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Approximate eigensolutions of the attractive potential via parametric Nikiforov-Uvarov method

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Abstract

The approximate analytical solutions of the non-relativistic Schrödinger equation for the Attractive potential model with the centrifugal term are investigated using the elegant methodology of the parametric Nikiforov-Uvarov. The energy equation and the corresponding un-normalized radial wave functions are obtained in a close and compact form after a proper Greene-Aldrich approximation scheme is applied. By changing the numerical values of some potential strengths, special cases of the Attractive potential are investigated in detail. The effects of the potential strengths and potential range respectively on the energy are also studied. The energy is found to be very sensitive to each of the potential parameters. Some theoretic quantities such as information energy, Rényi entropy and Tsallis entropy.

Keyword: Quantum mechanics

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1. Introduction

The solutions of the relativistic and non-relativistic quantum mechanical wave equations with various physical potentials have been regarded as an important area of research studies in quantum mechanics. Thus, the exact solutions of the wave equation with the various potential models have attracted much attention from many authors since they contain all the necessary information to study the quantum models. It is well known that there are various traditional techniques used to solve the relativistic and non-relativistic wave equations with the various quantum potential models [1, 2, 3, 4, 5, 6, 7, 8]. These techniques include asymptotic iteration method [9, 10, 11], supersymmetric approach [12, 13, 14, 15], factorization method [16, 17], Nikiforov-Uvarov method [18, 19, 20], exact/proper quantization rule [21], 1/N shifted expansion method [22]. Recently, Tezcan and Sever [23], developed parametric Nikiforov-Uvarov method from the conventional Nikiforv-Uvarov method. This method has been widely used because of its simplicity and accuracy. However, there are only few potentials that can be exactly solved for in three-dimensional space such as Coulomb and Harmonic potentials. Consequently, if we intend to use analytical techniques, the use of approximation schemes is inevitable. The choice of approximation scheme employed depends on the potential type under consideration. In this study, our aim is to investigate the three dimensional space of the Schrödinger equation with Attractive radial potential. The Attractive radial potential was introduced in 1993 by Williams and Poulios [24] as a four parameters exponential-type potential. This potential has received attention in the relativistic regime. For instance, Zou *et al.* [25], investigated the s-wave solutions of the Dirac equation with the Attractive radial potential; Eshghi and Hamzavi [26], studied symmetry in Dirac-Attractive radial problem and tensor potential. Recently, Ikot *et al.* [27], studied the symmetry limit of the Dirac equation with Attractive radial potential in the presence of Yukawa-like tensor potential. To the best of our knowledge, the Attractive radial potential has not receive any report in the non-relativistic regime. Thus, the motivation for this study. The Attractive radial potential is given as [24].

$$V(r) = \frac{Ae^{-2\alpha r} + B + Ce^{2\alpha r}}{e^{2\alpha r}(1 - e^{-2\alpha r})^2}, \tag{1}$$

where $A = \frac{\alpha^2}{4}$, $B = A(\lambda - 8)$ and $C = A(4 - \lambda)$. The parameter α and A are real and $\alpha > \frac{1}{2}$, $4 < \lambda < 8$. Obviously, the results of other useful potential can be obtained from the results of Attractive radial potential as will be seen later.

2. Theory/calculation

2.1. Bound state solution

In a spherical potential model $V(r)$, the time independent Schrödinger equation is given by [28, 29, 30, 31, 32, 33].

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$$\left[\frac{-\hbar^2\nabla^2}{2\mu}+V(r)\right]\psi_{n\ell m}(r,\theta,\phi)=E_{nk}\psi_{n\ell m}(r,\theta,\phi), \tag{2}$$

where ℓ and n are orbital angular momentum and radial quantum number respectively, $\psi_{n\ell m}(r,\theta,\phi)$ is the wave function, μ and $\hbar$ are reduced mass and Planck constant respectively, r is the internuclear separation and $E_{n\ell}$ is the non-relativistic energy. Setting the wave function $\psi_{n\ell m}(r,\theta,\phi)=\frac{R_{n\ell}(r)Y_{\ell m}(\theta,\phi)}{r}$ and substituting Eq. (1) into Eq. (2), we have

$$\left[\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}\left(E_{nk}-\frac{Ae^{-2\alpha r}+B+Ce^{2\alpha r}}{e^{2\alpha r}(1-e^{-2\alpha r})^2}-\frac{\hbar^2\ell(\ell+1)}{2\mu r^2}\right)\right]R_{n\ell}(r)=0. \tag{3}$$

Eq. (3) cannot be solved for $\ell=0$ because of the centrifugal term. Thus, we must apply an approximation scheme to deal with the centrifugal term. It is found that such approximation suggested by Greene and Aldrich is a good approximation to the centrifugal term $\frac{1}{r^2}$. For a short range potential, the following Greene-Aldrich approximation [34].

$$\frac{\ell(\ell+1)}{r^2}=\frac{4\ell(\ell+1)\alpha^2e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}, \tag{4}$$

is valid for $\alpha << 1$. Now, substituting Eq. (4) into Eq. (3) we have

$$\left[\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}\left(E_{nk}-\frac{Ae^{-2\alpha r}+B+Ce^{2\alpha r}}{e^{2\alpha r}(1-e^{-2\alpha r})^2}-\frac{2\ell(\ell+1)\alpha^2\hbar^2e^{-2\alpha r}}{\mu(1-e^{-2\alpha r})^2}\right)\right]R_{n\ell}(r)=0, \tag{5}$$

To solve Eq. (5) using the powerful parametric Nikiforov-Uvarov method, let us consider a second order differential equation of the form:

$$\left(\frac{d^2}{dr^2}+\frac{\alpha_1-\alpha_2}{s(1-\alpha_3 s)}\frac{d}{ds}+\frac{-\xi_1 s^2+\xi_2 s-\xi_3}{s^2(1-\alpha_3 s)^2}\right)\psi(s)=0. \tag{6}$$

According to parametric Nikiforov-Uvarov method, the condition for eigenvalues and eigenfunction are respectively [23].

$$n\alpha_2-(2n+1)\alpha_5+\alpha_7+2\alpha_3\alpha_8+n(n-1)\alpha_3+(2n+1)\sqrt{\alpha_9}+(2\sqrt{\alpha_9}+\alpha_3(2n+1))\sqrt{\alpha_8}=0, \tag{7}$$

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$$\psi_{n,\ell}(s) = N_{n,\ell} s^{\alpha_{12}} (1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}} \times P_n^{\left(\alpha_{10}-1, \frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1\right)}(1-2\alpha_3 s), \tag{8}$$

$$\left.\begin{array}{l}
\alpha_4 = \dfrac{1-\alpha_1}{2}, \alpha_5 = \dfrac{\alpha_2 - 2\alpha_3}{2}, \alpha_6 = \alpha_5^2 + \xi_1, \alpha_7 = 2\alpha_4\alpha_5 - \xi_2, \\
\alpha_8 = \alpha_4^2 + \xi_3, \alpha_9 = \alpha_3(\alpha_7 + \alpha_3\alpha_8) + \alpha_6, \alpha_{10} = \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \\
\alpha_{11} = \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}), \\
\alpha_{12} = \alpha_4 + \sqrt{\alpha_8}, \alpha_{13} = \alpha_5 - (\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})
\end{array}\right\}. \tag{9}$$

The parameters in Eqs. (7) and (8) and obtain using the relationships in Eq. (9). To obtain the solution of Eq. (5), we defining a variable of the form $y = e^{-2\alpha r}$ and substitute it into Eq. (5) to have

$$\left[\frac{d^2}{dy^2} + \frac{1-y}{y(1-y)}\frac{d}{dy} + \frac{Py^2 + Qy + T}{[y(1-y)]^2}\right] R_{n\ell}(y) = 0 \tag{10}$$

where

$$P = \frac{\mu A + \mu E_{n\ell}}{2\alpha^2\hbar^2}, \tag{11}$$

$$-Q = \frac{\mu B + 2\mu E_{n\ell}}{2\alpha^2\hbar^2} + \ell(\ell+1), \tag{12}$$

$$T = \frac{-\mu C + \mu E_{n\ell}}{2\alpha^2\hbar^2}. \tag{13}$$

Eqs. (11), (12), and (13) are parts of Eq. (10). Comparing Eq. (10) with Eq. (6), we have

$$\left.\begin{array}{l}
\alpha_1 = \alpha_2 = \alpha_3 = 1, \alpha_4 = 1 - \alpha_3, \alpha_5 = -\dfrac{\alpha_2}{2}, \alpha_6 = \dfrac{1}{4} - P, \alpha_7 = Q, \alpha_8 = -T, \alpha_9 = \dfrac{1}{4} + \dfrac{\mu(C+B-A)}{2\alpha^2\hbar^2} + \ell(\ell+1), \\
\alpha_{10} = 1 + 2\sqrt{-T}, \alpha_{11} = \sqrt{(1+2\ell)^2 + \dfrac{2\mu(C+B-A)}{\alpha^2\hbar^2}} + \alpha_{10} + 1, \alpha_{12} = \sqrt{-T}, \\
\alpha_{13} = -\dfrac{1}{2}\left(\sqrt{(1+2\ell)^2 + \dfrac{2\mu(C+B-A)}{\alpha^2\hbar^2}} + \alpha_{10}\right)
\end{array}\right\}. \tag{14}$$

Substituting Eq. (14) into Eqs. (7) and (8), we have the energy equation and the corresponding wave function respectively as

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$$E_{n\ell} = C - \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{n(n+1)+\ell(\ell+1)+\frac{1}{2}\left(1+\frac{\mu B+2\mu C}{\alpha^2\hbar^2}\right)+\sqrt{(1+2\ell)^2+\frac{2\mu(C+B-A)}{\alpha^2\hbar^2}}\left(n+\frac{1}{2}\right)}{1+2n+\sqrt{(1+2\ell)^2+\frac{2\mu(C+B-A)}{\alpha^2\hbar^2}}}\right]^2. \quad (15)$$

$$R_{n,\ell}(y) = N_{n,\ell}\, y^{\sqrt{-T}}(1-y)^{\frac{1}{2}+\frac{1}{2}\sqrt{(1+2\ell)^2+\frac{2\mu(C+B-A)}{\alpha^2\hbar^2}}} \times P_n^{\left(2\sqrt{-T},\sqrt{(1+2\ell)^2+\frac{2\mu(C+B-A)}{\alpha^2\hbar^2}}\right)}(1-2y), \quad (16)$$

2.2. Some theoretic quantities and the Coulomb-Hulthen-Pöschl-Teller potential (CHPT)

In this section, we calculate the Information energy, Tsallis entropy and Rényi entropy using the probability density obtained from the normalized radial wave function given in Eq. (16). To begin, we first calculate the normalized wave function. Given the radial wave function as

$$[R_{n\ell}(y)]^2 = N_{n\ell}^2 y^{2a}(1-y)^{2b}\left[P_n^{(2a,b)}(1-2y)\right]^2.\; y = e^{-2\alpha r}. \quad (17)$$

which is equal to the probability density $\rho(y)$, we can easily calculate the theoretic quantities mentioned above. Where $a = \sqrt{-T}$ and $b = \frac{1}{2} + \frac{1}{2}\sqrt{(1+2\ell)^2+\frac{2\mu(C+B-A)}{\alpha^2\hbar^2}}$. Normalizing the radial wave function, we have

$$\int_1^0 [R_{n\ell}(y)]^2 dy = 1,\; y = e^{-2\alpha}. \quad (18)$$

Substitute the value of $R_{n\ell}(y)$, Eqs. (17) and (18) after some simplifications turn to be

$$\frac{N_{n\ell}^2}{4\alpha}\int_{-1}^{1}(1-s)^{u-1}(1+s)^{\frac{1+v}{2}}\left[P_n^{(u,v)}(s)\right]^2 ds = 1, \quad (19)$$

where we have used: $2a = u$, $b = v$, $x = 1-2y$ and $\frac{1+x}{2} = 1-\left(\frac{1-s}{2}\right)$ Using integral of the form given in Eq. (20) below

$$\int_{-1}^{1}(1-x)^{t-1}(1+x)^k\left[P_n^{(t,k)}(x)\right]^2 dx = \frac{2^{t+k}\Gamma(t+n+1)\Gamma(k+n+1)}{n!\,t\,\Gamma(t+k+n+1)}, \quad (20)$$

we have the normalization constant in Eq. (19) as

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$$N_{n\ell}=2\sqrt{\frac{\alpha n!\mathrm{u}\Gamma\left(\frac{2u+v+3+2n}{2}\right)}{2^{\frac{2U+v+1}{2}}\Gamma(u+n+1)\Gamma\left(\frac{v+3+2n}{2}\right)}}. \tag{21}$$

Eq. (21) gives the normalization constant.

2.3. Information energy

Information energy is defined as

$$E(\rho)=4\pi\int_0^\infty \rho(r)dr. \tag{22}$$

$$E(\rho)=4\pi\int_1^0 \rho(y)dy.\, y=\exp(-2\alpha r). \tag{23}$$

$$E(\rho)=4\pi\int_{-1}^1 \rho(z)dz.\, z=1-2y. \tag{24}$$

Substitute the value of the probability density, the information energy given in Eq. (22) is simplified in Eqs. (23) and (24) to give

$$E(\rho)=4\pi N_{n\ell}^2\int_{-1}^1\left(\frac{1-z}{2}\right)^{u-1}\left(\frac{1+z}{2}\right)^{\frac{1+v}{2}}\left[P_n^{(u,v)}(z)\right]^2dz. \tag{25}$$

Using integral of the form given in Eq. (26) below,

$$\int_{-1}^1\left(\frac{1-y}{2}\right)^{c}\left(\frac{1+y}{2}\right)^{d}\left[P_n^{(u,v)}(y)\right]^2dy=\frac{2\Gamma(c+n+1)\Gamma(d+n+1)}{n!\Gamma(\mathrm{c}+d+2\mathrm{n}+1)\Gamma(\mathrm{c}+d+n+1)}, \tag{26}$$

and the normalization constant calculated in Eq. (21), the information energy in Eq. (25) turns to the form of Eq. (27) below

$$E(\rho)=\frac{100.544\alpha u\Gamma(u+n)}{2^{\frac{1+v+2u}{2}}\Gamma(u+n+1)\Gamma\left(\frac{2u+4n+v+3}{2}\right)}. \tag{27}$$

$$E(\rho)=\frac{221.68u\Gamma(u+n)}{2^{\frac{1+v+2u}{2}}\Gamma(u+n+1)\Gamma\left(\frac{2u+4n+v+3}{2}\right)}. \tag{28}$$

Eq. (28) is the information energy.

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2.4. Tsallis entropy

The Tsallis entropy is defined as

$$T(q)=\frac{1}{q-1}\left(1-4\pi\int_{0}^{\infty}\rho(r)^{q}dr\right).\, q\neq 1 \tag{29}$$

Substituting for the probability density into Eq. (29), we have

$$T(q)=\frac{1}{q-1}\left[1-4\pi\int_{-1}^{1}\left(N_{n\ell}^{2}\left(\frac{1-z}{2}\right)^{u-1}\left(\frac{1+z}{2}\right)^{\frac{1+v}{2}}\left[P_{n}^{(u,v)}(z)\right]^{2}\right)^{q}dz\right], \tag{30}$$

Using integral of the form in Eq. (26), the Tsallis entropy in Eq. (30) is obtain as

$$T(q)=\frac{1}{q-1}-\frac{12.568}{q-1}\left(\frac{4\alpha u\Gamma(u+n)}{2^{\frac{1+v+2u}{2}}\Gamma(u+n+1)\Gamma\left(\frac{2u+4n+v+3}{2}\right)}\right)^{q}. \tag{31}$$

Simplifying Eq. (31), we have

$$T(q)=\frac{1}{q-1}-\frac{12.568}{q-1}\left(\frac{8.8192u\Gamma(u+n)}{2^{\frac{1+v+2u}{2}}\Gamma(u+n+1)\Gamma\left(\frac{2u+4n+v+3}{2}\right)}\right)^{q}. \tag{32}$$

Eq. (32) is the Tsallis entropy of the system.

2.5. Rényi entropy

The Rényi entropy is defined as

$$R(q)=\frac{1}{1-q}\log 4\,\pi\int_{0}^{\infty}\rho(r)^{q}dr.\, q\neq 1 \tag{33}$$

Substituting for the probability density into Eq. (33), we have

$$R(q)=\frac{1}{1-q}\log 4\,\pi\int_{-1}^{1}\left(N_{n\ell}^{2}\left(\frac{1-z}{2}\right)^{u-1}\left(\frac{1+z}{2}\right)^{\frac{1+v}{2}}\left[P_{n}^{(u,v)}(z)\right]^{2}\right)^{q}dz, \tag{34}$$

Substituting for the normalization constant with the integral in Eq. (26) into Eq. (34), the Rényi entropy is obtain as

$$R(q)=\frac{1.0993}{1-q}\left(\frac{2^{\frac{5-2u-v}{2}}\alpha u\Gamma(u+n)}{\Gamma(u+n+1)\Gamma\left(\frac{2u+4n+v+3}{2}\right)}\right)^{q}. \tag{35}$$

Simplifying Eq. (35) a bit, we have

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$$R(q) = \frac{1.0993}{1-q}\left(\frac{17.6384 \times 2^{-\left(\frac{1+2u+v}{2}\right)}\alpha u\Gamma(u+n)}{\Gamma(u+n+1)\Gamma\left(\frac{2u+4n+v+3}{2}\right)}\right)^{q}. \tag{36}$$

Eq. (36) gives the Rényie entropy of the system.

3. Discussion

In Fig. 1, we plotted the approximate energy $E_{n,\ell}$ against the potential range α for four values of the quantum number n. It is seen from the figure that as the potential range α increases, the approximate energy increases negatively. Similarly, the energy increases negatively as the quantum number n increases. This implies that a particle subjected to this system is more attractive as it moves from the ground state level to the higher levels. Fig. 2 shows the variation of the exact energy E_n against the potential range. The effects observed in Fig. 1 are also observed in Fig. 2. In Figs. 3, 4, and 5, we plotted energy against each of the three potential strength. To achieve

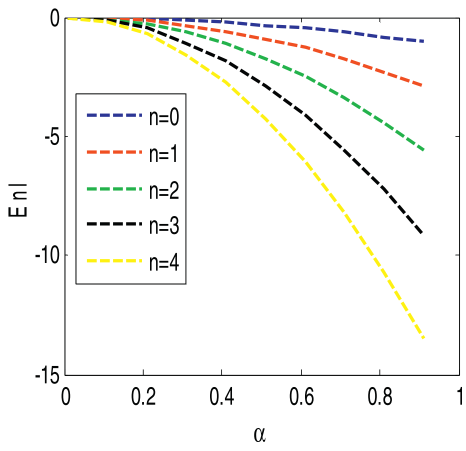

Fig. 1. Energy $E_{n\ell}$ against α for various n with $\ell = \mu = \hbar = 1$ and $\lambda = 6$.

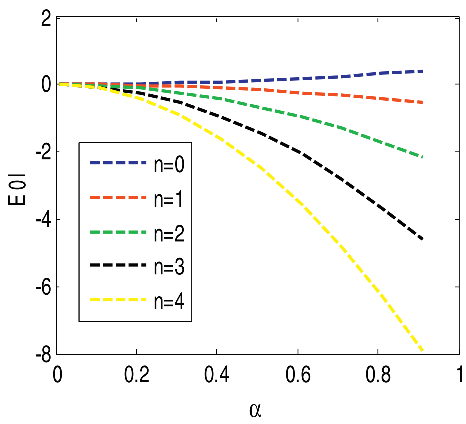

Fig. 2. Energy $E_{n\ell}$ against α for various n with $\ell = 0$, $\mu = \hbar = 1$ and $\lambda = 6$.

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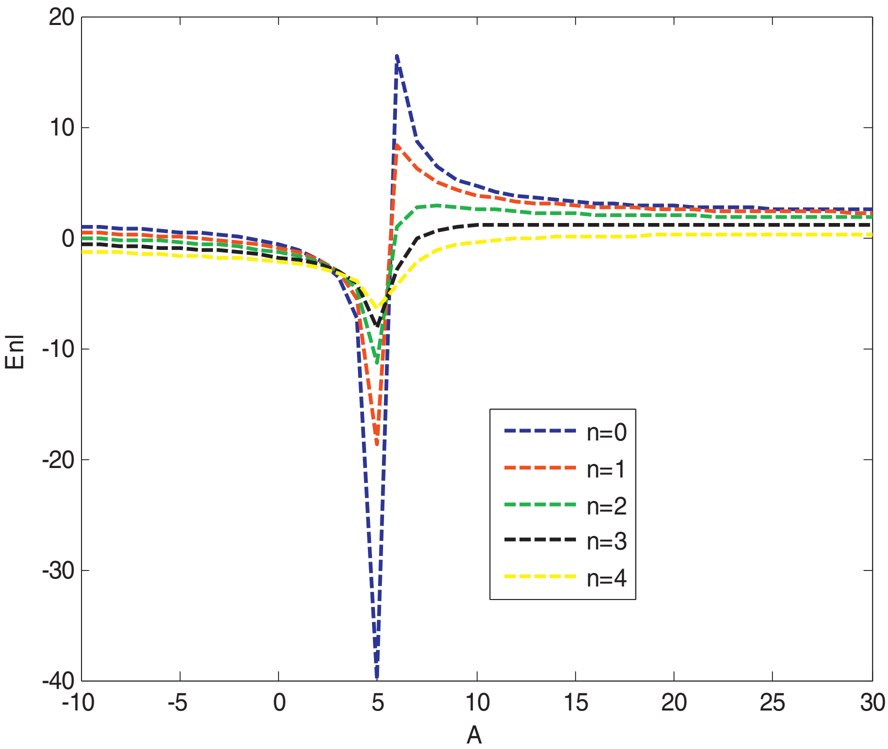

Fig. 3. Energy $E_{n\ell}$ against A for various n with $B = 3$, $B = 2$, $\ell = \mu = \hbar = 1$, and $\lambda = 0$.

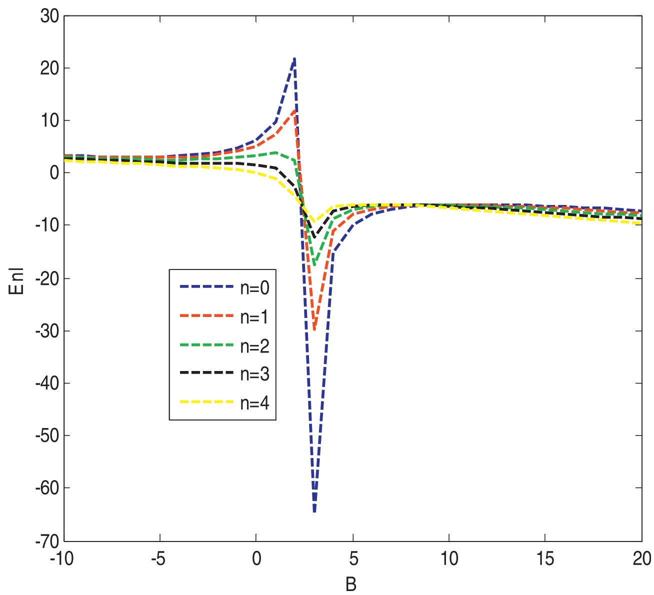

Fig. 4. Energy $E_{n\ell}$ against B for various n with $A = 6$, $C = A - 3$, $\ell = \mu = \hbar = 1$ and $\lambda = 0$.

this, we ignore the relation given between A, B and C in the text. It is observed that a particle under the influence of the Attractive potential exhibits the same behaviour as each of the potential strength increases respectively.

Special cases of the Attractive potential were considered.

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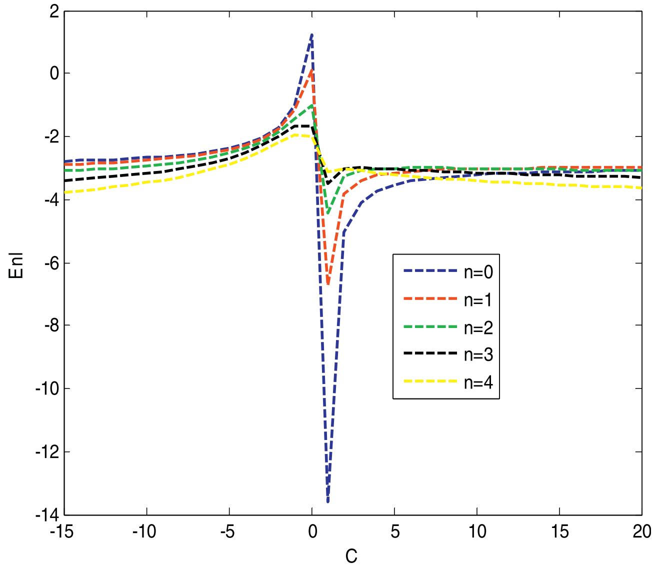

Fig. 5. Energy $E_{n\ell}$ against C for various n with $A = 3$, $B = A - 1$, $\ell = \mu = \hbar = 1$ and $\lambda = 0$.

Case 1: $\lambda = 8$. In this case, $B = 0$ and the potential (1) reduces to

$$V(r) = \frac{A(e^{-4\alpha r} - 4)}{(1 - e^{-2\alpha r})^2}, \tag{37}$$

Replacing Eq. (1) with (37), the energy Eq. (15) becomes

$$E_{n\ell} = -4A - \frac{2\alpha^2\hbar^2}{\mu}\left[\frac{n(n+1) + \ell(\ell+1) + \frac{1}{2}\left(1 - \frac{8\mu A}{\alpha^2\hbar^2}\right) + \left(n + \frac{1}{2}\right)\sqrt{(1+2\ell)^2 - \frac{10\mu A}{\alpha^2\hbar^2}}}{1 + 2n + \sqrt{(1+2\ell)^2 - \frac{10\mu A}{\alpha^2\hbar^2}}}\right]^2. \tag{38}$$

Eq. (38) is the energy equation for Eq. (37).

Case 2: $\lambda = 4$. In this case, $C = 0$ and the Attractive potential reduces to

$$V(r) = \frac{A(e^{-2\alpha r} - 4)e^{-2\alpha r}}{(1 - e^{-2\alpha r})^2}, \tag{39}$$

and the energy equation becomes

$$E_{n\ell} = -\frac{2\alpha^2\hbar^2}{\mu}\left[\frac{n(n+1) + \ell(\ell+1) + \frac{1}{2}\left(1 - \frac{4\mu A}{\alpha^2\hbar^2}\right) + \left(n + \frac{1}{2}\right)\sqrt{(1+2\ell)^2 - \frac{10\mu A}{\alpha^2\hbar^2}}}{1 + 2n + \sqrt{(1+2\ell)^2 - \frac{10\mu A}{\alpha^2\hbar^2}}}\right]^2. \tag{40}$$

Eq. (40) is the energy equation for Eq. (39).

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Case 3. A situation where the relation between the potential strengths is neglected when $B = 0$, as the

$$\lim_{C\to 0} V(r) = \frac{Ae^{-2\alpha r}}{(1-e^{-2\alpha r})^2}, \tag{41}$$

whose energy equation is given as

$$E_{n\ell} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\left[n+\frac{1}{2}+\frac{1}{2}\sqrt{(1+2\ell)^2+\frac{8\mu A}{\alpha^2\hbar^2}}\right]^2-\frac{2\mu A}{\alpha^2\hbar^2}}{n+\frac{1}{2}+\frac{1}{2}\sqrt{(1+2\ell)^2+\frac{8\mu A}{\alpha^2\hbar^2}}}\right]^2. \tag{42}$$

Eq. (42) is the energy equation for Eq. (41). Eq. (42) is equivalent to the solutions of the inversely quadratic Yukawa potential which has application in various areas of physics.

Table 1. Energy spectral $-E_{n\ell}$ for Attractive potential for three values of λ with $\mu = \hbar = 1$.

State	α	$\lambda = 7$	$\lambda = 6$	$\lambda = 5$
2p	0.55	0.6306517	0.5905882	0.5520254
	0.65	0.8808275	0.8248712	0.771010
	0.75	1.1726994	1.0982013	1.0264936
3p	0.55	1.3169916	1.2782588	1.2401388
	0.65	1.8394346	1.7853359	1.7320947
	0.75	2.4489513	2.3769265	2.3060432
3d	0.55	1.3997109	1.3610387	1.3229396
	0.65	1.9549681	1.9009549	1.8477421
	0.75	2.6027682	2.5308527	2.4600117
4p	0.55	2.3070406	2.2687305	2.2307521
	0.65	3.2222302	3.1687227	3.1156786
	0.75	4.2899515	4.2187137	4.1480928
4d	0.55	2.4191370	2.3808515	2.3428814
	0.65	3.3787946	3.3253215	3.2722889
	0.75	4.4983952	4.4272032	4.3565976
4f	0.55	2.4636560	2.4253796	2.3874125
	0.65	3.4409740	3.3875137	3.3344852
	0.75	4.5811785	4.5100034	4.4394034
5p	0.55	3.5999051	3.5617816	3.5238655
5d	0.55	3.7413464	3.7032352	3.6653231
5f	0.55	3.7973172	3.7592105	3.7213000
5g	0.55	3.8277719	3.7896677	3.7517580
6p	0.55	5.1953873	5.1573622	5.1194788
6d	0.55	5.3661640	5.3281459	5.2902649
6f	0.55	5.4335832	5.3955676	5.3576875
6g	0.55	5.4702302	5.4322161	5.3943364

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Table 2. Energy spectral $-E_{n\ell}$ for Attractive potential and its special cases with $\mu = \hbar = 1$.

State	α	λ=8 $B=0$	λ = 6	λ=4 $C=0$
2p	0.55	0.6722157	0.5905882	0.5149632
	0.65	0.9388798	0.8248712	0.7192462
	0.75	1.2499878	1.0982013	0.9575763
3p	0.55	1.3563388	1.2782582	1.2026332
	0.65	1.8943906	1.7853359	1.6797109
	0.75	2.5221176	2.3769265	2.2363015
3d	0.55	1.4389563	1.3610387	1.2854137
	0.65	2.0097819	1.9009549	1.7953299
	0.75	2.6757451	2.5308527	2.3902320
4p	0.55	2.3456824	2.2687305	2.1931055
	0.65	3.2762010	3.1687227	3.0630977
	0.75	4.3618060	4.2187137	4.0780887
4d	0.55	2.4577378	2.3808515	2.3052265
	0.65	3.4327081	3.3253215	3.2196965
	0.75	4.5701735	4.4272032	4.2865782
4f	0.55	2.5022416	2.4253796	2.3497546
	0.65	3.4948663	3.3875137	3.2818887
	0.75	4.6529285	4.5100034	4.3693784
5p	0.55	3.6382359	3.5617816	3.4861566
5d	0.55	3.7796568	3.7032352	3.6276102
5f	0.55	3.8356199	3.7592105	3.6835855
5g	0.55	3.8660706	3.7896677	3.7140427
6p	0.55	5.2335541	5.1573622	5.0817372
6d	0.55	5.4043193	5.3281459	5.2525209
6f	0.55	5.4717340	5.3955676	5.3199426
6g	0.55	5.5083787	5.4322161	5.3565911

Table 1 shows the numerical results for the Attractive potential with three values of the potential range α following the condition $4 < \lambda < 8$ and $\alpha > \frac{1}{2}$. We have taken the value of $\lambda = 5, 6$ and 7 respectively with $\alpha = 0.55, 0.65$ and 0.75 for 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states. As it can be seen from Table 1, the energy decreases as λ increases. Similarly, the energy decreases as the potential range increases.

This means that our choice of approximation is vallid for a small value of the potential range. In Table 2, we put B and C respectively to zero. The result obtained by putting them to zero is in fair agreement with the result of the Attractive potential. Thus, the result obtained for (A and $B \neq 0$) = (A and $C \neq 0$) = (A, B and $C \neq 0$).

4. Conclusion

The $\ell -$ state solutions of the Schrödinger equation with Attractive potential is obtained via a straightforward technique and an acceptable approximation scheme to

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the centrifugal term. The energy eigenvalues and the corresponding eigenfunctions are reported in terms of the Jacobi polynomial. It has been observed that the energy eigenvalues numerically obtained for the special cases of the Attractive potential are equivalent to the energy obtained for the real Attractive potential. However, the energy is highly sensitive to the potential strengths and the potential range. In a closed form, we have obtained equivalent energy for inversely quadratic Yukawa potential with various applications in physics. The information energy, Rẻnyi and Tsallis entropies were calculated. These are useful in information theoretic proof of the Central limit theory.

Declarations

Author contribution statement

C.A. Onate: Conceived and designed the analysis; Analyzed and interpreted the data; Wrote the paper.

O. Adebimpe, A.F. Lukman, I.J. Adama, J.O. Okoro, E.O. Davids: Analyzed and interpreted the data; Wrote the paper.

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Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

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