 2011. Iniciativa Digital Politècnica.
3. J. A. Bondy, Basic graph theory – paths and cycles, in: Handbook of Combinatorics,Vol. I, pp. 5–110, Elsevier, Amsterdam, 1995
4. J. A. Bondy and U. S. R. Murty, Graph Theory, Springer, New York, 2008.
5. G. Chartrand, On Hamiltonian line-graphs, *Trans. Amer. Math. Soc.* **134** (1968), 559–566.
6. Z-H. Chen, H-J. Lai, L. Xiong, H. Yan and M. Zhan, Hamilton-connected indices of graphs, *Discrete Math.* **309** (2009), 4819–4827.
7. C. C. Chen and N. Quimpo, On strongly Hamiltonian abelian group graphs, in: Combinatorial Mathematics VIII, K. L. McAvaney (ed), Lecture Notes in Mathematics, 884, Springer, Berlin, 1981, pp. 23–34.

8. R. Diestel, Graph Theory, Springer, New York, 4th edition, 2010.

9. A. Gardiner, C. E. Praeger and S. Zhou, Cross-ratio graphs, *J. London Math. Soc.* (2) **64** (2001), 257–272.

10. R. J. Gould, Advances on the Hamiltonian problem ?a survey, *Graphs and Combinatorics* **19** (2003), 7–52.

11. Z. Hu, F. Tian and B. Wei, Hamilton connectivity of line graphs and claw-free graphs, *J. Graph Theory* **50** (2005), 130–141.

12. M. A. Iranmanesh, C. E. Praeger and S. Zhou, Finite symmetric graphs with two-arc transitive quotients, *J. Combin. Theory (Ser. B)* **94** (2005), 79–99.

13. M. Knor, G. Xu and S. Zhou, A study of 3-arc graphs, *Discrete Appl. Math.* **159** (2011), 344–353.

14. M. Knor and S. Zhou, Diameter and connectivity of 3-arc graphs, *Discrete Math.* **310** (2010), 37–42.

15. K. Kutnar and D. Marušič, Hamilton cycles and paths in vertex-transitive graphs – current directions, *Discrete Math.* **309** (2009), 5491–5500.

16. D. Li, H-J. Lai and M. Zhan, Eulerian subgraphs and Hamilton-connected line graphs, *Discrete Appl. Math.* **145** (2005), 422–428.

17. C. H. Li, C. E. Praeger and S. Zhou, A class of finite symmetric graphs with 2-arc transitive quotients, *Math. Proc. Cambridge Phil. Soc.* **129** (2000), 19–34.

18. Z. Lu and S. Zhou, Finite symmetric graphs with 2-arc transitive quotients (II), *J. Graph Theory*, **56** (2007), 167–193.

19. C. E. Praeger, Finite symmetric graphs, in: Topics in Algebraic Graph Theory, Cambridge University Press, Cambridge, 2004, pp.179–202.

20. C. Thomassen, Reflections on graph theory, *J. Graph Theory* **10** (1986), 309–324.

21. C. Thomassen, Tilings of the torus and the Klein bottle and vertex-transitive graphs on a fixed surface, *Trans. Amer. Math. Soc.* **323** (1991), 605–635.

22. M. Zhan, Hamiltonicity of 6-connected line graphs, *Discrete Appl. Math.* **158** (2010), 1971–1975.

23. S. Zhou, Constructing a class of symmetric graphs, *European J. Combin.* **23** (2002), 741–760.

24. S. Zhou, Imprimitive symmetric graphs, 3-arc graphs and 1-designs, *Discrete Math.* **244** (2002), 521–537.

25. S. Zhou, Almost covers of 2-arc transitive graphs, *Combinatorica* **24** (2004), 731-745. [Erratum: **27** (2007), 745–746.]