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Two-point Diagonally Implicit Multistep Block Method for Solving Robin Boundary Value Problems Using Variable Step Size Strategy

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This study focuses on the multistep integration method for approximating directly the solutions of the second order boundary value problems (BVPs) with Robin boundary conditions. The derivation of the predictor and corrector formulas uses Lagrange interpolation polynomial in the form of Adam's method. Two numerical solutions are computed concurrently within a block method with non-uniformly step size. The implementation of multistep block method follows the $PE(CE)^m$ procedure via shooting technique. Newton divided difference interpolation method is used during the iterative process for estimating the guessing values. The properties including the order, zero-stable and stability region of the proposed method are discussed. Numerical examples are given to demonstrate the computational efficiency of the developed method.

Keywords: boundary value problems; direct block method; robin boundary conditions

I. INTRODUCTION

This study sheds light on the direct integration for solving higher order boundary value problems (BVPs) associated with two point Robin boundary conditions. In the general form, this type of BVPs is written as

$$y''(x) = f(x, y, y') \text{ for } a \le x \le b \tag{1}$$

with

$$c_1 y'(a) + c_2 y(a) = \alpha \quad \text{and} \quad c_3 y'(b) + c_4 y(b) = \beta \tag{2}$$

where $c_1, c_2, c_3, c_4, \alpha$ and β are constants. Numerous computational methods have been invented to express the solutions of (1) subject to (2) focusing mainly on obtaining high accuracy results. Finite difference scheme has been expressed in details by Cuomo and Marasco, (2008) to experimentally solve Robin BVPs. On the other hand, Bernoulli polynomials and Quintic B-spline were carried out in Islam and Shirin, (2011) and Lang and Xu, (2012), respectively. Meanwhile, scholars in Duan *et al.*, (2013) and Rach *et al.*, (2016) provided the approximate analytical solution for this particular BVPs in the form of recursive scheme using Adomian decomposition method. Recent work discussed in Anakira *et al.*, (2017) introduced the multistage optimal homotopy asymptotic method (MOHAM) by partitioning the domains for treating second-order Robin type BVPs.

In this paper, we are interested in directly solving (1) using two-point diagonally block methods with various step size. Direct approach for solving second order differential equations using multistep block method have evolved continuously due to its efficiency in computing the numerical results. This is supported in the study reported by several scholars where solving second order problems directly were taken into their consideration, see for example Awoyemi *et al.*, (2011) and Waeleh and Majid, (2017). Vital findings obtained from the study conducted by Phang *et al.*, (2013) which used block method for solving Dirichlet and Neumann type BVPs directly with variable step size strategy has motivated us in conducting this research. Following from there, this work is an extension to the proposed method in Nasir *et al.*, (2018) which limits their approaches to only constant step size.

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II. METHODOLOGY

A. *Formulation of the method*

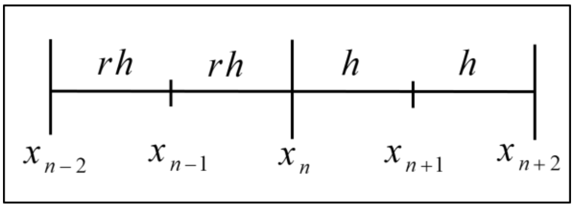

Figure 1. Two-point block method

Figure 1. visualizes the interval $[a,b]$ divided into a series of block where each block will compute two values simultaneously. The numerical solution for y_{n+1} and y_{n+2} at the point x_{n+1} and x_{n+2}, respectively are computed using the earlier values obtained at x_n, x_{n-1} and x_{n-2} from the previous block. The step size of the current block relies on the step size ratio, r defined by the previous block with the choices of r being $0.5, 1.0$ and 2.0. This implementation is known as the variable step size strategy.

The derivation of the formula for y_{n+1} and y_{n+2} involves the numerical integration and Lagrange interpolation polynomial process. Equation (1) is integrated once and twice over the interval $[x_n, x_{n+1}]$ and $[x_n, x_{n+2}]$ which yields the first and second point formula, respectively, given by

$$y'(x_{n+1}) - y'(x_n) = \int_{x_n}^{x_{n+1}} f(x,y,y')\,dx \tag{3}$$

$$y(x_{n+1}) - y(x_n) - hy'(x_n) = \int_{x_n}^{x_{n+1}} (x_{n+1} - x) f(x,y,y')\,dx \tag{4}$$

$$y'(x_{n+2}) - y'(x_n) = \int_{x_n}^{x_{n+2}} f(x,y,y')\,dx \tag{5}$$

$$y(x_{n+2}) - y(x_n) - 2hy'(x_n) = \int_{x_n}^{x_{n+2}} (x_{n+2} - x) f(x,y,y')\,dx. \tag{6}$$

Following from there, the integrand function, $f(x,y,y')$ in (3) to (6) will be approximated using Lagrange interpolation polynomial that interpolates the points $\{(x_{n+k}, f_{n+k})\}_{k=-2}^{k=1}$ for the first point and additional point (x_{n+2}, f_{n+2}) for the second point. Taking $s = \frac{x - x_{n+1}}{h}$ and replacing $dx = hds$, the evaluation of the integral from the limit -1 to 0 will be performed using MAPLE which yields the following corrector formula of y_{n+1}. The first point corrector formula is given by

$$\begin{aligned} y'_{n+1} = y'_n + \frac{h}{24(r+1)(r^2)}\big[&(r+1)f_{n-2} + (-2-8r)f_{n-1} \\ &+ (7r + 18r^2 + 12r^3 + 1)f_n + (12r^3 + 6r^2)f_{n+1}\big] \end{aligned} \tag{7}$$

$$\begin{aligned} y_{n+1} = y_n + hy'_n + \frac{h^2}{120r^2(r+1)(2r+1)}\big[&(7r + 5r^2 + 2)f_{n-2} \\ &+ (-28r - 40r^2 - 4)f_{n-1} + (89r^2 + 150r^3 + 80r^4 \\ &+ 21r + 2)f_n + (6r^2 + 30r^3 + 40r^4)f_{n+1}\big]. \end{aligned} \tag{8}$$

Now, by taking $s = \frac{x - x_{n+2}}{h}$ and substituting $dx = hds$, these replacements will change the limit of the integration to $[-2,0]$. Again, MAPLE is used to simplify the following corrector formula of y_{n+2}. The second point corrector formula yields the following

$$\begin{aligned} y'_{n+2} = y'_n + \frac{h}{15r^2(r+1)(2+r)(2r+1)}\big[&-(2+r)f_{n-2} \\ &+ (8r+4)f_{n-1} + (3r^2 + 35r^4 - 7r + 33r^3 + 10r^5 \\ &- 2)f_n + (48r^2 + 144r^3 + 140r^4 + 40r^5)f_{n+1} \\ &+ (33r^3 + 35r^4 + 10r^5 + 9r^2)f_{n+2}\big] \end{aligned} \tag{9}$$

$$\begin{aligned} y_{n+2} = y_n + 2hy'_n + \frac{h^2}{15r(2+r)(r+1)(2r+1)}\big[&(2+r)f_{n-2} \\ &- (16r+8)f_{n-1} + (91r^2 + 76r^3 + 41r + 20r^4 + 6)f_n \\ &+ (32r + 112r^2 + 128r^3 + 40r^4)f_{n+1} \\ &+ (6r^3 + 7r^2 + 2r)f_{n+2}\big]. \end{aligned} \tag{10}$$

The proposed method is called 2PDVS, which is designed via the combination of predictor and corrector formulas. The derivation of the predictor formula follows the simillar process as the corrector part but with the elimination of one interpolated point during the Lagrange approximation. Therefore, predictor formulas of y_{n+1} and y_{n+2} satisfies the explicit formulas. At the beginning of the computation, only one step method is used to provide a set of starting values to the proposed multistep method in order to initiate the computational procedure.

B. *Analysis of the method*

1. Order and error constant

Definition 1: Following the idea of hybrid multistep method as in Lambert, (1973) and Jator, (2010) the linear difference operator associated with (7) to (10) when substituting $r = 0.5$ is given by

$$\begin{aligned} L[y(x);h] = &\sum_{j=0}^{k}\big(\alpha_j y(x+jh) - h\beta_j y'(x+jh) - h^2\gamma_j y''(x+jh)\big) \\ &- h^2\sum_{j=1}^{k}\gamma_{vj} y''(x+jh) \end{aligned} \tag{11}$$

and the method satisfies order p if $C_0 = C_1 = \ldots = C_{p+1} = 0$ and $C_{p+2} \neq 0$.

By expanding and simplifying (11) using Taylor series about the x results in the following constant coefficients

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + \ldots + C_p h^p y^{(p)}(x) + \ldots \qquad (12)$$

where

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=0}^{k} (j\alpha_j - \beta_j)$$

$$C_2 = \sum_{j=0}^{k} \left(\frac{j^2}{2!}\alpha_j - j\beta_j - \gamma_j \right) - \sum_{j=0}^{k} \gamma_{v_j}$$

$$\vdots$$

$$C_p = \frac{1}{p!}\left[\sum_{j=0}^{k} \left(j^p \alpha_j - p j^{p-1}\beta_j - p(p-1) j^{p-2}(\gamma_j + \gamma_{v_j}) \right) \right], p = 3,4,\ldots$$

C_{p+2} is the error constant of the method while the local truncation error (LTE) of the method is given by

$$\text{LTE} = C_{p+2} h^{p+2} y^{(p+2)}(x_n) + o(h^{p+3}). \qquad (13)$$

Now, we apply (11) and (12) to our proposed corrector formulas with $r = 0.5$ which yields

$$C_0 = C_1 = C_2 = \ldots C_5 = [0,0,0,0]^T, C_6 = \left[-\frac{31}{2880}, -\frac{13}{2880}, 0, 0 \right]^T.$$

From Definition 1, the order of the proposed method is four with the error constant, C_6.

2. Convergence of method

Definition 2: The linear multistep method (LMM) is said to be consistent if it has an order of at least one (Lambert, 1973). The proposed method is consistent since the order of the method is $p = 4 > 1$.

Definition 3: According to Lambert, (1973), a LMM is zero-stable provided that the root $\xi_j, j = 0(1)k$ of the first characteristic polynomial $p(\xi)$ specified as $p(\xi) = \det\left| \sum_{j=0}^{k} A^{(j)} \xi^{(k-j)} \right| = 0$ satisfies $|\xi_j| \leq 1$ and for those roots with $|\xi_j| = 1$, the multiplicity must not exceed two.

We now transform the corrector formulas into matrix form where the first characteristic polynomial of 2PDVS is given by

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$p(\xi) = \det \begin{bmatrix} \xi & 0 & -1 & 0 \\ 0 & \xi & 0 & -1 \\ 0 & 0 & \xi - 1 & 0 \\ 0 & 0 & 0 & \xi - 1 \end{bmatrix} \qquad (14)$$

$$0 = \xi^2(\xi - 1)^2, \xi = 0,0,1,1.$$

According to Definition 3 and the roots obtained in (14), we conclude that the 2PDVS method is zero-stable.

Definition 4: The linear multistep method is convergent if and only if it is consistent and zero-stable (Lambert, 1973). Since the consistent and zero-stable conditions are satisfied, then the method is convergent.

3. Stability analysis

The following test equation

$$y'' = f = \theta y' + \lambda y \qquad (15)$$

is applied in order to calculate the stability polynomial of 2PDVS method. The stability polynomial of two-point block method are given as follows.

For $r = 0.5$:

$$t^8\left(1 + \frac{71}{675}H_1^2 + \frac{3881}{81000}H_1H_2 - \frac{146}{225}H_1 - \frac{89}{600}H_2 + \frac{217}{40500}H_2^2\right) +$$
$$t^7\left(-2 - \frac{18596}{10125}H_1H_2 + \frac{4}{25}H_1 - \frac{3317}{900}H_2 - \frac{17473}{20250}H_2^2 - \frac{1318}{675}H_1^2\right) +$$
$$t^6\left(1 - \frac{28}{75}H_1 - \frac{47}{360}H_2 - \frac{69263}{27000}H_1H_2 + \frac{271}{225}H_1^2 - \frac{811}{13500}H_2^2\right) +$$
$$t^5\left(\frac{194}{225}H_1 - \frac{8}{225}H_2 - \frac{7943}{20250}H_1H_2 + \frac{794}{10125}H_2^2 + \frac{442}{675}H_1^2\right) +$$
$$t^4\left(-\frac{4}{10125}H_1H_2 - \frac{76}{10125}H_2^2 - \frac{8}{675}H_1^2\right) = 0,$$

where $H_1 = h\theta,\ H_2 = h^2\lambda.$ (16)

For $r = 1.0$:

$$t^8\left(1 + \frac{1777}{32400}H_1H_2 - \frac{29}{180}H_2 - \frac{251}{360}H_1 + \frac{19}{3240}H_2^2 + \frac{29}{240}H_1^2\right) +$$
$$t^7\left(-2 - \frac{2173}{2025}H_1H_2 - \frac{331}{90}H_2 - \frac{131}{360}H_1 - \frac{2257}{3240}H_2^2 - \frac{1273}{1080}H_1^2\right) +$$
$$t^6\left(1 - \frac{29}{180}H_2 + \frac{59}{72}H_1 - \frac{1}{360}H_2^2 - \frac{5863}{5400}H_1H_2 + \frac{163}{180}H_1^2\right) +$$

$$t^5\left(\frac{29}{120}H_1+\frac{17}{3240}H_2^{\,2}-\frac{148}{2025}H_1H_2+\frac{11}{72}H_1^{\,2}\right)+$$
$$t^4\left(-\frac{23}{32400}H_1H_2-\frac{1}{1620}H_2^{\,2}-\frac{1}{2160}H_1^{\,2}\right)=0, \tag{17}$$

where $H_1=h\theta,\ H_2=h^2\lambda.$

For $r=2.0$:

$$t^8\left(1+\frac{9667}{16200}H_1H_2-\frac{17}{100}H_2-\frac{67}{90}H_1+\frac{113}{20250}H_2^{\,2}+\frac{59}{432}H_1^{\,2}\right)+$$
$$t^7\left(-2-\frac{242581}{324000}H_1H_2-\frac{533}{1200}H_1-\frac{38173}{43200}H_1^{\,2}-\frac{6581}{1800}H_2-\frac{26249}{40500}H_2^{\,2}\right)+$$
$$t^6\left(1-\frac{163}{900}H_2+\frac{673}{600}H_1+\frac{203}{27000}H_2^{\,2}+\frac{817}{1152}H_1^{\,2}-\frac{21463}{36000}H_1H_2\right)+$$
$$t^5\left(\frac{241}{3600}H_1+\frac{13}{1800}H_2-\frac{259}{162000}H_2^{\,2}+\frac{409}{10800}H_1^{\,2}-\frac{7363}{648000}H_1H_2\right)+$$
$$t^4\left(\frac{7}{162000}H_2^{\,2}-\frac{1}{86400}H_1^{\,2}-\frac{49}{648000}H_1H_2\right)=0,$$

where $H_1=h\theta,\ H_2=h^2\lambda.$ (18)

The boundary of the absolute stability region in H_1-H_2 plane is determined by substituting t with 1,-1 and $e^{i\theta}$ for $0\le\theta\le 2\pi$ in the stability polynomial which is done by using MAPLE. Figure 2 illustrates the region of the absolute stability with various values of r that lies inside the boundary traced by the dotted lines and the axes. The stability region gets bigger as the step size ratio increases. This indicates that the method provides a better accuracy with smaller step size.

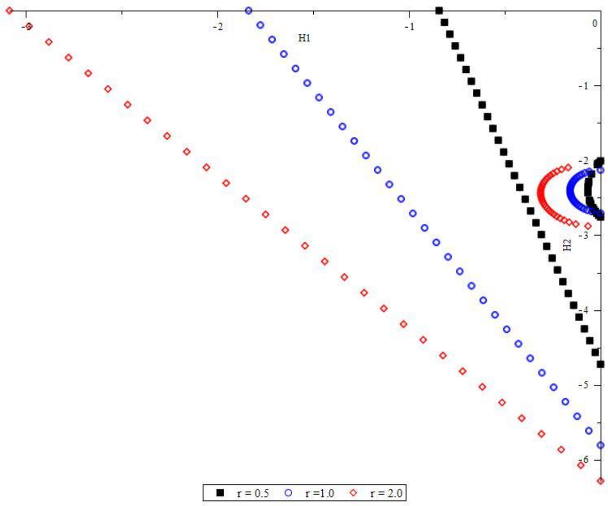

Figure 2. Stability region of two-point diagonally block method

C. *Implementation*

This study uses shooting technique for solving the BVPs of (1) with Robin conditions. The underlying concept of shooting technique is to transfrom BVPs into initial value problems (IVPs) which requires the initial guessing to represent the missing initial condition. Our shooting strategy works as follows. At first, we rewrite equation (1) into the following IVPs form

$$y''=f(x,y,y'),\ x\in[a,b]$$

with couple of the initial conditions

$$y_1(a)=s_1,\ y_1'(a)=V_1-Cy_1(a)$$

where $V_1=\frac{\alpha}{c_1}$, $C=\frac{c_1}{c_2}$ and s_1 is the guessing value. Next, we perform the computation using the proposed predictor and corrector formulas until end of the interval and verify the stopping condition given by

$$\left|g\left(y_1(b),y_1'(b)\right)-\beta\right|<TOL$$

where *TOL* is the tolerance that has been set while $g\left(y_1(b),y_1'(b)\right)=c_3y(b)+c_4y'(b)$. The iteration is repeated until we reach the prescribed stopping condition while the guessing value, s_j for $j=2,3,\ldots$ will be corrected using Newton divided difference interpolation technique. In this study, the first two guessing values are chosen as $s_1=0$ and $s_2=1$ because we adopt a strategy similar to Roberts, 1979 where both initial estimates were used to initialize the iterative solver.

In our code, the formulation to estimate the local truncation error, LTE, applied in the conditional statement while checking for the step size selection is based on the absolute difference between the derived corrector formula of order p with the corrector formula of order $p-1$, both at the first point. For example,

$$LTE=\left|\frac{h^2}{720}\left(58f_{n-2}-144f_{n-1}+108f_n-18f_{n+1}\right)\right|$$

is used as an estimator of *LTE* at x_{n+1} for the 2PDVS method when $r=0.5$. The choice of the next step size depends on the test comparison between *LTE* and *TOL* as follows:

- **Case 1**: If $LTE\le TOL$, the successful step is achieved. The step size ratio can be chosen as either $r=1.0$ or $r=0.5$. For $r=1.0$, the next step size is fixed. On the other hand, for $r=0.5$, the next step size will be doubled.
- **Case 2**: If $LTE>TOL$, then the failure step is achieved. At this stage, the next step size will be halved using the value of $r=2.0$. As a result, the computed solution in the current block will be recalculated again.

If the integration steps are successful, then the next step size prediction is given by

$$h_{new} = \delta \times h_{old} \times \left(\frac{TOL}{|LTE|} \right)^{\frac{1}{k+1}} \quad (19)$$

$$\text{if } (h_{new} > 2 \times h_{old}), \text{then } h_{new} = 2 \times h_{old}$$
$$\text{else } h_{new} = h_{old}$$

where $\delta = 0.5$ is a safety factor while k is the order of the LTE formula.

Algorithm of 2PDVS

Step 1 : Set TOL and calculate the initial guesses, $y_1(a) = s_1$, $y_1'(a) = V_1 - Cy_1(a)$.

Step 2: Calculate the initial step size.

Step 3: Compute a set of starting values using the direct Euler and modified Euler method.

Step 4: Compute the approximate values of y_p', y_p, f_p for $p = 3,4$ using the derived predictor and corrector formulas with $PE(CE)^m$ where $m = 1,2,...$ until it converges using the convergence test at each iteration.

Step 5: Calculate the LTE and determine for the step size selection; if $LTE \le TOL$, the step is a success. Apply the step size formula as given in (19) else halving the step size with $h_{new} = \frac{1}{2} \times h_{old}$.

Step 6: If $x_4 + 2h_{new} > b$, then $h_{final} = \left| \frac{b - x_4}{2} \right|$. Go to Step 7. Else, h_{new} remains as calculated. Set $x_i = x_{i+2}, y_i = y_{i+2}, y_i' = y_{i+2}', f_i = f_{i+2}$, for $i = 0,1,2$. Repeat Step 4.

Step 7: Reset the values of five back values using interpolation approach with h_{final} .

Step 8: At $x_4 = b$, verify the stopping condition. If $\left| g\left(y_j(b), y_j'(b)\right) - \beta \right| \le TOL$ is satisfied, then go to Step 10. Else, continue Step 9.

Step 9: Generate the new guessing values, $y_j(a) = s_j$ and $y_j'(a) = V_1 - Cy_j(a)$ for $j = 2,3,...$ based on the previous guesses using the Newton divided difference interpolation formula. Repeat Step 2.

Step 10: Execute the results. Complete.

In this study, the calculation of the maximum numerical error (MAXE) in the computed block is given by

$$MAXE = \max_{3 \le p \le 4} \left(\left| \frac{y(x_p) - y_p}{A + By(x_p)} \right| \right).$$

On the other hand, the convergence test formula is given by

$$\left| \frac{y_{n+1,m} - y_{n+1,m-1}}{A + B(y_{n+1,m})} \right| < 0.1 \times TOL.$$

We assigned the values of $A = 1$ and $B = 1$ in the above two formulas which corresponds to the mixed test.

III. RESULTS AND DISCUSSION

In this section, we consider two numerical tested problems to provide a clear view regarding the practical usefullness of the 2PDVS method. The following notation is used in the following results.

MAXE : Maximum error
h : step size
TOL : Tolerance
TS : Total step at last iteration
FS : Failure step
FCN : Total function call
TG : Total iteration of guess
Time : Time computation in second
2PDD4 : Direct two-point diagonal block method of order four developed in Nasir *et al.*, (2018)
2PDVS : Direct two-point diagonally block method with variable step size proposed in this study

Problem 1. Given the following linear second order differential equation

$$y'' = y - 2\cos(x), \ \frac{\pi}{2} \le x \le \pi$$

with $y'\left(\frac{\pi}{2}\right) + 3y\left(\frac{\pi}{2}\right) = -1$ and $y'(\pi) + 4y(\pi) = -4$.

Exact solution : $y(x) = \cos(x)$.

Source: Islam and Shirin, (2011)

Problem 2. Given the following nonlinear second order differential equation

$$y'' = \frac{1}{2} e^{-x} \left(y^2 + (y')^2 \right), \ 0 \le x \le 1$$

with $y(0) - y'(0) = 0$ and $y(1) + y'(1) = 2e$.

Exact solution : $y(x) = e^x$. Source: Duan *et al.*, (2013).

Table 1. Comparison of the numerical results for solving Problem 1

2PDVS							2PDD4					
TOL	TG	MAXE	TS	FS	FCN	TIME	*h*	TG	MAXE	TS	FCN	TIME
10(-2)	1	6.42(-7)	22	0	46	0.031	0.1	3	8.47(-7)	6	96	0.120
10(-4)	1	1.89(-7)	28	0	58	0.036	0.01	1	2.47(-10)	51	114	0.162
10(-6)	1	5.85(-9)	38	0	88	0.041						
10(-8)	1	9.85(-11)	56	0	144	0.045						
10(-10)	1	2.66(-12)	95	0	274	0.049						

Table 2. Comparison of the numerical results for solving Problem 2

2PDVS							2PDD4					
TOL	TG	MAXE	TS	FS	FCN	TIME	*h*	TG	MAXE	TS	FCN	TIME
10(-2)	2	2.48(-7)	19	0	81	0.036	0.1	4	4.90(-7)	6	120	0.131
10(-4)	2	1.61(-7)	26	0	109	0.042	0.01	2	1.90(-11)	51	228	0.182
10(-6)	2	2.17(-8)	34	0	145	0.047						
10(-8)	2	4.75(-11)	47	0	201	0.049						
10(-10)	2	1.37(-12)	74	0	315	0.052						

All the computation results for 2PDVS are computed using C language in Code::Blocks 16.01 platform where we have compared the performances of 2PDVS with 2PDD4 method. Both method satisfies the method of order four and in the form of diagonally block multistep method features. In addition, 2PDVS and 2PDD4 were implemented using the similar shooting strategy, but the latter method used the constant step size in its formulation.

In Table 1, tabulated data shows that 2PDVS requires only single iterations at TOL 10(-2) to satisfy the provided terminal value compare to 2PDD4 that acquires three initial guesses at $\mathrm{h}=0.1$ with comparable accuracy. At the same time, the total function calls for 2PDVS is lesser than 2PDD4. At TOL 10(-8), 2PDVS achieved high accuracy results with additional five steps than 2PDD4 at $\mathrm{h}=0.01$ for solving Problem 1.

2PDVS requires half number of total guesses at TOL 10(-2) compared to 2PDD4 at $\mathrm{h}=0.1$ for solving Problem 2 with comparable accuracy. As can be seen in Table 2, 2PDVS manages to achieve the same accuracy with the accuracy obtained by 2PDD4 at $\mathrm{h}=0.01$ but with less steps and less total function calls.

Overall observation from the numerical results displayed in Tables 1 - 2 show that the execution time for 2PDVS is faster than 2PDD4. This is expected since the algorithm of 2PDVS undergo the step size selection which allowed to double from the previous step size in the succesful block.

IV. CONCLUSION

In this research, we have shown that the proposed two-point diagonally block method is suitable for solving the second order Robin type BVPs directly with variable step size strategy. This method manages to preserve the accuracy of the numerical results, economically in terms of total steps and better in execution time when comparing with the existing method.

V. ACKNOWLEDGEMENTS

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VI. REFERENCES

Anakira, NR, Alomari, AK, Jameel, AF, & Hashim, I 2017, 'Multistage optimal homotopy asymptotic method for solving boundary value problems with robin boundary conditions', Far East Journal of Mathematical Sciences, vol. 102, no. 8, pp. 1727–1744.

Awoyemi, DO, Adebile, EA, Adesanya, AO & Anake, TA 2011, 'Modified block method for the direct solution of second order ordinary differential equations', International Journal of Applied Mathematics and Computation, vol. 3, no. 3, pp. 181–188.

Cuomo, S. & Marasco, A. 2008, 'A numerical approach to nonlinear two-point boundary value problems for ODEs', Computers & Mathematics with Applications, vol. 55, no. 11, pp. 2476–2489.

Duan, JS, Rach, R, Wazwaz, AM, Chaolu, T & Wang, Z 2013, 'A new modified Adomian decomposition method and its multistage form for solving nonlinear boundary value problems with Robin boundary conditions', Applied Mathematical Modelling, vol. 37, no. 20–21, pp. 8687–8708.

Islam, MS & Shirin, A 2011, 'Numerical Solutions of a Class of Second Order Boundary Value Problems on Using Bernoulli Polynomials', Applied Mathematics, vol. 2, no. 9, pp. 1059–1067.

Jator, SN 2010, 'Solving second order initial value problems by a hybrid multistep method without predictors', Applied Mathematics and Computation, vol. 217, no. 8, pp. 4036–4046.

Lambert, J.D., 1973. Computational Methods in Ordinary Differential Equations, London; New York: Wiley.

Lang, FG & Xu, XP 2012, 'Quintic B-spline collocation method for second order mixed boundary value problem', Computer Physics Communications, vol. 183, no. 4, pp. 913–921.

Nasir, NM, Majid, ZA, Ismail, F & Bachok, N 2018, 'Diagonal Block Method for Solving Two-Point Boundary Value Problems with Robin Boundary Conditions', Mathematical Problems in Engineering, 2018.

Phang, PS, Majid, ZA, Ismail, F, Othman, KI & Suleiman, M 2013, 'New algorithm of two-point block method for solving boundary value problem with dirichlet and Neumann boundary conditions', Mathematical Problems in Engineering, 2013.

Rach, R, Duan, JS & Wazwaz, AM 2016, 'Solution of Higher-Order, Multipoint, Nonlinear Boundary Value Problems with High-Order Robin-Type Boundary Conditions by the Adomian Decomposition Method', Applied Mathematics & Information Sciences, vol. 10, no. 4, pp. 1231–1242.

Roberts, C.E., 1979. Ordinary differential equations: a computational approach, Englewood Cliffs, NJ: Prentice-Hall.

Waeleh, N & Majid, ZA 2017, 'Numerical algorithm of block method for general second order ODEs using variable step size', Sains Malaysiana, vol. 46, no. 5, pp. 817–824.