A Mobility Determination Method for Parallel Platforms Based on the Lie Algebra of $SE(3)$ and Its Subspaces

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This contribution presents a screw theory-based method for determining the mobility of fully parallel platforms. The method is based on the application of three stages. The first stage involves the application of the intersection of subalgebras of Lie algebra, $se(3)$, of the special Euclidean group, $SE(3)$, associated with the legs of the platform. The second stage analyzes the possibility of the legs of the platform generating a sum or direct sum of two subalgebras of the Lie algebra, $se(3)$. The last stage, if necessary, considers the possibility of the kinematic pairs of the legs satisfying certain velocity conditions; these conditions reduce the platform's mobility analysis to one that can be solved using one of the two previous stages. Several examples are illustrated.

1 INTRODUCTION

One of the most challenging problems in theoretical and computational kinematics is that of determining the mobility of kinematic chains. The first contributions are due to Grübler and his disciple Kutzbach. However, even before the Kutzbach-Grübler criterion was formulated, there were already kinematic chains whose mobility could not be computed correctly by the criterion. The work of Gogu [1], related to the analysis and compilation of different mobility criteria can provide an idea of the complexity of the task.

Since the '60s of the last century, different approaches for determining the mobility of kinematic chains began to be developed:

1. **Group theory**. Within the group theory developments, Hervé [2], in a seminal contribution, enumerated the subgroups of the Euclidean group and classified the kinematic chains into trivial, exceptional, and paradoxical.[1] However, Hervé was unable to find a criterion for the mobility of exceptional chains. His results are very relevant for the mobility determination of kinematic chains, notwithstanding. Fanghella and Galleti also applied the composition of subgroups of the special Euclidean group, $SE(3)$, to the mobility of kinematic chains [4] and the synthesis of kinematotropic linkages [5]. In 2006, Rico et al. [6] presented a mobility criterion based on the theory of subgroups of the Euclidean group, $SE(3)$, generated by the legs of a fully parallel platform and their intersections. Even then, the authors knew the existence of parallel platforms that the mobility criterion failed to correctly determine their mobility.
2. **The theory of screws**. Screw theory was initiated by Ball, [7]. However, the knowledge remained forgotten until the '60s of the last century. On the one hand, the Romanian school represented by Vionea and Atanasiu, [8], presented a failed mobility criterion based on the dimension of the space generated by the screws of the kinematic chain. On the other hand, Hunt [9] presented a classification of screw systems, subspaces of the Lie algebra, $se(3)$, of $SE(3)$, including screw systems that allow full-cycle mobility. These last screw systems were characterized by Selig [10], as subalgebras of $se(3)$. In 2006, Rico et al. [11] presented a mobility criterion for fully parallel platforms, which considers the subalgebras of $se(3)$ generated by the legs and their intersections. This criterion is equivalent to that presented using group theory, [6].

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[1] In a recent contribution, Rico et al. [3] showed that Jordan discovered the subgroups of the Euclidean group one century before Hervé.

In this contribution, only two of the most relevant mobility criteria developed in the last fifteen years are considered:

1. Huang and his coworkers [12–15] developed a mobility criterion based on the subspace of reciprocal screws to the subspace generated by the screws that represent the kinematic pairs of a kinematic chain under consideration. However, the subspace of reciprocal screws is determined in a particular position, and the subspace and its dimension may change with a change of the position. For example, Huang and Ge [13] successfully applied their method to the Sarrus linkage; however, the same method determines, incorrectly that a Sarrus-like linkage presented by Rico et al. [16] has mobility 1 while it is actually a structure.
2. Yang and his coworkers [17–22] developed a method for the synthesis of kinematic chains, including parallel platforms, that also determines their mobility. At first sight, Yang's method uses neither group theory nor screw theory. Yang and his coworkers employ structural units formed by open chains, denoted by SOCs. These SOCs can be connected in a serial or parallel fashion. From these SOCs it is possible to obtain the matrix of position and orientation characteristics, POC. This matrix indicates the translation and rotation independent and dependent "elements". The intersection of the POC matrices requires 6 linear rules or criteria for rotational elements, and 6 linear rules or criteria for translational elements, [21]. Besides, the POC matrix composition requires 2 linear rules or criteria, and 1 nonlinear rule or criterion for translational elements, [22].
   Finally, Yang and his coworkers published a book, [20]. In such contribution, the authors indicate that serial chains require 8 linear and symbolic operation rules and 2 non-linear criteria; while parallel linkages require 12 linear and symbolic operation rules and 2 non-linear criteria. Yang et al. [20, 21] compare these number of rules or criteria with the number of conditions employed by Fanghella and Galletti [4, 5] who, as indicated by Yang et al. [23], employed 107 rules for the composition of subgroups. It can be implied that Yang's method is, at its core, an application of group theory. However, it seems that Yang and his coworkers are not aware that employing the Lie algebra, $se(3)$, of the special Euclidean group, $SE(3)$ —isomorphic to the screw algebra— it is possible to avoid the 107 rules of Fanghella and Galletti, and also and the 22 criteria proposed by Yang and his coworkers.

It is important to note that any local mobility criterion can, in general, only determine the mobility in a neighborhood of the kinematic chain in the analyzed position. The mere existence of kinematotropic linkages is proof of that. The method proposed in this paper provides the correct solution for fully parallel platforms, where there is no bracing between the legs, the analyzed position is not singular and no "paradoxical" linkage, as defined by Hervé, [2], is part of the parallel platform. Under these circumstances, the approach presented in this contribution provides the correct mobility of a wide range of configurations of fully parallel platforms. The approach uses only solid mathematical foundations from screw algebra.

The main contribution of this work is to present, a method based on screw theory, isomorphic to the Lie algebra $se(3)$ of $SE(3)$, that in three seamless stages, **correctly computes** the mobility of the great majority of fully parallel platforms. The method can be programmed using standard computer algebra programs. Unlike the methods reviewed in this section, the method developed here requires neither the analysis of reciprocal screws nor the use of a large number of conditions and definitions, other than those well established in spatial kinematics and screw theory.

This paper is structured as follows. Section 2 explains the details of each of the three steps of the method proposed in this paper. Section 3 presents a parallel platform whose mobility can be correctly determined using the first step of the method that involves the intersection of the subalgebras and subspaces of $se(3)$ generated by the platform's legs. Section 4 presents three examples of parallel platforms whose legs generate a sum of subalgebras of $se(3)$. Their mobility can be determined from the intersection of those sums. Section 5 presents a parallel platform whose legs contain closed loops. The kinematic analysis of these loops yields velocity conditions that allow mobility computations to be solved using one of the two previous steps. Finally, some conclusions are drawn in Section 6.

2 FUNDAMENTALS OF THE METHOD

The main fundamentals of the method for determining the mobility of fully parallel platforms proposed in this contribution, without indicating the details, will be presented in this section. As it was delineated in the abstract, the method has three stages:

1. **Stage I** of the method analyzes fully parallel platforms whose mobility can be explained and computed using simple mathematical structures associated with the screws of the serial connector chains, or legs. These legs connect the fixed and moving platforms. Usually, these mathematical structures are **subalgebras** of $se(3)$, the Lie algebra of the Euclidean group, $SE(3)$. However, in a few cases, the mathematical structures associated with the serial connector chains, or legs, that connect the fixed and moving platforms might be only subspaces of $se(3)$. This stage of the method has been already presented, with new small corrections, since 2003, see Aguilera [24], Rico et al. [6] and [11]. However, even then, it was recognized that this stage was not able to determine the mobility of a large number of parallel platforms. For example, those presented by Huang and Li [25] and [26].
2. **Stage II** of the method analyzes fully parallel platforms where the screws of the serial connector chains, or legs, are the sum, direct or not, of two subalgebras of $se(3)$ the Lie algebra of the Euclidean group, $SE(3)$. These mathematical structures are denominated **screw systems**

of locally constant rank and share many of the properties of the subalgebras of $se(3)$, the Lie algebra of the Euclidean group, $SE(3)$. Considering the intersection of these mathematical structures, this stage can explain and compute the mobility of a far larger group of fully parallel platforms than those successfully considered using only the first step. The details of this stage were presented since 2006, in Tadeo-Chávez and Pérez-Soto [27], Rico et al. [28], or Tadeo-Chávez [29]. However, the focus of these publications was the kinematic synthesis of fully parallel platforms.

3. **Stage III** of the method analyzes fully parallel platforms with serial connector chains, or legs, that either contain closed chains, or have parallel revolute or helical pairs. When the screws of these serial connector chains satisfy certain velocity conditions, the mathematical structures associated with the screws of the serial connector chain can be reduced to either a subalgebra or screw systems of locally constant rank. **The velocity conditions** are usually obtained by solving the velocity analysis of closed chains or by making the sum of angular velocities of revolute or helical joints with parallel axes equal to zero. Then, it is possible to explain and compute the mobility of these parallel platforms using the processes indicated either in the first or second stage. With this third step, it is possible to compute the mobility of even a greater group of fully parallel manipulators. Some details of this approach were illustrated in Tadeo-Chávez et al. [30] and Pérez-Soto [31]. Similarly, the focus of these publications was the kinematic synthesis of fully parallel platforms.

In this contribution, we present several examples that illustrate how stages II and III compute the mobility of parallel platforms. These computations were impossible using the techniques of stage I. The three stages must work iteratively; namely, after applying stage III, the results must be reanalyzed using the techniques developed in stages I or II.

3 Parallel platforms whose mobility can be determined using stage I of the method

This section presents a parallel platform whose mobility can be computed by the intersection of subalgebras and vector spaces of $se(3)$ associated with the platform's legs. Since this stage was discussed in great detail in Aguilera [24], Rico et al. [6, 11], only one example, which requires a modification of a definition in those manuscripts, is presented.

3.1 Parallel platform RRPS-2RPU

Figure 1 shows the parallel platform proposed by Álvarez-Pérez [32] as a possible alternative to rehabilitation therapy performed by skilled health workers. The application of the Kutzbach-Grübler criterion indicates that the platform has 2 DOF.

This platform has three connecting serial chains. The first one has an RRPS topology, while the remaining two legs have an RPU topology. The fixed and moving platforms are indicated by the numbers 1 and 11. The coordinate system $OXYZ$ is rigidly attached to the fixed platform with its origin, O, located at the center of the equilateral triangle formed by A_1, E_2, and E_3. Point P is located along the common rotation axis of the revolute pairs that connect the moving platform with the second and third legs. Its position vector with respect to the coordinate system in figure 1 is $\vec{r}_P = (\,0,\,100,\,0\,)^T$.

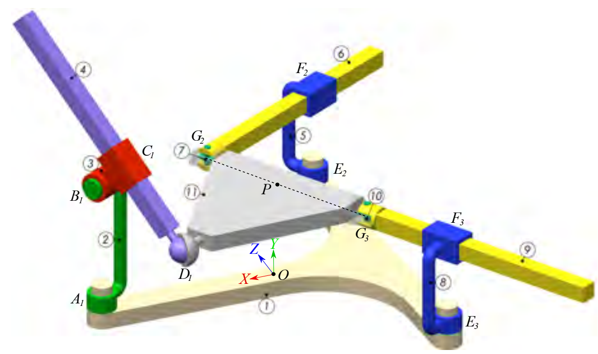

Fig. 1. The parallel platform proposed by Álvarez-Pérez, [32].

The architecture of the RRPS leg is shown on the left side of figure 2. The revolute axes associated with the points A_1 and B_1 are perpendicular and, together with the prismatic pair and the spherical joint located at point D_1, form a mechanical generator of the complete Lie algebra $se(3)$. In the remainder of the contribution, whenever there is no possibility of confusion, the points will be used also to indicate the corresponding kinematic pairs. The subscripts of the points or kinematic pairs indicate the platform's leg number.

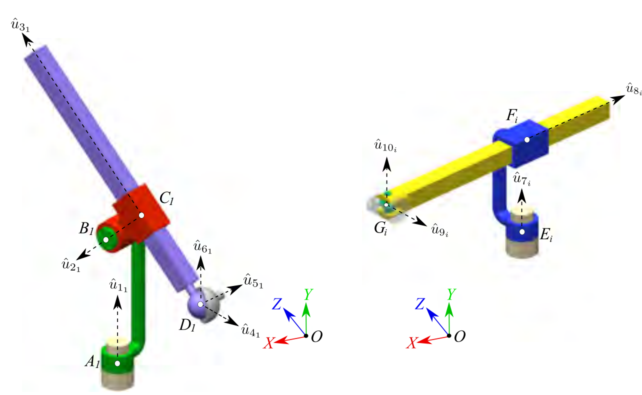

Fig. 2. The geometry of the connecting chains, R-R-P-S, and R-P-U.

The RPU architecture of leg 2 is shown on the right-hand side of figure 2. These characteristics are also valid for leg 3. The axis associated with the revolute joint located at point E_i and the axis of the first revolute pair of the Hooke joint with center at point G_i are parallel to the Y-axis. The direction of the prismatic pair, $\hat{u}_{8_i}$, is located in the plane $X-Z$, this result is also valid for the third leg. Finally, the rotation axis of the second revolute that forms the Hooke joint located at G_i is parallel to the unit vector $\hat{u}_{9_i}$. Furthermore, the last revolutes that form the Hooke joints, for the second and third legs, are coaxial.

The screws coordinates, reference points of the kinematic pairs of each connecting chain, and the corresponding Jacobian matrices are given now. The nomenclature follows figure 2 and the subscript, in the points and the Jacobian matrices, indicates the leg number. The rest of the paper will follow this convention.

For leg 1. $A_1 = (\frac{400}{\sqrt{3}}, 20, 0), {}^1\$^2; B_1 = (\frac{400}{\sqrt{3}}, 170, 0), {}^2\$^3; C_1, {}^3\$^4; D_1 = (100\sqrt{3}, 120, -100), {}^{4a}\$^{11a}; {}^{4b}\$^{11b}; {}^{4c}\$^{11c}$.

$$J_1 = \begin{bmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & -1 \\ 0 & -85 & \frac{\sqrt{2}}{4} & 100 & 0 & -120 \\ 0 & \frac{200}{\sqrt{3}} & \frac{1}{\sqrt{2}} & 0 & 100 & 100\sqrt{3} \\ \frac{400}{\sqrt{3}} & -85\sqrt{3} & \frac{\sqrt{6}}{4} & 100\sqrt{3} & 120 & 0 \end{bmatrix},$$

For leg 2. $E_2 = (-\frac{200}{\sqrt{3}}, 20, 200), {}^5\$^6; F_2, {}^6\$^7; G_2 = (\frac{100}{\sqrt{3}}, 120, 100), {}^{7a}\$^{11a}; {}^{7b}\$^{11b}$.

For leg 3. $E_3 = (-\frac{200}{\sqrt{3}}, 20, -200), {}^8\$^9; F_3, {}^9\$^{10}; G_3 = (-\frac{100}{\sqrt{3}}, 120, -100), {}^{10a}\$^{11a}; {}^{10b}\$^{11b}$.

$$J_2 = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{\sqrt{3}}{2} \\ -200 & -\frac{\sqrt{3}}{2} & -100 & -60\sqrt{3} \\ 0 & 0 & 0 & 0 \\ -\frac{200}{\sqrt{3}} & \frac{1}{2} & \frac{100}{\sqrt{3}} & 60 \end{bmatrix}, J_3 = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 200 & -\frac{1}{2} & 100 & 60\sqrt{3} \\ 0 & 0 & 0 & 0 \\ -\frac{200}{\sqrt{3}} & -\frac{\sqrt{3}}{2} & -\frac{100}{\sqrt{3}} & -60 \end{bmatrix}.$$

It should be noted that $[J_i]$ represents the column space of J_i; namely, the vector space spanned or generated by the columns of the matrix. This convention will be used for the rest of the paper.

The screw system given by the columns of the Jacobian matrix J_1 is the mechanical generator of the whole Lie algebra $se(3)$, of $SE(3)$. Thus, the dimension of the column space of the Jacobian matrix J_1, i.e. its rank, is equal to 6. Similarly, it can be proved that the dimension of the column space generated by the Jacobian matrices J_2 and J_3 is equal to 4. Moreover, by computing the intersection of the column spaces of these Jacobian matrices it follows that

$$[J_2] \cap [J_3] = [J_2] = [J_3] = \mathbf{V}_T^{m/f}. \quad (1)$$

Therefore, these column spaces are equal and they are not the mechanical generator of any subalgebra of $se(3)$, they only form a vector subspace, denoted by $\mathbf{V}_T^{m/f}$.

The next step is to determine the common closure algebra, $\mathbf{A}_{G^*}^{m/f}$, associated with all the infinitesimal mechanical liaisons,[2] $\mathbf{V}_j^{m/f}$ $j = 1, 2, 3$, or connections, associated with the legs. In this platform, the closure algebra of all legs is $se(3)$:

$$\mathbf{A}_{G^*}^{m/f} = \mathbf{A}_1^{m/f} = se(3) \quad (2)$$

Furthermore, for this parallel platform the subspace $\mathbf{V}_T^{m/f}$ is obviously contained in $\mathbf{A}_{G^*}^{m/f} = se(3)$:

$$\mathbf{V}_T^{m/f} < \mathbf{A}_{G^*}^{m/f} \quad (3)$$

Thus, stage I of the method identifies the parallel platform as a trivial of Tanev's type. Moreover, its mobility F can be determined using the following equation, see Rico et al. [11]:

$$F = dim(\mathbf{V}_T^{m/f}) = 4 \quad (4)$$

Stage I of the method shows that the subspace $V_T^{m/f}$ generated by the columns of the Jacobian matrices J_2 or J_3 represents the relative motion of the moving platform relative to the fixed platform. Rico et al. [6, 11] concluded that this is a trivial parallel platform Tanev's type with four degrees of freedom given by a planar motion in the plane $X - Z$, namely, $g_{\hat{u}_j}$, followed by a rotation around a fixed axis that passes through points G_{2b} and G_{3b}. *In Rico et al. [6, 11] it was assumed that a trivial parallel platform Tanev's type could only have one leg generating the subspace. This example shows that it is possible to have more than one leg generating the subspace.*

4 Parallel platforms whose mobility can be determined using stage II of the method

This section presents two examples of parallel platforms whose mobility is computed by determining the sum of subalgebras, of $se(3)$, associated with some of the legs of the platform.

4.1 Parallel platform RRPU-2RPU

The parallel platform shown in figure 3 is an improvement of the one proposed by Álvarez-Pérez [32], and analyzed in section 3.1, where the spherical pair that connects links 11 and 4 in figure 1 is replaced by a Hooke joint located at point D_1. The application of the Kutzbach-Grübler criterion indicates that the platform has 1 DOF.

[2] The infinitesimal mechanical liaisons is the subspace generated by all the screws that belong to a serial connection chain, while the closure algebra is the smallest subalgebra that contains the corresponding infinitesimal mechanical liaisons, for additional details see Rico et al. [16].

The platform has three legs. The first one has an RRPU topology, while the remaining two legs have an RPU topology. The fixed and moving platform are indicated by the numbers 1 and 12. The coordinate system $OXYZ$ is rigidly attached to the fixed platform and the origin, O, is located at the center of the equilateral triangle formed by A_1, E_2, and E_3. Finally, point P is attached to the moving platform and it is located along the common axis of the last revolute axes of legs 2 and 3.

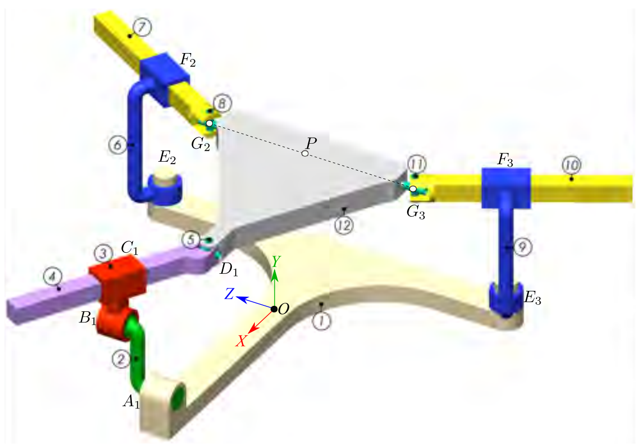

Fig. 3. Parallel Platform with three legs RRPU-2RPU.

The architecture of the RRPU leg is shown on the left side of figure 4. The axis of the revolute joint located at points A_1, B_1, and the axis of the first revolute pair of the Hooke joint with center at D_1 are parallel and their direction is that of $\hat{u}_{1_1}$; i.e. parallel to the Z-axis. The direction of the prismatic pair, $\hat{u}_{2_1}$, is parallel to the XZ plane. Finally, the axis of the second revolute joint that forms the Hooke joint with center at D_1, remains parallel to the Y-axis.

The architecture of the second leg RPU is shown on the right-hand side of figure 4. These characteristics are the same as those indicated in the platform of section 3.1.

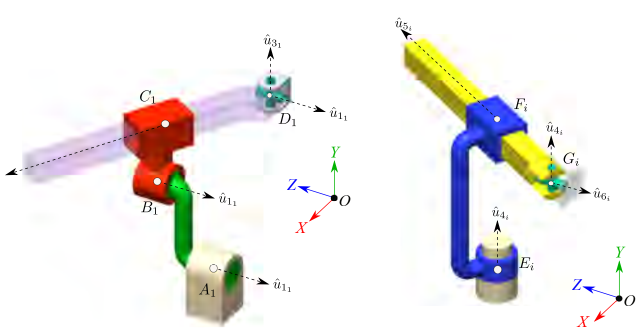

Fig. 4. The topology of the connecting chains R-R-P-U and R-P-U.

The screws coordinates, reference points of the kinematic pairs of each connecting chain, and the corresponding Jacobian matrices are given now. The nomenclature follows figure 3.

For leg 1. $A_1 = (\frac{400}{\sqrt{3}}, 30, 0), {}^1\$^2$; $B_1 = (\frac{400}{\sqrt{3}}, 100, 50), {}^2\$^3; C_1, {}^3\$^4; D_1 = (80\sqrt{3}, 150, 0), {}^4\$^5; {}^5\$^{12}$.

$$J_1 = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & -1 & 0 \\ -30 & -100 & \frac{\sqrt{3}}{2} & -150 & 0 \\ \frac{400}{\sqrt{3}} & \frac{400}{\sqrt{3}} & 0 & 80\sqrt{3} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 80\sqrt{3} \end{array}\right],$$

For leg 2. $E_2 = (\frac{-200}{\sqrt{3}}, 20, 200), {}^1\$^6; F_2, {}^6\$^7;$
$G_2 = (-40\sqrt{3}, 150, 120), {}^7\$^8; {}^8\$^{12}$.

For leg 3. $E_3 = (\frac{-200}{\sqrt{3}}, 20, -200), {}^1\$^9; F_3, {}^9\$^{10};$
$G_3 = (-40\sqrt{3}, 150, -120), {}^{10}\$^{11}; {}^{11}\$^{12}$.

$$J_2 = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -200 & -\frac{1}{2} & -120 & -150 \\ 0 & 0 & 0 & -40\sqrt{3} \\ -\frac{200}{\sqrt{3}} & \frac{\sqrt{3}}{2} & -40\sqrt{3} & 0 \end{array}\right], J_3 = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 200 & -\frac{1}{2} & 120 & 150 \\ 0 & 0 & 0 & 40\sqrt{3} \\ -\frac{200}{\sqrt{3}} & -\frac{\sqrt{3}}{2} & -40\sqrt{3} & 0 \end{array}\right].$$

In this section, the red dashed vertical lines indicate the separation of the Jacobian matrices in two parts; each of one forms a subalgebra.

It can be shown that the dimension of the column space, or rank, of J_2 and J_3 is 4. However, the screw systems associated with these legs are not the mechanical generators of any subalgebra of the Lie algebra, $se(3)$. Furthermore, the screw systems associated with J_2 and J_3 represent the same subspace $\mathbf{V}_2$. Similarly, the dimension of the column space or rank of J_1 is 5, and since there is no subalgebra of $se(3)$ of dimension 5 the screw system associated with J_1 is also a subspace $\mathbf{V}_1$. Moreover $\mathbf{V}_2 < \mathbf{V}_1$.

Since the screw systems associated with the three legs are not the mechanical generators of a subalgebra of the Lie algebra, $se(3)$, stage I of the method is unable to determine the mobility of the platform. However, stage II of the method determines that subspace $\mathbf{V}_2$ is the direct sum of two subalgebras $g_{\hat{j}} \oplus r_{G_{i_b}}$ of $se(3)$. The first subalgebra $g_{\hat{j}}$ represents the general planar motion in a plane perpendicular to the Y-axis. This subalgebra is generated by the three first columns of the Jacobian matrices J_2 and J_3. The second subalgebra, $r_{G_{i_b}}$, represents a rotation around the axis intersecting points G_{2_b} and G_{3_b}. The screw that generates subalgebra $r_{G_{i_b}}$ is located in the fourth column of both Jacobian matrices J_2 and J_3.

Similarly, stage II of the method determines that subspace $\mathbf{V}_1$, associated with the first leg, is the direct sum of two subalgebras $x_{\hat{k}} \oplus r_{D_{1_b}}$. The first subalgebra, $x_{\hat{k}}$ represents the subalgebra of Schönflies motions with rotation about the Z-axis. This subalgebra is generated by the first four columns. The second subalgebra $r_{D_{1_b}}$ represents a rotation around an axis parallel to the Y-axis intersecting point

D_1. The screw that generates this last subalgebra is located in the fifth column of the Jacobian matrix J_1.

Stage II of the method computes the intersection of the subspaces associated with the Jacobian matrices of the legs

$$\left[J_n^{m/f}\right] = \bigcap_{i=1}^{3} \left[J_i^{m/f}\right] = [J_1] \cap [J_2] \cap [J_3] = [J_2] = [J_3] = \mathbf{V}_2 \tag{5}$$

The intersection of subspaces provides the motion of the moving platform relative to the fixed platform. It is the column space of the matrix $J_n^{m/f}$ denoted $\left[J_n^{m/f}\right]$. It represents a planar translation in the plane XZ together with rotations around the X and Y-axes. Further, it can be proved that the mobility of the parallel platform is given by

$$F = dim\,[J_n^{m/f}] = rank\, J_n^{m/f} = rank\,(J_2) = rank\,(J_3) = 4 \tag{6}$$

The characteristics of this platform are similar to that of the previous section; i.e. a trivial platform of Tanev's type. However, in this case, none of the legs generate a subalgebra, but the first connecting chain generates a screw system of locally constant rank that contains the screw system of the remaining two serial chains. Furthermore, legs 2 and 3 generate the same screw system.

4.2 Parallel Platform 4-RRUR.

The 4-RRUR parallel platform shown on the right side of figure 5 was proposed by Fang and Tsai, [33]. The platform has 4 legs with the same topology indicated on the left side of figure 5. The kinematic joints located at the corresponding point C of each leg are Hooke joints that can be replaced by two intersecting revolute pairs whose directions are perpendicular. The platform has 18 links and 20 revolute pairs. The application of the Kutzbach-Grübler criterion indicates that the platform has 2 DOF. A Cartesian coordinate system $OXYZ$ is attached to the fixed platform for reference purposes.

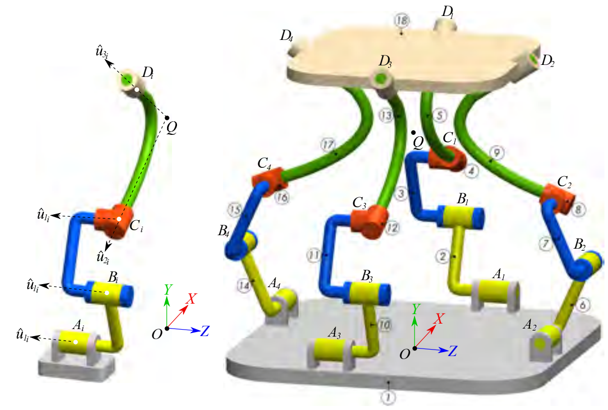

Fig. 5. The topology of the i-th leg of a parallel platform R-R-U-R and the complete platform. The leg illustrated corresponds to the third leg.

The characteristics of the kinematic pairs are:

1. The revolute joints located at points A_i, and B_i, and the first revolute joint of the Hooke joint at point C_i are parallel, and its unit vector is given by $\hat{u}_{1i}$. The unit vectors $\hat{u}_{1i}$ for legs 1 and 3 are parallel to the Z-axis; while the unit vectors $\hat{u}_{1i}$ for legs 2 and 4 are parallel to the X-axis.
2. The direction of the second revolute joint of the Hooke joint at point C_i, represented by the unit vector $\hat{u}_{2_i}$, is perpendicular to $\hat{u}_{1_i}$ and also to $\hat{u}_{3_i}$, which is associated to the revolute joint located at D_i. It should be noted that the axes of these two revolute joints intersect at point Q.

Finally, point Q is assumed to be attached to the moving platform and it is used to represent the relative motion between the moving and the fixed platforms. Its position vector relative to origin O of the coordinate system $OXYZ$ is given by $\vec{r}_Q = (0, 150, 0)^T$.

The screws coordinates, reference points of the kinematic pairs of each connecting chain, and the corresponding Jacobian matrices are given now. The nomenclature follows figure 5.

For leg 1. $A_1 = (100, 20, 30), {}^1\$^2; B_1 = (120, 65, 0), {}^2\$^3; C_1 = (100, 110, 0), {}^3\$^4; {}^4\$^5; D_1 = (0, 150, 0),\ {}^5\$^{18}$.

$$J_1 = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & \frac{5}{\sqrt{29}} & \frac{3}{\sqrt{13}} \\ 0 & 0 & 0 & -\frac{2}{\sqrt{29}} & \frac{2}{\sqrt{13}} \\ 1 & 1 & 1 & 0 & 0 \\ 20 & 65 & 110 & 0 & 0 \\ -100 & -120 & -100 & 0 & 0 \\ 0 & 0 & 0 & -\frac{750}{\sqrt{29}} & -\frac{450}{\sqrt{13}} \end{array}\right],$$

For leg 2. $A_2 = (-30, 20, 100), {}^1\$^6; B_2 = (0, 65, 120), {}^6\$^7; C_2 = (0, 110, 100), {}^7\$^8; {}^8\$^9; D_2 = (0, 150, 0), {}^9\$^{18}$.

$$J_2 = \left[\begin{array}{ccc|cc} -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{\sqrt{29}} & \frac{2}{\sqrt{13}} \\ 0 & 0 & 0 & \frac{5}{\sqrt{29}} & \frac{3}{\sqrt{13}} \\ 0 & 0 & 0 & \frac{750}{\sqrt{29}} & \frac{450}{\sqrt{13}} \\ -100 & -120 & -100 & 0 & 0 \\ 20 & 65 & 110 & 0 & 0 \end{array}\right],$$

For leg 3. $A_3 = (-100, 20, -30), {}^1\$^{10}; B_3 = (-120, 65, 0), {}^{10}\$^{11}; C_3 = (-100, 110, 0), {}^{11}\$^{12}; {}^{12}\$^{13}; D_3 = (0, 150, 0), {}^{13}\$^{18}$.

$$J_3 = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & -\frac{5}{\sqrt{29}} & -\frac{3}{\sqrt{13}} \\ 0 & 0 & 0 & -\frac{2}{\sqrt{29}} & \frac{2}{\sqrt{13}} \\ -1 & -1 & -1 & 0 & 0 \\ -20 & -65 & -110 & 0 & 0 \\ -100 & -120 & -100 & 0 & 0 \\ 0 & 0 & 0 & \frac{750}{\sqrt{29}} & \frac{450}{\sqrt{13}} \end{array}\right],$$

For leg 4. $A_4 = (30, 20, -100), {}^1\$^{14}; B_4 = (0, 65, -120), {}^{14}\$^{15}; C_4 = (0, 110, -100), {}^{15}\$^{16}; {}^{16}\$^{17}, D_4 = (0, 150, 0), {}^{17}\$^{18}$.

$$J_4 = \left[\begin{array}{ccc|cc} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{\sqrt{29}} & \frac{2}{\sqrt{13}} \\ 0 & 0 & 0 & -\frac{5}{\sqrt{29}} & -\frac{3}{\sqrt{13}} \\ 0 & 0 & 0 & -\frac{750}{\sqrt{29}} & -\frac{450}{\sqrt{13}} \\ -100 & -120 & -100 & 0 & 0 \\ -20 & -65 & -110 & 0 & 0 \end{array}\right].$$

The rank of all the Jacobian matrices is equal to 5. Since there are no subalgebras of dimension 5 in $se(3)$, none of the legs are the mechanical generators of any subalgebra of $se(3)$. The matrices only generate a subspace of $se(3)$. Hence, stage I of the method is unable to compute the mobility of the parallel platform.

Stage II of the method shows that the column space each of the Jacobian matrices is the non-direct sum of two subalgebras of $se(3)$ and, as a consequence, the column space forms screw systems of locally constant rank.

1. The screws of the revolute joints located in A_i, and B_i, and the first revolute joint of the Hooke joint indicated by C_{ia}, are parallel to the unit vector $\hat{u}_{1_i}$ appear in the first three columns of the Jacobian matrix. These screws generate the subalgebra of planar displacements $g_{\hat{u}_{1_i}}$ in a plane perpendicular to $\hat{u}_{1_i}$.
2. The screws associated with the second revolute of the Hooke joint indicated by C_{ib} and the revolute located in D_i appear in the last two columns of the Jacobian matrix. These screws generate a subspace $\mathbf{V}_{2_i}$ contained in the subalgebra s_Q of spherical displacements around point Q.

The red dashed line, in the Jacobian matrices, "separates" the complete subalgebra from the subspace.

Therefore, the column space of each Jacobian matrix forms a non-direct sum $g_{\hat{u}_{1_i}} + s_Q$, since $g_{\hat{u}_{1_i}} \cap s_Q \neq \emptyset$. The linear complement for the subspace $\mathbf{V}_{2_i}$ to generate the subalgebra s_Q is the screw of a revolute pair that passes through point Q and its direction is parallel to the unit vector $\hat{u}_{1_i}$. This screw is precisely the intersection of $g_{\hat{u}_{1_i}} \cap s_Q$.

The screws for the linear complement of $\mathbf{V}_{2_i}$ relative to s_Q for each of the legs are given by:

1. Leg 1. ${}^{4a}\$^{4b} = \left[0,\, 0,\, -1;\, -150,\, 0,\, 0\right]^T$
2. Leg 2. ${}^{8a}\$^{8b} = \left[1,\, 0,\, 0;\, 0,\, 0,\, -150\right]^T$
3. Leg 3. ${}^{12a}\$^{12b} = \left[0,\, 0,\, 1;\, 150,\, 0,\, 0\right]^T$
4. Leg 4. ${}^{16a}\$^{16b} = \left[-1,\, 0,\, 0;\, 0,\, 0,\, 150\right]^T$

If one wishes to exhibit the two subalgebras associated with each leg, the screw must be "virtually" placed in the middle of the third and fourth columns, in the corresponding matrix, J_i.

Then, the motion of the moving platform relative to the fixed platform is obtained by intersecting the column spaces of the Jacobian matrices of the legs J_i therefore

$$\left[J_n^{m/f}\right] = \bigcap_{i=1}^{4} [J_i] = [J_1] \cap [J_2] \cap [J_3] \cap [J_4] = \mathbf{V}. \qquad (7)$$

The result is given by

$$J_n{}^{m/f} = \begin{bmatrix} 0 & 0 & \frac{9}{13} & \frac{9}{26} \\ \frac{2}{\sqrt{13}} & \frac{2}{\sqrt{29}} & 0 & 0 \\ -\frac{3}{\sqrt{13}} & \frac{5}{\sqrt{29}} & 0 & 0 \\ -\frac{450}{13} & \frac{750}{\sqrt{29}} & 0 & 0 \\ 0 & 0 & \frac{-900}{13} & -\frac{710}{13} \\ 0 & 0 & -\frac{1350}{13} & -\frac{675}{13} \end{bmatrix} \qquad (8)$$

The columns of the matrix $J_n{}^{m/f}$, see equation (5), indicate that the parallel platform has mobility 4. However, the columns of $J_n{}^{m/f}$ do not provide a clear interpretation of the relative motion between the platforms. To address this issue, Rico and Duffy, [34], resorted to the definition of a screw system as a subspace $\mathbf{V}$ of the Lie algebra, $se(3)$, of $SE(3)$. Hence, the screw system has an infinite number of bases. Thus, the last step is to find another basis of the column space $\left[J_n^{m/f}\right] = [\$_1, \$_2, \$_3, \$_4]$ that provides a simpler interpretation of the relative motion between the moving and fixed platforms. This new basis is given by the column space of a new matrix $J_{n1}^{m/f}$ or, equivalently, the space generated by the matrix' columns:

$$\left[J_{n1}^{m/f}\right] = [\$_{1n}, \$_{2n}, \$_{3n}, \$_{4n}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{13}} & \frac{2}{\sqrt{29}} \\ 0 & 0 & -\frac{3}{\sqrt{13}} & \frac{5}{\sqrt{29}} \\ 0 & 0 & -\frac{450}{\sqrt{13}} & \frac{750}{\sqrt{29}} \\ 1 & 0 & 0 & 0 \\ 0 & -150 & 0 & 0 \end{bmatrix}, \qquad (9)$$

where[3]

$$\$_{1n} = \frac{13}{520}\$_3 - \frac{26}{520}\$_4,\ \$_{2n} = \frac{923}{234}\$_3 - 5\$_4,\ \$_{3n} = \$_1,\ \$_{4n} = \$_2.$$

From equation (9), it follows that the parallel platform 4RRUR features a **screw system of locally constant rank** and it has 4 degrees of freedom, the rank of the matrix $J_n^{m/f}$ and $J_{n1}^{m/f}$. Furthermore, the motion of the moving platform relative to the fixed platform consists of a translation along the Y-axis together with any spherical displacement centered in point Q.

4.3 Exechon Parallel Platform

The mobility of the Exechon parallel platform, shown in figure 6, has been studied by Bi and Jin, [36] and Zoppi et

[3]The process is not included due to space considerations. However, the step-by-step process is thoroughly described in Sánchez-García et al [35].

al. [37] using heuristic methods. The platform has 3 legs. Two of these legs have the same topology UPR, while the remaining leg has an SPR topology. The Hooke joints can be replaced by two intersecting revolute pairs whose directions are perpendicular. The platform has 10 links, 10 revolute pairs, and 1 spherical pair. The application of the Kutzbach-Grübler criterion indicates that the platform has 1 DOF. A Cartesian coordinate system $OXYZ$ is attached to the fixed platform for reference purposes.

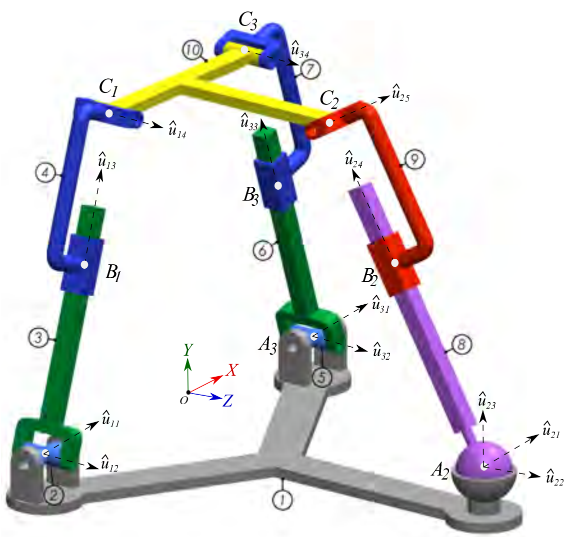

Fig. 6. Parallel platform Exechon.

The topology of the two UPR legs, denoted as 1 and 3, are similar. The axes of the first revolute joint of all the Hooke joints, located at points A_i, are coaxial and in the direction of the unit vector $\hat{u}_{i1}$ parallel to the X-axis. The second revolute joint of the Hooke joints located at point A_i and the revolute joints located at point C_i are parallel. These revolute joints have the direction indicated by the unit vector $\hat{u}_{i2}$, parallel to the Z-axis. The directions of the prismatic pairs located at B_i are different but both lie in the XY plane.

In the topology of the SPR leg, denoted as 2, one can assume that the spherical pair, located at point A_2, is replaced by three revolute joints whose directions are parallel to the X, Y, and Z-axes respectively. The prismatic pair, located at point B_2, has a general direction in the YZ plane. Finally, the revolute pair, located at point C_2, has the direction indicated by the unit vector $\hat{u}_{25}$, parallel to the X-axis.

The screws coordinates, reference points of the kinematic pairs of each connecting chain, and the corresponding Jacobian matrices are given now. The nomenclature follows figure 6.

For leg 1. $A_1 = (-10,0,0), {}^1\$^2; {}^2\$^3; B_1, {}^3\$^4; C_1 = (-5,15,0), {}^4\$^5$.

For leg 3. $A_3 = (10,0,0), {}^1\$^5; {}^5\$^6; B_3, {}^6\$^7; C_3 = (5,15,0), {}^7\$^{10}$.

$$J_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{\sqrt{10}} & 15 \\ 0 & 10 & \frac{3}{\sqrt{10}} & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad J_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -\frac{1}{\sqrt{10}} & 15 \\ 0 & -10 & \frac{3}{\sqrt{10}} & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

For leg 2. $A_2 = (0,0,10\sqrt{3}), {}^1\$^{2a}; {}^{2a}\$^{2b}; {}^{2b}\$^8; B_2, {}^8\$^9; C_2 = (0,15,5\sqrt{3}), {}^9\$^{10}$.

$$J_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -10\sqrt{3} & 0 & 0 \\ 10\sqrt{3} & 0 & 0 & \frac{\sqrt{3}}{2} & 5\sqrt{3} \\ 0 & 0 & 0 & -\frac{1}{2} & -15 \end{bmatrix}$$

The rank of both matrices J_1 and J_3 are 4. They do not generate a subalgebra of $se(3)$, since they include a rotation around two independent axes. Similarly, the rank of matrix J_2 is 5, and it does not generate a subalgebra of $se(3)$. Therefore, stage I of the method is unable to determine the platform's mobility. However, stage II of the method determines that the subspaces $\mathbf{V}_1 = \mathbf{V}_3$. These subspaces of $se(3)$, generated by legs 1 and 3, are the direct sum of two subalgebras $r_{O,\hat{i}} \oplus g_{\hat{k}}$ of $se(3)$; a subalgebra associated with a rotation around the X-axis that passes through point O and the subalgebra of planar displacements in a plane perpendicular to the Z-axis. The subspace generated by leg 2, $\mathbf{V}_2$, represents a non-direct sum of two subalgebras $s_{A_2} + g_{\hat{i}}$. The first subalgebra s_{A_2} represents the spherical subalgebra associated with point A_2. The second subalgebra $g_{\hat{i}}$, represents the subalgebra of planar displacements in a plane perpendicular to the X-axis.

Then, the motion of the moving platform relative to the fixed platform is obtained by intersecting the column spaces of the Jacobian matrices of the legs J_i therefore

$$\left[J_n^{m/f}\right] = \bigcap_{i=1}^{3}[J_i] = [J_1] \cap [J_2] \cap [J_3] = \mathbf{V} \tag{10}$$

The vector subspace $\mathbf{V}$ is given by the column space of the matrix J_n. The columns of J_n form a basis for $\mathbf{V}$

$$J_n{}^{m/f} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{11}$$

From this result, it follows that the moving platform of the Exechon can translate along the Z-axis and rotate along the X and Y-axes passing through the origin O. Thus, it has three degrees of freedom.

It should be noted that the intersection of the column spaces of the Jacobian matrices of the legs J_i cannot be blindly computed. The approach followed by Huang and his coworkers, see for example [13], rely on the constraint screws associated with each leg. Huang's approach is equivalent to employing only the first order or velocity analysis of parallel platforms; and it is, in some cases as reported in [16], unable to distinguish between mobile linkages or platforms and structures.

5 Parallel platforms whose mobility can be determined using stage III of the method.

In this section, an example of a parallel platform whose mobility is computed by determining the velocity conditions of the kinematic pairs of at least some connecting chains of the platform will be presented.

5.1 Parallel platform 3T-R1-B.b

Figure 7 presents the parallel platform 3T-R1-B.b proposed by Ting-Li [20]. The platform has 4 connecting chains or legs with the same topology; each leg contains a closed chain. The numbering of the legs is indicated by the common subscript of the points associated with the kinematic pair of the leg. The fixed and moving platforms are links 1 and 22 respectively. The platform has 22 links and 28 kinematic pairs, 8 of them spherical pairs, all the remainder revolute pairs. The application of the Kutzbach-Grübler criterion indicates that the platform has 2 DOF. A coordinate system $OXYZ$ has been rigidly attached to the fixed platform.

The kinematic topology of the i-th connecting chain is shown in figure 8, and it corresponds to leg 2 of the platform shown in figure 7. Moreover, all the characteristics and results are also valid for the remaining legs.

It should be noted that the axes of the revolute pairs A_i and B_i of legs 1, 3, and 2, 4 are parallel to $\hat{u}_{1_i}$, see figure 7. However, this direction changes for each pair of legs. On the contrary, the direction of all the revolute joints G_i, whose unit vector is $\hat{u}_{3_i}$, is parallel to the Y-axis.
As shown in figure 8 the kinematic pairs E_i and D_i are spherical joints that can be replaced by 3 revolute pairs whose axes are linearly independent. Therefore, the directions of the three revolute pairs that substitute the spherical pair are:

1. The unit vector $\hat{u}_{2_i}$ is parallel to the rotation axes of the revolute joints at C_i and F_i. The unit vector is different with each connecting chain.
2. Another revolute joint at E_i and D_i is parallel to the unit vector $\hat{u}_{1_i}$ of each extremity.
3. The final revolute joint at E_i and D_i is parallel to $\hat{u}_{3_i}$, and therefore to the Y-axis.

The screws coordinates, reference points of the kinematic pairs of each connecting chain, and the corresponding

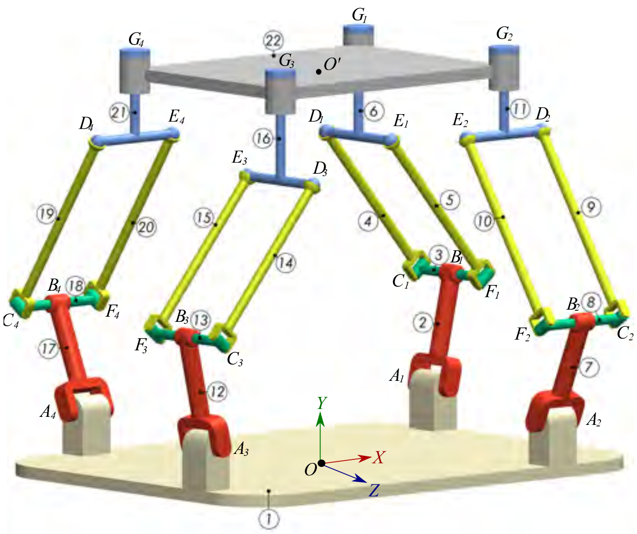

Fig. 7. Parallel platform 3T-R1-B.b of 4 DOF.

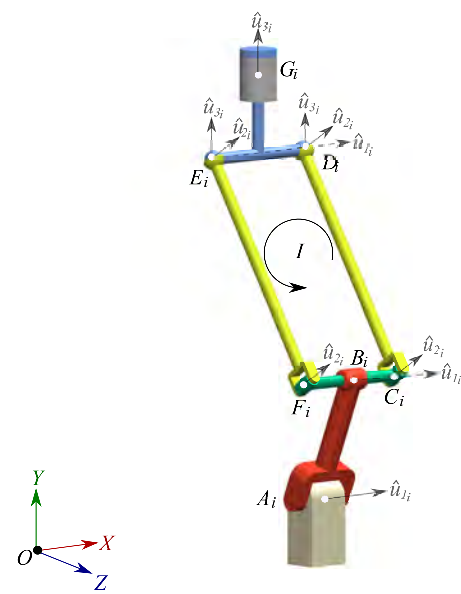

Fig. 8. Structure of the i-th serial connector chain of the parallel platform 3T-R1-B.b.

matrices are given now. The nomenclature follows figure 7.

For leg 1. $A_1 = (200, 40, -90), {}^{1}\$^{2}; B_1 = (220, 130, -90), {}^{2}\$^{3}; C_1 = (220, 130, -130), {}^{3}\$^{4}; D_1 = (120, 250, -130), {}^{4a}\$^{6a}; {}^{4b}\$^{6b}; {}^{4c}\$^{6c}; E_1 = (120, 250, -50), {}^{6a}\$^{5a}; {}^{6b}\$^{5b}; {}^{6c}\$^{5c}; F_1 = (220, 130, -50), {}^{5}\$^{3}; G_1 = (120, 350, -90), {}^{6}\$^{22}$

$$J_1 = \left[\begin{array}{cc|cccccccc|c} 0 & 0 & \frac{6}{\sqrt{61}} & 0 & \frac{6}{\sqrt{61}} & 0 & 0 & \frac{6}{\sqrt{61}} & 0 & \frac{6}{\sqrt{61}} & 0 \\ 0 & 0 & \frac{5}{\sqrt{61}} & 0 & \frac{5}{\sqrt{61}} & 1 & 0 & \frac{5}{\sqrt{61}} & 1 & \frac{5}{\sqrt{61}} & 1 \\ -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -40 & -130 & \frac{650}{\sqrt{61}} & -250 & \frac{650}{\sqrt{61}} & 130 & -250 & \frac{250}{\sqrt{61}} & 50 & \frac{250}{\sqrt{61}} & 90 \\ 200 & 220 & -\frac{780}{\sqrt{61}} & 120 & -\frac{780}{\sqrt{61}} & 0 & 120 & -\frac{300}{\sqrt{61}} & 0 & -\frac{300}{\sqrt{61}} & 0 \\ 0 & 0 & \frac{320}{\sqrt{61}} & 0 & -\frac{900}{\sqrt{61}} & 120 & 0 & -\frac{900}{\sqrt{61}} & 120 & \frac{320}{\sqrt{61}} & 120 \end{array}\right]$$

For leg 2. $A_2 = (120, 40, 150), {}^{2}\$^{7}; B_2 = (120, 124, 186), {}^{7}\$^{8}; C_2 = (160, 124, 186), {}^{8}\$^{9}; D_2 = (160, 268, 90), {}^{9a}\$^{11a};$

${}^{9b}\$^{11b};{}^{9c}\$^{11c};E_2=(80,268,90),{}^{11a}\$^{10c};{}^{11b}\$^{10b};{}^{11c}\$^{10c};$
$F_2=(80,124,186),{}^{10}\$^{8};G_2=(120,320,90),{}^{11}\$^{22}$

$$J_2=\left[\begin{array}{cc|cccccccc|c}
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{2}{\sqrt{13}} & 0 & \frac{2}{\sqrt{13}} & 1 & 0 & \frac{2}{\sqrt{13}} & 1 & \frac{2}{\sqrt{13}} & 1\\
0 & 0 & \frac{3}{\sqrt{13}} & 0 & \frac{3}{\sqrt{13}} & 0 & 0 & \frac{3}{\sqrt{13}} & 0 & \frac{3}{\sqrt{13}} & 0\\
0 & 0 & 0 & 0 & 48\sqrt{13} & -90 & 0 & 48\sqrt{13} & -90 & 0 & -90\\
150 & 186 & -\frac{480}{\sqrt{13}} & 90 & -\frac{480}{\sqrt{13}} & 0 & 90 & -\frac{240}{\sqrt{13}} & 0 & -\frac{240}{\sqrt{13}} & 0\\
-40 & -124 & \frac{320}{\sqrt{13}} & -268 & \frac{320}{\sqrt{13}} & 160 & -268 & \frac{160}{\sqrt{13}} & 80 & \frac{160}{\sqrt{13}} & 120
\end{array}\right]$$

For leg 3. $A_3=(-200,40,90),{}^{1}\$^{12};B_3=(-220,130,90),$
${}^{12}\$^{13};C_3=(-220,130,130),{}^{13}\$^{14};D_3=(-120,250,130),$
${}^{14a}\$^{16a};{}^{14b}\$^{16b};{}^{14c}\$^{16c};E_3=(-120,250,50),{}^{16a}\$^{15a};$
${}^{16b}\$^{15b};{}^{16c}\$^{15c};F_3=(-220,130,50),{}^{15}\$^{13};$
$G_3=(-120,320,90),{}^{16}\$^{22}$

$$J_3=\left[\begin{array}{cc|cccccccc|c}
0 & 0 & -\frac{6}{\sqrt{61}} & 0 & -\frac{6}{\sqrt{61}} & 0 & 0 & -\frac{6}{\sqrt{61}} & 0 & -\frac{6}{\sqrt{61}} & 0\\
0 & 0 & \frac{5}{\sqrt{61}} & 0 & \frac{5}{\sqrt{61}} & 1 & 0 & \frac{5}{\sqrt{61}} & 1 & \frac{5}{\sqrt{61}} & 1\\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
40 & 130 & -\frac{650}{\sqrt{61}} & 250 & -\frac{650}{\sqrt{61}} & -130 & 250 & -\frac{250}{\sqrt{61}} & -50 & -\frac{250}{\sqrt{61}} & -90\\
200 & 220 & -\frac{780}{\sqrt{61}} & 120 & -\frac{780}{\sqrt{61}} & 0 & 120 & -\frac{300}{\sqrt{61}} & 0 & -\frac{300}{\sqrt{61}} & 0\\
0 & 0 & -\frac{320}{\sqrt{61}} & 0 & \frac{900}{\sqrt{61}} & -120 & 0 & \frac{900}{\sqrt{61}} & -120 & -\frac{320}{\sqrt{61}} & -120
\end{array}\right]$$

For leg 4. $A_4=(-120,40,-150),{}^{1}\$^{17};$
$B_4=(-120,124,-186),{}^{17}\$^{18};C_4=(-160,124,-186),$
${}^{18}\$^{19};D_4=(-160,268,-90),{}^{19a}\$^{21a};{}^{19b}\$^{21b};{}^{19c}\$^{21c};$
$E_4=(-80,268,-90),{}^{21a}\$^{20a};{}^{21b}\$^{20b};{}^{21c}\$^{20c};$
$F_4=(-80,124,-186),{}^{20}\$^{28};G_4=(-120,320,-90),{}^{21}\$^{22}$

$$J_4=\left[\begin{array}{cc|cccccccc|c}
-1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{2}{\sqrt{13}} & 0 & \frac{2}{\sqrt{13}} & 1 & 0 & \frac{2}{\sqrt{13}} & 1 & \frac{2}{\sqrt{13}} & 1\\
0 & 0 & -\frac{3}{\sqrt{13}} & 0 & -\frac{3}{\sqrt{13}} & 0 & 0 & -\frac{3}{\sqrt{13}} & 0 & -\frac{3}{\sqrt{13}} & 0\\
0 & 0 & 0 & 0 & -48\sqrt{13} & 90 & 0 & -48\sqrt{13} & 90 & 0 & 90\\
150 & 186 & -\frac{480}{\sqrt{13}} & 90 & -\frac{480}{\sqrt{13}} & 0 & 90 & -\frac{240}{\sqrt{13}} & 0 & -\frac{240}{\sqrt{13}} & 0\\
40 & 124 & -\frac{320}{\sqrt{13}} & 268 & -\frac{320}{\sqrt{13}} & -160 & 268 & -\frac{160}{\sqrt{13}} & -80 & -\frac{160}{\sqrt{13}} & -120
\end{array}\right]$$

Pseudo Jacobian matrices of the connecting chains are shown next. Their common rank of these matrices is 5. They are not actual Jacobian matrices due to the presence of the screws associated with the kinematic pairs within the closed-loop of each connecting chain. These screws correspond from the third to the tenth columns "bracketed" by red and dashed lines.

All the Jacobian matrices associated with each of the connecting chains have rank 5. Since $se(3)$ does not have a subalgebra of dimension 5, stage I of the method is unable to determine the mobility of the parallel platform. Similarly, stage II of the method is unable to find screw systems of locally constant rank that can be expressed as a sum, direct or not, of two subalgebras of $se(3)$. One can assume that the presence of the closed-loop makes it impossible to find the sum of any subalgebras.

As already indicated, each connecting chain has a closed-loop formed by the parallelogram C_i, D_i, E_i, F_i. The corresponding Jacobian matrix is given by the 8 screws of the two revolute and two spherical pairs that belong to the loop. Therefore, stage III of the method is employed to find, if possible, velocity conditions that may reduce the pseudo-Jacobian matrices. In the case of closed-loops, the velocity conditions are obtained by carrying out the velocity analysis of the closed-loop given by

$$\begin{aligned}&\omega_{C_i}\$_{C_i}+\omega_{D_{ia}}\$_{D_{ia}}+\omega_{D_{ib}}\$_{D_{ib}}+\omega_{D_{ic}}\$_{D_{ic}}\\&+\omega_{E_{ia}}\$_{E_{ia}}+\omega_{E_{ib}}\$_{E_{ib}}+\omega_{E_{ic}}\$_{E_{ic}}+\omega_{F_i}\$_{F_i}=\vec{0}\end{aligned}\qquad(12)$$

where the actual screw in equation (12), depends on the corresponding leg. The velocity analysis yields the following conditions:

$$\omega_{D_{ib}}=\omega_{F_i}\;=\;-\omega_{C_i}=-\omega_{E_{ib}},\qquad(13)$$
$$\omega_{D_{ia}}=-\omega_{E_{ia}},\qquad \omega_{D_{ic}}=\omega_{E_{ic}}=0\qquad(14)$$

The reduced screw system associated with the closed loop becomes

$$\omega_{C_i}\left[\left(\$_{D_{ib}}-\$_{C_i}\right)-\left(\$_{F_i}-\$_{E_{ib}}\right)\right]+\omega_{D_{ia}}\left(\$_{D_{ia}}-\$_{E_{ia}}\right)\qquad(15)$$

Equation (13) represents the relationship between the angular velocities of the revolute pairs parallel to $\hat{u}_{2_i}$ that belong to loop I, shown in figure 8. Similarly, the first equation in (14) represents the relationship between the angular velocities of the revolute pairs parallel to $\hat{u}_{1_i}$ that belong to loop I. Finally, the second equation in equation (14) indicates that both angular velocities are zero. Hence the screws $\$_{D_{ic}}$ and $\$_{E_{ic}}$ will be discarded in the analysis. Furthermore

$$\$_{R1}=\$_{D_{ib}}-\$_{C_i}=\$_{F_i}-\$_{E_{ib}}\quad\text{and}\quad\$_{R2}=\$_{D_{ia}}=\$_{E_{ia}}$$

These two "equivalent" screws substitute the columns 3 to 10 of these pseudo-Jacobian matrices and also appear between red dashed lines. The results of this step are the following reduced Jacobian matrices whose ranks are all equal to 5.

$$J_{1r}=\left[\begin{array}{cc|cc|c}
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1\\
-1 & -1 & 0 & -1 & 0\\
-40 & -130 & 0 & -250 & 90\\
200 & 220 & 0 & 120 & 0\\
0 & 0 & 20\sqrt{61} & 0 & 120
\end{array}\right],\;J_{2r}=\left[\begin{array}{cc|cc|c}
1 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & -48\sqrt{13} & 0 & -90\\
150 & 186 & 0 & 90 & 0\\
-40 & -124 & 0 & -268 & 120
\end{array}\right]$$

$$J_{3r}=\left[\begin{array}{cc|cc|c}
0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1\\
1 & 1 & 0 & 1 & 0\\
40 & 130 & 0 & 250 & -90\\
200 & 220 & 0 & 120 & 0\\
0 & 0 & -20\sqrt{61} & 0 & -120
\end{array}\right],\;J_{4r}=\left[\begin{array}{cc|cc|c}
-1 & -1 & 0 & -1 & 0\\
0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & 48\sqrt{13} & 0 & 90\\
150 & 186 & 0 & 90 & 0\\
40 & 124 & 0 & 268 & -120
\end{array}\right]$$

Since the rank of all the pseudo-Jacobian matrices is 5, and there is no a subalgebra of $se(3)$ of dimension 5, it is clear that the connecting chains do not generate a subalgebra of $se(3)$. Therefore, the application of stage I of the method is, again, unable to compute the mobility of the parallel platform. However, stage II of the method reveals that the subspace generated by each connecting chain is given by the direct sum $x_{\hat{u}_{1_i}} \oplus r_{G_i, \hat{u}_{3_i}}$ of a Schönflies subalgebra and a revolute pair. The unit vector $\hat{u}_{1_i}$ is $\hat{k}$ for legs 1 and 3; and $\hat{i}$ for legs 2 and 4. The revolute $r_{G_i, \hat{u}_{3_i}}$ has, also the common direction, $\hat{j}$, but the location of the revolute axis is different for each chain.

In that case, the mobility of the platform can be found by the intersection of the subspaces of $se(3)$ associated with the matrices J_i obtained after applying the conditions revealed by the velocity analysis; namely

$$[J_f] = \bigcap_{i=1}^{4} [J_{ir}] = [J_{1r}] \cap [J_{2r}] \cap [J_{3r}] \cap [J_{4r}] = x_{\hat{j}} \qquad (16)$$

The relative motion between the platforms is given the subalgebra of $se(3)$ generated by the columns of matrix J_f. It should be noted that these columns are linearly independent

$$J_f = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -90 & 0 & -1 & 0 \\ 0 & -5 & 0 & 3 \\ 120 & -19 & 0 & -7 \end{bmatrix} \qquad (17)$$

It is straightforward to realize that the relative motion is a Schönflies $x_{\hat{j}}$ subalgebra that represents a spatial translation together with a rotation around the Y-axis. Thus, the mobility of the parallel platform, which is equal to the rank of J_f is 4. Hence, the parallel platform 3T-R1-B.b has 4 degrees of freedom. The same result was obtained by Yang et al. [20], but in this contribution, the result was obtained following a simpler process based only on the properties of the screw algebra.

6 Conclusions.

The method presented in this contribution provides a well-founded theoretical approach the correct determination of the mobility of a wide range of fully parallel platforms. The theoretical basis of the method is a careful application of the Lie algebra $se(3)$ of the special Euclidean group, $SE(3)$, isomorphic to screw algebra. The only requirements are a sound understanding of the subalgebras and subspaces of $se(3)$, the concept of sum of subalgebras, and the conditions imposed by the presence of closed chains in the legs of the parallel platforms. The method can be implemented using symbolic algebra software, so that tedious computations can be completely avoided. It must be noted that the method presented in this contribution determines, without any problem, the mobility of all parallel platforms proposed by Yang et al. [20], without resorting to the rather complex variety of conditions presented there. Finally, this contribution reveals the need for a more complete classification of fully parallel platforms, and for a renewed version of the work of Gogu, [1], that should review critically the newly developed mobility criteria. Both endeavors remain important, yet unfulfilled, tasks.

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References

[1] Gogu, G., 2005, Mobility of Mechanisms: A Critical Review, *Mech. Mach. Theory*, **40**, pp. 1068-1097.

[2] Hervé, J.M., 1978, Analyse Structurelle des Mécanismes par Groupe des Déplacements, *Mech. Mach. Theory*, **13**, pp. 437-450.

[3] Rico J.M., Jesús Cervantes-Sánchez J., Olivares-Conraud E., 2019, On the History of the Discovery of the Subgroups of the Euclidean Group. In: Uhl T. (eds) Advances in Mechanism and Machine Science. IFToMM WC 2019, Sringer, Cham, pp. 1233-1239

[4] Fanghella, P. and Galletti, C., 1995, Metric Relations and Displacement Groups in Mechanism and Robot Kinematics, *ASME J. Mech. Des.*, **117**, pp. 470-478.

[5] Fanghella, P. and Galletti, C., 1994, Mobility Analysis of Single-Loop Kinematic Chains: An Algorithmic Approach Based on Displacement Groups, *Mech. Mach. Theory*, **29**, pp. 1187-1204.

[6] Rico, J. M., Aguilera, L. D., Gallardo, J., Rodríguez, R., Orozco, H., and Barrera, J. M., 2006, A group theoretical derivation of a more general mobility criterion for parallel manipulators, *P*. I. Mech. Eng. C-J. Mech., **220**. pp. 969-987.

[7] Ball, R.S., 1900, *A Treatise on the Theory of Screws.* Cambridge: Cambridge University Press.

[8] Voinea, R. P. and Atanasiu, M. C., 1962, Théorie Géométrique des vis et Quelquels Applications á la Théorie des Mécanismes, *Revue Méc. appl. Buc.*, **7**, pp. 845-860.

[9] Hunt, K. H., 1978, *Kinematic Geometry of Mechanisms.* Oxford: Oxford University Press.

[10] Selig, J.M., 1996, *Geometrical Methods in Robotics*, New York: Springer Verlag.

[11] Rico, J. M., Aguilera, L. D., Gallardo, J., Rodríguez, R., Orozco, H. and Barrera J.M., 2006, A More General Mobility Criterion for Parallel Platforms, *ASME J. Mech. Des.* **128**, pp. 207-219.

[12] Huang, Z., Liu, J. F. and Zeng, D.X., 2009, A general methodology for mobility analysis of mechanisms based on constraint screw theory, *Sci. China Ser. E*, **52**, pp. 1337-1347.

[13] Huang, Z. and Ge. Q. J., 2006, A simple method for mobility analysis using reciprocal screws, ASME paper DETC-99677.

[14] Huang, Z. and Li, Q., 2008, A Unified Methodology for Mobility Analysis Based on Screw Theory, in *Smart Devices and Machines for Advanced Manufacturing*, Wang, L. and Xi, J. ed. Springer: London.

[15] Dai, J. S., Huang, Z. and Lipkin, H.,2006, Mobility of overconstrained parallel mechanisms, *ASME J. Mech. Des.*, **128**, pp. 220-229.

[16] Rico, J. M., Gallardo, J. and Ravani B., 2003, Lie Algebra and the Mobility of Kinematic Chains, *J. Robotic Syst.*, **17**, pp. 477-499.

[17] Yang, T.L., Liu A.X., Jin, Q., Lou, Y. F., Shen, H.P. and Hang, L.B., 2009, Position and orientation characteristic equation for topological design of robot mechanisms, *ASME J. Mech. Des.*, **131**, paper 02100b, p.p. 1-17.

[18] Yang, T.L. and Sun D.J., 2012, A general degree of freedom formula for parallel mechanisms and multiloop spatial mechanisms, *ASME J. Mech. Rob.*, Vol. 4, paper 011001, p.p. 1-17.

[19] Yang, T.L., Liu, A.X., Shen, H. P., Luo, Y. F. and Hang, L.B., 2013, On the correctness and strictness of the position and orientation characteristic equation for topological structure design of robot mechanisms, *ASME J. Mech. Rob.*, **5**, paper 021009, p.p. 1-18.

[20] Yang, T.L., Liu, A., Shen, H.P., Hang, L., Luo, Y., and Jin, Q., 2018, *Topology Design of Robot Mechanisms*, Springer: Singapore.

[21] Yang, T.L., Liu, A.X., Jin, Q., Lou, Y.F., Hang, L.B. and Shen, H.P., 2009, A systematical approach for structure synthesis of parallel mechanisms based on single-open-chain modules and its aplication, ASME paper DETC 2009-86106.

[22] Yang, T.L., Liu, A.X, Jin. Q, Lou, Y.F., Shen, H.P. and Hang L.B., 2009, Structure synthesis of rank-degenerated serial mechanisms and over-constrained single-loop kinematic chains, ASME paper DETC 2009-86105.

[23] Yang, T.L., Liu, A.X., Lou Y.F., Zhang, L.P., Shen, H.P. and Hang, L.B., 2009, Comparison study on three approaches for type synthesis of robot mechanisms, ASME paper DETC 2009-86107.

[24] Aguilera-Camacho D., 2003, *Mobility in Single-Loop Kinematic Chains and Parallel Platforms*, M. Sc. Thesis, Instituto Tecnólogico de Celaya, México, in Spanish.

[25] Huang, Z. and Li, Q. C., 2002, General Methodology for Type Synthesis of Symmetrical Lower-Mobility Parallel Manipulators and Several Novel, *Int. J. Rob. Res.*, **21**, pp. 131-145.

[26] Huang, Z. and Li, Q., 2003, Type Synthesis of Symmetrical Lower Mobility Parallel Mechanisms Using the Constraint Synthesis, *Int. J. Rob. Res.*, **22**, pp. 59-79.

[27] Tadeo-Chávez A. and Pérez-Soto G.I., 2006, *Number Synthesis of Kinematic Chains, a New Focus and New Mathematical Tools* M. Sc. Thesis, FIMEE, Universidad de Guanajuato, México, in Spanish.

[28] Rico, J. M., Cervantes, J. J., Tadeo, A., Pérez, G. I., and Rocha, J., 2008, New Considerations on the Theory of Type Synthesis of Fully Parallel Platforms, *ASME J. Mech. Des.* **130**, 112302 pp. 1-9.

[29] Tadeo-Chávez A., 2011, *Determination of the Subspaces of the Lie Algebra of the Euclidean Group that Preserve the Mobility of Kinematic Chains*, Ph. D. Dissertation, DICIS, Universidad de Guanajuato, México, in Spanish.

[30] Rico, J. M., Cervantes, J. J., Pérez-Soto, G. I., Tadeo, A. and A. Müller., 2011, Screw Systems Generated by Subalgebras: A Further Analysis, ASME Paper DETC2011-48304.

[31] Pérez-Soto G. I., 2013, *Mobility Analysis and Synthesis of Kinematic Chains by Means of Subspaces of Locally Constant Rank*, Ph. D. Dissertation, DICIS, Universidad de Guanajuato, México, in Spanish.

[32] Álvarez-Pérez M., 2018, *Design and Kinematic Analysis of a Parallel Robot for Ankle Rehabilitation*, M. Sc. Thesis, DICIS, Universidad de Guanajuato, México, in Spanish.

[33] Fang, Y., and Tsai, L., 2002, Structure Synthesis of a Class of 4-DoF and 5-DoF Parallel Manipulators with Identical Limb Structures, *Int. J. Rob. Res.*, **21**, pp. 799-810.

[34] Rico, J. M., and Duffy, J., 1998, A General Method for the Computation of the Canonical Form of Three-Systems of Infinitesimal Screws, *Robotica*, Vol. 16, pp. 37-45.

[35] Sánchez-García, A. J., Rico, J. M., Cervantes-Sánchez, J. J., and López-Custodio, P. C., 2020, A Mobility Determination Method for Parallel Platforms Based on the Lie Algebra of $SE(3)$ and its Subspaces, ASME Paper DETC2020/MECH-18676.

[36] Bi, Z. M., and Jin, Y., 2008, Kinematic Modeling of Exechon Parallel Kinematic Machine, *Rob. Comp. Integr. Manuf.*, **27**, pp. 186-193.

[37] Zoppi, M., Zlatanov, D. and Molfino, R., 2010, Kinematics Analysis of the Exechon Tripod, ASME Paper DETC2010-28668.