

EXPONENTIAL GROWTH OF PRODUCTS OF NON-STATIONARY MARKOV-DEPENDENT MATRICES

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ABSTRACT. Let $(\xi_j)_{j \geq 1}$, be a non-stationary Markov chain with phase space X and let $\mathbf{g}_j : X \mapsto \mathrm{SL}(m, \mathbb{R})$ be a sequence of functions on X with values in the unimodular group. Set $g_j = \mathbf{g}_j(\xi_j)$ and denote by $S_n = g_n \dots g_1$, the product of the matrices g_j . We provide sufficient conditions for exponential growth of the norm $\|S_n\|$ when the Markov chain is not supposed to be stationary. This generalizes the classical theorem of Furstenberg on the exponential growth of products of independent identically distributed matrices as well as its extension by Virtser to products of stationary Markov-dependent matrices.

1. INTRODUCTION

Let $(g_n)_{n \geq 1}$ be a sequence of matrices, $g_n \in \mathrm{SL}(m, \mathbb{R})$, and set

$$S_n = g_n \dots g_1.$$

In the seminal 1963 paper [5], H. Furstenberg proved the following theorem. (All relevant definitions can be found in section 2.1.2.)

Theorem 1.1 (Furstenberg, [5]). *Suppose that:*

- (a) $(g_n)_{n \geq 1}$ is a sequence of independent identically distributed (i.i.d.) random matrices satisfying $\int_{\mathrm{SL}(m, \mathbb{R})} \log \|g\| d\nu(g) < \infty$.
- (b) The group \mathbb{G}_ν generated by the support of ν does not preserve any probability measure on the unit sphere \mathcal{S} in \mathbb{R}^m .

Then the following limit (called the top Lyapunov exponent of the product S_n) exists with probability 1 and is strictly positive:

$$(1) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|S_n\| = \lambda > 0.$$

The existence of the limit in (1) was first proved by Furstenberg and Kesten in [4] for a stationary sequence $(g_n)_{n \geq 1}$. The main statement of Theorem 1.1 is the strict positivity of λ .

In the late 1970s A. Virtser [15] extended this result to products of stationary Markov-dependent sequences of matrices by which we mean that $g_n = \mathbf{g}(\xi_n)$, where $\xi = (\xi_n)_{n \geq 1}$ is a stationary Markov chain with a phase space X and $\mathbf{g} : X \mapsto \mathrm{SL}(m, \mathbb{R})$ is a ‘good’ function on X . The Markov chain ξ in [15] is supposed to satisfy the condition $\|K^0\| < 1$, where K^0 is the restriction of the transition operator of the chain to the subspace of functions on X orthogonal to constants (the exact

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statement of Virtser's theorem is given in section 3.3). The other conditions in [15] are as in Furstenberg's theorem.

The paper [15] was followed by a number of works of which we mention Royer [11], Guivarc'h [7], Ledrappier [10] where the stationarity of the chain plays a very important role.

In a very recent work [1], Avila, Damanik, and Zhang proved positivity of the Lyapunov exponent for products of the so called Schrödinger matrices (see section 3.1) with potentials generated by hyperbolic dynamical systems. In particular [1] deals with certain classes of potentials generated by stationary Markov chains. The matrices considered in this work are 2×2 , have the structure imposed on them by the Anderson model, and form a stationary sequence. It should be mentioned that the techniques used in [1] and in the present work are completely different.

The main goal of this work is to extend Virtser's result (and thus also Furstenberg's result) to the product of Markov-dependent matrices in the case when the underlying Markov chain is not supposed to be stationary. In contrast, stationarity is crucial for the technique used in the majority of previous work. Obviously, in this case the limit in (1) may not exist. What we shall show is that there is $\lambda > 0$ such that with probability 1 $\liminf_{n \rightarrow \infty} \frac{1}{n} \ln \|S_n\| \geq \lambda$. This will be done under very mild conditions. Namely:

- We remove the requirement for the Markov chain to be stationary.
- The functions defining the matrices g_n may be time-dependent: $g_n = \mathbf{g}_n(\xi_n)$, where $\mathbf{g}_n : X \mapsto \mathrm{SL}(m, \mathbb{R})$. Moreover, they may themselves be random.
- No moment condition is imposed on the distributions of matrices (and thus the case $\lambda = +\infty$ is not excluded).
- The conditions on the transition operators and the supports of the distributions of g_n are supposed to be satisfied only for a subsequence of indices of positive density.

The price we pay is that the group \mathbb{G}_ν appearing in assumption (b) of Furstenberg's theorem has to be replaced by a smaller group G_ν generated by all products $g\tilde{g}^{-1}$, where g and \tilde{g} are from the support of ν (in the non-stationary case, these groups depend on n). This means that in the stationary case our requirement is, formally speaking, more restrictive than the one in [15]. However, we show in Section 3.3 that Virtser's theorem can be deduced from our result.

Several important technical lemmas in the present work and in [15] are similar. We state these lemmas in the form which is convenient for us and we prove them to make this paper self-contained. The main innovation with respect to [15] is that we manage to upgrade the estimate on the spectral radius of certain operators to an estimate on the norm of a product of two operators of the same kind. This upgrade enables all the generalizations described above.

1.1. Motivation. First of all, the exponential growth is a fundamental property of products S_n and the task of extending it to wider classes of products is important in its own right.

Here are several examples of problems whose solutions depend, to a large extent, on the possibility to control the top Lyapunov exponent of the product S_n . The first of these problems will be addressed in this work. The other two problems will be discussed elsewhere.

1. Given a stationary sequence of matrices g_j , consider their perturbations of the form $a_j g_j$, where a_j is a non-random sequence of matrices, $a_j \in \text{SL}(m, \mathbb{R})$. The natural question is: what part of the theory of stationary products can be extended to this non-stationary case?

In section 3.2, we prove that the exponential growth of the product is preserved for the class of Markov-dependent matrices (which may not be stationary). In many applications this fact is more important than the existence of the limit in (1). (We remark that, unless a_n 's are chosen in some special way, this limit does not exist.)

2. In the theory of Anderson localization in dimension one, the exponential growth of S_n plays a crucial role. In particular, it implies the existence with probability 1 of a random vector $u \in \mathbb{R}^2$ such that $\limsup_{n \rightarrow \infty} n^{-1} \ln \|S_n u\| < 0$. Extending this theory to models with non-stationary potentials is an important problem and this work is a step in this direction. In the case of non-stationary potential with independent entries a solution to this problem was announced in [6].

3. One of the central questions in the general theory of products of independent random matrices is the existence of distinct Lyapunov exponents (see e.g. [14]). It turns out that this question can be reduced to the question about the positivity of the top Lyapunov exponent of a product of Markov-dependent matrices.

1.2. Some history: products of independent non-identically distributed matrices. There is extensive literature studying different aspects of the theory of Lyapunov exponents for products of a stationary sequence of matrices. We refer the interested reader to relatively recent books [14] and [2] and references therein.

In contrast, there are few papers dealing with products of non-identically distributed matrices most of which arise in the context of the spectral theory of random Schrödinger operators. We are aware of the following articles.

Works [3], [13], [9] deal with matrices arising in the theory of localization for Anderson model in dimension one with a potential decaying at infinity. These matrices are of the form

$$(2) \quad g_n = \begin{pmatrix} a_n q_n & -1 \\ 1 & 0 \end{pmatrix},$$

where q_n are i.i.d. random variables and the (deterministic) sequence $(a_n \in \mathbb{R})_{n \geq 1}$ satisfies $C_1 |n|^{-\alpha} < |a_n| < C_2 |n|^{-\alpha}$, where $n \neq 0$ and C_1, C_2, α are some positive constants. We note that for any $\alpha > 0$ the growth of the norm $\|S_n\|$ is at most sub-exponential.

The work [12] by Shubin-Vakilian-Wolff provides constructive estimates for the norm of an operator which is the average of a certain representation of $\text{SL}(2, \mathbb{R})$, where the average is computed over the distribution of the matrices. This result implies a constructive estimate for the exponential growth of products of matrices (2) with $a_n = 1$. With a bit of additional work, it is possible to extend this result to the case of independent non-identically distributed matrices of this form. Formally speaking, the latter has not been explicitly stated in [12] but it seems plausible that the authors were aware of this fact (see comments in [12, page 943]).

In [8], Y. Kifer proved a series of results concerned with different aspects of the theory of products of random matrices whose distributions form a stationary process. In particular, he proves under certain conditions the strict positivity of the top

Laypunov exponent. The stationarity condition is crucial for the technique used in [8] but is not satisfied for products of matrices considered in the present paper.

Recently, A. Gorodetski and V. Kleptsyn announced [6] a proof of exponential growth of a product of independent non-identically distributed matrices under conditions similar to those stated in Theorem 4.1 of this paper. For the case of 2×2 matrices, [6] contains additional results on the Law of Large Numbers and Large Deviations for such products.

1.3. Organization of the paper. In section 2 we recall some well known definitions and introduce the related notation in the form which is best suited for what follows; we then state the main results (Theorems 2.1 and 2.2) and provide some comments on them. The applications of the main results are considered in section 3; in particular, example 3 (section 3.3) explains how to deduce Virtser's theorem from Theorem 2.1. In section 4 we prove a particular case of Theorem 2.1. Namely, Theorem 4.1 considers the case of independent matrices. There are several reasons for that. First of all, products of independent matrices form a very important subclass in the theory of products of random matrices which deserves a separate consideration. Secondly, this allows us to explain some of the ideas in the case which is less technical and therefore more transparent than the general case. Finally, the proof in the Markov-dependent case makes use of Lemma 4.3 which is the main technical result needed for the independent case. In section 5 we introduce the technique which allows us to treat the products of Markov-dependent matrices and prove Theorem 2.1. The main parts of the proofs in the Markov-dependent and in the independent case differ significantly and this difference does not seem to be easily predictable (see Remark 5.9). In section 6, our second main result (Theorem 2.2) is derived from Lemma 6.1 which, in turn, is an extension of Theorem 2.1. Appendix contains three lemmas which we use in the main text of the paper.

1.4. Some notation and conventions. The following notation is used throughout the paper.

\mathcal{S} is the unit sphere in \mathbb{R}^m and $u \in \mathcal{S}$ is a unit vector. We write $\int_{\mathcal{S}} f(u) du$ for the integral over the uniform distribution on \mathcal{S} .

ξ and $(\xi_n)_{n \geq 1}$ denote the same Markov chain. A similar convention applies to all Markov chains which are introduced in the paper, such as $\tilde{\xi} = (\tilde{\xi}_n)_{n \geq 1}$, etc .

X is the phase space of ξ . The elements of X are denoted x, y, x_i, y_j , etc.

The term measure always means probability measure.

The notation $\|\cdot\|$ is mainly used for the norms of vectors and matrices; in those cases when it is used for norms of functions or operators, its exact meaning is always obvious from the context.

If f belongs to a space of functions \mathbb{H}_n , we write $\|f\|_{\mathbb{H}_n}$ for the norm of f when it is important to emphasize that $f \in \mathbb{H}_n$ and that the norm is the one with which \mathbb{H}_n is equipped.

2. MAIN RESULTS

2.1. The setup.

2.1.1. *The Markov chain.* Let (X, \mathcal{B}) be a measurable space (with \mathcal{B} being the sigma-algebra of measurable subsets of the set X). Consider a Markov chain $\xi = (\xi_n)_{n \geq 1}$, with the phase space X and the initial distribution μ_1 . For any $B \in \mathcal{B}$, set

$$k_n(x, B) = \mathbb{P}(\xi_{n+1} \in B \mid \xi_n = x).$$

We write $k_n(x, dy)$ for the corresponding transition kernels of the chain ξ .

Let μ_n be the distribution of ξ_n . As usual, for $n \geq 2$ and $B \in \mathcal{B}$ we have

$$(3) \quad \mu_n(B) = \mathbb{P}(\xi_n \in B) = \int_X \mu_{n-1}(dx) k_{n-1}(x, B).$$

We thus have a sequence of ‘Markov related’ measure spaces (X, \mathcal{B}, μ_n) . Denote by H_n the Hilbert space of μ_n -square integrable real valued functions,

$$(4) \quad H_n = \left\{ f : X \mapsto \mathbb{R}, \int_X |f(x)|^2 \mu_n(dx) < \infty \right\}$$

with the standard inner product: if $f, h \in H_n$ then $\langle f, h \rangle_{H_n} = \int_X f(x)h(x)\mu_n(dx)$. Set

$$(5) \quad H_n^0 = \left\{ f \in H_n : \int_X f(x)\mu_n(dx) = 0 \right\}.$$

The integral with respect to μ_n will be denote $\mathbb{E}_n : \mathbb{E}_n(f) = \int_X f(x)\mu_n(dx)$.

Let $K_n : H_{n+1} \mapsto H_n$ be the operator defined by

$$(K_n f)(x) = \int_X k_n(x, dy) f(y).$$

We remark that if $f \in H_{n+1}$ then $K_n f \in H_n$ which is a standard property of any Markov chain. Note that the operator K_n ‘computes’ the conditional expectation of $f(\xi_{n+1})$ conditioned on $\xi_n = x$.

Denote by K_n^0 the restriction of K_n to H_{n+1}^0 . It is easy to see that if $\mathbb{E}_{n+1}(f) = 0$ then $\mathbb{E}_n(K_n f) = 0$, that is $K_n^0 : H_{n+1}^0 \mapsto H_n^0$.

2.1.2. *The matrices.* Let $\mathfrak{g}_n : X \mapsto \text{SL}(m, \mathbb{R})$, $n \geq 1$, be a sequence of matrix-valued \mathcal{B} -measurable functions on X . Define a sequence of random matrices g_j by setting $g_j = \mathfrak{g}_j(\xi_j)$, $j \geq 1$. Let ν_j be the distribution of g_j , that is for a Borel set $\Gamma \subset \text{SL}(m, \mathbb{R})$ we define

$$(6) \quad \nu_j(\Gamma) = \mathbb{P}(\mathfrak{g}_j(\xi_j) \in \Gamma).$$

By $\text{supp}(\nu_j) \subset \text{SL}(m, \mathbb{R})$ we denote the support of ν_j .

Given a distribution ν on $\text{SL}(m, \mathbb{R})$ we define a group G_ν as follows:

$$(7) \quad G_\nu = \text{closed group generated by the set } \{g_1 g_2^{-1} : g_1, g_2 \in \text{supp}(\nu)\}.$$

Definition 1. For $g \in \text{SL}(m, \mathbb{R})$ and $u \in \mathcal{S}$ we define $g.u = gu/||gu||$.

The induced action of g on the set of probability measures on \mathcal{S} is defined by $(g\kappa)(B) = \kappa(g^{-1}B)$, where κ is a probability measure on \mathcal{S} and B is a Borel subset of \mathcal{S} .

We say that a probability measure κ on \mathcal{S} is preserved by g if $\kappa(B) = (g\kappa)(B)$ for any Borel B .

A group G preserves the measure κ on \mathcal{S} if every $g \in G$ preserves κ .

2.2. Main results. The following assumptions and their variations will be used throughout the paper.

I. There is a $c < 1$ such that for all $n \geq 1$

$$(8) \quad \|K_n^0\| \leq c.$$

II. There is a set M of probability measures on $\text{SL}(m, \mathbb{R})$ which is compact with respect to weak convergence and such that:

- (9) (a) all ν_n belong to M ,
 (b) all $\nu \in M$ are such that G_ν does not preserve any measure on \mathcal{S} .

Theorem 2.1. *Suppose that assumptions I and II are satisfied. Then there is $\lambda > 0$ such that with probability 1*

$$(10) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \|g_n \dots g_1\| \geq \lambda.$$

Our next result allows us to relax the assumptions of Theorem 2.1: it turns out that (10) holds when I and II are satisfied only for a subsequence of time moments. To state it, we need two more definitions. For integers $n \geq 1$ and $l \geq 1$ denote by $\nu_{n,l}$ the distribution of the product $\mathfrak{g}_{n+l-1}(\xi_{n+l-1}) \dots \mathfrak{g}_n(\xi_n)$: for a Borel set $\Gamma \subset \text{SL}(m, \mathbb{R})$ we set

$$\nu_{n,l}(\Gamma) = \mathbb{P}(\mathfrak{g}_{n+l-1}(\xi_{n+l-1}) \dots \mathfrak{g}_n(\xi_n) \in \Gamma) \quad \text{and}$$

$$(11) \quad G_{\nu_{n,l}} = \text{closure of the group generated by the set } \{g\tilde{g}^{-1} : g, \tilde{g} \in \text{supp}(\nu_{n,l})\}.$$

We note that $\nu_{n,1} = \nu_n$, $G_{\nu_{n,1}} = G_{\nu_n}$.

Theorem 2.2. *Suppose that there is a sequence of time intervals $[n_j, n_j + l_j]$, $j \geq 1$, with properties $n_1 \geq 1$, $l_j \geq 1$, $n_{j+1} \geq n_j + l_j$ and such that:*

- (i) *the inequalities $\|K_{n_j}^0\| \leq c$, $\|K_{n_j+l_j}^0\| \leq c$, where $c < 1$, hold for all $j \geq 1$;*
 (ii) *the distributions ν_{n_j+1, l_j} belong to a compact set M satisfying the requirement (9)(b).*

Then there is a (non-random) $\lambda > 0$ such that with probability 1

$$(12) \quad \liminf_{j \rightarrow \infty} \frac{1}{j} \ln \|g_{n_j+l_j} \dots g_1\| \geq \lambda$$

and λ in (12) does not depend on the choice of functions

$$(13) \quad \{\mathfrak{g}_i(\cdot) : i \in \bigcup_{j \geq 1} [n_j + l_j + 1, n_{j+1}]\}.$$

Theorem 2.1 is a particular case of Theorem 2.2 with $n_j = j$ and $l_j = 1$. However, we shall see in section 6 that the proof of Theorem 2.2 will be reduced to the proof of Theorem 2.1.

Here is one more useful particular case of Theorem 2.2.

Corollary 2.3. *Suppose that assumption I is satisfied and there is $k \geq 1$ such that all distributions $\nu_{nk+1, k}$, $n \geq 0$, belong to a compact set M satisfying (9)(b).*

Then there is a (non-random) $\lambda > 0$ such that with probability 1

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \ln \|g_n \dots g_1\| \geq \lambda.$$

Remarks.

1. If in Theorem 2.2 $n_j + l_j = n_{j+1}$ then, by convention, the interval $[n_j + l_j + 1, n_{j+1}]$ in (13) is empty .
2. If ξ is a finite Markov chain with $X = \{1, \dots, r\}$ then it is easy to see that (8) holds if there is $\delta > 0$ such that $\mathbb{P}(\xi_{n+1} = j | \xi_n = i) \geq \delta$ for all $n \geq 1$ and all $i, j \in X$. Similarly, (8) is satisfied if X is a compact metric space and $k_n(x, dy) = \bar{k}_n(x, y)dy$, where dy is a measure on X and $(\bar{k}_n(x, y))_{n \geq 1}$ is a sequens of equicontinuous functions of (x, y) such that $\bar{k}_n(x, y) \geq \delta > 0$.

In the case when ξ is a stationary ergodic Markov chain with finite phase space X , the necessary and sufficient condition for (8) can be easily established (see e.g. [15]).

3. If matrices g_n are independent then

$$(14) \quad G_{\nu_{n+1, l-1}} \subset G_{\nu_{n, l}} \quad \text{for any } n \geq 1, l \geq 2.$$

Indeed, definition (11) of g and \tilde{g} implies that if $g, \tilde{g} \in \text{supp}(\nu_{n+1, l-1})$ then $gg', \tilde{g}\tilde{g}' \in \text{supp}(\nu_{n, l})$, where $g', \tilde{g}' \in \text{supp}(\nu_n)$. Due to independence, we can choose $g' = \tilde{g}'$ (while g and \tilde{g} remain fixed) and hence if $g\tilde{g}^{-1} \in G_{\nu_{n+1, l-1}}$ then $g\tilde{g}^{-1} = gg'(\tilde{g}g')^{-1} \in G_{\nu_{n, l}}$.

In applications, it may happen that the group $G_{\nu_{n+1, l-1}}$ does preserve some probability measure on the unit sphere but the larger group $G_{\nu_{n, l}}$ doesn't.

In the case of Markov-dependent matrices, the same argument proves (14) if the support of the conditional distribution of $\{\mathbf{g}_{n+l+1}(\xi_{n+l+1}) | \xi_{n+l}\}$ does not depend on ξ_{n+l} . However, if this condition is not satisfied, (14) may fail.

The example considered in section 3.1 shows that in order to check that Condition II in above theorems is satisfied, it may be sufficient to establish that only a subgroup of $G_{\nu_{n+1, l-1}}$ belongs to $G_{\nu_{n, l}}$.

3. SOME APPLICATIONS OF THE MAIN RESULTS

3.1. The classical matrices (2) in the Markov setting. The example of the product of matrices (2) is the particularly well known one (see Introduction). Throughout this section, we suppose that in (2) all $a_n = 1$ and that

- (i) $q_n = \mathbf{g}_n(\xi_n)$, where $(\xi_n)_{n \geq 1}$ is a Markov chain satisfying assumption I
- (ii) there are $\delta > 0, C > 0$ such that $\text{Var}(q_n) \geq \delta$ and $|q_n| \leq C$ for all $n \geq 1$.

Let q and \tilde{q} denote two distinct points from the support of q_n and denote by g and \tilde{g} the matrices corresponding to q and \tilde{q} respectively. Then $g\tilde{g}^{-1} = \begin{pmatrix} 1 & q - \tilde{q} \\ 0 & 1 \end{pmatrix}$ and hence G_{ν_n} is a subgroup of the group of upper triangular matrices. Since $g\tilde{g}^{-1}e = e$, where $e = (1, 0)^T$, the action of G_{ν_n} on the unit sphere \mathcal{S} preserves any probability measure supported by the set $\{e, -e\} \subset \mathcal{S}$. Hence the exponential growth of S_n doesn't follow from Theorem 2.1 since assumption II of this theorem is not satisfied.

Remark 3.1. No other measure on \mathcal{S} is preserved by G_{ν_n} because if v is any non zero vector from \mathbb{R}^2 then $\lim_{j \rightarrow \infty} (g\tilde{g}^{-1})^j.v = \tilde{e}$, where \tilde{e} is either e or $-e$.

We shall now show that, in contrast, Corollary 2.3 with $k = 2$ implies the exponential growth of S_n under a very mild additional condition on the joint distribution of (q_{n-1}, q_n) . Namely, we suppose that

(iii) for all $n \geq 2$, the support of the joint distribution of (q_{n-1}, q_n) contains two points with the same first coordinate, say (y, z) and (y, \tilde{z}) (which may depend on n).

Proposition 3.2. *Suppose that conditions (i), (ii), and (iii) are satisfied. Then the product S_n grows exponentially.*

Proof. To be able to use Corollary 2.3, we first define the set M . Let \mathcal{Z} be the set of all two-dimensional distributions of pairs of random variables (z_1, z_2) such that $|z_i| \leq C$ and $\text{Var}(z_i) \geq \delta$, $i = 1, 2$. For each such pair (z_1, z_2) denote by $\nu_{(z_1, z_2)}$ the distribution on $\text{SL}(2, \mathbb{R})$ of the product $g_{z_2} g_{z_1}$, where g_{z_i} are matrices of the form (2) with $a_n q_n$ replaced by z_i , $i = 1, 2$. Finally,

$$M = \{\nu_{(z_1, z_2)} : \text{the distribution of } (z_1, z_2) \text{ belongs to } \mathcal{Z}\}.$$

In other words, M is the set all distributions $\nu_{(z_1, z_2)}$ described above. Conditions (ii) implies that M is a weakly compact set and that the distributions of products $g_n g_{n-1}$ belong to M for all $n \geq 2$. The first assumption of Corollary 2.3 is thus satisfied.

Our next Lemma 3.3 proves that also assumption II(b) is satisfied and hence the product S_n grows exponentially. \square

Lemma 3.3. *If conditions (ii) and (iii) are satisfied then no probability measure on \mathcal{S} is preserved by the group $G_{\nu_{n-1,2}}$.*

Proof. Let (g_y, g_z) , $(g_y, g_{\tilde{z}})$, and $(g_{\bar{y}}, g_{\bar{z}})$ be three pairs of matrices corresponding to the points (y, z) , (y, \tilde{z}) , and (\bar{y}, \bar{z}) respectively. Here the first two points are chosen from the support of the distribution of (q_{n-1}, q_n) as allowed by (iii) and (\bar{y}, \bar{z}) is one more point from the same support such that $\bar{y} \neq y$ (it exists due to condition (ii)).

The product $g_z g_y (g_{\tilde{z}} g_y)^{-1} = g_z g_{\tilde{z}}^{-1} \in G_{\nu_{n-1,2}}$ by the definition of $G_{\nu_{n-1,2}}$ (see (11)). Hence the group $G_{\nu_{n-1,2}}$ contains a non-trivial upper triangular subgroup. This, by Remark 3.1, implies that if there is a measure on \mathcal{S} preserved by $G_{\nu_{n-1,2}}$ then it should be supported by the set $\{e, -e\} \subset \mathcal{S}$.

But we also have that $g_z g_y (g_{\tilde{z}} g_{\bar{y}})^{-1} \in G_{\nu_{n-1,2}}$ and a straightforward calculation shows that

$$g_z g_y (g_{\tilde{z}} g_{\bar{y}})^{-1} e = \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & y - \bar{y} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & \bar{z} \end{pmatrix} e = \begin{pmatrix} z(\bar{y} - y) + 1 \\ \bar{y} - y \end{pmatrix}.$$

We thus see that the action of $G_{\nu_{n-1,2}}$ on \mathcal{S} does not map $\{e, -e\}$ into itself and therefore no measure on \mathcal{S} is preserved by $G_{\nu_{n-1,2}}$. Lemma is proved. \square

3.2. Perturbations of random products. Suppose that conditions of Theorem 2.1 are satisfied and consider a ‘‘distortion’’ of the product S_n of the form

$$\tilde{S}_n = a_n g_n a_{n-1} g_{n-1} \dots a_1 g_1,$$

where $a_j \in \text{SL}(m, \mathbb{R})$, $j \geq 1$, is a non-random sequence of bounded matrices, $\|a_n\| \leq C$ for some C and all $n \geq 1$, and arbitrary otherwise. We claim that then there is $\tilde{\lambda} > 0$ such that with probability 1

$$(15) \quad \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \|a_n g_n \dots a_1 g_1\| \geq \tilde{\lambda}.$$

Proof of (15). Set $\tilde{\mathbf{g}}_n = a_n \mathbf{g}_n$, where \mathbf{g}_n is the function defined in section 2.1.2. Denote $\tilde{g}_n = \tilde{\mathbf{g}}_n(\xi_n) = a_n g_n$; our product now is $\tilde{S}_n = \tilde{g}_n \tilde{g}_{n-1} \dots \tilde{g}_1$.

To be able to use Theorem 2.1 with functions \mathbf{g}_n replaced by $\tilde{\mathbf{g}}_n$, we shall replace M by \tilde{M} chosen so that to make assumption II to be satisfied for matrices \tilde{g}_n . (Note that assumption I is satisfied because the underlying Markov chain is the same.)

Namely, define

$$\tilde{M} = \bigcup_{b: b \in \text{SL}(m, \mathbb{R}), \|b\| \leq C} (bM),$$

where $b \in \text{SL}(m, \mathbb{R})$ and $bM = \{b\nu : \nu \in M\}$. Here $b\nu$ is the distribution on $\text{SL}(m, \mathbb{R})$ defined for a Borel set $\Gamma \subset \text{SL}(m, \mathbb{R})$ by $(b\nu)(\Gamma) = \nu(b^{-1}(\Gamma))$.

Denote by $\tilde{\nu}_n$ the distribution of \tilde{g}_n . Since $\text{supp}(\tilde{\nu}_n) = a_n \text{supp}(\nu_n) \in \tilde{M}$, assumption II is satisfied because of the following lemma.

Lemma 3.4. (i) \tilde{M} is a compact set.

(ii) For any $\tilde{\nu} \in \tilde{M}$ one has: no measure on \mathcal{S} is preserved by $G_{\tilde{\nu}}$.

Proof. Let $\bar{\nu}_n \in \tilde{M}$, $n \geq 1$, be a sequence of distributions such that $\lim_{n \rightarrow \infty} \bar{\nu}_n = \bar{\nu}$. Then there are $b_n \in \text{SL}(m, \mathbb{R})$ and $\nu'_n \in M$ such that $\bar{\nu}_n = b_n \nu'_n$. By passing, if necessary, to a subsequence, we can assume that $\lim_{n \rightarrow \infty} b_n = a$, where the convergence is in norm, and $\lim_{n \rightarrow \infty} \nu'_n = \nu' \in M$, where the convergence is understood in the weak sense. Hence $\bar{\nu} = a\nu' \in \tilde{M}$ and (i) is proved.

Since $G_{\bar{\nu}} = aG_{\nu'}a^{-1}$, no measure on \mathcal{S} is preserved by $G_{\bar{\nu}}$. This proves (ii). \square

And thus (15) now follows from Theorem 2.1. \square

3.3. Products of stationary Markov-dependent matrices. The goal of this section is to explain how to derive Virtser's (and thus also Furstenberg's) theorem from Theorem 2.1.

To state Virtser's theorem we start with a setup which is a simplified version of the one we saw in sections 2.1.1 and 2.1.2.

Let (X, \mathcal{B}, μ) be a probability space and let $\xi = (\xi_n)_{n \geq 1}$ be a stationary Markov chain with the phase space X , transition kernel $k(x, dy)$, and invariant measure μ .

Denote by H the Hilbert space of real valued functions on X which are square integrable with respect to μ and let $H^0 \subset H$ be the space of functions with zero mean (as in (5)). The related transition operator $K : H \mapsto H$ acts on $f \in H$ as follows: $(Kf)(x) = \int_X k(x, dy)f(y)$. The operator $K^0 : H^0 \mapsto H^0$ is the restriction of K to H^0 .

We recall that we are interested in the study of the growth of the product

$$S_n = \mathbf{g}(\xi_n) \dots \mathbf{g}(\xi_1), \text{ where } \mathbf{g} : X \mapsto \text{SL}(m, \mathbb{R}).$$

The distribution of $\mathbf{g}(\xi_j)$ on $\text{SL}(m, \mathbb{R})$ is denoted ν . The group \mathbb{G}_ν is as in Theorem 1.1 and G_ν is as in (7) (but now there is no dependence on j).

Theorem 3.5 (Virtser, [15]). *Suppose that:*

- (a) ξ is a stationary ergodic Markov chain .
- (b) $\|K^0\| = c$, where $c < 1$.
- (c) $\int_X \ln \|\mathbf{g}(x)\| \mu(dx) < \infty$.
- (d) No probability measure on \mathcal{S} is preserved by \mathbb{G}_ν .

Then there is $\lambda > 0$ such that with probability 1 $\lim_{n \rightarrow \infty} n^{-1} \ln \|S_n\| = \lambda$.

Proof. It follows from the definitions of \mathbb{G}_ν and G_ν that $G_\nu \subset \mathbb{G}_\nu$. If $G_\nu = \mathbb{G}_\nu$ then Theorem 3.5 is an immediate corollary of Theorem 2.1 with M in assumption II consisting of one point, $M = \{\nu\}$.

So, throughout this proof, we suppose that G_ν is a proper subgroup of \mathbb{G}_ν .

Let $I_m \in \text{SL}(m, \mathbb{R})$ be the $m \times m$ identity matrix. Note that if $I_m \in \text{supp}(\nu)$ then $G_\nu = \mathbb{G}_\nu$. We shall show that if the Markov chain ξ is stationary as in Theorem 3.5 then a stronger version of Theorem 2.1 holds for this chain. Namely, the group G_ν can be replaced by \mathbb{G}_ν . Virtser's theorem is again a corollary - but of this stronger version. It should be emphasized that the stationarity of ξ is crucial for the construction presented below.

3.3.1. *Definition of the Markov chain ζ and the function $\tilde{\mathbf{g}}$.* Given the chain ξ , we now define a new Markov chain ζ .

The phase space of ζ is $\tilde{X} = X \cup \bar{X}$ where $\bar{X} = X \times \{1\} = \{(x, 1) : x \in X\}$.

To define the corresponding sigma algebra $\tilde{\mathcal{B}}$ of subsets of \tilde{X} , we first define the sigma algebra $\bar{\mathcal{B}}$ of subsets of \bar{X} as the image of \mathcal{B} under the natural one to one correspondence $x \leftrightarrow (x, 1)$ between X and \bar{X} . We set $\tilde{\mathcal{B}} = \{B \cup \bar{B} : B \in \mathcal{B}, \bar{B} \in \bar{\mathcal{B}}\}$.

The transition probabilities of the chain ζ are defined as follows. Choose a p , $0 < p < 1$, and let $q = 1 - p$. Next, define

$$(16) \quad \mathbb{P}(\zeta_{n+1} = (x, 1) \mid \zeta_n = x) = \mathbb{P}(\zeta_{n+1} = (x, 1) \mid \zeta_n = (x, 1)) = p,$$

and, for $A \in \mathcal{B}$, $A \subset X \subset \tilde{X}$ define

$$(17) \quad \mathbb{P}(\zeta_{n+1} \in A \mid \zeta_n = x) = \mathbb{P}(\zeta_{n+1} \in A \mid \zeta_n = (x, 1)) = q\mathbb{P}(\xi_{n+1} \in A \mid \xi_n = x).$$

Define $\tilde{\mu} = (q\mu, p\mu)$ to be the initial distribution of ζ : if $B \cup \bar{B} \in \tilde{\mathcal{B}}$ then

$$\mathbb{P}(\zeta_1 \in B \cup \bar{B}) = q\mu(B) + p\mu(\bar{B}).$$

We shall see below (see Lemma 3.7) that $\tilde{\mu}$ is also the invariant measure of ζ .

Next, define $\tilde{\mathbf{g}} : \tilde{X} \mapsto \text{SL}(m, \mathbb{R})$ as follows: $\tilde{\mathbf{g}}(\tilde{x}) = \begin{cases} \mathbf{g}(x) & \text{if } \tilde{x} = x \in X, \\ I_m & \text{if } \tilde{x} \in \bar{X} \end{cases}$

3.3.2. *Derivation of Theorem 3.5.* Let $\tau_1 < \tau_2 < \dots < \tau_n < \dots$ be the sequence of all random consecutive time moments at which the chain ζ visits X . Set $\bar{\xi}_n = \zeta_{\tau_n}$. It is obvious from the definitions (16), (17) that the sequens $\bar{\xi} = (\bar{\xi}_n)_{n \geq 1}$ is a Markov chain which has the same transition probabilities and the same initial distribution as the chain ξ :

$$(18) \quad \xi \stackrel{d}{=} \bar{\xi}.$$

Here and below $\stackrel{d}{=}$ means the equality of distributions.

Set

$$\bar{S}_n = \bar{\mathbf{g}}(\bar{\xi}_n)\bar{\mathbf{g}}(\bar{\xi}_{n-1})\dots\bar{\mathbf{g}}(\bar{\xi}_1), \quad \tilde{S}_n = \tilde{\mathbf{g}}(\zeta_n)\dots\tilde{\mathbf{g}}(\zeta_1)$$

Equality (18) implies that $S_n \stackrel{d}{=} \bar{S}_n$. In turn $\bar{S}_n = \tilde{S}_{\tau_n}$ because the factors forming \tilde{S}_{τ_n} are either equal to I_m or coincide with one of the factors forming \bar{S}_n . Hence

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \|S_n\| = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|\bar{S}_n\| = \lim_{n \rightarrow \infty} \frac{\tau_n}{n} \frac{1}{\tau_n} \ln \|\tilde{S}_{\tau_n}\|,$$

where the existence of all limits follows from Kingman's sub-additive ergodic theorem. Set $\tilde{\lambda} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|\tilde{S}_n\|$. Taking into account that $\lim_{n \rightarrow \infty} \frac{\tau_n}{n} = q^{-1}$, we see that $\lambda = q^{-1}\tilde{\lambda}$. It remains to check that $\tilde{\lambda} > 0$.

3.3.3. *Theorem 2.1 in the setting of Theorem 3.5.* The definitions of $\bar{\mathfrak{g}}$ and the formula for the invariant measure $\tilde{\mu}$ imply that the distribution $\tilde{\nu}$ of $\bar{\mathfrak{g}}(\zeta_j)$ has the property $\text{supp}(\tilde{\nu}) = \text{supp}(\nu) \cup \{I_m\}$. So assumption II of Theorem 2.1 is satisfied with $M = \{\tilde{\nu}\}$ and because $G_{\tilde{\nu}} = \mathbb{G}_{\tilde{\nu}}$.

To see that assumption I is satisfied, consider the Hilbert space \tilde{H} of real-valued functions

$$\tilde{H} = \{f = (\varphi, \psi) : \|f\|^2 = q \int_X \varphi(x)^2 d\mu(x) + p \int_X \psi((x, 1))^2 d\mu(x) < \infty\},$$

where $\varphi : X \mapsto \mathbb{R}$ and $\psi : \bar{X} \mapsto \mathbb{R}$ are the restrictions of f to X and \bar{X} respectively.

Definitions (16) and (17) imply that the action of the transition operator \tilde{K} of the chain ζ on \tilde{H} is given by

$$(19) \quad \begin{aligned} (\tilde{K}f)(x) &= q \int_X k(x, dy) \varphi(y) + p\psi((x, 1)), \text{ where } x \in X \\ (\tilde{K}f)((x, 1)) &= q \int_X k(x, dy) \varphi(y) + p\psi((x, 1)), \text{ where } (x, 1) \in \bar{X}. \end{aligned}$$

Formulae (19) show that \tilde{K} maps \tilde{H} into its subspace which consists of functions $f = (\varphi, \psi)$ such that $\varphi(x) = \psi((x, 1))$ for all $x \in X$. Obviously this subspace is an invariant subspace of \tilde{K} . Denote this subspace \bar{H} and let $\bar{H}^0 \subset \bar{H}$ be the subspace of functions with zero mean. Finally let \tilde{K}^0 be the restriction of \tilde{K} to \bar{H}^0 .

Lemma 3.6. $\|\tilde{K}^0\|_{\bar{H}} \leq (qc + p) < 1$.

Proof. If $f = (\varphi, \varphi)$ then $\|f\|_{\bar{H}} = \|\varphi\|_H$, where the notations for the norms emphasize that $f \in \tilde{H}$ and $\varphi \in H$. If $f \in \bar{H}^0$ then $\mathbb{E}(f(\zeta)) = \int_X \varphi(x) d\mu(x) = 0$ and hence $\varphi \in H^0$. These two facts imply that for $f \in \bar{H}^0$

$$\|\tilde{K}^0 f\|_{\bar{H}} = \|qK^0 \varphi + p\varphi\|_H \leq q\|K^0\| \|\varphi\|_H + p\|\varphi\|_H = (qc + p)\|f\|_{\bar{H}}.$$

□

So, the chain ζ satisfies also assumption I of Theorem 2.1 with all H_n^0 replaced by \bar{H}^0 and hence $\tilde{\lambda} > 0$. □

We finish this section with a proof of the fact which we have already stated and used above.

Lemma 3.7. $\tilde{\mu} = (q\mu, p\mu)$ is an invariant measure of the chain ζ .

Proof. Formulae (19) show that $(\tilde{K}f)(x) = (\tilde{K}f)((x, 1))$. We also have that, by the definition of the invariant measure for ξ , $\int_X (K\varphi)(x) d\mu(x) = \int_X \varphi(x) d\mu(x)$. Now, the following calculation shows that

$$\int_{\bar{X}} (\tilde{K}f)(\tilde{x}) d\tilde{\mu}(\tilde{x}) = \int_X (\tilde{K}f)(x) d\mu(x) = q \int_X \varphi(x) d\mu(x) + p \int_X \psi((x, 1)) d\mu(x)$$

and hence $\tilde{\mu}$ is the invariant measure of the chain ζ . □

4. PROOF OF THEOREM 2.1 FOR PRODUCTS OF INDEPENDENT MATRICES

Suppose that matrices g_n , $n \geq 1$ are independent. In this setting, it is natural to assume that $X = \text{SL}(m, \mathbb{R})$. The kernels $k_n(x, dy)$ do not depend on x and $\mu_n(dy)$ is the distribution of g_n . Obviously, $\mu_n(dy) = k_n(dy) = \nu_n(dy)$ and $G_\nu = G_\mu$.

Theorem 4.1. *Suppose that M is a compact set of probability measures on $\text{SL}(m, \mathbb{R})$ and that for any $\nu \in M$ the group G_ν does not preserve any probability measure on the unit sphere \mathcal{S} . Then (10) holds with probability 1 for some non-random $\lambda > 0$.*

Proof. The proof of Theorem 4.1 will be carried out in three steps.

Step 1. Note that in order to prove (10) it suffices to show that there are constants $A > 0$ and $\mathbf{a} > 0$ such that

$$(20) \quad \mathbb{E}(\|S_n\|^{-\frac{m}{2}}) \leq Ae^{-\mathbf{a}n}.$$

Indeed, by the Markov inequality for any $\varepsilon > 0$

$$\mathbb{P}(\|S_n\| \leq e^{\varepsilon n}) = \mathbb{P}(\|S_n\|^{-\frac{m}{2}} \geq e^{-\frac{m}{2}\varepsilon n}) \leq e^{\varepsilon \frac{m}{2}n} \mathbb{E}(\|S_n\|^{-\frac{m}{2}}) \leq Ae^{(\varepsilon \frac{m}{2} - \mathbf{a})n}.$$

If $\varepsilon < 2\mathbf{a}/m$ then the Borel-Cantelli lemma implies that the set $\{n : \|S_n\| \leq e^{\varepsilon n}\}$ is a.s. finite. This means that for any $\lambda < 2\mathbf{a}/m$ (10) holds with probability 1.

Step 2. Let $L_2(\mathcal{S})$ be the Hilbert space of real valued functions on \mathcal{S} equipped with the Lebesgue measure du which is normalized to 1. The inner product of $f, h \in L_2(\mathcal{S})$ is given by

$$\langle f, h \rangle_{L_2} = \int_{\mathcal{S}} f(u)h(u)du.$$

Let \mathbb{V} be the set of unitary operators in $L_2(\mathcal{S})$. Consider a mapping $V : g \mapsto V_g$, where $g \in \text{SL}(m, \mathbb{R})$, $V_g \in \mathbb{V}$ and V_g is defined for $f \in L_2(\mathcal{S})$ as follows:

$$(21) \quad (V_g f)(u) = f(g.u) \|gu\|^{-\frac{m}{2}}.$$

The mapping V has the following properties:

$$(22) \quad \|f\|_{L_2} = \|V_g f\|_{L_2} \quad \text{and} \quad V_{g_1 g_2} = V_{g_2} V_{g_1}.$$

The first relation in (22) follows from the fact that $\|gu\|^{-m}$ is the Jacobian of the transformation $u \mapsto g.u$ (see [5, page 425, Lemma 8.8]). Therefore $\int_{\mathcal{S}} f(g.u)^2 \|gu\|^{-m} du = \int_{\mathcal{S}} f(u)^2 du$.

The second relation in (22) is verified by a straightforward calculation.

Remark 4.2. The mapping $g \mapsto V_{g^{-1}}$ is a representation of $\text{SL}(m, \mathbb{R})$ which was used in [15]. We prefer to work with V because it simplifies some formulae.

For a probability measure ν , put

$$(23) \quad W_\nu = \int_{\text{SL}(m, \mathbb{R})} V_g \nu(dg), \quad \text{that is} \quad (W_\nu f)(u) = \int_{\text{SL}(m, \mathbb{R})} f(g.u) \|gu\|^{-\frac{m}{2}} \nu(dg).$$

Lemma 4.3. *Let M be a weakly compact set of probability measures on $\text{SL}(m, \mathbb{R})$ such that each $\nu \in M$ has the property that no probability measure on \mathcal{S} is preserved by G_ν . Then there is a constant β , $0 \leq \beta < 1$, such that $\|W_\nu\| \leq \beta$ for all $\nu \in M$.*

We shall prove Lemma 4.3 after we finish the proof of our Theorem.

Step 3. Since $\|S_n\| \geq \|S_n u\|$, $u \in \mathcal{S}$, (20) would follow from

$$(24) \quad \int_{\mathcal{S}} \mathbb{E}(\|S_n u\|^{-\frac{m}{2}}) du \leq e^{-cn}.$$

Note next that

$$(25) \quad \|g_n \dots g_1 u\|^{-\frac{m}{2}} = (V_{g_n \dots g_1} \mathbf{1})(u) = (V_{g_1} \dots V_{g_n} \mathbf{1})(u),$$

where $\mathbf{1}$ is the function on \mathcal{S} which takes value 1 at every $u \in \mathcal{S}$. Therefore

$$(26) \quad \begin{aligned} \int_{\mathcal{S}} \mathbb{E}(\|S_n u\|^{-\frac{m}{2}}) du &= \mathbb{E} \left(\int_{\mathcal{S}} \|g_n \dots g_1 u\|^{-\frac{m}{2}} du \right) \\ &= \mathbb{E} \left(\int_{\mathcal{S}} (V_{g_1} \dots V_{g_n} \mathbf{1})(u) du \right) = \mathbb{E}(\langle V_{g_1} \dots V_{g_n} \mathbf{1}, \mathbf{1} \rangle) = \langle \mathbb{E}(V_{g_1} \dots V_{g_n}) \mathbf{1}, \mathbf{1} \rangle. \end{aligned}$$

Since the operators V_{g_1}, \dots, V_{g_n} are independent we obtain

$$\mathbb{E}(V_{g_1} \dots V_{g_n}) = \mathbb{E}(V_{g_1}) \dots \mathbb{E}(V_{g_n}) = W_{\nu_1} \dots W_{\nu_n}$$

Finally,

$$(27) \quad \int_{\mathcal{S}} \mathbb{E}(\|S_n u\|^{-\frac{m}{2}}) du = \langle W_{\nu_1} \dots W_{\nu_n} \mathbf{1}, \mathbf{1} \rangle \leq \|W_{\nu_1}\| \dots \|W_{\nu_n}\| \leq e^{-cn},$$

where $c = \inf_{\nu \in M} (-\ln \|W_{\nu}\|) > 0$ by Lemma 4.3.

This completes the proof of Theorem 4.1. \square

4.1. Proof of Lemma 4.3.

Proof. We shall prove the following statement which is equivalent to Lemma 4.3.

$$(28) \quad \begin{aligned} &\text{If } \sup_{\nu \in M} \|W_{\nu}\| = 1 \text{ then there is a } \nu \in M \text{ and a probability measure } \kappa \\ &\text{on } \mathcal{S} \text{ which is preserved by } G_{\nu}. \end{aligned}$$

From $\sup_{\nu \in M} \|W_{\nu}\| = 1$ it follows that there is a sequence of measures $\nu_n \in M$ and a sequence of functions $f_n \in L_2(\mathcal{S})$ with $\|f_n\| = 1$ and such that $\lim_{n \rightarrow \infty} \|W_{\nu_n} f_n\| = 1$. Define a sequence of probability measures κ_n on \mathcal{S} by setting $\kappa_n(du) = f_n(u)^2 du$.

We shall assume that both sequences of measures, ν_n and κ_n , have weak limits:

$$(29) \quad \lim_{n \rightarrow \infty} \nu_n = \nu, \text{ where } \nu \in M, \text{ and } \lim_{n \rightarrow \infty} \kappa_n = \kappa.$$

As usual, if (29) is not satisfied then the sequence (ν_n, κ_n) can be replaced by its subsequence which has these properties. For the sequence ν_n , this is possible because of the condition that M is a compact set. For the sequence κ_n , the existence of a converging subsequence follows from the fact that these measures are supported by the unit sphere \mathcal{S} which is a compact metric space.

We shall prove that κ and ν are such that

$$(30) \quad g_1^{-1} \kappa = g_2^{-1} \kappa \text{ for any } g_1, g_2 \in \text{supp}(\nu).$$

Thus $g_1 g_2^{-1} \kappa = \kappa$ and hence G_{ν} preserves κ .

Let us rewrite (30) in terms of test functions: we have to prove that

$$(31) \quad \int_{\mathcal{S}} \psi(u) (g_1^{-1} \kappa)(du) = \int_{\mathcal{S}} \psi(u) (g_2^{-1} \kappa)(du)$$

for any $g_1, g_2 \in \text{supp}(\nu)$ and any continuous function $\psi : \mathcal{S} \mapsto \mathbb{R}$. The last equality can be rewritten in the following equivalent form (see Appendix, section 7.2.):

$$(32) \quad \int_{\mathcal{S}} \psi(g_1^{-1}.u) \kappa(du) = \int_{\mathcal{S}} \psi(g_2^{-1}.u) \kappa(du).$$

Let $B(g_i, \delta) = \{g \in \text{SL}(m, \mathbb{R}) : \|g_i - g\| < \delta\}$ be open balls and let $S(g_i, \delta) = \{g \in \text{SL}(m, \mathbb{R}) : \|g_i - g\| = \delta\}$ be the spheres of radius δ centered at g_i , $i = 1, 2$.

It follows from the continuity of ψ and the compactness of \mathcal{S} that for a given $\varepsilon > 0$ there is a $\delta > 0$, $\delta = \delta(\varepsilon, g_1, g_2, \psi)$, such that for all $g'_1 \in B(g_1, \delta)$ and for all $g'_2 \in B(g_2, \delta)$

$$(33) \quad \sup_{u \in \mathcal{S}} |\psi(g_1^{-1}.u) - \psi(g_1'^{-1}.u)| \leq \varepsilon \quad \text{and} \quad \sup_{u \in \mathcal{S}} |\psi(g_2^{-1}.u) - \psi(g_2'^{-1}.u)| \leq \varepsilon.$$

In addition, we shall suppose that δ is such that

$$(34) \quad \nu(S(g_i, \delta)) = 0.$$

Note that $\nu(B(g_1, \delta)) > 0$ and $\nu(B(g_2, \delta)) > 0$ because $g_1, g_2 \in \text{supp}(\nu)$. The weak convergence of ν_n to ν together with (34) imply that

$$(35) \quad \lim_{n \rightarrow \infty} \nu_n(B(g_i, \delta)) = \nu(B(g_i, \delta)) \text{ for } i = 1, 2.$$

Next, we shall show that for a given $\epsilon > 0$ there is $N(\epsilon)$ such that for all $n \geq N(\epsilon)$ there are $\tilde{g}_1 \in B(g_1, \delta)$ and $\tilde{g}_2 \in B(g_2, \delta)$ such that the following inequalities hold:

$$(36) \quad I_i = \left| \int_{\mathcal{S}} \psi(g_i^{-1}.u) \kappa(du) - \int_{\mathcal{S}} \psi(\tilde{g}_i^{-1}.u) \kappa_n(du) \right| < \epsilon, \quad i = 1, 2,$$

$$(37) \quad I_3 = \left| \int_{\mathcal{S}} \psi(\tilde{g}_1^{-1}.u) \kappa_n(du) - \int_{\mathcal{S}} \psi(\tilde{g}_2^{-1}.u) \kappa_n(du) \right| < \epsilon.$$

It follows from (36) and (37) that

$$\left| \int_{\mathcal{S}} \psi(g_1^{-1}.u) \kappa(du) - \int_{\mathcal{S}} \psi(g_2^{-1}.u) \kappa(du) \right| \leq I_1 + I_3 + I_2 < 3\epsilon$$

and, since ϵ can be arbitrarily small, the last inequality proves (32).

It thus remains to prove (36) and (37). To prove (36) we write

$$(38) \quad \begin{aligned} & \left| \int_{\mathcal{S}} \psi(g_i^{-1}.u) \kappa(du) - \int_{\mathcal{S}} \psi(\tilde{g}_i^{-1}.u) \kappa_n(du) \right| \leq \\ & \left| \int_{\mathcal{S}} \psi(g_i^{-1}.u) \kappa(du) - \int_{\mathcal{S}} \psi(g_i^{-1}.u) \kappa_n(du) \right| + \left| \int_{\mathcal{S}} (\psi(g_i^{-1}.u) - \psi(\tilde{g}_i^{-1}.u)) \kappa_n(du) \right| \leq \\ & \left| \int_{\mathcal{S}} \psi(g_i^{-1}.u) \kappa(du) - \int_{\mathcal{S}} \psi(g_i^{-1}.u) \kappa_n(du) \right| + \varepsilon, \end{aligned}$$

where the last inequality is due to (33). This, together with the weak convergence of κ_n to κ , implies (36).

Note that (36) holds for all \tilde{g}_1 and \tilde{g}_2 from $B(g_1, \delta)$ and $B(g_2, \delta)$ respectively.

We now turn to (37). Note that for any $g \in \text{SL}(m, \mathbb{R})$

$$(39) \quad \int_{\mathcal{S}} \psi(g^{-1}.u) \kappa_n(du) = \int_{\mathcal{S}} \psi(g^{-1}.u) f_n(u)^2 du = \int_{\mathcal{S}} \psi(u) f_n(g.u)^2 \|gu\|^{-m} du.$$

The last equality in (39) follows from the change of variables $u \mapsto g.u$ since the corresponding Jacobian is $\|gu\|^{-m}$. (It can also be viewed as one more version of the definition of $g\kappa_n$ in the case when $|f_n|^2$ is the density function for κ_n .)

Using (39), we present the left hand side of (37) as

$$(40) \quad I_3 = \left| \int_{\mathcal{S}} \psi(u) (f_n(\tilde{g}_1.u)^2 \|\tilde{g}_1 u\|^{-m} - f_n(\tilde{g}_2.u)^2 \|\tilde{g}_2 u\|^{-m}) du \right|.$$

Denote by $f_n(g) = V_g f_n$ and define

$$\varphi_n = W_{\nu_n} f_n = \int_{\text{SL}(m, \mathbb{R})} f_n(g) \nu_n(dg).$$

Since $\|f_n(g)\|_{L_2} = 1$ and $\|\varphi_n\|_{L_2} \rightarrow 1$, Lemma 7.2 in the Appendix implies that for sufficiently large n

$$\nu_n \{g : \|\varphi_n - f_n(g)\|_{L_2} > \epsilon\} \leq \frac{2(1 - \|\varphi_n\|_{L_2})}{\epsilon^2 - 3(1 - \|\varphi_n\|_{L_2})}.$$

Therefore

$$(41) \quad \lim_{n \rightarrow \infty} \nu_n \{g : \|\varphi_n - f_n(g)\|_{L_2} > \epsilon\} = 0.$$

We thus can choose $N_1 = N_1(\epsilon, \delta, g_1, g_2)$ such that for $i = 1, 2$ and all $n \geq N_1$

$$\nu_n(\{g : \|\varphi_n - f_n(g)\|_{L_2} \leq \epsilon\} \cap B(g_i, \delta)) > 0.5\nu(B(g_i, \delta)).$$

Hence, for every $n \geq N_1$ there are $\tilde{g}_1 \in B(g, \delta)$ and $\tilde{g}_2 \in B(g_2, \delta)$ such that $\|f_n(\tilde{g}_1) - f_n(\tilde{g}_2)\|_{L_2} \leq \epsilon$. But then, for these \tilde{g}_1, \tilde{g}_2 (which may depend on n), we have

$$(42) \quad \begin{aligned} & \left| \int_{\mathcal{S}} \psi(u) (f_n(\tilde{g}_1.u)^2 \|\tilde{g}_1 u\|^{-m} - f_n(\tilde{g}_2.u)^2 \|\tilde{g}_2 u\|^{-m}) du \right| \\ & \leq \sup_{u \in \mathcal{S}} |\psi(u)| \int_{\mathcal{S}} |f_n(\tilde{g}_1.u)^2 \|\tilde{g}_1 u\|^{-m} - f_n(\tilde{g}_2.u)^2 \|\tilde{g}_2 u\|^{-m}| du \\ & = \sup_{u \in \mathcal{S}} |\psi(u)| \int_{\mathcal{S}} |f_n(\tilde{g}_1)^2 - f_n(\tilde{g}_2)^2| du \\ & = \sup_{u \in \mathcal{S}} |\psi(u)| \langle |f_n(\tilde{g}_1) - f_n(\tilde{g}_2)|, |f_n(\tilde{g}_1) + f_n(\tilde{g}_2)| \rangle_{L_2} \end{aligned}$$

Since $\| |f_n(\tilde{g}_1) + f_n(\tilde{g}_2)| \|_{L_2} \leq 2$ we obtain

$$\langle |f_n(\tilde{g}_1) - f_n(\tilde{g}_2)|, |f_n(\tilde{g}_1) + f_n(\tilde{g}_2)| \rangle_{L_2} \leq 2 \|f_n(\tilde{g}_1) - f_n(\tilde{g}_2)\|_{L_2} \leq 2\epsilon.$$

This proves (37) and completes the proof of the Lemma. \square

4.2. **Comments.** 1. In the context of products of matrices, operators W_ν were first explicitly defined in [15] where it was proved that the spectral radius of W_ν is less than 1. In the case of identically distributed independent g_n this fact implies Theorem 4.1. In fact, [15] starts with a more complicated version of this operator which allows one to control products of stationary Markov-dependent matrices and, once again, the positivity of the Lyapunov exponent follows from the fact that the corresponding spectral radius is less than 1.

2. We are now in a position to state in a more precise way the result from [12] mentioned in the Introduction. Namely, for the special case of independent matrices g_j given by (2), it was proved there that $\|W_\nu W_{\tilde{\nu}}\| \leq a < 1$, where a is explicitly expressed in terms of the variances of q_n 's.

Corollary 2.3 guarantees exponential growth for a more general class of potentials because the sequence q_n is only required to be Markov-dependent. On the other hand, obtaining a constructive estimate for $\|W_\nu W_{\tilde{\nu}}\|$ similar to the one in [12] requires additional work.

3. Furstenberg's theorem for the i.i.d. case can be derived directly from Theorem 4.3. This derivation is much more straightforward than the one for the Markov-dependent matrices discussed in section 3.3. Namely, we have again to consider two cases. If the identity matrix $I_m \in \text{supp}(\nu)$ then $G_\nu = \mathbb{G}_\nu$ and the positivity claimed by Theorem 1.1 follows from Theorem 4.3. If $I_m \notin \text{supp}(\nu)$, then we can apply our theorem to the measure $\tilde{\nu} = \frac{1}{2}\nu + \frac{1}{2}\delta_I$. Obviously, $I_m \in \text{supp}(\tilde{\nu})$ and by Theorem 4.3 the corresponding $\tilde{\lambda} > 0$. Since $\lambda = 2\tilde{\lambda}$ (as in section 3.3) the result follows. This completes the proof of the main claim of Furstenberg's theorem.

5. PROOF OF THEOREM 2.1

The plan of the proof is as follows.

As in the case of Theorem 4.1, our aim is to show that inequality (24) holds. To this end, we first introduce a sequence of Hilbert spaces \mathbb{H}_n which are extensions of the spaces H_n (defined by (4)) and operators \hat{K}_j and \hat{V}_{g_j} which are the analogues of K_j and V_{g_j} . We then compute $\int_{\mathcal{S}} \mathbb{E} (\|S_n u\|^{-1}) du$ in terms of products of these operators, state our main technical estimate (Theorem 5.5) and use it to prove Theorem 2.1. The proof of Theorem 5.8 is given after that, in section 5.2. As in the case of independent matrices, Lemma 4.3 plays an important role in the proof.

5.1. Auxiliary spaces and operators and proof of Theorem 2.1. Denote by \mathbb{H}_n the Hilbert space of $\mu_n \times du$ -square integrable real valued functions on $X \times \mathcal{S}$: with the standard inner product: if $f, h \in \mathbb{H}_n$ then $\langle f, h \rangle_{\mathbb{H}_n} = \int_{X \times \mathcal{S}} f(x, u)h(x, u)\mu_n(dx)du$.

The spaces H_n and $L_2(\mathcal{S})$ are naturally imbedded into \mathbb{H}_n . The image of the natural imbedding of $L_2(\mathcal{S})$ into \mathbb{H}_n will be denoted by \mathcal{L}_n . Obviously, \mathcal{L}_n consists of functions from \mathbb{H}_n which depend only on $u \in \mathcal{S}$.

5.1.1. *Operators \hat{K}_n and \hat{V}_{g_n} .* We first extend the action of K_n to \mathbb{H}_{n+1} . Namely, denote by $\hat{K}_n : \mathbb{H}_{n+1} \mapsto \mathbb{H}_n$ the operator which, for $f \in \mathbb{H}_{n+1}$, is defined by

$$(43) \quad (\hat{K}_n f)(x, u) = \int_X k_n(x, dy) f(y, u).$$

Remark 5.1. If $f \in \mathcal{L}_{n+1}$ (that is $f(x, u) \equiv f(u)$) then, by the definition of \hat{K}_n , $(\hat{K}_n f)(x, u) = f(u) \in \mathcal{L}_n$.

Next, we define unitary operators $\hat{V}_{\mathfrak{g}_n} : \mathbb{H}_n \mapsto \mathbb{H}_n$. Namely, for $f \in \mathbb{H}_n$ we set

$$(44) \quad (\hat{V}_{\mathfrak{g}_n} f)(x, u) = f(x, \mathfrak{g}_n(x).u) \|\mathfrak{g}_n(x)u\|^{-\frac{m}{2}},$$

where \mathfrak{g}_n are the functions on X introduced at the beginning of subsection 2.1.2.

Remark 5.2. The operators V_g defined by (21) act on \mathbb{H}_n in a natural way. Namely, $(V_g f)(x, u) = f(x, g.u) \|gu\|^{-\frac{m}{2}}$. Clearly, $\hat{V}_{\mathfrak{g}_n} = V_g$ if and only if $\mathfrak{g}_n(x) \equiv g$.

The following lemma and especially its corollary are a variation and an extension of Lemma 3 from [15] to the case of products of non-stationary Markov-dependent matrices.

Lemma 5.3. *Consider the Markov chain $(\xi_j)_{j \geq 1}$ and let $f \in \mathbb{H}_n$, $u \in \mathcal{S}$. Then for $n \geq 2$*

$$(45) \quad \mathbb{E}\{\|\mathfrak{g}_n(\xi_n) \dots \mathfrak{g}_2(\xi_2)u\|^{-\frac{m}{2}} f(\xi_n, \mathfrak{g}_n(\xi_n) \dots \mathfrak{g}_2(\xi_2).u) \mid \xi_1 = x\} = (\hat{K}_1 \hat{V}_{\mathfrak{g}_2} \dots \hat{K}_{n-1} \hat{V}_{\mathfrak{g}_n} f)(x, u).$$

Proof. By the above definitions

$$(\hat{K}_{n-1} \hat{V}_{\mathfrak{g}_n} f)(x, u) = \int_X k_{n-1}(x, dy) f(y, \mathfrak{g}_n(y).u) \|\mathfrak{g}_n(y)u\|^{-\frac{m}{2}}.$$

A straightforward induction argument now implies that

$$(46) \quad (\hat{K}_1 \hat{V}_{\mathfrak{g}_2} \dots \hat{K}_{n-1} \hat{V}_{\mathfrak{g}_n} f)(x, u) = \int_{X^{n-1}} k_1(x, dy_2) \dots k_{n-1}(y_{n-1}, dy_n) f(y_n, \mathfrak{g}_n(y_n) \dots \mathfrak{g}_2(y_2).u) \|\mathfrak{g}_n(y_n) \dots \mathfrak{g}_2(y_2)u\|^{-\frac{m}{2}},$$

where $X^{n-1} = X \times \dots \times X$ is the $(n-1)$ th direct power of X . The right hand side of the last formula coincides with the definition of the expectation in the left hand side of (45). This proves the Lemma. \square

Recall that $S_n = \mathfrak{g}_n(\xi_n) \dots \mathfrak{g}_2(\xi_2) \mathfrak{g}_1(\xi_1)$. Applying $\hat{V}_{\mathfrak{g}_1}$ to both parts of (45) we obtain:

$$(47) \quad \mathbb{E}\{\|S_n u\|^{-\frac{m}{2}} f(\xi_n, S_n.u) \mid \xi_1 = x\} = (\hat{V}_{\mathfrak{g}_1} \hat{K}_1 \hat{V}_{\mathfrak{g}_2} \dots \hat{K}_{n-1} \hat{V}_{\mathfrak{g}_n} f)(x, u).$$

In turn, (47) implies the following analogue of (26):

Corollary 5.4.

$$(48) \quad \int_{\mathcal{S}} \mathbb{E}(\|S_n u\|^{-\frac{m}{2}}) du = \left\langle \hat{V}_{\mathfrak{g}_1} \hat{K}_1 \hat{V}_{\mathfrak{g}_2} \dots \hat{K}_{n-1} \hat{V}_{\mathfrak{g}_n} \mathbf{1}_n, \mathbf{1}_1 \right\rangle_{\mathbb{H}_1},$$

where $\mathbf{1}_n \in \mathbb{H}_n$ and $\mathbf{1}_1 \in \mathbb{H}_1$ are functions taking the value 1 at all points of their respective domains.

Proof. Replace f in (47) by $\mathbf{1}_n$ and integrate both sides of (47) over $\mu_1(dx) \times du$. \square

The following theorem is the main technical result of this paper.

Theorem 5.5. *Suppose that assumptions I and II are satisfied. Then there is a positive constant $\alpha < 1$ such that for all n*

$$(49) \quad \|\hat{K}_{n-1}\hat{V}_{\mathfrak{g}_n}\hat{K}_n\hat{V}_{\mathfrak{g}_{n+1}}\| \leq \alpha.$$

Theorem 5.5 will be proved in the next section. We finish this section with the *Proof of Theorem 2.1*. Relations (48) and (49) imply for odd $n \geq 3$ that

$$(50) \quad \int_{\mathcal{S}} \mathbb{E}(\|S_n u\|^{-\frac{m}{2}}) du \leq \|\hat{V}_{\mathfrak{g}_1}\hat{K}_1\hat{V}_{\mathfrak{g}_2}\dots\hat{K}_{n-1}\hat{V}_{\mathfrak{g}_n}\| \leq \|\hat{K}_1\hat{V}_{\mathfrak{g}_2}\dots\hat{K}_{n-1}\hat{V}_{\mathfrak{g}_n}\| \\ \leq \|\hat{K}_1\hat{V}_{\mathfrak{g}_2}\hat{K}_2\hat{V}_{\mathfrak{g}_3}\dots\|\hat{K}_{n-2}\hat{V}_{\mathfrak{g}_{n-1}}\hat{K}_{n-1}\hat{V}_{\mathfrak{g}_n}\| \leq \alpha^{\frac{n-1}{2}}.$$

If $n \geq 5$ is even then, similarly,

$$\int_{\mathcal{S}} \mathbb{E}(\|S_n u\|^{-\frac{m}{2}}) du \leq \|\hat{K}_3\hat{V}_{\mathfrak{g}_4}\hat{K}_4\hat{V}_{\mathfrak{g}_5}\dots\|\hat{K}_{n-2}\hat{V}_{\mathfrak{g}_{n-1}}\hat{K}_{n-1}\hat{V}_{\mathfrak{g}_n}\| \leq \alpha^{\frac{n-2}{2}}.$$

We thus see that for all n , with the obvious choice of $A > 0$ and $\mathbf{a} > 0$,

$$\mathbb{E}(\|S_n\|^{-\frac{m}{2}}) \leq \int_{\mathcal{S}} \mathbb{E}(\|S_n u\|^{-\frac{m}{2}}) du \leq A e^{-\mathbf{a}n}$$

and hence exactly the same argument as in the Step 1 of the proof of Theorem 4.1 in section 4 finishes the proof. \square

5.2. Proof of Theorem 5.5. Since $\hat{V}_{\mathfrak{g}_{n+1}}$ is a unitary operator, the inequality (49) is equivalent to

$$(51) \quad \|\hat{K}_{n-1}\hat{V}_{\mathfrak{g}_n}\hat{K}_n\| \leq \alpha.$$

We shall prove this inequality in section 5.2.2. But first we prove some preparatory results.

5.2.1. Properties of \hat{K}_n and related operators. Let \mathbb{H}_n^0 be the orthogonal complement of \mathcal{L}_n in \mathbb{H}_n , that is $\mathbb{H}_n^0 = \{f \in \mathbb{H}_n : \langle f, h \rangle_{\mathbb{H}_n} = 0 \text{ for all } h \in \mathcal{L}_n\}$.

Denote by \mathcal{P}_n the orthogonal projector on \mathbb{H}_n^0 and by $\mathcal{Q}_n = \mathcal{I} - \mathcal{P}_n$ the orthogonal projector on the subspace \mathcal{L}_n (here \mathcal{I} is the identity operator in \mathbb{H}_n). We remark that if $f \in \mathbb{H}_n$ then

$$(52) \quad (\mathcal{P}_n f)(x, u) = f(x, u) - \int_X f(y, u) \mu_n(dy).$$

For the proof of (52) see Appendix, section 7.1. Obviously,

$$(53) \quad (\mathcal{Q}_n f)(x, u) = \int_X f(y, u) \mu_n(dy).$$

Set $\hat{K}_n^0 = \hat{K}_n \mathcal{P}_{n+1}$ and $\hat{K}_n^1 = \hat{K}_n \mathcal{Q}_{n+1}$. The following lemma lists several simple but useful properties of \hat{K}_n , \hat{K}_n^0 , \hat{K}_n^1 . We remark that property (iii) is a version of Lemma 2 from [15].

Lemma 5.6. (i) $\hat{K}_n^0(\mathbb{H}_{n+1}) \subset \mathbb{H}_n^0$.
(ii) $\hat{K}_n^1(\mathbb{H}_{n+1}) = \mathcal{L}_n$.
(iii) If $\|\hat{K}_n^0\| \leq c$ then also $\|\hat{K}_n^1\| \leq c$.

(iv) Suppose that $\|K_n^0\| \leq c < 1$ and let $f \in \mathbb{H}_{n+1}$ be such that $\|\hat{K}_n f\|^2 \geq \|f\|^2(1 - \epsilon)$, where $0 \leq \epsilon \leq 1$. Then

$$(54) \quad \|\mathcal{P}_{n+1}f\|^2 \leq \frac{\epsilon}{1-c^2}\|f\|^2 \text{ or, equivalently, } \|\mathcal{Q}_{n+1}f\|^2 \geq \left(1 - \frac{\epsilon}{1-c^2}\right)\|f\|^2.$$

Remark 5.7. The equivalence in (54) is due to $\|\mathcal{P}_{n+1}f\|^2 + \|\mathcal{Q}_{n+1}f\|^2 = \|f\|^2$. We state two inequalities because both of them will be referred to below.

Proof of (i). Consider an $f \in \mathbb{H}_{n+1}$. Set $\phi(y, u) = f(y, u) - \int_X f(z, u)\mu_{n+1}(dz)$. By the definition of \hat{K}_n^0

$$(\hat{K}_n^0 f)(x, u) = (\hat{K}_n \mathcal{P}_{n+1}f)(x, u) = \int_X \phi(y, u)k_n(x, dy).$$

If $h \in \mathcal{L}_n$ then

$$\left\langle \hat{K}_n^0 f, h \right\rangle_{\mathbb{H}_n} = \int_{X \times \mathcal{S}} \left(\int_X \phi(y, u)k_n(x, dy) \right) h(u)\mu_n(dx)du.$$

Changing the order of integration in the last formula, we obtain:

$$\begin{aligned} \left\langle \hat{K}_n^0 f, h \right\rangle_{\mathbb{H}_n} &= \int_{\mathcal{S}} \left(\int_X \left(\int_X \phi(y, u)k_n(x, dy) \right) \mu_n(dx) \right) h(u)du \\ &= \int_{\mathcal{S}} \left(\int_X \phi(y, u)\mu_{n+1}(dy) \right) h(u)du = 0, \end{aligned}$$

where the equality $\int_X \left(\int_X \phi(y, u)k_n(x, dy) \right) \mu_n(dx) = \int_X \phi(y, u)\mu_{n+1}(dy)$ is an equivalent version of (3) (with $n-1$ replaced by n). We also use that $\int_X \phi(y, u)\mu_{n+1}(dy) = 0$. \square

Proof of (ii). $\mathcal{Q}_{n+1}f \in \mathcal{L}_{n+1}$ for any $f \in \mathbb{H}_{n+1}$ by the definition \mathcal{Q}_{n+1} . Remark 5.1 implies (see also (43)) that $\hat{K}_n^1 f = \hat{K}_n \mathcal{Q}_{n+1}f \in \mathcal{L}_n$. \square

Proof of (iii). Since $\mathcal{P}_{n+1}f \in \mathbb{H}_{n+1}^0$ it suffices to prove that $\|\hat{K}_n f\| \leq c\|f\|$ for $f \in \mathbb{H}_{n+1}^0$. So for the rest of this proof we assume that $f \in \mathbb{H}_{n+1}^0$.

For such functions $\int_X f(y, u)\mu_{n+1}(dy) = 0$ for Lebesgue-a.e. $u \in \mathcal{S}$ (see Appendix, Lemma 7.1) which means that $f(\cdot, u) \in H_{n+1}^{(0)}$ for each u . Since $(\hat{K}_n^0 f)(x, u) = \int_X k_n(x, dy)f(y, u) \in H_n^{(0)}$ the condition of our Lemma implies that for these u

$$\int_X [(K_n f)(x, u)]^2 \mu_n(dx) \leq c^2 \int_X f(x, u)^2 \mu_{n+1}(dx).$$

Integrating both parts of this inequality over $u \in \mathcal{S}$ we obtain

$$\|\hat{K}_n^0 f\|_{\mathbb{H}_n}^2 \leq c^2 \int_{X \times \mathcal{S}} f(x, u)^2 \mu_{n+1}(dx)du = c^2 \|f\|_{\mathbb{H}_{n+1}}^2$$

which finishes the proof of part (iii). \square

Proof of (iv). $f = \mathcal{P}_{n+1}f + \mathcal{Q}_{n+1}f$ for any $f \in \mathbb{H}_{n+1}$ and hence $\hat{K}_n f = \hat{K}_n \mathcal{P}_{n+1}f + \hat{K}_n \mathcal{Q}_{n+1}f$. By properties (i) and (ii), the function $\hat{K}_n \mathcal{P}_{n+1}f = \hat{K}_n^0 f \in \mathbb{H}_n^0$ is orthogonal to $\hat{K}_n \mathcal{Q}_{n+1}f = \hat{K}_n^1 f \in \mathcal{L}_n$. Therefore

$$\|\hat{K}_n f\|^2 = \|\hat{K}_n \mathcal{P}_{n+1}f\|^2 + \|\hat{K}_n \mathcal{Q}_{n+1}f\|^2 = \|\hat{K}_n^0 f\|^2 + \|\mathcal{Q}_{n+1}f\|^2,$$

where the equality $\|\hat{K}_n \mathcal{Q}_{n+1} f\| = \|\mathcal{Q}_{n+1} f\|$ follows from (43). By (iii), $\|\hat{K}_n^0 f\| \leq c \|\mathcal{P}_{n+1} f\|$ and we obtain

$$(55) \quad \|\hat{K}_n f\|^2 \leq c^2 \|\mathcal{P}_{n+1} f\|^2 + \|\mathcal{Q}_{n+1} f\|^2 = (c^2 - 1) \|\mathcal{P}_{n+1} f\|^2 + \|f\|^2.$$

(We use here that $\|\mathcal{P}_{n+1} f\|^2 + \|\mathcal{Q}_{n+1} f\|^2 = \|f\|^2$.) By the condition of part (iv)

$$\|f\|^2(1 - \epsilon) \leq (c^2 - 1) \|\mathcal{P}_{n+1} f\|^2 + \|f\|^2 \text{ and hence } \|\mathcal{P}_{n+1} f\|^2 \leq \frac{\epsilon}{1 - c^2} \|f\|^2.$$

So $\|\mathcal{Q}_{n+1} f\|^2 = \|f\|^2 - \|\mathcal{P}_{n+1} f\|^2 \geq (1 - \frac{\epsilon}{1 - c^2}) \|f\|^2$. \square

5.2.2. *Proof of the main technical result.* In this section, we use the following simplified notation: $\hat{V}_{\mathfrak{g}_n} = \hat{V}_n$. Theorem 5.5 follows from the following statement.

Theorem 5.8. *Suppose that Condition I (inequality (8)) is satisfied and that*

$$(56) \quad \sup_n \|\hat{K}_{n-1} \hat{V}_n \hat{K}_n\| = 1.$$

Then there is a measure $\nu \in M$ and a probability measure κ on \mathcal{S} which is preserved by the group G_ν .

Remark 5.9. If matrices g_j are independent then $K_{n-1} V_n = W_{\nu_n}$ and Lemma 4.3 states that $\|W_{\nu_n}\| < 1$. In contrast, $\|\hat{K}_{n-1} \hat{V}_n\| = 1$. To see that, set $\mathfrak{f} = \hat{V}_n^{-1} \mathfrak{h}$, where $\mathfrak{h} \in \mathcal{L}_n$ and is arbitrary otherwise. Then $\hat{K}_{n-1} \hat{V}_n \mathfrak{f} = \hat{K}_{n-1} \mathfrak{h}$ and hence $\|\hat{K}_{n-1} \hat{V}_n \mathfrak{f}\| = \|\hat{K}_{n-1} \mathfrak{h}\| = \|\mathfrak{h}\| = \|\mathfrak{f}\|$ which proves the claim.

Proof of Theorem 5.8. Equality (56) implies that for a given (small) $\epsilon > 0$ there is $n(\epsilon)$ and a function $\varphi_\epsilon \in \mathbb{H}_{n+1}$ such that

$$(57) \quad \|\hat{K}_{n(\epsilon)-1} \hat{V}_{n(\epsilon)} \hat{K}_{n(\epsilon)} \varphi_\epsilon\|^2 \geq (1 - \epsilon) \|\varphi_\epsilon\|^2.$$

Since throughout this proof ϵ will be fixed, we shall from now on write φ for φ_ϵ and n for $n(\epsilon)$.

Set $\psi = \hat{K}_n \varphi$. Since $\|\hat{K}_{n-1}\| = \|\hat{V}_n\| = \|\hat{K}_n\| = 1$, it follows from (57) that

$$(58) \quad \|\psi\|^2 = \|\hat{K}_n \varphi\|^2 \geq (1 - \epsilon) \|\varphi\|^2.$$

(Otherwise, we would have had $\|\hat{K}_{n-1} \hat{V}_n \hat{K}_n \varphi\|^2 \leq \|\hat{K}_n \varphi\|^2 < (1 - \epsilon) \|\varphi\|^2$.)

Similarly, and using that \hat{V}_n is a unitary operator, we claim that

$$(59) \quad \|\hat{K}_{n-1} \hat{V}_n \psi\|^2 \geq (1 - \epsilon) \|\hat{V}_n \psi\|^2 = (1 - \epsilon) \|\psi\|^2.$$

To see this, suppose that to the contrary $\|\hat{K}_{n-1} \hat{V}_n \psi\|^2 < (1 - \epsilon) \|\psi\|^2$. Then

$$(1 - \epsilon) \|\psi\|^2 > \|\hat{K}_{n-1} \hat{V}_n \psi\|^2 = \|\hat{K}_{n-1} \hat{V}_n \hat{K}_n \varphi\|^2 \geq (1 - \epsilon) \|\varphi\|^2$$

and hence $\|\psi\|^2 > \|\varphi\|^2$ which contradicts $\|\psi\|^2 \leq \|\varphi\|^2$ and thus proves (59).

By Lemma 5.6(iv), the inequality in (58) implies that

$$(60) \quad \|\mathcal{P}_{n+1} \varphi\|^2 \leq \bar{\epsilon} \|\varphi\|^2 \text{ and } \|\mathcal{Q}_{n+1} \varphi\|^2 \geq (1 - \bar{\epsilon}) \|\varphi\|^2,$$

where $\bar{\epsilon} = \frac{\epsilon}{1 - c^2}$. It follows then that

$$\begin{aligned} \|\hat{K}_n^0 \varphi\|^2 &= \|\hat{K}_n \mathcal{P}_{n+1} \varphi\|^2 \leq \bar{\epsilon} \|\varphi\|^2 \text{ and} \\ \|\hat{K}_n^1 \varphi\|^2 &= \|\hat{K}_n \mathcal{Q}_{n+1} \varphi\|^2 = \|\mathcal{Q}_{n+1} \varphi\|^2 \geq (1 - \bar{\epsilon}) \|\varphi\|^2. \end{aligned}$$

Similarly, by Lemma 5.6(iv), the inequality in (59) implies that

$$(61) \quad \|\mathcal{Q}_n \hat{V}_n \psi\|^2 \geq (1 - \bar{\varepsilon}) \|\hat{V}_n \psi\|^2 = (1 - \bar{\varepsilon}) \|\psi\|^2 \geq (1 - \bar{\varepsilon})(1 - \varepsilon) \|\varphi\|^2,$$

where the last step is due to (58). It follows from (61) that, for $\bar{\varepsilon}, \varepsilon \in (0, 1)$,

$$(62) \quad \|\mathcal{Q}_n \hat{V}_n \psi\| \geq (1 - \bar{\varepsilon}) \|\varphi\|,$$

where $\tilde{\varepsilon} = (\bar{\varepsilon} + \varepsilon)$. Next, since $\psi = \hat{K}_n^1 \varphi + \hat{K}_n^0 \varphi$,

$$(63) \quad \|\mathcal{Q}_n \hat{V}_n \psi\| \leq \|\mathcal{Q}_n \hat{V}_n \hat{K}_n^1 \varphi\| + \|\mathcal{Q}_n \hat{V}_n \hat{K}_n^0 \varphi\| \leq \|\mathcal{Q}_n \hat{V}_n \hat{K}_n^1 \varphi\| + \bar{\varepsilon}^{\frac{1}{2}} \|\varphi\|.$$

Combining (62) and (63) we obtain:

$$(64) \quad \|\mathcal{Q}_n \hat{V}_n \hat{K}_n^1 \varphi\| \geq (1 - \tilde{\varepsilon} - \bar{\varepsilon}^{\frac{1}{2}}) \|\varphi\|.$$

Set $h = \hat{K}_n^1 \varphi = \hat{K}_n \mathcal{Q}_{n+1} \varphi$. Note that $h \in \mathcal{L}_n$ since $\mathcal{Q}_{n+1} \varphi \in \mathcal{L}_{n+1}$ and moreover $\|h\| = \|\mathcal{Q}_{n+1} \varphi\| \geq (1 - \bar{\varepsilon})^{\frac{1}{2}} \|\varphi\|$ (see Remark 5.1). It follows from (64) that

$$(65) \quad \|\mathcal{Q}_n \hat{V}_n h\| \geq (1 - \tilde{\varepsilon} - \bar{\varepsilon}^{\frac{1}{2}}) (1 - \bar{\varepsilon})^{-\frac{1}{2}} \|h\|.$$

By (53) and (44)

$$(66) \quad \begin{aligned} (\mathcal{Q}_n \hat{V}_n h)(u) &= \int_X (\hat{V}_n h)(x, u) \mu_n(dx) = \int_X h(\mathbf{g}_n(x).u) \|\mathbf{g}_n(x)u\|^{-\frac{m}{2}} \mu_n(dx) \\ &= \int_{\text{SL}(m, \mathbb{R})} h(g.u) \|gu\|^{-\frac{m}{2}} \nu_n(dg), \end{aligned}$$

where the last equality is due to the definition of ν_n . Equation (66) shows that the action of $\mathcal{Q}_n \hat{V}_n$ on \mathcal{L}_n is isomorphic to the action of W_{ν_n} on $L_2(\mathcal{S})$ (see (23)) and, in particular, $\|\mathcal{Q}_n \hat{V}_n\| = \|W_{\nu_n}\|$.

Since ε in (65) can be made arbitrarily small, it follows that $\sup_n \|\mathcal{Q}_n \hat{V}_n\| = 1$ and therefore also $\sup_{\nu \in M} \|W_{\nu_n}\| = 1$. This, by Lemma 4.3 (and (28)), implies the existence of a $\nu \in M$ and a κ on \mathcal{S} preserved by G_ν . Theorem 5.8 is proved. \square

6. PROOF OF THEOREM 2.2

Throughout this section, we suppose that the Markov chain ξ , the corresponding operators K_n^0 , and the sequence of functions $(\mathbf{g}_n)_{n \geq 1}$ are those defined in section 2.

Our plan is as follows. We first prove Lemma 6.1 (which is a particular case of Theorem 2.2) and then derive Theorem 2.2 from this Lemma.

As will be seen right now, Lemma 6.1 results from an easy analysis of the proof of Theorem 2.1.

Lemma 6.1. *Suppose that:*

- (i) *the inequalities $\|K_n^0\| \leq c$, where $c < 1$, are satisfied for all $n \geq 1$;*
- (ii) *all distributions ν_{2j} , $j \geq 1$ belong to a weakly compact set M of distributions satisfying (9)(b).*

Then there is a (non-random) $\lambda > 0$ such that with probability 1

$$(67) \quad \liminf_{j \rightarrow \infty} \frac{1}{j} \ln \|g_j \dots g_1\| \geq \lambda$$

and the estimate (67) does not depend on the choice of the subsequence of functions $(\mathbf{g}_{2j-1})_{j \geq 1}$.

Proof. An easy examination of the proof of Theorem 2.1 shows that this proof follows from the fact that inequalities

$$(68) \quad \|\hat{K}_{2j-1}\hat{V}_{\mathfrak{g}_{2j}}\hat{K}_{2j}\hat{V}_{\mathfrak{g}_{2j+1}}\| = \|\hat{K}_{2j-1}\hat{V}_{\mathfrak{g}_{2j}}\hat{K}_{2j}\| \leq \alpha < 1$$

hold for all $j \geq 1$.

Theorem 5.5 states that (51) holds, with $n = 2j$ if $\|\hat{K}_n^0\| \leq c < 1$, $\|\hat{K}_{n+1}^0\| \leq c < 1$, and the group G_{ν_n} does not preserve any probability measure on \mathcal{S} . The conditions of our Corollary thus imply that (67) follows from Theorem 2.1.

The uniformity of (67) with respect to the choice of the subsequence of functions $(\mathfrak{g}_{2j-1})_{j \geq 1}$ follows from the fact that the matrices $\mathfrak{g}_{2j-1}(\xi_{2j-1})$ have no impact on the inequalities (50) defining the value of λ . \square

Let us make one final observation concerning the proof of Theorem 2.1: it is not important for this proof that the phase space X of the chain ξ does not depend on n . Indeed, the definition $\xi = (\xi_j \in X^{(j)})_{j \geq 1}$, where $X^{(j)}$ is a sequence of phase spaces, is equivalent to the original definition with $X = \cup_{j=1}^{\infty} X^{(j)}$ - the union of $X^{(j)}$'s (which are considered as disjoint sets). Formally speaking, we also have to extend to X the definitions of functions $\mathfrak{g}_j : X^{(j)} \mapsto \text{SL}(m, \mathbb{R})$. This can be done e.g. by setting $\mathfrak{g}_j(y) = g \in \text{SL}(m, \mathbb{R})$ for all $y \notin X^{(j)}$; the choice of the g plays no role since the chain ξ at time j can take values only in $X^{(j)}$.

So, in the proof of Theorem 2.2 we use the version of Theorem 2.1 (and thus also of Lemma 6.1) which works for chains with phase spaces which depend on time.

Proof of Theorem 2.2. Set $n_0 = l_0 = 0$ and define a sequence of intervals $[a_j, b_j]$, $j \geq 1$, by

$$\begin{aligned} a_j &= n_{j-1} + l_{j-1} + 1, & b_j &= n_{j+1}, \text{ if } j \text{ is odd,} \\ a_j &= n_j + 1, & b_j &= n_j + l_j \text{ if } j \text{ is even.} \end{aligned}$$

Define a new Markov chain $\eta = (\eta_j)_{j \geq 1}$ by setting $\eta_j = (\xi_{a_j}, \xi_{a_j+1}, \dots, \xi_{b_j})$ with phase spaces $X^{(j)} = \overbrace{X \times \dots \times X}^{b_j - a_j + 1 \text{ times}}$ depending on j .

Define also a new sequence of functions $\bar{\mathfrak{g}}_j : X^{(j)} \mapsto \text{SL}(m, \mathbb{R})$ by

$$\bar{\mathfrak{g}}_j(\eta_j) = \mathfrak{g}_{b_j}(\xi_{b_j}) \dots \mathfrak{g}_{a_j}(\xi_{a_j}).$$

Let ν_j^η be the distribution of $\bar{\mathfrak{g}}_j(\eta_j)$. Note that condition (ii) of Theorem 2.2 and the definition of $[a_j, b_j]$ imply that if j is even then $\nu_j^\eta = \nu_{n_j+1, l_j}$ and hence condition (ii) of Lemma 6.1 is satisfied.

We shall now check that condition (i) of Lemma 6.1 is also satisfied.

Denote by \tilde{H}_j the Hilbert space of real-valued functions square integrable with respect to the measure on $X^{(j)}$ corresponding to the chain η . This measure is given by

$$\mathbb{P}(\eta_j \in dy_1 \times dy_2 \times \dots \times dy_{r_j}) = \mu_{a_j}(dy_1)k_{a_j}(y_1, dy_2) \dots k_{b_j-1}(y_{r_j-1}, dy_{r_j}),$$

where $r_j = b_j - a_j + 1$ (this notation will be used throughout this proof).

Let $\tilde{K}_j : \tilde{H}_{j+1} \mapsto \tilde{H}_j$, $j \geq 1$, be the transition operators of the chain η . Below, we use the notation $\mathbf{x} = (x_1, \dots, x_{r_j})$ and $\mathbf{y} = (y_1, \dots, y_{r_{j+1}})$ for elements of \tilde{H}_j and \tilde{H}_{j+1} respectively.

Denote the kernel of \tilde{K}_j by $\tilde{k}_j(\mathbf{x}, d\mathbf{y})$. Obviously,

$$\tilde{k}_j(\mathbf{x}, d\mathbf{y}) = k_{b_j}(x_{r_j}, dy_1)k_{a_{j+1}}(y_1, dy_2) \dots k_{r_{j+1}-1}(y_{r_{j+1}-1}, dy_{r_{j+1}}).$$

It is important that $\tilde{k}_j(\mathbf{x}, d\mathbf{y})$ depends only on x_{r_j} (and not other components of \mathbf{x}).

Denote by \tilde{K}_j^0 the restriction of \tilde{K}_j to the subspace \tilde{H}_j^0 of functions from \tilde{H}_j with zero mean.

Lemma 6.2. *Suppose that (as in Theorem 2.2) the inequalities $\|K_{n_j}^0\| \leq c$, $\|K_{n_j+l_j}^0\| \leq c$, where $c < 1$, hold for all $j \geq 1$. Then $\|\tilde{K}_j^0\| \leq c$ for all $j \geq 1$.*

Proof. Throughout this proof j and r_{j+1} are fixed and so we write r for r_{j+1} .

Let $f \in H_{j+1}^0$, that is

$$(69) \quad \int_{X^{(j+1)}} \mu_{a_{j+1}}(dy_1)k_{a_{j+1}}(y_1, dy_2) \dots k_{b_{j+1}-1}(y_{r-1}, dy_r)f(\mathbf{y}) = \int_X \mu_{a_{j+1}}(dy_1)\varphi(y_1) = 0,$$

where

$$(70) \quad \varphi(y_1) = \int_{y_2 \in X, \dots, y_r \in X} k_{a_{j+1}}(y_1, dy_2) \dots k_{b_{j+1}-1}(y_{r-1}, dy_r)f(y_1, y_2, \dots, y_r).$$

By the definition of the action of \tilde{K}_j ,

$$(71) \quad \begin{aligned} (\tilde{K}_j^0 f)(\mathbf{x}) &= \int_{X^{(j+1)}} k_{b_j}(x_{r_j}, dy_1)k_{a_{j+1}}(y_1, dy_2) \dots k_{b_{j+1}-1}(y_{r-1}, dy_r)f(\mathbf{y}) \\ &= \int_X k_{b_j}(x_{r_j}, dy_1)\varphi(y_1). \end{aligned}$$

We have to show that

$$(72) \quad \|\tilde{K}_j^0 f\|_{\tilde{H}_j} \leq c^2 \|f\|_{\tilde{H}_{j+1}}.$$

It follows from (71) that

$$\|\tilde{K}_j^0 f\|_{\tilde{H}_j}^2 = \int_X \left(\int_X k_{b_j}(x, dy)\varphi(y) \right)^2 \mu_{b_j}(dx).$$

On the other hand (69) means that $\varphi \in H_{a_{j+1}}^0$ and the right hand side of (71) coincides with $K_{b_j}^0 \varphi \in H_{b_j}^0$ and

$$\|K_{b_j}^0 \varphi\|_{H_{b_j}}^2 = \int_X \left(\int_X k_{b_j}(x, dy)\varphi(y) \right)^2 \mu_{b_j}(dx).$$

Note also that b_j 's are defined so that $\|K_{b_j}\| \leq c$ for all $j \geq 1$ by the condition of the Lemma. Therefore

$$(73) \quad \|\tilde{K}_j^0 f\|_{\tilde{H}_j}^2 = \|K_{b_j}^0 \varphi\|_{H_{b_j}}^2 \leq c^2 \|\varphi\|_{H_{b_{j+1}}}^2 = c^2 \int_X \mu_{a_{j+1}}(dx)\varphi(x)^2,$$

(where the last equality is due to $b_j + 1 = a_{j+1}$). It follows from (70) that, by the Cauchy-Schwartz inequality,

$$\varphi(y_1)^2 \leq \int_{y_2 \in X, \dots, y_r \in X} k_{a_{j+1}}(y_1, dy_2) \dots k_{b_{j+1}-1}(y_{r-1}, dy_r)[f(y_1, y_2, \dots, y_r)]^2.$$

The integral in the right hand side of (73) is now estimated by

$$\begin{aligned} & \int_X \mu_{a_{j+1}}(dx) \varphi(x)^2 \\ & \leq \int_X \mu_{a_{j+1}}(dy_1) \int_{y_2 \in X, \dots, y_r \in X} k_{a_{j+1}}(y_1, dy_2) \dots k_{b_{j+1}-1}(y_{r-1}, dy_r) [f(y_1, y_2, \dots, y_r)]^2 \\ & = \int_{X^{(j+1)}} \mu_{a_{j+1}}(dy_1) k_{a_{j+1}}(y_1, dy_2) \dots k_{b_{j+1}-1}(y_{r-1}, dy_r) [f(y_1, y_2, \dots, y_r)]^2 = \|f\|_{\tilde{H}_{j+1}}^2. \end{aligned}$$

The Lemma is proved. \square

We thus have shown that if the assumptions of Theorem 2.2 are satisfied then the chain η satisfies all assumptions of Lemma 6.1 and hence there is a non-random $\lambda > 0$ such that with probability 1

$$(74) \quad \liminf_{j \rightarrow \infty} \frac{1}{j} \ln \|\bar{\mathfrak{g}}_j(\eta_j) \dots \bar{\mathfrak{g}}_1(\eta_1)\| \geq \lambda.$$

Since $\bar{\mathfrak{g}}_j(\eta_j) \dots \bar{\mathfrak{g}}_1(\eta_1) = g_{n_j+l_j} \dots g_1$, where the product in the right side of this equality is the same as in (12), Theorem 2.2 is proved. \square

7. APPENDIX

7.1. Proof of (52). First, we have to show that \mathcal{P}_n defined by the rhs of (52) maps \mathbb{H}_n into \mathbb{H}_n^0 . Let $h \in \mathcal{L}_n$. Then

$$\begin{aligned} \langle \mathcal{P}_n f, h \rangle_{\mathbb{H}_n} &= \int_{X \times \mathcal{S}} f(x, u) h(u) \mu_n(dx) du - \int_{X \times \mathcal{S}} \left(\int_X f(y, u) \mu_n(dy) \right) h(u) \mu_n(dx) du \\ &= \int_{X \times \mathcal{S}} f(x, u) h(u) \mu_n(dx) du - \int_{\mathcal{S}} \left(\int_X f(y, u) \mu_n(dy) \right) h(u) du = 0. \end{aligned}$$

It remains to check that if $f \in \mathbb{H}_n^0$ then the integral in the rhs of (52) is vanishing. This follows from the following lemma.

Lemma 7.1. *If $f \in \mathbb{H}_n^0$ then $a(u) \equiv \int_X f(x, u) \mu_n(dx) = 0$ for Lebesgue - a.e. $u \in \mathcal{S}$.*

Proof. Note first that $a \in \mathcal{L}_n$ because $\int_{\mathcal{S}} a(u)^2 du \leq \int_{\mathcal{S}} \left(\int_X f(x, u)^2 \mu_n(dx) \right) du = \|f\|_{\mathbb{H}_n}^2$. By the definition of \mathbb{H}_n^0 , $\int_{X \times \mathcal{S}} f(x, u) a(u) \mu_{n+1}(dx) du = 0$. On the other hand,

$$\int_{X \times \mathcal{S}} f(x, u) a(u) \mu_{n+1}(dx) du = \int_{\mathcal{S}} \left(\int_X f(x, u) \mu_{n+1}(dx) \right) a(u) du = \int_{\mathcal{S}} a(u)^2 du.$$

The Lemma is proved. \square

7.2. Proof of the equivalence of (31) and (32). We have to check that for any $g \in \text{SL}(m, \mathbb{R})$

$$(75) \quad \int_{\mathcal{S}} \psi(u) (g^{-1} \kappa)(du) = \int_{\mathcal{S}} \psi(g^{-1} \cdot u) \kappa(du).$$

Obviously, it suffices to check (75) for characteristic functions of Borel subsets of \mathcal{S} . Let $A \subset \mathcal{S}$ be such a subset and χ_A be its characteristic function. Then

$$\int_{\mathcal{S}} \chi_A(u) (g^{-1} \kappa)(du) = (g^{-1} \kappa)(A) = \kappa(g \cdot A).$$

Since $\chi_A(g^{-1}.u) = \chi_{g.A}(u)$, we have

$$\int_S \chi_A(g^{-1}.u) \kappa(du) = \int_S \chi_{g.A}(u) \kappa(du) = \kappa(g.A)$$

and this proves (75).

7.3. One corollary of the uniform convexity of L_2 . Estimates similar to the one stated in Lemma 7.2 hold for a wide class of uniformly convex spaces. Here, we consider only the case of a Hilbert space because that is all we need to justify (41) which plays an important role in the proof of Lemma 4.3.

We use the following notation and definitions.

L_2 is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$. By \mathcal{U} we denote the unit sphere in L_2 .

(G, \mathcal{G}, ν) is a measure space: G is a set, \mathcal{G} is a sigma-algebra of measurable subsets of G , and ν is a probability measure on G .

$f : G \mapsto \mathcal{U}$ is a Bochner-measurable function with domain G and range \mathcal{U} . Set

$$\varphi = \int_{\text{SL}(m, \mathbb{R})} f(g) \nu(dg) \quad \text{and so} \quad \|\varphi\| \leq 1.$$

Let $\delta = 1 - \|\varphi\|$ and, for $\varepsilon > 0$, define $\Gamma = \{g : \|\varphi - f(g)\|_{L_2} > \varepsilon\}$.

Lemma 7.2. *If $\varepsilon^2 > 3\delta$ then $\nu(\Gamma) \leq \frac{2\delta}{\varepsilon^2 - 3\delta}$.*

Proof. We assume (with no loss of validity) that $1 > \nu(\Gamma) > 0$ and $\varepsilon \leq 0.5$.

Denote $p = \nu(\Gamma)$, $q = 1 - p$, and define

$$\bar{\varphi} = p^{-1} \int_{\Gamma} f(g) \nu(dg), \quad \bar{\bar{\varphi}} = q^{-1} \int_{G \setminus \Gamma} f(g) \nu(dg).$$

Set $\hat{\varphi} = \frac{\varphi}{\|\varphi\|}$ and let \bar{x} and $\bar{\bar{x}}$ be the ‘sizes’ of projections of $\bar{\varphi}$ and, respectively, $\bar{\bar{\varphi}}$ on the line $l_{\varphi} = \{a\hat{\varphi} : a \in \mathbb{R}\}$, that is $\bar{x} = \langle \bar{\varphi}, \hat{\varphi} \rangle$, $\bar{\bar{x}} = \langle \bar{\bar{\varphi}}, \hat{\varphi} \rangle$.

The equality $\varphi = p\bar{\varphi} + q\bar{\bar{\varphi}}$ implies

$$(76) \quad \|\varphi\|^2 = p\langle \bar{\varphi}, \varphi \rangle + q\langle \bar{\bar{\varphi}}, \varphi \rangle \quad \text{and hence} \quad \|\varphi\| = p\bar{x} + q\bar{\bar{x}}.$$

We then get

$$(77) \quad p = \frac{\bar{\bar{x}} - \|\varphi\|}{\bar{\bar{x}} - \bar{x}} \leq \frac{\delta}{\bar{\bar{x}} - \bar{x}}.$$

It remains to estimate $\bar{\bar{x}} - \bar{x}$. To this end, define $\mathcal{U}_{\varepsilon} = \{f \in \mathcal{U} : \|f - \varphi\| > \varepsilon\}$. It is easy to see that $\langle f, \hat{\varphi} \rangle < \|\varphi\| - b$ for any $f \in \mathcal{U}_{\varepsilon}$, where $b = \frac{\varepsilon^2 - 2\delta - \delta^2}{2(1-\delta)}$. Since $f(g) \in \mathcal{U}_{\varepsilon}$ when $g \in \Gamma$, we obtain

$$\bar{x} = p^{-1} \int_{\Gamma} \langle f(g), \hat{\varphi} \rangle \nu(dg) \leq \|\varphi\| - b \quad \text{and, because of (76),} \quad \bar{\bar{x}} \geq \|\varphi\|.$$

Hence $\bar{\bar{x}} - \bar{x} \geq b$ and it follows from (77) that $p \leq \frac{\delta}{b} = \frac{2\delta(1-\delta)}{\varepsilon^2 - 2\delta - \delta^2} \leq \frac{2\delta}{\varepsilon^2 - 3\delta}$. \square

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