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Elliptic Multiple Polylogarithms in Open String Theory

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Zusammenfassung

In dieser Dissertation wird eine Methode zur Berechnung der genus-eins Korrekturen von offenen Strings zu Feldtheorie-Amplituden konstruiert. Hierzu werden Vektoren von Integralen definiert, die ein elliptisches Knizhnik–Zamolodchikov–Bernard (KZB) System auf dem punktierten Torus erfüllen, und die entsprechenden Matrizen aus dem KZB System berechnet. Der elliptische KZB Assoziator erzeugt eine Relation zwischen zwei regulierten Randwerten dieser Vektoren. Die Randwerte enthalten die genus-null und genus-eins Korrekturen. Das führt zu einer Rekursion im Genus und der Anzahl externer Zustände, die einzig algebraische Operationen der bekannten Matrizen aus dem KZB System umfasst. Geometrisch werden zwei externe Zustände der genus-null Weltfläche der offenen Strings zu einer genus-eins Weltfläche zusammengeklebt.

Die Herleitung dieser genus-eins Rekursion und die Berechnung der relevanten Matrizen wird durch eine graphische Methode erleichtert, mit der die Kombinatorik strukturiert werden kann. Sie wurde durch eine erneute Untersuchung der auf Genus null bekannten Rekursion entwickelt, bei welcher der Drinfeld Assoziator Korrekturen offener Strings auf Genus null auf solche mit einem zusätzlichen externen Zustand abbildet. Diese genus-null Rekursion umfasst ebenfalls ausschliesslich Matrixoperationen und basiert auf einem Vektor von Integralen, der eine Knizhnik–Zamolodchikov (KZ) Gleichung erfüllt. Die in der Rekursion gebrauchten Matrizen aus der KZ Gleichung werden als Darstellungen einer Zopfgruppe identifiziert und rekursiv berechnet.

Der elliptische KZB Assoziator ist die Erzeugendenreihe der elliptischen Multiplen Zeta-Werte. Die Konstruktion der genus-eins Rekursion benötigt verschiedene Eigenschaften dieser Werte und ihren definierenden Funktionen, den elliptischen Multiplen Polylogarithmen. So werden Relationen verschiedener Klassen von elliptischen Polylogarithmen und Funktionalrelationen erzeugt durch elliptische Funktionen hergeleitet.

Abstract

In this thesis, a method to calculate the genus-one, open-string corrections to the field-theory amplitudes is constructed. For this purpose, vectors of integrals satisfying an elliptic Knizhnik–Zamolodchikov–Bernard (KZB) system on the punctured torus are defined and the matrices from the KZB system are calculated. The elliptic KZB associator is used to relate two regularised boundary values of these vectors. The boundary values are shown to contain the open-string corrections at genus zero and genus one. This yields a recursion in the genus and the number of external states, solely involving algebraic operations on the known matrices from the KZB system. Geometrically, two external states of the genus-zero, open-string worldsheet are glued together to form a genus-one, open-string worldsheet.

The derivation of this genus-one recursion and the calculation of the relevant matrices is facilitated by a graphical method to structure the combinatorics involved. It is motivated by the reinvestigation of the recursion in the number of external states known at genus zero, where the Drinfeld associator maps the genus-zero, open-string corrections to the corrections with one more external state. This genus-zero recursion also involves matrix operations only and is based on a vector of integrals satisfying a Knizhnik–Zamolodchikov (KZ) equation. The matrices in the KZ equation and used in the recursion are shown to be braid matrices and a recursive method for their calculation is provided.

The elliptic KZB associator is the generating series of elliptic multiple zeta values. The construction of the genus-one recursion requires various properties of these values and their defining functions, the elliptic multiple polylogarithms. Thus, the third part of this thesis consists of an analysis of elliptic multiple polylogarithms, which in particular leads to relations among different classes of elliptic polylogarithms and functional relations generated by elliptic functions.

Contents

Chapter 1

Introduction

1.1 Motivation

The main focus of this thesis lies on the investigation of properties of a class of functions, elliptic multiple polylogarithms (eMPLs), as well as special values thereof, elliptic multiple zeta values (eMZVs), and on the role they take in the construction of planar, one-loop amplitudes of open-string interactions. On the one hand, various functional relations among the eMPLs, derived in ref. [1], will be given and on the other hand, a recursive method to calculate the string corrections to the field-theory amplitudes in one-loop, open-string interactions, which involves the eMPLs, will be presented [2, 3]. The construction of this method was motivated by the results of ref. [4], where the analogous recursive method for tree-level interactions from ref. [5] has been reformulated in terms of twisted de Rham theory.

Elliptic multiple polylogarithms and elliptic multiple zeta values in high energy physics

For the last decades, various beautiful mathematical aspects of eMPLs and eMZVs have been known and investigated [6–8]. More recently, they have come to the attention of particle physicists and string theorists due to an increasing appearance in high-order calculations of scattering amplitudes in the standard model, $\mathcal{N}$=4 super-Yang–Mills theory and open string theory, for example in refs. [9–27]. Since higher and higher accuracies are achieved at collider experiments, such higher-order calculations become more and more relevant to compare theoretical predictions to the measured cross-sections.

The eMPLs and eMZVs are a natural class of functions and values appearing in such high-order calculations, generalising an important class at the lowest order, the multiple polylogarithms (MPLs) and multiple zeta values (MZVs): while MPLs and MZVs are iterated integrals on the Riemann sphere, a genus-zero Riemann surface, eMPLs and eMZVs are iterated integrals on a genus-one Riemann surface, the torus or elliptic curve, respectively. Understanding the origin of these iterated integrals in the corresponding theories along with their mathematical properties and numerical evaluation is crucial for theoretical predictions to keep up with the accuracy of experimental measurements.

String theory

The representation of MPLs and eMPLs on genus-zero and genus-one Riemann surfaces motivates to investigate these classes of iterated integrals in the context of string theory, where tree-level and one-loop interactions can be described in terms of worldsheets with these two geometries[1]. Besides this geometric motivation, string theory serves as a highly symmetric and pure laboratory to investigate properties of scattering amplitudes. High-order amplitudes leading to eMPLs and eMZVs can be investigated and efficient methods for their calculations can be constructed. These results may hopefully serve as a basis for future research projects in field theories.

In string perturbation theory, the full scattering amplitude is expanded in the genus of the worldsheet of the underlying string interactions and in the inverse string tension α'. At each genus in this double expansion of the string amplitudes, the corrections to the field-theory limit $\alpha'\to 0$ are referred to as string corrections. In particular it is the string corrections of massless open-(super)string states at genus zero and one, which are the main focus of this thesis. The genus-zero corrections to the (super-)Yang–Mills amplitudes are known to be given by the generating series of MZVs, the Drinfeld associator [28, 29]. This role of the MZVs is described in ref. [5] and facilitates a recursion in the number of external states to calculate the α'-expansion of genus-zero, open-string corrections solely using matrix operations. Further methods to recursively calculate the open-string, genus-zero corrections are known, such as a Berends–Giele recursion [30]. At genus zero, the string corrections of the closed-string states can be deduced from the open-string corrections by the beautiful Kawai–Lewellen–Tye (KLT) relations [31]. Again, this only requires matrix algebra and no integrals have to be calculated. More recently, further methods to obtain the closed-string corrections from the open-string corrections at genus zero have been described. A prominent example [32–34] uses a single-valued map [35, 36] which maps the MZVs in the string corrections to their single-valued analogues, the single-valued MZVs [37].

Research questions

In this thesis, the analogous role of eMPLs and eMZVs in open string theory at genus one is investigated, which ultimately leads to a recursive method to compute the α'-expansion of the string corrections at genus one. The recursion is described in refs. [2, 3] and only uses operations of matrices, which are explicitly calculated in the latter reference, such that none of the iterated integrals on the torus have to

[1] We will restrict ourselves to the two orientable worldsheets, the Riemann sphere and the torus, appearing in string interactions at genus zero and one. These are the relevant geometries for the description of MPLs and eMPLs.

be solved directly. At the same time, another method to calculate the α'-expansion of the genus-one, open-string corrections has been worked out in refs. [38, 39]. The relation between these two complementary approaches is discussed in ref. [3]. It is expected that both points of view are required to relate the string calculations to the corresponding mathematical literature, in particular to ref. [40].

The way to the genus-one recursion from refs. [2, 3] is presented in this thesis and split into three stages, where the following three questions are addressed:

> **Question 1:** How are the various notions of elliptic generalisations of MPLs interrelated, which functional relations do they satisfy and what are the properties of the generating series of these eMPLs?
>
> **Question 2:** What are the crucial ingredients in the genus-zero recursion from ref. [5] leading to the relation of genus-zero, open-string corrections to the MZVs and how can the relevant matrices be calculated?
>
> **Question 3:** Can the results of ref. [5] be generalised to genus one, i.e. are the genus-one, open-string integrals expressible in terms of the generating series of eMZVs and is there a recursive mechanism to calculate these integrals solely using matrix operations?

The first question has lead to the publication [1] and is of mathematical interest in its own right. The MPLs and MZVs are mathematically enormously rich objects leading to deep interconnections between various mathematical areas. A beautiful example is the Bloch–Wigner dilogarithm, a single-valued function on the Riemann sphere constructed by cancelling monodromies of the dilogarithm. Together with its functional relations, it admits a broad variety of mathematical interpretations and applications, ranging from periodicities of a cluster algebra [41–43], volumes in hyperbolic space [44, 45], the symbol calculus [46, 47], the relation to the Dedekind zeta function of an algebraic number field evaluated at two [48], to functional identities generated by rational functions on the Riemann sphere [49]. Thus, there is an intrinsic motivation to explore the properties of eMPLs, where at least part of this mathematical richness encountered at genus zero is expected to carry over to genus one. Moreover, a deep understanding of eMPLs and eMZVs is crucial to tackle questions two and three, which have lead to the publications [2–4].

Besides the importance of the second and third question for string amplitudes, answers are expected to lead to valuable insights into possible applications to quantum field theories. Recursive methods to calculate high-order contributions to scattering amplitudes are very valuable due to the (analytic and computational) complexity of the corresponding integrals. Moreover, special properties of the ingredients of string amplitudes may lead to deep insights into field-theory amplitudes. A prime example is the KLT relation at genus zero, which states that at genus zero closed-string corrections are, loosely speaking, squares of open-string corrections. Since the former describe gravitational states, while the latter lead to states in gauge theory, the field-theory limit of this relation corresponds to the tree-level gauge-gravity duality. A thorough investigation of the open-string corrections at genus one is crucial for a potential extension of this result to higher genera or loop orders, respectively. This is yet another motivation for this thesis: answering the three questions above may be an important step towards a KLT relation at genus one and are hoped to complement current efforts in this research field such as the progress presented in refs. [50–53].

1.2 Background

Particle interactions

In quantum field theory[2] particles are point-like excitations of vacuum states whose evolution in the four-dimensional Minkowski spacetime can be parametrised by lines, the so-called worldline of the particle. Hence, an interaction of several external particles, which may include exchanging particles, decaying into further particles and recombining from certain particles, is described in terms of graphs schematically encoding the trajectory of the interaction in spacetime. For a specific quantum field theory a dictionary associates to each of these so-called *Feynman graphs* an integral, the *Feynman integral*, which in turn, gives the contribution of the considered interaction to the *scattering amplitude* of all the possible interactions of the external particles. The modulus squared of the scattering amplitude is the differential cross-section of the interaction, which can be measured experimentally at particle colliders.

String interactions

In string theory[3] states are described by one-dimensional strings in spacetime (with a priori arbitrary integer dimensions) and excitations correspond to vibrational modes of the strings. The strings may either be closed or open. Thus, the evolution of a string in spacetime sweeps out a two-dimensional surface, a *worldsheet*, such as in figure 1.1. Therefore, a closed-string interaction can schematically be described by a two-dimensional worldsheet in spacetime with various handles or loops, respectively, which correspond to the splitting and recombination of strings, and open boundaries which are the external string states. Similarly, open-string interactions are described by two-dimensional worldsheets with boundaries at which the external string states appear in a certain order.

[2] See e.g. refs. [54–56] for thorough introductions to quantum field theory.

[3] See e.g. the lecture notes from ref. [57] for an accessible introduction to string theory with an emphasis on the mathematical structures. Other classical books are refs. [58–62].

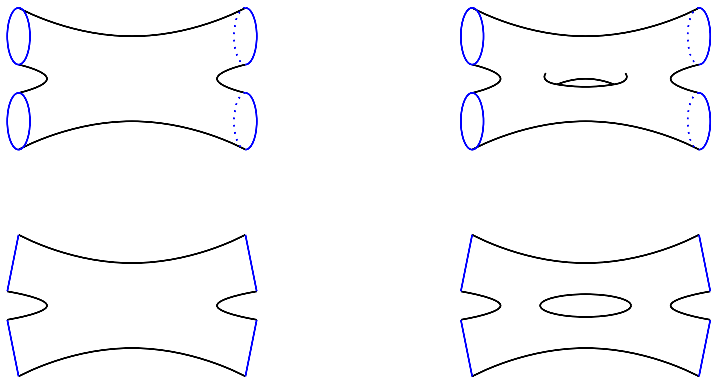


Figure 1.1: Worldsheets of an interaction with four (blue) external strings. On the top, closed worldsheets from four closed strings are depicted, where the boundaries correspond to the external strings. On the bottom, there are four open strings, leading to worldsheets with boundaries, on which the external strings appear in a certain order. The incoming strings can recombine and split up again to form the outgoing strings. Intermediate splittings and recombinations lead to worldsheets with handles (right-hand side), which are high-order (loop) corrections to the four-point interaction.

By conformal symmetry, these configurations can be mapped to punctured Riemann surfaces, where the external string states correspond to the punctures, cf. figure 1.2. In the language of conformal field theory, these punctures are the vertex insertion points of the external states. Correspondingly, closed string interactions are described by punctured Riemann surfaces without boundaries and the open interactions by Riemann surfaces with boundaries, at which the punctures are located. All possible configurations of the punctures on the Riemann surfaces contribute to the final amplitude. Thus, for closed strings all possible positions of the punctures on the Riemann surfaces without boundaries have to be integrated over. In the case of open strings, where the punctures are located at the boundaries in a certain order, all the configurations satisfying this order have to be added up, i.e. integrated along the boundaries. This procedure for open strings leads to iterated integrals on the Riemann surface. Since the punctured Riemann surfaces with boundaries can be embedded into the closed, punctured Riemann surfaces, the latter are the central geometry to describe closed as well as open string amplitudes. Correspondingly, the differential forms are defined[4] on these closed, punctured Riemann surfaces. However, the integration domains are different: for closed-string amplitudes it is the whole surface, while for open-string amplitudes it is only a path on the closed Riemann surface. From here on unless specified otherwise, we will restrict our focus to open strings and these iterated integrals.

[4]They may just be well-defined on universal covers of the surfaces, this is related to non-trivial monodromies of meromorphic functions, cf. chapter 2 and chapter 3.

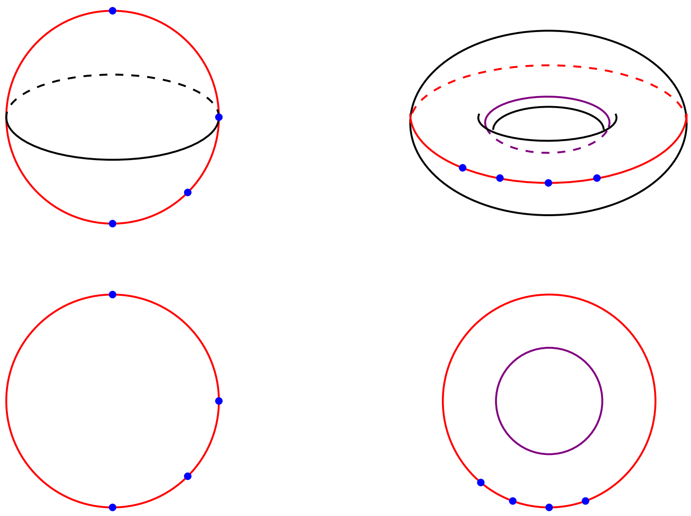


Figure 1.2: The punctured Riemann surfaces are the images of the worldsheets from figure 1.1, after applying a conformal transformation. The number of loops or handles in the worldsheet corresponds to the genus of the Riemann surface. The surfaces with boundaries from the open-string interactions at the bottom, the disk and the perforated disc (topologically a cylinder), can be embedded (red and violet contours) into the closed surfaces from the closed-string interactions on the top. The open-string amplitudes are obtained from integrating the ordered punctures (blue) along the red lines, which leads to the iterated integrals $\boldsymbol{F}^{\text{open}}_{4,0}(\alpha')$ and $\boldsymbol{F}^{\text{open}}_{4,1}(\alpha',\tau)$ appearing in eq. (1.4) below.

The n-point, genus-g, open-string corrections

For a given number of external string states n, the open-string scattering amplitude A^{open}_n obtains contributions from all possible interactions, including any number of loops. Thus, it can be written as a sum of contributions $A^{\text{open}}_{n,g}$ from two-dimensional worldsheets with an increasing number g of loops, which is the genus of the corresponding punctured Riemann surface:

$$A^{\text{open}}_n(\alpha') = \sum_{g\geq 0} A^{\text{open}}_{n,g}(\alpha')\,. \tag{1.1}$$

This is the expansion of the string amplitudes in the genus. For example, the first two terms $A^{\text{open}}_{4,0}(\alpha')$ and $A^{\text{open}}_{4,1}(\alpha')$ are the amplitudes which correspond to the punctured, genus-zero and genus-one Riemann surfaces at the bottom of figure 1.2.

At each genus, all possible configurations of the external states on the corresponding Riemann surface have to be summed up as well, which leads to an integral over the Riemann surface. Separating the colour and kinematic factors from the integral over the external states and the modular parameters of the Riemann surface yields a factorisation into field-theory contributions from particle amplitudes and open-string corrections[5]

$$A^{\text{open}}_{n,g}(\alpha') = \boldsymbol{A}^{\text{particle}}_{n,g} \cdot \boldsymbol{M}^{\text{open}}_{n,g}(\alpha')\,. \tag{1.2}$$

The string corrections in the vector $\boldsymbol{M}^{\text{open}}_{n,g}(\alpha')$ are integrals over the moduli space of the punctured, genus-g Riemann surface, stripped of any other factors which can be pulled out of the integral. Expanding them in the inverse string tension α', a double expansion of the string amplitude in genus and α' is obtained. These corrections depend on α' via the Mandelstam variables

$$s_{ij} = -\alpha'(k_i + k_j)^2\,, \tag{1.3}$$

where k_i is the momentum of the i-th external string state. Moreover, the *moduli-space integrals* $\boldsymbol{M}^{\text{open}}_{n,g}(\alpha')$ can further be decomposed into an integral over the modular parameters $\vec{\tau}$ of the Riemann surface and *configuration-space integrals* $\boldsymbol{F}^{\text{open}}_{n,g}(\alpha',\vec{\tau})$ over the punctures parametrising the external states on the boundary of the punctured Riemann surface with modular parameters $\vec{\tau}$:

$$\boldsymbol{M}^{\text{open}}_{n,g}(\alpha') = \int \mathrm{d}\vec{\tau}\, \boldsymbol{F}^{\text{open}}_{n,g}(\alpha',\vec{\tau})\,. \tag{1.4}$$

The integrals in the vector $\boldsymbol{F}^{\text{open}}_{n,g}(\alpha',\vec{\tau})$ are the iterated integrals mentioned at the end of the previous subsection, for example $\boldsymbol{F}^{\text{open}}_{4,0}(\alpha')$ and $\boldsymbol{F}^{\text{open}}_{4,1}(\alpha',\tau)$ are the iterated integrals obtained from integrating the four punctures in figure 1.2 along the red contours. The entries of the vector $\boldsymbol{F}^{\text{open}}_{n,g}(\alpha',\vec{\tau})$ correspond to the different possible orderings of arranging the n external states on the boundaries of the Riemann surface. These integrals encode the two-dimensionality of the open strings and are present irrelevant of the details of the theory such as compactification or the amount of supersymmetry. They will be referred to as *n-point, genus-g, open-string corrections* and are the central objects considered in this thesis. On the one hand, performing the integral over the modular parameters to obtain the moduli space integrals $\boldsymbol{M}^{\text{open}}_{n,g}(\alpha')$ is another delicate task which requires an extensive analysis. On the other hand, the integrals $\boldsymbol{F}^{\text{open}}_{n,g}(\alpha',\vec{\tau})$ and their dependence on the modular parameters are mathematically rich objects closely related to various geometric aspects of the corresponding Riemann surface.

As mentioned in the previous subsection, for closed strings, the string corrections $\boldsymbol{F}^{\text{closed}}_{n,g}(\alpha',\vec{\tau})$ are integrals over the full configuration space of punctured genus-g Riemann surfaces with modular parameter $\vec{\tau}$.

[5] In general, bold symbols denote vectors or matrices.

1.3 Results

Elliptic multiple polylogarithms

In ref. [1] several constructions of elliptic multiple polylogarithms have been related to the iterated integrals $\tilde{\Gamma}_w$ introduced in refs. [63, 64]. The eMPLs $\tilde{\Gamma}_w$, in turn, are based on the definitions in refs. [7,8]. Moreover, the functional relations from ref. [49] satisfied by the elliptic analogue of the Bloch–Wigner dilogarithm are reinvestigated. They are expressed in terms of the iterated integrals $\tilde{\Gamma}_w$, leading to more general functional relations. Thereby, an alternative derivation of the functional relations of the elliptic Bloch–Wigner dilogarithm is presented. This gives an answer to the first part of question 1.

Multiple polylogarithms in open-string corrections at genus zero

The integrands of the open-string corrections in the vector $\boldsymbol{F}^{\mathrm{open}}_{n,g}(\alpha', \vec{\tau})$ include meromorphic functions with simple poles, depending on the position of the punctures on the boundaries of the genus-g Riemann surface. However, meromorphic functions are not closed under integration. Therefore, iteratively integrating the meromorphic integrands of $\boldsymbol{F}^{\mathrm{open}}_{n,g}(\alpha', \vec{\tau})$ is not expected to yield meromorphic functions defined on the Riemann surface. More general classes of integrals have to be considered in order to describe the string corrections $\boldsymbol{F}^{\mathrm{open}}_{n,g}(\alpha', \vec{\tau})$.

For example at genus zero, the fraction $1/x$ appears in $\boldsymbol{F}^{\mathrm{open}}_{n,0}(\alpha')$. It is a meromorphic function on the Riemann sphere and has a non-vanishing residue, hence, its integral, the logarithm $\log(x)$, is not meromorphic due to the non-trivial monodromy at the origin. Thus, the genus-zero, open-string corrections $\boldsymbol{F}^{\mathrm{open}}_{n,0}(\alpha')$ contain logarithms and integrals thereof, the MPLs $G_w(x)$: they appear in the α'-expansion of the open-string corrections $\boldsymbol{F}^{\mathrm{open}}_{n,0}(\alpha')$ in the form of their values at one, the MZVs $\zeta_w = G_w(1)$. A few years ago, it was shown in ref. [5] that these MZVs appear in a particular form given by the genus-zero, open-string recursion from eq. (4.105) below. Schematically it reads for $n \geq 4$

$$\boldsymbol{F}^{\mathrm{open}}_{n,0}(\alpha') = \Phi_{n,0}(\alpha')\, \boldsymbol{F}^{\mathrm{open}}_{n-1,0}(\alpha')\,, \quad \boldsymbol{F}^{\mathrm{open}}_{3,0}(\alpha') = 1\,, \tag{1.5}$$

where the sum

$$\Phi_{n,0}(\alpha') = \sum_w \zeta_w\, \boldsymbol{e}_{w,n}(\alpha') \tag{1.6}$$

is the Drinfeld associator. It contains all MZVs ζ_w, which are labelled by words $w \in (e_0, e_1)^\times$ from a two-letter alphabet (e_0, e_1). For each n, the matrix $\boldsymbol{e}_{w,n}(\alpha')$, associated to a word w is homogeneous of degree the length $|w|$ of the word w in s_{ij}, thus proportional to $(\alpha')^{|w|}$. The investigation of question 2 in ref. [4] has lead to recursive expressions for the matrices $\boldsymbol{e}_{w,n}(\alpha')$ and a reformulation of the recursion in twisted de Rham theory. The relation (1.5) facilitates the calculation of the α'-expansion of the n-point, open-string corrections from the $(n-1)$-point corrections solely using matrix operations. Geometrically, it can be thought of as a gluing mechanism, where a trivalent interaction is glued to an external state of the $(n-1)$-point worldsheet. This is depicted in figure 1.3.

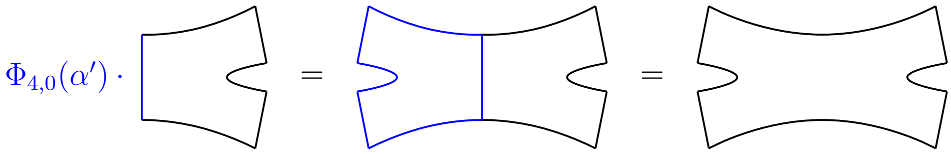


Figure 1.3: The genus-zero, open-string recursion can be interpreted as a gluing mechanism, where the Drinfeld associator $\Phi_{n,0}(\alpha')$ glues a trivalent (blue) interaction to a certain external (blue) state. The geometric interpretation of the four-point example $\Phi_{4,0}(\alpha')\,\boldsymbol{F}^{\text{open}}_{3,0}(\alpha') = \boldsymbol{F}^{\text{open}}_{4,0}(\alpha')$ is depicted.

Elliptic multiple polylogarithms in open-string corrections at genus one

At genus one, the integrands of $\boldsymbol{F}^{\text{open}}_{n,1}(\alpha',\tau)$ are meromorphic functions on the torus, i.e. meromorphic functions with two periods, one and τ. These are the elliptic functions. Their integrals lead to the class of eMPLs $\tilde{\Gamma}_w(z,\tau)$. Using the insights from the projects [1, 4] on the properties of eMPLs and the role of the MPLs in genus-zero, open-string corrections, the second part of question 1 could be answered and a generalisation of the genus-zero recursion (1.5) to genus one has been worked out, answering question 3 as well. The results of refs. [2,3], culminating in the genus-one, open-string recursion from eq. (5.98) below, can schematically be summarised as follows: for $n \geq 2$, the n-point, genus-one, open-string corrections $\boldsymbol{F}^{\text{open}}_{n,1}(\alpha',\tau)$ are obtained from the $(n+2)$-point, genus-zero corrections by

$$\boldsymbol{F}^{\text{open}}_{n,1}(\alpha',\tau) = \Phi_{n,1}(\alpha',\tau)\,\boldsymbol{F}^{\text{open}}_{n+2,0}(\alpha')\,, \tag{1.7}$$

where the sum

$$\Phi_{n,1}(\alpha',\tau) = \sum_w \omega_w(\tau)\,\boldsymbol{x}_{w,n}(\alpha') \tag{1.8}$$

is the generating series of eMZVs, the elliptic Knizhnik–Zamolodchikov–Bernard (KZB) associator [40]. It runs over all eMZVs $\omega_w(\tau) = \tilde{\Gamma}_w(1,\tau)$, which are labelled by words $w \in (x_0, x_1, x_2, \dots)^\times$ from an infinite alphabet $(x_0, x_1, x_2, \dots)$. The matrices $\boldsymbol{x}_{w,n}(\alpha')$ are again of degree $|w|$ in α' and explicitly known, cf. eq. (C.67) below. Even though eq. (1.7) by its one is at most the first step in a potential recursion in the genus, it is still called a recursion, since together with the genus-zero recursion (1.5), it can be used to recursively calculate the genus-one, open-string corrections for any number of external states n. The exact formulation of this genus-one, open-string recursion, including the calculation of the matrices $\boldsymbol{x}_{w,n}(\alpha')$ is the main focus of this thesis. Geometrically, eq. (1.7) expresses the gluing of two external states from a $(n+2)$-point, genus-zero worldsheet to an n-point, genus-one worldsheet, cf. figure 1.4.

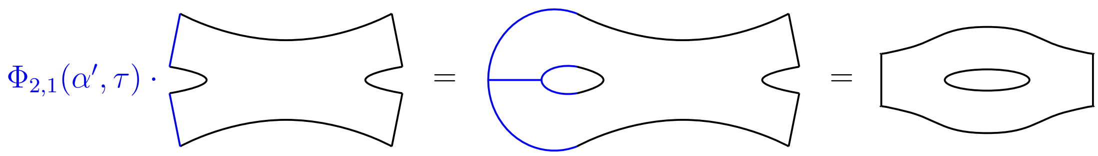


Figure 1.4: The genus-one, open-string recursion can be interpreted as a gluing mechanism, where the elliptic KZB associator $\Phi_{n,1}(\alpha', \tau)$ glues together two external states from a genus-zero, $(n+2)$-point interaction to from a genus-one interaction with n external states. For example, the action of $\Phi_{2,1}(\alpha', \tau)$ on the genus-zero, four-point corrections leads, according to $\Phi_{2,1}(\alpha', \tau)\, \boldsymbol{F}^{\text{open}}_{4,0}(\alpha') = \boldsymbol{F}^{\text{open}}_{2,1}(\alpha')$, to the two-point corrections at genus one, which is depicted in this figure on the level of the worldsheets.

Graphical method for generating series

The reinvestigation of the genus-zero recursion (1.5) in ref. [4] has lead to a graphical representation for products of a certain class of generating series of meromorphic functions. This graphical tool was crucial to structure the combinatorics in the calculation of the matrices $\boldsymbol{e}_{w,n}(\alpha')$ in the genus-zero recursion (1.5) and $\boldsymbol{x}_{w,n}(\alpha')$ in the genus-one recursion (1.7) in refs. [3, 4]. In this thesis, it is extended and defined in such a generality that it may hopefully be applied to future higher-genus calculations as well.

1.4 Outline

In chapter 2 various well-known aspects of MPLs are presented: integrals of meromorphic functions on the Riemann sphere are discussed in section 2.1, leading to the MPLs presented in section 2.2. In section 2.3 their monodromies are analysed and single-valued MPLs are constructed by cancelling these monodromies. The chapter finishes with section 2.4, where further mathematical concepts are introduced to describe functional relations of (single-valued) MPLs.

The analogous investigation at genus one is conducted in chapter 3: elliptic functions are discussed in section 3.1, while their integrals resulting in the eMPLs are investigated in section 3.2. The elliptic KZB associator, a crucial ingredient in the genus-one recursion (1.7), is introduced in section 3.3. Certain constructions of single-valued eMPLs are described in section 3.4. Finally, the results on eMPLs and their functional relations from ref. [1] are presented in section 3.5.

Having introduced and investigated MPLs and eMPLs, in chapter 4 the genus-zero recursion (1.5) is outlined: the precise form of the genus-zero, open-string corrections $\boldsymbol{F}^{\text{open}}_{n,0}(\alpha')$ is given in section 4.1. As a new result, the string corrections are generalised in section 4.2 to a class of integrals with integrands defined on the n-punctured Riemann sphere, where p punctures thereof remain unintegrated. This leads to a convenient formulation of the genus-zero recursion (1.5) in section 4.3. The latter two sections contain several results from ref. [4].

Chapter 5 is dedicated to the genus-one recursion (1.7) and follows the outline of the original construction in refs. [2, 3] and the previous genus-zero chapter: first, the open-string corrections $\boldsymbol{F}^{\text{open}}_{n,1}(\alpha', \tau)$ at genus one are introduced in section 5.1. Second, they are generalised to iterated integrals on the p-punctured torus with integrands defined on the n-punctured torus in section 5.2. This leads to the genus-one recursion in section 5.3.

The graphical method introduced in ref. [4] and used in refs. [2–4] to investigate products of generating series is extended, rigorously formulated and used to derive various identities in chapter 6. This is a crucial tool to structure the combinatorics in the derivation of the explicit expressions of the matrices $\boldsymbol{e}_{w,n}(\alpha')$ in the genus-zero recursion (1.5) and $\boldsymbol{x}_{w,n}(\alpha')$ in the genus-one recursion (1.7). The remaining chapters are formulated without this graphical method and can be understood without reading this chapter. At certain points, however, results obtained using the graphical derivation are stated with a reference to this chapter.

In the final chapter 7 the main results and their appearance in this thesis are summarised. Additionally, a brief outlook and a discussion on further questions and research directions are given.

The group structure of the elliptic curve is summarised in appendix A. In appendix B various calculations associated to elliptic polylogarithms, a subclass of the eMPLs, are explicitly given. In the last appendix C the derivation of various identities from chapter 6 is explicitly shown, which includes the closed formula for the matrices $\boldsymbol{x}_{w,n}(\alpha')$ in the genus-one recursion (1.7).

1.5 Publications and contributions by the author

The author of this thesis has composed or collaborated in the four publications [1–4] relevant for this thesis: the collaborative project [1] on functional relations of eMPLs and his single-author paper [4] on the genus-zero, open-string recursion (1.5) have motivated the two collaborative publications [2, 3] on the genus-one, open-string recursion (1.7). In section 5.2 certain results solely worked out by the author of this thesis from a current work in progress [65] are included as well. The content of chapter 6 and appendix C is new and composed by the author of this thesis.

Chapter 2

Multiple polylogarithms

To begin with, the essential mathematical objects required in the formulation of the genus-zero recursion (1.5) are introduced and discussed. In particular the MPLs and MZVs are motivated beginning with the integration of meromorphic functions on the Riemann sphere, following the outline of refs. [63, 66]. Afterwards, their single-valued versions are constructed according to ref. [37] and certain functional relations are mentioned, for example the results of ref. [49] along the lines of ref. [67].

The content of this chapter is well-known and introduced as a preliminary to the subsequent chapters. The only originality consists in the presented interconnections and structure, large parts of which are based on the references mentioned above. In section 2.1 meromorphic functions on the Riemann sphere are integrated, which leads to the MPLs in section 2.2. Single-valued versions are constructed in section 2.3. In section 2.4, some functional relations of these single-valued MPLs are reviewed.

2.1 Meromorphic functions on the sphere

The integrands of the genus-zero, open-string corrections $\boldsymbol{F}^{\text{open}}_{n,0}(\alpha')$ are meromorphic functions on the Riemann sphere. Therefore, in order to describe and investigate these string corrections, the class of functions obtained by iterated integration of such meromorphic functions has to be studied.

2.1.1 Rational functions

By definition, a *rational function* $f(x)$ on $\mathbb{C}$ in one variable[1] x, which is denoted by $f \in \mathbb{C}(x)$, can uniquely be expressed as quotient of two coprime polynomials of the

[1] We will usually use the variable x for meromorphic functions on the Riemann sphere, while the variable z is reserved for meromorphic functions on the torus introduced in subsequent chapters.

form

$$\begin{aligned} F(x) &= a\prod_{i=1}^{n}(x-b_i)^{n_i} \in \mathbb{C}[x]\,, \\ G(x) &= \prod_{i=1}^{m}(x-c_i)^{m_i} \in \mathbb{C}[x]^*\,, \end{aligned} \tag{2.1}$$

where the denominator $G(x)$ is not the zero polynomial. The non-negative integers

$$\deg(F) = n\,, \quad \deg(G) = m \tag{2.2}$$

are called the *degree* of F and G, respectively, while the positive integers $n_i, m_j \in \mathbb{Z}_{>0}$ are the *multiplicities* of the zeros b_i and poles c_i, respectively. Therefore, the rational function $f(x)$ can be written as a product

$$\begin{aligned} f(x) &= \frac{F(x)}{G(x)} \\ &= a\prod_{i=1}^{n+m}(x-a_i)^{d_i}\,, \end{aligned} \tag{2.3}$$

where the non-zero integers $d_i \in \mathbb{Z}_{\neq 0}$ are given by

$$d_i = \begin{cases} n_i & \text{if } a_i = b_i\,, \\ -m_i & \text{if } a_i = c_i\,. \end{cases} \tag{2.4}$$

The *order* of the rational function at some point x is defined in terms of these integers as follows

$$\mathrm{ord}_x(f) = \begin{cases} d_i & \text{if } x = a_i\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.5}$$

The representation (2.3) indicates that univariate rational functions $f(x) \in \mathbb{C}(x)$ are meromorphic functions on $\mathbb{C}$ and that eq. (2.5) defines the order of vanishing of f at x

$$\mathrm{ord}_x(f) = \mathrm{res}_x\left(\frac{f'}{f}\right)\,. \tag{2.6}$$

However, the inverse does not hold since for example e^x/x is meromorphic, but not rational on $\mathbb{C}$. Upon one-point compactification, the rational functions on $\mathbb{C}$ can naturally be extended to rational functions on the Riemann sphere[2] $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$,

[2] For the sake of simplicity, we choose to work with the dehomogenised form of the Riemann sphere $\mathbb{P}^1(\mathbb{C}) = \{[X:Y]\}$ using the non-homogeneous coordinates $x = X/Y \in \mathbb{C}$, $y = 1$ and $\infty = [1:0]$, such that $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$ can be identified by a minor abuse of notation.

denoted by

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}\,. \tag{2.7}$$

In terms of maps $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, this extension is obtained by assigning the value

$$f(\infty) = \begin{cases} a & \text{if } \deg(F) = \deg(G)\,, \\ 0 & \text{if } \deg(F) < \deg(G)\,, \\ \infty & \text{if } \deg(F) > \deg(G) \end{cases} \tag{2.8}$$

to the north pole ∞. Since the Riemann sphere is compact, Liouville's theorem now ensures that the meromorphic functions on the Riemann sphere $\hat{\mathbb{C}}$ are indeed exactly the rational functions on $\mathbb{C}$, naturally extended to $\hat{\mathbb{C}}$. Due to this fact, we will interchangeably call functions of the form (2.3) rational functions on the complex plane or meromorphic functions on the Riemann sphere. The ring of meromorphic functions on the Riemann sphere is denoted by

$$\begin{aligned} \hat{\mathbb{C}}(x) &= \{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \text{ meromorphic}\} \\ &= \{f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \mid f|_{\mathbb{C}} \in \mathbb{C}(x) \text{ and } f(\infty) \text{ given by (2.8)}\}\,. \end{aligned} \tag{2.9}$$

According to the representation (2.3), the rational functions are determined up to an overall constant by their zeros and poles, including the multiplicities. In order to facilitate the discussion on this dependence, the *group of divisors* of $\hat{\mathbb{C}}$ (or subsets thereof) is introduced: it is the free abelian group generated by points of $\hat{\mathbb{C}}$ and denoted by $\mathrm{Div}(\hat{\mathbb{C}})$. Thus, a divisor $D \in \mathrm{Div}(\hat{\mathbb{C}})$ is of the form

$$D = \sum_{x \in \hat{\mathbb{C}}} n_x (x)\,, \tag{2.10}$$

where all but finitely many of the integer coefficients $n_x \in \mathbb{Z}$ are zero. Using the order defined in eq. (2.5), a divisor can be associated to each rational function by

$$\mathrm{div}(f) = \sum_{x \in \hat{\mathbb{C}}} \mathrm{ord}_x(f)(x)\,, \tag{2.11}$$

where the order of f at infinity is given by the order of $g(x) = f(1/x)$ at zero. A divisor $D \in \mathrm{Div}(\hat{\mathbb{C}})$ is called *principal* if and only if there exists a rational function $f \in \mathbb{C}(x)$ such that $D = \mathrm{div}(f)$.

2.1.2 Integrating rational functions

The integral of a rational function $f(x) \in \mathbb{C}(x)$ as defined in eq. (2.3) with the singularities $S = \{c_1, \ldots, c_m\}$ leads to the following class of functions: using partial fractioning

$$\frac{1}{x_{ki}x_{kj}} = \frac{1}{x_{ki}x_{ij}} + \frac{1}{x_{kj}x_{ji}}\,, \tag{2.12}$$

where $x_i, x_j, x_k \in \mathbb{C}$ are distinct and

$$x_{ij} = x_{i,j} = x_i - x_j\,, \tag{2.13}$$

the integral of $f(x)$ is (up to an additive constant) a $\mathbb{C}$-linear combination (2.3) of integrals of the form

$$\int \frac{\mathrm{d}x}{(x-c_i)^k} = \begin{cases} \frac{(x-c_i)^{1-k}}{1-k} & \text{if } k \neq 1\,, \\ \log(x-c_i) & \text{if } k = 1\,, \end{cases} \tag{2.14}$$

where $k \in \mathbb{Z}$. For the case $k = 1$, logarithms are obtained, thus, rational functions are not closed under integration. This is expected from Cauchy's residue theorem: the integral of $1/(x-c_i)$ along a path from some $x_0 \neq c_i$ to some $x \neq c_i$ depends on the homotopy class of the integration path on $\mathbb{C}\backslash S$ and not only on the endpoint x. Thus, it is a *multi-valued*[3] function of x. However, rational functions are single-valued, hence the integral of $1/(x-c_i)$ can not be a rational function and is denoted by log.

In order to obtain a closed class of functions with respect to integration starting from rational functions, for $c_r \neq 0$ iterated integrals of the form

$$\begin{aligned} G(c_1, c_2, \dots, c_r; x) &= \int_0^x \mathrm{d}x' \frac{1}{x'-c_1} G(c_2, \dots, c_r; x')\,, \\ G(;x) &= 1 \end{aligned} \tag{2.15}$$

have to be considered, where $c_i \in S$ for some finite set of distinct points $S \subset \mathbb{C}$. These functions of $x \in \mathbb{C}\backslash S$ are called Goncharov polylogarithms (GPLs) [68, 69]. The non-negative integer r is called the *weight* of $G(c_1, c_2, \dots, c_r; x)$. For $c_r = 0$, the definition (2.15) would lead to divergent integrals due to the pole of the innermost integration kernel at $x = 0$ from the lower integration boundary. This may be accounted for by the following definition

$$G(\underbrace{0, \dots, 0}_{r}; x) = \frac{1}{r!} \log^r(x)\,, \quad \log(x) = \int_1^x \frac{\mathrm{d}x'}{x'} \tag{2.16}$$

[3] For a punctured Riemann surface M and $\pi : \tilde{M} \to M$ a universal cover, a holomorphic function $\tilde{f}$ on $\tilde{M}$ is called multi-valued if there is a holomorphic function f defined on a simply connected subset $U \subset M$, such that $\tilde{f}$ lifts f and there is a non-unique holomorphic continuation of f to M. Any such continuation of f to M is called a *multi-valued function on* M. In this sense, the logarithm and the GPLs defined below are multi-valued due to the path-dependent integration.

and using relations expected from the shuffle algebra of iterated integrals to reduce the general case $c_r = 0$ to the definitions (2.15) and (2.16). The resulting functions can as well be obtained from the definition (2.15) by a so-called tangential base-point regularisation [70]. This regularisation prescription ensures that for general sequences $A = (a_1, a_2, \ldots, a_r)$ and $B = (b_1, b_2, \ldots, b_s)$ the GPLs still satisfy the *shuffle algebra*

$$G(A;x)G(B;x) = G(A \sqcup\!\sqcup B;x) \tag{2.17}$$

even though not all of them are iterated integrals. Above, the identification of the GPLs by a sequence $A \mapsto G(A;x)$ is linearly extended to a formal sum of sequences and the shuffle of two sequences A and B is a linear combination of sequences, which is recursively defined by

$$\begin{aligned} A \sqcup\!\sqcup B &= (a_1, (a_2, \ldots a_r) \sqcup\!\sqcup B) + (b_1, A \sqcup\!\sqcup (b_2, \ldots, b_s))\,, \\ A \sqcup\!\sqcup \varnothing &= \varnothing \sqcup\!\sqcup A = A\,. \end{aligned} \tag{2.18}$$

Therefore, $A \sqcup\!\sqcup B$ is the sum of all permutations of $(a_1, \ldots, a_r, b_1, \ldots, b_s)$ such that the order of A and B is preserved. E.g. for $A = (1,2)$ and $B = (3,4)$, it is given by

$$\begin{aligned} (1,2) \sqcup\!\sqcup (3,4) = (1,2,3,4) &+ (1,3,2,4) + (3,1,2,4) \\ &+ (1,3,4,2) + (3,1,4,2) + (3,4,1,2)\,. \end{aligned} \tag{2.19}$$

To summarise, eqns. (2.15), (2.16) and (2.17) define the functions $G(c_1, \ldots, c_r; x)$ for any $c_i \in S$.

Due to the path-dependence of the logarithm $G(0;x) = \log(x)$, this class of functions is not single-valued. In this thesis, the branch cut of the logarithm is generally chosen to be $\mathbb{R}_{<0}$ and the identification $\log(-1) = i\pi$ is used.

The mentioned closure of the GPLs under integration can now be stated more precisely: let $\mathcal{R}(S) \subset \mathbb{C}(x)$ be the field of rational functions with poles at most at the points S and denote the $\mathcal{R}(S)$-algebra generated by all the GPLs G with singularities at most at S by $\mathcal{A}_{\mathrm{GPL}}(S)$. The multiplication is given by the shuffle product, which leads to a grading of $\mathcal{A}_{\mathrm{GPL}}(S)$ by the weight

$$\begin{aligned} \mathcal{A}_{\mathrm{GPL}}(S) &= \bigoplus_{r \geq 0} \mathcal{A}_{\mathrm{GPL},r}(S)\,, \\ \mathcal{A}_{\mathrm{GPL},r}(S) &= \langle G(c_1, c_2, \ldots, c_r; x) \mid c_i \in S \rangle_{\mathcal{R}(S)}\,. \end{aligned} \tag{2.20}$$

The algebra $\mathcal{A}_{\mathrm{GPL}}(S)$ is closed under differentiation and integration: for every $f \in \mathcal{A}_{\mathrm{GPL}}(S)$, there exists a $F \in \mathcal{A}_{\mathrm{GPL}}(S)$ such that $f = \partial_x F$ and vice-versa.

Let us point out that partial fractioning (2.12) is crucial for the closure under integration: it ensures that any product of two different integration kernels from eq. (2.15) can be written as a linear combination of such kernels

$$\frac{1}{x-c_i}\frac{1}{x-c_j} = \frac{1}{c_j-c_i}\frac{1}{x-c_j} + \frac{1}{c_i-c_j}\frac{1}{x-c_i}\,. \tag{2.21}$$

Similarly, integration by parts can be used to reduce any higher power of an integration kernel to $k=1$ and total derivatives

$$\frac{1}{(x-c_i)^k} = \frac{1}{1-k}\partial_x \frac{1}{(x-c_i)^{k-1}} \tag{2.22}$$

for $k \neq 1$.

2.2 Multiple polylogarithms

In the context of n-point, open-string amplitudes at genus zero, only a subclass of the GPLs needs to be considered. The reason is that there is a residual gauge symmetry, a global transformation on the Riemann sphere arising from diffeomorphisms which can be undone by Weyl transformations. The symmetry group is the conformal Killing group $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$, which can be used to gauge fix three of the n punctures $x_1, x_2, \dots, x_n$ at arbitrary points, e.g. $(x_1, x_2, x_n) = (0, 1, \infty)$. In order to include all the possible configurations of external string states contributing to the string amplitude, all the other (unfixed) punctures $x_3, x_4, \dots, x_{n-1}$ are integrated over. Hence, the unfixed punctures can be thought of as the integration variable x' in the definition (2.15), while the fixed punctures take the role of the singularities c_i. Accordingly, only the subclass of GPLs with $c_i \in S = \{0,1\}$ appears in the open-string corrections at genus zero.

The subclass of GPLs with singularities $c_i \in \{0,1\}$ is called *multiple polylogarithms* (MPLs) [71]. The set of all words generated by the two-letter alphabet $\mathcal{E} = (e_0, e_1)$, which is an ordered tuple of letters, is denoted by $\mathcal{E}^\times$. Hence, MPLs are multi-valued functions on $\mathbb{C}\backslash\{0,1\}$ indexed by words ending in e_1,

$$w = e_0^{n_r-1}e_1 \dots e_0^{n_1-1}e_1 \in \mathcal{E}^\times e_1\,, \tag{2.23}$$

where $n_i \geq 1$ and denoted by

$$G_w(x) = G(\underbrace{0,\dots,0}_{n_r-1}, 1, \dots, \underbrace{0,\dots,0}_{n_1-1}, 1; x) \tag{2.24}$$

and

$$G_\varnothing(x) = 1 \tag{2.25}$$

for the empty word $w = \varnothing$. For $|x| < 1$ the MPLs of the form (2.24) exhibit a well-known sum representation usually denoted by

$$\mathrm{Li}_{n_1,\ldots,n_r}(x) = \sum_{1\leq k_1<\cdots<k_r} \frac{x^{k_r}}{k_1^{n_1}\ldots k_r^{n_r}} = (-1)^r G_w(x)\,, \tag{2.26}$$

which can be analytically continued to $|x| \geq 1$. The subclass corresponding to $r = 1$ is called *polylogarithms* (PLs) of *weight* n_1

$$G_{e_0^{n_1-1}e_1}(x) = -\,\mathrm{Li}_{n_1}(x)\,. \tag{2.27}$$

While the definitions (2.24) and (2.25) only involve GPLs defined by eq. (2.15), one can also associate MPLs to words $w \in \mathcal{E}^\times e_0$ ending in e_0. This leads to the shuffle-regularised MPLs based on eqs. (2.16) and (2.17), such that for $n \geq 0$

$$G_{e_0^n}(x) = \frac{\log^n(x)}{n!}\,. \tag{2.28}$$

and for any words $w', w'' \in \mathcal{E}^\times$ the shuffle algebra

$$G_{w'}(x)G_{w''}(x) = G_{w'\sqcup\!\sqcup w''}(x) \tag{2.29}$$

is satisfied, i.e. inherited from eq. (2.17). This implies that for words of the form $w = e_0^n$, MPLs exhibit a logarithmic divergence in the limit[4] $x \to 0$, while they vanish for all the other words

$$\lim_{x\to 0} G_w(x) = 0\,, \quad \text{if} \quad w \neq e_0^n\,. \tag{2.30}$$

It is known that the MPLs are linearly independent over the ring $\mathbb{C}[x, \frac{1}{x}, \frac{1}{x-1}]$ of regular functions on $\mathbb{C}\backslash\{0,1\}$ [72, 73]. Moreover, according to the above definitions MPLs indexed by words $e_i w \in e_i\mathcal{E}^\times$ for $i \in \{0,1\}$ satisfy the differential equations

$$\partial_x G_{e_0 w}(x) = \frac{1}{x} G_w(x)\,, \quad \partial_x G_{e_1 w}(x) = \frac{1}{x-1} G_w(x)\,. \tag{2.31}$$

In fact, the MPLs are the unique family of holomorphic functions defined on the open set $U = \mathbb{C}\backslash((-\infty,0]\cup[1,\infty))$ satisfying the recursive differential equations (2.31) with initial condition (2.30) and the regularisation in eq. (2.28) [37, 74]. Thus, the MPLs may alternatively be defined in terms of a *generating series of MPLs*: it is the unique holomorphic solution $L_\mathcal{E}(x)$ of the differential equation

$$\partial_x L_\mathcal{E}(x) = \left(\frac{e_0}{x} + \frac{e_1}{x-1}\right) L_\mathcal{E}(x) \tag{2.32}$$

[4]Throughout this thesis, any limits to zero or one are generally taken within the unit interval unless stated otherwise.

such that

$$\lim_{x\to 0} x^{-e_0} L_{\mathcal{E}}(x) = 1\,. \tag{2.33}$$

Indeed, the function $L_{\mathcal{E}}(x)$ is the generating series of MPLs

$$L_{\mathcal{E}}(x) = \sum_{w\in\mathcal{E}^\times} w\, G_w(x)\,, \tag{2.34}$$

since the right-hand side of eq. (2.34) satisfies the differential equation (2.32) due to eq. (2.31) and its asymptotic behaviour as $x \to 0$ is according to eqs. (2.28) and (2.30) given by

$$\sum_{w\in\mathcal{E}^\times} w\, G_w(x) \sim \sum_{n\geq 0} e_0^n \frac{\log^n(x)}{n!} = x^{e_0}\,, \tag{2.35}$$

in agreement with the initial condition (2.33). The map $L_{\mathcal{E}} : x \to L_{\mathcal{E}}(x)$ is a multi-valued map from $\mathbb{C}\backslash\{0,1\}$ into the ring of formal power series in the words $\mathcal{E}^\times$ with coefficients in $\mathbb{C}$, denoted by

$$\mathbb{C}\langle\langle\mathcal{E}\rangle\rangle = \{\sum_{w\in\mathcal{E}^\times} c_w\, w \mid c_w \in \mathbb{C}\} \tag{2.36}$$

and equipped with the concatenation product of words.

A differential equation of the form (2.32) is called Knizhnik–Zamolodchikov (KZ) equation [75]. It will play an essential role in the construction of the recursion (1.5) for open-string corrections at genus zero, which will be summarised in subsection 2.2.2 and extensively discussed in chapter 4.

2.2.1 Multiple zeta values

In ref. [76] physicists realised that many scattering amplitudes can be expressed in terms of MPLs evaluated at certain points, which yields multiple zeta values. More than two hundred years ago, these numbers were investigated by Euler. More recently, they have again come to the attention of mathematicians, e.g. in ref. [77]. By now, they are known to be basic ingredients in a huge number of scattering amplitudes and their appearance therein is closely related to the properties of the corresponding quantum field or string theory. A prime example is open string theory, where the genus-zero string corrections involve the generating series of multiple zeta values, cf. the genus-zero, open-string recursion (1.5), which will be discussed in detail in chapter 4.

The *multiple zeta values* (MZVs) are defined by the value at one of the sum representation $\mathrm{Li}_{n_1,\dots,n_r}(x)$ with $n_r > 1$, such that Li converges:

$$\zeta_{n_1,\dots,n_r} = \mathrm{Li}_{n_1,\dots,n_r}(1)\,. \tag{2.37}$$

The sum $\sum_{i=1}^{r} n_i$ is called the *weight* and r the *depth* of $\zeta_{n_1,\dots,n_r}$. In terms of words, the MZVs are labelled by words of the form

$$w = e_0^{n_r-1}e_1 \dots e_0^{n_1-1}e_1 \in e_0\,\mathcal{E}^\times e_1, \quad n_r > 1\,, \tag{2.38}$$

i.e. beginning with e_0 and ending with e_1, such that $\zeta_{n_1,\dots,n_r}$ can be denoted by

$$\zeta_w = \zeta_{n_1,\dots,n_r} = (-1)^r G_w(1)\,. \tag{2.39}$$

Using the shuffle regularisation of G_w from eqs. (2.28) and (2.29), the definition of MZVs may be extended to also include words ending in e_0, this leads in particular to the value

$$\zeta_{e_0} = G_{e_0}(1) = G(0;1) = 0\,. \tag{2.40}$$

In addition to the pole of $\mathrm{d}x/x$ at the lower integration boundary for words ending in e_0 which required the regularisation of the MPLs, the integral $G_w(1)$ will also diverge at the upper integration boundary for words beginning with e_1 due to the pole of the differential form $\mathrm{d}x/(x-1)$ at $x = 1$. This issue has been circumvented so far by the requirement $n_r > 1$ in the definition of MZVs. The definition (2.39) can be extended to any word beginning with e_1. This regularisation of MZVs is again a tangential base-point regularisation [70, 78]: in negative direction at $x = 1$ (and along the positive direction for the regularisation of MPLs at $x = 0$) [66]. In analogy to the definition (2.40), it effectively amounts to setting the divergent integrals to zero

$$\zeta_{e_1} = -G_{e_1}(1) = -G(1;1) = 0\,, \tag{2.41}$$

and the use of the shuffle algebra

$$\zeta_{w'}\zeta_{w''} = \zeta_{w' ⧢ w''} \tag{2.42}$$

to reduce the remaining cases to the definitions (2.39), (2.40) and (2.41).

Multiple zeta values satisfy further relations than shuffle relations, a prominent example are the *stuffle relations*, which can be derived by a resummation of the sum representation from eq. (2.37). The simplest example of such a stuffle relation is

$$\zeta_m\zeta_n = \zeta_{m,n} + \zeta_{n,m} + \zeta_{m+n}\,. \tag{2.43}$$

The $\mathbb{Q}$-span of the MZVs is a subalgebra of $\mathbb{R}$, graded by the weight of the MZVs. There are conjectures for bases of this subalgebra at fixed weights and at fixed depths [77, 79, 80], cf. ref. [81] for a data mine of such bases at lowest weights. However, a rigorous mathematical treatment of the underlying algebraic structures of MZVs is so far only possible for a sophisticated model of the MZVs, based on a so-called f-alphabet and motivic MZVs [82–85]. This, in particular, leads to a decomposition of any motivic MZV into a non-canonical basis [83].

2.2.2 Drinfeld associator

The MZVs may also be described in terms of a generating series, derived from the generating series of MPLs $L_\mathcal{E}(x)$. Due to the divergence at $x = 1$ of the MPLs $G_{e_1 w}(x)$ mentioned above, the value of $L_\mathcal{E}(x)$ at one is divergent as well. Hence, in order to obtain the generating series of MZVs, the series $L_\mathcal{E}(x)$ has to be regularised as well before it can be evaluated at $x = 1$.

Let $\tilde{\mathcal{E}} = (e_1, e_0)$ denote the alphabet $\mathcal{E} = (e_0, e_1)$ with reversed order. By the symmetry $x \to 1-x$ of the KZ equation (2.32) (and interchanging the roles of e_0 and e_1), the function $L_{\tilde{\mathcal{E}}}(1-x)$ is a solution of the KZ equation (2.32) with asymptotic behaviour for $x \to 1$ given by

$$L_{\tilde{\mathcal{E}}}(1-x) \sim \sum_{n\geq 0} e_1^n \frac{\log^n(1-x)}{n!} = (1-x)^{e_1} \,. \tag{2.44}$$

The product

$$\Phi_\mathcal{E}(x) = (L_{\tilde{\mathcal{E}}}(1-x))^{-1} L_\mathcal{E}(x) \tag{2.45}$$

relates the two solutions according to

$$L_\mathcal{E}(x) = L_{\tilde{\mathcal{E}}}(1-x)\Phi_\mathcal{E}(x) \tag{2.46}$$

and is independent of x, which holds for any product of the form $(L_1(x))^{-1}L_2(x)$ where $L_1(x)$ and $L_2(x)$ are solutions of the same KZ equation (i.e. with the same letters e_0 and e_1). This independence can readily be shown by differentiating both sides of eq. (2.46). The product $\Phi_\mathcal{E}(x)$ is known as the *Drinfeld associator* [28, 29] and due to the x-independence usually denoted by

$$\Phi_\mathcal{E} = \Phi_\mathcal{E}(x) \in \mathbb{C}\langle\langle \mathcal{E} \rangle\rangle \,. \tag{2.47}$$

The Drinfeld associator is the generating series of MZVs [86], which can be seen using the x-independence to evaluate the product $\Phi_\mathcal{E}(x)$ in the limit $x \to 1$, where the asymptotic behaviour (2.44) can be exploited:

$$\begin{aligned}\Phi_{\mathcal{E}} &= \lim_{x\to 1}\Phi_{\mathcal{E}}(x)\\ &= \lim_{x\to 1}(1-x)^{e_1}L_{\mathcal{E}}(x)\\ &= \sum_{w\in\mathcal{E}^{\times}} w\,\zeta_w\\ &= 1-\zeta_2[e_0,e_1]-\zeta_3[e_0+e_1,[e_0,e_1]]\\ &\quad+\zeta_4([e_1,[e_1,[e_1,e_0]]]+\tfrac{1}{4}[e_1,[e_0,[e_1,e_0]]]\\ &\quad-[e_0,[e_0,[e_0,e_1]]]+\tfrac{5}{4}[e_0,e_1]^2)+\dots\,. \end{aligned} \tag{2.48}$$

Therefore, the regulating prefactor $(1-x)^{-e_1}$ implements the regularisation from eqs. (2.41) and (2.42), leading to the generating series of MZVs. For the last equality in the calculation (2.48), relations among MZVs have been implemented [81].

The Drinfeld associator does not only relate the two solutions $L_{\mathcal{E}}(x)$ and $L_{\tilde{\mathcal{E}}}(1-x)$ of the KZ equation according to eq. (2.46), but actually relates two regularised boundary values of any solution of the KZ equation as follows: let us consider an arbitrary solution $L(x)$ of the KZ equation (2.32) and define the two regularised boundary values

$$C_0(L)=\lim_{x\to 0}x^{-e_0}L(x)\,,\quad C_1(L)=\lim_{x\to 1}(1-x)^{-e_1}L(x)\,. \tag{2.49}$$

These boundary values are connected by the Drinfeld associator according to the *genus-zero associator equation*

$$C_1(L)=\Phi_{\mathcal{E}}\,C_0(L)\,. \tag{2.50}$$

This equation is obtained using the asymptotics (2.35) and (2.44) as well as the x-independence of an inverse solution times a solution of the KZ equation (mentioned below eq. (2.46)):

$$\begin{aligned}\Phi_{\mathcal{E}}\,C_0(L) &= \lim_{x\to 0}\Phi_{\mathcal{E}}(x)x^{-e_0}L(x)\\ &= \lim_{x\to 0}(L_{\tilde{\mathcal{E}}}(1-x))^{-1}L(x)\\ &= \lim_{x\to 1}(L_{\tilde{\mathcal{E}}}(1-x))^{-1}L(x)\\ &= \lim_{x\to 1}(1-x)^{-e_1}L(x)\\ &= C_1(L)\,. \end{aligned} \tag{2.51}$$

The associator equation (2.50) is the backbone of the open-string recursion (1.5) at genus zero. The main result given in eq. (4.105) below and derived in ref. [5] is that for each $n\geq 5$ a $(n-3)!$-dimensional vector of iterated integrals $\hat{\boldsymbol{F}}_n$ satisfying a KZ equation with some generators (square matrices) $\boldsymbol{\mathcal{E}}_n = (\boldsymbol{e}_{0,n}, \boldsymbol{e}_{1,n})$ has been constructed. The regularised boundary values include the $(n-2)$-point and $(n-1)$-point open-string corrections $\boldsymbol{F}^{\text{open}}_{n-2,0}$ and $\boldsymbol{F}^{\text{open}}_{n-1,0}$, respectively:

$$\boldsymbol{C}_0(\hat{\boldsymbol{F}}_n) = \begin{pmatrix} \boldsymbol{F}^{\text{open}}_{n-2,0} \\ 0 \end{pmatrix} , \quad \boldsymbol{C}_1(\hat{\boldsymbol{F}}_n) = \begin{pmatrix} \boldsymbol{F}^{\text{open}}_{n-1,0} \\ \vdots \end{pmatrix} . \tag{2.52}$$

Hence, eq. (2.50) yields a recursion solely using matrix algebra to calculate the $(n-1)$-point integrals $\boldsymbol{F}_{n-1}$ from the $(n-2)$-point integrals $\boldsymbol{F}_{n-2}$,

$$\begin{pmatrix} \boldsymbol{F}^{\text{open}}_{n-1,0} \\ \vdots \end{pmatrix} = \boldsymbol{\Phi}_{\boldsymbol{\mathcal{E}}_n} \begin{pmatrix} \boldsymbol{F}^{\text{open}}_{n-2,0} \\ 0 \end{pmatrix} , \tag{2.53}$$

which is simply a more precise formulation of the genus-zero recursion (1.5). In ref. [4] a combinatorial recursion to obtain the matrices $\boldsymbol{e}_{0,n}, \boldsymbol{e}_{1,n}$ has been given and it is shown that they are (up to a basis transformation) the genus-zero braid matrices, which will be elaborated on in section 4.2. Moreover, in refs. [2, 3] an analogous mechanism for open-string integrals at genus one has been constructed, which will be discussed in chapter 5.

2.3 Single-valued multiple polylogarithms

Due to the simple poles of the integration kernels in eq. (2.15), MPLs depend on the homotopy class of the integration path and, thus, are multi-valued. According to the residue theorem single-valued versions of MPLs, whose integration kernels have simple poles as well, can only be defined by giving up holomorphicity. This leads to the single-valued MPLs, which are introduced in this section. They are particularly relevant for closed-string corrections and functional relations of MPLs.

2.3.1 Single-valued polylogarithms

Generally, the MPLs are multi-valued on $\mathbb{C}\setminus\{0,1\}$. They depend on the homotopy class of the integration path due to the simple poles of the integration kernels at zero and one. However, they may be modified using complex conjugate MPLs to cancel the corresponding monodromies. Giving up the holomorphicity of the MPLs leads to non-holomorphic combinations of iterated integrals with simple poles, which however are single-valued on $\mathbb{C}\setminus\{0,1\}$.

The simplest example is the logarithm. Let γ_x denote a path on $U = \mathbb{C}\setminus(-\infty, 0]$

from 1 to $x \in U$ and evaluate the integral $G_{e_0}(x)$ along γ_x

$$G_{e_0}(x) = \log(x) = \int_{\gamma_x} \frac{\mathrm{d}x}{x} \,. \tag{2.54}$$

Since U is simply connected, this integral is uniquely defined by x: it is single-valued on U. However, if paths on $\mathbb{C}\backslash\{0\}$ are allowed, $G_{e_0}(x)$ might be chosen to be evaluated along a path encircling the origin. For example by first going around the circle σ_1 of radius one centred at the origin, then going along γ_x, one obtains the following integral

$$G_{e_0}(x) = \log(x) = \int_{\gamma_x \sigma_1} \frac{\mathrm{d}x}{x} = \int_{\gamma_x} \frac{\mathrm{d}x}{x} + \int_{\sigma_1} \frac{\mathrm{d}x}{x} = \int_{\gamma_x} \frac{\mathrm{d}x}{x} + 2\pi i \,. \tag{2.55}$$

A residue is picked up, which leads to the *monodromy* $2\pi i$ in the value of $G_{e_0}(x)$. A monodromy operator is defined by acting on an integral adding a (sufficiently small) loop around the point p (with positive orientation) of the integration domain before travelling along the default path and denoted by $\mathcal{M}_p$. In other words, $\mathcal{M}_p$ acts on a local branch of a possibly multi-valued function, giving the value of its analytic continuation along a small circle around p. In particular, $\mathcal{M}_p$ is multiplicative and commutes with differentiation. The above calculation and the monodromy of G_{e_0} can therefore be expressed as

$$\mathcal{M}_0 G_{e_0}(x) = G_{e_0}(x) + 2\pi i \,. \tag{2.56}$$

Using Chen series [87] the multi-valuedness of any MPL may be expressed in terms of the homotopy class of the corresponding integration path, which leads to the monodromies of the generating series of MPLs around zero and one. The result is [72]

$$\begin{aligned} \mathcal{M}_0 L_{\mathcal{E}}(x) &= L_{\mathcal{E}}(x) e^{2\pi i e_0} \,, \\ \mathcal{M}_1 L_{\mathcal{E}}(x) &= L_{\mathcal{E}}(x) \Phi_{\mathcal{E}}^{-1} e^{2\pi i e_1} \Phi_{\mathcal{E}} \,. \end{aligned} \tag{2.57}$$

These monodromy relations can be derived using the Drinfeld associator as follows: the monodromy from $\mathcal{M}_0$ follows from the monodromy of the logarithm (2.56) and the asymptotics (2.35). The second monodromy follows from eq. (2.46) and the asymptotics (2.44)

$$\begin{aligned} \mathcal{M}_1 L_{\mathcal{E}}(x) &= \mathcal{M}_1 L_{\tilde{\mathcal{E}}}(1-x) \Phi_{\mathcal{E}}(x) \\ &= \mathcal{M}_1 L_{\tilde{\mathcal{E}}}(1-x) \Phi_{\mathcal{E}}(x) \\ &= L_{\tilde{\mathcal{E}}}(1-x) e^{2\pi i e_1} \Phi_{\mathcal{E}}(x) \\ &= L_{\mathcal{E}}(x) \Phi_{\mathcal{E}}^{-1} e^{2\pi i e_1} \Phi_{\mathcal{E}} \,. \end{aligned} \tag{2.58}$$

Note that the last equality makes use of the invertibility of $\Phi_{\mathcal{E}}$, which naturally follows from the construction in terms of Chen series [87].

With the monodromies of the MPLs at hand, a class of single-valued MPLs may be constructed by adding the appropriate complex conjugates to cancel these monodromies. For the logarithm, one can deduce from eq. (2.56) that for the complex conjugate $\overline{G}_{e_0}(x) = G_{e_0}(\bar{x})$

$$\mathcal{M}_0 G_{e_0}(\bar{x}) = G_{e_0}(\bar{x}) - 2\pi i\,, \tag{2.59}$$

such that the linear combination

$$G_{e_0}(x) + G_{e_0}(\bar{x}) = 2\,\mathrm{Re}(G_{e_0}(x)) = \log(|x|^2) \tag{2.60}$$

has a trivial monodromy

$$\mathcal{M}_0 \log(|x|^2) = \log(|x|^2) \tag{2.61}$$

and is therefore single-valued. A similar situation is found for the weight-one polylogarithm $\mathrm{Li}_1(x) = -G_{e_1}(x) = -\log(1-x)$ and the monodromy at one. Generalisations to higher-weight polylogarithms $\mathrm{Li}_n(x) = -G_{e_0^{n-1}e_1}(x)$ with $n \geq 1$ have for example been introduced by Ramakrishnan [88], which has lead to the definition of the following family of real and single-valued functions on $\mathbb{C}\backslash\{0,1\}$

$$\mathcal{L}_n(x) = \mathrm{Re}_n\left(\sum_{k=0}^{n-1} \frac{2^k B_k}{k!} \log^k(|x|)\,\mathrm{Li}_{n-k}(x)\right)\,, \tag{2.62}$$

where Re_n denotes Re for even n and Im for odd n, and B_k is the k-th Bernoulli number. The k-th Beurnoulli number is the constant term $B_k = B_k(0)$ of the Bernoulli polynomial $B_k(x)$ defined in the expansion in t of the following generating series

$$\frac{te^{xt}}{e^t - 1} = \sum_{k\geq 0} B_k(x)\frac{t^k}{k!}\,, \tag{2.63}$$

the first examples are

$$\begin{aligned}
B_0(x) &= 1\,,\\
B_1(x) &= x - \frac{1}{2}\,,\\
B_2(x) &= x^2 - x + \frac{1}{6}\,,\\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x\,.
\end{aligned}$$

The functions $\mathcal{L}_n(x)$ above are linear combinations of a more general class of functions $\mathrm{D}_{a,b}(x)$ introduced by Zagier [89]. These are the *single-valued polylogarithms*

$$\mathrm{D}_{a,b}(x) = (-1)^{a-1}\sum_{n=a}^{a+b-1}\binom{n-1}{a-1}\frac{(-2\log(|x|))^{a+b-1-n}}{(a+b-1-n)!}\,\mathrm{Li}_n(x) + (-1)^{b-1}\sum_{n=b}^{a+b-1}\binom{n-1}{b-1}\frac{(-2\log(|x|))^{a+b-1-n}}{(a+b-1-n)!}\overline{\mathrm{Li}_n(x)}, \tag{2.64}$$

which satisfy $\overline{\mathrm{D}_{a,b}(x)} = \mathrm{D}_{b,a}(x)$. Yet another family of single-valued polylogarithms has for example been introduced by Wojtkowiak [90], the functions $\mathcal{L}_n(x)$ are simple modifications of this class [91].

Before the class of single-valued multiple polylogarithms is introduced, generalising the above construction for polylogarithms to multiple polylogarithms, an important function from the class $\mathcal{L}_n(x)$ is pointed out. Some of its rich and beautiful properties will be discussed in section 2.4. It is the single-valued version of the dilogarithm $\mathrm{Li}_2(x)$, the second simplest polylogarithm after the logarithm or $\mathrm{Li}_1(x)$, respectively. Its single-valued version is called the *Bloch–Wigner dilogarithm* and deserves a distinct notation given by

$$\mathrm{D}(x) = \mathrm{Im}\left(\mathrm{Li}_2(x) - \log(|x|)\,\mathrm{Li}_1(x)\right) = \mathcal{L}_2(x) = \frac{1}{2}\,\mathrm{Im}\left(\mathrm{D}_{1,2}(x)\right)\,. \tag{2.65}$$

The Bloch–Wigner dilogarithm is continuous on the Riemann sphere and real analytic except at the points 0, 1 and ∞, where it is defined to vanish.

2.3.2 Single-valued multiple polylogarithms

In ref. [37], single-valued versions of MPLs have been constructed by cancelling the monodromies of all MPLs as in example (2.61) using the monodromies (2.57) of the generating series. It is also shown that every single-valued version of (univariate) multiple polylogarithms, including the single-valued polylogarithms $\mathcal{L}_n(x)$ and $\mathrm{D}_{a,b}(x)$ from eqs. (2.62) and (2.64), is contained in this class.

The single-valued multiple polylogarithms are constructed from the generating series of MPLs $L_{\mathcal{E}}(x)$ defined in eq. (2.34). For a word $w \in \mathcal{E}^\times$, let $\widetilde{w}$ denote the reverse word and linearly extend the action of the operator $\widetilde{\ }$, which inverts words, to $\mathbb{C}\langle\langle\mathcal{E}\rangle\rangle$. Let $\mathcal{E}' = (e_0', e_1')$ be an alphabet to be determined below with $e_0', e_1' \in \mathbb{C}\langle\langle\mathcal{E}\rangle\rangle$, i.e. e_0', e_1' are formal series in e_0, e_1. Then, the following product of generating series of MPLs is defined:

$$\mathcal{L}_{\mathcal{E}}(x) = L_{\mathcal{E}}(x)\widetilde{L}_{\mathcal{E}'}(\bar{x})\,. \tag{2.66}$$

The monodromies of $\mathcal{L}_{\mathcal{E}}(x)$ can be deduced from the ones of $L_{\mathcal{E}}(x)$ given in eq. (2.57), which leads to

$$\begin{aligned}\mathcal{M}_0\mathcal{L}_{\mathcal{E}}(x) &= L_{\mathcal{E}}(x)e^{2\pi i e_0}e^{-2\pi i e'_0}\widetilde{L}_{\mathcal{E}'}(\bar{x})\,,\\ \mathcal{M}_1\mathcal{L}_{\mathcal{E}}(x) &= L_{\mathcal{E}}(x)\Phi_{\mathcal{E}}^{-1}e^{2\pi i e_1}\Phi_{\mathcal{E}}\Phi_{\mathcal{E}'}e^{-2\pi i e'_1}\Phi_{\mathcal{E}'}^{-1}\widetilde{L}_{\mathcal{E}'}(\bar{x})\,.\end{aligned} \tag{2.67}$$

Upon requiring vanishing monodromies, these equations impose the conditions

$$\begin{aligned}e'_0 &= e_0\,,\\ \Phi_{\mathcal{E}'}e'_1\Phi_{\mathcal{E}'}^{-1} &= \Phi_{\mathcal{E}}^{-1}e_1\Phi_{\mathcal{E}}\end{aligned} \tag{2.68}$$

on the alphabet $\mathcal{E}'$. These constraint equations can be solved for $e'_1 \in \mathbb{C}\langle\langle\mathcal{E}\rangle\rangle$ recursively in the length of words in e_0, e_1, leading to a unique solution [36]. Therefore, choosing $\mathcal{E}' = (e'_0, e'_1)$ such that the conditions (2.68) hold (which is assumed from here on), the monodromies of $\mathcal{L}_{\mathcal{E}}(x)$ are trivial, i.e.

$$\begin{aligned}\mathcal{M}_0\mathcal{L}_{\mathcal{E}}(x) &= \mathcal{L}_{\mathcal{E}}(x)\,,\\ \mathcal{M}_1\mathcal{L}_{\mathcal{E}}(x) &= \mathcal{L}_{\mathcal{E}}(x)\,,\end{aligned} \tag{2.69}$$

such that the series

$$\mathcal{L}_{\mathcal{E}}(x) = \sum_{w\in\mathcal{E}^\times} w\,\mathcal{L}_w(x) \tag{2.70}$$

and its coefficients $\mathcal{L}_w(x)$, are single-valued functions on $\mathbb{C}\backslash\{0,1\}$. The latter are called *single-valued multiple polylogarithms* and $\mathcal{L}_{\mathcal{E}}(x)$ their generating series. A theorem in ref. [37] states that the single-valued MPLs satisfy analogous properties as the MPLs, described around eq. (2.31): the single-valued MPLs form the unique family of single-valued linear combinations of products $G_{w_1}(x)G_{w_2}(\bar{x})$ for words $w_1, w_2 \in \mathcal{E}^\times$, which satisfies

$$\partial_x\mathcal{L}_{e_0w}(x) = \frac{1}{x}\mathcal{L}_w(x)\,,\quad \partial_x\mathcal{L}_{e_1w}(x) = \frac{1}{x-1}\mathcal{L}_w(x)\,, \tag{2.71}$$

such that for all $n \geq 0$

$$\mathcal{L}_{e_0^n}(x) = \frac{\log^n(|x|^2)}{n!} \tag{2.72}$$

and for $w \neq e_0^n$

$$\lim_{x\to 0}\mathcal{L}_w(x) = 0\,. \tag{2.73}$$

The functions $\mathcal{L}_w(x)$ satisfy shuffle relations and are linearly independent over $\mathcal{O}\overline{\mathcal{O}}$, where $\mathcal{O} = \mathbb{C}[x, \frac{1}{x}, \frac{1}{x-1}]$ denotes the ring of regular functions on $\mathbb{C}\backslash\{0,1\}$. Furthermore, any single-valued linear combination of functions $G_{w_1}(x)G_{w_2}(\bar{x})$ can be expressed as a unique linear combination of functions $\mathcal{L}_w(x)$.

2.3.3 Single-valued multiple zeta values

Values of the single-valued MPLs at one are called *single-valued multiple zeta values* [36] and defined in analogy to MZVs (cf. eq. (2.39)) for words of the form

$$w = e_0^{n_r-1}e_1 \dots e_0^{n_1-1}e_1, \quad n_r > 1\,, \tag{2.74}$$

as

$$\zeta_{sv}(w) = (-1)^r \mathcal{L}_w(1)\,. \tag{2.75}$$

Single-valued MZVs associated to words ending with e_1 are defined using the same shuffle regularisation as for MZVs and

$$\zeta_{sv}(e_1) = 0\,. \tag{2.76}$$

Similar to the MZVs, single-valued MZVs satisfy shuffle, stuffle and further relations [36]. A rigorous framework to study their algebraic properties is provided by the motivic versions of the single-valued MZVs introduced in ref. [36], where for example the generating series of single-valued MZVs was shown to be given by the Deligne associator [78].

2.4 Functional relations

The properties of MPLs and in particular their functional relations are an active field of research. The motivic versions of MPLs form a graded Hopf algebra [69, 82, 92], which can be exploited to study their functional relations [93]. Moreover, the Duval algorithm [94] yields a basis of MPLs evaluated at the same argument with respect to the shuffle product. Further relations between MPLs with different arguments can be investigated using the coproduct or symbol map, respectively, see e.g. refs. [68, 95–100]. Such functional relations may simplify calculations in scattering amplitudes and in particular the corresponding numerical evaluations [101–105].

The functional relations discussed below are constant linear combinations of MPLs $G_w(x_i)$, where the arguments of the MPLs are rational functions in one or more variables $x_i \in \mathbb{C}(t, s, \dots)$. An important object to study these functional relations are the Bloch groups [91, 106, 107]. However, for the sake of brevity, we will not discuss Bloch groups here and instead consider functional relations of the Bloch–Wigner dilogarithm $D(x)$. On the one hand, this is a very important and mathematically rich example of such relations, on the other hand, it has been generalised to the elliptic version of the Bloch–Wigner dilogarithm [49] and to the elliptic multiple polylogarithms [1], which will be discussed in chapter 3. Recently, such elliptic analogues of MPLs have appeared in higher-loop calculations of string corrections and Feynman integrals. Thus, the study of their functional relations became relevant for physics, which motivated the project [1]. In this section, the foundations at genus zero for the discussion of the elliptic functions and functional relations further below are introduced.

2.4.1 The five-term identity

The functional relations of the Bloch–Wigner dilogarithm $\mathrm{D}(x)$ defined in eq. (2.65) are well-understood, highly non-trivial and offer a beautiful glimpse into the mathematical richness of this topic ranging from cluster algebras [41–43], hyperbolic volumes [44, 45] and symbol calculus [46, 47] to functional identities generated by rational functions on the Riemann sphere [49]. The latter will be discussed in the subsequent subsection, while in this subsection the simplest examples and in particular the important five-term identity are presented.

The simplest functional relations of the Bloch–Wigner dilogarithm are the symmetry relations

$$\begin{aligned}\mathrm{D}(t) &= \mathrm{D}\left(1-\frac{1}{t}\right) = \mathrm{D}\left(\frac{1}{1-t}\right) = -\mathrm{D}\left(\frac{1}{t}\right) = -\mathrm{D}\left(1-t\right)\\ &= -\mathrm{D}\left(\frac{-t}{1-t}\right)\end{aligned} \tag{2.77}$$

and the duplication relation

$$\mathrm{D}(t^2) = 2\,\mathrm{D}\left(t\right) + 2\,\mathrm{D}\left(-t\right) . \tag{2.78}$$

They can be derived from the fundamental properties of the logarithm and the dilogarithm.

The next-to-simplest and very non-trivial functional relation is the *five-term identity*

$$\mathrm{D}\left(t\right) + \mathrm{D}\left(s\right) + \mathrm{D}\left(\frac{1-t}{1-ts}\right) + \mathrm{D}\left(1-ts\right) + \mathrm{D}\left(\frac{1-s}{1-ts}\right) = 0 \tag{2.79}$$

for $s, t, st \notin \{0, 1\}$. This important identity appears in various fields of mathematics. For example, it follows from the periodicity of a cluster algebra [43] or it expresses an equality between volumes of hyperbolic three-simplices [44, 45]. This geometric interpretation leads to an illustrative construction [67, 99] of the Bloch group [91, 106, 107]. The Bloch group, in turn, is related to the Dedekind zeta function of an algebraic number field [106, 108] and, thus, a crucial object in algebraic

K-theory[5] [106, 108]. Moreover, many functional relations of the dilogarithm or Bloch–Wigner dilogarithm, respectively, can be obtained from a finite number of applications of the five-term identity [91, 97, 99]. In physics, on the other hand, the five-term identity and in particular its interpretation in terms of splitting a volume into several polyhedra has been used in calculations associated to various Feynman diagrams, see e.g. refs. [103, 104]. This importance of the Bloch–Wigner dilogarithm and its five-term identity motivates the investigation of analogous structures at genus one, presented in section 3.5.

2.4.2 The classical Bloch relation

A formalised concept to generate functional relations of the Bloch–Wigner dilogarithm (and its elliptic generalisation) has been put forward in ref. [49]. The identities are parametrised by principal divisors and, thus, brings us back to rational functions.

The functional relations are generated by non-trivial rational functions on the Riemann sphere $f \in \hat{\mathbb{C}}(x)$ satisfying

$$f(0) = f(\infty) = 1\,. \tag{2.80}$$

According to eq. (2.3), these are finite products of linear factors to some integer power

$$f(x) = \prod_i (x - a_i)^{d_i}\,, \tag{2.81}$$

where the distinct zeros and poles $a_i \in \mathbb{C}$ and the corresponding multiplicities $d_i \in \mathbb{Z}\backslash\{0\}$ are restricted by

$$\sum_i d_i = 0\,, \quad \prod_i a_i^{d_i} = 1\,. \tag{2.82}$$

The product representation of $1 - f(x)$ is of the form

$$1 - f(x) = b \prod_j (x - b_j)^{e_j}\,, \tag{2.83}$$

where $b, b_j \in \mathbb{C}$ and $e_j \in \mathbb{Z}$ depend non-trivially on a_i and d_i, respectively. The corresponding divisors (2.11) of these functions are

$$\mathrm{div}(f) = \sum_i d_i(a_i)\,, \quad \mathrm{div}(1 - f) = \sum_j e_j(b_j)\,. \tag{2.84}$$

In ref. [49] the following statement is shown: for any rational function f as above,

[5] See e.g. ref. [67] for an extensive overview of the five-term identity, Bloch groups and the connection to algebraic K-theory.

the Bloch–Wigner dilogarithm satisfies

$$\sum_{i,j} d_i e_j \, \mathrm{D}\left(\frac{a_i}{b_j}\right) = 0 \,. \tag{2.85}$$

Equation (2.85) is called the *classical Bloch relation*.

Varying the zeros and poles a_i of f subject to the constraint (2.82), the classical Bloch relation is a functional relation in a_i. For distinct rational functions satisfying eq. (2.80) in the first place, a whole class of functional relations for the Bloch–Wigner dilogarithm is generated. However, this class is not independent: it is conjectured to be generated by the single example of the five-term identity, cf. the discussion below eq. (2.79).

In ref. [1], the classical Bloch relation (2.85) has been extended to holomorphic generalisations of the dilogarithm at genus one, which involves elliptic functions instead of rational functions. The corresponding results are discussed in section 3.5.

Generating the five-term identity

As an example, the rational function which leads to the five-term identity is presented following ref. [67]: let $a, b \in \mathbb{C}$ be a zero and a pole parametrising the rational function

$$f(x) = \frac{(x-a)(x-a')(x-bb')}{(x-b)(x-b')(x-aa')} \,, \tag{2.86}$$

with $f(0) = f(\infty) = 1$, where $a' = 1-a$ and $b' = 1-b$. Then

$$1 - f(x) = \frac{(bb'-aa')x^2}{(x-b)(x-b')(x-aa')} \,, \tag{2.87}$$

such that the principal divisors

$$\begin{aligned} \mathrm{div}(f) &= (a) + (a') + (bb') - (b) - (b') - (aa') \,, \\ \mathrm{div}(1-f) &= 2(0) + (\infty) - (b) - (b') - (aa') \end{aligned} \tag{2.88}$$

can be read off. Bloch's relation can be applied, which yields the identity

$$\mathrm{D}\left(\frac{a}{b}\right) + \mathrm{D}\left(\frac{a'}{b'}\right) + \mathrm{D}\left(\frac{a}{b'}\right) + \mathrm{D}\left(\frac{bb'}{aa'}\right) + \mathrm{D}\left(\frac{a'}{b}\right) = 0 \,, \tag{2.89}$$

where $\mathrm{D}(0) = \mathrm{D}(1) = \mathrm{D}(\infty) = 0$ and the symmetry relations (2.77) have been used to simplify and cancel various terms. Finally, the change of variables $t = a/b$, $s = a'/b'$ leads to the five-term identity (2.79).

Chapter 3

Elliptic multiple polylogarithms

While MPLs and MZVs are sufficient to describe open-string corrections at genus zero, further classes of functions and values appear at higher order. In particular at genus one, the integrands of the string corrections are defined on the torus and, thus, elliptic. Upon integration, this leads to elliptic MPLs and elliptic MZVs, which are motivated and described in this chapter.

It essentially follows the same outline as the previous chapter: in section 3.1, the torus and its relation to the elliptic curve are introduced. Afterwards, the meromorphic functions on the torus, i.e. elliptic functions, and their integrals are reviewed, which leads to the elliptic MPLs in section 3.2. In section 3.3, some properties of their values at one, the elliptic MZVs, and their generating series, the elliptic KZB associator, are presented. A few examples of single-valued elliptic MPLs are given in section 3.4. Finally, in section 3.5, some results from ref. [1] are presented, giving an answer to question 1: the single-valued elliptic MPLs are related to the elliptic MPLs leading to functional relations among different elliptic generalisations of MPLs. Additionally, the classical Bloch relation (2.85) is stated in terms of the elliptic Bloch–Wigner dilogarithm and, thereby, extended to genus one.

3.1 Meromorphic functions on the torus

In this section, the parametrisation of the torus on the complex plane, meromorphic functions defined on the torus and its relation to the elliptic curve are reviewed[1]. Moreover, having introduced the torus and the elliptic curve, a third equivalent description is given, the so-called Tate curve.

[1]See ref. [109] for a thorough review of these well-known mathematical concepts. The outline of the following discussion is closely related to the presentation in ref. [63], where these mathematical concepts are related to the physical objects described in this thesis.

3.1.1 The torus

A *torus* $\mathbb{C}/\Lambda$ is the complex plane modulo a two-dimensional lattice

$$\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}\,, \tag{3.1}$$

where the *periods* $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$. For simplicity, we often identify a point $z + \Lambda$ on the torus by its representative $z = a\,\omega_1 + b\,\omega_2$ with $0 \leq a, b < 1$ in the *fundamental parallelogram*

$$P_\Lambda = \{a\,\omega_1 + b\,\omega_2 \mid 0 \leq a, b < 1\}\,. \tag{3.2}$$

Two distinct representatives z_1 and z_2 of $z + \Lambda$ are identified using the notation $z_1 \equiv z_2$. The complex plane may always be rescaled by one of the periods, say $1/\omega_1$, without changing the geometry of the torus. This defines an isomorphic torus with periods

$$\tau = \omega_2/\omega_1 \tag{3.3}$$

and one, the corresponding lattice is denoted by

$$\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}\,. \tag{3.4}$$

Without loss of generality[2] the *modular parameter* τ is assumed to be an element of the upper half plane

$$\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}\,, \tag{3.5}$$

such that the fundamental parallelogram can be depicted as in figure 3.1. The straight line $[0,1] + \Lambda_\tau$ is called the *A-cycle* and $[0,\tau] + \Lambda_\tau$ the *B-cycle* of the torus.

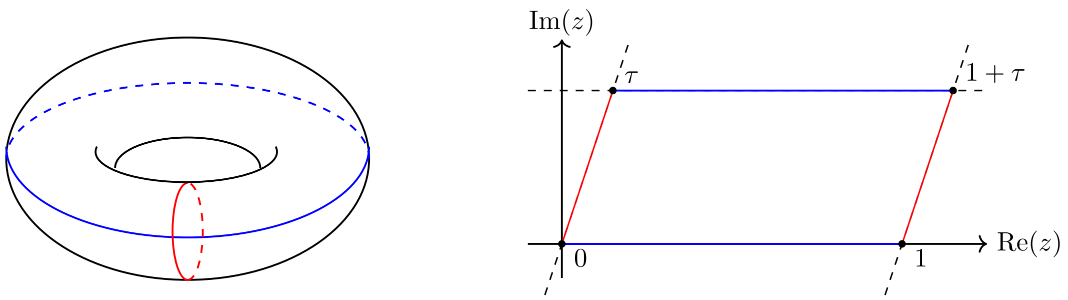


Figure 3.1: A torus can be parametrised by the modular parameter $\tau \in \mathbb{H}$ as the quotient $\mathbb{C}/\Lambda_\tau$ of the complex plane divided by the lattice $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$. In the parametrisation on the right-hand side, the fundamental parallelogram P_{Λ_τ} is depicted, which is bounded by the A-cycle (blue) and the B-cycle (red).

[2] Otherwise, the role of ω_1 and ω_2 can be interchanged.

Two tori defined by modular parameters τ and τ' are isomorphic if they are related by a *modular transformation*[3]

$$\tau \mapsto \tau' = \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}). \tag{3.6}$$

Such a transformation may define another fundamental parallelogram, but does not change the geometry of the torus.

3.1.2 Elliptic functions

In this thesis, we generally work with properly rescaled tori defined by a lattice satisfying eqs. (3.4) and (3.5). Similarly, functions with two independent periods are always pulled-back with the appropriate rescaling, such that their periods are one and $\tau \in \mathbb{H}$. Such a function is called Λ_τ*-periodic*. Moreover, a function is called *elliptic* on $\mathbb{C}$ if it is Λ_τ-periodic and meromorphic. The set of elliptic functions (relative to the lattice Λ_τ) is denoted by

$$\mathbb{C}(\Lambda_\tau) = \{f : \mathbb{C} \to \mathbb{C} \text{ meromorphic} \mid f(z + \omega) = f(z) \text{ for all } \omega \in \Lambda_\tau\}. \tag{3.7}$$

Therefore, the canonical extension of an elliptic function to the quotient space $\mathbb{C}/\Lambda_\tau$ is single-valued on the torus and, thus, meromorphic functions on the torus are simply the elliptic[4] functions. They are the genus-one analogues of the rational functions, which are the meromorphic functions on the Riemann sphere, cf. eq. (2.9).

The two canonical examples of elliptic functions are the *Weierstrass $\wp$-function*[5]

$$\wp(z) = \wp(z, \tau) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z + m + n\tau)^2} - \frac{1}{(m + n\tau)^2} \right) \tag{3.8}$$

and its derivative $\wp'(z) = \partial_z \wp(z)$. The Weierstrass $\wp$-function is even and has a double pole at the lattice points. The derivative $\wp'$, in turn, is odd and has a triple pole at the lattice points. The three zeros of $\wp'$ in the fundamental domain are the half periods $1/2$, $\tau/2$ and $(1+\tau)/2$. The two zeros of $\wp$ depend highly non-trivial on τ (see e.g. [110]). The Weierstrass $\wp$-function and its derivative satisfy the partial

[3] Despite the richness of this mathematical topic, we will not investigate the modular properties of the functions introduced below. Various aspects related to modular transformation can be found in the corresponding references.

[4] By a slight abuse of conventions, the generalisations of MPLs to the torus, leading to the meromorphic functions introduced in section 3.2 are also called elliptic (MPLs). This is justified by the fact that a multi-valuedness is introduced anyway from non-vanishing residues as in the case of the logarithm at genus zero. Thus, the quasi-periodicity of the corresponding integration kernels $g^{(k)}$ of the elliptic MPLs with respect to τ is only an additional source of multi-valuedness.

[5] While most of the quantities related to tori depend via the modular parameter τ on the particular geometry, the explicit argument τ is often omitted for the sake of notational simplicity.

differential equation

$$\wp'(z)^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3) = 4\wp(z)^3 - g_2\wp(z) - g_3\,, \tag{3.9}$$

where the *roots* e_1, e_2, e_3 are defined by the values of $\wp$ at the half periods[6]

$$e_1 = \wp\left(\frac{1}{2}\right)\,, \quad e_2 = \wp\left(\frac{\tau}{2}\right)\,, \quad e_3 = \wp\left(\frac{1+\tau}{2}\right)\,. \tag{3.10}$$

The *Weierstrass invariants* g_2 and g_3 can be expressed in terms of the holomorphic *Eisenstein series*[7]

$$\mathrm{G}_0 = -1\,, \quad \mathrm{G}_{2k} = \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^{2k}}\,, \quad \mathrm{G}_{2k-1} = 0\,, \tag{3.12}$$

where $k \in \mathbb{Z}_{>0}$. They are given by

$$g_2 = 60\,\mathrm{G}_4\,, \quad g_3 = 140\,\mathrm{G}_6 \tag{3.13}$$

and satisfy together with the roots the relations

$$e_1 + e_2 + e_3 = 0\,, \quad e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{4}g_2\,, \quad e_1e_2e_3 = \frac{1}{4}g_3\,. \tag{3.14}$$

In fact, even the Weierstrass $\wp$-function can be expressed in terms of G_k: the holomorphic Eisenstein series appear in the coefficients of the Laurent expansion around $z = 0$:

$$\wp(z) = \frac{1}{z^2} + \sum_{k\geq 4}(k-1)\,\mathrm{G}_k\; z^{k-2}\,. \tag{3.15}$$

Having recalled the definition and two important examples of elliptic functions, let us summarise some of their properties. Ellipticity restricts[8] the zeros and poles of an elliptic function F in the fundamental parallelogram P_{Λ_τ} according to

$$\sum_{z\in P_{\Lambda_\tau}} \mathrm{ord}_z\,(F) = 0\,, \quad \sum_{z\in P_{\Lambda_\tau}} \mathrm{ord}_z\,(F)\, z \in \Lambda_\tau\,, \tag{3.16}$$

[6] The six possible permutations of the indices of the roots correspond to the six possible assignments of the half-periods in eq. (3.10) to each root. This and further such redundancies are related to the modular transformations from eq. (3.6), see e.g. ref. [1, 109]. For the numerical examples in this thesis, the explicit choices from the presented definitions are used.

[7] For G_2 the Eisenstein summation prescription is assumed

$$\sum_{(m,n)\neq(0,0)} a_{m,n} = \lim_{N\to\infty}\lim_{M\to\infty}\sum_{n=-N}^{N}\sum_{m=-M}^{M} a_{m,n}\,. \tag{3.11}$$

[8] See e.g. ref. [109] for the derivation of equation (3.16) and the following statements about elliptic and Weierstrass functions.

where $\mathrm{ord}_z(F)$ is the order of vanishing of F at z from eq. (2.6). Moreover, due to Cauchy's residue theorem an elliptic function can not have a single simple pole: integration along the fundamental parallelogram, where the (reversed) parallel paths cancel pairwise due to the Λ_τ-periodicity, the sum of the residues of an elliptic function has to vanish. This excludes single simple poles. The restrictions (3.16) can be shown by the same cancellation of the integration along the fundamental parallelogram using the generalised argument principle. Another property expressing the restrictive nature of ellipticity and the close relation to meromorphic functions on the Riemann sphere or rational functions, respectively, is the fact that elliptic functions are determined up to scaling by their zeros and poles: Liouville's theorem can be used to deduce that since the quotient of two elliptic functions with the same zeros and poles including their multiplicities is an entire bounded function, this quotient is constant. Thus, the two elliptic functions are proportional to each other. Furthermore, this implies that any elliptic function is a rational function in the Weierstrass $\wp$-function and its derivative $\wp'$: on the one hand, since $\wp$ and $\wp'$ are elliptic, any rational function in $\wp$ and $\wp'$ is elliptic as well. On the other hand, any elliptic function F can be decomposed into an even F^+ and an odd part F^-:

$$\begin{aligned} F^+(z) &= \frac{1}{2}(F(z)+F(-z))\,, \quad F^-(z) = \frac{1}{2}(F(z)-F(-z))\,, \\ F(z) &= F^+(z)+F^-(z)\,. \end{aligned} \tag{3.17}$$

If A_i^+ are the zeros and poles of $F^+(z)$ with multiplicities d_i^+, then

$$F^+(z) = a^+\prod_i(\wp(z)-\wp(A_i^+))^{d_i^+} \tag{3.18}$$

for some constant $a^+\in\mathbb{C}$, since the right-hand side is elliptic and has by construction the same zeros and poles as $F^+(z)$. A similar statement holds for the even function $F^-(z)/\wp'(z)$, such that

$$F^-(z) = a^-\wp'(z)\prod_i(\wp(z)-\wp(A_i^-))^{d_i^-}\,, \tag{3.19}$$

for some $a^-\in\mathbb{C}$ and the zeros and poles A_i^- of $F^-(z)$. Therefore, the elliptic functions are indeed the rational functions in $\wp$ and $\wp'$

$$\mathbb{C}(\Lambda_\tau) = \mathbb{C}(\wp(z),\wp'(z)) \tag{3.20}$$

and can be written in the form

$$F(z) = R_1(\wp(z)) + \wp'(z)R_2(z)\,, \quad R_1,R_2\in\mathbb{C}(x)\,. \tag{3.21}$$

Equation (3.20) is analogous to eq. (2.9) at genus zero, which states that any meromorphic function on the Riemann sphere can be written as a rational function on the complex plane.

Another description of elliptic functions is based on the *Weierstrass ζ-function*

$$\zeta(z) = \frac{1}{z} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{z-m-n\tau} + \frac{1}{m+n\tau} + \frac{z}{(m+n\tau)^2} \right) , \tag{3.22}$$

which is the negative odd primitive of the Weierstrass $\wp$-function, i.e. $\zeta(-z) = -\zeta(z)$ and

$$\partial_z \zeta(z) = -\wp(z) \, . \tag{3.23}$$

It defines the *Weierstrass σ-function*

$$\sigma(z) = s_0 \exp\left(\int_{z_0}^{z} \mathrm{d}z' \, \zeta(z') \right) , \tag{3.24}$$

where the scaling factor and base point $s_0, z_0 \in \mathbb{C}$ are chosen such that $\sigma'(0) = 1$. The Weierstrass ζ-function is the logarithmic derivative of the Weierstrass σ-function

$$\partial_z \log(\sigma(z)) = \zeta(z) \, . \tag{3.25}$$

The functions ζ and σ are meromorphic, but neither of them is Λ_τ-periodic, thus, they are not elliptic: while the Weierstrass ζ-function has only a single simple pole in the fundamental parallelogram, the Weierstrass σ-function has only one simple zero at the lattice points but no poles, such that the conditions (3.16) are violated. The non-periodicity of the Weierstrass ζ-function follows from integrating the equation $\wp(z+\omega) = \wp(z)$, where $\omega \in \{1, \tau\}$ is one of the periods, which implies that ζ changes by some integration constant

$$\zeta(z+\omega) = \zeta(z) + 2\eta_\omega \tag{3.26}$$

i.e. it is *quasi-periodic* with the *quasi-periods* $\eta_\omega = \zeta(\omega/2)$. This quasi-periodicity of the Weierstrass ζ-function determines the behaviour of the Weierstrass σ-function under a lattice displacement, which reads

$$\sigma(z+\omega) = \exp\Big(2\eta_\omega z + \xi_\omega \Big) \sigma(z) \, , \tag{3.27}$$

where ξ_ω is a further integration constant[9] (see e.g. ref. [109]).

As an alternative to the representation (3.21) in terms of rational functions in $\wp$ and $\wp'$, the functions ζ, σ and in particular eq. (3.27) can be used to represent any elliptic function F as follows: let $\{A_i\}$ be representatives of the zeros and poles of

[9]Note that the integration constants ξ_ω with $\omega \in \{1, \tau\}$ in eq. (3.27) determine s_0 and z_0 in eq. (3.24) and vice-versa.

F such that the conditions (3.16) are of the form

$$\sum_i d_i = 0\,, \quad \sum_i d_i A_i = 0\,, \tag{3.28}$$

where $d_i = \mathrm{ord}_{A_i}(F)$. Then, the product[10]

$$\prod_i \sigma\left(z - A_i\right)^{d_i} = \exp\left(\sum_i d_i \int_0^{z-A_i} \mathrm{d}z'\,\zeta(z')\right) \tag{3.29}$$

is elliptic. First, it is meromorphic since the Weierstrass σ-function is meromorphic. Second, for $\omega \in \{1, \tau\}$, the transformation behaviour (3.27) implies that

$$\begin{aligned}\prod_i \sigma\left(z + \omega - A_i\right)^{d_i} &= \exp\Big(\sum_i d_i(2\eta_\omega(z - A_i) + \xi_\omega)\Big) \prod_i \sigma\left(z - A_i\right)^{d_i} \\ &= \prod_i \sigma\left(z - A_i\right)^{d_i}\,,\end{aligned} \tag{3.30}$$

where we have used the conditions (3.28) on the zeros and poles for the last equality. Thus, this product of Weierstrass σ-functions is also Λ_τ-periodic and, hence, indeed elliptic. Moreover, the function $\sigma(z - A_i)$ has only simple zeros at $A_i + \Lambda_\tau$, such that the elliptic product in eq. (3.29) has exactly the same zeros and poles, including multiplicities, as the original elliptic function F. Therefore, the former has to be proportional to the latter, i.e. the elliptic function F is given by

$$F(z) = s_A \prod_i \sigma\left(z - A_i\right)^{d_i} = s_A \exp\left(\sum_i d_i \int_0^{z-A_i} \mathrm{d}z'\,\zeta(z')\right) \tag{3.31}$$

for some scaling factor $s_A \in \mathbb{C}$.

Similar to the rational functions from the previous chapter, according to the representation (3.31), the elliptic functions are also determined up to an overall constant by their zeros and poles. Again, this motivates the introduction of the *group of divisors* of $\mathbb{C}/\Lambda_\tau$ (or subsets thereof), which is the free abelian group generated by points of the fundamental parallelogram P_{Λ_τ} and denoted by $\mathrm{Div}(\mathbb{C}/\Lambda_\tau)$. Thus, in analogy to eq. (2.10) at genus zero, a divisor $D \in \mathbb{C}/\Lambda_\tau$ is of the form

$$D = \sum_{z \in P_{\Lambda_\tau}} n_z(z)\,, \tag{3.32}$$

where all but finitely many of the integer coefficients $n_z \in \mathbb{Z}$ are zero. As in eq. (2.11) for rational functions, a divisor can be associated to each elliptic function $F \in \mathbb{C}(\Lambda_\tau)$

[10] Note that compared to the definition of the Weierstrass σ-function (3.24), the factors of s_0 from the product on the left-hand side of eq. (3.29) multiply to one and the base point z_0 of the integrals in the exponential can be shifted to zero due to the condition $\sum_i d_i = 0$ in eq. (3.28).

according to[11]

$$\operatorname{div}(F) = \sum_{z \in P_{\Lambda_\tau}} \operatorname{ord}_z(F)(z) \tag{3.33}$$

and a divisor $D \in \operatorname{Div}(\mathbb{C}/\Lambda_\tau)$ is called *principal* if and only if there exists an elliptic function $F(x) \in \mathbb{C}(\Lambda_\tau)$ such that $D = \operatorname{div}(F)$. Due to the conditions (3.16), this is equivalent to being of the form $D = \sum_i d_i(A_i)$, where

$$\sum_i d_i = 0\,, \quad \sum_i d_i A_i \in \Lambda_\tau\,. \tag{3.34}$$

Therefore, two elliptic functions are equal up to scaling if and only if they have the same divisors.

3.1.3 The elliptic curve

The description of elliptic functions as rational functions in $\wp$ and $\wp'$ leads to another representation in terms of rational functions on a complex projective algebraic curve, the *elliptic curve*[12]

$$E\left(\mathbb{C}\right) = \{(x,y) \mid y^2 = 4x^3 - g_2 x - g_3\} \cup \{\infty\}\,, \tag{3.35}$$

where g_2 and g_3 are the Weierstrass invariants (3.13), which defines the relation to the tori via the τ-dependence of g_2, g_3. Modular transformations (3.6) of the torus are related to changes of variables of the algebraic equation defining the elliptic curve and, thus, represent the same geometry and elliptic curve, respectively. This algebraic equation is of *Weierstrass form* and called *Weierstrass equation*. It is exactly the same as the differential equation (3.9) of the Weierstrass $\wp$-function.

This observation leads to the following isomorphism of Riemann surfaces from the torus to the elliptic curve

$$\phi_{\tau,E} : \begin{cases} \mathbb{C}/\Lambda_\tau & \to E\left(\mathbb{C}\right)\,, \\ 0 \neq z + \Lambda_\tau & \mapsto \left(\wp(z), \wp'(z)\right), \\ 0 + \Lambda_\tau & \mapsto \infty\,. \end{cases} \tag{3.36}$$

The addition on the elliptic curve, with the unity being ∞, is given by the *chord-tangent construction* described in appendix A.

[11] By a slight abuse of notation, any non-vanishing term $\operatorname{ord}_z(F)(z)$ in a divisor $\operatorname{div}(F)$ of an elliptic function can be represented by any other representative $z' \equiv z$ of $z + \Lambda_\tau$ than $z \in P_{\Lambda_\tau}$ in the fundamental parallelogram, by identifying $\operatorname{ord}_z(F)(z)$ with $\operatorname{ord}_{z'}(F)(z')$ if and only if $z' \equiv z$.

[12] We choose to work with the dehomogenised form of the projective elliptic curve $E\left(\mathbb{C}\right) = \{[X : Y : Z] \in \mathbb{C}^2 | Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3\} \subset \mathbb{P}^2(\mathbb{C})$ using the non-homogeneous coordinates $x = X/Z$, $y = Y/Z$, $Z = 1$ and denoting $\infty = [0 : 1 : 0]$.

Solving the differential equation (3.9) of the Weierstrass $\wp$-function leads to an integral representation of the inverse of the isomorphism $\phi_{\tau,E}$, called *Abel's map*: let $P=(x_P,y_P)\in E(\mathbb{C})$. If $y_P=0$, then $z_P+\Lambda_\tau$ with $(\wp(z_P),\wp'(z_P))=(x_P,0)$ is one of the half periods $\omega_i/2\in\{1/2,\tau/2,(1+\tau)/2\}$ modulo the lattice Λ_τ and, according to eq. (3.10), the x-coordinate of P has to be one of the roots $e_i\in\{e_1,e_2,e_3\}$. The i-th root satisfying $x_P=e_i$ in turn, determines the appropriate half period $\omega_i/2$ by $\wp(\omega_i/2)=e_i$. If $y_P\neq 0$, the differential equation has to be solved, leading to

$$z_P=\pm\int_\infty^{x_P}\frac{\mathrm{d}x}{y}+\Lambda_\tau\,, \tag{3.37}$$

where the sign is determined by the requirement $\wp'(z_P)=y_P$.

To summarise, according to the identification

$$x=\wp(z)\,,\quad y=\wp'(z) \tag{3.38}$$

and the representation (3.20) of elliptic functions in terms of rational functions in $\wp$ and $\wp'$, the elliptic functions are the rational functions in x and y on the elliptic curve $E(\mathbb{C})$, i.e.

$$\begin{aligned}\mathbb{C}(\Lambda_\tau)&=\mathbb{C}(x,y)/\{y^2=4x^3-g_2x-g_3\}\\&=\{f\in\mathbb{C}(x,y)\mid y^2=4x^3-g_2x-g_3\}\,.\end{aligned} \tag{3.39}$$

At this point, let us make a comment on how to obtain the modular parameter τ of an appropriately rescaled torus given an arbitrary elliptic curve in Weierstrass form $E:y^2=4x^3-g_2x-g_3$. In general, the Weierstrass invariants g_2 and g_3 are not chosen such that there exists a $\tau\in\mathbb{H}$ satisfying eq. (3.13). Rather, they correspond to a not yet rescaled lattice $\Lambda=\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$ such that

$$g_2=60\sum_{(m,n)\neq(0,0)}\frac{1}{(m\omega_1+n\omega_2)^4}\,,\quad g_3=140\sum_{(m,n)\neq(0,0)}\frac{1}{(m\omega_1+n\omega_2)^6}\,. \tag{3.40}$$

The roots e_i of the Weierstrass equation still satisfy the relations (3.14), which can be used to determine the roots from the Weierstrass invariants. Having the roots $e_i=e_i(g_2,g_3)$ at hands, Abel's map (3.37) yields the following relation[13] to calculate the lattice periods from the roots [111]

$$\omega_1=2\int_{e_3}^{e_2}\frac{\mathrm{d}x}{y}\,,\quad \omega_2=2\int_{e_1}^{e_2}\frac{\mathrm{d}x}{y}+\omega_1\,. \tag{3.41}$$

This, in turn, leads to the modular parameter $\tau=\omega_2/\omega_1$ of the appropriately rescaled torus.

[13] This is valid for the specific choices of the roots in eq. (3.10). See also the footnote above eq. (3.10) for a brief note on possible other choices.

3.1.4 The Tate curve

Another equivalent description of the torus, which nicely relates to the concepts introduced so far, can be obtained by considering the exponential of the modular parameter

$$q = e^{2\pi i\tau}\,. \tag{3.42}$$

This exponential map induces the isomorphism

$$\phi_{\tau,q} : \begin{cases} \mathbb{C}/\Lambda_\tau & \to \mathbb{C}^*/q^{\mathbb{Z}} \\ z+\Lambda_\tau & \mapsto e^{2\pi i z}\cdot q^{\mathbb{Z}}\,. \end{cases} \tag{3.43}$$

The codomain $\mathbb{C}^*/q^{\mathbb{Z}}$, with a multiplicative group structure inherited from addition on the torus via the exponential, is called *Tate curve*[14].

The description of elliptic functions on the Tate curve offers yet another connection to rational functions than their description on the torus and on the elliptic curve via eqs. (3.20) and (3.39), respectively. One of its advantages is that the genus-zero limit $\tau \to i\infty$ can be implemented using the q-expansion of functions depending on the modular parameter, i.e. expanding them in q, and taking $q \to 0$.

The relation of elliptic functions to functions on the Tate curve is thoroughly described in ref. [49] and based on the same class of rational functions discussed in subsection 2.4.2, which appears in the classical Bloch relation. These are the non-trivial meromorphic functions on the Riemann sphere $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $f(0) = f(\infty) = 1$, which are of the form (2.81) and their zeros and poles a_i with multiplicities d_i satisfy eq. (2.82). Forming the infinite product

$$F(x) = \prod_{l\in\mathbb{Z}} f(xq^l) \tag{3.44}$$

of such a function f yields a meromorphic function which obeys the transformed Λ_τ-periodicity condition (pulled back via the exponential to the Tate curve)

$$F(xq) = F(x)\,. \tag{3.45}$$

Therefore, F is a well-defined meromorphic function on the Tate curve

$$F : \begin{cases} \mathbb{C}^*/q^{\mathbb{Z}} & \to \hat{\mathbb{C}} \\ x\cdot q^{\mathbb{Z}} & \mapsto F(x)\,. \end{cases} \tag{3.46}$$

We refer to the function F as *elliptic average* of f and call any meromorphic func-

[14]Thorough introductions to the Tate curve can be found in ref. [112], appendix A.1.2, or ref. [113], section 4.3.

tion satisfying eq. (3.45) elliptic (on the Tate curve), which is justified in the next paragraph. Properties of such functions on the Tate curve are discussed in detail in ref. [49].

The condition $f(0) = f(\infty) = 1$ ensures on the one hand that in the genus-zero limit $q \to 0$ the original rational function is recovered

$$\lim_{q\to 0} F(x) = f(x)\,. \tag{3.47}$$

On the other hand it implies that representatives A_i of the zeros and poles of the function

$$F \circ \phi_{\tau,q}(z) = F(e^{2\pi i z}) \tag{3.48}$$

can be found, which satisfy the condition (3.28) and such that $a_i = e^{2\pi i A_i}$. As argued in subsection 3.1.2, these two conditions (modulo lattice displacements) are not only necessary, but also sufficient to be the zeros and poles of some elliptic function. Therefore, the function $F \circ \phi_{\tau,q}$ in eq. (3.48) is elliptic and any elliptic function with divisor $\sum_i d_i(A_i)$ can be written up to scaling in the form (3.48) where F is the elliptic average of the rational function f with $\mathrm{div}(f) = \sum_i d_i(e^{2\pi i A_i})$ and $f(0) = f(\infty) = 1$. The divisor $\mathrm{div}(F) = \sum_i d_i(A_i)$ of an elliptic function expressed on the Tate curve is simply the divisor of the corresponding rational function $\mathrm{div}(f) = \sum_i d_i(e^{2\pi i A_i})$.

3.2 Elliptic multiple polylogarithms

In this section the class of functions obtained from integrating elliptic functions is identified. It turns out that the description of elliptic functions as rational functions on the elliptic curve is a convenient tool for this purpose. This will lead to the elliptic multiple polylogarithms and elliptic multiple zeta values. While the elliptic MPLs considered are meromorphic and multi-valued on the torus, other elliptic generalisations of MPLs have been proposed. Certain classes of elliptic MPLs on the Tate curve, including a particular single-valued but non-meromorphic class, will be discussed later in this chapter. In particular, in section 3.5 some results of ref. [1] are presented, where singled-valued elliptic MPLs are related to meromorphic elliptic MPLs and the genus-zero Bloch relation is generalised to the torus.

3.2.1 Integrating elliptic functions

Identifying the class of functions obtained from (iterated) integrations of elliptic functions is more subtle than for the rational functions on the Riemann sphere. The approach shown below is similar to the construction of elliptic multiple polylogarithms in ref. [63] and leads to integrals closely related to the functions introduced in ref. [8].

According to eq. (3.21) or eq. (3.39), respectively, any elliptic function can be written as

$$F(z) = R_1(\wp(z)) + \wp'(z)R_2(z) = R_1(x) + yR_2(x)\,, \quad R_1, R_2 \in \mathbb{C}(x)\,, \tag{3.49}$$

where $x = \wp(z)$ and $y = \wp'(z)$. Since

$$\frac{\mathrm{d}x}{y} = \frac{\mathrm{d}\wp(z)}{\wp'(z)} = \mathrm{d}z \tag{3.50}$$

integrating such an elliptic function or rational function on the elliptic curve, respectively, leads to integrals which are linear combinations of

$$\int R_1(\wp(z))\mathrm{d}z = \int \frac{1}{y}R_1(x)\mathrm{d}x \tag{3.51}$$

and

$$\int \wp'(z)R_2(\wp(z))\mathrm{d}z = \int R_2(x)\mathrm{d}x\,. \tag{3.52}$$

As for the rational functions, which lead to eq. (2.14), the second integral (3.52) can be reduced using partial fractioning to a $\mathbb{C}$-linear combination of integrals of the form

$$\int \frac{\mathrm{d}x}{(x-c_i)^k} = \begin{cases} \frac{(x-c_i)^{1-k}}{1-k} = \frac{(\wp(z)-\wp(z_i))^{1-k}}{1-k} & \text{if } k \neq 1\,, \\ \log(x-c_i) & \text{if } k = 1\,, \end{cases} \tag{3.53}$$

where $\wp(z_i) = c_i$. Thus, for $k \neq 1$, an elliptic function is recovered. Again, for $k = 1$, the class of elliptic function is surpassed, such that integrals of the differential forms

$$\varphi_1(c_i, x)\mathrm{d}x = \frac{\mathrm{d}x}{(x-c_i)} = \frac{\wp'(z)}{\wp(z)-\wp(z_i)}\mathrm{d}z \tag{3.54}$$

have to be included in any class extending the elliptic functions which is closed under integration. The function on the right-hand side can be expressed as a linear combination of Weierstrass ζ-functions

$$\frac{\wp'(z)}{\wp(z)-\wp(z_i)} = \zeta(z-z_i) + \zeta(z-z_i) - 2\zeta(z) \tag{3.55}$$

since the linear combination of Weierstrass ζ-functions on the right-hand side is odd, elliptic and has simple poles at $z = \pm z_i$ and $z = 0$ with $\mathrm{res}_{\pm z_i} = 1$ and $\mathrm{res}_0 = -2$, respectively. The same holds for the left-hand side, such that the difference between the left- and the right-hand side has to vanish and, thus, both agree. For reasons explained below, it is more convenient to express the above linear combination in terms of a slight modification of the Weierstrass ζ-function,

$$g^{(1)}(z) = g^{(1)}(z, \tau) = \zeta(z) - 2\eta_1 z \,, \tag{3.56}$$

which leads to

$$\frac{\wp'(z)}{\wp(z) - \wp(z_i)} = g^{(1)}(z - z_i) + g^{(1)}(z - z_i) - 2g^{(1)}(z) \,. \tag{3.57}$$

Subtracting a linear shift including the half-period η_1 from $\zeta(z)$ ensures that, according to eq. (3.26), $g^{(1)}(z)$ is one-periodic

$$g^{(1)}(z + 1) = g^{(1)}(z) \tag{3.58}$$

in contrast to $\zeta(z)$. Moreover, it is still an odd function

$$g^{(1)}(-z) = -g^{(1)}(z) \,. \tag{3.59}$$

However, a τ-periodicity can not be restored without violating the meromorphicity: since $\zeta(z)$ or $g^{(1)}(z)$, respectively, is meromorphic and has only a simple pole at each lattice point, it can not be elliptic. But it shows the best possible behaviour: it is quasi-periodic in τ

$$g^{(1)}(z + \tau) = g^{(1)}(z) - 2\pi i \,. \tag{3.60}$$

Its q-expansion is given by [27]

$$g^{(1)}(z, \tau) = \pi \cot(\pi z) + 4\pi \sum_{k,l>0} \sin(2\pi k z) q^{kl} \,. \tag{3.61}$$

To summarise, iterated integrals of the differential form (3.54) have to be included to describe (iterated) integrals of elliptic functions of the form (3.52). They are linear combinations of iterated integrals of the integration kernel $g^{(1)}(z)$ and are defined in analogy to the Goncharov polylogarithms (2.15) for $z_r \neq 0$ by

$$\begin{aligned} \tilde{\Gamma}\left(\begin{smallmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_r \end{smallmatrix} ; z, \tau \right) &= \int_0^z \mathrm{d}z' \, g^{(1)}(z' - z_1, \tau) \, \tilde{\Gamma}\left(\begin{smallmatrix} 1 & \cdots & 1 \\ z_2 & \cdots & z_r \end{smallmatrix} ; z', \tau \right) , \\ \tilde{\Gamma}(; z, \tau) &= 1 \,. \end{aligned} \tag{3.62}$$

Indeed, the integration kernel $g^{(1)}(z, \tau)$ is the best possible genus-one generalisation of the genus-zero kernel $1/x$: it is odd, meromorphic, has a simple pole at each lattice point, is one-periodic and quasi-periodic in τ. The simple poles lead to the same necessity for a regularisation of the functions (3.84) in the case $z_r = 0$ at the lower integration boundary of the innermost kernel, which already appeared for Goncharov polylogarithms, cf. eq. (2.28). This can be implemented using a similar subtraction of the corresponding logarithmic divergence from the lower integration boundary, which lead to eq. (2.16), giving in this case

$$\begin{aligned}\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau) &= \lim_{\epsilon\to 0}\left(\int_{\epsilon}^{z}\mathrm{d}z'\,g^{(1)}(z',\tau)+\log(2\pi i\epsilon)\right)\\ &= \log(1-e^{2\pi iz})-\pi iz+4\pi\sum_{k,l>0}\frac{1}{2\pi k}\left(1-\cos(2\pi kz)\right)q^{kl}\,. \end{aligned} \tag{3.63}$$

The q-expansion on the second line can be deduced from the q-expansion (3.61) of the integration kernel. Moreover, for multiple labels the definition

$$\tilde{\Gamma}(\underbrace{\begin{smallmatrix}1&\cdots&1\\0&\cdots&0\end{smallmatrix}}_{r};z,\tau)=\frac{1}{r!}\left(\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau)\right)^{r} \tag{3.64}$$

is used. Thus, the function $\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau)$ is the genus-one analogue of the logarithm

$$\lim_{\epsilon\to 0}\left(\int_{\epsilon}^{x}\frac{\mathrm{d}x}{x}+\log(\epsilon)\right)=\log(x)\,. \tag{3.65}$$

Having discussed integrals of the form (3.52), the remaining integrals (3.51) have to be considered to determine if even more integration kernels are required. With integration by parts and partial fractioning on the elliptic curve, such an integral can be reduced to integrals of the form (3.52) and integrals of the three new differential forms

$$\begin{aligned}\varphi_{-1}(c_i,x)\mathrm{d}x &= \frac{\mathrm{d}x}{y(x-c_i)}=\frac{\mathrm{d}z}{\wp(z)-\wp(z_i)}\,,\\ \varphi_0(x)\mathrm{d}x &= \frac{\mathrm{d}x}{y}=\mathrm{d}z\,,\\ \phi(x)\mathrm{d}x &= \frac{x\mathrm{d}x}{y}=\wp(z)\mathrm{d}z\,. \end{aligned} \tag{3.66}$$

The function appearing in the first differential form can be expressed in terms of $g^{(1)}(z)$ as [63]

$$\frac{1}{\wp(z)-\wp(z_i)}=\frac{1}{\wp'(z_i)}\left(g^{(1)}(z+z_i)-g^{(1)}(z-z_i)+2g^{(1)}(z_i)\right)\,, \tag{3.67}$$

which follows from a similar analysis as eq. (3.55). Upon integration, the second differential form leads to powers of z and the corresponding integration kernel is simply denoted by

$$g^{(0)}(z) = 1\,. \tag{3.68}$$

The third differential form includes the derivative of $g^{(1)}(z)$

$$\wp(z) = -\partial_z g^{(1)}(z) - 2\eta_1\,. \tag{3.69}$$

Hence, in addition to the differential form $g^{(1)}(z)\mathrm{d}z$ in the family of integrals (3.84), the forms $g^{(0)}(z)\mathrm{d}z = \mathrm{d}z$ and $\partial_z g^{(1)}(z)\mathrm{d}z$ have to be included as well. While $g^{(0)}(z)$ is genuinely new, using integration by parts on the torus, the derivative $\partial_z g^{(1)}(z)$ can be expressed in terms of a product of kernels $g^{(1)}(z)$, for example

$$\begin{aligned}
&\int \mathrm{d}z' \left(\partial_{z'} g^{(1)}(z'-z_0,\tau)\right) \tilde{\Gamma}\left(\begin{smallmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_r \end{smallmatrix}; z',\tau\right) \\
&= g^{(1)}(z-z_0,\tau)\,\tilde{\Gamma}\left(\begin{smallmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_r \end{smallmatrix}; z,\tau\right) \\
&\quad - \int \mathrm{d}z' g^{(1)}(z'-z_0;\tau) g^{(1)}(z'-z_1;\tau)\,\tilde{\Gamma}\left(\begin{smallmatrix} 1 & \cdots & 1 \\ z_2 & \cdots & z_r \end{smallmatrix}; z',\tau\right).
\end{aligned} \tag{3.70}$$

Here, a major difference to the construction of Goncharov polylogarithms can be observed: in the genus-zero case, partial fractioning (2.21) ensures that any such product of two kernels is again a linear combination of the kernels $1/(x-c_i)$. However, the function $g^{(1)}(z-z_i,\tau)$ does not satisfy partial fractioning, but a similar identity which involves infinitely many functions $g^{(k)}(z-z_j,\tau)$ for $k \geq 0$. Thus, all of these infinitely many kernels have to be included as well.

The functions $g^{(k)}(z,\tau)$ for $k \geq 0$ are generated by the *Eisenstein–Kronecker series*[15] $F(z,\eta,\tau)$ [8, 114]

$$F(z,\eta,\tau) = \frac{\theta_1'(0,\tau)\theta_1(z+\eta,\tau)}{\theta_1(z,\tau)\theta_1(\eta,\tau)}\,, \tag{3.71}$$

where

$$\theta_1(z,\tau) = q^{\frac{1}{12}}(z^{\frac{1}{2}} - z^{-\frac{1}{2}}) \prod_{j\geq 1}(1-q^j z) \prod_{j\geq 1}(1-q^j z^{-1}) \tag{3.72}$$

is the odd Jacobi θ-function and $\theta_1'(z,\tau) = \partial_z \theta_1(z,\tau)$. The periodicity properties of the Jacobi θ-function imply the quasi-periodicity of the Eisenstein–Kronecker series

$$F(z+1,\eta,\tau) = F(z,\eta,\tau)\,, \quad F(z+\tau,\eta,\tau) = e^{-2\pi i\eta} F(z,\eta,\tau)\,. \tag{3.73}$$

[15] Even though the same symbol F is used as for a generic elliptic function, the Eisenstein–Kronecker series is not elliptic, which will be explained below. It is always clear from the context or the arguments of the function, whether F denotes a generic elliptic function or the Eisenstein–Kronecker series.

Similarly, the antisymmetry

$$F(z,\eta,\tau) = -F(-z,-\eta,\tau) \tag{3.74}$$

can be shown. Moreover, the Eisenstein–Kronecker series satisfies the mixed heat equation

$$2\pi i\partial_\tau F(z,\eta,\tau) = \partial_z\partial_\eta F(z,\eta,\tau)\,. \tag{3.75}$$

The functions $g^{(k)}(z,\tau)$ are the coefficients in the expansion with respect to η

$$\eta F(z,\eta,\tau) = \sum_{k\geq 0} g^{(k)}(z,\tau)\eta^k\,, \tag{3.76}$$

where the index k is called the *weight* of $g^{(k)}$. The first two functions $g^{(0)} = 1$ and $g^{(1)}$ indeed agree with eqs. (3.68) and (3.56), respectively. For general k, the functions $g^{(k)}$ inherit various properties from the Eisenstein–Kronecker series. For example, they are one-periodic

$$g^{(k)}(z+1,\tau) = g^{(k)}(z,\tau)\,, \tag{3.77}$$

but not periodic with respect to τ and satisfy the symmetry property

$$g^{(k)}(-z,\tau) = (-1)^k g^{(k)}(z,\tau) \tag{3.78}$$

as well as the mixed heat equation

$$2\pi i\partial_\tau g^{(k)}(z,\tau) = k\partial_z g^{(k+1)}(z,\tau)\,. \tag{3.79}$$

Their q-expansions can be deduced from the q-expansion of the Eisenstein–Kronecker series and can be found in appendix B.1. For $k \geq 2$, the functions $g^{(k)}$ are polynomials of degree k in $g^{(1)}$ where the subleading terms are such that the poles at the integers of the leading term $(g^{(1)})^k/k!$ cancel. Thus, they are holomorphic on the real line, in contrast to $g^{(1)}$ which has a simple pole at each lattice point.

The identity analogous to partial fractioning (2.12) at genus one, which was mentioned above, is the *Fay identity*

$$F_{ki}(\eta_1)F_{kj}(\eta_2) = F_{ki}(\eta_1+\eta_2)F_{ij}(\eta_2) + F_{kj}(\eta_1+\eta_2)F_{ji}(\eta_1)\,, \tag{3.80}$$

where we have used the short notation

$$F_{ij}(\eta) = F(z_{ij},\eta,\tau)\,, \quad z_{ij} = z_{i,j} = z_i - z_j\,. \tag{3.81}$$

Expanding both sides of eq. (3.80) leads to the Fay identity for the integration kernels

$$g^{(m)}_{kj}g^{(n)}_{ki} = (-1)^{m+1}g^{(m+n)}_{ji} + \sum_{r=0}^{n}\binom{m+r-1}{m-1}g^{(m+r)}_{kj}g^{(n-r)}_{ji} + \sum_{r=0}^{m}(-1)^{m-r}\binom{n+r-1}{n-1}g^{(n+r)}_{ki}g^{(m-r)}_{ji}\,, \tag{3.82}$$

where

$$g^{(n)}_{ij} = g^{(n)}(z_{ij},\tau)\,. \tag{3.83}$$

By means of the Fay identity and integration by parts, the iterated integral in eq. (3.70) can be expressed as a $\mathbb{Q}$-linear combination of iterated integrals over the kernels $g^{(k)}(z-z_i,\tau)$ with $k\in\{0,1,2\}$ times the kernels themselves. This statement turns out to be true in general and leads to the following generalisation of the definition (3.62) to iterated integrals over the kernels $g^{(k)}(z-z_i,\tau)$ with $k\geq 0$ [8, 27]: for $k_r\neq 1$, the *elliptic multiple polylogarithms* with shifts z_i (eMPLs with shifts) are defined[16] by

$$\begin{aligned}\tilde\Gamma\left(\begin{smallmatrix}k_1 & \dots & k_r\\ z_1 & \dots & z_r\end{smallmatrix};z,\tau\right) &= \int_0^z \mathrm{d}z'\; g^{(k_1)}(z'-z_1,\tau)\,\tilde\Gamma\left(\begin{smallmatrix}k_2 & \dots & k_r\\ z_2 & \dots & z_r\end{smallmatrix};z',\tau\right),\\ \tilde\Gamma(;z,\tau) &= 1\,.\end{aligned} \tag{3.84}$$

The number r is called the *length* and $\sum_{i=1}^r k_i$ the *weight* of the eMPL. In the case $k_r=1$, the functions $\tilde\Gamma$ are defined by eqs. (3.63) and (3.64), respectively, and all the remaining cases can recursively be defined using tangential base-point regularisation analogously to the Goncharov polylogarithms, which results in the following prescription: they are polynomials of eMPLs (with shifts) of the form (3.84) or (3.64) and of lower length such that the shuffle algebra of iterated integrals

$$\begin{aligned}&\tilde\Gamma(A_1,A_2,\dots,A_j;z,\tau)\,\tilde\Gamma(B_1,B_2,\dots,B_k;z,\tau)\\ &=\tilde\Gamma\left((A_1,A_2,\dots,A_j)\sqcup\!\sqcup\,(B_1,B_2,\dots,B_k);z,\tau\right)\end{aligned} \tag{3.85}$$

is preserved for any combined letters $A_i = \begin{smallmatrix}k_i\\ z_i\end{smallmatrix}$.

In ref. [63], it was shown that the eMPLs with shifts including their integration kernels indeed close the elliptic functions under integration and differentiation: let

$$\mathcal{A}^\tau_{\text{eMPL}} = \langle g^{(k_0)}(z-z_0,\tau)\,\tilde\Gamma\left(\begin{smallmatrix}k_1 & \dots & k_r\\ z_1 & \dots & z_r\end{smallmatrix};z,\tau\right)\mid r\geq 0, k_i\geq 0, z_i\in\mathbb{C}\rangle_{\mathbb{C}(\Lambda_\tau)} \tag{3.86}$$

denote the $\mathbb{C}(\Lambda_\tau)$-algebra generated by the eMPLs with shifts and their integration

[16] For modified definitions based on Λ_τ-periodic, but non-holomorphic integration kernels see e.g. refs. [8, 27, 115]. These different definitions can be related to each other by forming linear combinations of the eMPLs and their complex conjugates.

kernels over the field of elliptic functions $\mathbb{C}(\Lambda_\tau)$ associated to a torus with modular parameter τ with the multiplication given by the shuffle product. Then, the algebra $\mathcal{A}^\tau_{\text{eMPL}}$ is closed under differentiation and integration, i.e. for every $f \in \mathcal{A}^\tau_{\text{eMPL}}$, there exists a $F \in \mathcal{A}^\tau_{\text{eMPL}}$ such that $f = \partial_x F$ and vice-versa. Moreover, it is graded by the *total length* $l = k_0 + r$

$$\begin{aligned} \mathcal{A}^\tau_{\text{eMPL}} &= \bigoplus_{l\geq 0} \mathcal{A}^\tau_{\text{eMPL},l}\,, \\ \mathcal{A}^\tau_{\text{eMPL},l} &= \langle g^{(k_0)}(z-z_0,\tau)\,\tilde{\Gamma}(\begin{smallmatrix} k_1 & \cdots & k_r \\ z_1 & \cdots & z_r \end{smallmatrix};z,\tau) \mid r\geq 0, k_i\geq 0, z_i \in \mathbb{C}, k_0+r=l\rangle_{\mathbb{C}(\Lambda_\tau)}\,. \end{aligned} \tag{3.87}$$

3.2.2 Elliptic multiple polylogarithms

While the eMPLs with shifts (3.84) are the genus-one analogues of the Goncharov polylogarithms (2.15), the eMPLs without shifts ($z_i = 0$), simply called eMPLs from here on, are the analogues of the MPLs (2.24). It is exactly this class of iterated integrals that appears in the open-string integrals at genus one, since the torus symmetry can be gauged in the string corrections by choosing one external state to be at $z_1 = 0 \equiv 1$, while the others are integrated over and rather correspond to the integration variables such as z' in the iterated integrals (3.84).

These *elliptic multiple polylogarithms* (eMPLs) can be denoted by the set of all words $\mathcal{X}^\times$ generated by an infinite alphabet $\mathcal{X} = (x_0, x_1, x_2, \dots)$ as follows: the empty word corresponds to the empty integral $\tilde{\Gamma}(;z,\tau) = 1$ and a non-trivial word

$$w = x_{k_1}x_{k_2}\dots x_{k_1}x_{k_r} \in \mathcal{X}^\times \tag{3.88}$$

to the iterated integral defined by eq. (3.84) and the corresponding shuffle regularisation, i.e.

$$\tilde{\Gamma}_w(z) = \tilde{\Gamma}(\begin{smallmatrix} k_1 & \cdots & k_r \\ z_1 & \cdots & z_r \end{smallmatrix};z,\tau)\,. \tag{3.89}$$

In analogy to the polylogarithms (2.27), the subclass of *elliptic polylogarithms* (ePLs) is defined by words of the form $w = x_0^{n-1}x_m$ with $n \geq 1$ and denoted by

$$\tilde{\Gamma}_n(m;z) = \tilde{\Gamma}_n(m;z,\tau) = \tilde{\Gamma}_{x_0^{n-1}x_m}(z) = \tilde{\Gamma}(\underbrace{\begin{smallmatrix} 0 & \cdots & 0 & m \\ 0 & \cdots & 0 & 0 \end{smallmatrix}}_{n};z,\tau)\,, \tag{3.90}$$

where the subscript n indicates the number of integrations over the integration kernel $g^{(m)}$. For latter convenience, the integration kernels are sometimes denoted as ePLs as well, with $n = 0$ number of integrations

$$\tilde{\Gamma}_0(m;z) = g^{(m)}(z)\,. \tag{3.91}$$

The q-expansions of the ePLs can be calculated by multiple integrations of the q-expansion of the integration kernels, the results are given in appendix B.1. For any words $w', w'' \in \mathcal{X}^\times$, the shuffle algebra (3.85) reduces for the eMPLs to the algebra

$$\tilde{\Gamma}_{w'}(z)\,\tilde{\Gamma}_{w''}(z) = \tilde{\Gamma}_{w' \sqcup\!\sqcup w''}(z)\,. \tag{3.92}$$

The eMPLs associated to words of the form $x_k w \in \mathcal{X}^\times$ satisfy the partial differential equation

$$\partial_z\,\tilde{\Gamma}_{x_k w}(z) = g^{(k)}(z)\,\tilde{\Gamma}_w(z)\,. \tag{3.93}$$

Moreover, the regularisation of the eMPLs implies that

$$\lim_{z\to 0}\tilde{\Gamma}_w(z) = 0\,, \quad \text{if} \quad w \neq x_1^n \tag{3.94}$$

for any $n \geq 0$, while words of the form x_1^n lead to the products

$$\tilde{\Gamma}_{x_1^n}(z) = \frac{1}{n!}\left(\tilde{\Gamma}_{x_1}(z)\right)^n = \frac{1}{n!}\left(\tilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau\right)\right)^n\,, \tag{3.95}$$

whose asymptotic behaviour for $z \to 0$ is dominated by the simple pole of the integration kernel $g^{(1)}$ at zero and can be deduced from its q-expansion (3.63):

$$\tilde{\Gamma}_{x_1^n}(z) \sim \frac{1}{n!}\log^n(-2\pi i z)\,. \tag{3.96}$$

3.2.3 Elliptic multiple zeta values

The genus-one analogues of the MZVs from eq. (2.39) can be defined by certain values of the eMPLs. These *elliptic multiple zeta values* (eMZVs) [27,115,116] are defined by the regularised iterated integrals $\tilde{\Gamma}_w$ evaluated at $z = 1$, associated to words $w = x_{n_1}\dots x_{n_k} \in \mathcal{X}\backslash x_1\mathcal{X}$, with $n_1 \neq 1$:

$$\omega_{\tilde{w}} = \omega_{\tilde{w}}(\tau) = \omega(n_k,\dots,n_1;\tau) = \tilde{\Gamma}_w(1,\tau) = \tilde{\Gamma}\left(\begin{smallmatrix}n_1 & \cdots & n_k\\ 0 & \cdots & 0\end{smallmatrix};1,\tau\right)\,, \tag{3.97}$$

where $\tilde{w}$ is the word w reversed and $\omega(;\tau) = 1$ is assigned to the empty word. Similar to the MZVs, this definition can be extended to all words $w \in \mathcal{X}$, including words of the form $x_1 w \in x_1\mathcal{X}$ by regularising the singularity of $\tilde{\Gamma}_{x_1 w}(z,\tau)$ at $z = 1$, see e.g. ref. [1]. The result is that for $n \geq 1$

$$\omega_{x_1^n} = 0 \tag{3.98}$$

and the remaining cases can recursively be related to (already) well-defined eMZVs by the shuffle algebra

$$\omega_{w'}\omega_{w''} = \omega_{w' \sqcup\!\sqcup w''}\,. \tag{3.99}$$

This regularisation preserves the shuffle algebra, the properties implied by the Fay identity and some further properties inherited from the eMPLs such as the reflection identity

$$\omega(n_k,\dots,n_1)=(-1)^{n_1+\dots+n_k}\omega(n_1,\dots,n_k) \tag{3.100}$$

due to the symmetry (3.78). Further relations between eMZVs are given in ref. [117].

The eMZVs defined by the ePLs (3.90) are called *elliptic zeta value* (eZV) and denoted by

$$\omega_n(m)=\omega_n(m;\tau)=\tilde{\Gamma}_n(m;1)=\tilde{\Gamma}(\underbrace{\begin{smallmatrix}0 & \dots & 0 & m\\ 0 & \dots & 0 & 0\end{smallmatrix}}_{n};1,\tau)=\omega(\underbrace{m,0,\dots,0}_{n})\,. \tag{3.101}$$

The even zeta values are recovered from the elliptic zeta values with $n=1$

$$\omega_1(2m)=-2\zeta_{2m}, \tag{3.102}$$

which can be seen from the q-expansion in eq. (B.7).

3.2.4 Further elliptic generalisations of polylogarithms

The definition of the eMPLs $\tilde{\Gamma}$ introduced in the context of high energy physics in ref. [27] is based on the definitions and concepts from ref. [8]. Further elliptic generalisations of (multiple) polylogarithms have been proposed both, in mathematics and physics literature, e.g. in refs. [6,7,15,118–120] to only mention a few of them. In this section a certain class of elliptic polylogarithms introduced and investigated in the context of the sunrise and kite integral in refs. [15,17,19] is discussed, which happens to appear in the connection of single-valued elliptic polylogarithms to elliptic polylogarithms established in ref. [1].

This elliptic generalisation of the polylogarithms is based on the sum representation (2.26) of the multiple polylogarithms $\mathrm{Li}_{n_1,\dots,n_r}$ and defined for two variables t,s (and q) by

$$\mathrm{ELi}_{n,m}(t,s,q)=\sum_{k,l>0}\frac{t^k}{k^n}\frac{s^l}{l^m}q^{kl} \tag{3.103}$$

and similarly for more variables. Note that this defines a multi-valued function on the Tate curve. Since the eMPLs $\tilde{\Gamma}_w$ only depend on z (and τ), we will restrict ourselves to one variable (and q) using the following combination of $\mathrm{ELi}_{n,m}$

$$\mathrm{E}_{n,m}(x,q)=-\left(\mathrm{ELi}_{n,m}(x,1,q)-(-1)^{n+m}\,\mathrm{ELi}_{n,m}(x,1,q)\right). \tag{3.104}$$

The functions $\mathrm{E}_{n,m}(x,q)$ satisfy the partial differential equation [19]

$$\frac{\partial}{\partial x}\,\mathrm{E}_{n,m}(x,q) = \frac{1}{x}\,\mathrm{E}_{n-1,m}(x,q)\,. \tag{3.105}$$

In ref. [1], it is shown that for $n > 1$ and $m < 0$ the value at $x = 1$ is finite and a linear combination of eZVs:

$$\mathrm{E}_{n,-|m|}(1,q) = \begin{cases} |m|!(2\pi i)^{n-1-|m|}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} d_{2k+1}\omega_{n+1-2k}(|m|+1;\tau) & n+m \text{ odd},\\ 0 & n+m \text{ even}. \end{cases} \tag{3.106}$$

The coefficients $d_k \in \mathbb{Q}$ are determined by the sequence

$$d_k = \begin{cases} -1 & k=1\,,\\ 0 & k \text{ even},\\ -\frac{d_1}{k!} - \frac{d_3}{(k-2)!} - \cdots - \frac{d_{k-2}}{3!} & k \text{ odd}, \end{cases} \tag{3.107}$$

such that e.g.

$$d_1 = -1\,,\quad d_3 = \frac{1}{3!}\,,\quad d_5 = \frac{1}{5!} - \frac{1}{3!3!}\,,\quad d_7 = \frac{1}{7!} - \frac{1}{5!3!} - \frac{1}{3!5!} + \frac{1}{3!3!3!}\,. \tag{3.108}$$

Not only the values $\mathrm{E}_{n,-|m|}(1,q)$, but the full functions $\mathrm{E}_{n,-|m|}(x,q)$ can be expressed on the torus in terms or ePLs. A derivation of the corresponding result in eq. (3.175) below from [1] is summarised in subsection 3.5.2.

For latter convenience, we define for $m = 0$ the following subclass

$$\mathrm{E}_n(x,q) = -\left(\frac{1}{2}\,\mathrm{Li}_n(x) - (-1)^n\frac{1}{2}\,\mathrm{Li}_n(x)\right) + \mathrm{E}_{n,0}(x,q)\,. \tag{3.109}$$

The functions E_n still satisfy the partial differential equation

$$\frac{\partial}{\partial x}\,\mathrm{E}_n(x,q) = \frac{1}{x}\,\mathrm{E}_{n-1}(x,q)\,, \tag{3.110}$$

with finite initial value for $n > 1$ given by a linear combination of eZVs [1]:

$$\begin{aligned} \mathrm{E}_n(1,q) &= -\left(\frac{1}{2}(1-(-1)^n)\,\mathrm{Li}_n(1) + (1-(-1)^n)\,\mathrm{ELi}_{n,0}(1,1,q)\right)\\ &= \begin{cases} (2\pi i)^{n-1}\sum_{k=0}^{\frac{(n-1)}{2}} d_{2k+1}\omega_{n+1-2k}(1;\tau) & n \text{ odd},\\ 0 & n \text{ even}. \end{cases} \end{aligned} \tag{3.111}$$

Again, also the functions $\mathrm{E}_n(x,q)$ can be written on the torus using the ePLs [1]. The derivation of the resulting eq. (3.171) is given in subsection 3.5.2.

3.3 The elliptic KZB associator

Similar to the MPLs and MZVs, the eMPLs and eMZVs can also be described in terms of generating series [115, 121]. On the one hand, such a description offers a possibility to rather deal with only one function instead of an infinite class. This may considerably simplify the derivation of certain properties of all functions in the corresponding class. On the other hand, an auxiliary alphabet has to be introduced to define the generating series. The relevant definitions and conventions are given in this section.

3.3.1 Generating series of elliptic multiple polylogarithms

The construction of the generating series of eMPLs is completely analogous to the construction in eq. (2.32) and below for the MPLs at genus zero: the *generating series of eMPLs* [121] is the unique holomorphic solution $\tilde{\Gamma}_{\mathcal{X}}(z) = \tilde{\Gamma}_{\mathcal{X}}(z,\tau)$ on the fundamental parallelogram P_{Λ_τ} with values in $\mathbb{C}\langle\langle\mathcal{X}\rangle\rangle$[17] of the differential equation

$$\partial_z \tilde{\Gamma}_{\mathcal{X}}(z) = \left(\sum_{k\geq 0} x_k\, g^{(k)}(z)\right) \tilde{\Gamma}_{\mathcal{X}}(z) \tag{3.112}$$

such that

$$\lim_{z\to 0}(-2\pi i z)^{-x_1}\, \tilde{\Gamma}_{\mathcal{X}}(z) = 1\,. \tag{3.113}$$

The function $\tilde{\Gamma}_{\mathcal{X}}(z)$ is indeed the generating series of eMPLs

$$\tilde{\Gamma}_{\mathcal{X}}(z) = \sum_{w\in\mathcal{X}^\times} w\, \tilde{\Gamma}_w(z)\,, \tag{3.114}$$

[17] In contrast to the original construction in ref. [121], the alphabet $\mathcal{X}$ consists of a priori independent letters, which e.g. are neither nested commutators of certain Lie algebra generators nor do they satisfy any non-trivial relations.

since the right-hand side of eq. (3.114) satisfies the partial differential eq. (2.32) due to the partial differential eq. (3.93) of the eMPLs:

$$\begin{aligned}\partial_z \sum_{w\in\mathcal{X}^\times} w\,\tilde{\Gamma}_w(z) &= \sum_{w\in\mathcal{X}^\times} w\,\partial_z\,\tilde{\Gamma}_w(z)\\ &= \sum_{k\geq 0}\sum_{w\in\mathcal{X}^\times} x_k w\,\partial_z\,\tilde{\Gamma}_{x_k w}(z)\\ &= \sum_{k\geq 0}\sum_{w\in\mathcal{X}^\times} x_k w\, g^{(k)}(z)\,\tilde{\Gamma}_w(z)\\ &= \left(\sum_{k\geq 0} x_k\, g^{(k)}(z)\right)\sum_{w\in\mathcal{X}^\times} w\,\tilde{\Gamma}_w(z)\,. \end{aligned} \tag{3.115}$$

Moreover, its asymptotic behaviour as $z\to 0$ is according to eqs. (3.94) and (3.96) also given by

$$\sum_{w\in\mathcal{X}^\times} w\,\tilde{\Gamma}_w(z) \sim \sum_{n\geq 0} x_1^n \frac{\log^n(-2\pi i z)}{n!} = (-2\pi i z)^{x_1}\,, \tag{3.116}$$

which agrees with the initial condition (3.113).

3.3.2 The elliptic KZB equation and associator

The differential equation (3.112) is called *elliptic Knizhnik–Zamolodchikov–Bernard (KZB) equation* [122, 123], it is the genus-one analogue of the KZ equation (2.32). In ref. [2], the following fact has been exploited: two regularised boundary values of a solution of the elliptic KZB equation can be related to each other via the generating series of eMZVs, the elliptic KZB associator, which is exactly the same mechanism that lead to the genus-zero associator equation (2.50). The derivation according to ref. [2] is reviewed in this subsection.

Consider the alphabet $\mathcal{Y}=(y_0,y_1,y_2,\dots)=(-x_0,x_1,-x_2,x_3,-x_4,\dots)$ obtained from $\mathcal{X}=(x_0,x_1,x_2)$ by $y_k=(-1)^{k+1}x_k$ and the corresponding function

$$\tilde{\Gamma}_{\mathcal{Y}}(1-z) = \sum_{w\in\mathcal{Y}^\times} w\,\tilde{\Gamma}_w(1-z)\,. \tag{3.117}$$

Its asymptotic behaviour for $z\to 1$ can be deduced from eqs. (3.94) and (3.96), it is given by

$$\tilde{\Gamma}_{\mathcal{Y}}(1-z) \sim \sum_{n\geq 0} x_1^n \frac{\log^n(-2\pi i(1-z))}{n!} = (-2\pi i(1-z))^{x_1}\,. \tag{3.118}$$

The function $\tilde{\Gamma}_{\mathcal{Y}}(1-z)$ satisfies the partial differential equation

$$\begin{aligned}
\partial_z \tilde{\Gamma}_{\mathcal{Y}}(1-z) &= \sum_{w\in\mathcal{Y}^\times} w\partial_z \tilde{\Gamma}_w(1-z) \\
&= -\sum_{k\geq 0}\sum_{w\in\mathcal{Y}^\times} (-1)^k x_k w \partial_z \tilde{\Gamma}_{y_k w}(1-z) \\
&= \sum_{k\geq 0}\sum_{w\in\mathcal{Y}^\times} (-1)^k x_k w\, g^{(k)}(1-z)\, \tilde{\Gamma}_w(1-z) \\
&= \sum_{k\geq 0}\sum_{w\in\mathcal{Y}^\times} x_k w\, g^{(k)}(z)\, \tilde{\Gamma}_w(1-z) \\
&= \left(\sum_{k\geq 0} x_k g^{(k)}(z)\right) \sum_{w\in\mathcal{Y}^\times} w\ \tilde{\Gamma}_w(1-z)\,, \qquad (3.119)
\end{aligned}$$

were we have used the one-periodicity (3.77) and the symmetry property (3.78) of $g^{(k)}(z)$ such that $g^{(k)}(1-z) = (-1)^k g^{(k)}(z)$. Thus, the function $\tilde{\Gamma}_{\mathcal{Y}}(1-z)$ satisfies the same elliptic KZB equation (3.112) as $\tilde{\Gamma}_{\mathcal{X}}(z)$. The product

$$\Phi^\tau_{\mathcal{X}} = (\tilde{\Gamma}_{\mathcal{Y}}(1-z))^{-1}\, \tilde{\Gamma}_{\mathcal{X}}(z) \qquad (3.120)$$

is independent of z and called *elliptic KZB associator*. It was originally constructed in ref. [40] for a specific alphabet $\mathcal{X}$. The z-independence is true for any product of an inverse of a solution multiplied with any other solution of the same elliptic KZB equation (with the same alphabet $\mathcal{X}$), which can be shown the same way as in the genus-zero scenario mentioned below eq. (2.46). Thus, the elliptic KZB associator $\Phi^\tau_{\mathcal{X}}$ can be rewritten by evaluating the right-hand side of eq. (3.120) in the limit $z \to 1$. Using the asymptotics (3.118), it turns out that the result is exactly the generating series of eMZVs [115, 116]: the inverse of $(\tilde{\Gamma}_{\mathcal{Y}}(1-z))^{-1}$ implements the appropriate regularisation (3.98) of the eMZVs (3.97), such that

$$\begin{aligned}
\Phi^\tau_{\mathcal{X}} &= \lim_{z\to 1}(\tilde{\Gamma}_{\mathcal{Y}}(1-z))^{-1}\, \tilde{\Gamma}_{\mathcal{X}}(z) \\
&= \lim_{z\to 1}(-2\pi i(1-z))^{-x_1}\, \tilde{\Gamma}_{\mathcal{X}}(z) \\
&= \sum_{w\in\mathcal{X}^\times} w\, \omega_w \\
&= 1 + x_0 - 2\zeta_2 x_2 + \frac{1}{2}x_0x_0 - [x_0,x_1]\omega(0,1) - \zeta_2\{x_0,x_2\} \\
&\quad + [x_1,x_2]\big(\omega(0,3) - 2\zeta_2\omega(0,1)\big) - [x_0,x_3]\,\omega(0,3) \\
&\quad + \zeta_4(-\{x_0,x_4\} + 5x_2x_2 - 2x_4) + \dots\,, \qquad (3.121)
\end{aligned}$$

where for the last equality, relations among eMZVs have been used [117].

Similar to the Drinfeld associator, the elliptic KZB associator relates two regularised boundary values of any solution $L^\tau(z)$ of the elliptic KZB equation (3.112).

The two regularised boundary values

$$C_0^\tau(L^\tau) = \lim_{z\to 0}(-2\pi i z)^{-x_1} L^\tau(z)\,, \quad C_1^\tau(L^\tau) = \lim_{z\to 1}(-2\pi i(1-z))^{-x_1} L^\tau(z) \tag{3.122}$$

are connected by the elliptic KZB associator according to the *genus-one associator equation*

$$C_1^\tau(L^\tau) = \Phi^\tau_{\mathcal{X}}\, C_0^\tau(L^\tau)\,. \tag{3.123}$$

In analogy to the genus-zero associator eq. (2.50), this equation is obtained using the asymptotics (3.116) and (3.118) as well as the z-independence of an inverse solution times a solution of the elliptic KZB equation (cf. the explanation below eq. (2.46)):

$$\begin{aligned}
\Phi^\tau_{\mathcal{X}}\, C_0^\tau(L^\tau) &= \lim_{z\to 0} \Phi^\tau_{\mathcal{X}} (-2\pi i z)^{-x_1} L^\tau(z)\\
&= \lim_{z\to 0} (\tilde{\Gamma}_{\mathcal{Y}}(1-z))^{-1} L^\tau(z)\\
&= \lim_{z\to 1} (\tilde{\Gamma}_{\mathcal{Y}}(1-z))^{-1} L^\tau(z)\\
&= \lim_{z\to 1} (-2\pi i(1-z))^{-x_1} L^\tau(z)\\
&= C_1^\tau(L^\tau)\,.
\end{aligned} \tag{3.124}$$

This genus-one associator equation has been used in refs. [2,3] to obtain the α'-expansion of open-string integrals at genus one from the genus-zero string integrals, given in full detail in eq. (5.98) below and schematically described in eq. (1.7), which is one of the main result of this thesis and explained in detail in chapter 5. Using a slightly more sophisticated notation than in eq. (1.7), it can be summarised as follows: for each $n \geq 3$ a $(n{-}1)!$-dimensional vector of iterated integrals $\boldsymbol{Z}^\tau_{n,2}$ satisfying an elliptic KZB equation with some generators $\boldsymbol{\mathcal{X}}_n = (\boldsymbol{x}_{0,n}, \boldsymbol{x}_{1,n}, \boldsymbol{x}_{2,n}, \dots)$ (square matrices) has been constructed, such that the regularised boundary values include the $(n{+}1)$-point, genus-zero and the $(n{-}1)$-point, genus-one string integrals $\boldsymbol{F}^{\text{open}}_{n+1,0}$ and $\boldsymbol{F}^{\text{open}}_{n-1,1}$, respectively

$$\boldsymbol{C}_0^\tau(\boldsymbol{Z}^\tau_{n,2}) = \boldsymbol{F}^{\text{open}}_{n+1,0}\,, \quad \boldsymbol{C}_1^\tau(\boldsymbol{Z}^\tau_{n,2}) = \boldsymbol{F}^{\text{open}}_{n-1,1}\,. \tag{3.125}$$

Hence, eq. (3.123) yields a relation using matrix algebra to calculate the $(n{-}1)$-point, genus-one integrals $\boldsymbol{F}^{\text{open}}_{n-1,1}$ from the $(n{+}1)$-point, genus-zero integrals $\boldsymbol{F}^{\text{open}}_{n+1,0}$

$$\boldsymbol{F}^{\text{open}}_{n-1,1} = \boldsymbol{\Phi}^\tau_{\boldsymbol{\mathcal{X}}_n}\, \boldsymbol{F}^{\text{open}}_{n+1,0}\,. \tag{3.126}$$

The explicit expression of the matrices $\boldsymbol{x}_{k,n}$ and the detailed construction as well as the fully appropriate notation will be discussed in section 5.3. Together with the genus-zero recursion (2.53), eq. (3.126) can be used to calculate the n-point, genus-one string integrals by the elliptic KZB associator times a product of Drinfeld associators. This reveals the origin of eMZVs and MZVs in the genus-one corrections.

3.4 Single-valued elliptic polylogarithms

The eMPLs presented so far are not single-valued. First, simple poles of the integration kernels $g^{(k)}$ at the lattice points lead to a dependence on the integration path. This is analogous to the multi-valuedness of the genus-zero MPLs. However, at genus one, there is a second issue: the kernels $g^{(k)}$ are not well-defined on the torus since they are only one-periodic, but not τ-periodic. The advantage of the kernels $g^{(k)}$, on the other hand, is that they are meromorphic and have simple poles. Certain constructions of generalisations are based on the opposite choice: they define single-valued eMPLs, which however are not meromorphic. Two such classes are presented in this subsection and related to the multi-valued ePLs $\tilde{\Gamma}_n(m;z)$ from eq. (3.90) in subsection 3.5.2. An approach to define single-valued eMPLs generalising the construction from ref. [37] of single-valued MPLs, cf. subsection 2.3.2, to genus one is not yet known.

The first class of single-valued elliptic polylogarithms is obtained by an elliptic average over the Tate curve of Ramakrishnan's single-valued polylogarithms $\mathcal{L}_n(x)$ from eq. (2.62). Since the functions $\mathcal{L}_n(x)$ are neither rational nor do they satisfy the conditions (2.80), the elliptic average can not be constructed multiplicatively as in eq. (3.44). But a summation over the Tate curve leads to the elliptic generalisation

$$\begin{aligned}\mathcal{L}_n^\tau(x) &= \sum_{l\in\mathbb{Z}} \mathcal{L}_n(xq^l)\\ &= \sum_{k=0}^{n-1} \frac{2^k B_k}{k!} \operatorname{Re}_n \Big(\sum_{l>0} \log^k(|xq^l|) \operatorname{Li}_{n-k}(xq^l)\\ &\quad + (-1)^{n-1} \sum_{l>0} \log^k(|x^{-1}q^l|) \operatorname{Li}_{n-k}(x^{-1}q^l)\Big)\\ &\quad + \mathcal{L}_n(x)\,, \end{aligned} \tag{3.127}$$

introduced in ref. [89] and investigated in ref. [67]. In particular, the function

$$\mathrm{D}^\tau(x) = \mathcal{L}_2^\tau(x) = \sum_{l\in\mathbb{Z}} \mathrm{D}(xq^l) \tag{3.128}$$

is called *elliptic Bloch–Wigner dilogarithm* and plays an analogous role in the construction of elliptic dilogarithm functional relations [49] as its genus-zero cousin $\mathrm{D}(x)$, cf. subsection 3.5.1. Certain properties of the classical Bloch–Wigner dilogarithm carry over to the elliptic version, in particular from eqs. (2.77) and (2.78) the inversion

$$\mathrm{D}^\tau(x^{-1}) = -\,\mathrm{D}^\tau(x) \tag{3.129}$$

and the duplication relation

$$\mathrm{D}^\tau(x^2) = 2\left(\mathrm{D}^\tau(x) + \mathrm{D}^\tau(x\sqrt{q}) + \mathrm{D}^\tau(-x) + \mathrm{D}^\tau(-x\sqrt{q})\right)\,, \tag{3.130}$$

respectively.

In ref. [1], it has been shown that the functions $\mathcal{L}_n^\tau(x)$ turn out to be related to the functions $\mathrm{E}_{n,-m}(x,q)$, defined in eq. (3.104), according to

$$\begin{aligned}\mathcal{L}_n^\tau(x) = -\sum_{m=0}^{k}\binom{k}{m}\frac{2^k B_k}{k!}\log^{k-m}(|x|)\log^m(|q|)\,\mathrm{Re}_n\left(\mathrm{E}_{n-k,-m}(x,q)\right)\\ +\,\mathcal{L}_n(x)\,.\end{aligned} \tag{3.131}$$

For example, the elliptic Bloch–Wigner dilogarithm is given by

$$\mathrm{D}^\tau(x) = -\,\mathrm{Im}\left(\mathrm{E}_2(x,q)\right) + \log(|x|)\,\mathrm{Im}\left(\mathrm{E}_1(x,q)\right) + \log(|q|)\,\mathrm{Im}\left(\mathrm{E}_{1,-1}(x,q)\right)\,. \tag{3.132}$$

Equation (3.131) and the relation of the functions $\mathrm{E}_{n-k,-m}(x,q)$ to the meromorphic ePLs $\tilde{\Gamma}_n(m;z)$, which will be discussed in subsection 3.5.2 and given in eqs. (3.171) and (3.175), ultimately relates the single-valued ePLs $\mathcal{L}_n^\tau(x)$ to the meromorphic ePLs $\tilde{\Gamma}_n(m;z)$.

A more general class of single-valued elliptic polylogarithms was introduced in ref. [89]. This class has been extensively used in the context of closed-string amplitudes at one loop [124]. It is based on the single-valued elliptic polylogarithms $\mathrm{D}_{a,b}(x)$ from eq. (2.64). Again, it is a sum over the Tate curve and given by [89]

$$\begin{aligned}\mathrm{D}_{a,b}^\tau(x) = \sum_{l\geq 0}\mathrm{D}_{a,b}(xq^l) + (-1)^{a+b}\sum_{l>0}\mathrm{D}_{a,b}(x^{-1}q^l)\\ +\,\frac{(4\pi\,\mathrm{Im}(\tau))^{a+b-1}}{(a+b)!}B_{a+b}(u)\,,\end{aligned} \tag{3.133}$$

where B_k is the k-th Bernoulli polynomial (2.63), u is defined by $x = e^{2\pi i z}$ with $z = u\tau + v$ and $u, v \in [0,1]$. The functions $\mathcal{L}_n^\tau$ are contained in the class $\mathrm{D}_{a,b}^\tau$, which is called *single-valued elliptic polylogarithms* (single-valued ePLs). For example, the elliptic Bloch–Wigner dilogarithm is given by

$$\mathrm{D}^\tau(x) = -\frac{1}{2}\,\mathrm{Im}(\mathrm{D}_{2,1}^\tau(x))\,. \tag{3.134}$$

The single-valued ePLs $\mathrm{D}_{a,b}^\tau$ can also be expressed in terms of the sums $\mathrm{E}_{n,-m}$, which yields the expression [1]

$$\begin{aligned}
\mathrm{D}^{\tau}_{a,b}(x,q) = (-1)^a \sum_{n=a}^{a+b-1} \binom{n-1}{a-1} \frac{(-2)^{a+b-1-n}}{(a+b-1-n)!} \\
\sum_{m=0}^{a+b-1-n} \binom{a+b-1-n}{m} \log^{a+b-1-n-m}(|x|) \log^m(|q|)\, \mathrm{E}_{n,-m}(x,1,q) \\
+ (-1)^b \sum_{n=b}^{a+b-1} \binom{n-1}{b-1} \frac{(-2)^{a+b-1-n}}{(a+b-1-n)!} \\
\sum_{m=0}^{a+b-1-n} \binom{a+b-1-n}{m} \log^{a+b-1-n-m}(|x|) \log^m(|q|) \overline{\mathrm{E}_{n,-m}}(x,1,q) \\
+ \mathrm{D}_{a,b}(x) + \frac{(4\pi \,\mathrm{Im}(\tau))^{a+b-1}}{(a+b)!} B_{a+b}(u)\,. \qquad (3.135)
\end{aligned}$$

This ultimately leads to a relation to the meromorphic ePLs $\tilde{\Gamma}_n(m;z)$. Thus, the single-valued ePLs $\mathrm{D}^{\tau}_{a,b}$ are a linear combination of the functions $\mathrm{E}_{n,-m}$ and complex conjugates thereof. Again, using eqs. (3.171) and (3.175) below leads to an expression of $\mathrm{D}^{\tau}_{a,b}$ in terms of the ePLs $\tilde{\Gamma}_n(m;z)$ on the torus.

3.5 Functional relations of elliptic polylogarithms

Functional relations of elliptic multiple polylogarithms may be derived using the symbol map [125]. Another approach used to define elliptic analogues of the Bloch group has been considered in [67]. The latter approach is motivated by the elliptic analogue of the classical Bloch relation, the elliptic Bloch relation, which yields a class of relations for the elliptic Bloch–Wigner dilogarithm [49]. While the elliptic symbol calculus has been extensively discussed in ref. [125], we will focus on the elliptic Bloch relation and derive it essentially using methods from the symbol calculus of iterated integrals. The main results of ref. [1] are presented in this section: after formulating the elliptic Bloch relation, the single-valued ePLs, including the elliptic Bloch–Wigner dilogarithm, are related to the multi-valued ePLs. This leads to an expression of the elliptic Bloch relation in terms of ePLs, which can be used to give a very compact proof and motivates holomorphic analogues of the elliptic Bloch relation in terms of multi-valued ePLs.

3.5.1 The elliptic Bloch relation

The elliptic version of the classical Bloch relation (2.85) can be stated as follows: let $\kappa \in \mathbb{C}$ and F be an elliptic function on the Tate curve, i.e. a function of the form (3.44), with divisors

$$\operatorname{div}(F) = \sum_i d_i(a_i) \tag{3.136}$$

and[18]

$$\operatorname{div}(\kappa - F) = \sum_j e_j(b_j)\,. \tag{3.137}$$

Then, the following sum vanishes

$$\sum_{i,j} d_i e_j \, \mathrm{D}^\tau\left(\frac{a_i}{b_j}\right) = 0\,. \tag{3.138}$$

This identity is referred to as the *elliptic Bloch relation* and can be proven via the classical Bloch relation and the representation (3.44) of the elliptic function in terms of rational functions using sophisticated limits [49]. We will give an alternative proof in subsection 3.5.3. This relation can be stated for elliptic functions on the torus via composition with the exponential map, i.e. the isomorphism (3.43), and on the elliptic curve via an additional composition with Abel's map $\phi_{\tau,E}^{-1}$, the inverse of the isomorphism (3.36). The former leads to the *elliptic Bloch relation on the torus*: for $F \in \mathbb{C}(\Lambda_\tau)$ with

$$\operatorname{div}(F) = \sum_i d_i(A_i) \tag{3.139}$$

and

$$\operatorname{div}(\kappa - F) = \sum_j e_j(B_j)\,, \tag{3.140}$$

where $a_i = e^{2\pi i A_i}$ and $b_j = e^{2\pi i B_i}$, the following expression vanishes

$$\sum_{i,j} d_i e_j \, \mathrm{D}^\tau\left(e^{2\pi i (A_i - B_j)}\right) = 0\,. \tag{3.141}$$

For a rational function on the elliptic curve $F \in \mathbb{C}(x,y)/\{y^2 = 4x^3 - g_2 x - g_3\}$ with

$$\operatorname{div}(F) = \sum_i d_i(P_i) \tag{3.142}$$

[18] Note that in eq. (3.137), the number κ is not restricted to $\kappa = 1$, in contrast to eq. (2.84) used for the classical Bloch relation (2.85). This discrepancy illustrates that taking a limit such as $\tau \to i\infty$ of the elliptic Bloch relation to recover the classical Bloch relation is highly non-trivial, since the validity for $\kappa \neq 1$ is lost in this limit.

and

$$\operatorname{div}(\kappa - F) = \sum_j e_j(Q_j)\,, \tag{3.143}$$

where $A_i = \phi^{-1}_{\tau,E}(P_i)$ and $B_j = \phi^{-1}_{\tau,E}(Q_j)$ the same identity holds: the *elliptic Bloch relation on the elliptic curve* states that

$$\sum_{i,j} d_i e_j \, \mathrm{D}^\tau\left(e^{2\pi i\,\phi^{-1}_{\tau,E}(P_i - Q_j)}\right) = 0\,. \tag{3.144}$$

It is expected that the elliptic Bloch relation (3.138), the inversion relation (3.129) and the duplication relation (3.130) generate the functional relations of the elliptic Bloch–Wigner dilogarithm [67]. However, unlike for the classical Bloch relation and the five-term identity, the functional identities generated by the elliptic Bloch relation are generally independent of each other and parametrised by the elliptic functions upon varying their zeros and poles.

A divisor on $y^2 = 4x^3 - 4x + 1$

A representative example of the elliptic Bloch relation (3.144) on the elliptic curve from ref. [67] is the elliptic curve $E(\mathbb{C})$ defined by the Weierstrass equation

$$E : y^2 = 4x^3 - 4x + 1 \tag{3.145}$$

with $g_2 = 4$ and $g_3 = -1$, the number $\kappa = 1$ (appearing in eq. (3.137)), and the rational function

$$F(x,y) = \frac{y+1}{2} \tag{3.146}$$

on $E(\mathbb{C})$. The three zeros of F are

$$P_1 = (0,-1)\,, \quad P_2 = (1,-1)\,, \quad P_3 = (-1,-1) \tag{3.147}$$

and since F expressed on the torus $F(\wp(z),\wp'(z)) = \frac{\wp'(z)+1}{2}$ has a triple pole at the lattice points, the order of the only pole at

$$P_4 = \infty \tag{3.148}$$

is $\mathrm{ord}_\infty(F) = -3$. Similarly, the zeros of $\kappa - F = 1 - F$ are

$$Q_1 = -P_1\,, \quad Q_2 = -P_2\,, \quad Q_3 = -P_3 \tag{3.149}$$

and the only pole of order three is

$$Q_4 = \infty \,. \tag{3.150}$$

According to the addition on the elliptic curve from appendix A, the zeros of F and $1 - F$ are multiples of each other, since (more generally)

$$\begin{aligned} -3P_1 &= (-1,-1) = P_3\,, & -2P_1 &= (1,1) = Q_2 & -P_1 &= (0,1) = Q_1\,, \\ P_1 &= (0,-1)\,, & 2P_1 &= (1,-1) = P_2\,, & 3P_1 &= (-1,1)\ , \\ 4P_1 &= (2,5)\ , & 5P_1 &= \left(\frac{1}{4},\frac{1}{4}\right)\ , & 6P_1 &= (6,-29)\ . \end{aligned} \tag{3.151}$$

Thus, the only point that has to be mapped to the torus and ultimately to the Tate curve is $P_1 = (0,-1)$. This can be done as follows: the roots of

$$y^2 = 4x^3 - 4x + 1 = 4(x-e_1)(x-e_2)(x-e_3) \tag{3.152}$$

are

$$e_1 = 0.8375654352\,, \quad e_2 = 0.2695944364\,, \quad e_3 = -1.1071598716\,. \tag{3.153}$$

Therefore, the periods of the corresponding torus can be calculated according to eq. (3.41), which yields

$$\omega_1 = 2.9934586462\,, \quad \omega_2 = 2.9934586462 + 2.4513893819i \tag{3.154}$$

with the modular parameter

$$\tau = \frac{\omega_2}{\omega_1} = 1 + 0.8189153991i \in \mathbb{H}\,. \tag{3.155}$$

The point z'_P in the fundamental parallelogram P_Λ of the lattice $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ which corresponds to $P_1 = (0,-1)$ on the elliptic curve $E(\mathbb{C})$ is determined by Abel's map from eq. (3.37) and given by

$$\begin{aligned} z'_{P_1} &= \int_0^\infty \frac{dx}{\sqrt{4(x-e_1)(x-e_2)(x-e_3)}} + \omega_2 - \omega_1 \\ &= 2.0638659408 + 1.2256947056i\,. \end{aligned} \tag{3.156}$$

Its representative in the fundamental parallelogram of the rescaled torus with modular parameter τ is

$$z_{P_1} = \frac{z'_{P_1}}{\omega_1} = 0.6894586481 + 0.4094577022i\,. \tag{3.157}$$

By exponentiation, this point is mapped to the representative

$$x_{P_1} = e^{2\pi i z_{P_1}} = -0.0283399159 - 0.0708731874i \tag{3.158}$$

on the Tate curve $\mathbb{C}^*/q^{\mathbb{Z}}$. The parameter q of the Tate curve, in turn, is given by

$$q = e^{2\pi i\tau} = 0.0058261597\,. \tag{3.159}$$

The elliptic Bloch relation (3.138) states that

$$-8\,\mathrm{D}^\tau\left(x_{P_1}, q\right) - 7\,\mathrm{D}^\tau\left(x_{P_1}^2, q\right) + 8\,\mathrm{D}^\tau\left(x_{P_1}^3, q\right) + \mathrm{D}^\tau\left(x_{P_1}^4, q\right) - \mathrm{D}^\tau\left(x_{P_1}^6, q\right) = 0\,, \tag{3.160}$$

where the inversion relation (3.129) has been used to cancel several terms. This can be tested numerically by truncating the series defining the elliptic Bloch–Wigner dilogarithm and approximating it by

$$\mathrm{D}_k^\tau(x, q) = \sum_{l=-k}^{k} D(xq^l). \tag{3.161}$$

Already for $k = 10$, eq. (3.160) can be shown to hold with a precision of up to 10^{-7}.

Lines on the projective elliptic curve

In the above example, the elliptic Bloch relation is evaluated for a fix elliptic function, leading to a vanishing linear combination (3.160) of complex numbers. If classes of elliptic functions are considered, rather than numeric relations, functional relations are obtained. A prime example are lines on the elliptic curve $E : y^2 = 4x^3 - g_2 x - g_3$, which are rational functions of the form

$$L_{a,b,c}(x, y) = ax + by + c \tag{3.162}$$

with $(a, b) \neq (0, 0)$. The poles of the lines are located at ∞ and of multiplicity two if $b = 0$ and three otherwise, i.e. $m_b = 2\delta_{b,0} + 3\delta_{b\neq 0}$. The cubic equations

$$\left(\frac{a}{b}x + \frac{c}{b}\right)^2 = 4x^3 - g_2 x - g_3\,, \quad \left(\frac{a}{b}x + \frac{c-\kappa}{b}\right)^2 = 4x^3 - g_2 x - g_3 \tag{3.163}$$

lead to expressions of the zeros of $L_{a,b,c}$ and $\kappa - L_{a,b,c}$ which depend algebraically on a, b, c and κ, i.e.

$$P_i = P_i(a, b, c)\,, \quad Q_i = Q_i(a, b, c, \kappa) \tag{3.164}$$

for $i = 1, 2, 3$. The divisors are therefore given by

$$\begin{aligned} \mathrm{div}(L_{a,b,c}) &= (P_1) + (P_2) + (P_3) - m_\infty(\infty)\,, \\ \mathrm{div}(\kappa - L_{a,b,c}) &= (Q_1) + (Q_2) + (Q_3) - m_\infty(\infty)\,. \end{aligned} \tag{3.165}$$

Hence, the elliptic Bloch relation on the elliptic curve (3.141) leads in this case to a functional relation with variables a, b, c, κ. However, these variables are not independent, but restricted by the requirement of being the zeros of $L_{a,b,c}$ and of $\kappa - L_{a,b,c}$, respectively.

3.5.2 Relations among classes of elliptic polylogarithms

Functional relation between the ePLs $\tilde{\Gamma}_n(m; z)$ from eq. (3.90) and the functions $\mathrm{E}_{n,m}(x, q)$ defined in eq. (3.104) on the Tate curve have been derived in ref. [1] with the following result: the functions $\mathrm{E}_{n,-m}(x, q) = \mathrm{E}_{n,-m}(e^{2\pi i z}, e^{2\pi i \tau})$ with $m > 0$, and E_n for $m = 0$, defined in eq. (3.109) are up to polynomials in z equal to the n-fold iterated integral of the integration kernels $g^{(m+1)}(z, \tau)$, which are the ePLs $\tilde{\Gamma}_n(m + 1; z)$. Below, a summary of the main steps in the derivation is given, the details can be found in ref. [1].

This statement can be proven starting with the case $m = 0$ and the functions $\mathrm{E}_n(x, q)$ defined in eq. (3.109). In order to do so, the Eisenstein–Kronecker series is expressed on the Tate curve in the variables

$$x = e^{2\pi i z}\,, \quad q = e^{2\pi i \tau}\,, \quad w = e^{2\pi i \eta} \tag{3.166}$$

as follows [126]:

$$F(z, \eta, \tau) = -2\pi i \left(\frac{x}{1 - x} + \frac{1}{1 - w} + \sum_{k,l>0} (x^k w^l - x^{-k} w^{-l}) q^{kl} \right)\,. \tag{3.167}$$

Then, the integration kernel of weight one can be represented by the limit

$$g^{(1)}(z, \tau) = \lim_{\eta \to 0} \left(F(z, \eta, \tau) - 1/\eta \right)\,, \tag{3.168}$$

which leads to[19]

$$\mathrm{E}_0(x, q) = \frac{1}{2\pi i} g^{(1)}(z, \tau)\,. \tag{3.169}$$

The weight-one ePLs on the torus are recovered from eq. (3.169) and iteratively integrating the differential equation (3.110) composed with the exponential map (3.166),

[19] This connection has been shown in ref. [127]. It was the motivation to consider the generalisations for $\mathrm{E}_{n,-m}(t, 1, q)$ with $n, m > 0$ in the subsequent paragraphs. Similar calculations have been worked out in ref. [118].

i.e.

$$\frac{\partial}{\partial z}\,\mathrm{E}_n(e^{2\pi i z},q) = 2\pi i\,\mathrm{E}_{n-1}(e^{2\pi i z},q)\,. \tag{3.170}$$

The result is

$$\mathrm{E}_n(x,q) = (2\pi i)^{n-1}\Big(\tilde{\Gamma}_n(1;z,\tau) + \sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{k=0}^{j} d_{2k+1}\omega_{2j+2-2k}(1;\tau)\,\tilde{\Gamma}_{n-1-2j}(0;z,\tau)\Big)\,, \tag{3.171}$$

where by definition

$$\tilde{\Gamma}_n(0;z,\tau) = \frac{z^n}{n!} \tag{3.172}$$

and the coefficients $d_k \in \mathbb{Q}$ are given by the sequence defined in eq. (3.107). Equation (3.171) expresses the z dependence of $\mathrm{E}_n(x,q)$ on the torus in terms of ePLs with at most weight one.

A similar result can be derived for $m>0$ to relate the functions $\mathrm{E}_{n,-m}(x,q)$ to ePLs up to weight $m{+}1$. From the q-expansions of $g^{(m+1)}$ given in appendix B.1, the following relation is obtained

$$\mathrm{E}_{0,-m}(x,q) = \frac{m!}{(2\pi i)^{m+1}}\left(g^{(m+1)}(z,\tau) + (1+(-1)^{m+1})\zeta_{m+1}\right)\,. \tag{3.173}$$

Starting from eq. (3.173) and iteratively integrating the partial differential equation (3.105) composed with the exponential map, which yields

$$\frac{\partial}{\partial z}\,\mathrm{E}_{n,m}(e^{2\pi i z},q) = 2\pi i\,\mathrm{E}_{n-1,m}(e^{2\pi i z},q)\,, \tag{3.174}$$

the results for $n>0$ are obtained. This leads to the identity

$$\mathrm{E}_{n,-m}(x,q) = \begin{cases} m!(2\pi i)^{n-1-m}\Big(\tilde{\Gamma}_n(m+1;z,\tau) \\ \qquad + \sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\sum_{k=0}^{j} d_{2k+1}\omega_{2j+1-2k}(m+1;\tau)\,\tilde{\Gamma}_{n-2j}(0;z,\tau)\Big) & m \text{ odd}, \\ m!(2\pi i)^{n-1-m}\Big(\tilde{\Gamma}_n(m+1;z,\tau) \\ \qquad + \sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\sum_{k=0}^{j} d_{2k+1}\omega_{2j+2-2k}(m+1;\tau)\,\tilde{\Gamma}_{n-1-2j}(0;z,\tau)\Big) & m \text{ even}, \end{cases} \tag{3.175}$$

which establishes a connection between the sums $\mathrm{E}_{n,-m}(x,q)$ on the Tate curve and the ePLs which are iterated integrals on the torus.

The examples

$$\mathrm{E}_1(x,q) = \tilde{\Gamma}_1(1;z,\tau) - \omega_2(1;\tau) \tag{3.176}$$

and

$$\mathrm{E}_2(x,q) = 2\pi i \left(\tilde{\Gamma}_2(1;z,\tau) - \omega_2(1;\tau)z\right) \tag{3.177}$$

as well as

$$\mathrm{E}_{1,-1}(x,q) = \frac{1}{2\pi i}\left(\tilde{\Gamma}_1(2;z,\tau) - \omega_1(2;\tau)z\right) = \frac{1}{2\pi i}\tilde{\Gamma}_1(2;z,\tau) + \frac{1}{\pi i}\zeta_2 z \tag{3.178}$$

can be used to rewrite the elliptic Bloch–Wigner dilogarithm from eq. (3.132) in terms of ePLs as follows:

$$\begin{aligned}\mathrm{D}^\tau(x) = {} & \mathrm{Im}(\tau)\,\mathrm{Re}\left(\tilde{\Gamma}_1(2;z)\right) - 2\pi\,\mathrm{Re}\left(\tilde{\Gamma}_2(1;z)\right) - 2\pi\,\mathrm{Im}(z)\,\mathrm{Im}\left(\tilde{\Gamma}_1(1;z)\right)\\ & + 2\,\mathrm{Re}(z)\left(\pi\,\mathrm{Re}\left(\omega_2(1;\tau)\right) + \zeta_2\,\mathrm{Im}(\tau)\right)\,.\end{aligned} \tag{3.179}$$

Using the identity

$$\begin{aligned}&\mathrm{Re}\left(\tilde{\Gamma}_2(1;z)\right) + \mathrm{Im}(z)\,\mathrm{Im}\left(\tilde{\Gamma}_1(1;z)\right)\\ &= -\,\mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix};z,\tau\right)\right) + \mathrm{Re}(z)\,\mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix}1\\ 0\end{smallmatrix};z,\tau\right)\right)\end{aligned} \tag{3.180}$$

a slightly modified version including an eMPL, which is not explicitly an ePL is obtained

$$\begin{aligned}\mathrm{D}^\tau(x) = {} & \mathrm{Im}(\tau)\,\mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix}2\\ 0\end{smallmatrix};z,\tau\right)\right) + 2\pi\,\mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix};z,\tau\right)\right) - 2\pi\,\mathrm{Re}(z)\,\mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix}1\\ 0\end{smallmatrix};z,\tau\right)\right)\\ & + 2\,\mathrm{Re}(z)\left(\pi\,\mathrm{Re}\left(\omega_2(1;\tau)\right) + \zeta_2\,\mathrm{Im}(\tau)\right)\,.\end{aligned} \tag{3.181}$$

This second representation of $\mathrm{D}^\tau(x)$ is used in ref. [1] to give an alternative proof of the elliptic Bloch relation, which will be outlined in the next subsection.

Not only the elliptic Bloch–Wigner dilogarithm, but the whole class of single-valued ePLs $\mathcal{L}_n^\tau(x)$ and $\mathrm{D}_{a,b}^\tau(x)$ can be expressed in terms of the meromorphic ePLs on the torus [1]: starting from eq. (3.131) and using eqs. (3.171) and (3.175) leads to the desired relation for $\mathcal{L}_n^\tau(x)$. The more general class $\mathrm{D}_{a,b}^\tau(x)$ can be rewritten similarly, starting from eq. (3.135).

In [63], the eMPLs $\tilde{\Gamma}_w(z)$ defined in eq. (3.89) on the torus have been expressed on the elliptic curve. This direct translation can be used to further translate the single-valued ePLs $\mathcal{L}_n^\tau(x)$ and $\mathrm{D}_{a,b}^\tau(x)$ and all the corresponding identities to the elliptic curve as well [1].

3.5.3 The holomorphic elliptic Bloch relation on the torus

To finish this chapter, the versions of the elliptic Bloch relation (3.141) in terms of the meromorphic eMPLs $\tilde{\Gamma}_w$ on the torus are presented. These relations do not include complex conjugates of eMPLs and are therefore expressed in terms of holomorphic quantities on the appropriate domain and, thus, called holomorphic elliptic Bloch relations. They have been derived in ref. [1], the proof is based on the differential calculus of the iterated integrals, which is a special case of the elliptic symbol calculus [125]. While the results are summarised below, an outline of the derivation is given in appendix B.2.

The *holomorphic elliptic Bloch relations on the torus* state the following: for an elliptic function F and a complex number $\kappa \in \mathbb{C}$ with divisors

$$\operatorname{Div}(F) = \sum_i d_i(A_i)\,, \quad \sum_i d_i = 0\,, \quad \sum_i d_i A_i = 0\,, \tag{3.182}$$

and

$$\operatorname{Div}(\kappa - F) = \sum_j e_j(B_j)\,, \quad \sum_j e_j = 0\,, \quad \sum_j e_j B_j = 0\,, \tag{3.183}$$

the following two sums of eMPLs vanish:

$$\sum_{i,j} d_i e_j\, \tilde{\Gamma}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; A_i - B_j, \tau\right) = 0 \tag{3.184}$$

and

$$\sum_{i,j} d_i e_j\, \tilde{\Gamma}\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}; A_i - B_j, \tau\right) = 0\,. \tag{3.185}$$

Moreover, the following two non-holomorphic combinations vanish as well:

$$\sum_{i,j} d_i e_j \operatorname{Re}(A_i - B_j) \operatorname{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_i - B_j, \tau\right)\right) = 0 \tag{3.186}$$

and

$$\sum_{i,j} d_i e_j \operatorname{Im}(A_i - B_j) \operatorname{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_i - B_j, \tau\right)\right) = 0. \tag{3.187}$$

According to the representation (3.181) of the elliptic Bloch–Wigner dilogarithm, the elliptic Bloch relation (3.141) on the torus is given by

$$
\begin{aligned}
0 &= \sum_{i,j} d_i e_j \, \mathrm{D}^\tau \left(e^{2\pi i (A_i - B_j)}, q\right) \\
&= -2\pi \sum_{i,j} d_i e_j \Bigg(\mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; A_i - B_j, \tau\right)\right) && \text{(3.188a)} \\
&\qquad + \mathrm{Re}\left(\frac{\tau}{2\pi i}\right) \mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}; A_i - B_j, \tau\right)\right) && \text{(3.188b)} \\
&\qquad - \mathrm{Re}(A_i - B_j)\, \mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_i - B_j, \tau\right)\right) \Bigg) . && \text{(3.188c)}
\end{aligned}
$$

Therefore, the holomorphic elliptic Bloch relations (3.184) and (3.185) together with the non-holomorphic combination (3.186) allow for an alternative proof of the elliptic Bloch relation showing that the sums (3.188a), (3.188b) and (3.188c) vanish separately. This proof of the elliptic Bloch relation does not use any sophisticated limits from rational to elliptic functions on the Tate curve compared to the original proof in ref. [49] and is based on the elliptic symbol calculus of eMPLs, cf. appendix B.2.

Chapter 4

Open-string corrections at genus zero

The aim of this chapter is to investigate and properly formulate the recursion from ref. [5] to calculate open-string corrections of massless string states at genus zero, schematically given in eqs. (1.5) and (2.53): it is based on a vector of integrals with an auxiliary point on the Riemann sphere, which satisfies the KZ equation. The boundary values of this solution of the KZ equation are the $(n-1)$- and n-point, open-string corrections. They are related by the genus-zero associator equation (2.50). Therefore, the Drinfeld associator offers a method to calculate the α'-expansion of the n-point corrections from the $(n-1)$-point corrections, which only requires matrix operations.

The results from ref. [5] will be complemented by various properties of the corresponding class of integrals, which, in turn, are based on the investigations of ref. [4], where the recursion has been related to twisted de Rham theory. While the connection to twisted forms and twisted de Rham cohomologies is only briefly mentioned, it is shown that the matrices in the KZ equation (2.53) are representations of the generators of the braid group. In particular, they can be computed combinatorially, such that the same holds for the n-point, open-string corrections. A convenient tool for these combinatorial properties is a graphical representation of certain products of rational functions, which has been introduced in ref. [4]. Lengthy calculations are outsourced to chapter 6, where this powerful graphical machinery is introduced and exploited.

In section 4.1 open-string corrections at genus zero are presented and related to two classes of integrals, the Z_n- and Selberg integrals, which are referred to as open-string integrals. In the subsequent section 4.2 these classes of integrals are generalised even beyond the scope of ref. [4]. This ultimately leads to the recursion in section 4.3, where the main result from ref. [5] is presented with some supplements from ref. [4].

4.1 Genus-zero, open-string corrections

The genus-zero worldsheet of n external open-string states is a disk with n punctures located at the boundary. The punctures are the vertex insertion points and correspond to the external states. This punctured disk can be described by one hemisphere of the Riemann sphere with the punctures sitting on the real line plus infinity $\mathbb{P}^1(\mathbb{R}) \subset \mathbb{P}^1(\mathbb{C})$, cf. figure 1.2. The $\mathrm{SL}(2,\mathbb{R})$-redundancy from a residual gauge freedom can be used to fix three of these insertion points, in our case the points

$$(x_1, x_2, x_n) = (0, 1, \infty) \,. \tag{4.1}$$

The remaining punctures are assumed[1] to be ordered according to

$$\Delta_{n,3} = \Delta_{n,3}(x_i) = \{0 = x_1 < x_{n-1} < x_{n-2} < \cdots < x_3 < x_2 = 1\} \,, \tag{4.2}$$

cf. figure 4.1.

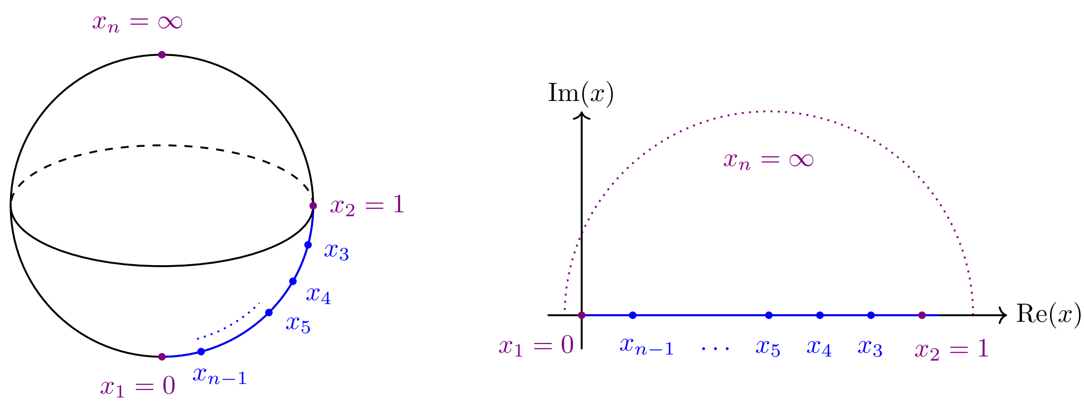


Figure 4.1: The punctured Riemann sphere with the three fixed (violet) punctures x_1, x_2, x_n and its parametrisation on the complex plane (right-hand side). The $n-3$ unfixed (blue) punctures $x_3, \ldots, x_{n-1}$ will be integrated over in the sting corrections respecting the order $x_{i+1} < x_i$. This defines the corresponding integration cycle $\Delta_{n,3}(x_i)$ (blue line).

The genus-zero string corrections $F^{\text{open}}_{n,0}(\sigma;\alpha')$ are the integral contributions to the colour-ordered, tree-level superstring amplitudes of n massless, open-string states

$$A^{\text{open}}_{n,0}(\alpha') = \sum_{\sigma \in S_{n-3}} F^{\text{open}}_{n,0}(\sigma;\alpha') A^{\text{particle}}_{n,0}(\sigma) \,, \tag{4.3}$$

[1] Different orderings lead to permutations of the corresponding labels in the integrals associated to the integration cycle $\Delta_{n,3}$, see e.g. ref. [128] for the dependence on the order of the punctures in such integrals.

obtained from integrating the unfixed insertion points $x_3, \dots, x_{n-1}$ over all possible configurations satisfying the order (4.2) [129, 130]. These are iterated integrals depending on the Mandelstam variables[2]

$$s_{i_1 \dots i_p} = s_{i_1, \dots, i_p} = -\alpha'(k_{i_1} + \dots + k_{i_p})^2 , \quad 1 \leq i_k \leq n , \tag{4.4}$$

where α' is the inverse string tension and k_i the momentum associated to the i-th insertion point x_i. The *n-point, open-string corrections at genus zero* are the $(n-3)!$ integrals [129, 130, 133]

$$F^{\text{open}}_{n,0}(\sigma; \alpha') = F^{\text{open}}_{n,0}(\sigma; \{s_{ij}\}) = \int_{\Delta_{n,3}} \prod_{i=3}^{n-1} \mathrm{d}x_i \ \mathrm{KN}_{12\dots n-1} \ \sigma \left(\prod_{k=3}^{n-1} \sum_{l=2}^{k-1} \frac{s_{lk}}{x_{lk}} \right) , \tag{4.5}$$

where the permutation[3] $\sigma \in S_{n-3}$ acts on the labels $3, \dots, n-1$ of the unfixed punctures, whose differences is denoted by

$$x_{ij} = x_{i,j} = x_i - x_j . \tag{4.6}$$

The Koba–Nielsen factor

$$\mathrm{KN}_{i_1 \dots i_p} = \mathrm{KN}_{i_1 \dots i_p}(x_{i_1}, \dots, x_{i_p}; \{s_{ij}\}) = \prod_{\substack{i,j \in \{i_1 \dots i_p\} \\ i<j}} |x_{ij}|^{-s_{ij}} \tag{4.7}$$

introduces the dependence on the Mandelstam variables and α', respectively. It originates from the plane wave contributions to the vertex operators. Expanding the Koba–Nielsen factor in the string corrections (4.5) in α', an infinite sum of integrals weighted by powers of the Mandelstam variables is obtained. A comparison with the definition (2.39) shows that these integrals are MZVs. The recursion summarised in section 4.3 affirms this result and offers a method to calculate the α'-expansion of the open-string corrections where the integrals are already evaluated and assigned to the appropriate MZVs.

For example, the four-point, open-string correction at genus zero is the Veneziano

[2] For the sake of generality, momentum conservation is not imposed on the Mandelstam variables. However, in order to ensure convergence of the integrals in this and the subsequent chapter, the condition $\mathrm{Re}(s_{i_1 \dots i_p}) < 0$ is imposed for consecutive points $i_1, \dots, i_p$ on the boundary of the genus-zero and genus-one Riemann surfaces [131, 132]. Other regions of the parameter space can be reached by analytic continuation.

[3] For $0 \leq k < p \leq n$, permutations $\sigma \in S_{p-k}$ acting on the set $\{k+1, k+2, \dots, p\}$ are often implicitly extended to permutations $\sigma \in S_n$ by the trivial action $\sigma(i) = i$ for $1 \leq i \leq k$ and $p+1 \leq i \leq n$, and identified with their image on the sequence $(k+1, k+2, \dots, p)$ or $(1, 2, \dots, n)$, respectively.

amplitude [130, 134]

$$
\begin{aligned}
F^{\text{open}}_{4,0}(\alpha') &= \int_0^1 \mathrm{d}x_3\, |x_{13}|^{-s_{13}} |x_{23}|^{-s_{23}} \frac{s_{23}}{x_{23}} = \frac{\Gamma(1-s_{13})\Gamma(1-s_{23})}{\Gamma(1-s_{13}-s_{23})} \\
&= 1 - \zeta_2 s_{12}s_{23} - \zeta_3 s_{12}s_{23}(s_{12}+s_{23}) + \mathcal{O}((\alpha')^4)\,.
\end{aligned} \tag{4.8}
$$

4.1.1 Z_n-integrals

Another well-known representation of the string corrections is a linear combination in terms of so-called Z_n-integrals defined by the Koba–Nielsen factor and the Parke–Taylor forms [135]

$$
\mathrm{PT}(\sigma) = \frac{\mathrm{d}x_{n-1}\wedge \mathrm{d}x_{n-2}\wedge\cdots\wedge \mathrm{d}x_3}{x_{2\sigma(3)}x_{\sigma(3)\sigma(4)}\cdots x_{\sigma(n-2)\sigma(n-1)}}\,, \tag{4.9}
$$

where $\sigma \in S_{n-3}$ is acting on $\{3,4,\dots,n-1\}$. The coefficient, i.e. the product of fractions, in the Parke–Taylor form has a chain-like structure, which facilitates a convenient graphical representation, which in turn is enormously helpful to structure and derive various properties of such products. This graphical representation has been introduced in ref. [4] and is summarised in chapter 6. In the spirit of the graphical representation, this chain of fractions (beginning at a_p and ending at a_1) is denoted by

$$
\begin{aligned}
\varphi(a_1,a_2,\dots,a_p) &= \frac{1}{x_{a_1a_2}x_{a_2a_3}\cdots x_{a_{p-1}a_p}} = \prod_{i=2}^{p}\frac{1}{x_{a_{i-1}a_i}}\,,\\
\varphi(a_1) &= 1\,,
\end{aligned} \tag{4.10}
$$

where $a_i \in \{1,2,\dots,n-1\}$ and $a_i \neq a_j$ for $i\neq j$. Products of this form are called *chain products* and if the factors are fractions such as above, it is called a *chain of fractions*. A short-hand notation for a sequence $A=(a_1,\dots,a_p)$ is

$$
\varphi(A) = \varphi(a_1,\dots,a_p)\,. \tag{4.11}
$$

Note that the partial fractioning identity (2.12) generalises to the following shuffle identity for chains of fractions: for two disjoint sequences B and C, each having pairwise distinct elements, and $a \notin B, C$[4]

$$
\varphi(a,B)\varphi(a,C) = \varphi(a, B \sqcup\!\sqcup\, C)\,, \tag{4.12}
$$

which can be shown via the recursive definition (2.18) of the shuffle product. Further such properties of chain products can be found in chapter 6, where they are heavily

[4] A subset or element of a tuple is a subset or element, respectively, of the set of elements of the tuple.

used in various calculations and related to graphical identities.

Thus, the Parke–Taylor forms can be written as

$$\mathrm{PT}(\sigma) = \varphi(2, \sigma(3, 4, \dots, n-1))\mathrm{d}x_{n-1} \wedge \mathrm{d}x_{n-2} \wedge \cdots \wedge \mathrm{d}x_3 \,. \tag{4.13}$$

They define the following Z_n-*integrals* [30]

$$\begin{aligned} Z_n(\sigma; \{s_{ij}\}) &= \int_{\Delta_{n,3}} \mathrm{KN}_{12\dots n-1}\, \mathrm{PT}(\sigma) \\ &= \int_{\Delta_{n,3}} \prod_{i=3}^{n-1} \mathrm{d}x_i \, \mathrm{KN}_{12\dots n-1}\, \varphi(2, \sigma(3, 4, \dots, n-1)) \,. \end{aligned} \tag{4.14}$$

Assembling the string corrections and the Z_n-integrals in $(n-3)!$-dimensional vectors

$$\boldsymbol{F}^{\mathrm{open}}_{n,0} = \boldsymbol{F}^{\mathrm{open}}_{n,0}(\{s_{ij}\}) = \Big(F^{\mathrm{open}}_{n,0}(\sigma; \{s_{ij}\})\Big)_{\sigma \in S_{n-3}} \tag{4.15}$$

and

$$\boldsymbol{Z}_n = \boldsymbol{Z}_n(\{s_{ij}\}) = \Big(Z_n(\sigma; \{s_{ij}\})\Big)_{\sigma \in S_{n-3}} \,, \tag{4.16}$$

they can be related by an invertible transformation[5]

$$\boldsymbol{F}^{\mathrm{open}}_{n,0} = \boldsymbol{B}^{\mathrm{cha}}_n \boldsymbol{Z}_n \,, \tag{4.17}$$

where $\boldsymbol{B}^{\mathrm{cha}}_n$ is (up to integration by parts) known as momentum kernel [136, 137]. The matrix $\boldsymbol{B}^{\mathrm{cha}}_n$ can be found up to integration by parts in ref. [133]. Alternatively, it can be calculated using the algorithm from appendix C.2.2 or simply via eq. (4.46) below.

The $(n-3)!$ Parke–Taylor forms $\mathrm{PT}(\sigma)$ are a basis for the open-string corrections at genus zero [129, 130]. The reason why they span all genus-zero, open-string corrections is based on the fact that they represent a basis of the twisted de Rham cohomology[6] of the moduli space of n-punctured Riemann spheres [142], which is the configuration space of n-punctured Riemann spheres with three fixed coordinates

$$\begin{aligned} \mathcal{M}_{0,n} &= \mathrm{Conf}_n(\mathbb{P}^1(\mathbb{C}))/\,\mathrm{SL}(2, \mathbb{C}) \\ &= \{(x_3, x_4, \dots, x_{n-1}) \in \mathbb{C} | x_i \neq 0, 1, x_j \text{ for } i \neq j\} \,. \end{aligned} \tag{4.18}$$

[5] The superscript cha stands for chain, which refers to the chain product from eq. (4.10) defining the Z_n-integrals in eq. (4.14).

[6] See ref. [138] for a thorough introduction to twisted de Rham theory and e.g. refs. [139–141] for various applications in string amplitudes and Feynman integrals.

Its twisted de Rham cohomology is given by

$$H^{n-3}(\mathcal{M}_{0,n},\nabla_{n-3}) = \ker(\nabla_{n-3})/\operatorname{Im}(\nabla_{n-3})\,,$$
$$\nabla_{n-3} = \mathrm{d} + \mathrm{d}\log\left(\mathrm{KN}_{12\ldots n-1}\right)|_{\mathrm{d}x_1=\mathrm{d}x_2=0}\wedge\,, \tag{4.19}$$

where ∇_{n-3} is an integrable connection [143]. The equivalence classes of the Parke–Taylor forms $\mathrm{PT}(\sigma)+\operatorname{Im}(\nabla_{n-3})$ are a basis of $H^{n-3}(\mathcal{M}_{0,n},\nabla_{n-3})$. In other words, the Z_n-integrals are on the one hand linearly independent with respect to partial fractioning and integration by parts and on the other hand, all string corrections can be written in terms of linear combinations of these integrals.

4.1.2 Selberg integrals

Yet another basis of integrals spanning the space of string corrections at genus zero are the Selberg integrals [144]. We will define Selberg integrals following ref. [142], where they were originally used to define a basis of $H^{n-3}(\mathcal{M}_{0,n},\nabla_{n-3})$. Instead of chains of fractions, Selberg integrals involve products of the form

$$\varphi\left(\begin{smallmatrix} a_1 & \cdots & a_p \\ e(a_1) & \cdots & e(a_p)\end{smallmatrix}\right) = \prod_{i=1}^{p}\frac{1}{x_{e(a_i)a_i}}\,, \tag{4.20}$$

where

$$1 \leq e(a_i) < a_i\,, \tag{4.21}$$

$a_i \in \{2,3,\ldots,n-1\}$ and $a_i \neq a_j$ for $i \neq j$. For a set $A \subset \{1,2,\ldots,n-1\}$, the map

$$e : A \to \{1,2,\ldots,n-1\} \tag{4.22}$$

is called *admissible* (with respect to A), if condition (4.21) holds. In this case, the product in eq. (4.20) is called *admissible product (of fractions)*. It is independent of the order of $(a_1,\ldots,a_p)$, such that for a set $A=\{a_1,\ldots,a_p\}$, the notation

$$\varphi\left(\begin{smallmatrix} A \\ e(A)\end{smallmatrix}\right) = \varphi\left(\begin{smallmatrix} a_1 & \cdots & a_p \\ e(a_1) & \cdots & e(a_p)\end{smallmatrix}\right) \tag{4.23}$$

is well-defined. Let us define for a set $A=\{a_1,\ldots,a_p\}$, an admissible map e and $i \in \{1,2,\ldots,n-1\}$, the possibly empty subset

$$A_e(i) = \{j \in A \mid \exists m > 0 : e^m(j) = i\} \subset A\,, \tag{4.24}$$

where $e^m(j) = e^{m-1}(e(j))$ is the m-fold application of e to j. Then, the following disjoint decomposition of A is obtained:

$$A = \bigcup_{i \in e(A)} A_e(i) \,. \tag{4.25}$$

Therefore, the admissible product (4.20) can be factorised into several admissible products

$$\varphi \left({}^{A}_{e(A)} \right) = \prod_{i \in e(A)} \varphi \left({}^{A_e(i)}_{e(A_e(i))} \right) \,. \tag{4.26}$$

For example given the set $A = \{4, 5, 6, 7, 8\}$ and admissible map e defined by the double sequence

$$\left({}^{A}_{e(A)} \right) = \left({}^{4\,5\,6\,7\,8}_{2\,1\,3\,6\,1} \right) \,, \tag{4.27}$$

the corresponding admissible product is

$$\varphi \left({}^{4\,5\,6\,7\,8}_{2\,1\,3\,6\,1} \right) = \frac{1}{x_{24} x_{15} x_{36} x_{67} x_{18}} \,. \tag{4.28}$$

The decomposition

$$A_e(1) = \{5, 8\} \,, \quad A_e(2) = \{4\} \,, \quad A_e(3) = \{6, 7\} \,, \tag{4.29}$$

such that $A_2(1) \cup A_2(2) \cup A_2(3) = \{4, 5, 6, 7, 8\}$, leads to the decomposition of the admissible product into three admissible products

$$\varphi \left({}^{4\,5\,6\,7\,8}_{2\,1\,3\,6\,1} \right) = \varphi \left({}^{4}_{2} \right) \varphi \left({}^{5\,8}_{1\,1} \right) \varphi \left({}^{6\,7}_{3\,6} \right) \,. \tag{4.30}$$

These constructions are related to the string corrections as follows: for the trivial permutation $\sigma = \mathrm{id}$ with $\sigma(3, 4, \ldots, n-1) = (3, 4, \ldots, n-1)$, the integrand without the Koba–Nielsen factor in the string correction (4.5) is a sum of admissible products

$$\prod_{k=3}^{n-1} \sum_{l=2}^{k-1} \frac{s_{lk}}{x_{lk}} = \sum_{\substack{e \text{ adm} \\ 1 < e(k) < k}} \prod_{k=3}^{n-1} s_{e(k),k} \, \varphi \left({}^{3}_{e(3)} \, {}^{\cdots}_{\cdots} \, {}^{n-1}_{e(n-1)} \right) \,, \tag{4.31}$$

where the sum on the right-hand side runs over all the admissible maps w.r.t. $\{3, \ldots, n-1\}$ with $e(k) \neq 1$. In the case of an arbitrary permutation σ the same holds: the admissible products can nicely be described in terms of directed tree graphs, leading to the recursive algorithm in appendix C.2.1. The algorithm can be applied to obtain a linear combination of admissible products

$$
\begin{aligned}
\sigma\left(\prod_{k=3}^{n-1}\sum_{l=2}^{k-1}\frac{s_{lk}}{x_{lk}}\right) &= \sum_{\substack{e \text{ adm}\\ 1<e(k)<k}} \prod_{k=3}^{n-1} s_{e^\sigma(k),k}\, \varphi\left(\begin{smallmatrix} 3 & \cdots & n-1 \\ e^\sigma(3) & \cdots & e^\sigma(n-1)\end{smallmatrix}\right) \\
&= \sum_{\substack{e \text{ adm}\\ 1<e(k)<k}} \prod_{k=3}^{n-1} s_{e^\sigma(k),k} \sum_{\substack{e' \text{ adm}\\ 1<e'(k)<k}} b_{e^\sigma,e'}\,\varphi\left(\begin{smallmatrix} 3 & \cdots & n-1 \\ e'(3) & \cdots & e'(n-1)\end{smallmatrix}\right)\,, \qquad (4.32)
\end{aligned}
$$

where

$$
e^\sigma(k) = \sigma(e(\sigma^{-1}(k)))\,. \qquad (4.33)
$$

The integers $b_{e^\sigma,e'} \in \mathbb{Z}$ in the transformation of the possibly non-admissible product on the right-hand side of the first line of eq. (4.32) to admissible products, i.e.

$$
\varphi\left(\begin{smallmatrix} 3 & \cdots & n-1 \\ e^\sigma(3) & \cdots & e^\sigma(n-1)\end{smallmatrix}\right) = \sum_{\substack{e' \text{ adm}\\ 1<e'(k)<k}} b_{e^\sigma,e'}\,\varphi\left(\begin{smallmatrix} 3 & \cdots & n-1 \\ e'(3) & \cdots & e'(n-1)\end{smallmatrix}\right)\,, \qquad (4.34)
$$

are determined by the algorithm in appendix C.2.1. In particular, $b_{e^{\mathrm{id}},e'} = \delta_{e,e'}$ for $\sigma = \mathrm{id}$, in agreement with eq. (4.31).

Analogously to the Z_n-integrals in eq. (4.14), the *Selberg integrals* [142, 144]

$$
S_n\left(\begin{smallmatrix} 3 & \cdots & n-1 \\ e(3) & \cdots & e(n-1)\end{smallmatrix}; \{s_{ij}\}\right) = \int_{\Delta_{n,3}} \prod_{i=3}^{n-1} \mathrm{d}z_i \ \mathrm{KN}_{23\ldots n}\ \varphi\left(\begin{smallmatrix} 3 & \cdots & n-1 \\ e(3) & \cdots & e(n-1)\end{smallmatrix}\right) \qquad (4.35)
$$

with $2 \leq e(k) < k$ span the vector space generated by the string corrections: the $(n-3)!$-dimensional vector

$$
\boldsymbol{S}_n = \boldsymbol{S}_n(\{s_{ij}\}) = \left(S_n\left(\begin{smallmatrix} 3 & \cdots & n-1 \\ e(3) & \cdots & e(n-1)\end{smallmatrix}; \{s_{ij}\}\right)\right)_{2\leq e(k)<k} \qquad (4.36)
$$

is related by a basis transformation to the string corrections

$$
\boldsymbol{F}^{\mathrm{open}}_{n,0} = \boldsymbol{B}^{\mathrm{adm}}_n\, \boldsymbol{S}_n\,, \qquad (4.37)
$$

where the entries of the $(n-3)!$-dimensional square matrix[7] $\boldsymbol{B}^{\mathrm{adm}}_n$ are given by the coefficients of the admissible products in the linear combination (4.32) [4]. The admissible maps e with $e(k) = 1$ for some $3 \leq k \leq n-1$ can be excluded from the basis in eq. (4.36), since any such integral can be related to the integrals in $\boldsymbol{S}_n$ using integration by parts and partial fractioning.

Moreover, there is also a basis transformation between the Selberg integrals and

[7]The superscript adm stands for admissible, which refers to the admissible product from eq. (4.20) defining the Selberg integrals in eq. (4.35).

the Z_n-integrals from eq. (4.14). Given a sequence (or set) A and an admissible map e w.r.t. A, starting from the double sequence $\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right)$, let us recursively define the following formal sum of sequences

$$\begin{aligned} \left(i, \begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) &= \Bigg(i, \bigsqcup_{\substack{j\in A \\ e(j)=i}} \left(j, \begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right)\Bigg)\,, \\ (A, \varnothing) &= (A)\,. \end{aligned} \tag{4.38}$$

For the example $\left(\begin{smallmatrix} 4&5&6&7&8 \\ 2&1&3&6&1 \end{smallmatrix}\right)$ from eq. (4.27), the following sums of sequences are obtained:

$$\begin{aligned} &\left(1, \begin{smallmatrix} 4&5&6&7&8 \\ 2&1&3&6&1 \end{smallmatrix}\right) = \left(1, \left(5, \begin{smallmatrix} 4&5&6&7&8 \\ 2&1&3&6&1 \end{smallmatrix}\right) \sqcup \left(8, \begin{smallmatrix} 4&5&6&7&8 \\ 2&1&3&6&1 \end{smallmatrix}\right)\right) = (1, 5 \sqcup 8) = (1,5,8) + (1,8,5)\,, \\ &\left(2, \begin{smallmatrix} 4&5&6&7&8 \\ 2&1&3&6&1 \end{smallmatrix}\right) = \left(2, \left(4, \begin{smallmatrix} 4&5&6&7&8 \\ 2&1&3&6&1 \end{smallmatrix}\right)\right) = (2,4)\,, \\ &\left(3, \begin{smallmatrix} 4&5&6&7&8 \\ 2&1&3&6&1 \end{smallmatrix}\right) = \left(3, \left(6, \begin{smallmatrix} 4&5&6&7&8 \\ 2&1&3&6&1 \end{smallmatrix}\right)\right) = \left(3, \left(6, \left(7, \begin{smallmatrix} 4&5&6&7&8 \\ 2&1&3&6&1 \end{smallmatrix}\right)\right)\right) = (3,6,7)\,. \end{aligned} \tag{4.39}$$

Then, the admissible product from eq. (4.26) can be written as a linear combination of chain products defined in eq. (4.10) as follows

$$\varphi\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) = \prod_{i\in e(A)} \varphi\left(\begin{smallmatrix} A_e(i) \\ e(A_e(i)) \end{smallmatrix}\right) = \prod_{i\in e(A)} \varphi\left(i, \begin{smallmatrix} A_e(i) \\ e(A_e(i)) \end{smallmatrix}\right)\,, \tag{4.40}$$

cf. appendix 6.1.4. For our example from eq. (4.30), this leads to the following representation of the admissible product in terms of chain products

$$\begin{aligned} \varphi\left(\begin{smallmatrix} 4&5&6&7&8 \\ 2&1&3&6&1 \end{smallmatrix}\right) &= \varphi\left(\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right) \varphi\left(\begin{smallmatrix} 5&8 \\ 1&1 \end{smallmatrix}\right) \varphi\left(\begin{smallmatrix} 6&7 \\ 3&6 \end{smallmatrix}\right) \\ &= \varphi\left(2, \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right) \varphi\left(1, \begin{smallmatrix} 5&8 \\ 1&1 \end{smallmatrix}\right) \varphi\left(3, \begin{smallmatrix} 6&7 \\ 3&6 \end{smallmatrix}\right) \\ &= \varphi(2,4) \left(\varphi(1,5,8) + \varphi(1,8,5)\right) \varphi(3,6,7)\,, \end{aligned} \tag{4.41}$$

which is nothing but a well-structured notation for the partial-fractioning identity

$$\frac{1}{x_{24}x_{15}x_{36}x_{67}x_{18}} = \frac{1}{x_{24}} \left(\frac{1}{x_{15}}\frac{1}{x_{58}} + \frac{1}{x_{18}}\frac{1}{x_{85}}\right) \frac{1}{x_{36}x_{67}}\,. \tag{4.42}$$

In particular, the admissible product in the Selberg integrals S_n in eq. (4.35) can be written as a linear combination of chain products ending at 2:

$$\varphi\left(\begin{smallmatrix} 3 & \dots & n-1 \\ e(3) & \dots & e(n-1) \end{smallmatrix}\right) = \varphi\left(2, \begin{smallmatrix} 3 & \dots & n-1 \\ e(3) & \dots & e(n-1) \end{smallmatrix}\right)\,. \tag{4.43}$$

This linear combination defines the entries of the basis transformation $\boldsymbol{B}_n$ between the vectors of Selberg integrals and Z_n-integrals, respectively:

$$\boldsymbol{S}_n = \boldsymbol{B}_n\, \boldsymbol{Z}_n\,. \tag{4.44}$$

Using the transformation $\boldsymbol{B}_n$ and eq. (4.37), the representation

$$\boldsymbol{F}_{n,0}^{\text{open}} = \boldsymbol{B}_n^{\text{adm}} \boldsymbol{S}_n = \boldsymbol{B}_n^{\text{adm}} \boldsymbol{B}_n \boldsymbol{Z}_n \tag{4.45}$$

of the string corrections shows that the momentum kernel from eq. (4.17) can combinatorially be obtained from

$$\boldsymbol{B}_n^{\text{cha}} = \boldsymbol{B}_n^{\text{adm}} \boldsymbol{B}_n \,. \tag{4.46}$$

According to eq. (4.45), the string corrections, the Selberg integrals as well as the Z_n-integrals are linear combinations of each other. We often use the convention that the integrals $\boldsymbol{F}_{n,0}^{\text{open}}$ are called *open-string corrections at genus zero*, while the Selberg integrals $\boldsymbol{S}_n$ and and the Z_n-integrals $\boldsymbol{Z}_n$ are called *(admissible and chain) open-string integrals at genus zero*. Since the Riemann sphere has no moduli, the genus-zero, open-string corrections are actually already the moduli-space integrals from eq. (1.4). Therefore, the Z_n-integrals and Selberg integrals are often referred to as moduli-space or configuration-space integrals in the context of string amplitudes.

4.2 Genus-zero, type-(n,p) integrals

The recursion in ref. [5] involves more general integrals than the Z_n-integrals and the Selberg integrals introduced in the previous section. The differential forms of the Z_n-integrals and the Selberg integrals define two different bases of the twisted de Rham cohomology $H^{n-3}(\mathcal{M}_{0,n}, \nabla_{n-3})$ of the configuration space of n-punctured Riemann spheres with three fixed coordinates. The corresponding generalisations of these integrals are introduced in the following subsections and defined by bases of the twisted de Rham cohomology

$$\begin{aligned} H^{n-p}(\mathcal{F}_{n,p}, \nabla_{n-p}) &= \ker(\nabla_{n-p}) / \operatorname{Im}(\nabla_{n-p}) \,, \\ \nabla_{n-p} &= \mathrm{d} + \mathrm{d}\log\left(\mathrm{KN}_{12\ldots n-1}\right)|_{\mathrm{d}z_1 = \cdots = \mathrm{d}z_{p-1} = 0} \wedge \end{aligned} \tag{4.47}$$

of the configuration space of n-punctured Riemann spheres with p fixed coordinates [140],

$$\mathcal{F}_{n,p} = \{(x_p, x_{p+1}, \ldots, x_{n-1}) \in \mathbb{C}^{n-p} \mid x_i \neq x_1, \ldots, x_{p-1}, x_j \text{ for } i \neq j\} \,, \tag{4.48}$$

originally described in ref. [142] and introduced in ref. [140] in the context of string amplitudes. Note that for $p = 3$, the moduli space of punctured Riemann spheres is recovered $\mathcal{F}_{n,3} = \mathcal{M}_{0,3}$. We will still work with the fixed punctures $(x_1, x_2, x_n) = (0, 1, \infty)$, assume the ordering from the integration cycle $\Delta_{n,3}$ given in eq. (4.2), i.e. $x_i < x_{i-1}$ for $3 \leq i \leq n-1$, and consider the integration cycles

$$\Delta_{n,p} = \Delta_{n,p}(x_i) = \{0 = x_1 < x_{n-1} < x_{n-2} < \cdots < x_p < x_{p-1}\}\,, \tag{4.49}$$

cf. figure 4.2. Therefore, integrating a basis of $H^{n-p}(\mathcal{F}_{n,p}, \nabla_{n-p})$ along $\Delta_{n,p}$ defines a class of

$$\dim\left(H^{n-p}(\mathcal{F}_{n,p}, \nabla_{n-p})\right) = \frac{(n-3)!}{(p-3)!} \tag{4.50}$$

independent integrals. The two relevant classes in the context of string amplitudes, denoted by $Z_{n,p}$ and $S_{n,p}$, are introduced in the following two subsections.

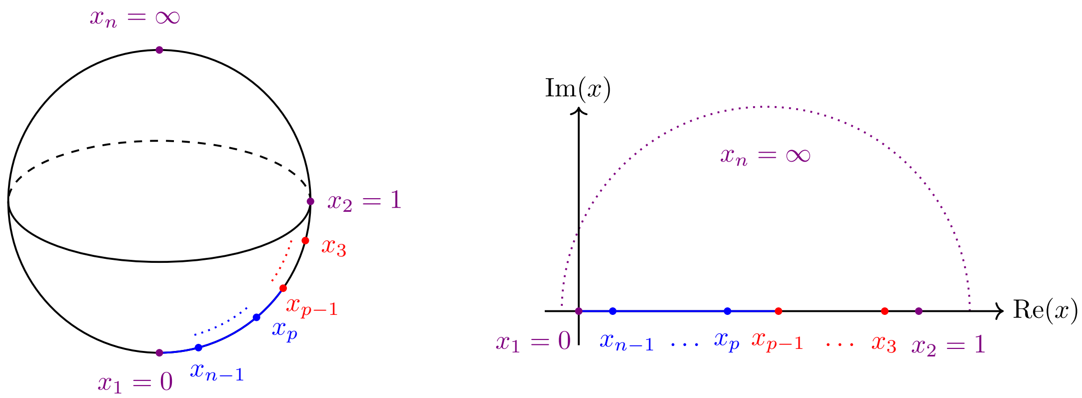


Figure 4.2: The p-punctured Riemann sphere associated to the integrals $Z_{n,p}$ and $S_{n,p}$ introduced in the next subsections. The p unintegrated (red and violet) punctures are $x_1, x_2, \ldots, x_{p-1}, x_n$. Its parametrisation on the complex plane is depicted on the right-hand side. The $n-p$ integrated (blue) punctures $x_p, \ldots, x_{n-1}$ define the integration cycle $\Delta_{n,p}(x_i)$ (blue line) from eq. (4.49) and the integrands of $Z_{n,p}$ and $S_{n,p}$ are defined on the n-punctured Riemann sphere (violet, red and blue punctures). The three unintegrated (violet) punctures $(x_1, x_2, x_n) = (0, 1, \infty)$ are canonically fixed, while the $p-3$ (red) additional punctures $x_3, \ldots, x_{p-1}$ can be varied on the interval $0 = x_1 < x_{p-1} < \cdots < x_3 < x_2 = 1$.

4.2.1 Type-(n, p) chain integrals $Z_{n,p}$

The first class generalising the Z_n-integrals are the *chain integrals of type* (n, p)

$$\begin{aligned} &Z_{n,p}\left((1, A^1), \ldots, (p-1, A^{p-1}); x_3, \ldots, x_{p-1}; \{s_{ij}\}\right) \\ &= \int_{\Delta_{n,p}} \prod_{i=p}^{n-1} \mathrm{d}z_i \; \mathrm{KN}_{12\ldots n-1} \prod_{k=1}^{p-1} \varphi(k, A^k)\,, \end{aligned} \tag{4.51}$$

where A^k are possibly empty subsequences of a permutation of $(p, \dots, n{-}1)$,

$$(A^1, A^2, \dots, A^{p-1}) = \sigma(p, \dots, n-1)\,, \quad \sigma \in S_{n-p}\,, \tag{4.52}$$

with σ acting on $p, \dots, n{-}1$. Equation (4.52) defines a partition of the permuted sequence $\sigma(p, \dots, n{-}1)$ into at most $p{-}1$ non-trivial subsequences. The twisted forms

$$\prod_{k=2}^{p-1} \varphi(k, A^k) \mathrm{d}x_{n-1} \wedge \mathrm{d}x_{n-2} \wedge \dots \wedge \mathrm{d}x_p + \mathrm{Im}(\nabla_{n-p}) \tag{4.53}$$

for all possible partitions A^k from eq. (4.52) with $A^1 = \varnothing$ form a basis of the twisted de Rham cohomology $H^{n-p}(\mathcal{F}_{n,p}, \nabla_{n-p})$. Using integration by parts and partial fractioning, any other subsequence $A^j = \varnothing$ than A^1 may be chosen to be empty, which leads to different bases.

The completeness of the basis (4.53) has an echo in the following differential equation: the $(n{-}3)!/(p{-}3)!$-dimensional vector[8]

$$\begin{aligned}
&\boldsymbol{Z}_{n,p}(x_3, \dots, x_{p-1}) \\
&= \boldsymbol{Z}_{n,p}(x_3, \dots, x_{p-1}; \{s_{ij}\}) \\
&= \Big(Z_{n,p}\left((2, A^2), \dots, (p-1, A^{p-1}); x_3, \dots, x_{p-1}; \{s_{ij}\}\right) \Big)_{\substack{(A^2, \dots, A^{p-1}) = \sigma(p, \dots, n-1) \\ \sigma \in S_{n-p}}}
\end{aligned} \tag{4.54}$$

of type-(n, p) chain integrals satisfies w.r.t. x_i for $3 \leq i \leq p-1$ a partial differential equation of Fuchsian type [142]

$$\partial_{x_i} \boldsymbol{Z}_{n,p}(x_3, \dots, x_{p-1}) = \left(\sum_{\substack{r=1 \\ r \neq i}}^{p-1} \frac{r_n^{\mathrm{cha}}(\boldsymbol{e}_p^{ir})}{x_{ir}} \right) \boldsymbol{Z}_{n,p}(x_3, \dots, x_{p-1})\,. \tag{4.55}$$

The square matrices $r_n^{\mathrm{cha}}(\boldsymbol{e}_p^{ir})$ are $(n{-}3)!/(p{-}3)!$-dimensional representations of the genus-zero braid group with p strands, which is the algebra generated by $\boldsymbol{e}_p^{ij}$ for $1 \leq i < j \leq p$ satisfying the infinitesimal pure braid relations [142]

$$\begin{aligned}
\boldsymbol{e}_p^{ij} &= \boldsymbol{e}_p^{ji}\,, \\
[\boldsymbol{e}_p^{ir}, \boldsymbol{e}_p^{jq}] &= 0 \quad \text{if } |\{i, j, q, r\}| = 4\,, \\
[\boldsymbol{e}_p^{ij} + \boldsymbol{e}_p^{jq}, \boldsymbol{e}_p^{iq}] &= 0 \quad \text{if } |\{i, j, q\}| = 3\,.
\end{aligned} \tag{4.56}$$

As shown in eq. (6.122) the representations $r_n^{\mathrm{cha}}(\boldsymbol{e}_p^{ir})$ can be calculated recursively. In particular, they are homogeneous of degree one in the Mandelstam variables s_{ij} and,

[8]Without loss of generality and unless stated otherwise, the entries of the vectors constructed in this thesis are assumed to be ordered lexicographically.

therefore, proportional to α'. The Schwarz integrability condition $[\partial_{x_i}, \partial_{x_j}]\boldsymbol{Z}_{n,p} = 0$ can be used to show that the commutation relations (4.56) are indeed satisfied by $r_n^{\text{cha}}(\boldsymbol{e}_p^{ir})$, cf. appendix 6.2.4. Knowing the explicit form of the matrices $r_n^{\text{cha}}(\boldsymbol{e}_p^{ir})$, the integrals $\boldsymbol{Z}_{n,p}(x_3, \dots, x_{p-1})$ can be determined by solving eq. (4.55) using Picard iteration. Recently, this has been worked out for certain linear combinations of $\boldsymbol{Z}_{n,p}(x_3, \dots, x_{p-1})$ in the context of open-string amplitudes in [128]. For our purposes, we do not need to solve for the full integrals, but rather only use the genus-zero associator eq. (2.50) to relate two boundary values in the case of $p = 2$, cf. section 4.3.

The chain integrals of type $(n, 3)$ contain the Z_n-integrals (4.14) appearing in the n-point, open-string corrections at genus zero

$$Z_{n,3}\left((2, A^2); \{s_{ij}\}\right) = Z_n(\sigma; \{s_{ij}\}), \tag{4.57}$$

where $\sigma(2, 3, \dots, n-1) = (2, A^2)$.

4.2.2 Type-(n, p) admissible integrals $S_{n,p}$

The second class of integrals are the integrals obtained from integrating the fibration basis [140] of $H^{n-p}(\mathcal{F}_{n,p}, \nabla_{n-p})$ formed by the equivalence classes of the twisted forms defined by the admissible products

$$\varphi\begin{pmatrix} p & \cdots & n-1 \\ e(p) & \cdots & e(n-1) \end{pmatrix} \mathrm{d}x_p \wedge \mathrm{d}x_{p+1} \wedge \cdots \wedge \mathrm{d}x_{n-1} + \mathrm{Im}(\nabla_{n-p}), \quad 2 \le e(k) < k \tag{4.58}$$

for $p \le k \le n-1$.

The resulting integrals are called *admissible integrals of type* (n, p) and defined by [142]

$$S_{n,p}\left(\begin{smallmatrix} p & \cdots & n-1 \\ e(p) & \cdots & e(n-1) \end{smallmatrix}; x_3, \dots, x_{p-1}; \{s_{ij}\}\right) = \int_{\Delta_{n,3}} \prod_{i=p}^{n-1} \mathrm{d}z_i \; \mathrm{KN}_{23\dots n}\, \varphi\left(\begin{smallmatrix} p & \cdots & n-1 \\ e(p) & \cdots & e(n-1) \end{smallmatrix}\right), \tag{4.59}$$

where e is admissible w.r.t $\{p, \dots, n-1\}$. Note that the admissibility ensures that these integrals can alternatively be defined recursively by

$$\begin{aligned} &S_{n,p}\left(\begin{smallmatrix} p & \cdots & n-1 \\ e(p) & \cdots & e(n-1) \end{smallmatrix}; x_3, \dots, x_{p-1}; \{s_{ij}\}\right) \\ &= \int_0^{x_{p-1}} \frac{\mathrm{d}x_p}{x_{e(p)p}} S_{n,p+1}\left(\begin{smallmatrix} p+1 & \cdots & n-1 \\ e(p+1) & \cdots & e(n-1) \end{smallmatrix}; x_3, \dots, x_{p-1}; \{s_{ij}\}\right) \end{aligned} \tag{4.60}$$

and

$$S_{n,n}(x_3, \dots, x_{n-1}; \{s_{ij}\}) = \mathrm{KN}_{12\dots n-1}\,. \tag{4.61}$$

The $(n-2)!/(p-2)!$ integrals in eq. (4.59) can be reduced to a basis of $(n-3)!/(p-3)!$ integrals using integration by parts and partial fractioning, which we usually choose to be the ones with $e(k) \neq 1$ for $p \leq k \leq n-1$. This reproduces the integrals defined by the fibration basis (4.58) along the cycle $\Delta_{n,p}$.

The vector of type-(n,p) admissible integrals

$$\begin{aligned}\boldsymbol{S}_{n,p}(x_3,\dots,x_{p-1}) &= \boldsymbol{S}_{n,p}(x_3,\dots,x_{p-1};\{s_{ij}\})\\ &= \left(S_{n,p}\left(\begin{smallmatrix} p & \dots & n-1\\ e(p) & \dots & e(n-1)\end{smallmatrix};x_3,\dots,x_{p-1};\{s_{ij}\}\right)\right)_{2\leq e(k)<k}\end{aligned} \tag{4.62}$$

is related by a basis transformation to the vector of type-(n,p) chain integrals (4.54):

$$\boldsymbol{S}_{n,p} = \boldsymbol{B}_{n,p}\,\boldsymbol{Z}_{n,p}\,. \tag{4.63}$$

According to eq. (4.40), the entries of the transformation matrix $\boldsymbol{B}_{n,p}$ are determined by the linear combination of the corresponding admissible product appearing in the definition (4.59) in terms of chain products, i.e.

$$\varphi\left(\begin{smallmatrix} A\\ e(A)\end{smallmatrix}\right) = \prod_{k=2}^{p-1}\varphi\left(k,\begin{smallmatrix} A_e(k)\\ e(A_e(k))\end{smallmatrix}\right)\,, \tag{4.64}$$

where $A=(p,\dots,n-1)$ and $1<e(i)<i$. This leads to a linear combination of any admissible integral in terms of chain integrals

$$\begin{aligned}&S_{n,p}\left(\begin{smallmatrix} A\\ e(A)\end{smallmatrix};x_3,\dots,x_{p-1};\{s_{ij}\}\right)\\ &= Z_{n,p}\left(\left(1,\begin{smallmatrix} A\\ e(A)\end{smallmatrix}\right),\dots,\left(p-1,\begin{smallmatrix} A\\ e(A)\end{smallmatrix}\right);x_3,\dots,x_{p-1};\{s_{ij}\}\right)\,,\end{aligned} \tag{4.65}$$

which yields the basis transformation (4.63).

Similar to the chain integrals from above, the completeness of the fibration basis leads to the following result for the admissible integrals: the vector of type-(n,p) admissible integrals $\boldsymbol{S}_{n,p}$ satisfies w.r.t. x_i for $3\leq i\leq p-1$ a partial differential equation of Fuchsian type [142]

$$\partial_{x_i}\boldsymbol{S}_{n,p}(x_3,\dots,x_{p-1}) = \left(\sum_{\substack{r=1\\ r\neq i}}^{p-1}\frac{r_n^{\text{adm}}(\boldsymbol{e}_p^{ir})}{x_{ir}}\right)\boldsymbol{S}_{n,p}(x_3,\dots,x_{p-1})\,. \tag{4.66}$$

The $(n-3)!/(p-3)!$-dimensional representation $r_n^{\text{adm}}(\boldsymbol{e}_p^{ir})$ of the genus-zero braid group with p strands is related to the representation $r_n^{\text{cha}}(\boldsymbol{e}_p^{ir})$ appearing in eq. (4.55) via the transformation matrix $\boldsymbol{B}_{n,p}$ from eq. (4.63) as follows:

$$r_n^{\text{adm}}(\boldsymbol{e}_p^{ir}) = \boldsymbol{B}_{n,p}\,r_n^{\text{cha}}(\boldsymbol{e}_p^{ir})\,\boldsymbol{B}_{n,p}^{-1}\,. \tag{4.67}$$

The matrices $r_n^{\text{adm}}(\boldsymbol{e}_p^{ir})$ are homogeneous of degree one in the Mandelstam variables s_{ij}, for example $r_n^{\text{adm}}(\boldsymbol{e}_n^{ir}) = s_{ir}$. They can be calculated[9] recursively in p using the iterative definition (4.60) [140, 142, 145], cf. eq. (6.122). Similarly, validity of the commutation relations (4.56) satisfied by $r_n^{\text{adm}}(\boldsymbol{e}_p^{ir})$ can be shown recursively in p or directly, using the Schwarz integrability condition $[\partial_{x_i}, \partial_{x_j}]\boldsymbol{S}_{n,p} = 0$.

The admissible integrals of type $(n, 3)$ contain the Selberg integrals (4.35) forming a basis for the n-point, open-string corrections at genus zero

$$S_{n,3}\left(\begin{smallmatrix} 3 & \dots & n-1 \\ e(3) & \dots & e(n-1) \end{smallmatrix}; \{s_{ij}\}\right) = S_n\left(\begin{smallmatrix} 3 & \dots & n-1 \\ e(3) & \dots & e(n-1) \end{smallmatrix}; \{s_{ij}\}\right) . \tag{4.68}$$

4.2.3 Examples

Type-(5,4) integrals: $\boldsymbol{Z}_{5,4}$ and $\boldsymbol{S}_{5,4}$

Let us consider a simple example, which will be used in subsection 4.3.5 to calculate the four-point, open-string corrections. It is the class defined by the type-$(5, 4)$ integrals for which the vectors of chain and admissible integrals from eqs. (4.54) and (4.62) agree:

$$\begin{aligned} \boldsymbol{Z}_{5,4}(x_3) &= \boldsymbol{S}_{5,4}(x_3) \\ &= \int_0^{x_3} \mathrm{d}x_4 \ \mathrm{KN}_{1234} \begin{pmatrix} \frac{1}{x_{24}} \\ \frac{1}{x_{34}} \end{pmatrix} , \end{aligned} \tag{4.69}$$

such that the transformation matrix $\boldsymbol{B}_{5,4}$ from eq. (4.63) is the identity

$$\boldsymbol{B}_{5,4} = \mathbb{1}_2 \, . \tag{4.70}$$

The differential eq. (4.55) for the vector $\boldsymbol{Z}_{5,4}(x_3)$ is

$$\partial_{x_3} \boldsymbol{Z}_{5,4}(x_3) = \left(\frac{r_5^{\text{cha}}(\boldsymbol{e}_4^{31})}{x_3} + \frac{r_5^{\text{cha}}(\boldsymbol{e}_4^{32})}{x_3 - 1} \right) \boldsymbol{Z}_{5,4}(x_3) \, , \tag{4.71}$$

where the matrices are

$$\begin{aligned} r_5^{\text{cha}}(\boldsymbol{e}_4^{31}) = r_5^{\text{adm}}(\boldsymbol{e}_4^{31}) &= \begin{pmatrix} -s_{13} & 0 \\ -s_{24} & -s_{134} \end{pmatrix} , \\ r_5^{\text{cha}}(\boldsymbol{e}_4^{32}) = r_5^{\text{adm}}(\boldsymbol{e}_4^{32}) &= \begin{pmatrix} -s_{23} - s_{34} & s_{34} \\ s_{24} & -s_{23} - s_{24} \end{pmatrix} , \end{aligned} \tag{4.72}$$

cf. eq. (6.122) for their recursive construction.

[9] See ref. [4] for the exact relation to the matrices in ref. [140].

Type-$(6,4)$ integrals: $Z_{6,4}$ and $S_{6,4}$

The type-$(6,4)$ integrals are relevant for the five-point amplitudes. They define the vectors

$$\begin{aligned} \boldsymbol{Z}_{6,4}(x_3) &= \int_0^{x_3} \mathrm{d}x_4 \int_0^{x_4} \mathrm{d}x_5 \ \mathrm{KN}_{12345} \begin{pmatrix} \varphi(2,4,5)\varphi(3) \\ \varphi(2,5,4)\varphi(3) \\ \varphi(2,4)\varphi(3,5) \\ \varphi(2,5)\varphi(3,4) \\ \varphi(2)\varphi(3,4,5) \\ \varphi(2)\varphi(3,5,4) \end{pmatrix} \\ &= \int_0^{x_3} \mathrm{d}x_4 \int_0^{x_4} \mathrm{d}x_5 \ \mathrm{KN}_{12345} \begin{pmatrix} \frac{1}{x_{24}x_{45}} \\ \frac{1}{x_{25}x_{54}} \\ \frac{1}{x_{24}x_{35}} \\ \frac{1}{x_{25}x_{34}} \\ \frac{1}{x_{34}x_{45}} \\ \frac{1}{x_{35}x_{54}} \end{pmatrix} \end{aligned} \tag{4.73}$$

and

$$\begin{aligned} \boldsymbol{S}_{6,4}(x_3) &= \int_0^{x_3} \mathrm{d}x_4 \int_0^{x_4} \mathrm{d}x_5 \ \mathrm{KN}_{12345} \begin{pmatrix} \varphi\left(\begin{smallmatrix} 4 & 5 \\ 2 & 2 \end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix} 4 & 5 \\ 2 & 3 \end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix} 4 & 5 \\ 2 & 4 \end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix} 4 & 5 \\ 3 & 2 \end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix} 4 & 5 \\ 3 & 3 \end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix} 4 & 5 \\ 3 & 4 \end{smallmatrix}\right) \end{pmatrix} \\ &= \int_0^{x_3} \mathrm{d}x_4 \int_0^{x_4} \mathrm{d}x_5 \ \mathrm{KN}_{12345} \begin{pmatrix} \frac{1}{x_{24}x_{25}} \\ \frac{1}{x_{24}x_{35}} \\ \frac{1}{x_{24}x_{45}} \\ \frac{1}{x_{34}x_{25}} \\ \frac{1}{x_{34}x_{35}} \\ \frac{1}{x_{34}x_{45}} \end{pmatrix} . \end{aligned} \tag{4.74}$$

The admissible products in $\boldsymbol{S}_{6,4}(x_3)$ can be rewritten in terms of the chain products from $\boldsymbol{Z}_{6,4}(x_3)$ using eq. (4.64), which defines the transformation matrix $\boldsymbol{B}_{6,4}$ in

$$\boldsymbol{S}_{6,4} = \boldsymbol{B}_{6,4}\, \boldsymbol{Z}_{6,4} \, . \tag{4.75}$$

It is determined by the calculation

$$\begin{pmatrix} \varphi\left(\begin{smallmatrix}4&5\\2&2\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\2&3\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\2&4\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&2\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&3\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&4\end{smallmatrix}\right) \end{pmatrix} = \begin{pmatrix} \varphi\left(\begin{smallmatrix}4&5\\2&2\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4\\2\end{smallmatrix}\right)\varphi\left(\begin{smallmatrix}5\\3\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\2&4\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4\\3\end{smallmatrix}\right)\varphi\left(\begin{smallmatrix}5\\2\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&3\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&4\end{smallmatrix}\right) \end{pmatrix} = \begin{pmatrix} \varphi\left(2,\begin{smallmatrix}4&5\\2&2\end{smallmatrix}\right) \\ \varphi\left(2,\begin{smallmatrix}4\\2\end{smallmatrix}\right)\varphi\left(3,\begin{smallmatrix}5\\3\end{smallmatrix}\right) \\ \varphi\left(2,\begin{smallmatrix}4&5\\2&4\end{smallmatrix}\right) \\ \varphi\left(3,\begin{smallmatrix}4\\3\end{smallmatrix}\right)\varphi\left(2,\begin{smallmatrix}5\\2\end{smallmatrix}\right) \\ \varphi\left(3,\begin{smallmatrix}4&5\\3&3\end{smallmatrix}\right) \\ \varphi\left(3,\begin{smallmatrix}4&5\\3&4\end{smallmatrix}\right) \end{pmatrix} = \begin{pmatrix} \varphi(2, 4 \shuffle 5) \\ \varphi(2,4)\varphi(3,5) \\ \varphi(2,4,5) \\ \varphi(3,4)\varphi(2,5) \\ \varphi(3, 4 \shuffle 5) \\ \varphi(3,4,5) \end{pmatrix}, \tag{4.76}$$

such that

$$\begin{pmatrix} \varphi\left(\begin{smallmatrix}4&5\\2&2\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\2&3\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\2&4\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&2\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&3\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&4\end{smallmatrix}\right) \end{pmatrix} = \underbrace{\begin{pmatrix} 1&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&0&1&0 \end{pmatrix}}_{\boldsymbol{B}_{6,4}} \begin{pmatrix} \varphi(2,4,5)\varphi(3) \\ \varphi(2,5,4)\varphi(3) \\ \varphi(2,4)\varphi(3,5) \\ \varphi(2,5)\varphi(3,4) \\ \varphi(2)\varphi(3,4,5) \\ \varphi(2)\varphi(3,5,4) \end{pmatrix}. \tag{4.77}$$

The vector of admissible integrals $\boldsymbol{S}_{6,4}$ satisfies the KZ equation

$$\partial_{x_3}\boldsymbol{S}_{6,4}(x_3) = \left(\frac{r_6^{\text{adm}}(\boldsymbol{e}_4^{31})}{x_3} + \frac{r_6^{\text{adm}}(\boldsymbol{e}_4^{32})}{x_3 - 1}\right)\boldsymbol{S}_{6,4}(x_3), \tag{4.78}$$

where the matrices

$$r_6^{\text{adm}}(\boldsymbol{e}_4^{31}) = \begin{pmatrix} -s_{13} & 0 & 0 & 0 & 0 & 0 \\ -s_{25} & -s_{135} & -s_{45} & 0 & 0 & 0 \\ 0 & 0 & -s_{13} & 0 & 0 & 0 \\ -s_{24}-s_{45} & 0 & s_{45} & -s_{134} & 0 & 0 \\ 0 & -s_{24} & 0 & -s_{25} & -s_{1345} & 0 \\ s_{25} & 0 & -s_{24}-s_{25} & -s_{25} & 0 & -s_{1345} \end{pmatrix} \tag{4.79}$$

and

$$r_6^{\text{adm}}(\boldsymbol{e}_4^{32}) = \begin{pmatrix} -s_{3,245} & s_{35} & 0 & s_{34} & 0 & 0 \\ s_{25} & -s_{24,35} & s_{45} & 0 & s_{34}+s_{45} & -s_{45} \\ 0 & s_{35} & -s_{3,245} & 0 & -s_{35} & s_{34}+s_{35} \\ s_{24}+s_{45} & 0 & -s_{45} & -s_{25,34} & s_{35} & s_{45} \\ 0 & s_{24} & 0 & s_{25} & -s_{2,345} & 0 \\ -s_{25} & 0 & s_{24}+s_{25} & s_{25} & 0 & -s_{2,345} \end{pmatrix} \tag{4.80}$$

are given by eq. (6.122), where for two non-empty sequences $P = (p_1, p_2, \ldots, p_l)$ and

$Q = (q_1, q_2, \ldots, q_m)$ the following sum of Mandelstam variables is defined:

$$s_{p_1p_2\ldots p_l,q_1q_2\ldots q_m} = s_{P,Q} = \sum_{i=1}^{l}\sum_{j=1}^{m} s_{p_iq_j}\,. \tag{4.81}$$

These matrices, in turn, can be used to calculate the matrices in the KZ equation of the vector of chain integrals

$$\partial_{x_3}\boldsymbol{Z}_{6,4}(x_3) = \left(\frac{r_6^{\text{cha}}(\boldsymbol{e}_4^{31})}{x_3} + \frac{r_6^{\text{cha}}(\boldsymbol{e}_4^{32})}{x_3-1}\right)\boldsymbol{Z}_{6,4}(x_3)\,. \tag{4.82}$$

They are given by

$$\begin{aligned} r_6^{\text{cha}}(\boldsymbol{e}_4^{31}) &= \boldsymbol{B}_{6,4}^{-1}\, r_6^{\text{adm}}(\boldsymbol{e}_4^{31})\,\boldsymbol{B}_{6,4} \\ &= \begin{pmatrix} -s_{13} & 0 & 0 & 0 & 0 & 0 \\ 0 & -s_{13} & 0 & 0 & 0 & 0 \\ -s_{25}-s_{45} & -s_{25} & -s_{135} & 0 & 0 & 0 \\ -s_{24} & -s_{24}-s_{45} & 0 & -s_{134} & 0 & 0 \\ -s_{24} & s_{25} & 0 & -s_{25} & -s_{1345} & 0 \\ s_{24} & -s_{25} & -s_{24} & 0 & 0 & -s_{1345} \end{pmatrix} \end{aligned} \tag{4.83}$$

and

$$\begin{aligned} r_6^{\text{cha}}(\boldsymbol{e}_4^{32}) &= \boldsymbol{B}_{6,4}^{-1}\, r_6^{\text{adm}}(\boldsymbol{e}_4^{32})\,\boldsymbol{B}_{6,4} \\ &= \begin{pmatrix} -s_{3,245} & 0 & s_{35} & 0 & s_{34} & -s_{35} \\ 0 & -s_{3,245} & 0 & s_{34} & -s_{34} & s_{35} \\ s_{25}+s_{45} & s_{25} & -s_{24,35} & 0 & s_{34} & s_{34}+s_{45} \\ s_{24} & s_{24}+s_{45} & 0 & -s_{25,34} & s_{35}+s_{45} & s_{35} \\ s_{24} & -s_{25} & 0 & s_{25} & -s_{2,345} & 0 \\ -s_{24} & s_{25} & s_{24} & 0 & 0 & -s_{2,345} \end{pmatrix} \end{aligned} \tag{4.84}$$

in agreement with eqs. (4.67) and (6.123).

4.3 Genus-zero, open-string recursion

In ref. [5] a recursion[10] to calculate the α'-expansion of the open-string corrections $\boldsymbol{F}_{n,0}^{\text{open}}$ from eq. (4.15) has been constructed. It is based on a vector of integrals with an auxiliary unintegrated puncture, such that differentiating with respect to this puncture leads to a KZ equation. The corresponding associator equation (2.50)

[10] A further recursive method based on a Berends–Giele recursion has been introduced in ref. [30].

relates the two regularised boundary values of this vector via the Drinfeld associator. The boundary values, in turn, contain the n-point and $(n-1)$-point, open-string corrections, such that the associator equation facilitates a recursion in the number of external states. This recursion has been reformulated in twisted de Rham theory and related to the fibration basis (4.58) in ref. [4]. The recursion and the results of the reformulation from the latter reference are summarised in this subsection.

4.3.1 KZ equation

In order to calculate the string corrections from eq. (4.5), which is a linear combination of type-$(n,3)$ admissible or chain integrals, the linear combinations [5]

$$\hat{F}_n^{\nu}(\sigma; x_3) = \int_{\Delta_{n,4}} \prod_{i=4}^{n-1} \mathrm{d}x_i \; \mathrm{KN}_{12\ldots n-1} \; \sigma \left(\prod_{k=n-\nu+1}^{n-1} \sum_{j=2}^{k-1} \frac{s_{jk}}{x_{jk}} \prod_{m=4}^{n-\nu} \left(\sum_{l=4}^{m-1} \frac{s_{lm}}{x_{lm}} + \frac{s_{2m}}{x_{2m}} \right) \right) \tag{4.85}$$

of type-$(n,4)$ integrals, where $1 \leq \nu \leq n-3$ and $\sigma \in S_{n-4}$ acting on the labels $4, 5, \ldots, n-1$ are introduced. The integration domain is bounded by the additional unintegrated puncture x_3

$$\Delta_{n,4} = \Delta_{n,4}(x_i) = \{0 < x_{n-1} < x_{n-2} < \cdots < x_4 < x_3\}\,, \tag{4.86}$$

cf. figure 4.3.

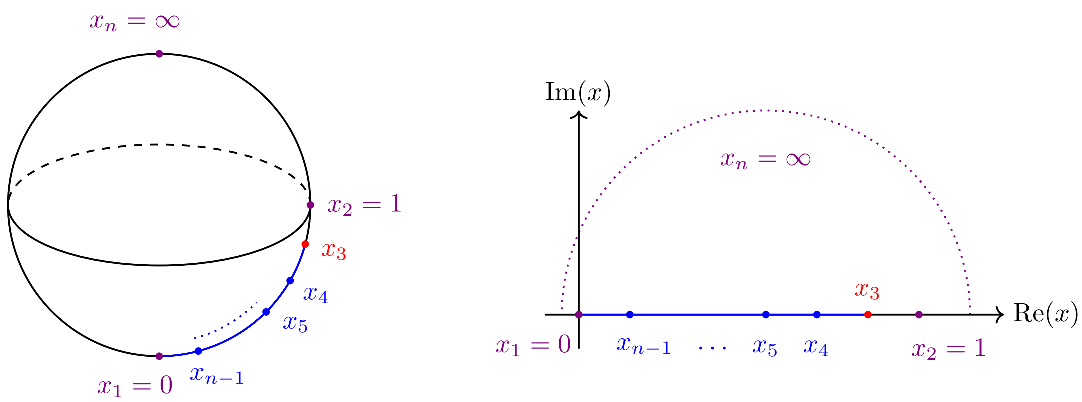


Figure 4.3: The four-punctured Riemann sphere associated to the integrals $\hat{F}_n^{\nu}(\sigma; x_3)$ from eq. (4.85). The four unintegrated (red and violet) punctures are x_1, x_2, x_3, x_n. Its parametrisation on the complex plane is depicted on the right-hand side. The $n-4$ integrated (blue) punctures $x_4, \ldots, x_{n-1}$ define the integration cycle $\Delta_{n,4}(x_i)$ (blue line) from eq. (4.86) and the integrands of $\hat{F}_n^{\nu}(\sigma; x_3)$ are defined on the n-punctured Riemann sphere (violet, red and blue punctures). The three unintegrated (violet) punctures $(x_1, x_2, x_n) = (0, 1, \infty)$ are canonically fixed, while the fourth (red) additional puncture x_3 can be varied on the interval $0 = x_1 < x_3 < x_2 = 1$.

The integrals $\hat{F}_n^{\nu}(\sigma; x_3)$ are the starting point of ref. [5]. Using transformation (4.34), one can rewrite

$$\begin{aligned} &\sigma\left(\prod_{k=n-\nu+1}^{n-1}\sum_{j=2}^{k-1}\frac{s_{jk}}{x_{jk}}\prod_{m=4}^{n-\nu}\left(\sum_{l=4}^{m-1}\frac{s_{lm}}{x_{lm}}+\frac{s_{2m}}{x_{2m}}\right)\right) \\ &= \sum_{\substack{e \text{ adm} \\ 1<e(k)<k \\ k\leq n-\nu:\ e(k)\neq 3}} \prod_{k=4}^{n-1} s_{e^\sigma(k),k}\, \varphi\begin{pmatrix} 3 & \cdots & n-1 \\ e^\sigma(3) & \cdots & e^\sigma(n-1)\end{pmatrix} \\ &= \sum_{\substack{e \text{ adm} \\ 1<e(k)<k \\ k\leq n-\nu:\ e(k)\neq 3}} \prod_{k=4}^{n-1} s_{e^\sigma(k),k} \sum_{\substack{e' \text{ adm} \\ 1<e'(k)<k}} b_{e^\sigma,e'}\varphi\begin{pmatrix} 3 & \cdots & n-1 \\ e'(3) & \cdots & e'(n-1)\end{pmatrix}, \end{aligned} \tag{4.87}$$

where the sum in the second line runs over all admissible maps

$$e : \{4, \ldots, n-1\} \to \{2, 3, \ldots, n-2\} \tag{4.88}$$

with $1 < e(k) < k$ and $e(k) \neq 3$ for $4 \leq k \leq n-\nu$. Therefore, the $(n-3)!$-dimensional vector

$$\hat{\boldsymbol{F}}_n(x_3) = \begin{pmatrix} \hat{\boldsymbol{F}}_n^{n-3}(x_3) \\ \vdots \\ \hat{\boldsymbol{F}}_n^{1}(x_3)\end{pmatrix}, \quad \hat{\boldsymbol{F}}_n^{\nu}(x_3) = \left(\hat{F}_n^{\nu}(\sigma; x_3)\right)_{\sigma\in S_{n-4}} \tag{4.89}$$

is related by an invertible transformation matrix to the type-$(n, 4)$ admissible and chain integrals, respectively:

$$\hat{\boldsymbol{F}}_n(x_3) = \hat{\boldsymbol{B}}_n\, \boldsymbol{S}_{n,4}(x_3) = \hat{\boldsymbol{B}}_n\, \boldsymbol{B}_{n,4}\, \boldsymbol{Z}_{n,p}(x_3)\,, \tag{4.90}$$

where the entries of $\hat{\boldsymbol{B}}_n$ are determined by the coefficients in eq. (4.87) and $\boldsymbol{B}_{n,4}$ is the transformation matrix from eq. (4.63).

According to eqs. (4.66) and (4.55), the vector $\hat{\boldsymbol{F}}_n(x_3)$ satisfies a KZ equation (2.32)

$$\partial_{x_3}\hat{\boldsymbol{F}}_n(x_3) = \left(\frac{\hat{r}_n(\boldsymbol{e}_4^{31})}{x_3} + \frac{\hat{r}_n(\boldsymbol{e}_4^{32})}{x_3-1}\right)\hat{\boldsymbol{F}}_n(x_3)\,, \tag{4.91}$$

where the matrices $\hat{r}_n(\boldsymbol{e}_4^{3j})$ are related by the corresponding basis transformation to the matrices $r_n^{\text{adm}}(\boldsymbol{e}_4^{3j})$ and $r_n^{\text{cha}}(\boldsymbol{e}_4^{3j})$ with $j = 1, 2$ from eqs. (4.55) and (4.66), respectively [4]:

$$\begin{aligned} \hat{r}_n(\boldsymbol{e}_4^{31}) &= \hat{\boldsymbol{B}}_n\, r_n^{\text{adm}}(\boldsymbol{e}_4^{31})(\hat{\boldsymbol{B}}_n)^{-1} = \hat{\boldsymbol{B}}_n\, \boldsymbol{B}_{n,4}\, r_n^{\text{cha}}(\boldsymbol{e}_4^{31})(\hat{\boldsymbol{B}}_n\, \boldsymbol{B}_{n,4})^{-1}\,, \\ \hat{r}_n(\boldsymbol{e}_4^{32}) &= \hat{\boldsymbol{B}}_n\, r_n^{\text{adm}}(\boldsymbol{e}_4^{32})(\hat{\boldsymbol{B}}_n)^{-1} = \hat{\boldsymbol{B}}_n\, \boldsymbol{B}_{n,4}\, r_n^{\text{cha}}(\boldsymbol{e}_4^{32})(\hat{\boldsymbol{B}}_n\, \boldsymbol{B}_{n,4})^{-1}\,. \end{aligned} \tag{4.92}$$

Hence, they are representations of the genus-zero braid group. In ref. [5] it was shown via direct calculations using integration by parts and partial fractioning that for $n \leq 10$ the derivative $\partial_{x_3}\hat{\boldsymbol{F}}_n(x_3)$ indeed satisfies the KZ equation (4.91). The method derived in ref. [4] and presented above is a well-structured method based on integration by parts and partial fractioning to recursively calculate the matrices $\hat{r}_n(\boldsymbol{e}_4^{31})$ and $\hat{r}_n(\boldsymbol{e}_4^{32})$ for any n. The results of ref. [5] for $n \leq 10$ can be found in ref. [146].

In order to determine the regularised boundary values (2.49) for $x_3 \to 0$ and $x_3 \to 1$, the asymptotic behaviour of the integrals $\hat{F}_n^{\nu}(\sigma; x_3)$ is determined in the following subsections according to ref. [5] (see also ref. [4] for the explicit calculation).

4.3.2 Lower boundary value

For the lower boundary value, a change of variables $x_i = x_3 w_i$ leads to

$$\lim_{x_3\to 0} x_3^{s_{134\dots n-1}} \hat{F}_n^{\nu}(\sigma; x_3) \tag{4.93}$$

$$= \begin{cases} \int_{\Delta_{n,5}(w_i)} \prod_{i=5}^{n-1} \mathrm{d}w_i \ \mathrm{KN}_{145\dots n-1}(w_i) \\ \qquad \times \sigma\left(\prod_{k=5}^{n-1}\sum_{l=4}^{k-1}\frac{s_{lk}}{w_{lk}}\right) + \mathcal{O}(s_{3j}) & \text{if } \nu = n-3\,, \sigma(4) = 4\,, \\ \mathcal{O}(s_{3j}) & \text{otherwise,} \end{cases} \tag{4.94}$$

where

$$w_1 = 0 < w_{n-1} < w_{n-2} < \cdots < w_5 < w_4 = w_3 = 1 < w_2 = w_n = \infty\,, \tag{4.95}$$

cf. figure 4.4.

Upon comparing with the string corrections (4.5), the integral above is the $(n-2)$-point, open-string correction at genus zero for the $n-2$ distinct punctures in eq. (4.95). Note that the additional merging of w_4 with w_3 comes from the fact that for $s_{3j} \to 0$, the integrand in $\lim_{x_3\to 0} x_3^{s_{134\dots n-1}} \hat{F}_n^{\nu}(\sigma; x_3)$ with $\nu = n-3$ and $\sigma(4) = 4$ is a total derivative in w_4, such that the only non-vanishing contribution originates from the boundary $w_4 = w_3$ [4]. Moreover, the regulating factor $x_3^{-\hat{r}_n(\boldsymbol{e}_4^{31})}$ in the regularised boundary value

$$C_0(\hat{\boldsymbol{F}}_n) = \lim_{x_3\to 0} x_3^{-\hat{r}_n(\boldsymbol{e}_4^{31})} \hat{\boldsymbol{F}}_n(x_3) \tag{4.96}$$

acts on the non-vanishing integrals in the limit $s_{3j} \to 0$ in eq. (4.93) simply by projecting out the eigenvalue $x_3^{s_{134\ldots n-1}}$ [4, 5], such that

$$\begin{aligned} C_0(\hat{\boldsymbol{F}}_n) &= C_0(\hat{\boldsymbol{F}}_n(\{s_{ij}|i,j=1,2,\ldots,n-1\})) \\ &= \begin{pmatrix} \boldsymbol{F}^{\text{open}}_{n-2,0}(\{s_{ij}|i,j=1,4,5,\ldots,n-1\}) + \mathcal{O}(s_{3j}) \\ \mathcal{O}(s_{3j}) \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{F}^{\text{open}}_{n-2,0} + \mathcal{O}(s_{3j}) \\ \mathcal{O}(s_{3j}) \end{pmatrix} , \end{aligned} \tag{4.97}$$

where

$$\begin{aligned} &\boldsymbol{F}^{\text{open}}_{n-2,0}(\{s_{ij}|i,j=1,4,5,\ldots,n-1\}) \\ &= \left(\int_{\Delta_{n,5}(w_i)} \prod_{i=5}^{n-1} \mathrm{d}w_i \ \mathrm{KN}_{145\ldots n-1}(w_i)\, \sigma \left(\prod_{k=5}^{n-1} \sum_{l=4}^{k-1} \frac{s_{lk}}{w_{lk}}\right)\right)_{\sigma\in S_{n-5}} , \end{aligned} \tag{4.98}$$

with $\sigma \in S_{n-5}$ acting on the labels $5, 6, \ldots, n-1$.

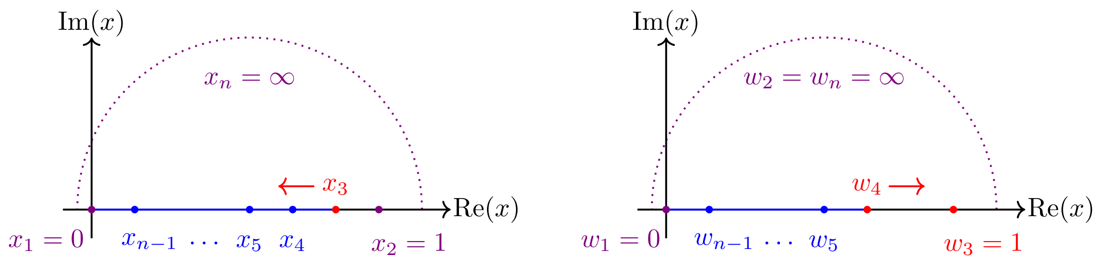


Figure 4.4: The change of variables $x_i = x_3 w_i$ and the limit $x_3 \to 0$ degenerates the four unintegrated (red and violet) punctures x_1, x_2, x_3, x_n in the integration domain $\Delta_{n,4}(x_i)$ (blue line) on the left-hand side to the three punctures w_1, w_3 and $w_2 = w_n$ depicted on the right-hand side. In the additional limit $s_{3j} \to 0$, only certain boundary values survive such that the puncture w_4 is not integrated over and merges with $w_3 = 1$. This yields the integration domain (blue) $\Delta_{n,5}(w_i)$ on the right-hand side, given in eq. (4.95).

4.3.3 Upper boundary value

For the upper boundary value as $x_3 \to 1 = x_2$ and $\nu = n-3$, the asymptotic behaviour

$$\lim_{x_3\to 1} x_3^{s_{23}} \hat{F}_n^{n-3}(\sigma; x_3) = \int_{\Delta_{n,4}} \prod_{i=4}^{n-1} \mathrm{d}x_i \ \mathrm{KN}_{1245\ldots n-1} \prod_{\substack{j=1 \\ j\neq 2,3}}^{n-1} |x_{3j}|^{-s_{3j}}\, \sigma \left(\prod_{k=4}^{n-1} \sum_{j=2}^{k-1} \frac{s_{jk}}{x_{jk}}\right) \tag{4.99}$$

is observed. These are the $(n-1)$-point string corrections for the $n-1$ distinct punctures

$$x_1 = 0 < x_{n-1} < x_{n-2} < \cdots < x_4 < x_3 = x_2 = 1 < x_n = \infty\,, \tag{4.100}$$

cf. figure 4.5, where the effective Mandelstam variables associated to the puncture at $1 = x_2 = x_3$ are the sum $\tilde{s}_{2j} = s_{2j} + s_{3j}$. Thus, in the limit $s_{3j} \to 0$, the $(n-1)$-point string corrections with Mandelstam variables s_{ij} for $i,j \in \{1,2,4,5,\ldots,n-1\}$ are recovered. Again, the regulating factor $x_3^{-\hat{r}_n(\boldsymbol{e}_4^{32})}$ in

$$C_1(\hat{\boldsymbol{F}}_n) = \lim_{x_3\to 1} x_3^{-\hat{r}_n(\boldsymbol{e}_4^{32})} \hat{\boldsymbol{F}}_n(x_3) \tag{4.101}$$

projects out the correct eigenvalue $x_3^{s_{23}}$ [4,5], such that

$$\begin{aligned} C_1(\hat{\boldsymbol{F}}_n) &= C_1(\hat{\boldsymbol{F}}_n(\{s_{ij}|i,j=1,2,\ldots,n-1\})) \\ &= \begin{pmatrix} \boldsymbol{F}^{\text{open}}_{n-1,0}(\{s_{ij}|i,j=1,2,4,5,\ldots,n-1\}) + \mathcal{O}(s_{3j}) \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{F}^{\text{open}}_{n-1,0} + \mathcal{O}(s_{3j}) \\ \vdots \end{pmatrix}\,, \end{aligned} \tag{4.102}$$

where

$$\begin{aligned} &\boldsymbol{F}^{\text{open}}_{n-1,0}(\{s_{ij}|i,j=1,2,4,5,\ldots,n-1\}) \\ &= \left(\int_{\Delta_{n,4}} \prod_{i=4}^{n-1} \mathrm{d}x_i \; \mathrm{KN}_{1245\ldots n-1}\; \sigma\left(\prod_{k=4}^{n-1} \sum_{\substack{j=2\\ j\neq 3}}^{k-1} \frac{s_{jk}}{x_{jk}} \right) \right)_{\sigma\in S_{n-4}}\,. \end{aligned} \tag{4.103}$$

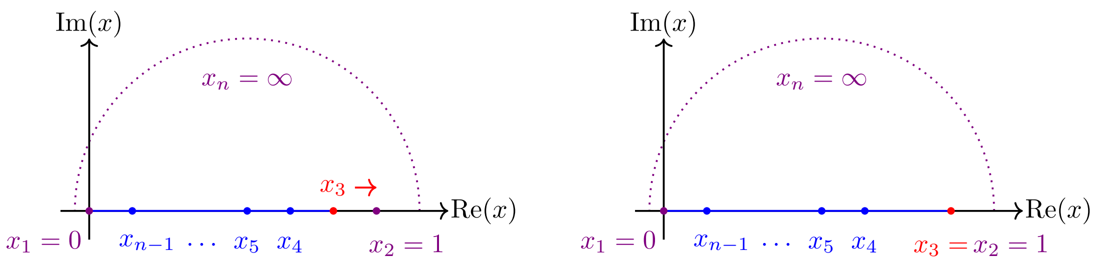


Figure 4.5: In the limit $x_3 \to 1$, the four unintegrated (red and violet) punctures x_1, x_2, x_3, x_n in the integration domain (blue) $\Delta_{n,4}(x_i)$ on the left-hand side merge to the three punctures x_1, x_n and $x_2 = x_3$ depicted on the right-hand side. The result is the integration domain (blue) $\Delta_{n-1,3}(x_i)$ on the right-hand side, given in eq. (4.100).

4.3.4 Recursion in number of external states

Having a solution $\hat{\boldsymbol{F}}_n(x_3)$ of the KZ equation and its regularised boundary values $C_0(\hat{\boldsymbol{F}}_n)$ and $C_1(\hat{\boldsymbol{F}}_n)$ at hand, the associator equation (2.50)

$$C_1(\hat{\boldsymbol{F}}_n) = \Phi_{\mathcal{E}_n} C_0(\hat{\boldsymbol{F}}_n) \tag{4.104}$$

with the alphabet $\mathcal{E}_n = (\hat{r}_n(\boldsymbol{e}_4^{31}), \hat{r}_n(\boldsymbol{e}_4^{32}))$ can be formed. Its limit $s_{3j} \to 0$ yields the *genus-zero, open-string recursion*

$$\begin{pmatrix} \boldsymbol{F}^{\text{open}}_{n-1,0} \\ \vdots \end{pmatrix} = \Phi_{\mathcal{E}_n}|_{s_{3j}=0} \begin{pmatrix} \boldsymbol{F}^{\text{open}}_{n-2,0} \\ 0 \end{pmatrix}, \tag{4.105}$$

which is the correct and complete formulation of the schematic eqs. (1.5) and (2.53). The recursion (4.105) is the main result of ref. [5] and can be used to calculate the α'-expansion of the $(n-1)$-point string corrections from the α'-expansion of the $(n-2)$-point corrections and the α'-expansion of the Drinfeld associator in eq. (2.48): since the matrices $\hat{r}_n(\boldsymbol{e}_4^{31})$ and $\hat{r}_n(\boldsymbol{e}_4^{32})$ are proportional to α', the expansion of the associator in the word length is simply its expansion in α', i.e.

$$\begin{aligned} \Phi_{\mathcal{E}_n}|_{s_{3j}=0} = \mathbb{1}_{(n-3)!} &- \zeta_2[\hat{r}_n(\boldsymbol{e}_4^{31}), \hat{r}_n(\boldsymbol{e}_4^{32})]|_{s_{3j}=0} \\ &- \zeta_3[\hat{r}_n(\boldsymbol{e}_4^{31}) + \hat{r}_n(\boldsymbol{e}_4^{32}), [\hat{r}_n(\boldsymbol{e}_4^{31}), \hat{r}_n(\boldsymbol{e}_4^{32})]]|_{s_{3j}=0} + \mathcal{O}((\alpha')^4) \,. \end{aligned} \tag{4.106}$$

Geometrically, the Drinfeld associator glues a trivalent open-string worldsheet to some external state of the $(n-2)$-point interaction leading to an effective $(n-1)$-point worldsheet associated to the $(n-1)$-point interaction, cf. figure 1.3 for an illustration[11] of the four-point calculation. The auxiliary variable x_3 in $\hat{\boldsymbol{F}}_n(x_3)$ parametrises between the $(n-2)$-point and $(n-1)$-point string corrections, but is not a puncture associated to an external state since in particular $s_{3j} = 0$ in the recursion (4.105). On the one hand, in the lower boundary value $C_0(\hat{\boldsymbol{F}}_n)$ the Mandelstam variables s_{i4} are associated to the puncture $w_4 = 1$, while the variables s_{i2} associated to $w_2 = \infty$ are absent, cf. eq. (4.97) and figure 4.4. On the other hand, in the upper boundary value $C_1(\hat{\boldsymbol{F}}_n)$ the Mandelstam variables s_{i2} are present and correspond to the puncture $x_2 = 1$, cf. eq. (4.102) and figure 4.5. Thus, the trivalent interaction is glued to the puncture $w_4 = 1$ of the $(n-2)$-point worldsheet, leading to the two punctures $x_4 < x_2 = 1$ of the $(n-1)$-point worldsheet.

[11] With $\Phi_{n,0} = \Phi_{\mathcal{E}_{n+1}}|_{s_{3j}=0}$.

4.3.5 Examples

Four-point, open-string correction

In order to calculate the four-point string correction (4.8), the linear combinations

$$\hat{\boldsymbol{F}}_5(x_3) = \begin{pmatrix} \hat{F}_5^2(x_3) \\ \hat{F}_5^1(x_3) \end{pmatrix} = \int_0^{x_3} \mathrm{d}x_4 \ \mathrm{KN}_{1234} \begin{pmatrix} \frac{s_{24}}{x_{24}} + \frac{s_{34}}{x_{34}} \\ \frac{s_{24}}{x_{24}} \end{pmatrix} = \hat{\boldsymbol{B}}_5 \, \boldsymbol{Z}_{5,4}(x_3) \,, \tag{4.107}$$

where

$$\hat{\boldsymbol{B}}_5 = \begin{pmatrix} s_{24} & s_{34} \\ s_{24} & 0 \end{pmatrix} , \tag{4.108}$$

of the type-$(5,4)$ integrals $\boldsymbol{Z}_{5,4}$ from eq. (4.69) have to be considered. The matrix $\hat{\boldsymbol{B}}_5$ is determined by the coefficients in eq. (4.87). The vector $\hat{\boldsymbol{F}}_5(x_3)$ satisfies the KZ equation

$$\partial_{x_3} \hat{\boldsymbol{F}}_5(x_3) = \left(\frac{\hat{r}_5(\boldsymbol{e}_4^{31})}{x_3} + \frac{\hat{r}_5(\boldsymbol{e}_4^{32})}{x_3 - 1} \right) \hat{\boldsymbol{F}}_5(x_3) \,, \tag{4.109}$$

where the matrices [4]

$$\hat{r}_5(\boldsymbol{e}_4^{31}) = \hat{\boldsymbol{B}}_5 \, r_5^{\mathrm{adm}}(\boldsymbol{e}_4^{31}) (\hat{\boldsymbol{B}}_5)^{-1} = \begin{pmatrix} -s_{134} & s_{14} \\ 0 & -s_{13} \end{pmatrix} \tag{4.110}$$

and

$$\hat{r}_5(\boldsymbol{e}_4^{32}) = \hat{\boldsymbol{B}}_5 \, r_5^{\mathrm{adm}}(\boldsymbol{e}_4^{32}) (\hat{\boldsymbol{B}}_5)^{-1} = \begin{pmatrix} -s_{23} & 0 \\ s_{24} & -s_{234} \end{pmatrix} \tag{4.111}$$

can be calculated from the type-$(5,4)$ matrices from eq. (4.72). The regularised boundary values (4.97) and (4.101) are

$$C_0(\hat{\boldsymbol{F}}_5) = \begin{pmatrix} 1 + \mathcal{O}(s_{3j}) \\ \mathcal{O}(s_{3j}) \end{pmatrix} \tag{4.112}$$

and

$$\begin{aligned} C_1(\hat{\boldsymbol{F}}_5) &= \begin{pmatrix} \int_0^1 \mathrm{d}x_4 \, |x_{14}|^{-s_{14}} |x_{24}|^{-s_{24}} \frac{s_{24}}{x_{24}} + \mathcal{O}(s_{3j}) \\ \dots \end{pmatrix} \\ &= \begin{pmatrix} \frac{\Gamma(1-s_{14})\Gamma(1-s_{24})}{\Gamma(1-s_{14}-s_{24})} + \mathcal{O}(s_{3j}) \\ \dots \end{pmatrix} , \end{aligned} \tag{4.113}$$

respectively. Therefore, in the limit $s_{3j} \to 0$, the recursion (4.105) yields the relation

$$\begin{pmatrix} \frac{\Gamma(1-s_{14})\Gamma(1-s_{24})}{\Gamma(1-s_{14}-s_{24})} \\ \dots \end{pmatrix} = \Phi_{\mathcal{E}_5}|_{s_{3j}=0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{4.114}$$

where [5]

$$\mathcal{E}_5|_{s_{3j}=0} = \left(\hat{r}_5(\boldsymbol{e}_4^{31})|_{s_{3j}=0}, \hat{r}_5(\boldsymbol{e}_4^{32})|_{s_{3j}=0}\right) = \left(\begin{pmatrix} -s_{14} & s_{14} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ s_{24} & -s_{24} \end{pmatrix} \right) . \tag{4.115}$$

Equation (4.114) is the example depicted[12] in figure 1.3. The expansion of the Drinfeld associator from eq. (2.48) and the matrices from eq. (4.115) lead to the following expansion of the right-hand side of eq. (4.114)

$$\begin{aligned} &\left(\mathbb{1}_2 - \zeta_2 [r_5(\boldsymbol{e}_4^{31}), r_5(\boldsymbol{e}_4^{32})] - \zeta_3 \left([r_5(\boldsymbol{e}_4^{31}) + r_5(\boldsymbol{e}_4^{32}), [r_5(\boldsymbol{e}_4^{31}), r_5(\boldsymbol{e}_4^{32})]]\right) + \dots \right)|_{s_{3j}=0} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \zeta_2 \begin{pmatrix} s_{14}s_{24} \\ s_{14}s_{24} \end{pmatrix} - \zeta_3 \begin{pmatrix} s_{14}^2 s_{24} + s_{14}s_{24}^2 \\ s_{14}^2 s_{24} + s_{14}s_{24}^2 \end{pmatrix} + \dots . \end{aligned} \tag{4.116}$$

The first entry indeed agrees with the left-hand side, the Veneziano amplitude, whose expansion is given in eq. (4.8). Higher-point examples can be found in ref. [5].

Type-$(6,4)$ matrices

The calculation of the five-point corrections involves the type-$(6, 4)$ vector

$$\begin{aligned} \hat{\boldsymbol{F}}_6(x_3) = \begin{pmatrix} \hat{\boldsymbol{F}}_6^3(x_3) \\ \hat{\boldsymbol{F}}_6^2(x_3) \\ \hat{\boldsymbol{F}}_6^1(x_3) \end{pmatrix} &= \begin{pmatrix} \hat{F}_6^3(4,5;x_3) \\ \hat{F}_6^3(5,4;x_3) \\ \hat{F}_6^2(4,5;x_3) \\ \hat{F}_6^2(5,4;x_3) \\ \hat{F}_6^1(4,5;x_3) \\ \hat{F}_6^1(5,4;x_3) \end{pmatrix} \\ &= \int_0^{x_3} \mathrm{d}x_4 \int_0^{x_4} \mathrm{d}x_5 \ \mathrm{KN}_{12345} \begin{pmatrix} \left(\frac{s_{24}}{x_{24}} + \frac{s_{34}}{x_{34}}\right)\left(\frac{s_{25}}{x_{25}} + \frac{s_{35}}{x_{35}} + \frac{s_{45}}{x_{45}}\right) \\ \left(\frac{s_{25}}{x_{25}} + \frac{s_{35}}{x_{35}}\right)\left(\frac{s_{24}}{x_{24}} + \frac{s_{34}}{x_{34}} + \frac{s_{45}}{x_{54}}\right) \\ \left(\frac{s_{25}}{x_{25}} + \frac{s_{35}}{x_{35}} + \frac{s_{45}}{x_{45}}\right)\frac{s_{24}}{x_{24}} \\ \left(\frac{s_{24}}{x_{24}} + \frac{s_{34}}{x_{34}} + \frac{s_{45}}{x_{54}}\right)\frac{s_{25}}{x_{25}} \\ \frac{s_{24}}{x_{24}}\left(\frac{s_{25}}{x_{25}} + \frac{s_{45}}{x_{45}}\right) \\ \frac{s_{25}}{x_{25}}\left(\frac{s_{24}}{x_{24}} + \frac{s_{45}}{x_{54}}\right) \end{pmatrix} . \end{aligned} \tag{4.117}$$

[12] With $\Phi_{4,0} = \Phi_{\mathcal{E}_5}|_{s_{3j}=0}$.

The matrix $\hat{\boldsymbol{B}}_6$ from eq. (4.90), i.e.

$$\hat{\boldsymbol{F}}_6(x_3) = \hat{\boldsymbol{B}}_6\, \boldsymbol{S}_{6,4}(x_3) = \hat{\boldsymbol{B}}_6\, \boldsymbol{B}_{6,4}\, \boldsymbol{Z}_{6,4}(x_3)\,, \tag{4.118}$$

can be calculated according to eq. (4.87). The vectors of chain $\boldsymbol{Z}_{6,4}$ and admissible $\boldsymbol{S}_{6,4}$ integrals are given in eqs. (4.73) and (4.74), and the transformation matrix $\boldsymbol{B}_{6,4}$ between them in eq. (4.77). The result is

$$\begin{pmatrix} \left(\frac{s_{24}}{x_{24}}+\frac{s_{34}}{x_{34}}\right)\left(\frac{s_{25}}{x_{25}}+\frac{s_{35}}{x_{35}}+\frac{s_{45}}{x_{45}}\right) \\ \left(\frac{s_{25}}{x_{25}}+\frac{s_{35}}{x_{35}}\right)\left(\frac{s_{24}}{x_{24}}+\frac{s_{34}}{x_{34}}+\frac{s_{45}}{x_{54}}\right) \\ \left(\frac{s_{25}}{x_{25}}+\frac{s_{35}}{x_{35}}+\frac{s_{45}}{x_{45}}\right)\frac{s_{24}}{x_{24}} \\ \left(\frac{s_{24}}{x_{24}}+\frac{s_{34}}{x_{34}}+\frac{s_{45}}{x_{54}}\right)\frac{s_{25}}{x_{25}} \\ \frac{s_{24}}{x_{24}}\left(\frac{s_{25}}{x_{25}}+\frac{s_{45}}{x_{45}}\right) \\ \frac{s_{25}}{x_{25}}\left(\frac{s_{24}}{x_{24}}+\frac{s_{45}}{x_{54}}\right) \end{pmatrix} = \hat{\boldsymbol{B}}_6 \begin{pmatrix} \varphi\left(\begin{smallmatrix}4&5\\2&2\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\2&3\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\2&4\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&2\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&3\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}4&5\\3&4\end{smallmatrix}\right) \end{pmatrix}, \tag{4.119}$$

where [4]

$$\hat{\boldsymbol{B}}_6 = \begin{pmatrix} s_{24}s_{25} & s_{24}s_{35} & s_{24}s_{45} & s_{25}s_{34} & s_{34}s_{35} & s_{34}s_{45} \\ s_{24}s_{25}+s_{25}s_{45} & s_{24}s_{35} & -s_{25}s_{45} & s_{25}s_{34} & s_{34}s_{35}+s_{35}s_{45} & -s_{35}s_{45} \\ s_{24}s_{25} & s_{24}s_{35} & s_{24}s_{45} & 0 & 0 & 0 \\ s_{24}s_{25}+s_{25}s_{45} & 0 & -s_{25}s_{45} & s_{25}s_{34} & 0 & 0 \\ s_{24}s_{25} & 0 & s_{24}s_{45} & 0 & 0 & 0 \\ s_{24}s_{25}+s_{25}s_{45} & 0 & -s_{25}s_{45} & 0 & 0 & 0 \end{pmatrix}. \tag{4.120}$$

This matrix determines the matrices in the KZ equation

$$\partial_{x_3}\hat{\boldsymbol{F}}_6(x_3) = \left(\frac{\hat{r}_6(\boldsymbol{e}_4^{31})}{x_3} + \frac{\hat{r}_6(\boldsymbol{e}_4^{32})}{x_3-1}\right)\hat{\boldsymbol{F}}_6(x_3)\,, \tag{4.121}$$

according to eq. (4.92), which can be calculated using the matrices $r_6^{\mathrm{adm}}(\boldsymbol{e}_4^{31})$ and $r_6^{\mathrm{adm}}(\boldsymbol{e}_4^{32})$ from eqs. (4.79) and (4.2.3), respectively. They are given by [4]

$$\begin{aligned} \hat{r}_6(\boldsymbol{e}_6^{31}) &= \hat{\boldsymbol{B}}_5\, r_6^{\mathrm{adm}}(\boldsymbol{e}_4^{31})(\hat{\boldsymbol{B}}_6)^{-1} \\ &= \begin{pmatrix} -s_{1345} & 0 & s_{14}+s_{45} & s_{15} & s_{15} & -s_{15} \\ 0 & -s_{1345} & s_{14} & s_{15}+s_{45} & -s_{14} & s_{14} \\ 0 & 0 & -s_{135} & 0 & s_{15} & 0 \\ 0 & 0 & 0 & -s_{134} & 0 & s_{14} \\ 0 & 0 & 0 & 0 & -s_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & -s_{13} \end{pmatrix} \end{aligned} \tag{4.122}$$

and

$$\hat{r}_6(\boldsymbol{e}_4^{32}) = \hat{\boldsymbol{B}}_6\, r_6^{\mathrm{adm}}(\boldsymbol{e}_4^{32})(\hat{\boldsymbol{B}}_6)^{-1}$$
$$= \begin{pmatrix} -s_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & -s_{23} & 0 & 0 & 0 & 0 \\ s_{24} & 0 & -s_{234} & 0 & 0 & 0 \\ 0 & s_{25} & 0 & -s_{235} & 0 & 0 \\ s_{24} & -s_{24} & s_{25}+s_{45} & s_{24} & -s_{2345} & 0 \\ -s_{25} & s_{25} & s_{25} & s_{24}+s_{45} & 0 & -s_{2345} \end{pmatrix}. \tag{4.123}$$

For the special case $s_{3j} = 0$ appearing in the genus-zero recursion (4.105) to calculate the five-point, open-string corrections, the matrices $\hat{r}_6(\boldsymbol{e}_4^{31})$ and $\hat{r}_6(\boldsymbol{e}_4^{32})$ indeed degenerate to the matrices found in ref. [5] via a direct calculation using integration by parts and partial fractioning. The method derived in ref. [4] and presented above is a well-structured method to recursively calculate the matrices $\hat{r}_n(\boldsymbol{e}_4^{31})$ and $\hat{r}_n(\boldsymbol{e}_4^{32})$ for any type n. They are required to calculate the n-point, open-string corrections at genus zero via the genus-zero recursion from ref. [5], where so far only the matrices up to $n = 10$ have been determined by explicit calculations [146].

Chapter 5

Open-string corrections at genus one

The genus-zero recursion from ref. [5] discussed in the previous chapter yields a method to calculate the α'-expansion of the open-string corrections at genus zero solely using matrix operations. Having the combinatorial algorithms at hand to determine the relevant matrices [4], the calculation is purely combinatorial and can be implemented straightforwardly in computer algebra systems. Geometrically, it relates the n-point worldsheet of the tree-level, open-string interaction to the $(n-1)$-point worldsheet. The action of the Drinfeld associator on the latter can be interpreted as a gluing mechanism, where a trivalent interaction is glued to the external states of the $(n-1)$-point worldsheet, such that an n-point worldsheet is obtained.

An extension of the genus-zero recursion to genus one will be shown to incorporate gluing together two external open-string states of a genus-zero, $(n+2)$-point worldsheet, to obtain an n-point worldsheet at genus one. Starting from the genus-zero recursion from ref. [5] and the corresponding reformulation in twisted de Rham theory in ref. [4], such a mechanism has been constructed in ref. [2], which is schematically given in eqs. (1.7) and (3.126): acting with the elliptic KZB associator on $(n+2)$-point, open-string corrections at genus zero yields the n-point, open-string corrections at genus one. Again, the algorithm involves matrix operations exclusively. Moreover, the splitting of the corresponding momenta supports the geometric interpretation in terms of gluing together two external states to obtain a genus-one geometry. This mechanism will be presented in this chapter, supplemented by additional calculations and generalisations.

During the same time, another mechanism to calculate the α'-expansion of open-string integrals at genus one has been identified in refs. [38, 39]. It offers another perspective on the algorithm of ref. [2]. In ref. [3] both mechanisms have been related to each other. In particular, the resulting conventions and formulations have been used to calculate explicit formulas of the relevant matrices and to determine their properties. Therefore, a purely combinatorial algorithm is available to calculate the α'-expansion of the open-string integrals at genus one from the open-string integrals at genus zero.

The structure of this chapter is similar to the previous chapter where the genus-zero objects have been discussed: in section 5.1 the open-string corrections at genus one are presented. In section 5.2 these integrals are generalised and some properties of the resulting classes are summarised. These results are used in section 5.3 to state the recursion from ref. [2], mainly in terms of the conventions and notation introduced in ref. [3].

5.1 Genus-one, open-string corrections

A planar[1], genus-one interaction of n external open-string states leads to a world-sheet of cylinder topology with the vertex insertion points on one of the cylinder boundaries. In analogy to the genus-zero scenario, this punctured cylinder can be described by one half of a punctured torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with modular parameter τ. Due to the rotational symmetry of the torus one of the punctures can be fixed at the origin

$$z_1 = 0 \tag{5.1}$$

or lattice points $\mathbb{Z}+\tau\mathbb{Z}$, respectively. The remaining $n-1$ punctures are arranged in a fixed order on the A-cycle $(0,1) \subset \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, which we chose to be

$$\Delta_{n,1} = \Delta_{n,1}(z_i) = \{z_1 = 0 < z_n < z_{n-1} < \cdots < z_2 < 1 \equiv z_1\}, \tag{5.2}$$

cf. figure 5.1.

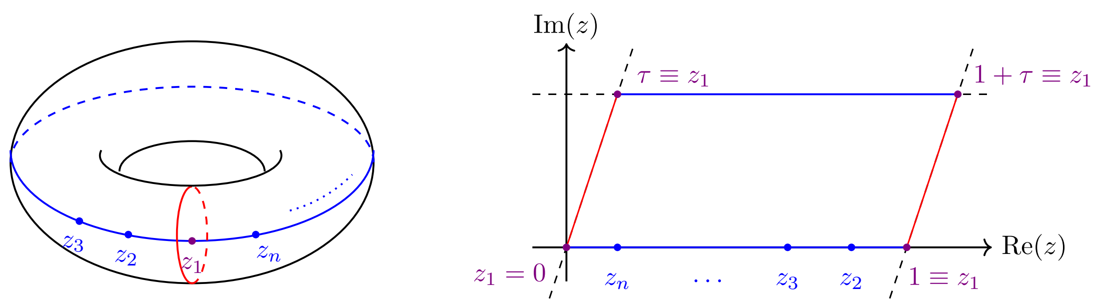


Figure 5.1: The punctured torus with one fixed (violet) puncture z_1 and its parametrisation on the complex plane (right-hand side). The $n-1$ unfixed (blue) punctures $z_2, \ldots, z_n$ will be integrated over in the sting corrections respecting the order $z_{i+1} < z_i$. This defines the corresponding integration cycle $\Delta_{n,1}(z_i)$ (blue line).

[1]Non-planar interactions lead to vertex insertion points on both cylinder boundaries, i.e. also on the violet boundary of the topology on the right-hand side at the bottom of figure 1.2. They may analogously be described as the planar interactions by replacing the elliptic multiple polylogarithms with twisted elliptic multiple polylogarithms, see e.g. ref. [147]. In this thesis, we restrict ourselves to planar interactions to omit further technicalities. See e.g. refs. [38, 39] for the necessary steps to include non-planar interactions in this framework.

The *n-point, open-string corrections at genus one* of massless string states are linear combinations of the configuration-space integrals obtained from integrating the $n-1$ unfixed punctures in the genus-one, open-string amplitudes, cf. eq. (1.4). They depend on the Mandelstam variables and the modular parameter τ. Accordingly, they are the integrals obtained from integrating over the integration cycle $\Delta_{n,1}$, but before integrating over the modular parameter. These integrals are of the form [27, 148, 149]

$$\begin{aligned} F^{\text{open}}_{n,1,(k_2,k_3,\ldots,k_n)}(\sigma;\alpha') &= F^{\text{open}}_{n,1,(k_2,k_3,\ldots,k_n)}(\sigma;\{s_{ij}\}) \\ &= \int_{\Delta_{n,1}} \prod_{i=2}^{n} \mathrm{d}z_i \ \mathrm{KN}^{\tau}_{12\ldots n} \prod_{i=2}^{n} g^{(k_i)}_{\sigma(i-1)\sigma(i)}\,, \end{aligned} \tag{5.3}$$

where $k_i \geq 0$ and

$$g^{(k)}_{ij} = g^{(k)}_{i,j} = g^{(k)}(z_{ij},\tau) \tag{5.4}$$

are the integration kernels appearing in the eMPLs, which are the η-coefficients of the Eisenstein–Kronecker series, cf. eq. (3.76). The permutation $\sigma \in S_{n-1}$ acts on the labels $i = 2, 3, \ldots, n$ of the punctures z_i. The genus-one Koba–Nielsen factor is

$$\begin{aligned} \mathrm{KN}^{\tau}_{i_1\ldots i_p} &= \mathrm{KN}^{\tau}_{i_1\ldots i_p}(z_{i_1},\ldots,z_{i_p};\{s_{ij}\}) = \prod_{\substack{i,j\in\{i_1,\ldots,i_p\}\\ i<j}} \exp\Big(-s_{ij}\left(\tilde{\Gamma}_{ij} - \omega(1,0)\right)\Big)\,, \\ \tilde{\Gamma}_{ij} &= \tilde{\Gamma}\left(\begin{smallmatrix}1\\0\end{smallmatrix};|z_{ij}|,\tau\right) \end{aligned} \tag{5.5}$$

and comes from the plane-wave contribution of the vertex operators. The genus-one Green's function $\tilde{\Gamma}_{ij} - \omega(1,0)$ in the exponent involves the real branch of the regularised ePL $\tilde{\Gamma}_1(1;z)$, which is the genus-one analogue of the logarithm. The latter, in turn, is the genus-zero Green's function.

For example, the four-point, open-string amplitude for massless gluons is [148, 149]

$$\begin{aligned} A^{\text{open}}_{4,1}(1,2,3,4;\alpha') &= s_{12}s_{23}A^{\text{YM}}_{4,0}\int_0^{\infty} \mathrm{d}\tau\, F^{\text{open}}_{4,1,(0,0,0)}(1,2,3,4;\alpha')\,, \\ F^{\text{open}}_{4,1,(0,0,0)}(1,2,3,4;\alpha') &= \int_{\Delta_{4,1}} \prod_{i=2}^{4} \mathrm{d}z_i \ \mathrm{KN}^{\tau}_{1234}\,. \end{aligned} \tag{5.6}$$

5.1.1 Z^{τ}_n-integrals

Since there are infinitely many functions $g^{(0)}_{ij}, g^{(1)}_{ij}, g^{(2)}_{ij}, \ldots$ defined by the series expansion of the Eisenstein–Kronecker series, eq. (5.3) defines infinitely many integrals $F^{\text{open}}_{n,1,(k_2,k_3,\ldots,k_n)}$ labelled by $(k_2,\ldots,k_n) \in \mathbb{Z}^{n-1}_{\geq 0}$. It turned out [3, 38, 39] to be convenient instead of working with such an infinite family, to use finitely many generating functions obtained upon replacing the functions $g_{ij}^{(k)}$ in eq. (5.3) by the Eisenstein–Kronecker series $F_{ij}(\eta_k)$ at the expense of introducing some auxiliary variables $\vec{\eta} = (\eta_2, \eta_3, \dots, \eta_n)$. This defines the $(n-1)!$ distinct *n-point, open-string integrals at genus one*, or Z_n^τ*-integrals* [39] for short,

$$\begin{aligned} Z_n^\tau(\sigma) &= Z_n^\tau(\sigma; \vec{\eta}; \{s_{ij}\}) \\ &= \int_{\Delta_{n,1}} \prod_{i=2}^{n} \mathrm{d}z_i \; \mathrm{KN}_{12\dots n}^\tau \; \varphi^\tau(1, \sigma(2,3,\dots,n))\,, \end{aligned} \tag{5.7}$$

where $\sigma \in S_{n-1}$ acts on the labels $2, 3, \dots, n$. The integrand involves a *chain of Eisenstein-Kronecker series*

$$\begin{aligned} \varphi^\tau(a_1, a_2, \dots, a_p) &= \varphi^\tau(a_1, a_2, \dots, a_p; \vec{\eta}) \\ &= F_{a_1 a_2}(\eta_{a_2} + \dots + \eta_{a_p}) \dots F_{a_{p-2} a_{p-1}}(\eta_{a_{p-1}} + \eta_{a_p}) F_{a_{p-1} a_p}(\eta_{a_p}) \\ &= \prod_{i=2}^{p} F_{a_{i-1} a_i}(\eta_{a_i \dots a_p})\,, \\ \varphi^\tau(a_1) &= 1\,, \end{aligned} \tag{5.8}$$

associated to a sequence $A = (a_1, a_2, \dots, a_p)$ with the convention

$$\eta_A = \eta_{a_1 a_2 \dots a_p} = \eta_{a_1} + \eta_{a_2} + \dots + \eta_{a_p}\,. \tag{5.9}$$

The chain of Eisenstein–Kronecker series is a chain product, the genus-one analogue of the chain of fractions from eq. (4.10) satisfying analogous identities such as the shuffle identity (4.12), which generalises the Fay identity (3.80). These Z_n^τ-integrals are the genus-one analogues of the genus-zero Z_n-integrals defined in eq. (4.14). In refs. [38, 39] it is shown how their α'-expansion can be calculated using Picard iteration. An alternative method based on the genus-one associator equation (3.123) is introduced in refs. [2,3], where all the required ingredients are explicitly calculated. It is this latter approach, which will be presented in this chapter and in particular in section 5.3.

The string corrections from eq. (5.3) are recovered[2] upon expanding all the Eisenstein–Kronecker series as the coefficient of $\eta_{\sigma(23\dots n)}^{k_2-1} \eta_{\sigma(3\dots n)}^{k_3-1} \dots \eta_{\sigma(n)}^{k_n-1}$, which is denoted by

$$F_{n,1,(k_2,k_3,\dots,k_n)}^{\text{open}}(\sigma; \alpha'\}) = [Z_n^\tau(\sigma; \vec{\eta}; \{s_{ij}\})]_{\eta_{\sigma(23\dots n)}^{k_2-1} \eta_{\sigma(3\dots n)}^{k_3-1} \dots \eta_{\sigma(n)}^{k_n-1}}\,. \tag{5.10}$$

[2]See ref. [39] for a discussion about the extraction of the appropriate coefficient.

The $(n-1)!$-dimensional vector of all the Z_n^τ-integrals is denoted by

$$\begin{aligned}\boldsymbol{Z}_n^\tau &= \boldsymbol{Z}_n^\tau(\vec{\eta};\{s_{ij}\})\\ &= \Big(Z_n^\tau(\sigma;\vec{\eta};\{s_{ij}\})\Big)_{\sigma\in S_{n-1}}\,.\end{aligned} \tag{5.11}$$

Two-point example

If momentum conservation is not imposed, the simplest non-trivial example is the two-point, open-string integral, which is the single integral

$$\begin{aligned}Z_2^\tau(1,2) &= \int_0^1 \mathrm{d}z_2\ \mathrm{KN}_{12}^\tau\ F_{12}(\eta_2) = \sum_{k_2\geq 0}\eta_2^{k_2-1}F^{\text{open}}_{n,1,(k_2)}(1,2;\alpha')\,,\\ F^{\text{open}}_{n,1,(k_2)}(1,2;\alpha') &= \int_0^1 \mathrm{d}z_2\ \mathrm{KN}_{12}^\tau\, g_{12}^{(k_2)}\,,\\ \mathrm{KN}_{12}^\tau &= \exp\Big(-s_{12}\Big(\tilde{\Gamma}_{12}-\omega(1,0)\Big)\Big)\,.\end{aligned} \tag{5.12}$$

Its expansion in α' and the auxiliary variable η_2 facilitates MZVs and eMZVs [39]

$$\begin{aligned}&Z_2^\tau(1,2)\\ &=\frac{1}{\eta_2}\Big[1+s_{12}^2\Big(\frac{\omega(0,0,2)}{2}+\frac{5\zeta_2}{12}\Big)+s_{12}^3\Big(\frac{\omega(0,0,3,0)}{18}-\frac{4\zeta_2}{3}\omega(0,0,1,0)+\frac{\zeta_3}{12}\Big)+\mathcal{O}(s_{12}^4)\Big]\\ &\quad+\eta_2\Big[-2\zeta_2+s_{12}\omega(0,3)+s_{12}^2\Big(3\zeta_2\omega(0,0,2)-\frac{\omega(0,0,4)}{2}+\frac{13\zeta_4}{12}\Big)+\mathcal{O}(s_{12}^3)\Big]\\ &\quad+\eta_2^3\Big[-2\zeta_4+s_{12}\Big(\omega(0,5)-2\zeta_2\omega(0,3)\Big)+\mathcal{O}(s_{12}^2)\Big]+\mathcal{O}(\eta^5)\,.\end{aligned} \tag{5.13}$$

This turns out to be a general feature: expanding the genus-one Koba–Nielsen factor in the string corrections (5.3) in α', integrals of the same form as the eMZVs defined in eq. (3.97) are obtained. The actual structure of the eMZVs and MZVs appearing in the open-string integrals at genus one can nicely be read off from the recursion presented in section 5.3.

5.2 Genus-one, type-(n,p) integrals

Similarly to the genus-zero scenario, the genus-one recursion in refs. [2, 3] involves more general integrals than the Z_n^τ-integrals. For a fixed kinematic configuration[3] $\{s_{ij}\}$, their differential forms are defined on the configuration space of n-punctured

[3] As in the genus-zero case, rather than momentum conservation, the condition $\mathrm{Re}(s_{i_1\dots i_p})<0$ is imposed for consecutive points $i_1,\dots,i_p$ on the boundary of the cylinder to ensure convergence of the genus-one integrals [3].

tori associated to the modular parameter τ with p fixed coordinates, denoted by

$$\mathcal{F}^{\tau}_{n,p} = \{(z_{p+1}, z_{p+2}, \dots, z_n) \in (\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}))^{n-p} | z_i \neq z_1, \dots, z_p, z_j \text{ for } i \neq j\}\,, \tag{5.14}$$

where

$$z_1 = 0 \tag{5.15}$$

is fixed by rotational symmetry. The corresponding integrals $Z^{\tau}_{n,p}$ and $S^{\tau}_{n,p}$ introduced in the next two subsections are defined for the integration cycle

$$\Delta_{n,p} = \Delta_{n,p}(z_i) = \{0 = z_1 < z_n < z_{n-1} < \dots < z_{p+1} < z_p\}\,, \tag{5.16}$$

cf. figure 5.2.

It is unclear how these genus-one integrals, the corresponding forms and integration cycles can be understood in terms of twisted de Rham theory. [4] However, the closure of the various partial differential equations discussed below suggests that the considered classes of integrals may be understood as a complete set of integrals.

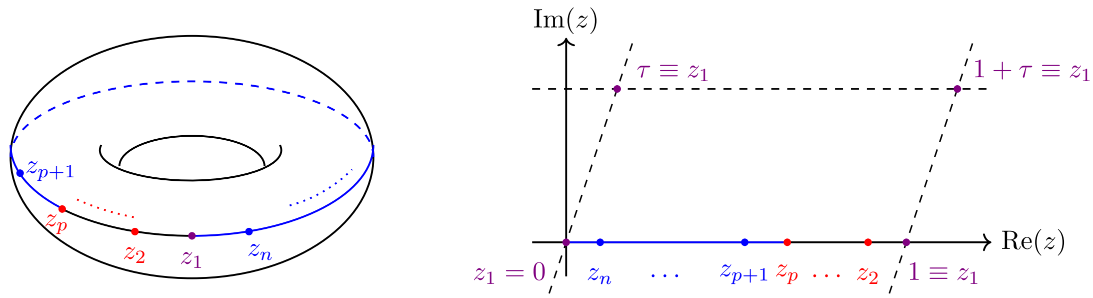


Figure 5.2: The p-punctured torus associated to the integrals $Z^{\tau}_{n,p}$ and $S^{\tau}_{n,p}$ introduced in the next subsections. The p unintegrated (red and violet) punctures are $z_1, z_2, \dots, z_p$. Its parametrisation on the complex plane is depicted on the right-hand side. The $n-p$ integrated (blue) punctures $z_{p+1}, \dots, z_n$ define the integration cycle $\Delta_{n,p}(z_i)$ (blue line) from eq. (5.16). The integrands of $Z^{\tau}_{n,p}$ and $S^{\tau}_{n,p}$ are defined on the n-punctured torus (violet, red and blue punctures). The unintegrated (violet) puncture $z_1 = 0$ is canonically fixed, while the other $p-1$ unintegrated (red) punctures $z_2, \dots, z_p$ can be varied on the A-cycle interval $0 = z_1 < z_p < \dots < z_2 < 1 \equiv z_1$.

[4] In ref. [39], it is conjectured that the vector $\boldsymbol{Z}^{\tau}_{n}$ defined in eq. (5.11) contains a basis of the genus-one twisted forms for $p = 1$.

5.2.1 Genus-one, type-(n, p) chain integrals $Z^\tau_{n,p}$

The generalisation of the open-string integrals at genus one Z^τ_n to p unintegrated punctures leads to the *genus-one chain integrals of type* (n, p)[5]

$$\begin{aligned} &Z^\tau_{n,p}\left((1, A^1), \dots, (p, A^p); z_2, \dots, z_p; \vec{\eta}; \{s_{ij}\}\right) \\ &= \int_{\Delta_{n,p}} \prod_{i=p+1}^{n} \mathrm{d}z_i \ \mathrm{KN}^\tau_{12\dots n} \prod_{k=1}^{p} \varphi^\tau(k, A^k)\,, \end{aligned} \tag{5.17}$$

where A^k are possibly empty subsequences of a permutation of $(p+1, \dots, n)$

$$(A^1, A^2, \dots, A^p) = \sigma(p+1, p+2, \dots, n)\,, \quad \sigma \in S_{n-p}\,, \tag{5.18}$$

with σ acting on $p+1, p+2, \dots, n$. These are the genus-one analogues of the type-(n, p) chain integrals from eq. (4.51)

Equation (5.17) defines[6]

$$d_{n,p} = \frac{(n-1)!}{(p-1)!} \tag{5.19}$$

independent integrals. The $d_{n,p}$-dimensional vector

$$\begin{aligned} &\boldsymbol{Z}^\tau_{n,p}(z_2, \dots, z_p) \\ &= \boldsymbol{Z}^\tau_{n,p}(z_2, \dots, z_p; \vec{\eta}; \{s_{ij}\}) \\ &= \left(Z^\tau_{n,p}\left((1, A^1), \dots, (p, A^p); z_2, \dots, z_p; \vec{\eta}; \{s_{ij}\}\right)\right)_{\substack{(A^1, A^2, \dots, A^p) = \sigma(p+1, p+2, \dots, n) \\ \sigma \in S_{n-p}}} \end{aligned} \tag{5.20}$$

of these integrals satisfies a closed system of differential equations upon varying the unintegrated punctures z_i for $i = 2, \dots, p$ and the modular parameter τ. It is given

[5] The construction of these integrals, the investigation of their differential systems below and the remaining results in section 5.2 have been worked out by the author of this thesis and will be published in ref. [65]. In this thesis, no other results from ref. [65] are included except for the ones solely obtained by the author of this thesis.

[6] The number $d_{n,p}$ is algebraically very similar to the number of type-(n, p) chain and admissible integrals at genus zero, which is nothing but the dimension of $H^{n-p}(\mathcal{F}_{n,p}, \nabla_{n-p})$ from eq. (4.50), i.e. $d_{n-2,p-2}$. The shift of two comes from an integration-by-parts redundancy at genus zero and the fixed puncture at infinity, see also eq. (6.114) for a discussion about this similarity. However, the number $d_{n,p}$ of genus-one integrals has not yet been interpreted as a dimension of a certain (twisted) de Rham cohomology in a similar context as at genus zero.

by

$$\begin{aligned}\partial_{z_i}\boldsymbol{Z}^{\tau}_{n,p}\left(z_2,\ldots,z_p\right) &= \left(r_n^{\text{cha}}(\boldsymbol{x}_p^0)+\sum_{k>0}\sum_{\substack{j=1\\ j\neq i}}^{p} r_n^{\text{cha}}(\boldsymbol{x}_p^{ij,k})g_{ij}^{(k)}\right)\boldsymbol{Z}^{\tau}_{n,p}\left(z_2,\ldots,z_p\right)\,,\\ 2\pi i\partial_{\tau}\boldsymbol{Z}^{\tau}_{n,p}\left(z_2,\ldots,z_p\right) &= \Bigg(-r_n^{\text{cha}}(\boldsymbol{\epsilon}_p^0)+\sum_{k\geq 4}(1-k)\,\mathrm{G}_k\, r_n^{\text{cha}}(\boldsymbol{\epsilon}_p^k)\\ &\quad+\sum_{k\geq 2}(k-1)\sum_{\substack{r,q=1\\ r>q}}^{p} r_n^{\text{cha}}(\boldsymbol{x}_p^{rq,k-1})g_{rq}^{(k)}\Bigg)\boldsymbol{Z}^{\tau}_{n,p}\left(z_2,\ldots,z_p\right)\,.\end{aligned} \tag{5.21}$$

The $d_{n,p}$-dimensional matrices $r_n^{\text{cha}}(\boldsymbol{x}_p^{ij,k})$ are homogeneous of degree one in the Mandelstam variables s_{ij}, hence, linear in α', and of degree $k-1$ in the auxiliary variables $\vec{\eta}=(\eta_2,\ldots,\eta_n)$. The matrix $r_n^{\text{cha}}(\boldsymbol{x}_p^0)$ is also homogeneous of degree one in s_{ij} and contains first derivatives with respect to the auxiliary variables $\vec{\eta}$. Counting these derivatives as degree minus one in η_i, $r_n^{\text{cha}}(\boldsymbol{x}_p^0)$ is of total degree minus one in the variables $\vec{\eta}$. Similarly, the matrices $r_n^{\text{cha}}(\boldsymbol{\epsilon}_p^k)$ are of degree one in s_{ij} and degree $k-2$ in $\vec{\eta}$, if the second-order derivatives and ζ_2 appearing in the diagonal of $r_n^{\text{cha}}(\boldsymbol{\epsilon}_p^0)$ are counted as degree minus two. The explicit form of the matrices is given by the closed formulæ of the corresponding derivatives in eqs. (C.66) and (C.73), derived in appendix C.3. As in the genus-zero case, having the explicit form of the matrices at hand, the partial differential equations (5.21) might be solved by Picard iteration, see e.g. ref. [39] for the case $p=1$ and ref. [65] for general p. In order to calculate the genus-one, open-string corrections, this is not necessary and can simply be done using the genus-one associator eq. (3.123) and the type-$(n,2)$ integrals, which is shown in section 5.3.

The system (5.21) is reminiscent of the elliptic KZB system on the p-punctured torus investigated in refs. [40, 116], where a certain set of Lie algebra generators $\boldsymbol{x}_p^i$, $\boldsymbol{y}_p^j$ and $\boldsymbol{\epsilon}_p^k$ is considered: they generate the elliptic braid group $\bar{\mathfrak{t}}_{1,p}$ of the p-punctured torus. In particular, the generators of the genus-zero braid group satisfying the infinitesimal braid relations (4.56) are recovered from the commutators $\boldsymbol{e}_p^{ij}=[\boldsymbol{x}_p^i,\boldsymbol{y}_p^j]$. Moreover, the elliptic KZB system in ref. [40] incorporates the k-fold commutator $\boldsymbol{x}_p^{ij,k}=[\boldsymbol{x}_p^i,\ldots,[\boldsymbol{x}_p^i,\boldsymbol{y}_p^j]\ldots]$, in the place of our matrices $r_n^{\text{cha}}(\boldsymbol{x}_p^{ij,k})$. However, in our case, we do not a priori impose any restrictions on the matrices $r_n^{\text{cha}}(\boldsymbol{x}_p^{ij,k})$ and $r_n^{\text{cha}}(\boldsymbol{\epsilon}_p^k)$. In particular, our investigations suggest that $r_n^{\text{cha}}(\boldsymbol{x}_p^{ij,k})$ is not a nested commutator such as the elements $\boldsymbol{x}_p^{ij,k}$. Thus, it is doubtful and far from clear whether $r_n^{\text{cha}}(\boldsymbol{x}_p^{ij,k})$ and $r_n^{\text{cha}}(\boldsymbol{\epsilon}_p^k)$ are indeed representations of the genus-one braid group. Commutation relations of these matrices may be derived from commuting partial derivatives, which lead in the genus-zero case to an a-posteriori confirmation that the corresponding matrices satisfy the infinitesimal braid relations (4.56).

At genus one, this approach is much more technical and only leads to a subset of the commutation relations satisfied by the elliptic braid group $\bar{\mathfrak{t}}_{1,p}$, see for example ref. [3] for the case of two punctures $p = 2$ and ref. [65] for the commutation relations for general p. Despite these open questions regarding its embedding into the mathematical literature, the system (5.21) will simply be called an elliptic KZB system on the p-punctured torus.

The genus-one chain integrals of type $(n, 1)$ are the n-point, open-string integrals Z^τ_n at genus one defined in eq. (5.7), i.e.

$$Z^\tau_{n,1}\left((1, A^1); \vec{\eta}; \{s_{ij}\}\right) = Z^\tau_n(\sigma; \vec{\eta}; \{s_{ij}\})\,, \tag{5.22}$$

where $\sigma(2, \ldots, n) = A^1$, such that the corresponding vectors agree:

$$\boldsymbol{Z}^\tau_{n,1}\,(\vec{\eta}; \{s_{ij}\}) = \boldsymbol{Z}^\tau_n(\vec{\eta}; \{s_{ij}\})\,. \tag{5.23}$$

5.2.2 Genus-one, type-(n, p) admissible integrals $S^\tau_{n,p}$

As for the Z^τ_n-integrals and the more general type-(n, p) chain integrals $Z^\tau_{n,p}$, the genus-one analogues of the Selberg and admissible integrals can also either be described in terms of infinitely many integrals or finitely many generating series at the expense of introducing the auxiliary variables $\vec{\eta}$. The former approach has been chosen in ref. [2], which lead to a recursive definition of type-(n, p) integrals with n punctures, p of which are fixed at distinct values, including the gauge fixed puncture $z_1 = 0$ as well as $z_2, \ldots, z_p$. These *genus-one Selberg integrals of type* (n, p) are given by

$$\begin{aligned} &S^\tau_{n,p,(k_{p+1},\ldots,k_n)}\left(\begin{smallmatrix} p+1 & \cdots & n \\ e(p+1) & \cdots & e(n) \end{smallmatrix}; z_2, \ldots, z_p; \{s_{ij}\}\right) \\ &= \int_0^{z_p} \mathrm{d}z_{p+1}\, g^{(k_{p+1})}_{e(p+1)p+1}\, S^\tau_{n,p+1,(k_{p+2},\ldots,k_n)}\left(\begin{smallmatrix} p+2 & \cdots & n \\ e(p+2) & \cdots & e(n) \end{smallmatrix}; z_2, \ldots, z_{p+1}; \{s_{ij}\}\right)\,, \end{aligned} \tag{5.24}$$

where e is again an admissible map, i.e.

$$1 \leq e(k) < k \tag{5.25}$$

for $k = 2, \ldots, n$, and the empty integral is the genus-one Koba–Nielsen factor

$$S^\tau_{n,n}\,(z_2, \ldots, z_n; \{s_{ij}\}) = \mathrm{KN}^\tau_{12\ldots n}\,. \tag{5.26}$$

Their generating series contain the auxiliary variables $\vec{\eta} = (\eta_2, \ldots, \eta_n)$ and are the $d_{n,p}$ distinct *genus-one admissible integrals of type* (n, p)

$$\begin{aligned} &S^\tau_{n,p}\left(\begin{smallmatrix} p+1 & \ldots & n \\ e(p+1) & \ldots & e(n) \end{smallmatrix} ; z_2, \ldots, z_p; \vec{\eta}; \{s_{ij}\}\right) \\ &= \int_0^{z_p} \mathrm{d}z_{p+1}\, F_{e(p+1)p+1}(\eta_e(p+1)))\, S^\tau_{n,p+1}\left(\begin{smallmatrix} p+2 & \ldots & n \\ e(p+2) & \ldots & e(n) \end{smallmatrix} ; z_2, \ldots, z_{p+1}; \vec{\eta}; \{s_{ij}\}\right) \\ &= \int_{\Delta_{n,p}} \prod_{i=p+1}^{n} \mathrm{d}z_i\, \mathrm{KN}^\tau_{12\ldots n}\, \varphi^\tau\left(\begin{smallmatrix} p+1 & \ldots & n \\ e(p+1) & \ldots & e(n) \end{smallmatrix} ; \vec{\eta}\right), \end{aligned} \tag{5.27}$$

where for $k = 2, \ldots, n$, the variable

$$\eta_e(k) = \sum_{\substack{i=k \\ \exists m \geq 0:\, e^m(i)=k}}^{n} \eta_i \tag{5.28}$$

is a sum of the auxiliary variables η_i such that there exists an integer $m \geq 0$ with $e^m(i) = e^{m-1}(e(i)) = k$. The product

$$\varphi^\tau\left(\begin{smallmatrix} a_1 & \ldots & a_p \\ e(a_1) & \ldots & e(a_p) \end{smallmatrix}\right) = \varphi^\tau\left(\begin{smallmatrix} a_1 & \ldots & a_p \\ e(a_1) & \ldots & e(a_p) \end{smallmatrix} ; \vec{\eta}\right) = \prod_{i=2}^{p} F_{e(a_i)a_i}(\eta_e(a_i)) \tag{5.29}$$

with $1 \leq e(a_i) < a_i$ is the genus-one analogue of the admissible product of fractions defined in eq. (4.20), an *admissible product of Eisenstein–Kronecker series*. The particular linear combinations $\eta_e(k)$ of the variables η_i in the definition (5.27) follow straightforwardly from the graphical representation of the corresponding products of Eisenstein–Kronecker series described in appendix 6.1.2. The components of the genus-one admissible integrals in the expansion with respect to the auxiliary variables are the genus-one Selberg integrals of type (n, p):

$$\begin{aligned} &S^\tau_{n,p,(k_{p+1},\ldots,k_n)}\left(\begin{smallmatrix} p+1 & \ldots & n \\ e(p+1) & \ldots & e(n) \end{smallmatrix} ; z_2, \ldots, z_p; \{s_{ij}\}\right) \\ &= \left[S^\tau_{n,p}\left(\begin{smallmatrix} p+1 & \ldots & n \\ e(p+1) & \ldots & e(n) \end{smallmatrix} ; z_2, \ldots, z_p; \vec{\eta}; \{s_{ij}\}\right)\right]_{\eta_e^{k_{p+1}-1}(p+1)\ldots\eta_e^{k_n-1}(n)}. \end{aligned} \tag{5.30}$$

In analogy to the genus-zero integrals leading to eqs. (4.64) and (4.65), the genus-one admissible integrals $S^\tau_{n,p}$ may be expressed as a linear combination of chain integrals based on the identity

$$\varphi^\tau\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) = \prod_{k=1}^{p} \varphi^\tau\left(\begin{smallmatrix} A_e(k) \\ e(A_e(k)) \end{smallmatrix}\right) = \prod_{k=1}^{p} \varphi^\tau\left(k,\, \begin{smallmatrix} A_e(k) \\ e(A_e(k)) \end{smallmatrix}\right) \tag{5.31}$$

of admissible and chain products formed by the Eisenstein–Kronecker series, where $A = (p+1, \ldots, n)$ and $A_e(k)$ is defined in eq. (4.24). The linear combination $\eta_e(k)$ of the auxiliary variables η_i in eq. (5.28) ensures that this identity of Eisenstein–Kronecker series holds. This leads to the following relation between genus-one admissible and chain integrals:

$$\begin{aligned} &S^{\tau}_{n,p}\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}; z_2, \dots, z_p; \vec{\eta}; \{s_{ij}\}\right) \\ &= Z^{\tau}_{n,p}\left(\left(1, \begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right), \dots, \left(p, \begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right); z_2, \dots, z_1; \vec{\eta}; \{s_{ij}\}\right) . \end{aligned} \tag{5.32}$$

Combining the $d_{n,p}$ integrals on both sides into a vector, the linear combination (5.32) can be written as a matrix transformation

$$\boldsymbol{S}^{\tau}_{n,p} = \boldsymbol{B}^{\tau}_{n,p}\, \boldsymbol{Z}^{\tau}_{n,p} \tag{5.33}$$

where

$$\begin{aligned} \boldsymbol{S}^{\tau}_{n,p}(z_2, \dots, z_p) &= \boldsymbol{S}^{\tau}_{n,p}(z_2, \dots, z_p; \vec{\eta}; \{s_{ij}\}) \\ &= \left(S^{\tau}_{n,p}\left(\begin{smallmatrix} p+1 & \dots & n \\ e(p+1) & \dots & e(n) \end{smallmatrix}; z_2, \dots, z_p; \vec{\eta}; \{s_{ij}\}\right)\right)_{1 \leq e(k) < k} \end{aligned} \tag{5.34}$$

is the vector of admissible genus-one integrals and $\boldsymbol{Z}^{\tau}_{n,p}$ the vector of chain integrals from eq. (5.20). The transformation (5.33) translates the two classes of integrals defined in refs. [2,3] into each other. Since the conversion of the genus-one products φ^{τ} from admissible to chain products in eq. (5.31) is exactly the same as in eq. (4.64) for the genus-zero products φ, the genus-one transformation matrix from eq. (5.33) agrees with the genus-zero matrix from eq. (4.63):

$$\boldsymbol{B}^{\tau}_{n,p} = \boldsymbol{B}_{n+2,p+2} . \tag{5.35}$$

The shift in the labels comes from a redundancy of the puncture at infinity and integration-by-parts identities at genus zero discussed in appendix 6.2.4.

The vector $\boldsymbol{S}^{\tau}_{n,p}$ also satisfies an elliptic KZB system on the p-punctured torus:

$$\begin{aligned} \partial_{z_i} \boldsymbol{S}^{\tau}_{n,p}(z_2, \dots, z_p) &= \left(r^{\mathrm{adm}}_n(\boldsymbol{x}^0_p) + \sum_{k>0} \sum_{\substack{j=1 \\ j \neq i}}^{p} r^{\mathrm{adm}}_n(\boldsymbol{x}^{ij,k}_p) g^{(k)}_{ij} \right) \boldsymbol{S}^{\tau}_{n,p}(z_2, \dots, z_p) , \\ 2\pi i \partial_{\tau} \boldsymbol{S}^{\tau}_{n,p}(z_2, \dots, z_p) &= \Bigg(- r^{\mathrm{adm}}_n(\boldsymbol{\epsilon}^0_p) + \sum_{k \geq 4} (1-k)\, \mathrm{G}_k\, r^{\mathrm{adm}}_n(\boldsymbol{\epsilon}^k_p) \\ &\quad + \sum_{k \geq 2} (k-1) \sum_{\substack{r,q=1 \\ r>q}}^{p} r^{\mathrm{adm}}_n(\boldsymbol{x}^{rq,k-1}_p) g^{(k)}_{rq} \Bigg) \boldsymbol{S}^{\tau}_{n,p}(z_2, \dots, z_p) . \end{aligned} \tag{5.36}$$

The corresponding matrices are related by the transformation matrix $\boldsymbol{B}^{\tau}_{n,p}$ to the system (5.21) satisfied by the chain integrals according to

$$\begin{aligned} r_n^{\text{adm}}(\boldsymbol{x}_p^0) &= \boldsymbol{B}_{n,p}^\tau\, r_n^{\text{cha}}(\boldsymbol{x}_p^0)\, (\boldsymbol{B}_{n,p}^\tau)^{-1}\,, \\ r_n^{\text{adm}}(\boldsymbol{x}_p^{ij,k}) &= \boldsymbol{B}_{n,p}^\tau\, r_n^{\text{cha}}(\boldsymbol{x}_p^{ij,k})\, (\boldsymbol{B}_{n,p}^\tau)^{-1}\,, \\ r_n^{\text{adm}}(\boldsymbol{\epsilon}_p^k) &= \boldsymbol{B}_{n,p}^\tau\, r_n^{\text{cha}}(\boldsymbol{\epsilon}_p^k)\, (\boldsymbol{B}_{n,p}^\tau)^{-1}\,. \end{aligned} \tag{5.37}$$

Again, the matrices $r_n^{\text{adm}}(\boldsymbol{x}_p^{ij,k})$ and $r_n^{\text{adm}}(\boldsymbol{\epsilon}_p^k)$ are linear in s_{ij} and, up to $r_n^{\text{adm}}(\boldsymbol{\epsilon}_p^0)$ with ζ_2 in the diagonal, of degree $k-1$ and $k-2$, respectively, in the auxiliary variables $\vec{\eta}$. In ref. [2], the z_i-derivatives of the system (5.37) have been expressed in terms of the genus-one Selberg integrals $S^\tau_{n,p,(k_{p+1},\ldots,k_n)}$ from eq. (5.24).

5.2.3 Examples

Type-$(3,2)$ integrals: $\boldsymbol{Z}^\tau_{3,2}$ and $\boldsymbol{S}^\tau_{3,2}$

A simple example is the basis vector $\boldsymbol{Z}^\tau_{3,2} = \boldsymbol{S}^\tau_{3,2}$ of type-$(3,2)$, which will lead to the two-point, genus-one string integral from eq. (5.12), cf. eq. (5.106) below. Note that for this example, the transformation matrix $\boldsymbol{B}^\tau_{3,2}$ from eq. (5.33) is the identity:

$$\boldsymbol{B}^\tau_{3,2} = \mathbb{1}_2\,. \tag{5.38}$$

The vector is given by

$$\boldsymbol{Z}^\tau_{3,2}(z_2) = \begin{pmatrix} Z^\tau_{3,2}((1,3),(2);z_2) \\ Z^\tau_{3,2}((1),(2,3);z_2) \end{pmatrix} = \int_0^{z_2} \mathrm{d}z_3\ \mathrm{KN}^\tau_{123} \begin{pmatrix} F_{13}(\eta_3) \\ F_{23}(\eta_3) \end{pmatrix} \tag{5.39}$$

and satisfies, according to the closed formulæ (C.67) and (C.74) below, the following partial differential equations

$$\begin{aligned} \partial_{z_2}\boldsymbol{Z}^\tau_{3,2}(z_2) &= \begin{pmatrix} -(s_{12}+s_{23})g_{21}^{(1)} - s_{23}\partial_{\eta_3} & s_{23}F_{21}(-\eta_3) \\ s_{13}F_{21}(\eta_3) & -(s_{12}+s_{13})g_{21}^{(1)} + s_{13}\partial_{\eta_3} \end{pmatrix} \boldsymbol{Z}^\tau_{3,2}(z_2)\,, \\ 2\pi i\partial_\tau \boldsymbol{Z}^\tau_{3,2}(z_2) &= \left(\frac{1}{2}(s_{13}+s_{23})\partial_{\eta_3^2} - 2\zeta_2 s_{123}\right)\boldsymbol{Z}^\tau_{3,2}(z_2) \\ &\quad + \begin{pmatrix} -(s_{12}+s_{23})g_{21}^{(2)} - s_{13}\wp(\eta_3) & s_{23}\partial_{\eta_3}F_{21}(-\eta_3) \\ s_{13}\partial_{\eta_3}F_{21}(\eta_3) & -(s_{12}+s_{13})g_{21}^{(2)} - s_{23}\wp(\eta_3) \end{pmatrix} \boldsymbol{Z}^\tau_{3,2}(z_2)\,. \end{aligned} \tag{5.40}$$

Expanding the Eisenstein–Kronecker series and Weierstrass $\wp$-functions in the matrices above leads to the elliptic KZB system (5.21) (with $\boldsymbol{x}_2^0 = \boldsymbol{x}_2^{21,0}$) on the twice-punctured torus:

$$\begin{aligned}\partial_{z_2}\boldsymbol{Z}^{\tau}_{3,2}(z_2) &= \left(\sum_{k\geq 0} r_3^{\mathrm{cha}}(\boldsymbol{x}_2^{21,k})g^{(k)}_{21}\right)\boldsymbol{Z}^{\tau}_{3,2}(z_2)\,,\\ 2\pi i\partial_\tau \boldsymbol{Z}^{\tau}_{3,2}\,(z_2) &= \Bigg(-r_3^{\mathrm{cha}}(\boldsymbol{\epsilon}_2^0)+\sum_{k\geq 4}(1-k)\,\mathrm{G}_k\,r_3^{\mathrm{cha}}(\boldsymbol{\epsilon}_2^k)\\ &\qquad +\sum_{k\geq 2}(k-1)r_3^{\mathrm{cha}}(\boldsymbol{x}_2^{21,k-1})g^{(k)}_{21}\Bigg)\boldsymbol{Z}^{\tau}_{3,2}\,(z_2)\;,\end{aligned} \tag{5.41}$$

where

$$\begin{aligned} r_3^{\mathrm{cha}}(\boldsymbol{x}_2^{21,0}) &= r_3^{\mathrm{adm}}(\boldsymbol{x}_2^{21,0}) = \begin{pmatrix} -s_{23}\partial_{\eta_3} & -s_{23}/\eta_3\\ s_{13}/\eta_3 & s_{13}\partial_{\eta_3}\end{pmatrix}\,,\\ r_3^{\mathrm{cha}}(\boldsymbol{x}_2^{21,1}) &= r_3^{\mathrm{adm}}(\boldsymbol{x}_2^{21,1}) = \begin{pmatrix} -(s_{12}+s_{23}) & s_{23}\\ s_{13} & -(s_{12}+s_{13})\end{pmatrix}\,,\\ r_3^{\mathrm{cha}}(\boldsymbol{x}_2^{21,k}) &= r_3^{\mathrm{adm}}(\boldsymbol{x}_2^{21,k}) = \eta_3^{k-1}\begin{pmatrix} 0 & (-1)^{k-1}s_{23}\\ s_{13} & 0\end{pmatrix}\,,\quad k\geq 2\,,\end{aligned} \tag{5.42}$$

and

$$\begin{aligned} r_3^{\mathrm{cha}}(\boldsymbol{\epsilon}_2^0) &= r_3^{\mathrm{adm}}(\boldsymbol{\epsilon}_2^0) = \frac{1}{\eta_3^2}\begin{pmatrix} s_{13} & s_{23}\\ s_{13} & s_{23}\end{pmatrix} + \left(2\zeta_2 s_{123} - \frac{1}{2}(s_{13}+s_{23})\partial^2_{\eta_3}\right)\mathbb{1}_2\,,\\ r_3^{\mathrm{cha}}(\boldsymbol{\epsilon}_2^k) &= r_3^{\mathrm{adm}}(\boldsymbol{\epsilon}_2^k) = \eta_3^{k-2}\begin{pmatrix} s_{13} & 0\\ 0 & s_{23}\end{pmatrix}\,,\quad k\geq 4\,.\end{aligned} \tag{5.43}$$

Type-$(4,2)$ integrals: $\boldsymbol{Z}^\tau_{4,2}$ and $\boldsymbol{S}^\tau_{4,2}$

The type-$(4,2)$ integrals are relevant for the genus-one, three-point amplitudes. They define the vectors

$$\begin{aligned}\boldsymbol{Z}^\tau_{4,2}(z_2) &= \int_0^{z_2} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_4\ \mathrm{KN}^\tau_{1234} \begin{pmatrix} \varphi^\tau(1,3,4)\varphi^\tau(2) \\ \varphi^\tau(1,4,3)\varphi^\tau(2) \\ \varphi^\tau(1,3)\varphi^\tau(2,4) \\ \varphi^\tau(1,4)\varphi^\tau(2,3) \\ \varphi^\tau(1)\varphi^\tau(2,3,4) \\ \varphi^\tau(1)\varphi^\tau(2,4,3) \end{pmatrix} \\ &= \int_0^{x_2} \mathrm{d}x_3 \int_0^{x_3} \mathrm{d}x_4\ \mathrm{KN}^\tau_{1234} \begin{pmatrix} F_{13}(\eta_{34})F_{34}(\eta_4) \\ F_{14}(\eta_{34})F_{43}(\eta_3) \\ F_{13}(\eta_3)F_{24}(\eta_4) \\ F_{14}(\eta_4)F_{23}(\eta_3) \\ F_{23}(\eta_{34})F_{34}(\eta_4) \\ F_{24}(\eta_{34})F_{43}(\eta_3) \end{pmatrix}\end{aligned} \tag{5.44}$$

and

$$\begin{aligned}\boldsymbol{S}^\tau_{4,2}(z_2) &= \int_0^{z_2} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_4\ \mathrm{KN}^\tau_{1234} \begin{pmatrix} \varphi^\tau\left(\begin{smallmatrix}3&4\\1&1\end{smallmatrix}\right) \\ \varphi^\tau\left(\begin{smallmatrix}3&4\\1&2\end{smallmatrix}\right) \\ \varphi^\tau\left(\begin{smallmatrix}3&4\\1&3\end{smallmatrix}\right) \\ \varphi\left(\begin{smallmatrix}3&4\\2&1\end{smallmatrix}\right) \\ \varphi^\tau\left(\begin{smallmatrix}3&4\\2&2\end{smallmatrix}\right) \\ \varphi^\tau\left(\begin{smallmatrix}3&4\\2&3\end{smallmatrix}\right) \end{pmatrix} \\ &= \int_0^{z_2} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_4\ \mathrm{KN}^\tau_{1234} \begin{pmatrix} F_{13}(\eta_3)F_{14}(\eta_4) \\ F_{13}(\eta_3)F_{24}(\eta_4) \\ F_{13}(\eta_3)F_{34}(\eta_4) \\ F_{23}(\eta_3)F_{14}(\eta_4) \\ F_{23}(\eta_3)F_{24}(\eta_4) \\ F_{23}(\eta_3)F_{34}(\eta_4) \end{pmatrix} .\end{aligned} \tag{5.45}$$

Repeating the calculation (4.77) with the genus-zero admissible and chain products φ replaced by the genus-one products φ^τ, the transformation matrix in

$$\boldsymbol{S}^\tau_{4,2} = \boldsymbol{B}^\tau_{4,2}\,\boldsymbol{Z}^\tau_{4,2} \tag{5.46}$$

turns out to be exactly the same as in the genus-zero calculation,

$$\boldsymbol{B}^\tau_{4,2} = \boldsymbol{B}_{6,4}\,, \tag{5.47}$$

in agreement with eq. (5.35). The explicit elliptic KZB system on the twice-punctured torus satisfied by $\boldsymbol{Z}^{\tau}_{4,2}$ and the matrices appearing therein can be found in ref. [3].

5.3 Genus-one, open-string recursion

Having introduced the above classes of integrals, the recursion from refs. [2, 3] can be conveniently formulated and the schematic eqs. (1.7) and (3.126) can be written out properly. Ultimately, it relates the $(n-1)$-point, genus-one Z^{τ}_{n-1}-integrals to the $(n+1)$-point, genus-zero Z_{n+1}-integrals. This recursion is the genus-one analogue of the recursion from section 4.3 and based on the genus-one associator eq. (3.123) involving the elliptic KZB associator. It can either be formulated in terms of the admissible integrals [2] or the chain integrals [3], in this thesis, the latter approach is presented. The corresponding translation between the two equivalent methods follows from the results in subsection 5.2.2.

5.3.1 Elliptic KZB system on the twice-punctured torus

Similar to the genus-zero recursion, instead of working with the genus-one integrals of type $(n-1, 1)$, which yield according to eq. (5.22) the $(n-1)$-point, open-string integrals at genus one, they are augmented by an additional unintegrated puncture in refs. [2, 3]. Thus, the $d_{n,2} = (n-1)!$ genus-one chain integrals of type $(n, 2)$ are considered,

$$\begin{aligned} Z^{\tau}_{n,2}\left((1, A^1), (2, A^2)\right) &= Z^{\tau}_{n,2}\left((1, A^1), (2, A^2); z_2; \vec{\eta}; \{s_{ij}\}\right) \\ &= \int_{\Delta_{n,2}} \prod_{i=3}^{n} \mathrm{d}z_i \ \mathrm{KN}^{\tau}_{12\ldots n} \ \varphi^{\tau}(1, A^1)\varphi^{\tau}(2, A^2)\,, \end{aligned} \tag{5.48}$$

where

$$(A^1, A^2) = \sigma(3, 4, \ldots, n) \tag{5.49}$$

is a partition into two possibly empty subsequences with $\sigma \in S_{n-2}$ acting on $3, 4, \ldots, n$. The integration domain

$$\Delta_{n,2} = \Delta_{n,2}(z_2) = \{0 < z_n < z_{n-1} < \cdots < z_3 < z_2\}\,, \tag{5.50}$$

cf. figure 5.3, does not cover the whole A-cycle of the torus and is parametrised by the additional unintegrated puncture $0 < z_2 < 1$.

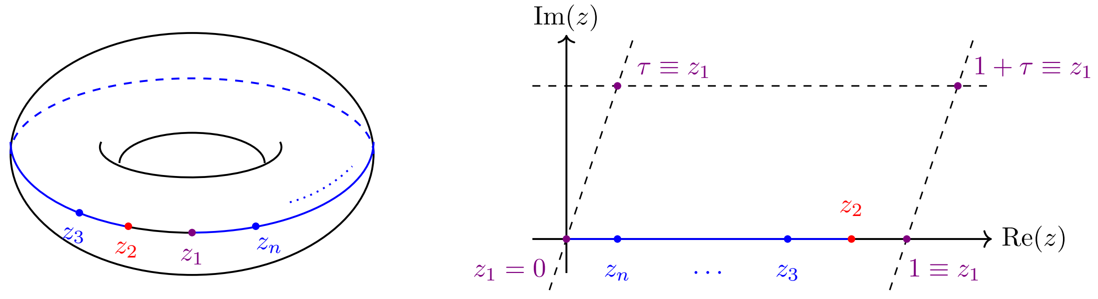


Figure 5.3: The twice-punctured torus associated to the integrals $Z^\tau_{n,2}$ from eq. (5.48). The two unintegrated (red and violet) punctures are z_1 and z_2. Its parametrisation on the complex plane is depicted on the right-hand side. The $n-2$ integrated (blue) punctures $z_3, \ldots, z_n$ define the integration cycle $\Delta_{n,2}(z_i)$ (blue line) from eq. (5.50) and the integrands of $Z^\tau_{n,2}$ are defined on the n-punctured torus (violet, red and blue punctures). The unintegrated (violet) puncture $z_1 = 0$ is canonically fixed, while the second unintegrated (red) puncture z_2 can be varied on the A-cycle $0 = z_1 < z_2 < 1 \equiv z_1$.

The vector $\boldsymbol{Z}^\tau_{n,2}(z_2)$ formed by these integrals satisfies an elliptic KZB system on the twice-punctured torus [3]

$$\partial_{z_2} \boldsymbol{Z}^\tau_{n,2}(z_2) = \left(\sum_{k\geq 0} r^{\text{cha}}_n(\boldsymbol{x}^{21,k}_2) g^{(k)}_{21} \right) \boldsymbol{Z}^\tau_{n,2}(z_2) \, , \tag{5.51}$$

$$\begin{aligned} 2\pi i \partial_\tau \boldsymbol{Z}^\tau_{n,2}(z_2) = \Bigg(&- r^{\text{cha}}_n(\boldsymbol{\epsilon}^0_2) + \sum_{k\geq 4} (1-k)\, \mathrm{G}_k \, r^{\text{cha}}_n(\boldsymbol{\epsilon}^k_2) \\ &+ \sum_{k\geq 2} (k-1) r^{\text{cha}}_n(\boldsymbol{x}^{21,k-1}_2) g^{(k)}_{21} \Bigg) \boldsymbol{Z}^\tau_{n,2}(z_2) \, , \end{aligned} \tag{5.52}$$

which follows from eq. (5.21). The matrices $r^{\text{cha}}_n(\boldsymbol{x}^{21,k}_2)$ and $r^{\text{cha}}_n(\boldsymbol{\epsilon}^k_2)$ are explicitly determined by the partial derivatives (C.67) and (C.74) derived in appendix C.3. They are homogeneous of degree one in the Mandelstam variables and, thus, linear in α':

$$\deg_{\alpha'}(r^{\text{cha}}_n(\boldsymbol{x}^{21,k}_2)) = 1 \, . \tag{5.53}$$

Counting the k-th derivative with respect to an auxiliary variable $\vec{\eta}$ as degree minus k, $r^{\text{cha}}_n(\boldsymbol{x}^{21,k}_2)$ and $r^{\text{cha}}_n(\boldsymbol{\epsilon}^k_2)$ are homogeneous of degree $k-1$ and $k-2$, respectively, in the auxiliary variables $\vec{\eta}$ appearing in the chains of Eisenstein–Kronecker series φ^τ. Again, one exception is the diagonal of $r^{\text{cha}}_n(\boldsymbol{\epsilon}^0_2)$, which contains factors of ζ_2.

In particular, eq. (5.51) is an elliptic KZB equation of the form (3.112), such that the whole machinery from section 3.3 can be applied: the regularised boundary values $C_0^\tau(\boldsymbol{Z}_{n,2}^\tau)$ and $C_1^\tau(\boldsymbol{Z}_{n,2}^\tau)$ are related by the genus-one associator equation

$$C_1^\tau(\boldsymbol{Z}_{n,2}^\tau) = \Phi^\tau_{\mathcal{X}_n} C_0^\tau(\boldsymbol{Z}_{n,2}^\tau) \tag{5.54}$$

with the explicitly known alphabet $\mathcal{X}_n = \big(r_n^{\text{cha}}(\boldsymbol{x}_2^{21,0}), r_n^{\text{cha}}(\boldsymbol{x}_2^{21,1}), \dots\big)$. The second partial differential equation (5.52) with respect to τ will be used to calculate the boundary values. In the next two subsections, the lower boundary value $C_0^\tau(\boldsymbol{Z}_{n,2}^\tau)$ will be shown to contain the $(n{+}1)$-point, open-string integrals at genus zero and the upper boundary value $C_1^\tau(\boldsymbol{Z}_{n,2}^\tau)$ to contain the $(n{-}1)$-point, open-string integrals at genus one. Thus, using the linearity (5.53) and expanding the associator equation (5.54) in α', it can be used to calculate the α'-expansion of the open-string integrals at genus one from the open-string integrals at genus zero. The calculation only involves matrix operations of explicitly known matrices.

5.3.2 Lower boundary value

The asymptotic behaviour as $z_2 \to 0$ of the integrals in eq. (5.48) can be determined using the change of variables $z_i = z_2 x_i$ for $1 \leq i \leq n$, such that $x_1 = 0$ and $x_2 = 1$. Considering the logarithmic divergence of $\tilde{\Gamma}_{ij}$ in eq. (3.96), the Koba–Nielsen factor behaves as

$$\begin{aligned}
&\mathrm{KN}^\tau_{12\dots n}(z_1, \dots, z_n; \{s_{ij}\}) \\
&= (-2\pi i z_2)^{-s_{12\dots n}} e^{s_{12\dots n}\omega(1,0)} \prod_{1\leq i<j\leq n} |x_{ij}|^{s_{ij}} (1 + \mathcal{O}(z_2))\,,
\end{aligned} \tag{5.55}$$

such that in the following regularised limit, a $(n{+}1)$-point, genus-zero Koba–Nielsen factor (4.7) is recovered:

$$\begin{aligned}
&\lim_{z_2\to 0} (-2\pi i z_2)^{s_{12\dots n}}\, \mathrm{KN}^\tau_{12\dots n}(z_1, \dots, z_n; \{s_{ij}\}) \\
&= e^{s_{12\dots n}\omega(1,0)}\, \mathrm{KN}_{12\dots n}(x_1, \dots, x_n; \{s_{ij}\})\,.
\end{aligned} \tag{5.56}$$

The corresponding $n{+}1$ punctures on the Riemann sphere are

$$x_1 = 0 < x_n < x_{n-1} < \dots < x_2 = 1 < x_{n+1} = \infty\,, \tag{5.57}$$

where an additional puncture x_{n+1} at infinity has appeared. Geometrically, by rescaling $z_i = z_2 x_i$ and letting $z_2 \to 0$, the lattice points $\mathbb{Z} + \tau\mathbb{Z}$ of the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ split into the two punctures $x_1 = 0$ and $x_{n+1} = \infty$ on the Riemann sphere: the lattice point at the origin becomes x_1 and stays at the origin, while all the other lattice points merge to infinity, which results in the puncture x_{n+1}, cf. figure 5.4.

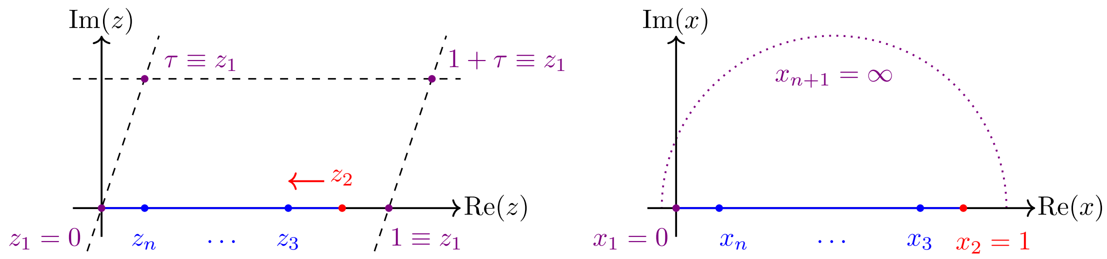


Figure 5.4: The change of variables $z_i = z_2 x_i$ and the limit $z_2 \to 0$ degenerate the unintegrated (red and violet) punctures z_1 and z_2 in the integration domain $\Delta_{n,2}(z_i)$ (blue line) on the left-hand side to the two unintegrated (red and violet) punctures x_1 and $x_2 = 1$ in the integration domain $\Delta_{n+1,3}(x_i)$ (blue line) on the right-hand side. The other representatives of the congruence class of the puncture $z_1 + \mathbb{Z} + \tau\mathbb{Z}$ on the torus degenerate in this limit to infinity, which yields the third unintegrated (violet) puncture x_{n+1}. Hence, a three-punctured Riemann sphere is recovered with the canonically fixed (violet and red) punctures $(x_1, x_2, x_{n+1}) = (0, 1, \infty)$. Correspondingly, the Mandelstam variables s_{i2} associated to the auxiliary puncture z_2 is the genus-zero Mandelstam variable of the puncture $x_2 = 1$.

The asymptotic behaviour of the differential form in the integral from eq. (5.48) without the Koba–Nielsen factor is governed by the simple pole of the integration kernel

$$g^{(k)}(z,\tau) = \begin{cases} \frac{1}{z} + \mathcal{O}(z) & \text{if } k = 1\,, \\ \mathcal{O}(1) & \text{otherwise,} \end{cases} \tag{5.58}$$

for $k = 1$, such that

$$F(z,\eta,\tau) = \frac{1}{z} + \mathcal{O}(1)\,. \tag{5.59}$$

These poles ensure that the genus-zero differential forms, e.g. the Parke–Taylor forms (4.9), are recovered for $z_2 \to 0$: the factor z_2^{n-2} from the change of variables $z_i = z_2 x_i$ in the integration measure can only be compensated by the simple poles in the Eisenstein–Kronecker series in the chains φ^τ. This leads to the genus-zero chains φ with respect to the new variables x_i, i.e.

$$\varphi^\tau(1, A^1)\varphi^\tau(2, A^2) \prod_{k=3}^{n} \mathrm{d}z_k = \varphi(1, A^1)\varphi(2, A^2) \prod_{k=3}^{n} \mathrm{d}x_k\,. \tag{5.60}$$

To conclude, the genus-one string integral in eq. (5.48) has the following asymptotic behaviour

$$\begin{aligned}
&\lim_{z_2\to 0}(-2\pi i z_2)^{s_{12\ldots n}} Z^{\tau}_{n,2}\left((1,A^1),(2,A^2);z_2;\vec{\eta};\{s_{ij}\}\right)\\
&= e^{s_{12\ldots n}\omega(1,0)}\int_{\Delta_{n,2}}\prod_{k=3}^{n}\mathrm{d}x_k\ \mathrm{KN}_{12\ldots n}\ \varphi(1,A^1)\varphi(2,A^2)\\
&= e^{s_{12\ldots n}\omega(1,0)} Z_{n+1,2}\left((1,A^1),(2,A^2);\{s_{ij}\}\right)\ ,
\end{aligned} \tag{5.61}$$

where $Z_{n+1,2}$ is a genus-zero chain integral of type $(n{+}1,2)$ as defined in eq. (4.51). Using integration by parts and partial fractioning, it can be written as a linear combination of such integrals with $A^1=\varnothing$. Therefore, the vector $\boldsymbol{Z}^{\tau}_{n,2}$ degenerates to the basis vector $\boldsymbol{Z}_{n+1,2}$ from eq. (4.54) of the genus-zero chain integrals of type $(n{+}1,2)$ with $A^1=\varnothing$, which in turn is nothing but the vector $\boldsymbol{Z}_{n+1}$ of genus-zero Z_{n+1}-integrals from eq. (4.16):

$$\begin{aligned}
\lim_{z_2\to 0}(-2\pi i z_2)^{s_{12\ldots n}}\boldsymbol{Z}^{\tau}_{n,2}(z_2;\vec{\eta};\{s_{ij}\}) &= e^{s_{12\ldots n}\omega(1,0)}\boldsymbol{U}_{0,n+1}\,\boldsymbol{Z}_{n+1,2}(\{s_{ij}\})\\
&= e^{s_{12\ldots n}\omega(1,0)}\boldsymbol{U}_{0,n+1}\,\boldsymbol{Z}_{n+1}(\{s_{ij}\})\,,
\end{aligned} \tag{5.62}$$

where $\boldsymbol{U}_{0,n+1}$ is a $(n{-}1)!\times(n{-}2)!$-dimensional matrix implementing the integration-by-parts and partial-fractioning identities among $Z_{n+1,2}\left((1,A^1),(2,A^2);\{s_{ij}\}\right)$ with $A^1\neq\varnothing$ and $A^1=\varnothing$. Moreover, the vector $\boldsymbol{Z}_{n+1}(\{s_{ij}\})$ is related to the $(n{+}1)$-point genus-zero string corrections by the invertible transformation in eq. (4.17).

In the remainder of this subsection, it is shown that the regularised boundary value

$$C^{\tau}_0(\boldsymbol{Z}^{\tau}_{n,2}) = \lim_{z_2\to 0}(-2\pi i z_2)^{-r^{\mathrm{cha}}_n(\boldsymbol{x}^{21,1}_2)}\boldsymbol{Z}^{\tau}_{n,2}(z_2;\vec{\eta};\{s_{ij}\}) \tag{5.63}$$

indeed degenerates to the limit in eq. (4.54), such that

$$C^{\tau}_0(\boldsymbol{Z}^{\tau}_{n,2}) = e^{s_{12\ldots n}\omega(1,0)}\boldsymbol{U}_{0,n+1}\,\boldsymbol{Z}_{n+1}(\{s_{ij}\})\,, \tag{5.64}$$

reproducing the $(n{+}1)$-point genus-zero string integrals. This is where the partial differential equation (5.52) with respect to τ is used. Equation (5.64) can be shown by proving the following eigenvalue equations satisfied by the matrix $\boldsymbol{U}_{0,n+1}$ [3]:

$$\begin{aligned}
r^{\mathrm{cha}}_n(\boldsymbol{x}^{21,1}_2)\boldsymbol{U}_{0,n+1} &= -s_{12\ldots n}\boldsymbol{U}_{0,n+1}\,,\\
r^{\mathrm{cha}}_n(\boldsymbol{\epsilon}^{0}_2)\boldsymbol{U}_{0,n+1} &= 2\zeta_2\, s_{12\ldots n}\boldsymbol{U}_{0,n+1}\,,\\
\left(r^{\mathrm{cha}}_n(\boldsymbol{\epsilon}^{k}_2)+r^{\mathrm{cha}}_n(\boldsymbol{x}^{21,k-1}_2)\right)\boldsymbol{U}_{0,n+1} &= 0\,,\qquad k\geq 4\,.
\end{aligned} \tag{5.65}$$

The first of these equations shows that the exponent $-r^{\mathrm{cha}}_n(\boldsymbol{x}^{21,1}_2)$ in the regularisation in eq. (5.63) projects out the appropriate eigenvalue $s_{12\ldots n}$ in the exponent of the regulating factor in eq. (5.62). The proof of the eigenvalue equations (5.65) makes use of the continuity of $(-2\pi i z_2)^{s_{12\dots n}} 2\pi i \partial_\tau \boldsymbol{Z}^\tau_{n,2}$ at $z_2 = 0$ based on the absence of singular terms in the partial differential equation (5.52). This allows to interchange the order of taking the limit $z_2 \to 0$ and the derivative $2\pi i \partial_\tau$ of $(-2\pi i z_2)^{s_{12\dots n}} \boldsymbol{Z}^\tau_{n,2}$, which leads to the two equations

$$\begin{aligned}
&\lim_{z_2\to 0} (-2\pi i z_2)^{s_{12\dots n}} 2\pi i \partial_\tau \boldsymbol{Z}^\tau_{n,2}(z_2) \\
&= \Bigg(- r_n^{\text{cha}}(\boldsymbol{\epsilon}_2^0) - \mathrm{G}_2\, r_n^{\text{cha}}(\boldsymbol{x}_2^{21,k-1}) + \sum_{k\geq 4} (1-k)\, \mathrm{G}_k \left(r_n^{\text{cha}}(\boldsymbol{\epsilon}_2^k) + r_n^{\text{cha}}(\boldsymbol{x}_2^{21,k-1}) \right) \Bigg) \\
&\quad e^{s_{12\dots n}\omega(1,0)} \boldsymbol{U}_{0,n+1}(\{s_{ij}\})\, \boldsymbol{Z}_{n+1}(\{s_{ij}\})
\end{aligned} \tag{5.66}$$

and

$$\begin{aligned}
&2\pi i \partial_\tau \lim_{z_2\to 0} (-2\pi i z_2)^{s_{12\dots n}} \boldsymbol{Z}^\tau_{n,2}(z_2) \\
&= s_{12\dots n} (\mathrm{G}_2 - 2\zeta_2) e^{s_{12\dots n}\omega(1,0)} \boldsymbol{U}_{0,n+1}(\{s_{ij}\})\, \boldsymbol{Z}_{n+1}(\{s_{ij}\})\,,
\end{aligned} \tag{5.67}$$

respectively, where for $k \geq 2$ the relation

$$\lim_{z_2\to 0} g_{21}^{(k)} = -\,\mathrm{G}_k \tag{5.68}$$

among the Eisenstein series G_k and the integration kernels $g^{(k)}$ has been used. Requiring that both lead to the same result and comparing the coefficients of G_k indeed yields the eigenvalue equations (5.65).

Example for $\boldsymbol{Z}^\tau_{3,2}$

The above results can nicely be exemplified on the type-$(3,2)$ basis vector from eq. (5.39). The limit (5.62) results in

$$\begin{aligned}
\lim_{z_2\to 0} (-2\pi i z_2)^{s_{123}} \boldsymbol{Z}^\tau_{3,2}(z_2) &= e^{s_{123}\omega(1,0)} \int_0^1 \mathrm{d}x_3\, \mathrm{KN}_{123} \begin{pmatrix} \frac{1}{x_{13}} \\ \frac{1}{x_{23}} \end{pmatrix} \\
&= e^{s_{123}\omega(1,0)} \begin{pmatrix} Z_{4,2}((1,3),(2);z_2) \\ Z_{4,2}((1),(2,3);z_2) \end{pmatrix} \\
&= e^{s_{123}\omega(1,0)} \begin{pmatrix} -\frac{s_{23}}{s_{13}} \\ 1 \end{pmatrix} Z_4(\{s_{ij}\})\,,
\end{aligned} \tag{5.69}$$

where Z_4 is the genus-zero chain integral that is proportional to the Veneziano amplitude in eq. (4.8):

$$Z_4(\{s_{ij}\}) = -\frac{1}{s_{23}} F_4(\alpha') = -\frac{1}{s_{23}} \frac{\Gamma(1-s_{13})\Gamma(1-s_{23})}{\Gamma(1-s_{13}-s_{23})}\,. \tag{5.70}$$

Hence, the lower boundary value from eq. (5.64) evaluates to

$$C_0^\tau(\boldsymbol{Z}_{3,2}^\tau) = e^{s_{123}\omega(1,0)}\boldsymbol{U}_{0,4}\left(-\frac{1}{s_{23}}\frac{\Gamma(1-s_{13})\Gamma(1-s_{23})}{\Gamma(1-s_{13}-s_{23})}\right), \tag{5.71}$$

where the genus-zero, four-point integration-by-parts matrix is given by

$$\boldsymbol{U}_{0,4} = \begin{pmatrix} -\frac{s_{23}}{s_{13}} \\ 1 \end{pmatrix}. \tag{5.72}$$

5.3.3 Upper boundary value

The upper boundary value $C_1^\tau(\boldsymbol{Z}_{n,2}^\tau)$ as $z_2 \to 1$ can be calculated analogously. First, the Koba–Nielsen factor has the following asymptotic behaviour

$$\begin{aligned}&\mathrm{KN}_{12\ldots n}^\tau(z_1,\ldots,z_n;\{s_{ij}\})\\&= (-2\pi i(1-z_2))^{-s_{12}}e^{s_{12}\omega(1,0)}\\&\quad\prod_{\substack{i,j\in\{1,3,4,\ldots,n\}\\ i<j}}\exp\left(-\tilde{s}_{ij}\left(\tilde{\Gamma}_{ij}-\omega(1,0)\right)\right)(1+\mathcal{O}(1-z_2)),\end{aligned} \tag{5.73}$$

where for $i,j\in\{1,3,4,\ldots,n\}$

$$\tilde{s}_{ij} = \begin{cases} s_{ij} & \text{if } i<j \text{ and } i\neq 1,\\ s_{1j}+s_{2j} & \text{if } i<j \text{ and } i=1,\\ \tilde{s}_{ji} & \text{if } i>j. \end{cases} \tag{5.74}$$

Thus, a $(n-1)$-point, genus-one Koba–Nielsen factor with effective Mandelstam variables $\tilde{s}_{ij}$ is obtained,

$$\begin{aligned}&\lim_{z_2\to 1}(-2\pi i(1-z_2))^{s_{12}}\,\mathrm{KN}_{12\ldots n}^\tau(z_1,\ldots,z_n;\{s_{ij}\})\\&= e^{s_{12}\omega(1,0)}\,\mathrm{KN}_{134\ldots n}^\tau(z_1,z_3,z_4,\ldots,x_n;\{\tilde{s}_{ij}\}),\end{aligned} \tag{5.75}$$

which is associated to the $(n-1)$ distinct punctures on the torus

$$z_1 = 0 < z_n < z_{n-1} < \cdots < z_2 = 1 \equiv 0 = z_1, \tag{5.76}$$

cf. figure 5.5.

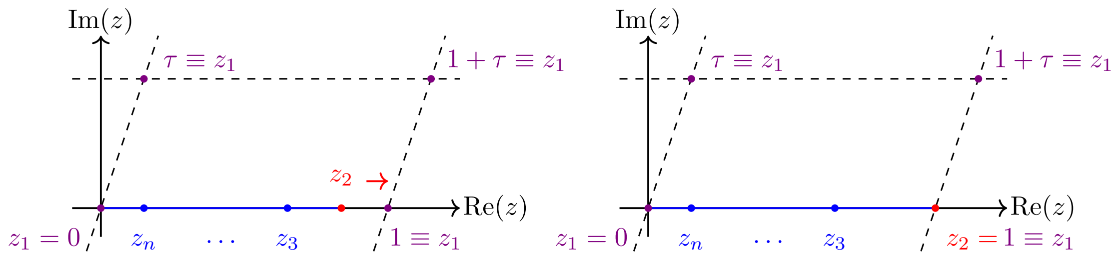


Figure 5.5: In the limit $z_2 \to 1$, the (representatives of the) two unintegrated (red and violet) punctures z_1 and z_2 in the integration domain $\Delta_{n,2}(z_i)$ (blue line) on the left-hand side merge to the single puncture $z_1 + \mathbb{Z} + \tau\mathbb{Z}$ on the torus. The result is that the integration domain $\Delta_{n,2}(z_i)$ (blue line) with $z_2 = 1$ on the right-hand side, given in eq. (5.76), closes on the A-cycle. Therefore, the effective genus-one Mandelstam variable $\tilde{s}_{1j} = s_{1j} + s_{2j}$ associated to the puncture $z_1 + \mathbb{Z} + \tau\mathbb{Z}$ acquires an additional contribution to s_{1j} from the genus-zero Mandelstam variable s_{2j}. This summation of the genus-zero Mandelstam variables to the effective genus-one Mandelstam variables motivates the geometric interpretation mentioned below eq. (5.99) and depicted in figure 1.4.

Second, due to the one-periodicity of the Eisenstein–Kronecker series and the shuffle identity (4.12) of chain products, the product of Eisenstein–Kronecker chains in $Z^\tau_{n,2}$ from eq. (5.48) merge to a linear combination of single chains

$$\lim_{z_2\to 1} \varphi^\tau(1, A^1)\varphi^\tau(2, A^2) = \varphi^\tau(1, A^1 \shuffle A^2)\,. \tag{5.77}$$

Therefore, the following asymptotic behaviour of the genus-one chain integrals of type $(n, 2)$ is observed[7]

$$\begin{aligned}
&\lim_{z_2\to 1}(-2\pi i(1-z_2))^{s_{12}} Z^\tau_{n,2}\left((1, A^1), (2, A^2); z_2; \vec{\eta}; \{s_{ij}\}\right)\\
&= e^{s_{12}\omega(1,0)} \int_{\Delta_{n,2}} \prod_{i=3}^{n} \mathrm{d}z_i\ \mathrm{KN}^\tau_{134\dots n}(\{\tilde{s}_{ij}\})\,\varphi^\tau(1, A^1 \shuffle A^2)\\
&= e^{s_{12\dots n}\omega(1,0)} Z^\tau_{n-1,1}\left((1, A^1 \shuffle A^2); \{\tilde{s}_{ij}\}\right)\,,
\end{aligned} \tag{5.78}$$

which yields a linear combination of type-$(n-1, 1)$ integrals, cf. eq. (5.17). Thus, in this regularised limit the vector $\boldsymbol{Z}^\tau_{n,2}$ degenerates to the basis vector $\boldsymbol{Z}^\tau_{n-1,1}$ of genus-one chain integrals of type $(n-1, 1)$, which in turn is the vector of $(n-1)$-point,

[7] It is assumed that $\mathrm{Re}(s_{12}) < 0$ is sufficiently small, such that for any consecutive points $z_{i_1}, \dots, z_{i_p}$ on the A-cycle, the following inequality is satisfied: $\mathrm{Re}(s_{i_1\dots i_p}) < \mathrm{Re}(s_{12}) < 0$. This ensures that no rest term appear in this calculation, cf. appendix E in [3]. This assumption holds without loss of generality, since s_{12} is only an auxiliary parameter associated to the additional momentum from the puncture at the point z_2. In the recursion below, it will merge with the genus-one momentum of z_1 and, therefore, only implements an artificial splitting of the effective genus-one momentum at z_1.

genus-one string integrals $\boldsymbol{Z}^{\tau}_{n}$, cf. eq. (5.23). They are related according to

$$\begin{aligned}\lim_{z_2\to 1}(-2\pi i(1-z_2))^{s_{12}}\boldsymbol{Z}^{\tau}_{n,2}(z_2;\vec{\eta};\{s_{ij}\}) &= e^{s_{12}\omega(1,0)}\boldsymbol{U}_{1,n-1}\,\boldsymbol{Z}^{\tau}_{n-1,1}(\vec{\eta};\{\tilde{s}_{ij}\})\\ &= e^{s_{12}\omega(1,0)}\boldsymbol{U}_{1,n-1}\,\boldsymbol{Z}^{\tau}_{n-1}(\vec{\eta};\{\tilde{s}_{ij}\})\,, \qquad (5.79)\end{aligned}$$

where $\boldsymbol{U}_{1,n-1}$ is a $(n-1)!\times(n-2)!$-dimensional matrix implementing the linear combinations in the shuffle $Z^{\tau}_{n-1,1}\left((1,A^1\sqcup A^2);\{\tilde{s}_{ij}\}\right)$.

The upper boundary value

$$C^{\tau}_{1}(\boldsymbol{Z}^{\tau}_{n,2}) = \lim_{z_2\to 1}(-2\pi i(1-z_2))^{-r^{\mathrm{adm}}_{n}(\boldsymbol{x}^{21,1}_{2})}\boldsymbol{Z}^{\tau}_{n,2}(z_2;\vec{\eta};\{s_{ij}\}) \qquad (5.80)$$

degenerates to the limit in eq. (5.79), i.e.

$$C^{\tau}_{1}(\boldsymbol{Z}^{\tau}_{n,2}) = e^{s_{12}\omega(1,0)}\boldsymbol{U}_{1,n-1}\,\boldsymbol{Z}^{\tau}_{n-1}(\vec{\eta};\{\tilde{s}_{ij}\})\,, \qquad (5.81)$$

which is a consequence of the first of the following eigenvalue equations of the matrix $\boldsymbol{U}_{1,n-1}$ [3]:

$$\begin{aligned} r^{\mathrm{adm}}_{n}(\boldsymbol{x}^{21,1}_{2})\boldsymbol{U}_{1,n-1} &= -s_{12}\boldsymbol{U}_{1,n-1}\,,\\ r^{\mathrm{adm}}_{n}(\boldsymbol{\epsilon}^{0}_{2})\boldsymbol{U}_{1,n-1} &= \boldsymbol{U}_{1,n-1}\left(2\zeta_2\,s_{12}+r^{\mathrm{adm}}_{n-1}(\boldsymbol{\epsilon}^{0}_{1})|_{\tilde{s}_{ij}}\right)\,,\\ \left(r^{\mathrm{adm}}_{n}(\boldsymbol{\epsilon}^{k}_{2})+r^{\mathrm{adm}}_{n}(\boldsymbol{x}^{21,k-1}_{2})\right)\boldsymbol{U}_{1,n-1} &= \boldsymbol{U}_{1,n-1}r^{\mathrm{adm}}_{n-1}(\boldsymbol{\epsilon}^{k}_{1})|_{\tilde{s}_{ij}}\,, \qquad k\geq 4\,, \qquad (5.82)\end{aligned}$$

where $r^{\mathrm{adm}}_{n-1}(\boldsymbol{\epsilon}^{k}_{1})|_{\tilde{s}_{ij}}$ are the matrices appearing in the partial differential equation (5.21) with respect to τ of the genus-one chain integrals $Z_{n-1,1}\left((1,A^1);\{\tilde{s}_{ij}\}\right)$ of type $(n-1,1)$ with the effective Mandelstam variables $\tilde{s}_{ij}$. These matrices also appear in the KZB system of refs. [38, 39]. Similar to the lower boundary value, the first equation ensures that also in the upper boundary value, the exponent $-r^{\mathrm{adm}}_{n}(\boldsymbol{x}^{21,1}_{2})$ in the regularisation in eq. (5.80) projects out the appropriate eigenvalue s_{12} in the exponent of the regulating factor in eq. (5.79). The proof is based on the continuity of $(-2\pi i(1-z_2))^{s_{12}}2\pi i\partial_\tau\boldsymbol{Z}^{\tau}_{n,2}$ at $z_2=1$, such that the order of taking the limit $z_2\to 1$ and the derivative $2\pi i\partial_\tau$ of $(-2\pi i(1-z_2))^{s_{12}}\boldsymbol{Z}^{\tau}_{n,2}$ can be interchanged. This leads to the two equations

$$\begin{aligned}&\lim_{z_2\to 1}(-2\pi i(1-z_2))^{s_{12}}2\pi i\partial_\tau\boldsymbol{Z}^{\tau}_{n,2}(z_2)\\ &=\left(-r^{\mathrm{cha}}_{n}(\boldsymbol{\epsilon}^{0}_{2})-\mathrm{G}_2\,r^{\mathrm{cha}}_{n}(\boldsymbol{x}^{21,k-1}_{2})+\sum_{k\geq 4}(1-k)\,\mathrm{G}_k\left(r^{\mathrm{cha}}_{n}(\boldsymbol{\epsilon}^{k}_{2})+r^{\mathrm{cha}}_{n}(\boldsymbol{x}^{21,k-1}_{2})\right)\right)\\ &\quad e^{s_{12}\omega(1,0)}\boldsymbol{U}_{1,n-1}\,\boldsymbol{Z}^{\tau}_{n-1}(\{s_{ij}\}) \qquad (5.83)\end{aligned}$$

and

$$\begin{aligned}
&2\pi i\partial_\tau \lim_{z_2\to 0}(-2\pi i z_2)^{s_{12\dots n}}\boldsymbol{Z}^\tau_{n,2}(z_2)\\
&=\boldsymbol{U}_{1,n-1}\Bigg(-\left(r^{\text{cha}}_{n-1}(\boldsymbol{\epsilon}^0_1)|_{\tilde{s}_{ij}}+2\zeta_1 s_{12}\right)+s_{12}\,\mathrm{G}_2+\sum_{k\geq 4}(1-k)\,\mathrm{G}_k\,r^{\text{cha}}_{n-1}(\boldsymbol{\epsilon}^k_1)|_{\tilde{s}_{ij}}\Bigg)\\
&\quad e^{s_{12}\omega(1,0)}\boldsymbol{Z}^\tau_{n-1}(\{s_{ij}\})\,, \qquad (5.84)
\end{aligned}$$

which follows from the system of differential equations (5.21). Both results can be equated and the corresponding coefficients of G_k compared, leading to the eigenvalue equations (5.82).

In order to extract the string integrals $\boldsymbol{Z}^\tau_{n-1}$ from the boundary value $C^\tau_1(\boldsymbol{Z}^\tau_{n,2})$ in eq. (5.81) a projection can be constructed as follows: the first $(n-2)!$ and last $(n-2)!$ column vectors of the basis transformation $\boldsymbol{U}_n$ which diagonalises the matrix $r^{\text{cha}}_n(\boldsymbol{x}^{21,1}_2)$ are determined by the the eigenvalue equations (5.82) and (5.65). They are the $(n-1)!\times(n-2)!$-dimensional matrices $\boldsymbol{U}_{1,n-1}$ and $\boldsymbol{U}_{0,n+1}$, respectively:

$$\begin{aligned}
r^{\text{cha}}_n(\boldsymbol{x}^{21,1}_2)&=\boldsymbol{U}_n\,\mathrm{diag}(\underbrace{-s_{12},\dots,-s_{12}}_{(n-2)!},\dots,\underbrace{-s_{12\dots n},\dots,-s_{12\dots n}}_{(n-2)!})\,\boldsymbol{U}^{-1}_n\,,\\
\boldsymbol{U}_n&=\begin{pmatrix}\boldsymbol{U}_{1,n-1} & \dots & \boldsymbol{U}_{0,n+1}\end{pmatrix}\,. \qquad (5.85)
\end{aligned}$$

Accordingly, the $(n-2)!$ first row vectors of $\boldsymbol{U}^{-1}_n$ are the dual vectors of the column vectors of $\boldsymbol{U}_{1,n-1}$, i.e.

$$\begin{aligned}
\boldsymbol{U}^{-1}_n&=\begin{pmatrix}\boldsymbol{P}_{1,n-1}\\ \vdots\end{pmatrix}\,,\\
\boldsymbol{P}_{1,n-1}\boldsymbol{U}_{1,n-1}&=\mathbb{1}_{(n-2)!}\,. \qquad (5.86)
\end{aligned}$$

Therefore, the $(n-2)!\times(n-1)!$-dimensional matrix $\boldsymbol{P}_{1,n-1}$ is the correct projection to extract the $(n-1)$-point, open-string integrals:

$$\boldsymbol{P}_{1,n-1}\,C^\tau_1(\boldsymbol{Z}^\tau_{n,2})=e^{s_{12}\omega(1,0)}\boldsymbol{Z}^\tau_{n-1}(\vec{\eta};\{\tilde{s}_{ij}\})\,. \qquad (5.87)$$

Example for $\boldsymbol{Z}^\tau_{3,2}$

The upper boundary value of the type-$(3,2)$ basis vector from eq. (5.39) leads to the two-point, open-string integral given in eq. (5.12): the limit (5.79) results in

$$\begin{aligned}
&\lim_{z_2\to 1}(-2\pi i(1-z_2))^{s_{12}}\boldsymbol{Z}^\tau_{3,2}(z_2;\vec{\eta};\{s_{ij}\})\\
&=e^{s_{12}\omega(1,0)}\int_0^1 \mathrm{d}z_3\;\mathrm{KN}^\tau_{13}(\tilde{s}_{13})\begin{pmatrix}F_{13}(\eta_3)\\ F_{13}(\eta_3)\end{pmatrix}\,, \qquad (5.88)
\end{aligned}$$

where we have used that for $z_2 = 1 \equiv 0 = z_1$, due to the one-periodicity, the two Eisenstein–Kronecker series $F_{23}(\eta_3) = F_{13}(\eta_3)$ agree and the Koba–Nielsen factor is given by

$$\mathrm{KN}^{\tau}_{13}(\tilde{s}_{13}) = \exp\left(-\tilde{s}_{13}\left(\tilde{\Gamma}_{13} - \omega(1,0)\right)\right)\,, \quad \tilde{s}_{13} = s_{12} + s_{13}\,. \tag{5.89}$$

A comparison with the two-point integral

$$Z^{\tau}_{2}(1,3;\tilde{s}_{13}) = \int_0^1 \mathrm{d}z_2\; \mathrm{KN}^{\tau}_{13}(\tilde{s}_{13})\, F_{13}(\eta_3) \tag{5.90}$$

from eq. (5.12) shows that it is indeed recovered in the upper boundary value (5.81):

$$C^{\tau}_{1}(\boldsymbol{Z}^{\tau}_{3,2}) = e^{s_{12}\omega(1,0)}\boldsymbol{U}_{1,2}\, Z^{\tau}_{2}(1,3;\tilde{s}_{13}) \tag{5.91}$$

and the two-point, genus-one shuffle matrix $\boldsymbol{U}_{1,2}$ is simply

$$\boldsymbol{U}_{1,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,. \tag{5.92}$$

Together with the four-point, genus-zero integration-by-parts matrix $\boldsymbol{U}_{0,4}$ from eq. (5.72), the eigenvalue decomposition (5.85) of the matrix $r^{\mathrm{cha}}_{3}(\boldsymbol{x}^{21,1}_{2})$ from eq. (5.42) is given by

$$\begin{pmatrix} -(s_{12}+s_{23}) & s_{23} \\ s_{13} & -(s_{12}+s_{13}) \end{pmatrix} = \boldsymbol{U}_3 \begin{pmatrix} -s_{12} & 0 \\ 0 & -s_{123} \end{pmatrix} \boldsymbol{U}_3^{-1}\,, \tag{5.93}$$

where

$$\boldsymbol{U}_3 = \begin{pmatrix} \boldsymbol{U}_{1,2} & \boldsymbol{U}_{0,4} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{s_{23}}{s_{13}} \\ 1 & 1 \end{pmatrix}\,. \tag{5.94}$$

Therefore, the inverse of the transformation matrix $\boldsymbol{U}_3$ is

$$\boldsymbol{U}_3^{-1} = \frac{1}{s_{13}+s_{23}} \begin{pmatrix} s_{13} & s_{23} \\ -s_{13} & s_{13} \end{pmatrix} = \begin{pmatrix} \boldsymbol{P}_{1,2} \\ \dots \end{pmatrix} \tag{5.95}$$

which includes the projection

$$\boldsymbol{P}_{1,2} = \frac{1}{s_{13}+s_{23}} \begin{pmatrix} s_{13} & s_{23} \end{pmatrix}\,, \tag{5.96}$$

from eq. (5.86), such that

$$\begin{aligned}\boldsymbol{P}_{1,2}\, C_1^\tau(\boldsymbol{Z}_{3,2}^\tau) &= \frac{1}{s_{13}+s_{23}} \begin{pmatrix} s_{13} & s_{23}\end{pmatrix} \begin{pmatrix}1\\1\end{pmatrix} e^{s_{12}\omega(1,0)} Z_2^\tau(1,3;\tilde{s}_{13})\\ &= e^{s_{12}\omega(1,0)} Z_2^\tau(1,3;\tilde{s}_{13}) \end{aligned} \tag{5.97}$$

indeed yields the two-point, open-string integral at genus one with effective Mandelstam variable $\tilde{s}_{13} = s_{12}+s_{13}$, in agreement with eq. (5.87).

5.3.4 Recursion in genus and number of external states

The genus-one recursion from refs. [2, 3] is based on the genus-one associator equation (5.54). Using the results (5.63) and (5.81) for the limits $C_0^\tau(\boldsymbol{Z}_{n,2}^\tau)$ and $C_1^\tau(\boldsymbol{Z}_{n,2}^\tau)$, respectively, as well as the projection $\boldsymbol{P}_{1,n-1}$ in eq. (5.87), the associator equation expresses the $(n-1)$-point, genus-one string integrals in terms of the $(n+1)$-point, genus-zero string integrals according to the *genus-one, open-string recursion*

$$\boldsymbol{Z}_{n-1}^\tau(\vec{\eta};\{\tilde{s}_{ij}\}) = e^{(s_{12\dots n}-s_{12})\omega(1,0)} \boldsymbol{P}_{1,n-1}\, \Phi_{\mathcal{X}_n}^\tau\, \boldsymbol{U}_{0,n+1}\, \boldsymbol{Z}_{n+1}(\{s_{ij}\})\,, \tag{5.98}$$

where $\mathcal{X}_n = (r_n^{\text{cha}}(\boldsymbol{x}_2^{21,0}), r_n^{\text{cha}}(\boldsymbol{x}_2^{21,1}),\dots)$ is the alphabet formed by the matrices determined by the closed formula (C.67). Due to the linearity (5.53) of the matrices $r_n^{\text{cha}}(\boldsymbol{x}_2^{21,k})$ in α', the α'-expansion of the elliptic KZB associator is given by its expansion in word length[8]

$$\Phi_{\mathcal{X}_n}^\tau = \sum_{l\geq 0}\ \sum_{m_1,\dots,m_l\geq 0} r_n^{\text{cha}}(\boldsymbol{x}_2^{21,m_1}\dots \boldsymbol{x}_2^{21,m_l})\omega(m_l,\dots,m_1)\,. \tag{5.99}$$

Equation (5.98) is the exact formulation of the genus-one, open-string recursion schematically described in eqs. (1.7) and (3.126). Geometrically, it can be interpreted as a gluing mechanism on the level of the worldsheets, depicted[9] in figure 1.4 [2]: the elliptic KZB associator effectively glues together two external states of the $(n+1)$-point, genus-zero worldsheet to form a $(n-1)$-point, genus-one worldsheet. The two external states glued together are the ones that correspond to the punctures z_1 and z_2, which follows from their merging in the upper limit $z_2 \to 1$ around the A-cycle, depicted in figure 5.5. This interpretation is supported by the form $\tilde{s}_{i1} = s_{i1}+s_{i2}$ of the effective genus-one Mandelstam variables associated to the puncture $z_1 \equiv z_2$ from eq. (5.74), where s_{i1} and s_{i2} are the genus-zero Mandelstam variables, cf. figure 5.4.

In practice, the recursion (5.98) can be employed to calculate the α'-expansion

[8] For notational simplicity, the action of r_n^{cha} on $\boldsymbol{x}_2^{21,k}$ is extended to an algebra homomorphism to the algebra generated by the symbols $\boldsymbol{x}_2^{21,k}$. This is in agreement with its interpretation in terms of a representation of the underlying algebra generated by $\boldsymbol{x}_2^{21,k}$

[9] With $\Phi_{n,1}(\alpha',\tau) = e^{(s_{12\dots n+1}-s_{12})\omega(1,0)} \boldsymbol{P}_{1,n}\, \Phi_{\mathcal{X}_{n+1}}^\tau\, \boldsymbol{U}_{0,n+2}$.

of the $(n-1)$-point, open-string corrections at genus one as follows:

- First, the minimal order in α' of the genus-zero string integrals is [30]

$$o_{\alpha'}^{\min}(\boldsymbol{Z}_{n+1}) = 2 - n\,. \tag{5.100}$$

- Second, the maximal order of the associator in eq. (5.99) up to words of maximal length $l_{\max}$ is simply $l_{\max}$, i.e.

$$o_{\alpha'}^{\max}\left(\sum_{l=0}^{l_{\max}} \sum_{m_1,\dots,m_l\geq 0} r_n^{\mathrm{cha}}(\boldsymbol{x}_2^{21,m_1}\dots\boldsymbol{x}_2^{21,m_l})\omega(m_l,\dots,m_1)\right) = l_{\max}\,. \tag{5.101}$$

Thus, in order to calculate the integrals $\boldsymbol{Z}_{n-1}^{\tau}(\vec{\eta};\{\tilde{s}_{ij}\})$ up to a desired order $o_{\alpha'}$ using the recursion (5.98), the words up to order $o_{\alpha'}+n-2$ in the associator as well as the integrals $\boldsymbol{Z}_{n+1}(\{s_{ij}\})$ up to order $o_{\alpha'}$ have to be calculated, i.e.

$$\begin{aligned}
&\boldsymbol{Z}_{n-1}^{\tau}(\vec{\eta};\{\tilde{s}_{ij}\}) + \mathcal{O}((\alpha')^{o_{\alpha'}+1})\\
&= e^{(s_{12\dots n}-s_{12})\omega(1,0)}\\
&\quad \boldsymbol{P}_{1,n-1}\left(\sum_{l=0}^{o_{\alpha'}+n-2}\sum_{m_1,\dots,m_l\geq 0} r_n^{\mathrm{cha}}(\boldsymbol{x}_2^{21,m_1}\dots\boldsymbol{x}_2^{21,m_l})\omega(m_l,\dots,m_1)\right)\boldsymbol{U}_{0,n+1}\\
&\quad \left(\boldsymbol{Z}_{n+1}(\{s_{ij}\}) + \mathcal{O}((\alpha')^{o_{\alpha'}+1})\right)\,.
\end{aligned} \tag{5.102}$$

- Third, the sum in eq. (5.102) has still infinitely many terms due to the sum over $m_1,\dots,m_l$. It has to be reduced to a finite sum as follows: on the one hand, according to eq. (5.10) the actual open-string corrections $F^{\mathrm{open}}_{n-1,1,(k_3,k_4,\dots,k_n)}(\sigma;\{s_{ij}\})$ appear in $\boldsymbol{Z}_{n-1}^{\tau}$ as the coefficients of the variables $\vec{\eta}=(\eta_3,\dots,\eta_n)$ at total degree

$$d = k_3 + \dots + k_n - n + 2\,. \tag{5.103}$$

On the other hand, the matrices $r_n(\boldsymbol{x}_2^{21,k})$ are of degree $k-1$ in $\vec{\eta}$, such that for each word length l, only words $r_n(\boldsymbol{x}_2^{21,m_1}\dots\boldsymbol{x}_2^{21,m_l})$ satisfying

$$m_1 + \dots + m_l - l = d \tag{5.104}$$

can contribute non-trivially to $F^{\mathrm{open}}_{n-1,1,(k_3,k_4,\dots,k_n)}(\sigma;\{s_{ij}\})$.

Putting all together, the $(n-1)$-point, genus-one open-string corrections can be calculated up to the order $o_{\alpha'}$ using finitely many words of the associator and the genus-zero string integrals up to the same order $o_{\alpha'}$:

$$\begin{aligned}
&F^{\text{open}}_{n-1,1,(k_3,k_4,\dots,k_n)}(\sigma;\{\tilde{s}_{ij}\}) + \mathcal{O}((\alpha')^{o_{\alpha'}+1}) \\
&= P_{(k_3,\dots,k_n)}(\sigma)\Bigg[e^{(s_{12\dots n}-s_{12})\omega(1,0)} \\
&\boldsymbol{P}_{1,n-1}\left(\sum_{l=0}^{o_{\alpha'}+n-2}\sum_{\substack{m_1,\dots,m_l\geq 0\\ m_1+\dots+m_l-l=k_3+\dots+k_n-n+2}} r_n^{\text{cha}}(\boldsymbol{x}_2^{21,m_1}\dots\boldsymbol{x}_2^{21,m_l})\omega(m_l,\dots,m_1)\right) \\
&\boldsymbol{U}_{0,n+1}\left(\boldsymbol{Z}_{n+1}(\{s_{ij}\}) + \mathcal{O}((\alpha')^{o_{\alpha'}+1})\right)\Bigg], \qquad (5.105)
\end{aligned}$$

where the projection $P_{(k_3,\dots,k_n)}(\sigma)$ implements the extraction[10] of the component labelled by the permutation σ from the vector in the square brackets, form the expansion in $\vec{\eta}$ of the corresponding component and takes the coefficient of the monomial $\eta^{k_3-1}_{\sigma(34\dots n)}\eta^{k_4-1}_{\sigma(4\dots n)}\dots\eta^{k_n-1}_{\sigma(n)}$.

5.3.5 Examples

Two-point, open-string correction

To finish this chapter, the two-point, open-string correction $Z^\tau_{2,(0)}(\tilde{s}_{13})$ at genus one is calculated by means of the genus-one recursion (5.98):

$$Z^\tau_2(\eta_3;\tilde{s}_{13}) = e^{(s_{13}+s_{23})\omega(1,0)}\boldsymbol{P}_{1,2}\,\Phi^\tau_{\mathcal{X}_3}\,\boldsymbol{U}_{0,4}\,Z_4(\{s_{ij}\})\,, \qquad (5.106)$$

which is the example depicted[11] in figure 1.4. The matrices appearing in the reduced elliptic KZB associator $\boldsymbol{P}_{1,2}\,\Phi^\tau_{\mathcal{X}_3}\,\boldsymbol{U}_{0,4}$ are known from the type-$(3,2)$ calculations in eqs. (5.96) and (5.72), and the alphabet $\mathcal{X}_3$ from eq. (5.42). The four-point, genus-zero chain integral $Z_4(\{s_{ij}\})$ is given in eq. (5.70) in terms of the Veneziano amplitude. Therefore, the two-point, open-string integral $Z^\tau_2(\eta_3;\tilde{s}_{13})$ can be calculated from the expansion

$$\begin{aligned}
Z^\tau_2(\eta_3;\tilde{s}_{13}) = \frac{1}{s_{13}+s_{23}}\begin{pmatrix}s_{13} & s_{23}\end{pmatrix}\sum_{l\geq 0}\sum_{m_1,\dots,m_l\geq 0} r_2^{\text{cha}}(\boldsymbol{x}_2^{21,m_1}\dots\boldsymbol{x}_2^{21,m_l})\omega(m_l,\dots,m_1) \\
\begin{pmatrix}-\frac{s_{23}}{s_{13}}\\ 1\end{pmatrix}\left(-\frac{1}{s_{23}}\frac{\Gamma(1-s_{13})\Gamma(1-s_{23})}{\Gamma(1-s_{13}-s_{23})}\right). \qquad (5.107)
\end{aligned}$$

[10] See ref. [39] for a discussion on the extraction of the corresponding component. The approach from [2] works directly with the components and does not require any such extraction or projection $P_{(k_3,\dots,k_n)}(\sigma)$, respectively, on the expense of operating with formally infinite matrices and vectors.

[11] With $\Phi_{2,1}(\alpha',\tau) = e^{(s_{13}+s_{23})\omega(1,0)}\boldsymbol{P}_{1,2}\,\Phi^\tau_{\mathcal{X}_3}\,\boldsymbol{U}_{0,4}$.

The four-point, genus-zero chain integral $Z_4(\{s_{ij}\})$ from eq. (5.70), this reduced associator acts upon, is proportional to the Veneziano amplitude. Thus, its α'-expansion is

$$
\begin{aligned}
Z_4(\{s_{ij}\}) &= -\frac{1}{s_{23}}\frac{\Gamma(1-s_{13})\Gamma(1-s_{23})}{\Gamma(1-s_{13}-s_{23})} \\
&= -\frac{1}{s_{23}} + \zeta_2 s_{12} + \zeta_3 s_{12}(s_{12}+s_{23})) + \mathcal{O}((\alpha')^3\,,
\end{aligned} \tag{5.108}
$$

which can be read of from eq. (4.8). It comprises MZVs and its minimal order in α' is $o^{\min}_{\alpha'}(\boldsymbol{Z}_4) = -1$, in agreement with eq. (5.100). Therefore, the expansion (5.107) reveals the origin of the eMZVs and MZVs in the integral Z_2^τ written out in eq. (5.13): the eMZVs enter via the elliptic KZB associator and the MZVs via the genus-zero chain integrals. However, using identities among eMZVs to reduce them to a basis may introduce further MZVs [117].

The actual string correction $F^{\text{open}}_{2,1,(0)}(\tilde{s}_{13})$ to the string amplitude is the coefficient of η_3^{-1} in the two-point, open-string integral $Z_2^\tau(\eta_3;\tilde{s}_{13})$, cf. eq. (5.10). In order to calculate it up to and including second order in α', i.e. $o_{\alpha'}=2$, eq. (5.105) can be used. The projection P_{k_3} simply extracts the coefficient of η_3^{-1}, which leads to

$$
\begin{aligned}
&F^{\text{open}}_{2,1,(0)}(\tilde{s}_{13}) + \mathcal{O}((\alpha')^3) \\
&= e^{(s_{13}+s_{23})\omega(1,0)} \\
&\quad \frac{1}{s_{13}+s_{23}}\begin{pmatrix} s_{13} & s_{23}\end{pmatrix}\left[\sum_{l=0}^{3}\sum_{\substack{m_1,\dots,m_l\geq 0\\ m_1+\cdots+m_l-l=-1}} r_2^{\text{cha}}(\boldsymbol{x}_2^{21,m_1}\dots\boldsymbol{x}_2^{21,m_l})\omega(m_l,\dots,m_1)\right]_{\eta_3^{-1}} \\
&\quad \begin{pmatrix} -\frac{s_{23}}{s_{13}} \\ 1\end{pmatrix}\left(-\frac{1}{s_{23}} + \zeta_2 s_{12} + \zeta_3 s_{12}(s_{12}+s_{23})) + \mathcal{O}((\alpha')^3\right).
\end{aligned} \tag{5.109}
$$

The contributing sum of the elliptic KZB associator is given by

$$
\begin{aligned}
&\left[\sum_{l=0}^{3}\sum_{\substack{m_1,\dots,m_l\geq 0\\ m_1+\cdots+m_l-l=-1}} r_2^{\text{cha}}(\boldsymbol{x}_2^{21,m_1}\dots\boldsymbol{x}_2^{21,m_l})\omega(m_l,\dots,m_1)\right]_{\eta_3^{-1}} \\
&= \Big[r_2^{\text{cha}}(\boldsymbol{x}_2^{21,0})\,\omega(0) + r_2^{\text{cha}}([\boldsymbol{x}_2^{21,1},\boldsymbol{x}_2^{21,0}])\omega(0,1) \\
&\quad + r_2^{\text{cha}}\left(\boldsymbol{x}_2^{21,2}\boldsymbol{x}_2^{21,0}\boldsymbol{x}_2^{21,0} + \boldsymbol{x}_2^{21,0}\boldsymbol{x}_2^{21,0}\boldsymbol{x}_2^{21,2}\right)\omega(0,0,2) + r_2^{\text{cha}}(\boldsymbol{x}_2^{21,0}\boldsymbol{x}_2^{21,2}\boldsymbol{x}_2^{21,0})\omega(0,2,0) \\
&\quad + r_2^{\text{cha}}\left(\boldsymbol{x}_2^{21,1}\boldsymbol{x}_2^{21,1}\boldsymbol{x}_2^{21,0} + \boldsymbol{x}_2^{21,0}\boldsymbol{x}_2^{21,1}\boldsymbol{x}_2^{21,1}\right)\omega(0,1,1) + r_2^{\text{cha}}(\boldsymbol{x}_2^{21,1}\boldsymbol{x}_2^{21,0}\boldsymbol{x}_2^{21,1})\omega(1,0,1)\Big]_{\eta_3^{-1}} \\
&= \lim_{\eta_3\to 0}\eta_3\Big[r_2^{\text{cha}}(\boldsymbol{x}_2^{21,0}) + r_2^{\text{cha}}([\boldsymbol{x}_2^{21,1},\boldsymbol{x}_2^{21,0}])\omega(0,1) \\
&\quad + r_2^{\text{cha}}\left([\boldsymbol{x}_2^{21,0},[\boldsymbol{x}_2^{21,0},\boldsymbol{x}_2^{21,2}]]\right)\omega(0,0,2) - r_2^{\text{cha}}(\boldsymbol{x}_2^{21,0}\boldsymbol{x}_2^{21,2}\boldsymbol{x}_2^{21,0})\zeta_2 \\
&\quad + r_2^{\text{cha}}\left([\boldsymbol{x}_2^{21,1},[\boldsymbol{x}_2^{21,1},\boldsymbol{x}_2^{21,0}]]\right)\left(\frac{5}{12}\zeta_2 + \frac{1}{2}\omega(0,1)^2 + \frac{1}{2}\omega(0,0,2)\right)\Big]\,,
\end{aligned} \tag{5.110}
$$

where identities among eMZVs have been used and the multiplication with η_3 from the left and the limes $\eta_3 \to 0$ simply pick the coefficients of η_3^{-1}, while trivialising any remaining derivatives. From eq. (5.42), the commutators appearing above can be evaluated. For example, the first two yield – if sandwiched between the matrices $\boldsymbol{P}_{1,2}$ and $\boldsymbol{U}_{0,4}$ – the following results:

$$\frac{1}{s_{13}+s_{23}}\begin{pmatrix} s_{13} & s_{23}\end{pmatrix} \lim_{\eta_3\to 0} \eta_3\, r_2^{\mathrm{cha}}(\boldsymbol{x}_2^{21,0}) \begin{pmatrix} -\frac{s_{23}}{s_{13}} \\ 1 \end{pmatrix} = -s_{23}\,,$$
$$\frac{1}{s_{13}+s_{23}}\begin{pmatrix} s_{13} & s_{23}\end{pmatrix} \lim_{\eta_3\to 0} \eta_3\, r_2^{\mathrm{cha}}([\boldsymbol{x}_2^{21,1},\boldsymbol{x}_2^{21,0}]) \begin{pmatrix} -\frac{s_{23}}{s_{13}} \\ 1 \end{pmatrix} = -s_{23}(s_{13}+s_{23})\,. \tag{5.111}$$

Using the sum (5.110) in the expansion (5.110) leads to the α'-expansion

$$F^{\mathrm{open}}_{2,1,(0)}(\tilde{s}_{13}) = 1 + \tilde{s}_{13}^2\left(\frac{\omega(0,0,2)}{2} + \frac{5\zeta_2}{12}\right) + \mathcal{O}((\alpha'))^3\,, \tag{5.112}$$

where $\tilde{s}_{13} = s_{12} + s_{13}$. These are indeed the lowest order terms appearing in the coefficient of η^{-1} in eq. (5.13). Further examples up to the genus-one, four-point integrals can be found in refs. [2, 3, 39].

Chapter 6

Graph products and integrals

In this chapter a mathematical tool is constructed which is crucial to conveniently formulate and derive various results presented in this thesis. The content and conventions are new, but based on the graphical formulation introduced in ref. [4] and heavily used in ref. [3]. It is a graphical tool to describe products of functions which are antisymmetric and satisfy an identity of the same structure as the Fay identity (3.80), this e.g. includes partial fractioning (2.12). Two examples of such functions are the fraction $1/x_{ij}$ and the Eisenstein–Kronecker series $F_{ij}(\eta)$ from eq. (3.81) depending on the difference $x_{ij} = x_i - x_j$ of some variables. This tool captures the combinatoric structure if these identities are applied recursively on such products. On the one hand, this chapter is intended for readers interested in the technicalities appearing in the derivation of the closed formulæ for the differential systems satisfied by the genus-zero and genus-one, type-(n, p) integrals introduced in section 4.2 and section 5.2. On the other hand, it is shown how these classes of integrals, constructed from the fraction $1/x_{ij}$ or the Eisenstein–Kronecker series, respectively, and their properties are generalised to generic totally antisymmetric functions satisfying a Fay identity. Such a generalisation is expected to be beneficial for the description of open-string recursions at higher genera and maybe even for other theories.

In section 6.1 the graphical representation is introduced and two bases of such products are identified which generalise the chain and admissible products from eqs. (4.10) and (4.20) appearing in the genus-zero, open-string corrections and from eqs. (5.8) and (5.29) at genus one. Moreover, various identities of such products are shown and the two bases are related to each other. The generalised admissible and chain integrals of type-(n, p) constructed from totally antisymmetric functions satisfying a Fay identity are described in section 6.2. In particular, closed formulæ for their partial derivatives are given, which yields a differential system similar to the elliptic KZB systems from eqs. (5.21) and (5.36).

6.1 Graphical representation of generating series

Let us consider a function $f(x, \eta)$, which is meromorphic in both variables $x, \eta \in \mathbb{C}$, totally antisymmetric[1]

$$f(x, \eta) = -f(-x, -\eta) \tag{6.1}$$

and satisfies the *Fay identity*

$$f(x_{ki}, \eta_i)f(x_{kj}, \eta_j) = f(x_{ki}, \eta_{ij})f(x_{ij}, \eta_j) + f(x_{kj}, \eta_{ij})f(x_{ji}, \eta_i)\,, \tag{6.2}$$

where x_{ij} is the difference $x_i - x_j$ and

$$\eta_{ij\dots k} = \eta_i + \eta_j + \dots \eta_k\,. \tag{6.3}$$

In practice, such a function will be considered to be a meromorphic function of an auxiliary variable η_j associated to some puncture x_j, while the first argument is a difference x_{ij} of two points $x_i \neq x_j$. This will be denoted by

$$f_{ij}(\eta_j) = f(x_{ij}, \eta_j)\,, \tag{6.4}$$

such that the Laurent series of $f_{ij}(\eta_j)$ with respect to η_j generates a family of meromorphic functions $g^{(k)}_{ij} = g^{(k)}(x_{ij})$ of the difference x_{ij}:

$$f_{ij}(\eta_j) = \sum_{k\geq 0} g^{(k)}_{ij}\eta_j^{k-m}\,, \quad m \in \mathbb{Z}\,. \tag{6.5}$$

Two examples of such functions f, relevant for this thesis, are the Eisenstein–Kronecker series F defined in eq. (3.71) and the fraction $1/x$ with trivial η-dependence.

6.1.1 Graphs of generating series

A graphical representation for products of functions $f_{ij}(\eta)$ satisfying the antisymmetry (6.1) and Fay identity (6.2) can be constructed as follow: first, a finite tuple of distinct punctures $\vec{x} = (x_1, \dots, x_n) \in \mathbb{C}^n$ defines the vertices, to each of which an auxiliary variable $\eta_j = \eta(x_j)$ is assigned. The map η is defined by its image

$$\vec{\eta} = (\eta_1, \dots, \eta_n) = (\eta(x_1), \dots, \eta(x_n)) = \eta(\vec{x}) \tag{6.6}$$

on the tuple $\vec{x}$. Second, each tree graph γ with vertices $x(\gamma) \subset \vec{x}$ has edges $e_{ij}(\gamma)$ from x_j to $x_i \neq x_j$, which have a certain weight $w_{ij}(\gamma)$. The weights $w_{ij}(\gamma)$ are

[1]The same construction can also be defined for totally symmetric functions $f(x, \eta) = f(-x, -\eta)$. Additionally, the meromorphicity in x can be given up as well.

defined by the auxiliary variables $\eta(x(\gamma)) \subset \vec{\eta}$ associated to the vertices $x(\gamma)$ as follows: the weight of a non-existing edge is zero and for each connected component $C_l(\gamma)$ of $\gamma = \bigcup_l C_l(\gamma)$, the outgoing weights $w_{ij}(\gamma)$ and incoming weights $w_{jk}(\gamma)$ at a vertex $x_j \in x(C_l(\gamma))$ in the component $C_l(\gamma)$ satisfy the *continuity condition*

$$\sum_{x_i \in x(C_l(\gamma))} w_{ij}(\gamma) = \eta_j + \sum_{x_k \in x(C_l(\gamma))} w_{jk}(\gamma)\,, \tag{6.7}$$

where the sums run over all vertices $x(C_l(\gamma))$ of the connected component $C_l(\gamma)$. Thus, the vertex x_j acts as a source and contributes η_j plus all the incoming weights to the outgoing weights: the weights accumulate as one traverses the edges according to their direction, cf. for example the graphs depicted in eqs. (6.13) and (6.22) below. Summing over all vertices $x_j \in x(C_l(\gamma))$, the condition (6.7) leads for each connected component $C_l(\gamma)$ to an additional condition on the auxiliary variables

$$\sum_{\eta_j \in \eta(x(C_i(\gamma)))} \eta_j = 0\,, \tag{6.8}$$

where the sum runs over all auxiliary variables η_j associated to the vertices in the connected component $C_i(\gamma)$. The set of directed tree graphs γ with such weights satisfying the continuity condition (6.7) is denoted by $\mathcal{G}(\vec{x}, \vec{\eta})$.

Graph products

To each graph $\gamma \in \mathcal{G}(\vec{x}, \vec{\eta})$, a product of functions f_{ij} can be associated. The resulting product, defined and denoted by

$$f(\gamma) = \prod_{\substack{1 \le i,j \le n \\ w_{ij}(\gamma) \neq 0}} f_{ij}(w_{ij}(\gamma))\,, \tag{6.9}$$

is called *graph product (associated to γ)*. A graph consisting of one vertex only, without any edges, is mapped to one and the function $f_{ij}(\eta_j)$ is for example obtained by[2]

$$f_{ij}(\eta_j) = f\left(i \bullet \xleftarrow{\;\eta_j\;} \bullet\, j \right). \tag{6.10}$$

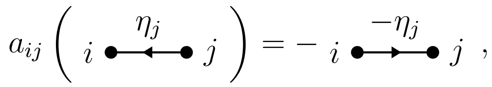
For a graph γ with more edges, the definition (6.9) instructs to apply eq. (6.10) to each edge separately to obtain the graph product $f(\gamma)$. The free abelian group

[2]If the vertices $\vec{x}$ are known or generic variables, any vertex $x_i \in \vec{x}$ in the graphs is simply represented by its index i.

generated by the image of $\mathcal{G}(\vec{x},\vec{\eta})$ under f is denoted[3] by

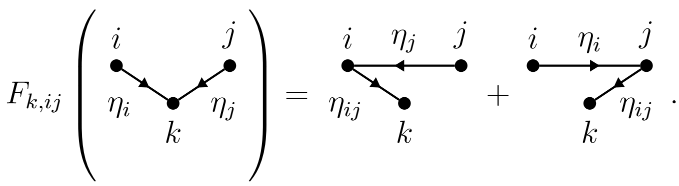
$$G_f(\vec{x},\vec{\eta}) = \langle f(\gamma) \mid \gamma \in \mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}}\,. \tag{6.11}$$

Now, let us consider the free abelian group $\langle\mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}}$ generated by the graphs and the following two actions on any edge and its weight. For a graph $\gamma \in \langle\mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}}$ the first operation is the reversal $a_{ij} : \langle\mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}} \to \langle\mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}}$ of an edge e_{ij}:



(6.12)

and the second operation is the shuffling $F_{k,ij} : \langle\mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}} \to \langle\mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}}$ of two incoming edges e_{ki} and e_{kj}:



(6.13)

Both, the reversal a_{ij} and the shuffling $F_{k,ij}$ preserve the condition (6.7) on the weights of the graphs and, thus, are indeed maps to $\langle\mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}}$: the reversal of an edge of a graph γ leads to the weight $w_{ji}(a_{ij}(\gamma)) = -w_{ij}(\gamma)$, and the shuffling to $w_{ki}(F_{k,ij}(\gamma)) = w_{ki}(\gamma) + w_{kj}(\gamma)$ and $w_{kj}(F_{k,ij}(\gamma)) = 0$ for the first graph on the right-hand side of eq. (6.13) and for the second graph the same with i and j exchanged. Thus, the result of both operations preserves the condition (6.7) and can be applied to any subgraph of a given graph. If the map f is linearly extended to $\langle\mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}}$, the operations (6.12) and (6.13) represent the antisymmetry (6.1) and the Fay identity (6.2): these identities of f are equivalent to

$$f \circ a_{ij} = f\,, \quad f \circ F_{k,ij} = f \tag{6.14}$$

as maps $\langle\mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}} \to G_f(\vec{x},\vec{\eta})$. Thus, the operations a_{ij} and $F_{k,ij}$ are also called antisymmetry and Fay identity (on graphs).

Graphs of f

An equivalence relation on graphs $\gamma_1, \gamma_2 \in \langle\mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}}$ can be defined by whether γ_1 and γ_2 are related by a finite number of applications of the antisymmetry a_{ij} and $F_{k,ij}$. In order to capture possibly further identities of a specific choice for f in terms of operations on graphs as in eq. (6.14), the more restrictive definition

$$[\gamma_1]_f = [\gamma_2]_f \quad \Leftrightarrow \quad f(\gamma_1) = f(\gamma_2) \tag{6.15}$$

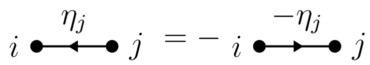
[3]The calligraphic letter $\mathcal{G}$ is used to denote objects consisting of graphs, e.g. $\mathcal{G}(\vec{x},\vec{\eta})$, and the non-calligraphic G for products of functions, e.g. $G_f(\vec{x},\vec{\eta})$.

is used to construct equivalence classes $[\gamma]_f$ on $\langle \mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}}$. However, unless specific examples are considered, it is assumed that f does not satisfy further relations than the antisymmetry and Fay identity, such that the statements derived below are valid for generic functions f. The free abelian group generated by all equivalence classes is denoted by

$$\mathcal{G}_f(\vec{x},\vec{\eta}) = \langle [\gamma]_f \mid \gamma \in \mathcal{G}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}} \,. \tag{6.16}$$

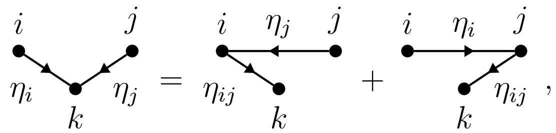
By definition, the action of $f([\gamma]_f)$ on any element $[\gamma]_f \in \mathcal{G}_f(\vec{x},\vec{\eta})$ is well-defined by its action $f(\gamma)$ on any representative $\gamma \in [\gamma]_f$. The group $\mathcal{G}_f(\vec{x},\vec{\eta})$ is called the *graphs of* f and an element $[\gamma]_f$ is referred to as the graph of the graph product $f(\gamma)$.

If it is known which generating function f is considered, by abuse of notation, an element $[\gamma]_f$ is usually just identified by its representatives γ, such that we write $\gamma_1 = \gamma_2$ for $[\gamma_1]_f = [\gamma_2]_f$ as an identity in $\mathcal{G}_f(\vec{x},\vec{\eta})$. With this identification, the antisymmetry (6.1) and Fay identity (6.2) can be expressed as the following conditions on the edges of the (equivalence classes of) graphs in $\mathcal{G}_f(\vec{x},\vec{\eta})$:



(6.17)

and



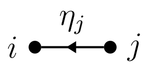
(6.18)

respectively.

By construction, f is a group isomorphism

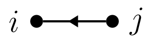
$$\mathcal{G}_f(\vec{x},\vec{\eta}) \overset{f}{\simeq} G_f(\vec{x},\vec{\eta}) \tag{6.19}$$

and we often also identify the graphs γ or $[\gamma]_f$, respectively, with their image under f. With this identification, the function $f_{ij}(\eta_j)$ can be denoted by a weighted, directed graph

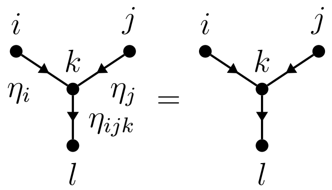
$f_{ij}(\eta_j) \simeq$ 

. (6.20)

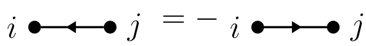
If in addition to f, the assignment η is known, such that it is clear which auxiliary variable $\eta_j = \eta(x_j)$ is related to each vertex $x_j \in \vec{x}$, the weight is sometimes omitted from the graphical representation, since it is uniquely determined by the continuity condition (6.7). In this case, the function $f_{ij}(\eta_j)$ is represented by

$$f_{ij}(\eta_j) \simeq$$  . (6.21)

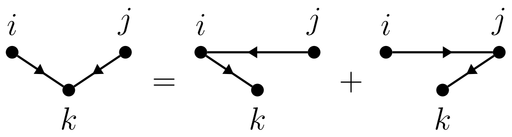
An example for a graph product with three factors and the corresponding graph is

$$f_{lk}(\eta_{ijk})f_{ki}(\eta_i)f_{kj}(\eta_j) \simeq$$  . (6.22)

The antisymmetry takes the form

 (6.23)

and the Fay identity

 . (6.24)

For the vertices $x(\gamma)$ of a graph $\gamma \in \mathcal{G}(\vec{x}, \vec{\eta})$, the following conventions are used: a vertex with no incoming and one outgoing edge is called *loose point* (e.g. x_i and x_j of the graph (6.22)), a vertex with at least one incoming edge and no outgoing edge *base point* (e.g. x_l in the graph (6.22)), a vertex with more than two attached (either at least two incoming or two outgoing) edges *branch point* (e.g. x_k in the graph (6.22)).

In the next two subsections, two sets of generators of the group of the directed, weighted tree graphs $\mathcal{G}_f(\vec{x}, \vec{\eta})$ satisfying the antisymmetry (6.12) and Fay identity (6.13) are presented. Both of these sets of generators and the corresponding images under f are used to describe the structure of the products of functions appearing in the string amplitudes at genus zero and one.

6.1.2 Admissible graphs and products

The first set of generators are the *admissible graphs*. These are graphs $\gamma \in \mathcal{G}(\vec{x}, \vec{\eta})$, such that their edges satisfy the *admissibility condition*, which states that for each vertex x_j, there is at most one outgoing edge e_{ij} and, if existing, it points to a lower vertex $i < j$:

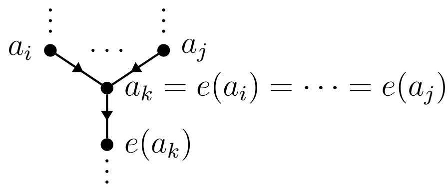
$$\forall j : w_{ij}(\gamma) = 0 \text{ except for at most one } i < j\,. \tag{6.25}$$

An admissible graph γ defines an admissible map[4] e as follows: for each vertex $x_j \in x(\gamma)$ which has an outgoing edge, the value $e(j) = i$ is defined by the unique outgoing edge e_{ij}:

$$e(j) = i < j \text{ such that } w_{ij}(\gamma) \neq 0\,. \tag{6.26}$$

Thus, the admissible graphs are of the following form and denoted by

$$\varphi\left(\begin{smallmatrix} a_1 & \dots & a_p \\ e(a_1) & \dots & e(a_p) \end{smallmatrix}\right) = \tag{6.27}$$



They are labelled by a double sequence

$$\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) = \left(\begin{smallmatrix} a_1 & \dots & a_p \\ e(a_1) & \dots & e(a_p) \end{smallmatrix}\right)\,, \quad e(a_i) < a_i\,, \tag{6.28}$$

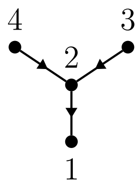
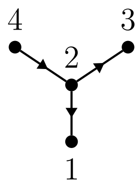
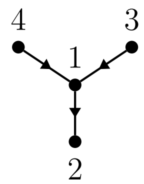
where $A = (a_1, \dots, a_p)$ are the indices of the vertices $x(\gamma)$ with one outgoing edge. The set of all such admissible graphs is denoted by

$$\mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta}) = \{\gamma \in \mathcal{G}(\vec{x}, \vec{\eta}) | \gamma \text{ satisfies admissibility condition (6.25)}\} \tag{6.29}$$

and the corresponding free abelian group of equivalence classes by

$$\mathcal{G}_f^{\mathrm{adm}}(\vec{x}, \vec{\eta}) = \langle [\gamma]_f | \gamma \in \mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta}) \rangle_{\mathbb{Z}}\,. \tag{6.30}$$

Examples of admissible and non-admissible graphs are for $\vec{x} = (x_1, x_2, x_3, x_4)$ and $\vec{\eta} = (\eta_1, \eta_2, \eta_3, \eta_4)$

$\varphi\left(\begin{smallmatrix} 2 & 3 & 4 \\ 1 & 2 & 2 \end{smallmatrix}\right) =$  $\in \mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta})\,,$  ,  $\in \mathcal{G}(\vec{x}, \vec{\eta}) \backslash \mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta})\,.$ (6.31)

In appendix C.2, it is shown that the admissible graphs indeed generate the free abelian group of all the graphs satisfying the antisymmetry and Fay identity:

$$\mathcal{G}_f^{\mathrm{adm}}(\vec{x}, \vec{\eta}) = \mathcal{G}_f(\vec{x}, \vec{\eta})\,. \tag{6.32}$$

The proof is constructive and leads to the actual linear combination, i.e. it is an algorithm which can be applied to any graph $\gamma \in \mathcal{G}_f(\vec{x}, \vec{\eta})$ and yields a linear com-

[4]See eq. (4.21) for the definition of admissible maps.

bination in terms of admissible graphs in $\mathcal{G}_f^{\text{adm}}(\vec{x},\vec{\eta})$:

$$\gamma = \sum_{\gamma' \in \mathcal{G}^{\text{adm}}(\vec{x},\vec{\eta})} b_{\gamma'}^{\text{adm}} \gamma' \in \mathcal{G}_f^{\text{adm}}(\vec{x},\vec{\eta}) \,, \quad b_{\gamma'}^{\text{adm}} \in \mathbb{Z} \,. \tag{6.33}$$

The result of the above algorithm, i.e. the right-hand side of eq. (6.33), is denoted by $b^{\text{adm}}(\gamma)$.

The analogous statements hold for the images under f: the graph products of admissible graphs are denoted by

$$\varphi_f \left(\begin{smallmatrix} a_1 & \dots & a_p \\ e(a_1) & \dots & e(a_p) \end{smallmatrix} \right) = f \left(\varphi \left(\begin{smallmatrix} a_1 & \dots & a_p \\ e(a_1) & \dots & e(a_p) \end{smallmatrix} \right) \right) \tag{6.34}$$

and called *admissible products of* f. For example, the admissible graph in eq. (6.31) is mapped by f to the product

$$\varphi_f \left(\begin{smallmatrix} 2 & 3 & 4 \\ 1 & 2 & 2 \end{smallmatrix} \right) = f_{12}(\eta_{234}) f_{23}(\eta_3) f_{24}(\eta_4) \,. \tag{6.35}$$

Further examples are the admissible products of fractions $f(x,\eta) = 1/x$ and the admissible products of Eisenstein–Kronecker series F defined in eqs. (4.20) and (5.29), respectively (where the simplified notation $\varphi = \varphi_{1/x}$ and $\varphi^\tau = \varphi_F$ has been used).

Defining the isomorphic free abelian group of admissible products

$$G_f^{\text{adm}}(\vec{x},\vec{\eta}) = \langle f(\gamma) | \gamma \in \mathcal{G}^{\text{adm}}(\vec{x},\vec{\eta}) \rangle_{\mathbb{Z}} \simeq \mathcal{G}_f^{\text{adm}}(\vec{x},\vec{\eta}) \,, \tag{6.36}$$

eq. (6.32) states that any linear combination of products $f(\gamma) \in G_f(\vec{x},\vec{\eta})$ can be written as a linear combination of admissible products $f(\gamma) \in G_f^{\text{adm}}(\vec{x},\vec{\eta})$, i.e.

$$G_f^{\text{adm}}(\vec{x},\vec{\eta}) = G_f(\vec{x},\vec{\eta}) \,. \tag{6.37}$$

This statement follows from mapping eq. (6.33) via f to graph products, which leads to exactly the same linear combination: any graph product $f(\gamma) \in G_f(\vec{x},\vec{\eta})$ can be written in terms of admissible products as

$$f(\gamma) = \sum_{\gamma' \in \mathcal{G}^{\text{adm}}(\vec{x},\vec{\eta})} b_{\gamma'}^{\text{adm}} f(\gamma') \in G_f^{\text{adm}}(\vec{x},\vec{\eta}) \,, \quad b_{\gamma'}^{\text{adm}} \in \mathbb{Z} \,, \tag{6.38}$$

where $b_{\gamma'}^{\text{adm}}$ are the integers from eq. (6.33). For example, eq. (4.34) is nothing but the transformation (6.38) applied to fractions $f(x,\eta) = 1/x$. The right-hand side of eq. (6.38) is denoted by $b^{\text{adm}}(f(\gamma))$.

6.1.3 Chain graphs and products

The second set of generators are the *chain graphs*, which are graphs $\gamma \in \mathcal{G}(\vec{x},\vec{\eta})$ having at most one incoming and at most one outgoing edge at each vertex x_j:

$$\forall j : w_{ij}(\gamma) = 0 \text{ and } w_{jk}(\gamma) = 0 \text{ except for at most one } i \neq j \\ \text{and at most one } k \neq i, j\,. \tag{6.39}$$

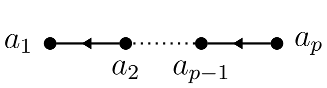
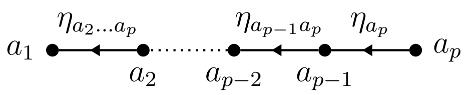
These chain graphs are denoted by

$$\mathcal{G}^{\mathrm{cha}}(\vec{x},\vec{\eta}) = \{\gamma \in \mathcal{G}(\vec{x},\vec{\eta}) | \gamma \text{ satisfies chain condition (6.39)}\} \tag{6.40}$$

and the free abelian group of its equivalence classes by

$$\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta}) = \langle [\gamma]_f | \gamma \in \mathcal{G}^{\mathrm{cha}}(\vec{x},\vec{\eta}) \rangle_{\mathbb{Z}}\,. \tag{6.41}$$

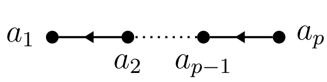
Thus, the connected components of chain graphs are of the form of a *chain*, denoted by

$\varphi(A) =$  $=$  (6.42)

and labelled by a permutation $A = (a_1, \ldots, a_p)$ of a subtuple of the indices of the vertices $\vec{x}$. To each chain, the associated product of functions f is denoted by

$$\varphi_f(A) = f(\varphi(A)) = \prod_{i=2}^{p} f_{a_{i-1},a_i}(\eta_{a_i \ldots a_p})\,, \tag{6.43}$$

such that the identification of $\varphi_f(A)$ with $\varphi(A)$ in $\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta})$ leads to

$\prod_{i=2}^{p} f_{a_{i-1},a_i}(\eta_{a_i \ldots a_p}) \simeq$  . (6.44)

Two examples of chain products are the chains of fractions φ and the chains of Eisenstein–Kronecker series φ^τ defined in eqs. (4.10) and (5.8), respectively.

It turns out that similar to eq. (6.32), all the graphs $\mathcal{G}_f(\vec{x},\vec{\eta})$ can alternatively be represented and, thus, are generated by chain graphs

$$\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta}) = \mathcal{G}_f(\vec{x},\vec{\eta})\,, \tag{6.45}$$

which is proven in appendix C.2. The images of these graphs under f form the free abelian group of *chain products of f*

$$G_f^{\text{cha}}(\vec{x},\vec{\eta}) = \langle f(\gamma) | \gamma \in \mathcal{G}^{\text{cha}}(\vec{x},\vec{\eta})\rangle_{\mathbb{Z}} \simeq \mathcal{G}_f^{\text{cha}}(\vec{x},\vec{\eta})\,, \tag{6.46}$$

such that eq. (6.45) implies that any linear combination of products $f(\gamma) \in G_f(\vec{x},\vec{\eta})$ can be written as a linear combination of chain products $f(\gamma) \in G_f^{\text{adm}}(\vec{x},\vec{\eta})$:

$$G_f^{\text{adm}}(\vec{x},\vec{\eta}) = G_f(\vec{x},\vec{\eta})\,. \tag{6.47}$$

Again, for each graph $\gamma \in \mathcal{G}_f(\vec{x},\vec{\eta})$, the algorithm from appendix C.2 leads to a linear combination in terms of admissible graphs:

$$\gamma = \sum_{\gamma' \in \mathcal{G}^{\text{cha}}(\vec{x},\vec{\eta})} b_{\gamma'}^{\text{cha}} \gamma' \in \mathcal{G}_f^{\text{cha}}(\vec{x},\vec{\eta})\,, \quad b_{\gamma'}^{\text{cha}} \in \mathbb{Z}\,. \tag{6.48}$$

The same linear combination holds for the graph products:

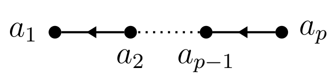
$$f(\gamma) = \sum_{\gamma' \in \mathcal{G}^{\text{cha}}(\vec{x},\vec{\eta})} b_{\gamma'}^{\text{cha}} f(\gamma') \in G_f^{\text{cha}}(\vec{x},\vec{\eta})\,, \quad b_{\gamma'}^{\text{cha}} \in \mathbb{Z}\,. \tag{6.49}$$

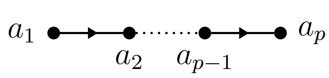
The right-hand sides of eqs. (6.48) and (6.49) are denoted by $b^{\text{cha}}(\gamma)$ and $b^{\text{cha}}(f(\gamma))$, respectively. If for example the graph γ is admissible, this is nothing but the transformation in eqs. (4.64) and (5.31) from admissible to chain products, which will be explained in more detail in subsection 6.1.4.

Graph identities

The chain and admissible graphs satisfy various identities if the antisymmetry and Fay identity are applied recursively. The following three identities are particularly useful in subsequent calculations, including the derivation in appendix C.3 of the closed formulæ for the elliptic KZB system (5.21). Their derivation can be found in appendix C.1 and ref. [3]. They can not only be applied to the whole graph, but to any subgraph and to the corresponding graph products and factors thereof.

The first identity is the reflection property:

$\varphi(A) =$ 

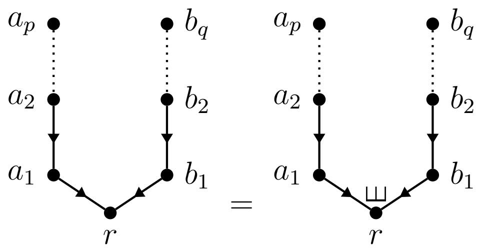
$= (-1)^{p-1}$ 

$$= (-1)^{|A|-1}\varphi(\tilde{A})\,, \tag{6.50}$$

where $\tilde{A} = (a_p, \dots, a_1)$ is the reversed sequence $A = (a_1, \dots, a_p)$ and $|A| = p$ its length. It is based on the antisymmetry (6.17) and the condition (6.8) on the weights. Moreover, the same identity holds for the image under f, which is the following identity for chain products:

$$\varphi_f(A) = (-1)^{|A|-1}\varphi_f(\tilde{A})\,. \tag{6.51}$$

The second identity is the shuffling of two branches with a branch point at the vertex x_r:



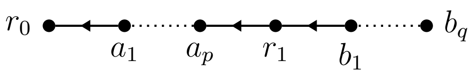
, (6.52)

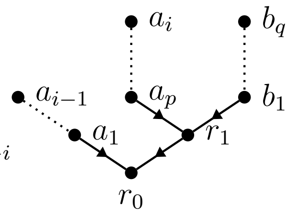
where the sum on the right-hand side is an element of $\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta})$ obtained from iteratively applying the Fay identity to the initial graph until a sum of chains is left. The corresponding identity on products of functions is

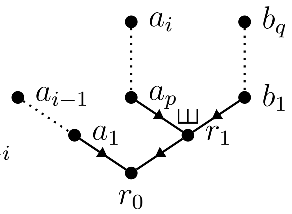
$$\varphi_f(r,A)\varphi_f(r,B) = \varphi_f(r, A \sqcup\!\sqcup B) \tag{6.53}$$

for $A = (a_1,\ldots,a_p)$ and $B = (b_1,\ldots,b_q)$.

The third identity can be used to shift two labels r_0 and r_1 in a chain $\varphi(r_0,A,r_1,B)$ next to each other:

$\varphi(r_0,A,r_1,B) =$ 

$= \sum_{i=1}^{p+1}(-1)^{p+1-i}$ 

$= \sum_{i=1}^{p+1}(-1)^{p+1-i}$ 

, (6.54)

where for the last equality, the shuffle identity (6.52) has been applied. The analogous identity on chain products is

$$\varphi_f(r_0, A, r_1, B) = \sum_{i=1}^{p+1} (-1)^{p+1-i} \varphi_f(r_0, a_1, \ldots, a_{i-1}) \varphi_f(r_0, r_1, (a_p, a_{p-1}, \ldots, a_i) \sqcup B) . \tag{6.55}$$

6.1.4 Admissible and chain products

According to eq. (6.45), the admissible graphs can be written as a sum of chain graphs. This can be achieved by applying the shuffle identity (6.52) to any branch point of the admissible graph, effectively folding back all the branches. This process is described by eq. (4.40): given a double sequence $\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right)$ with $e(a_i) < a_i$ defining an admissible graph $\varphi\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) \in \mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta})$ with base point $i = e(a_1)$, let us recursively define the sum of sequences

$$\left(i, \begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) = \left(i, \bigsqcup_{\substack{j \in A \\ e(j)=i}} \left(j, \begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right)\right) . \tag{6.56}$$

For the example $\left(\begin{smallmatrix} 2&3&4&5&6&7 \\ 1&2&2&4&4&6 \end{smallmatrix}\right)$, the following sum of sequences beginning at 1 is obtained:

$$\begin{aligned} \left(1, \begin{smallmatrix} 2&3&4&5&6&7 \\ 1&2&2&4&4&6 \end{smallmatrix}\right) &= \left(1, \left(2, \begin{smallmatrix} 2&3&4&5&6&7 \\ 1&2&2&4&4&6 \end{smallmatrix}\right)\right) \\ &= \left(1, 2, \left(3, \begin{smallmatrix} 2&3&4&5&6&7 \\ 1&2&2&4&4&6 \end{smallmatrix}\right) \sqcup \left(4, \begin{smallmatrix} 2&3&4&5&6&7 \\ 1&2&2&4&4&6 \end{smallmatrix}\right)\right) \\ &= \left(1, 2, 3 \sqcup \left(5, \begin{smallmatrix} 2&3&4&5&6&7 \\ 1&2&2&4&4&6 \end{smallmatrix}\right) \sqcup \left(6, \begin{smallmatrix} 2&3&4&5&6&7 \\ 1&2&2&4&4&6 \end{smallmatrix}\right)\right) \\ &= \left(1, 2, 3 \sqcup 5 \sqcup \left(6, \left(7, \begin{smallmatrix} 2&3&4&5&6&7 \\ 1&2&2&4&4&6 \end{smallmatrix}\right)\right)\right) \\ &= (1, 2, 3 \sqcup 5 \sqcup (6, 7)) \end{aligned} \tag{6.57}$$

Further examples are given in eq. (4.39). Then, the admissible graph with one single base point i can be written as the following linear combination of chain graphs:

$$\varphi\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) = \varphi\left(i, \begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) \in \mathcal{G}_f^{\mathrm{cha}}(\vec{x}, \vec{\eta}) . \tag{6.58}$$

The same identity holds for graph products: an admissible product with base point i can be expressed as a linear combination of chain graphs according to

$$\varphi_f\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) = \varphi_f\left(i, \begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) \in G_f^{\mathrm{cha}}(\vec{x}, \vec{\eta}) . \tag{6.59}$$

Graphs and products with base points

The above translation from admissible to chain graphs and products, respectively, can be generalised to admissible graphs with more than one base point. For this purpose, graphs with n vertices $\vec{x} = (x_1, x_2, \ldots, x_n)$ and the first p base points

$B = \{x_1, x_2, \dots, x_p\}$ for $1 \le p \le n$ are considered. The subset of *admissible graphs with base points* B is denoted by

$$\mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta}; B) \subset \mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta}) \tag{6.60}$$

and the *chain graphs with base points* B by

$$\mathcal{G}^{\mathrm{cha}}(\vec{x}, \vec{\eta}; B) \subset \mathcal{G}^{\mathrm{cha}}(\vec{x}, \vec{\eta})\,. \tag{6.61}$$

Their equivalence classes under the equivalence relation (6.15) are denoted by

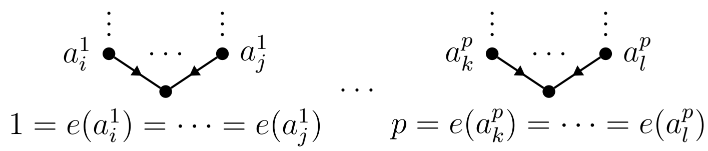
$$\mathcal{G}_f^{\mathrm{adm}}(\vec{x}, \vec{\eta}; B) \subset \mathcal{G}_f^{\mathrm{adm}}(\vec{x}, \vec{\eta}) \tag{6.62}$$

and

$$\mathcal{G}_f^{\mathrm{cha}}(\vec{x}, \vec{\eta}; B) \subset \mathcal{G}_f^{\mathrm{cha}}(\vec{x}, \vec{\eta})\,. \tag{6.63}$$

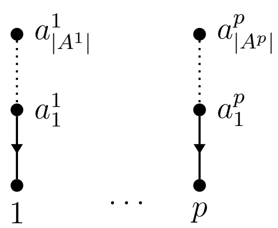
Therefore, generic elements of the admissible graphs with base points B, i.e. $\mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta}; B)$, are of the form

$$\varphi\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) = \varphi\left(\begin{smallmatrix} A_e(1) \\ e(A_e(1)) \end{smallmatrix}\right) \cdots \varphi\left(\begin{smallmatrix} A_e(p) \\ e(A_e(p)) \end{smallmatrix}\right)$$

$=$ 

$$\in \mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta}; B)\,, \tag{6.64}$$

where e is an admissible map, i.e. $1 \le e(a_j^i) < a_j^i \le n$, and the p disjoint, possibly empty sequences $A_e(i) = (a_1^i, \dots, a_{|A_e(i)|}^i)$ are defined in eq. (4.24) in terms of e and the sequence $A = (p+1, p+2, \dots, n)$. On the other hand, an element of $\mathcal{G}^{\mathrm{cha}}(\vec{x}, \vec{\eta}; B)$, the chain graphs with base points B, is of the form

$\varphi\left(1, A^1\right) \dots \varphi\left(p, A^p\right) =$ 

$\in \mathcal{G}^{\mathrm{cha}}(\vec{x}, \vec{\eta}; B)\,,$ (6.65)

where this time, the p sequences A^i are defined as in eq. (5.18) by a partition of the image of some permutation $\sigma \in S_{n-p}$ on $A = (p+1, p+2, \dots, n)$, i.e.

$$(A^1, A^2, \dots, A^p) = \sigma(p+1, p+2, \dots, n)\,. \tag{6.66}$$

The admissible graphs $\mathcal{G}^{\mathrm{adm}}(\vec{x},\vec{\eta};B)$ can be written in terms of chain graphs in $\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta};B)$ by the application of eq. (6.58) to each connected component with base point $i\in\{1,2,\dots,p\}$:

$$\varphi\left(\begin{smallmatrix}A_e(i)\\ e(A_e(i))\end{smallmatrix}\right)=\varphi\left(i,\begin{smallmatrix}A_e(i)\\ e(A_e(i))\end{smallmatrix}\right)\in\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta};B)\,. \tag{6.67}$$

From a combinatorial analysis, both of the sets $\mathcal{G}^{\mathrm{adm}}(\vec{x},\vec{\eta};B)$ and $\mathcal{G}^{\mathrm{cha}}(\vec{x},\vec{\eta};B)$ turn out to have

$$d_{n,p}=\frac{(n-1)!}{(p-1)!} \tag{6.68}$$

disjoint elements, which are independent generators of the groups $\mathcal{G}_f^{\mathrm{adm}}(\vec{x},\vec{\eta};B)$ and $\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta};B)$ (assuming f only satisfies the antisymmetry and the Fay identity, but no further relations which might decrease the number of generators). Note that this agrees with the number $d_{n,p}$ in eq. (5.19) of type-(n,p) chain and admissible integrals at genus one and the dimension of the twisted de Rham cohomology of $\mathcal{F}_{n+2,p+2}$ in eq. (4.50) (the shift of two comes from an integration-by-parts redundancy and from the point at infinity on the Riemann sphere, see also subsection 6.2.4).

Assembling the elements of $\mathcal{G}^{\mathrm{adm}}(\vec{x},\vec{\eta};B)$ and $\mathcal{G}^{\mathrm{cha}}(\vec{x},\vec{\eta};B)$ given in eqs. (6.64) and (6.65), respectively, into $d_{n,p}$-dimensional vectors

$$\begin{aligned}\boldsymbol{\varphi}^{\mathrm{adm}}(\vec{x},\vec{\eta};B)&=\left(\varphi\left(\begin{smallmatrix}A\\ e(A)\end{smallmatrix}\right)\right)_{1\leq e(k)<k}\\ &=\left(\varphi\left(\begin{smallmatrix}A_e(1)\\ e(A_e(1))\end{smallmatrix}\right)\dots\varphi\left(\begin{smallmatrix}A_e(p)\\ A_e(p)\end{smallmatrix}\right)\right)_{1\leq e(k)<k}\end{aligned} \tag{6.69}$$

and

$$\boldsymbol{\varphi}^{\mathrm{cha}}(\vec{x},\vec{\eta};B)=\Big(\varphi\left(1,A^1\right)\dots\varphi\left(p,A^p\right)\Big)_{\substack{(A^1,\dots,A^p)=\sigma(A)\\ \sigma\in S_{n-p}}}\,, \tag{6.70}$$

the translation (6.67) leads to a basis transformation

$$\boldsymbol{\varphi}^{\mathrm{adm}}(\vec{x},\vec{\eta};B)=\boldsymbol{B}_{n,p}\,\boldsymbol{\varphi}^{\mathrm{cha}}(\vec{x},\vec{\eta};B)\in(\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta};B))^{d_{n,p}} \tag{6.71}$$

between the (generators of the) free abelian groups $\mathcal{G}^{\mathrm{adm}}(\vec{x},\vec{\eta};B)$ and $\mathcal{G}^{\mathrm{cha}}(\vec{x},\vec{\eta};B)$. This transformation is invertible, since any graph can be written in terms of admissible graphs via the algorithm in appendix C.2, such that

$$\boldsymbol{\varphi}^{\mathrm{cha}}(\vec{x},\vec{\eta};B)=\boldsymbol{B}_{n,p}^{-1}\,\boldsymbol{\varphi}^{\mathrm{cha}}(\vec{x},\vec{\eta};B)\in(\mathcal{G}_f^{\mathrm{adm}}(\vec{x},\vec{\eta};B))^{d_{n,p}}\,. \tag{6.72}$$

The above definitions and statements have an echo under the image of f: the subgroups of the admissible and chain products with base points B, which are the images of $\mathcal{G}_f^{\text{adm}}(\vec{x},\vec{\eta};B)$ and $\mathcal{G}_f^{\text{cha}}(\vec{x},\vec{\eta};B)$ under f, are denoted by

$$G_f^{\text{adm}}(\vec{x},\vec{\eta};B) = f(\mathcal{G}_f^{\text{adm}}(\vec{x},\vec{\eta};B)) \subset G_f^{\text{adm}}(\vec{x},\vec{\eta}) \tag{6.73}$$

and

$$G_f^{\text{cha}}(\vec{x},\vec{\eta};B) = f(\mathcal{G}_f^{\text{cha}}(\vec{x},\vec{\eta};B)) \subset G_f^{\text{cha}}(\vec{x},\vec{\eta})\,. \tag{6.74}$$

They are generated by the entries of the $d_{n,p}$-dimensional vectors

$$\boldsymbol{\varphi}_f^{\text{adm}}(\vec{x},\vec{\eta};B) = f(\boldsymbol{\varphi}^{\text{adm}}(\vec{x},\vec{\eta};B)) = \left(\varphi_f\begin{pmatrix}A_e(1)\\ e(A_e(1))\end{pmatrix}\dots\varphi_f\begin{pmatrix}A_e(p)\\ A_e(p)\end{pmatrix}\right)_{1\leq e(k)<k} \tag{6.75}$$

and

$$\boldsymbol{\varphi}_f^{\text{cha}}(\vec{x},\vec{\eta};B) = f(\boldsymbol{\varphi}^{\text{cha}}(\vec{x},\vec{\eta};B)) = \Big(\varphi_f\,(1,A^1)\dots\varphi_f\,(p,A^p)\Big)_{\substack{(A^1,\dots,A^p)=\sigma(A)\\ \sigma\in S_{n-p}}}\,, \tag{6.76}$$

respectively, where $A = (p+1,p+2,\dots,n)$. These vectors are related by the invertible basis transformation

$$\boldsymbol{\varphi}_f^{\text{adm}}(\vec{x},\vec{\eta};B) = \boldsymbol{B}_{n,p}\,\boldsymbol{\varphi}_f^{\text{cha}}(\vec{x},\vec{\eta};B) \in (\mathcal{G}_f^{\text{cha}}(\vec{x},\vec{\eta};B))^{d_{n,p}}\,. \tag{6.77}$$

The matrix $\boldsymbol{B}_{n,p}$ is the same as in the graphical eq. (6.71) and also the same matrix as in the transformation (5.33) between the genus-one, admissible and chain integrals and with a shift of two in n and p the matrix from eq. (4.63) at genus zero. Examples for the calculation (6.77) and the matrix $\boldsymbol{B}_{n,p}$ for fractions $f(x,\eta) = 1/x$ and Eisenstein–Kronecker series are given in eqs. (4.77) and (5.47).

6.2 Graph integrals

The construction of the type-(n,p) admissible and chain integrals at genus zero in section 4.2 and genus one in section 5.2, which are based on admissible and chain products of fractions and Eisenstein–Kronecker series, respectively, can be generalised using other generating functions $f(x,\eta)$ satisfying the antisymmetry (6.1) and Fay identity (6.2).

6.2.1 Admissible and chain integrals

For this purpose, functions $f(x,\eta)$ satisfying the antisymmetry (6.1) and Fay identity (6.2) are considered, which have an expansion $f(x,\eta) = \sum_{k\geq 0} g^{(k)}(x)\eta^{k-m}$ of the form (6.5), where $g^{(m)}(x)$ is meromorphic with a simple pole at $x=0$ and the other functions $g^{(k)}(x)$ for $k\neq m$ are holomorphic. Note that the antisymmetry of

f implies that $g^{(m)}(x)$ is odd.

The following two vectors of *graph integrals* can be constructed with such functions: first, one factor of the integrands consists of the admissible $\boldsymbol{\varphi}_f^{\mathrm{adm}}(\vec{x},\vec{\eta};B)$ or chain products $\boldsymbol{\varphi}_f^{\mathrm{cha}}(\vec{x},\vec{\eta};B)$ with vertices $\vec{x}=(x_1,\dots,x_n)$, auxiliary variables $\vec{\eta}=(\eta_1,\dots,\eta_n)$ and base points $B=\{x_1,\dots,x_p\}$. The other factor of the integrand is a function $u_f(\vec{x})$ satisfying the partial differential equation

$$\begin{aligned}\partial_{x_i}u_f(\vec{x}) &= -\sum_{j\neq i}\alpha_{ij}g_{ij}^{(m)}u_f(\vec{x})\\ &= \left[-\sum_{j\neq i}\alpha_{ij}f_{ij}(\eta)u_f(\vec{x})\right]_{\eta^0}\,,\end{aligned} \tag{6.78}$$

valid for $x_i\in\vec{x}$, where α_{ij} are complex parameters with $\mathrm{Re}(\alpha_{ij})<0$ and which involves the function $g^{(m)}$. Moreover, for any $x_i,x_j\in\vec{x}$, the boundary condition

$$\lim_{x_j\to x_k}u_f(\vec{x})=0 \tag{6.79}$$

has to be satisfied. For example, a product of exponentials of the form

$$\begin{aligned}u_f(\vec{x}) &= u_f(\vec{x},\{\alpha_{ij}\})\\ &= \prod_{1\leq i<j\leq n}\exp\left(-\alpha_{ij}\lim_{\epsilon\to 0}\left(\int_\epsilon^{|x_{ji}|}\mathrm{d}x\,g^{(m)}(x)+\log(\epsilon)\right)\right)\end{aligned} \tag{6.80}$$

satisfies these conditions. Other solutions, e.g. associated to different conventions to subtract the logarithmic divergencies in the integrals appearing in the exponentials, are possible. Second, the integration domain is the real simplex

$$\Delta_{n,p}=\Delta_{n,p}(x_k)=\{0<x_n<x_{n-1}<\cdots<x_{p+1}<x_p\}\,, \tag{6.81}$$

cf. eq. (5.16).

This defines two types of graph integrals: the *type-(n,p) admissible integrals associated to f* are defined by

$$\begin{aligned}\boldsymbol{S}_{n,p}^f(B) &= \boldsymbol{S}_{n,p}^f(B;\vec{\eta};\{\alpha_{ij}\})\\ &= \int_{\Delta_{n,p}}\prod_{k=p+1}^n\mathrm{d}x_k\,u_f(\vec{x})\,\boldsymbol{\varphi}_f^{\mathrm{adm}}(\vec{x},\vec{\eta};B)\end{aligned} \tag{6.82}$$

and the *type-(n,p) chain integrals associated to f* by

$$\begin{aligned}\boldsymbol{Z}^f_{n,p}(B) &= \boldsymbol{Z}^f_{n,p}(B;\vec{\eta};\{\alpha_{ij}\})\\ &= \int_{\Delta_{n,p}} \prod_{k=p+1}^{n} \mathrm{d}x_k\, u_f(\vec{x})\, \boldsymbol{\varphi}^{\mathrm{cha}}_f(\vec{x},\vec{\eta};B)\,. \end{aligned}\tag{6.83}$$

The boundary condition (6.79) ensures that these integrals converge[5] and that a total derivative with respect to $x_i \in \vec{x}\backslash B$ vanishes, i.e.

$$\begin{aligned}&\int_{\Delta_{n,p}} \partial_{x_i}\left(\prod_{k=p+1}^{n} \mathrm{d}x_k\, u_f(\vec{x})\, \boldsymbol{\varphi}^{\mathrm{adm}\,/\,\mathrm{cha}}_f(\vec{x},\vec{\eta};B)\right)\\ &= \int_{\partial_i\Delta_{n,p}} \prod_{\substack{k=p+1\\ k\neq i}}^{n} \mathrm{d}x_k\, u_f(\vec{x})\, \boldsymbol{\varphi}^{\mathrm{adm}\,/\,\mathrm{cha}}_f(\vec{x},\vec{\eta};B)\\ &= 0\,,\end{aligned}\tag{6.84}$$

where

$$\begin{aligned}\partial_i\Delta_{n,p} &= \{0<x_n<\cdots<x_{i+1}<x_{i-1}<\cdots<x_p \text{ and } x_i=x_{i-1}\}\\ &\quad -\{0<x_n<\cdots<x_{i+1}<x_{i-1}<\cdots<x_p \text{ and } x_i=x_{i+1}\}\,.\end{aligned}\tag{6.85}$$

6.2.2 Closed differential equation

The admissible and chain integrals $\boldsymbol{S}^f_{n,p}$ and $\boldsymbol{Z}^f_{n,p}$ satisfy a closed partial differential equation when differentiated with respect to a basis point $x_i \in B$. It can be expressed explicitly using the chain identities from the previous section. The derivation is based on the calculations in refs. [2–4], generalising the $(n,p)=(n,2)$ result from ref. [3], and can be found in eq. (C.61), derived in appendix C.3. The result is the differential equation

$$\partial_i \boldsymbol{Z}^f_{n,p}(B) = \left(\sum_{k\geq 0}\sum_{\substack{r=1\\ r\neq i}}^{p} r^f_{n,p}(\boldsymbol{x}^{ri}_k)\, g^{(k)}_{ri}\right)\boldsymbol{Z}^f_{n,p}(B)\,,\tag{6.86}$$

where the components of the matrices $r^f_{n,p}(\boldsymbol{x}^{ri}_k)$ are explicitly given by the coefficients in the linear combination in eq. (C.61), from which certain properties of these

[5]The domain of convergence depending on the real part of the variables α_{ij} from eq. (6.78) is the same as for the genus-zero integrals from chapter 4. In both cases, the only singularities are due to the simple poles in $g^{(m)}(x)$ or $1/x$, respectively. Therefore, the condition (6.79) is sufficient to ensure convergence [132].

matrices may be deduced: they are homogeneous of degree one in the variables α_{ij},

$$\deg_\alpha(r^f_{n,p}(\boldsymbol{x}^{ri}_k)) = 1\,, \tag{6.87}$$

and homogeneous of degree $k-m$ in the variables $\eta_i \in \vec{\eta}$,

$$\deg_\eta(r^f_{n,p}(\boldsymbol{x}^{ri}_k)) = k-m\,, \tag{6.88}$$

where a l-th derivative $\partial^l_{\eta_i}$ is counted to be of degree $-l$. The matrices with $k<m$ are differential-operator-valued, while for the matrices $k,l\geq m$ with distinct labels r,i,q,j and any k,l, the commutation relation

$$[r^f_{n,p}(\boldsymbol{x}^{ri}_k), r^f_{n,p}(\boldsymbol{x}^{qj}_l)] = 0\,, \quad k,l\geq m, |\{r,i,q,j\}| = 4 \tag{6.89}$$

holds. By the basis transformation $\boldsymbol{B}_{n,p}\in \mathrm{Mat}_{d_{n,p}}(\mathbb{Z})$ from eq. (6.77) with

$$\deg_\alpha(\boldsymbol{B}_{n,p}) = \deg_\eta(\boldsymbol{B}_{n,p}) = 0\,, \tag{6.90}$$

the corresponding result for the admissible integrals is immediately obtained and given by

$$\partial_i \boldsymbol{S}^f_{n,p}(B) = \left(\sum_{k\geq 0}\sum_{\substack{r=1\\ r\neq i}}^{p} \boldsymbol{B}_{n,p}\, r^f_{n,p}(\boldsymbol{x}^{ri}_k)\, \boldsymbol{B}^{-1}_{n,p}\, g^{(k)}_{ri}\right) \boldsymbol{S}^f_{n,p}(B)\,. \tag{6.91}$$

Having the explicit form of the matrices $\boldsymbol{B}_{n,p}$ and $r^f_{n,p}(\boldsymbol{x}^{ri}_k)$ at hand, the above partial differential equations can in principle be solved by Picard iteration. The only task left is the calculation and regularisation of the initial values for converging base points $x_i\to x_j$, which depend on the concrete form of the function f.

6.2.3 Genus-one example

A prime example of an antisymmetric function satisfying the Fay identity and having a simple pole in its expansion is the Eisenstein–Kronecker series

$$f(z,\eta) = F(z,\eta,\tau) = \sum_{k\geq 0} g^{(k)}(z,\tau)\eta^{k-1} \tag{6.92}$$

from eq. (3.71), at a fixed value of τ, where only $g^{(1)}$ has a simple pole. Denoting the vertices by $\vec{z} = (z_1,\dots,z_n)$, this leads to the corresponding graph products in $G^{\mathrm{adm}}_F(\vec{z},\vec{\eta})$ and $G^{\mathrm{cha}}_F(\vec{z},\vec{\eta})$, and integrals discussed in section 5.2. The chain and admissible products of Eisenstein–Kronecker series are defined in eqs. (5.8) and (5.29) and are denoted by

$$\varphi^\tau(A) = \varphi_F(A) = \prod_{i=2}^{p} F_{a_{i-1}a_i}(\eta_{a_i\dots a_p}) \tag{6.93}$$

and

$$\varphi^\tau\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) = \varphi_F\left(\begin{smallmatrix} A \\ e(A) \end{smallmatrix}\right) = \prod_{i=1}^{p} F_{e(a_i)a_i}(\eta_{a_i\dots a_p}) \,, \tag{6.94}$$

respectively, where $A = (a_1, \dots, a_p)$, e is an admissible map, i.e. $1 \leq e(a_i) < a_i \leq n$, and

$$F_{ij}(\eta) = F(z_{ij}, \eta, \tau) = \sum_{k\geq 0} g_{ij}^{(k)} \eta^{k-1} = \sum_{k\geq 0} g^{(k)}(z_{ij}, \tau)\eta^{k-1} \,. \tag{6.95}$$

In agreement with the genus-one string amplitudes, the torus symmetry is used to fix one of the coordinates

$$z_1 = 0 \,. \tag{6.96}$$

If additionally the parameters $\alpha_{ij} = s_{ij}$ are identified with the Mandelstam variables s_{ij} and the factor u_F from eq. (6.80) is the genus-one Koba–Nielsen factor

$$u_F(1, z_2, \dots, z_n) = \mathrm{KN}^\tau_{12\dots n} \,, \tag{6.97}$$

the type-(n, p) chain and admissible integrals associated to F are simply the genus-one chain and admissible integrals of type (n, p) defined in eqs. (5.17) and (5.27). Thus, the corresponding vectors from eqs. (5.20) and (6.83) as well as eqs. (5.34) and (6.82) agree:

$$\begin{aligned} \boldsymbol{Z}^\tau_{n,p}(z_2, \dots, z_p) &= \boldsymbol{Z}^F_{n,p}(B) \,, \\ \boldsymbol{S}^\tau_{n,p}(z_2, \dots, z_p) &= \boldsymbol{S}^F_{n,p}(B) \,, \end{aligned} \tag{6.98}$$

where the base points are $B = \{0, z_2, \dots, z_p\}$. Accordingly, also the matrices in the differential eqs. (5.21) and (6.86) as well as eqs. (5.36) and (6.91) agree, i.e.

$$r^{\mathrm{cha}}_{n,p}(\boldsymbol{x}^{ri}_k) = r^F_{n,p}(\boldsymbol{x}^{ri}_k) \,, \tag{6.99}$$

$$r^{\mathrm{adm}}_{n,p}(\boldsymbol{x}^{ri}_k) = \boldsymbol{B}_{n,p}\, r^F_{n,p}(\boldsymbol{x}^{ri}_k)\, \boldsymbol{B}^{-1}_{n,p} \,. \tag{6.100}$$

6.2.4 Genus-zero example

The second example of a function $f(x,\eta)$ satisfying the total antisymmetry (6.1) and the Fay identity (6.2) is the fraction

$$f(x,\eta) = \frac{1}{x}\,. \tag{6.101}$$

Since it is independent of η, the weights $\vec{\eta}$ do not have to be specified. The corresponding graph products have already been introduced in eqs. (4.10) and (4.20), these are the chains of fractions

$$\varphi(A) = \varphi_{\frac{1}{x}}(A) = \frac{1}{x_{a_1a_2}x_{a_2a_3}\dots x_{a_{p-1}a_p}} \tag{6.102}$$

and admissible products of fractions

$$\varphi\left(\begin{smallmatrix}A\\e(A)\end{smallmatrix}\right) = \varphi_{\frac{1}{x}}\left(\begin{smallmatrix}A\\e(A)\end{smallmatrix}\right) = \prod_{i=1}^{p}\frac{1}{x_{e(a_i),a_i}}\,, \tag{6.103}$$

where $A=(a_1,\dots,a_p)$ and e is admissible, i.e. $1\leq e(a_i)<a_i\leq n$.

Describing a basis of the corresponding graph integrals $\boldsymbol{Z}^{\frac{1}{x}}_{n,p}$ and $\boldsymbol{S}^{\frac{1}{x}}_{n,p}$ is not as straightforward as in the previous genus-one case: first, in order to recover the genus-zero string integrals, three of the punctures x_1, x_2 and x_3 are fixed according to[6]

$$(x_1,x_2,x_3) = (\infty,0,1)\,. \tag{6.104}$$

The fixing of $x_1=\infty$ introduces a redundancy, since any graph product containing the label one will vanish, i.e. for $1\in\{a_1,\dots,a_p\}$

$$\varphi(a_1,a_2,\dots,a_p) = 0 \tag{6.105}$$

and for $1\in\{a_1,\dots,a_p\}\cup e(\{a_1,\dots,a_p\})$

$$\varphi\left(\begin{smallmatrix}a_2 & \dots & a_p\\ e(a_2) & \dots & e(a_p)\end{smallmatrix}\right) = 0\,. \tag{6.106}$$

Second, for the n vertices $\vec{x}=(x_1,\dots,x_n)$, the p base points $B=\{\infty,0,1,x_4,\dots,x_p\}$ and choosing $\alpha_{ij}=s_{ij}$ as well as the genus-zero Koba–Nielsen factor for the factor

[6]Note that this is a slightly different convention to eq. (4.1) in section 4.1. They can be related to each other with the corresponding shift $x_i\to x_{i+1}$ for $1\leq i<n$ and $x_n\to x_1$. The convention here is more natural in the construction of the general class of graph integrals and also used in the context of string amplitudes in refs. [4, 140]. However, the conventions from section 4.1 are closer to the genus-one conventions in chapter 5.

from eq. (6.80),

$$u_{\frac{1}{x}} = \mathrm{KN}_{23\ldots n}\,, \tag{6.107}$$

the admissible and chain integrals in $\boldsymbol{Z}^{\frac{1}{x}}_{n,p}$ and $\boldsymbol{S}^{\frac{1}{x}}_{n,p}$ are not independent of each other: since $f(x,\eta) = 1/x$ is independent of η, there are integration-by-parts relations among the integrals due to eq. (6.78). Thus, the vectors $\boldsymbol{Z}^{\frac{1}{x}}_{n,p}$ and $\boldsymbol{S}^{\frac{1}{x}}_{n,p}$ can be reduced even more, which is e.g. implemented by the matrix $\boldsymbol{U}_{0,n}$ from eq. (5.62).

To summarise, all integrals with the factors

$$\varphi\left(\begin{smallmatrix} A_e(1) \\ e(A_e(1)) \end{smallmatrix}\right)\ldots\varphi\left(\begin{smallmatrix} A_e(p) \\ e(A_e(p)) \end{smallmatrix}\right) \tag{6.108}$$

and

$$\varphi\,(1, A_1)\ldots\varphi\,(p, A_p) \tag{6.109}$$

defined in eqs. (6.69) and (6.70), such that $A_e(1), A_1 \neq \varnothing$ vanish. Moreover, integration by parts can be used to relate for a particular $2 \leq i_0 \leq p$ an integral with a non-empty sequence $A_e(i_0)$ or A_{i_0}, respectively, to the integrals with empty sequence $A_e(i_0) = A_{i_0} = \varnothing$. Therefore, for the choice $i_0 = 2$ the independent integrals are the ones which have a factor of (6.108) or (6.109) with

$$A_e(1) = A_e(2) = A_1 = A_2 = \varnothing\,. \tag{6.110}$$

This reduces the $d_{n,p}$-dimensional vectors $\boldsymbol{Z}^{\frac{1}{x}}_{n,p}$ and $\boldsymbol{S}^{\frac{1}{x}}_{n,p}$ to the $d_{n-2,p-2}$-dimensional vectors

$$\tilde{\boldsymbol{Z}}^{\frac{1}{x}}_{n,p} = \int_{\Delta_{n,p}} \prod_{k=p+1}^{n} \mathrm{d}x_k\ \mathrm{KN}_{23\ldots n}\left(\varphi_f\left(3, \sigma_3(A_3)\right)\ldots\varphi_f\left(p, \sigma_p(A_p)\right)\right)_{\substack{(A^3,\ldots,A^p)=\sigma(A)\\ \sigma\in S_{n-p}}} \tag{6.111}$$

and

$$\tilde{\boldsymbol{S}}^{\frac{1}{x}}_{n,p} = \int_{\Delta_{n,p}} \prod_{k=p+1}^{n} \mathrm{d}x_k\ \mathrm{KN}_{23\ldots n}\left(\varphi_f\left(\begin{smallmatrix} A_e(3) \\ e(A_e(3)) \end{smallmatrix}\right)\ldots\varphi_f\left(\begin{smallmatrix} A_e(p) \\ e(A_e(p)) \end{smallmatrix}\right)\right)_{3\leq e(k)<k}\,, \tag{6.112}$$

where $A = (p+1, p+2, \ldots, n)$. Up to the shift $(1, 2, \ldots, n) \to (2, 3, \ldots, n, 1)$ in the indices mentioned in footnote 6, these are exactly the basis vectors $\boldsymbol{Z}_{n,p}$ and $\boldsymbol{S}_{n,p}$ of the genus-zero chain and admissible integrals of type-(n,p) defined in eqs. (4.54) and (4.62).

Note that the reduction of the vector $\boldsymbol{S}^{\frac{1}{x}}_{n,p}$ to $\tilde{\boldsymbol{S}}^{\frac{1}{x}}_{n,p}$ trivially carries over to the transformation matrix $\boldsymbol{B}_{n,p}$ from eq. (6.71) by simply deleting the corresponding components. The same holds for the reduction from $\boldsymbol{Z}^{\frac{1}{x}}_{n,p}$ to $\tilde{\boldsymbol{Z}}^{\frac{1}{x}}_{n,p}$. Thus, the matrix

$\tilde{\boldsymbol{B}}_{n,p}$ in

$$\tilde{\boldsymbol{S}}^{\frac{1}{x}}_{n,p} = \tilde{\boldsymbol{B}}_{n,p}\,\tilde{\boldsymbol{Z}}^{\frac{1}{x}}_{n,p} \tag{6.113}$$

can be obtained from $\boldsymbol{B}_{n,p}$ by deleting the corresponding rows and columns, or, alternatively, according to the same recursive procedure given in eq. (6.67).

Each of the vectors $\tilde{\boldsymbol{S}}^{\frac{1}{x}}_{n,p}$ and $\tilde{\boldsymbol{Z}}^{\frac{1}{x}}_{n,p}$ contains a basis of the twisted de Rham cohomology $H^{n-p}(\mathcal{F}_{n,p},\nabla_{n-p})$ defined in eq. (4.47), with dimension

$$\dim\big(H^{n-p}(\mathcal{F}_{n,p},\nabla_{n-p})\big) = d_{n-2,p-2} = \frac{(n-3)!}{(p-3)!}\,, \tag{6.114}$$

cf. eq. (4.50). As pointed out in ref. [4], the differential forms

$$\prod_{i=3}^{p}\varphi\left(\begin{smallmatrix}A_i\\ e(A_i)\end{smallmatrix}\right)\mathrm{d}x_n\wedge\cdots\wedge\mathrm{d}x_{p+1} \tag{6.115}$$

in the admissible integrals are exactly the fibration basis introduced in ref. [139]. Thus, the reduced matrices $\tilde{\boldsymbol{B}}_{n,p}\,\tilde{r}^{\frac{1}{x}}_{n,p}(\boldsymbol{x}_0^{ri})\,\tilde{\boldsymbol{B}}^{-1}_{n,p}$ in the differential equation (6.91) of $\tilde{\boldsymbol{S}}^{\frac{1}{x}}_{n,p}(x_4,\ldots,x_p)$, i.e.

$$\partial_i\tilde{\boldsymbol{S}}^{\frac{1}{x}}_{n,p}(\infty,0,1,x_4,\ldots,x_p) = \left(\sum_{\substack{r=1\\ r\neq i}}^{p}\frac{\tilde{\boldsymbol{B}}_{n,p}\,\tilde{r}^{\frac{1}{x}}_{n,p}(\boldsymbol{x}_0^{ri})\,\tilde{\boldsymbol{B}}^{-1}_{n,p}}{x_{ri}}\right)\tilde{\boldsymbol{S}}^{\frac{1}{x}}_{n,p}(\infty,0,1,x_4,\ldots,x_p)\,. \tag{6.116}$$

for $4\leq i\leq p$, are the braid matrices [140, 142, 145]

$$\tilde{\boldsymbol{B}}_{n,p}\,\tilde{r}^{\frac{1}{x}}_{n,p}(\boldsymbol{x}_0^{ri})\,\tilde{\boldsymbol{B}}^{-1}_{n,p} = \boldsymbol{\Omega}^{ri}_{n,p}\,. \tag{6.117}$$

The braid matrices $\boldsymbol{\Omega}^{ri}_{n,p}$, in turn, are $d_{n-2,p-2}$-dimensional representations of the genus-zero braid group and recursively defined[7] as follows [140,142,145]: $\boldsymbol{\Omega}^{ij}_{(n,n)} = -s_{ij}$

[7] The minus sign in $\boldsymbol{\Omega}^{ij}_{(n,n)} = -s_{ij}$ is due to the sign in the definition of the Mandelstam variables in eq. (1.3).

and for $q, r = 3, 4, \ldots, p-1$ by the components

$$(\mathbf{\Omega}^{ij}_{(n,p-1)})_{qr} = \begin{cases} \mathbf{\Omega}^{ij}_{(n,p)} & \text{if } q = r, q \neq i, j, \\ \mathbf{\Omega}^{pj}_{(n,p)} + \mathbf{\Omega}^{qj}_{(n,p)} & \text{if } q = r = i, j \neq 1, 2, \\ \mathbf{\Omega}^{pj}_{(n,p)} + \mathbf{\Omega}^{qj}_{(n,p)} + \mathbf{\Omega}^{pq}_{(n,p)} & \text{if } q = r = i, j = 1, 2, \\ \mathbf{\Omega}^{ip}_{(n,p)} + \mathbf{\Omega}^{iq}_{(n,p)} & \text{if } q = r = j, \\ -\mathbf{\Omega}^{pj}_{(n,p)} & \text{if } q = i, r = j, \\ -\mathbf{\Omega}^{ip}_{(n,p)} & \text{if } q = j, r = i, \\ \mathbf{\Omega}^{ip}_{(n,p)} & \text{if } j = 1, r = i, q \neq i, \\ \mathbf{\Omega}^{pr}_{(n,p)} & \text{if } j = 2, q = i, r \neq i, \\ 0 & \text{otherwise}, \end{cases} \tag{6.118}$$

where $4 \leq i \leq p-1$, $1 \leq j \leq p-1$ and $i \neq j$.

For example, the two independent type-$(5,4)$ matrices are

$$\mathbf{\Omega}^{42}_{(5,4)} = \begin{pmatrix} -s_{24} & 0 \\ -s_{35} & -s_{245} \end{pmatrix}, \quad \mathbf{\Omega}^{43}_{(5,4)} = \begin{pmatrix} -s_{34} - s_{45} & s_{45} \\ s_{35} & -s_{34} - s_{35} \end{pmatrix}. \tag{6.119}$$

These are (up to the shift in the labels) the matrices appearing in the example (4.72), which agrees with eq. (6.117) since $\boldsymbol{B}_{5,4} = \mathbb{1}_2$. For $(n,p) = (6,4)$, they are of the form

$$\mathbf{\Omega}^{42}_{(6,4)} = \begin{pmatrix} -s_{24} & 0 & 0 & 0 & 0 & 0 \\ -s_{36} & -s_{246} & -s_{65} & 0 & 0 & 0 \\ 0 & 0 & -s_{24} & 0 & 0 & 0 \\ -s_{35} - s_{56} & 0 & s_{56} & -s_{245} & 0 & 0 \\ 0 & -s_{35} & 0 & -s_{36} & -s_{2456} & 0 \\ s_{36} & 0 & -s_{35} - s_{36} & -s_{36} & 0 & -s_{2456} \end{pmatrix} \tag{6.120}$$

as well as

$$\mathbf{\Omega}^{43}_{(6,4)} = \begin{pmatrix} -s_{4,356} & s_{46} & 0 & s_{45} & 0 & 0 \\ s_{36} & -s_{35,46} & s_{56} & 0 & s_{45} + s_{56} & -s_{56} \\ 0 & s_{46} & -s_{4,356} & 0 & -s_{46} & s_{45} + s_{46} \\ s_{35} + s_{56} & 0 & -s_{56} & -s_{36,45} & s_{46} & s_{56} \\ 0 & s_{35} & 0 & s_{36} & -s_{3,456} & 0 \\ -s_{36} & 0 & s_{35} + s_{36} & s_{36} & 0 & -s_{3,456} \end{pmatrix}, \tag{6.121}$$

which are the matrices from eqs. (4.79) and (4.2.3) with the corresponding shift in the labels.

Therefore, from rearranging eq. (6.117) the matrices $\tilde{r}^{\frac{1}{x}}_{n,p}(\boldsymbol{x}_0^{ri})$ are explicitly known, which yields, upon shifting the labels by one, explicit expressions for the matrices

$$r_n^{\mathrm{cha}}(\boldsymbol{e}_p^{ir}) = \tilde{r}^{\frac{1}{x}}_{n,p}(\boldsymbol{x}_0^{ri}) = \tilde{\boldsymbol{B}}_{n,p}^{-1}\,\boldsymbol{\Omega}_{n,p}^{ri}\,\tilde{\boldsymbol{B}}_{n,p} \tag{6.122}$$

and

$$r_n^{\mathrm{adm}}(\boldsymbol{e}_p^{ir}) = \boldsymbol{\Omega}_{n,p}^{ri} \tag{6.123}$$

from the KZ equations (4.55) and (4.66) of the genus-zero vectors $\boldsymbol{Z}_{n,p}(x_3,\dots,x_{p-1})$ and $\boldsymbol{S}_{n,p}(x_3,\dots,x_{p-1})$, respectively.

The braid matrices satisfy the infinitesimal pure braid relations (4.56). This can for example be derived recursively from eq. (6.118) or from the Schwarz integrability condition. The latter calculation goes as follows for a solution of the KZ equation

$$\partial_i \boldsymbol{S} = \sum_{\substack{j=1\\ j\neq i}}^{p} \frac{\boldsymbol{x}_{ij}}{x_{ij}} \boldsymbol{S}\,, \tag{6.124}$$

where $\boldsymbol{x}_{ij}$ is homogeneous of degree one in some parameter α': for $i{\neq}j{\in}\{1,\dots,p\}$, the following second order derivative is given by

$$\begin{aligned}
\partial_i\partial_j \boldsymbol{S} &= \partial_i \left(\sum_{\substack{q=1\\ q\neq j}}^{p} \frac{\boldsymbol{x}_{jq}}{x_{jq}} \boldsymbol{S} \right)\\
&= \sum_{\substack{q=1\\ q\neq j,i}}^{p} \sum_{\substack{r=1\\ r\neq i,j,q}}^{p} \frac{\boldsymbol{x}_{jq}\boldsymbol{x}_{ir}}{x_{jq}x_{ir}} \boldsymbol{S} + \sum_{\substack{q=1\\ q\neq j,i}}^{p} \left(\frac{\boldsymbol{x}_{jq}\boldsymbol{x}_{ij}}{x_{jq}x_{ij}} + \frac{\boldsymbol{x}_{jq}\boldsymbol{x}_{iq}}{x_{jq}x_{iq}} + \frac{\boldsymbol{x}_{ji}\boldsymbol{x}_{iq}}{x_{ji}x_{iq}} \right) \boldsymbol{S}\\
&\quad + \frac{\boldsymbol{x}_{ji} - \boldsymbol{x}_{ji}\boldsymbol{x}_{ij}}{x_{ji}^2} \boldsymbol{S}\,.
\end{aligned} \tag{6.125}$$

Thus, using the functional dependence on the various punctures, the independence of the integrals $S_{n,p}$ and the fact that $\boldsymbol{x}_{ij}$ is proportional to α', the vanishing of the commutator

$$\begin{aligned}
0 &= [\partial_i, \partial_j]\boldsymbol{S} \\
&= \sum_{\substack{q=1\\ q\neq j,i}}^{p} \sum_{\substack{r=1\\ r\neq i,j,q}}^{p} \frac{[\boldsymbol{x}_{jq}, \boldsymbol{x}_{ir}]}{x_{jq}x_{ir}}\boldsymbol{S} \\
&\quad + \sum_{\substack{q=1\\ q\neq j,i}}^{p} \left(\frac{[\boldsymbol{x}_{jq}, \boldsymbol{x}_{ij}]}{x_{jq}x_{ij}} + \frac{[\boldsymbol{x}_{jq}, \boldsymbol{x}_{iq}]}{x_{ji}x_{iq} + x_{ij}x_{jq}} + \frac{[\boldsymbol{x}_{ji}, \boldsymbol{x}_{iq}]}{x_{ji}x_{iq}} \right) \boldsymbol{S} \\
&\quad + \frac{\boldsymbol{x}_{ji} - \boldsymbol{x}_{ij} + [\boldsymbol{x}_{ij}, \boldsymbol{x}_{ji}]}{x_{ji}^2}\boldsymbol{S} \\
&\quad + \sum_{\substack{q=1\\ q\neq j,i}}^{p} \left(\frac{[\boldsymbol{x}_{jq}, \boldsymbol{x}_{ij} + \boldsymbol{x}_{iq}]}{x_{jq}x_{ij}} + \frac{[\boldsymbol{x}_{jq} + \boldsymbol{x}_{ji}, \boldsymbol{x}_{iq}]}{x_{ji}x_{iq}} \right) \boldsymbol{S} \\
&\quad + \frac{\boldsymbol{x}_{ji} - \boldsymbol{x}_{ij} + [\boldsymbol{x}_{ij}, \boldsymbol{x}_{ji}]}{x_{ji}^2}\boldsymbol{S}
\end{aligned} \tag{6.126}$$

leads to the infinitesimal pure braid relations

$$\begin{aligned}
\boldsymbol{x}_{ij} &= \boldsymbol{x}_{ji}\,, \\
[\boldsymbol{x}_{ir}, \boldsymbol{x}_{jq}] &= 0 \quad \text{if } |\{i,j,q,r\}| = 4\,, \\
[\boldsymbol{x}_{ij} + \boldsymbol{x}_{jq}, \boldsymbol{x}_{iq}] &= 0 \quad \text{if } |\{i,j,q\}| = 3\,.
\end{aligned} \tag{6.127}$$

Chapter 7

Conclusion

7.1 Summary of results

In this thesis, the main results of the publications [1–4] have been presented, interrelated and embedded into the appropriate mathematical background.

Elliptic multiple polylogarithms

Question 1 was addressed in refs. [1, 2]. In the former reference, various notions of elliptic generalisations of polylogarithms have been formulated in terms of the ePLs $\tilde{\Gamma}_n(m; z)$ defined in eq. (3.90) as iterated integrals on the torus. The corresponding expressions are functional relations among different elliptic generalisations of polylogarithms. Having such relations at hand paves the way to canonical expressions for elliptic scattering amplitudes. Additional functional relations among dilogarithmic ePLs have been derived, leading to a class of functional relations parametrised by elliptic functions. Moreover, this approach yields an alternative proof of the elliptic Bloch relation (3.138). In the latter ref. [2], the elliptic KZB associator $\Phi^\tau_{\mathcal{X}}$ as defined in eq. (3.120) has been identified to satisfy the genus-one associator equation (3.123). Concretely, the following results have been shown:

- The elliptic Bloch–Wigner dilogarithm D^τ defined in eq. (3.128) has been translated from the Tate curve to the torus, leading to the expression (3.179) in terms of ePLs $\tilde{\Gamma}_n(m; z)$.
- Generalising the above result for the elliptic Bloch–Wigner dilogarithm, the single-valued ePLs $\mathrm{D}^\tau_{a,b}$ defined in eq. (3.133) on the Tate curve and originally constructed in ref. [89] have been expressed in terms of ePLs $\tilde{\Gamma}_n(m; z)$ as well. Again, this yields a translation from the Tate curve to the torus, the final expressions are obtained from eq. (3.135). Similarly, the subclass of single-valued ePLs $\mathcal{L}^\tau_n$ from eq. (3.127) can be expressed on the torus via eq. (3.131).
- Using the translation in ref. [63] of the eMPLs $\tilde{\Gamma}_w(z)$ defined in eq. (3.89) from the torus to the elliptic curve, the single-valued ePLs $\mathcal{L}^\tau_n(x)$ and $\mathrm{D}^\tau_{a,b}(x)$ as well as all the corresponding identities can be translated to the elliptic curve.

- These translations yield a formulation of the elliptic Bloch relation on the torus given by the three lines (3.188a), (3.188b) and (3.188c). However, these lines already vanish separately, which yields the holomorphic elliptic Bloch relations (3.184) and (3.185) as well as the non-holomorphic combinations (3.186) and (3.187). These identities provide an alternative prove of the elliptic Bloch relation.

- The elliptic KZB associator $\tilde{\Gamma}_{\mathcal{X}}(z)$ as defined in eq. (3.114) with an a-priori unspecified alphabet $\mathcal{X}$ satisfies the genus-one associator equation (3.123). This is the genus-one generalisation of the genus-zero associator equation (2.50).

Multiple polylogarithms in open-string corrections at genus zero

Question 2 has been answered in ref. [4] by giving for any n a recursive algorithm to derive the matrices appearing in the alphabet $\mathcal{E}_n$ required for the genus-zero, open-string recursion (4.105), which has already been given schematically in eqs. (1.5) and (2.53). The derivation of these expressions involved a reformulation of the recursion in twisted de Rham theory. In this thesis, a slightly different notation and more general classes of integrals have been introduced to present these results. In particular:

- The vector $\hat{\boldsymbol{F}}_n$ of integrals from eq. (4.85) and originally constructed in ref. [5] to formulate the genus-zero, open-string recursion (4.105) has been translated to the type-$(n, 4)$ chain and admissible integrals $\boldsymbol{Z}_{n,4}$ and $\boldsymbol{S}_{n,4}$ in eq. (4.90). The corresponding transformation matrix can be calculated by the algorithm from appendix C.2.

- The vectors $\boldsymbol{Z}_{n,p}$ and $\boldsymbol{S}_{n,p}$ of the type-(n,p) chain and admissible integrals from eqs. (4.54) and (4.62), respectively, contain a basis of $H^{n-p}(\mathcal{F}_{n,p}, \nabla_{n-p})$, the twisted de Rham cohomology of the n-punctured Riemann spheres with p fixed coordinates. In particular, the vector $\boldsymbol{S}_{n,p}$ of admissible integrals contains the fibration basis from ref. [140]. Both vectors satisfy a differential equation of Fuchsian type given in eqs. (4.55) and (4.66). The involved matrices are shown to be genus-zero braid matrices which can explicitly be calculated by the recursion from eq. (6.118) and the algorithm from appendix C.2. Thus, the integrals $\boldsymbol{Z}_{n,p}$ and $\boldsymbol{S}_{n,p}$ can in principle be solved via Picard iteration.

- The above investigations have motivated the introduction of the graphical representation properly formulated in chapter 6. It was crucial to structure the combinatorial problems involved in the calculations.

Elliptic multiple polylogarithms in open-string corrections at genus one

Deriving an answer to question 3 by generalising the genus-zero, open-string recursion from ref. [5] to calculate the open-string corrections at genus one was the main focus of ref. [2] and this thesis. In ref. [3] the results have been supplemented by the alternative approach from refs. [38, 39] to derive the genus-one, open-string corrections. This has lead to the genus-one, open-string recursion (5.98), which has already been given schematically in eqs. (1.7) and (3.126). The main steps towards this recursion were the following:

- Genus-one, type-(n,p) chain and admissible integrals have been defined in eqs. (5.17) and (5.27), respectively. The vectors $\boldsymbol{Z}^{\tau}_{n,p}$ and $\boldsymbol{S}^{\tau}_{n,p}$ from eqs. (5.20) and (5.34) of these integrals are related by the transformation in eq. (5.33), where the corresponding transformation matrix is determined by eq. (5.32).

- On the one hand, the genus-one Selberg integrals of type-(n,p) introduced in ref. [2] are the coefficients in the $\vec{\eta}$-expansion of the admissible integrals in $\boldsymbol{S}^{\tau}_{n,p}$, cf. eq. (5.30). On the other hand, the Z^{τ}_{n}-integrals in $\boldsymbol{Z}^{\tau}_{n}$ from eq. (5.11) introduced in ref. [39] are exactly the type-(n,1) chain integrals $\boldsymbol{Z}^{\tau}_{n,1}$, cf. eq. (5.23). Both, the genus-one, type-$(n,1)$ Selberg integrals and the coefficients in the $\vec{\eta}$-expansion of the Z^{τ}_{n}-integrals contain the n-point, genus-one, open-string corrections. Therefore, the type-$(n,1)$ vectors $\boldsymbol{Z}^{\tau}_{n,1}$ and $\boldsymbol{S}^{\tau}_{n,1}$ are generating series of the n-point, open-string corrections at genus one.

- The vectors $\boldsymbol{Z}^{\tau}_{n,p}$ and $\boldsymbol{S}^{\tau}_{n,p}$ satisfy the elliptic KZB systems on the p-punctured torus given in eqs. (5.21) and (5.36), respectively. For the chain integrals $\boldsymbol{Z}^{\tau}_{n,p}$, the corresponding matrices are explicitly calculated in appendix C.3.1. The matrices appearing in the elliptic KZB system of the admissible integrals $\boldsymbol{S}^{\tau}_{n,p}$ can be obtained from the corresponding transformation given in eq. (5.37). These matrices can be used to solve the integrals in $\boldsymbol{Z}^{\tau}_{n,p}$ and $\boldsymbol{S}^{\tau}_{n,p}$ via Picard iteration, cf. ref. [65].

- The type-$(n,2)$ integrals $\boldsymbol{Z}^{\tau}_{n,2}$ and $\boldsymbol{S}^{\tau}_{n,2}$ satisfy an elliptic KZB system on the twice punctured torus, where the z_2-derivative is an elliptic KZB equation (3.112). Thus, the two regularised boundary values C^{τ}_{0} and C^{τ}_{1} for $z_2 \to 0$ and $z_2 \to 1$ can be calculated and related to each other via the genus-one associator equation (3.123). Since $\boldsymbol{Z}^{\tau}_{n,p}$ and $\boldsymbol{S}^{\tau}_{n,p}$ are related by an invertible matrix, it is sufficient to only consider one of the vectors. In this thesis, we have focused on $\boldsymbol{Z}^{\tau}_{n,p}$.

- The lower boundary value $C^{\tau}_{0}(\boldsymbol{Z}^{\tau}_{n,2})$ has been shown to contain the genus-zero type-$(n{+}1,3)$ vector $\boldsymbol{Z}_{n+1,3}$ which is nothing but $\boldsymbol{Z}_{n+1}$, the vector of Z_{n+1}-integrals appearing in the genus-zero, open-string corrections of $(n{+}1)$-point interactions, cf. eq. (4.57). The final expression of the boundary value $C^{\tau}_{0}(\boldsymbol{Z}^{\tau}_{n,2})$ is given in eq. (5.64). The correct regularisation has been calculated using the τ-derivative (5.52) of the elliptic KZB system satisfied by $\boldsymbol{Z}^{\tau}_{n,2}$.

- The upper boundary value $C_1^\tau(\boldsymbol{Z}_{n,2}^\tau)$ has also been calculated using the τ-derivative of the elliptic KZB system satisfied by $\boldsymbol{Z}_{n,2}^\tau$, resulting in eq. (5.81). It contains the genus-one, type-$(n-1,1)$ integrals $\boldsymbol{Z}_{n-1,1}^\tau$ and, thus, the generating series of the genus-one, open-string corrections $\boldsymbol{Z}_{n-1}^\tau$ for $(n-1)$-point interactions. The appropriate projection to extract $\boldsymbol{Z}_{n-1}^\tau$ from the boundary value $C_1^\tau(\boldsymbol{Z}_{n,2}^\tau)$ is given in eq. (5.87).

- Putting all together, the genus-one, open-string recursion (5.98) has been formulated, where all the involved matrices and expressions are known. It can be used to calculate the α'-expansion of the open-string corrections at genus one solely using matrix operations.

- The formulation of the genus-one recursion and the corresponding interpretation of the Mandelstam variables associated to the auxiliary puncture z_2 of $\boldsymbol{Z}_{n,2}^\tau(z_2)$ merging in the limits $C_0^\tau(\boldsymbol{Z}_{n,2}^\tau)$ and $C_1^\tau(\boldsymbol{Z}_{n,2}^\tau)$ to the puncture $z_1 = 0$ and around the A-cycle of the torus to $1 \equiv z_1$, respectively, motivates a geometric interpretation: the recursion implements a gluing mechanism, where the two external states parametrised by z_1 and z_2 in $C_0^\tau(\boldsymbol{Z}_{n,2}^\tau)$ of a genus-zero worldsheet are glued together to form a genus-one worldsheet with two fewer external states.

Graph products and integrals

In chapter 6 a graphical method to describe products of meromorphic functions $f(x,\eta)$ which satisfy the antisymmetry (6.1) and Fay identity (6.2) has been introduced. This construction generalises the graphical representation of fractions from ref. [4] and was used in refs. [3,4] to structure the combinatorics of recursive applications of the antisymmetry and partial fractioning or the Fay identity to products of fractions or Eisenstein–Kronecker series, respectively. In this thesis, the following results were obtained:

- The graphical method has been defined rigorously, based on directed, weighted, tree graphs $\mathcal{G}(\vec{x},\vec{\eta})$ with vertices $\vec{x}$. The weights of these graphs satisfy the continuity condition (6.7) and are linear combinations of the auxiliary variables $\vec{\eta}$.

- A product of $f(x,\eta)$ is assigned to each such graph by eq. (6.9). The free abelian group $G_f(\vec{x},\vec{\eta})$ generated by such products, which are images of graphs from $\mathcal{G}(\vec{x},\vec{\eta})$, is defined.

- Two sets of generators of $G_f(\vec{x},\vec{\eta})$ are presented, the admissible products and the chain products. Two algorithms to write any elements from $G_f(\vec{x},\vec{\eta})$ in terms of admissible products or chain products, respectively, are presented in appendix C.2.

- Generalisations of the type-(n, p) chain and admissible integrals from genus zero $\boldsymbol{Z}_{n,p}$, $\boldsymbol{S}_{n,p}$ and genus one $\boldsymbol{Z}^{\tau}_{n,p}$, $\boldsymbol{S}^{\tau}_{n,p}$ based on antisymmetric functions $f(x, \eta)$, which satisfy the Fay identity and have simple poles, are constructed. This has lead to vectors $\boldsymbol{Z}^{f}_{n,p}$, $\boldsymbol{S}^{f}_{n,p}$ of integrals in eqs. (6.83) and (6.82), which satisfy closed differential systems (6.86) and (6.91) similar to the elliptic KZB system.

- The closed formulæ for the matrices appearing in the differential systems (6.86) and (6.91) are calculated explicitly in eq. (C.61) using the graphical representation of products of $f(x, \eta)$ and graphical identities.

- Choosing $f(x, \eta)$ to be the Eisenstein–Kronecker series, these differential systems are the elliptic KZB systems satisfied by $\boldsymbol{Z}^{\tau}_{n,p}$, $\boldsymbol{S}^{\tau}_{n,p}$, such that the corresponding matrices relevant for the genus-one, open-string recursion (5.98) are readily obtained from the more general expression in eq. (C.61).

7.2 Outlook

The results from refs. [1–4] and this thesis open various discussions and directions in further research areas, ranging from pure mathematics to particle physics. Some of the most important aspects are briefly presented in the following paragraphs.

Loop KLT and elliptic single-valued map

Comparing with the genus-zero string corrections, an important question to pursue in further research projects is the relation of the genus-one, open-string corrections to the closed-string corrections. As mentioned in the introduction, various efforts such as in refs. [50–53] have been put forward recently to express the latter in terms of the former. However, a complete picture such as a genus-one KLT relation or an elliptic single-valued map is still missing. Extending the knowledge from genus zero to higher genera is essential to gain insights into the nature of quantum gravity and its relation to gauge theories.

A possible connection of refs. [2, 3] to genus-one, closed-string corrections might be the following: the appearance of elliptic multiple zeta values in the genus-one, open-string corrections via the elliptic KZB associator in the genus-one recursion (5.98) is reminiscent of the role of the generating series of multiple zeta values, the Drinfeld associator, in the genus-zero, open-string corrections given in the genus-zero recursion (4.105). The Drinfeld associator gives rise to a generating series of single-valued multiple zeta values, the Deligne associator, which can ultimately be related to closed-string amplitudes [32, 132]. This sheds another light on the (tree-level) KLT relation and leads to the question whether the same game can be played at genus one. An analogous mechanism at genus one would reveal valuable aspects of a conjectural genus-one KLT relation. Moreover, this would nicely fit into the picture of the geometric interpretation of the recursions (4.105) and (5.98) in terms of a gluing mechanism on the worldsheets: gluing together two open-string worldsheets at the boundaries would yield a closed-string worldsheet, cf. figure 1.2.

Higher-genus, open-string recursion

The canonical form of the genus-zero and genus-one, open-string recursions from eqs. (4.105) and (5.98), respectively, and their geometric interpretations suggest that such recursions can be found at higher genera as well. It is desirable to have a recursive method to calculate the open-string corrections $\boldsymbol{F}^{\text{open}}_{n,g}(\alpha', \vec{\tau})$ for each genus g and number of external states n in terms of simpler integrals at lower genera. Thus, a full recursion in the genus, where the genus-one recursion (5.98) is the first step is expected to be constructible. The corresponding mathematical structures and open-string corrections beyond genus one need to be investigated. The graphical method from chapter 6 might be applicable to higher genera as well and helpful to derive various results analogous to the ones presented in this thesis.

Feynman integrals

The method of refs. [2, 3] to compute (loop) corrections via the recursions from eqs. (4.105) and (5.98) might be applicable to amplitudes in field theories as well. A convenient connection is the (ambi)twistor string which leads to field-theory amplitudes, while the underlying interactions are still describable in terms of vertex-operation insertions on worldsheets [150, 151]. Another point to interconnect the string mechanism to field theories should be its relation to the Berends–Giele recursion [152], which has a particularly promising origin from a minimal model in the Batalin–Vilkovisky (BV) formalism [153, 154].

Mathematical aspects

While various mathematical questions have been settled in this thesis, further questions remain open, which may be crucial for progress concerning the physical aspects mentioned in the previous paragraphs.

For example, the vectors of iterated integrals constructed in this thesis and the matrices appearing in their differential systems should be properly embedded into the existing mathematical literature. This is a crucial step to extract more information from the presented results, which might lead to further physical insights. The relation of the integrals presented in section 5.2 to the theory of the elliptic KZB system on the p-punctured torus investigated in refs. [40, 116] is of particular interest.

An open question concerning the generalisation of the genus-one, open-string recursion (5.98) to higher genera is its connection to topological recursions [155,156].

The genus-one, open-string recursion presented in this thesis may be an example of one recursive step in a topological recursion. However, its proper mathematical description in the framework of topological recursions is not clear yet, but may be crucial for the construction of further recursive steps at higher genera.

The genus-one, open-string recursion (5.98) might be expressible in the BV formalism, which may lead to a connection between the integrals in this thesis and well-known algebraic structures. One starting point to establish such a connection might be the relation of the genus-zero, open-string recursion (4.105) to the method to calculate the genus-zero, open-string corrections via a Berends–Giele recursion from ref. [30]. Such a relation might be generalisable to the genus-one recursion as well and ultimately lead to a relation to the recursions from ref. [154] formulated in the BV formalism.

Moreover, in order to calculate the full open-string corrections $\boldsymbol{M}^{\text{open}}_{n,g}(\alpha')$ introduced in eq. (1.4), the integrals over the modular parameters have to be understood as well.

Regarding the results of ref. [1] on the functional relations of eMPLs, there is still a lot to do. We have only scraped on the boundary of the space of functional relations. Various connections to the algebraic and number-theoretic aspects nicely summarised and investigated in ref. [67] might be constructible.

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A necessary condition for progress in research is the curiosity, dedication and passion of countless people for developing new ideas and expanding the boundaries of knowledge. For thousands of years, innovative people have advanced societies based on more and more sophisticated scientific insights which have lead to the prosperity we encounter today. May this evolution continue and always aim for the well-being of all the inhabitants of this planet.

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Declaration

"Hiermit erkläre ich, die Dissertation selbstständig und nur unter Verwendung der angegebenen Hilfen und Hilfsmittel angefertigt zu haben. Ich habe mich nicht anderwärts um einen Doktorgrad in dem Promotionsfach beworben und besitze keinen entsprechenden Doktorgrad. Die Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 42 am 11. Juli 2018, habe ich zur Kenntnis genommen."

"I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 42 on July 11 2018."

Signature: ______________________________

Date: ______________________________

Chapter A

Group addition on elliptic curve

In this section, the additive group structure on the elliptic curve is described. The following paragraphs are up to a few minor changes formulated in appendix A of ref. [1].

The geometric picture of the addition on the elliptic curve is that two distinct points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ with $y_1 \neq \pm y_2$ form a line which intersects the elliptic curve $y^2 = 4x^3 - g_2 x - g_3$ at a third point $-P_3 = (x_3, -y_3)$. The sum $P_3 = P_1 + P_2$ is defined as being the projection of $-P_3 = (x_3, -y_3)$ to the negative y-coordinate $P_3 = (x_3, y_3)$, cf. the top left graph in figure A.1. Thus, two points with their y-coordinate being of the opposite sign are indeed inverse to each other with ∞ being the unit element since the line defined by P_3 and $-P_3$ intersects the elliptic curve only at infinity, cf. the top right graph in figure A.1.

The algebraic description is the following: for P_1 and P_2 as above, the line intersecting them is given by $y = \lambda x + \mu$, where

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1}, \qquad \mu = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}. \tag{A.1}$$

The x-coordinate of the third point $-P_3 = (x_3, -y_3)$ intersecting the line and the elliptic curve is the third solution (besides x_1 and x_2) of the cubic equation

$$(\lambda x + \mu)^2 = 4x^3 - g_2 x - g_3 , \tag{A.2}$$

which is in terms of x_1 and x_2 given by

$$x_3 = -x_1 - x_2 + \frac{\lambda^2}{4}. \tag{A.3}$$

The y coordinate of P_3 is then simply the negative of the y coordinate determined by the line and x_3,

$$y_3 = -\lambda x_3 - \mu . \tag{A.4}$$

The last case we need to consider is if the points P_1 and P_2 are identical and not

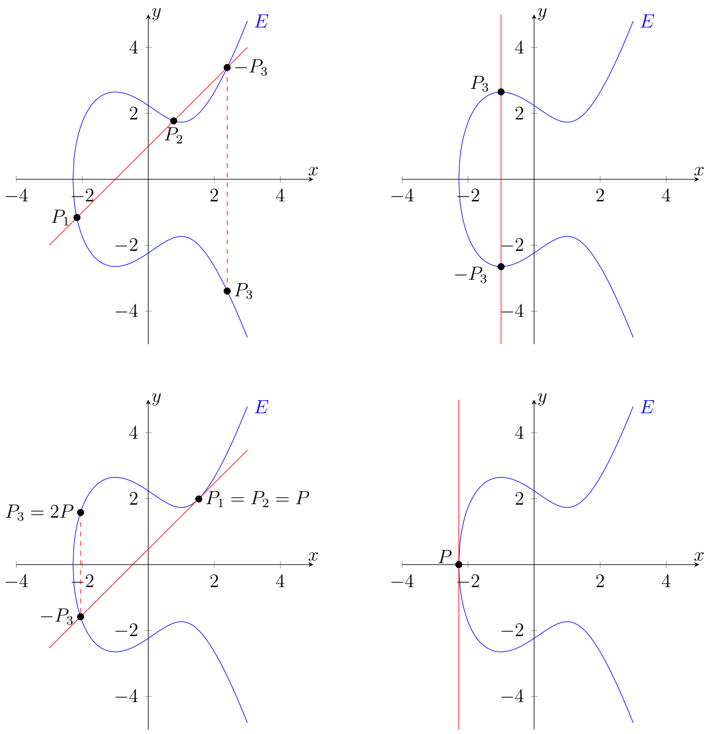


Figure A.1: The geometric picture of the addition on the elliptic curve for the example $E : y^2 = x^3 - 3x + 5$. On the top left, the sum of two distinct punctures $P_1 + P_2 = P_3$ is given by the projection (red dashed line) of $-P_3$ to the negative y-component. $-P_3$ in turn is the third point of intersection of the (red) line, which goes through P_1 and P_2. As shown on the bottom left, if the two punctures are the same $P_1 = P_2 = P$, the corresponding line is the tangent to E at P, which determines $2P$. Similarly, the inverse of a point P_3 is simply the point with the y-component of the opposite sign and the unit element is ∞, cf. top right. If the y-component of P is zero, then it is its own inverse, as depicted on the bottom right.

the unit element, i.e. $P_1 = P_2 = P = (x_P, y_P)$. For $y_P \neq 0$, the above description of taking the line intersecting P_1 and P_2 degenerates to taking the tangent on the elliptic curve at P, cf. the bottom left graph in figure A.1. The sum of twice the point P, i.e. $2P = P + P = (x_{2P}, y_{2P})$, is then again the projection of the second point lying on this tangent and the elliptic curve with respect to the x-coordinate. Algebraically, this corresponds to

$$\lambda = \frac{12x_P^2 - g_2}{2y_P}\,, \qquad \mu = y_P - \lambda x_P \tag{A.5}$$

and

$$x_{2P} = -2x_P + \frac{\lambda^2}{4}\,, \qquad y_{2P} = -\lambda x_{2P} - \mu \tag{A.6}$$

as before. In the case of $y_P = 0$, the point P is inverse to itself, such that in particular $P + P = P - P = \infty$, cf. the bottom right graph in figure A.1.

These addition rules exactly agree with the well-known addition formula of the

Weierstrass $\wp$-function

$$\wp(x_1 + x_2) = -\wp(x_1) - \wp(x_2) + \frac{1}{4}\left(\frac{\wp'(x_2) - \wp'(x_1)}{\wp(x_2) - \wp(x_1)}\right)^2 \tag{A.7}$$

for $x_1 \neq x_2$ and similar for its derivative. This ensures that $\phi_{\tau,E}$ defined in eq. (3.36) is indeed a homomorphism.

Chapter B

Elliptic polylogarithms

In this chapter various properties of ePLs are given following the lines of ref. [1]. They complement certain aspects discussed in chapter 3.

In section B.1 the q-expansion of the integration kernels $g^{(m)}$ and the ePLs $\tilde{\Gamma}_n(m;z)$ are given. In section B.2 some calculations and derivations of to the main results from section 3.5 are written out explicitly.

B.1 q-expansions

The q-expansion (3.72) of the Jacobi θ-function can be used to deduce the corresponding expansion of the integration kernels via eqs. (3.71) and (3.76). This yields [27]

$$\begin{aligned} g^{(0)}(z,\tau) &= 1\,, \\ g^{(1)}(z,\tau) &= \pi\cot(\pi z) + 4\pi \sum_{k,l>0} \sin(2\pi k z) q^{kl}\,, \\ g^{(2m)}(z,\tau) &= -2\zeta_{2m} - 2\frac{(2\pi i)^{2m}}{(2m-1)!} \sum_{k,l>0} \cos(2\pi k z) l^{2m-1} q^{kl}\,, \\ g^{(2m+1)}(z,\tau) &= -2i\frac{(2\pi i)^{2m+1}}{(2m)!} \sum_{k,l>0} \sin(2\pi k z) l^{2m} q^{kl}\,, \end{aligned} \tag{B.1}$$

where $m > 0$.

Integrating these q-expansions n times and additionally integrating $n-1$ times the q-expansion (3.63) of the regularised integral $\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix};z,\tau)$ results in the q-expansions of the ePLs $\tilde{\Gamma}_n(m;z)$ defined in eq. (3.90). They have been calculated in ref. [1] based on the following iterated integrals of $\sin(2\pi kz)$ with $k \in \mathbb{Z}$

$$\begin{aligned}
\int_0^z \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2 \cdots \int_0^{z_{2n-1}} \mathrm{d}z_{2n} \sin(2\pi k z_{2n}) &= \frac{(-1)^n}{(2\pi k)^{2n}} \sin(2\pi kz) \\
&\quad + \sum_{j=1}^{n} \frac{(-1)^{n-j}}{(2\pi k)^{2n+1-2j}} \frac{z^{2j-1}}{(2j-1)!}\,, \\
\int_0^z \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2 \cdots \int_0^{z_{2n}} \mathrm{d}z_{2n+1} \sin(2\pi k z_{2n+1}) &= \frac{(-1)^{n+1}}{(2\pi k)^{2n+1}} \cos(2\pi kz) \\
&\quad + \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(2\pi k)^{2n+1-2j}} \frac{z^{2j}}{(2j)!}
\end{aligned} \tag{B.2}$$

and of $\cos(2\pi kz)$

$$\begin{aligned}
\int_0^z \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2 \cdots \int_0^{z_{2n-1}} \mathrm{d}z_{2n} \cos(2\pi k z_{2n}) &= \frac{(-1)^n}{(2\pi k)^{2n}} \cos(2\pi kz) \\
&\quad + \sum_{j=0}^{n-1} \frac{(-1)^{n+1-j}}{(2\pi k)^{2n-2j}} \frac{z^{2j}}{(2j)!}\,, \\
\int_0^z \mathrm{d}z_1 \int_0^{z_1} \mathrm{d}z_2 \cdots \int_0^{z_{2n}} \mathrm{d}z_{2n+1} \cos(2\pi k z_{2n+1}) &= \frac{(-1)^n}{(2\pi k)^{2n+1}} \sin(2\pi kz) \\
&\quad + \sum_{j=1}^{n} \frac{(-1)^{n-j}}{(2\pi k)^{2n+2-2j}} \frac{z^{2j-1}}{(2j-1)!}\,,
\end{aligned} \tag{B.3}$$

where $n \geq 0$. The results for the regularised iterated integral of $g^{(1)}$ are

$$\begin{aligned}
\tilde{\Gamma}_{2n}(1;z) &= \tilde{\Gamma}(\underbrace{\begin{smallmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{smallmatrix}}_{2n}; z, \tau) \\
&= -\frac{1}{(2\pi i)^{2n-1}} \mathrm{Li}_{2n}(e^{2\pi i z}) + \sum_{j=1}^{2n-1} \frac{\zeta_{j+1}}{(2\pi i)^j} \frac{z^{2n-1-j}}{(2n-1-j)!} - \pi i \frac{z^{2n}}{(2n)!} \\
&\quad + (-1)^n 4\pi \sum_{k,l>0} \frac{1}{(2\pi k)^{2n}} \left(\sin(2\pi kz) + \sum_{j=1}^{n} \frac{(-1)^j}{(2\pi k)^{1-2j}} \frac{z^{2j-1}}{(2j-1)!} \right) q^{kl}
\end{aligned} \tag{B.4}$$

and

$$\begin{aligned}
\tilde{\Gamma}_{2n+1}(1;z) &= \tilde{\Gamma}(\underbrace{\begin{smallmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{smallmatrix}}_{2n+1}; z, \tau) \\
&= -\frac{1}{(2\pi i)^{2n}} \mathrm{Li}_{2n+1}(e^{2\pi i z}) + \sum_{j=1}^{2n} \frac{\zeta_{j+1}}{(2\pi i)^j} \frac{z^{2n-j}}{(2n-j)!} - \pi i \frac{z^{2n+1}}{(2n+1)!} \\
&\quad + (-1)^{n+1} 4\pi \sum_{k,l>0} \frac{1}{(2\pi k)^{2n+1}} \left(\cos(2\pi kz) + \sum_{j=0}^{n} \frac{(-1)^{j+1}}{(2\pi k)^{-2j}} \frac{z^{2j}}{(2j)!} \right) q^{kl}\,.
\end{aligned} \tag{B.5}$$

For $m \geq 1$, the following q-expansions are obtained:

$$\begin{aligned}
&\tilde{\Gamma}_{2n}(2m;z)\\
&=\tilde{\Gamma}(\underbrace{\begin{smallmatrix}0 & \dots & 0 & 2m\\ 0 & \dots & 0 & 0\end{smallmatrix}}_{2n};z,\tau)\\
&=-2\zeta_{2m}\frac{z^{2n}}{(2n)!}\\
&\quad+(-1)^{n+1}2\frac{(2\pi i)^{2m}}{(2m-1)!}\sum_{k,l>0}\frac{1}{(2\pi k)^{2n}}\left(\cos(2\pi kz)+\sum_{j=0}^{n-1}\frac{(-1)^{1+j}}{(2\pi k)^{-2j}}\frac{z^{2j}}{(2j)!}\right)l^{2m-1}q^{kl} \quad \text{(B.6)}
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\Gamma}_{2n+1}(2m;z)\\
&=\tilde{\Gamma}(\underbrace{\begin{smallmatrix}0 & \dots & 0 & 2m\\ 0 & \dots & 0 & 0\end{smallmatrix}}_{2n+1};z,\tau)\\
&=-2\zeta_{2m}\frac{z^{2n+1}}{(2n+1)!}\\
&\quad+(-1)^{n+1}2\frac{(2\pi i)^{2m}}{(2m-1)!}\sum_{k,l>0}\frac{1}{(2\pi k)^{2n+1}}\left(\sin(2\pi kz)+\sum_{j=1}^{n}\frac{(-1)^{j}}{(2\pi k)^{1-2j}}\frac{z^{2j-1}}{(2j-1)!}\right)l^{2m-1}q^{kl}
\end{aligned} \quad \text{(B.7)}$$

as well as

$$\begin{aligned}
&\tilde{\Gamma}_{2n}(2m+1;z) \quad \text{(B.8)}\\
&=\tilde{\Gamma}(\underbrace{\begin{smallmatrix}0 & \dots & 0 & 2m+1\\ 0 & \dots & 0 & 0\end{smallmatrix}}_{2n};z,\tau)\\
&=(-1)^{n+1}2i\frac{(2\pi i)^{2m+1}}{(2m)!}\sum_{k,l>0}\frac{1}{(2\pi k)^{2n}}\left(\sin(2\pi kz)+\sum_{j=1}^{n}\frac{(-1)^{j}}{(2\pi k)^{1-2j}}\frac{z^{2j-1}}{(2j-1)!}\right)l^{2m}q^{kl}
\end{aligned} \quad \text{(B.9)}$$

and

$$\begin{aligned}
&\tilde{\Gamma}_{2n+1}(2m+1;z)\\
&=\tilde{\Gamma}(\underbrace{\begin{smallmatrix}0 & \dots & 0 & 2m+1\\ 0 & \dots & 0 & 0\end{smallmatrix}}_{2n+1};z,\tau)\\
&=(-1)^{n}2i\frac{(2\pi i)^{2m+1}}{(2m)!}\sum_{k,l>0}\frac{1}{(2\pi k)^{2n+1}}\left(\cos(2\pi kz)+\sum_{j=0}^{n}\frac{(-1)^{1+j}}{(2\pi k)^{-2j}}\frac{z^{2j}}{(2j)!}\right)l^{2m}q^{kl}\,, \quad \text{(B.10)}
\end{aligned}$$

where the convention $\tilde{\Gamma}_0(m,z)=g^{(m)}(z,\tau)$ is used.

B.2 The elliptic Bloch relation on the torus

In this section, the proofs of the holomorphic Bloch relations (3.184) and (3.185) on the torus and the non-holomorphic relations (3.186) and (3.187) are given. They have been described in ref. [1], where the following paragraphs can be found and only minor changes have been implemented in this version.

The connections between the different notions of elliptic polylogarithms found in section 3.4 and subsection 3.5.2 can be exploited to translate and to compare various concepts and structures among them. Below, it is shown how the elliptic Bloch relation (3.138) translates to the torus, which leads to the more general relations (3.184)-(3.187) thereon and thereby provides an alternative proof of the elliptic Bloch relation. In doing so, it will be shown how the Bloch relation can be interpreted in terms of differentials of iterated integrals or, more generally, in terms of the elliptic symbol calculus introduced in ref. [125].

Let F be an elliptic function on the Tate curve, cf. eq. (3.44), with the following divisor

$$\mathrm{Div}(F) = \sum_i d_i(a_i)\,, \quad \sum_i d_i = 0\,, \quad \prod_i a_i^{d_i} = 1\,. \tag{B.11}$$

Formulated on the torus via eq. (3.48), the above equations translate to

$$\mathrm{Div}(F) = \sum_i d_i(A_i)\,, \quad \sum_i d_i = 0\,, \quad \sum_i d_i A_i = 0\,. \tag{B.12}$$

where $a_i = e^{2\pi i A_i}$ and A_i are representatives of the zeros of the elliptic function F such that the sum $\sum_i d_i A_i$ vanishes. Using eq. (3.31), one can express F in terms of a product of Weierstrass σ-functions

$$F(z) = s_A \prod_i \sigma(z-A_i)^{d_i} = s_A \exp\left(\sum_i d_i \int_0^{z-A_i} \mathrm{d}z'\,\zeta(z')\right) \tag{B.13}$$

for some scaling $s_A \in \mathbb{C}^*$ of F. Similarly, for a given $\kappa \in \mathbb{C}^*$, $\kappa - F$ can be represented by

$$\kappa - F(z) = s_B \prod_j \sigma(z-B_j)^{e_j} = s_B \exp\left(\sum_j e_j \int_0^{z-B_j} \mathrm{d}z'\,\zeta(z')\right), \tag{B.14}$$

where $s_B \in \mathbb{C}^*$. For notational convenience, let us split the set of zeros and poles of F and $\kappa - F$, denoted by I and J, respectively, into the zeros of F, $I' = \{A_i | d_i > 0\}$, the zeros of $\kappa - F$, $J' = \{B_j | e_j > 0\}$, and the common set of poles of F and $\kappa - F$, $K = \{A_i | d_i < 0\} = \{B_j | e_j < 0\}$. Using these conventions, the elliptic Bloch relation

(3.141) on the torus can be rewritten by means of eq. (3.181) as

$$
\begin{aligned}
0 &= \sum_{i,j} d_i e_j \, \mathrm{D}^\tau \left(e^{2\pi i(A_i - B_j)}, q\right) \\
&= -2\pi \sum_{i,j} d_i e_j \Bigg(\mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}; A_i - B_j, \tau\right)\right) && \text{(B.15a)} \\
&\qquad + \mathrm{Re}\left(\frac{\tau}{2\pi i}\right) \mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix}2\\ 0\end{smallmatrix}; A_i - B_j, \tau\right)\right) && \text{(B.15b)} \\
&\qquad - \mathrm{Re}(A_i - B_j)\, \mathrm{Re}\left(\tilde{\Gamma}\left(\begin{smallmatrix}1\\ 0\end{smallmatrix}; A_i - B_j, \tau\right)\right) \Bigg), && \text{(B.15c)}
\end{aligned}
$$

where the summation indices (i, j) run over $I \times J$, unless mentioned otherwise. Equation (B.15) is called the elliptic Bloch relation on the torus.

An alternative proof of equation (B.15) is given in the following paragraphs by showing that the sums over the single iterated integrals $\tilde{\Gamma}$ occurring in the above formula vanish separately (and for the first two also their imaginary parts, yielding two holomorphic analogues of the elliptic Bloch relation). Note that since we are interested in generating functional relations we consider the zeros and poles A_i and B_j as well as the scaling factors s_A and s_B to be (not independent) variables, e.g. depending on variable coefficients of the rational function on the elliptic curve that determine F, cf. the examples in subsection 3.5.1.

Let us start with the first term of the elliptic Bloch relation on the torus, eq. (B.15a): naturally, the zeros and poles satisfy the constraints $\sum_i d_i A_i = 0$ and $\sum_j e_j B_j = 0$ as functional identities. Hence, the functional identity

$$
\kappa = \kappa - F(A_i) = s_B \prod_j \sigma(A_i - B_j)^{e_j} \tag{B.16}
$$

holds for $i \in I'$, such that taking the total differential of both sides and using eq. (3.56), i.e. $\zeta(z) = g^{(1)}(z, \tau) + 2\eta_1 z$, as well as the representations (B.13) and (B.14), the differential equation

$$
\sum_j e_j g^{(1)}(A_i - B_j) \mathrm{d}(A_i - B_j) = -\mathrm{d}\log(s_B) - c_1 \sum_j e_j B_j \mathrm{d}B_j \tag{B.17}
$$

can be obtained. For $k \in K$, a functional identity involving the residue instead of the infinite value $\kappa - F(A_k)$ can be used for a similar calculation: since by convention $\sigma'(0) = 1$, the residue of $\kappa - F$ at A_k is

$$
\mathrm{res}_{A_k}(\kappa - F) = s_B \prod_{j \neq k} \sigma(A_k - B_j)^{e_j}, \tag{B.18}
$$

which implies that

$$\sum_{j\neq k} e_j g^{(1)}(A_k - B_j)\mathrm{d}(A_k - B_j) = \mathrm{d}\log\left(\mathrm{res}_{A_k}(\kappa - F)\right) - \mathrm{d}\log(s_B) - c_1 \sum_j e_j B_j \mathrm{d}B_j . \tag{B.19}$$

Two similar differential equations for sums over I can be found, the first one starting from $\kappa = F(B_j)$, where $j \in J'$,

$$\sum_i d_i g^{(1)}(A_i - B_j)\mathrm{d}(A_i - B_j) = -\mathrm{d}\log(s_A) - c_1 \sum_i d_i A_i \mathrm{d}A_i . \tag{B.20}$$

With $k \in K$ and using that $\mathrm{res}_{A_k}(F) = -\mathrm{res}_{A_k}(\kappa - F)$, the last such differential equation turns out to be

$$\sum_{i\neq k} d_i g^{(1)}(A_k - A_i)\mathrm{d}(A_k - A_i) = \mathrm{d}\log\left(\mathrm{res}_{A_k}(\kappa - F)\right) - \mathrm{d}\log(s_A) - c_1 \sum_i d_i A_i \mathrm{d}A_i . \tag{B.21}$$

Going through the calculations of ref. [1], the four differential equations (B.17), (B.19), (B.20) and (B.21) can be combined into the differential equation

$$\sum_{i,j} d_i e_j (A_i - B_j) g^{(1)}(A_i - B_j, \tau)\mathrm{d}(A_i - B_j) = 0 . \tag{B.22}$$

For integration paths with $\mathrm{d}\tau = 0$, the differential of the iterated integral $\tilde{\Gamma}(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}; z, \tau)$ is given by

$$\mathrm{d}\,\tilde{\Gamma}(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}; z, \tau) = z g^{1}(z, \tau)\mathrm{d}z . \tag{B.23}$$

Accordingly, eq. (B.22) implies that

$$\sum_{i,j} d_i e_j \, \tilde{\Gamma}(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}; A_i - B_j, \tau) = c_2 \tag{B.24}$$

for some constant $c_2 \in \mathbb{C}$. In general, the zeros and poles of F are only constrained by $\sum_i d_i A_i = 0 = \sum_i d_i$, thus, it may be assumed that they can be split in a way such that the divisor of F consists of triplets with two of them being unconstrained and the third one being given by $A_3 = -A_1 - A_2$. An alternative way of saying this is that divisors of the form $(A_1) + (A_2) - (0) - (A_1 + A_2)$ span the set of principal divisors. Thus, by continuity, the above equation can be evaluated at the point where all $A_i = 0$ to determine

$$c_2 = \sum_j e_j \, \tilde{\Gamma}(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}; -B_j, \tau) \sum_i d_i = 0. \tag{B.25}$$

Therefore, we find a holomorphic analogue of the elliptic Bloch relation

$$\sum_{i,j} d_i e_j \, \tilde{\Gamma}(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}; A_i - B_j, \tau) = 0. \tag{B.26}$$

Similar arguments apply for the term (B.15c) involving the iterated integral $z\,\tilde{\Gamma}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z, \tau)$ in the elliptic Bloch relation on the torus (B.15). Let $i \in I'$ and write

$$\kappa = \kappa - F(A_i) = s_B \exp\left(\sum_j e_j \int_0^{A_i - B_j} \mathrm{d}z\, g^{(1)}(z,\tau) + \frac{c_1}{2} \sum_j e_j B_j^2\right), \tag{B.27}$$

such that

$$\sum_j e_j \, \tilde{\Gamma}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_i - B_j, \tau) = \log(\kappa) - \log(s_B) - \frac{c_1}{2} \sum_j e_j B_j^2 - 2\pi i m_1, \tag{B.28}$$

for some $m_1 \in \mathbb{Z}$. For $k \in K$ and with $\sigma(z) = s_0 \exp\left(\int_{z_0}^z dz' \, \zeta(z')\right)$ such that $\sigma'(0) = 1$, the same calculation as before leads to

$$\begin{aligned} \mathrm{res}_{A_k}(\kappa - F) &= s_B s_0 \exp\left(\sum_{j \neq k} e_j \int_{z_0}^{A_k - B_j} \mathrm{d}z\, \zeta(z)\right) \\ &= s_B s_0 \exp\left(\sum_{j \neq k} e_j \int_0^{A_k - B_j} \mathrm{d}z\, g^{(1)}(z,\tau) + \frac{c_1}{2} \sum_j e_j B_j^2 + \int_0^{z_0} \mathrm{d}z\, \zeta(z)\right) \end{aligned} \tag{B.29}$$

which implies that

$$\begin{aligned} \sum_{j \neq k} e_j \int_0^{A_k - B_j} \mathrm{d}z\, g^{(1)}(z,\tau) = \log(\mathrm{res}_{A_k}(\kappa - F)) - \log(s_0) - \log(s_B) - \frac{c_1}{2} \sum_j e_j B_j^2 \\ - \int_0^{z_0} \mathrm{d}z\, \zeta(z) - 2\pi i m_2, \end{aligned} \tag{B.30}$$

where $m_2 \in \mathbb{Z}$, and analogously for the sum over $I \backslash \{k\}$

$$\begin{aligned} \sum_{i \neq k} d_i \int_0^{A_k - A_i} \mathrm{d}z\, g^{(1)}(z,\tau) = \log(\mathrm{res}_{A_k}(F)) - \log(s_0) - \log(s_A) - \frac{c_1}{2} \sum_i d_i A_i^2 \\ - \int_0^{z_0} \mathrm{d}z\, \zeta(z) - 2\pi i m_3, \end{aligned} \tag{B.31}$$

for $m_3 \in \mathbb{Z}$. A similar result holds for $j \in J'$,

$$\sum_i d_i \, \tilde{\Gamma}(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; A_i - B_j, \tau) = \log(\kappa) - \log(s_A) - \frac{c_1}{2} \sum_i d_i A_i^2 - 2\pi i m_4, \tag{B.32}$$

where $m_4 \in \mathbb{Z}$. Since $\log(\mathrm{res}_{A_k}(F)) = \log(\mathrm{res}_{A_k}(1 - F)) + i\pi$, equations (B.30) and

(B.31) lead to

$$\sum_j e_j \, \tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; A_k - B_j, \tau) - \sum_i d_i \, \tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; A_k - A_i, \tau)$$
$$= -i\pi(1 + 2m_2 - 2m_3) + \log(s_A) + \frac{c_1}{2} \sum_i d_i A_i^2 - \log(s_B) - \frac{c_1}{2} \sum_j e_j B_j^2. \quad \text{(B.33)}$$

Finally, using the equations (B.28), (B.30), (B.31) and (B.32) all together, the identities

$$\sum_{i,j} d_i e_j \,\mathrm{Re}\,(A_i - B_j)\,\mathrm{Re}\left(\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; A_i - B_j, \tau)\right) = 0 \quad \text{(B.34)}$$

and

$$\sum_{i,j} d_i e_j \,\mathrm{Im}\,(A_i - B_j)\,\mathrm{Re}\left(\tilde{\Gamma}(\begin{smallmatrix}1\\0\end{smallmatrix}; A_i - B_j, \tau)\right) = 0. \quad \text{(B.35)}$$

can be obtained [1].

Now, we are left with the term (B.15b) involving $\tilde{\Gamma}(\begin{smallmatrix}2\\0\end{smallmatrix}; z, \tau)$. Let us take the partial derivative of eq. (B.28) with respect to τ and use the mixed heat equation (3.79) of the integration kernel, i.e. $2\pi i \frac{\partial}{\partial\tau} g^{(1)}(z,\tau) = \frac{\partial}{\partial z} g^{(2)}(z,\tau)$, to find

$$\sum_j e_j g^{(2)}(A_i - B_j, \tau) = -2\pi i \frac{\partial}{\partial\tau} \frac{c_1}{2} \sum_j e_j B_j^2$$
$$- 2\pi i \sum_j e_j g^{(1)}(A_i - B_j, \tau) \frac{\partial}{\partial\tau}(A_i - B_j), \quad \text{(B.36)}$$

valid for $i \in I'$. A similar result holds for $j \in J'$

$$\sum_i d_i g^{(2)}(A_i - B_j, \tau) = -2\pi i \frac{\partial}{\partial\tau} \frac{c_1}{2} \sum_i d_i A_i^2$$
$$- 2\pi i \sum_i d_i g^{(1)}(A_i - B_j, \tau) \frac{\partial}{\partial\tau}(A_i - B_j) \quad \text{(B.37)}$$

and for $k \in K$

$$\sum_j e_j g^{(2)}(A_k - B_j) - \sum_i d_i g^{(2)}(A_k - A_i)$$
$$= -2\pi i \frac{\partial}{\partial\tau} \frac{c_1}{2} \sum_i d_i A_i^2 - 2\pi i \frac{\partial}{\partial\tau} \frac{c_1}{2} \sum_j e_j B_j^2$$
$$- 2\pi i \sum_j e_j g^{(1)}(A_k - B_j) \frac{\partial}{\partial\tau}(A_k - B_j) + 2\pi i \sum_i d_i g^{(1)}(A_k - A_i) \frac{\partial}{\partial\tau}(A_k - B_j).$$
(B.38)

The equations (B.36), (B.37) and (B.38) imply that for paths with $\mathrm{d}\tau = 0$ the differential equation [1]

$$\mathrm{d}\sum_{ij} d_i e_j \, \tilde{\Gamma}\left(\begin{smallmatrix}2\\0\end{smallmatrix}; A_i - B_j, \tau\right) = 0 \tag{B.39}$$

holds. By the same argument as for eq. (B.26), we therefore find another functional identity which can be interpreted as a holomorphic analogue of the elliptic Bloch relation on the torus:

$$\sum_{i,j} d_i e_j \, \tilde{\Gamma}\left(\begin{smallmatrix}2\\0\end{smallmatrix}; A_i - B_j, \tau\right) = 0. \tag{B.40}$$

To summarise, the elliptic Bloch relation (B.15) has been expressed in terms of iterated integrals on the torus.

Let us comment on the two holomorphic functional equations (B.26) and (B.40) respectively, in terms of the iterated integrals $\tilde{\Gamma}$ on the torus which have the same structure as the original elliptic Bloch relation: in the language of ref. [49], it turns out that the iterated integrals $\tilde{\Gamma}\left(\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}; z, \tau\right)$ and $\tilde{\Gamma}\left(\begin{smallmatrix}2\\0\end{smallmatrix}; z, \tau\right)$ are Steinberg functions. However, we have to be careful when using these functional identities: these iterated integrals are multi-valued and in order to reproduce eqs. (B.26) and (B.40) they have to be evaluated on the representatives of the zeros and poles of F and $\kappa - F$ which satisfy $\sum_i d_i A_i = 0 = \sum_j e_j B_j$, and not only such that these sums lie in the lattice Λ_τ. These equations have been obtained by differential calculus of iterated integrals, which is simply the symbol calculus of an iterated integral with depth one. Thus, together with eq. (B.34) an interpretation of the elliptic Bloch relation using the elliptic symbol calculus of the iterated integrals $\tilde{\Gamma}$ on the torus is provided.

Chapter C

Closed differential system

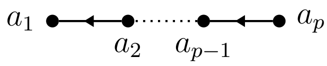
In this appendix, various statements from chapter 6 are proven. This includes the derivation of the closed formulæ for the elliptic KZB system (5.21) satisfied by the type-(n,p), genus-one chain integrals $\boldsymbol{Z}^{\tau}_{n,p}(z_2,\dots,z_p)$.

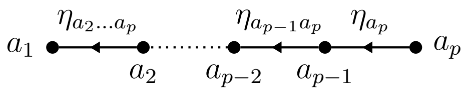
First, in section C.1 the graph identities (6.50), (6.52) and (6.54) are discussed in more detail. Second, in section C.2 the algorithm to express any graph in terms of chain or admissible graphs leading to eqs. (6.33) and (6.48) is given. This shows that the chain and admisisble graphs are generators of the group $\mathcal{G}_f(\vec{x},\vec{\eta})$, proving eqs. (6.32) and (6.41). Third, the closed formulæ for the differential system (6.86) of the chain integrals and, as a special case, the elliptic KZB system from the genus-one recursion in eq. (5.21) is derived and given in section C.3.

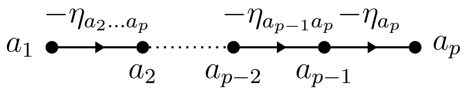
C.1 Graph identities

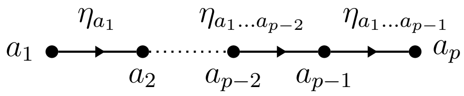
The antisymmetry (6.17) and Fay identity (6.18) on edges of graphs in $\mathcal{G}_f(\vec{x},\vec{\eta})$ lead to various useful identities on (sub)graphs. In this subsection, the derivation of the three identities (6.50), (6.52) and (6.54) are summarised. The detailed proofs can be found in ref. [3], formulated for their images in $G_f(\vec{x},\vec{\eta})$ under f.

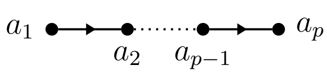
The first identity is the reflection property of a chain, which can be proven by the application of the antisymmetry (6.17) to each edge and the condition (6.8) on the weights. The result is

$\varphi(A) =$ 

$=$ 

$= (-1)^p$ 

$= (-1)^p$ 

$= (-1)^p$ 

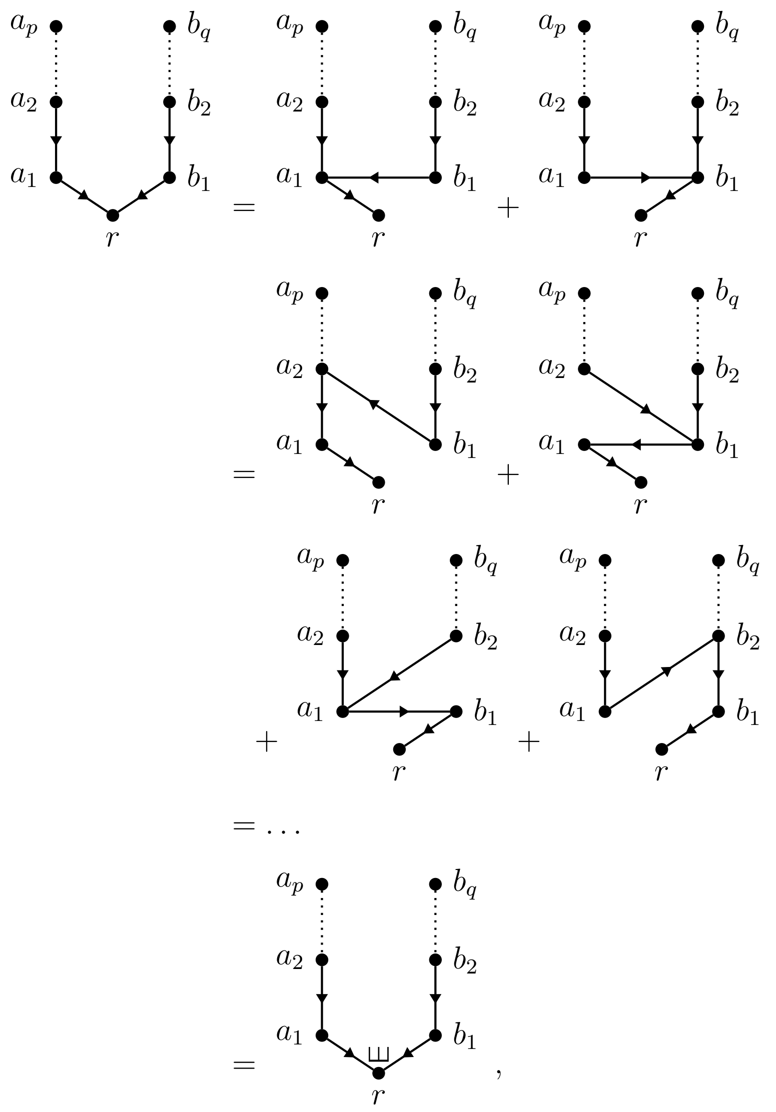
$$= (-1)^{|A|}\varphi(\tilde{A})\,, \tag{C.1}$$

where $\tilde{A} = (a_p, \ldots, a_1)$ is the reversed sequence $A = (a_1, \ldots, a_p)$ and $|A| = p$ its length. Since the antisymmetry (6.17) is compatible with the continuity condition (6.7) of the weights, the reflection identity (C.1) can be applied to any subgraph which is of the form of a chain. Thus, the same identity holds for the chain products

$$\varphi_f(A) = (-1)^{|A|}\varphi_f(\tilde{A})\,. \tag{C.2}$$

The second identity is the shuffling of two branches with a branch point at x_r.

An iterative application of the Fay identity (6.24) leads to the shuffle identity



(C.3)

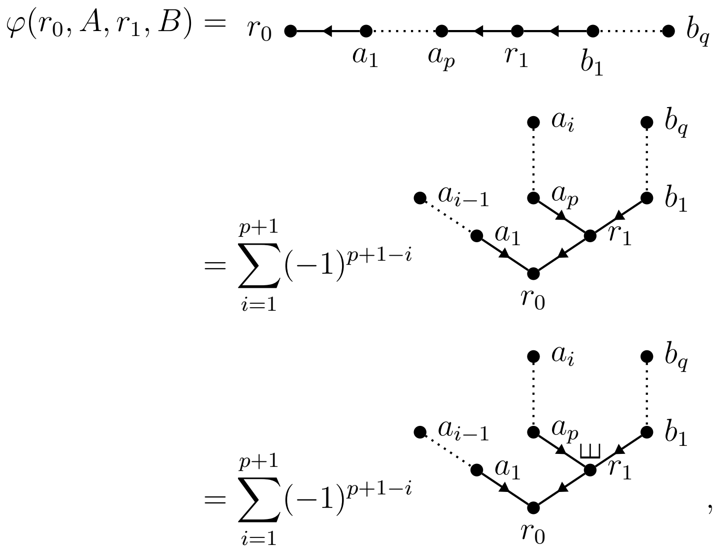
where the sum on the right-hand side is an element of $\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta})$ obtained from iteratively applying the Fay identity until a sum of chains is left. Due to the compatibility of the Fay identity with the continuity condition (6.7) of the weights, the shuffle identity (C.3) can be applied to all the subgraphs of any graph $\gamma \in \mathcal{G}_f(\vec{x},\vec{\eta})$. The corresponding identity on products of functions is

$$\varphi_f(r,A)\varphi_f(r,B) = \varphi_f(r, A \shuffle B) \tag{C.4}$$

for $A = (a_1,\dots,a_p)$ and $B = (b_1,\dots,b_q)$.

The third identity can be used to shift two labels r_0 and r_1 in a chain $\varphi(r_0, A, r_1, B)$

next to each other. It takes the form



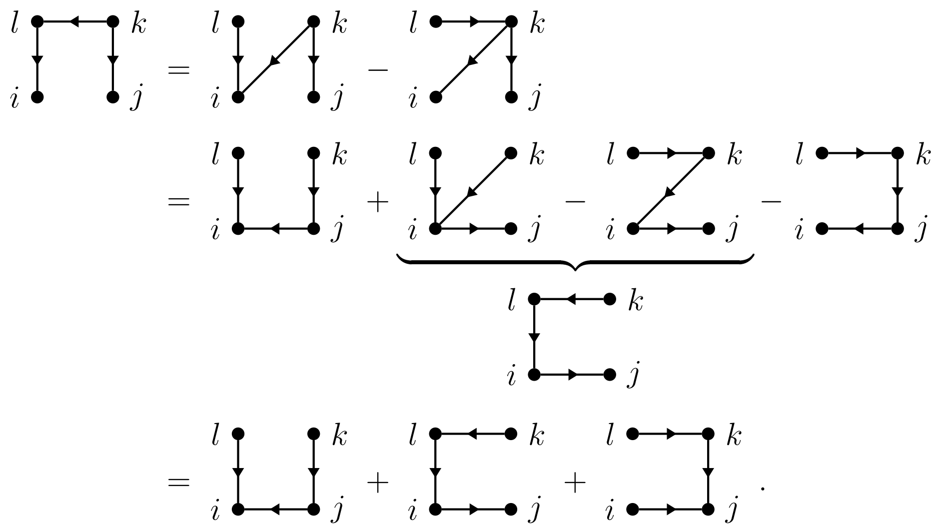
, (C.5)

where for the last equality, we have applied the shuffle identity (C.3). This manipulation is called *solving for* r_0 *and* r_1 and follows from the antisymmetry, the Fay identity and combinatorial identities. The proof in terms of its image under f in $G_f(\vec{x}, \vec{\eta})$ has been derived in ref. [3]. The result is the identity

$$\varphi_f(r_0, A, r_1, B) = \sum_{i=1}^{p+1} (-1)^{p+1-i} \varphi_f(r_0, a_1, \dots, a_{i-1}) \varphi_f(r_0, r_1, (a_p, a_{p-1}, \dots, a_i) \shuffle B) \,. \tag{C.6}$$

And again, due to the compatibility of the continuity condition (6.7) of the weights, solving for two vertices can be applied to any chain-like subgraph of any chain in $\mathcal{G}_f(\vec{x}, \vec{\eta})$.

Of course, further identities may be derived upon combining the antisymmetry and Fay identity. One specific example which will be used below is the identity



. (C.7)

C.2 Generators of graphs

In this section, it is shown that the admissible graphs $\mathcal{G}_f^{\mathrm{adm}}(\vec{x},\vec{\eta})$ and the chain graphs $\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta})$ indeed span the set of graphs $\mathcal{G}_f(\vec{x},\vec{\eta})$, such that the same holds for the graph products under the image of f. A constructive proof is given, which yields an algorithm to actually calculate the corresponding linear combinations.

In subsection C.2.1 an algorithm to write any graph in terms of admissible graphs is given, which proves eq. (6.32), i.e.

$$\mathcal{G}_f^{\mathrm{adm}}(\vec{x},\vec{\eta}) = \mathcal{G}_f(\vec{x},\vec{\eta})\,. \tag{C.8}$$

In subsection C.2.2 the analogous eq. (6.45) for chain graphs, i.e.

$$\mathcal{G}_f^{\mathrm{cha}}(\vec{x},\vec{\eta}) = \mathcal{G}_f(\vec{x},\vec{\eta})\,, \tag{C.9}$$

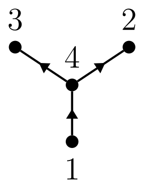
is derived.

C.2.1 Admissible graphs

Equation (C.8) is proven by a recursive calculation leading to the linear combination (6.33), i.e.

$$\gamma = \sum_{\gamma'\in\mathcal{G}^{\mathrm{adm}}(\vec{x},\vec{\eta})} b_{\gamma'}^{\mathrm{adm}}\gamma' \in \mathcal{G}_f^{\mathrm{adm}}(\vec{x},\vec{\eta})\,, \quad b_{\gamma'}^{\mathrm{adm}} \in \mathbb{Z}\,, \tag{C.10}$$

for any $\gamma \in \mathcal{G}_f(\vec{x},\vec{\eta})$. The algorithm is based on a similar construction as in refs. [4, 157]. It acts separately on each connected component, thus, without loss of generality the graph $\gamma \in \mathcal{G}_f(\vec{x},\vec{\eta})$ is assumed to be connected. The various steps of the algorithm are exemplified on the graph

$\gamma_{\mathrm{ex}} =$  $\in \mathcal{G}(\vec{x},\vec{\eta})\backslash\mathcal{G}^{\mathrm{adm}}(\vec{x},\vec{\eta})\,.$ (C.11)

The algorithm consists of two parts:

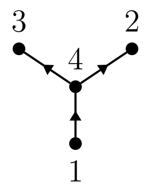
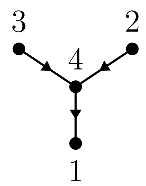
1. First, using the antisymmetry (6.23), i.e.

$$i \bullet\!\!\longleftarrow\!\!\bullet\, j \;=\; -\, i \bullet\!\!\longrightarrow\!\!\bullet\, j\,, \tag{C.12}$$

   the graph $\gamma \in \mathcal{G}_f(\vec{x},\vec{\eta})$ has to be rewritten such that it has only one base point and each vertex has at most one outgoing edge. The base point is canonically chosen to be the vertex $x_{i_0} \in x(\gamma)$ with the smallest index i_0

$$i_0 = i_0(\gamma) = \min\{i \mid x_i \in x(\gamma)\}\,, \tag{C.13}$$

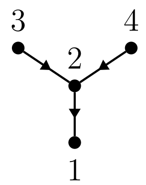
such that the reversal of the remaining edges to obtain one outgoing edge at each vertex except at the base point x_{i_0} is fixed: starting from the base point and going up the edges $e_{i_0 j}$, any edge e_{kj} with $k \neq i_0$ is reversed. Repeating this process iteratively going up the edges from the vertex j leads to the desired form, after a finite number m of reversing edges. For our example, we obtain $i_0 = 1$ and

$\gamma_{\text{ex}} =$  $= (-1)^3$  . (C.14)

This procedure proves that there is a non-negative integer $m \in \mathbb{Z}_{\geq 0}$, an admissible graph $\gamma^{\text{adm}} \in \mathcal{G}^{\text{adm}}(\vec{x}, \vec{\eta})$ and a permutation σ with $\sigma(i_0) = i_0$ acting on the labels of the vertices and weights of the graph γ^{adm}, such that

$$\gamma = (-1)^m \sigma(\gamma^{\text{adm}}) \in \mathcal{G}_f(\vec{x}, \vec{\eta})\,, \tag{C.15}$$

where m is the number of reversals of a single edge. The graph $\sigma(\gamma^{\text{adm}})$ is called *σ-permuted admissible.* For γ_{ex}, one can determine

$\gamma_{\text{ex}} = (-1)^3 \tau_{24}(\gamma_{\text{ex}}^{\text{adm}})\,, \quad \gamma_{\text{ex}}^{\text{adm}} =$  $\in \mathcal{G}^{\text{adm}}(\vec{x}, \vec{\eta})\,,$ (C.16)

where τ_{24} transposes the indices two and four. If σ can be chosen to be the identity or, equivalently, all non-vanishing edges $e_{ij}(\sigma(\gamma^{\text{adm}}))$ satisfy $i < j$, the admissibility condition (6.25), i.e.

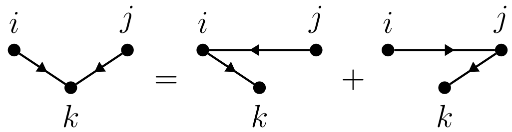
$$\forall j : w_{ij}(\sigma(\gamma^{\text{adm}})) = 0 \text{ except for at most one } i < j\,, \tag{C.17}$$

is satisfied and we are done, since then $\sigma(\gamma^{\text{adm}}) \in \mathcal{G}^{\text{adm}}(\vec{x}, \vec{\eta})$ and

$$\gamma = (-1)^m \sigma(\gamma^{\text{adm}}) \in \mathcal{G}_f^{\text{adm}}(\vec{x}, \vec{\eta})\,. \tag{C.18}$$

Otherwise, the following, second step of the algorithm has to be applied.

2. Second, the antisymmetry (6.23) and the Fay identity (6.24), i.e.

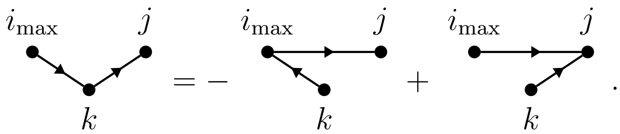


(C.19)

are used iteratively as follows: it starts at the vertex with the highest label $i_{\max}$ of the graph $\sigma(\gamma^{\mathrm{adm}})$ with a non-admissible outgoing edge

$$\begin{aligned} i_{\max} &= i_{\max}(\sigma(\gamma^{\mathrm{adm}})) \\ &= \max\{i \mid x_i \in x(\sigma(\gamma^{\mathrm{adm}})) \text{ and there exists } k > i \\ &\qquad\qquad \text{such that } w_{ki}(\sigma(\gamma^{\mathrm{adm}})) \neq 0\}\,. \end{aligned} \tag{C.20}$$

Due to the application of the first step of the algorithm, the vertex k with $w_{ki_{\max}}(\sigma(\gamma^{\mathrm{adm}})) \neq 0$ is unique and $i_{\max}, k > i_0$. Thus, there is exactly one $j < k$, such that $w_{jk}(\sigma(\gamma^{\mathrm{adm}})) \neq 0$. Then, using the antisymmetry and Fay identity, the subgraph of $\sigma(\gamma^{\mathrm{adm}})$ consisting of the three vertices k, $i_{\max} < k$ and $j < k$ and the two edges $w_{ki_{\max}}(\sigma(\gamma^{\mathrm{adm}}))$ and $w_{jk}(\sigma(\gamma^{\mathrm{adm}}))$ can be rewritten as follows:



(C.21)

Each vertex of the resulting two subgraphs on the right-hand side has at most one outgoing edge.

(a) If $j < i_{\max}$, then they are even admissible.

(b) If $i_{\max} < j$, we can repeat this procedure with k replaced by $j < k$ for each graph in the above linear combination, until in each graph γ' in the resulting linear combination, the single outgoing edge of the vertex $i_{\max}$ to the unique vertex j' is admissible $w_{j' i_{\max}}(\gamma') \neq 0$ with $j' < i_{\max}$. Note that this iteration terminates after finitely many steps since we only consider tree graphs, such that $j' \neq i_{\max}$ and at each iteration, j' is decreased.

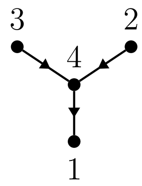
Once the outgoing edge of $i_{\max}$ is admissible in each graph γ' of the resulting linear combination, either each graph γ' is admissible or there are non-admissible graphs γ' with $i_{\max}(\gamma') < i_{\max}(\sigma(\gamma^{\mathrm{adm}}))$. In the latter case, we can repeat the whole second step of the algorithm with $\sigma(\gamma^{\mathrm{adm}})$ replaced by γ'. Again, this terminates after finitely many steps.

This algorithm leads to the representation of $\gamma = (-1)^m \sigma(\gamma^{\mathrm{adm}}) \in \mathcal{G}_f(\vec{x}, \vec{\eta})$ in terms of a $\mathbb{Z}$-linear combination of admissible graphs $\gamma' \in \mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta})$ of the form (C.10).

Thus, the admissible graphs indeed generate $\mathcal{G}_f(\vec{x},\vec{\eta})$, which proofs eq. (C.8).

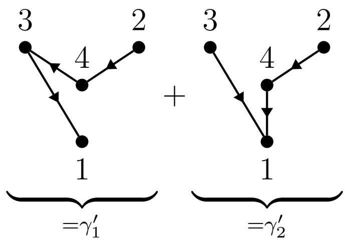
Let us complete the second step of the above algorithm for $\gamma_{\mathrm{ex}} = -\tau_{24}(\gamma_{\mathrm{ex}}^{\mathrm{adm}})$, where

$$\tau_{24}(\gamma_{\mathrm{ex}}^{\mathrm{adm}}) = \quad . \qquad \text{(C.22)}$$



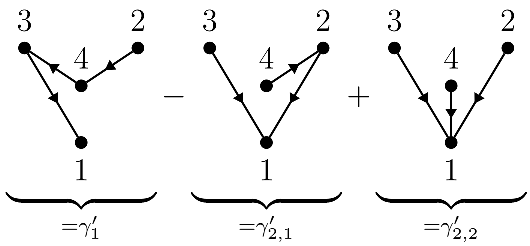
The maximal label with a non-admissible outgoing edge is $i_{\max}(\tau_{24}(\gamma_{\mathrm{ex}}^{\mathrm{adm}})) = 3$. Thus, applying eq. (C.21) to $i_{\max}(\tau_{24}(\gamma_{\mathrm{ex}}^{\mathrm{adm}})) = 3$, $k = 4$ and $j = 1$ leads to

$$\tau_{24}(\gamma_{\mathrm{ex}}^{\mathrm{adm}}) = - \quad + \quad . \qquad \text{(C.23)}$$



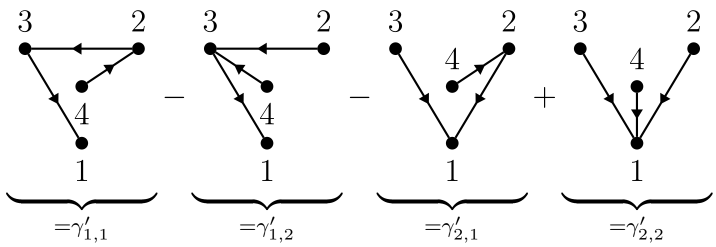
In both graphs γ_1' and γ_2', the vertex $i_{\max}(\tau_{24}(\gamma_{\mathrm{ex}}^{\mathrm{adm}})) = 3$ is now admissible (i.e. the point (2.a) of the algorithm applies) and we can proceed with the new $i_{\max}(\gamma_1') = 2$ and $i_{\max}(\gamma_2') = 2$ repeating step (2) of the algorithm for both graphs γ_1' and γ_2'. For the second graph γ_2', eq. (C.21) can be applied to $i_{\max}(\gamma_2') = 2$, $k = 4$ and $j = 1$, such that

$$\tau_{24}(\gamma_{\mathrm{ex}}^{\mathrm{adm}}) = - \quad - \quad + \qquad \text{(C.24)}$$



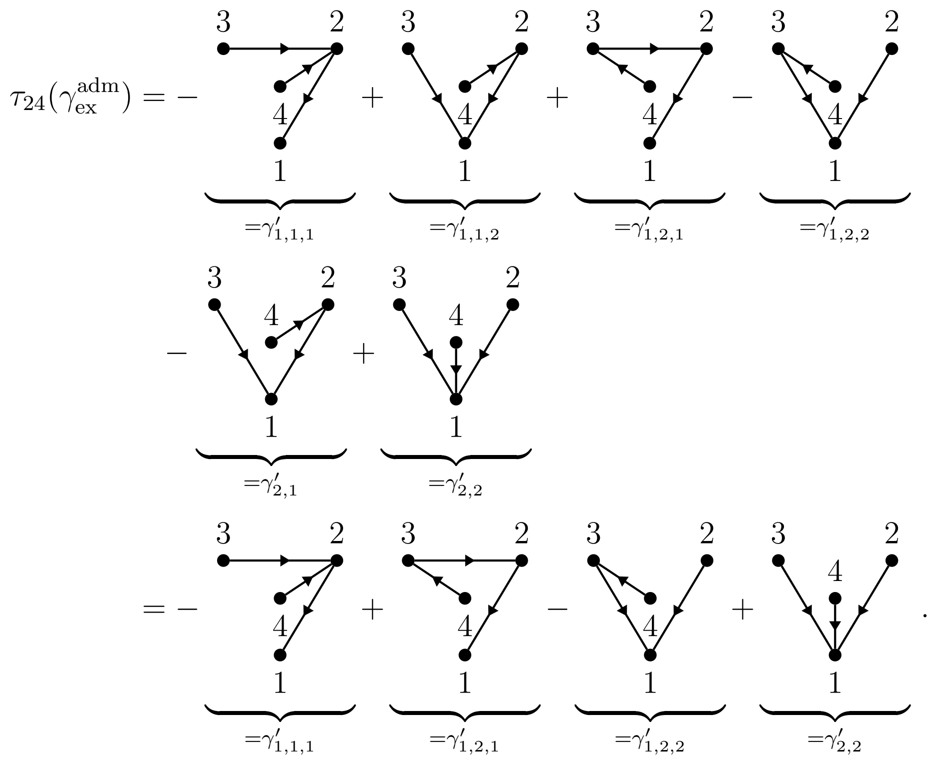
with the admissible graphs $\gamma_{2,1}', \gamma_{2,2}' \in \mathcal{G}^{\mathrm{adm}}(\vec{x},\vec{\eta})$ (the point (2.a) applies and the graphs have no further non-admissible edge). Thus, we are left with the graph γ_1' with $i_{\max}(\gamma_1') = 2$, $k = 4$ and $j = 3$, such that eq. (C.21) leads to

$$\tau_{24}(\gamma_{\mathrm{ex}}^{\mathrm{adm}}) = \quad - \quad - \quad + \quad . \qquad \text{(C.25)}$$



But still, the vertex $i_{\max}(\gamma_1') = 2$ is not admissible in $\gamma_{1,1}'$ and $\gamma_{1,2}'$ due to the inequality $i_{\max}(\gamma_1') = 2 < 3 = j$, thus step (2.b) has to be applied, i.e. eq. (C.21) on

$k = 3, j' = 1$ and $i_{\max}(\gamma'_1) = 2$, such that

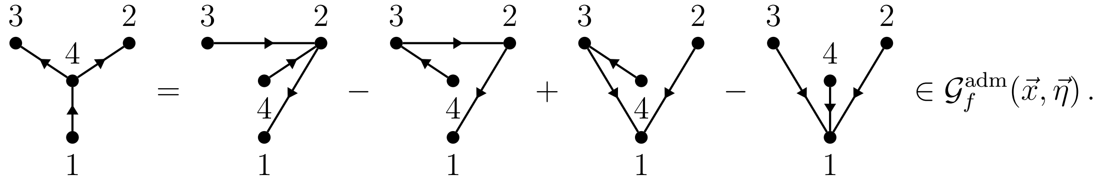


(C.26)

The right-hand side of this equation in $\mathcal{G}_f(\vec{x}, \vec{\eta})$ is a linear combination of admissible graphs $\gamma'_{1,1,1}, \gamma'_{1,2,1}, \gamma'_{1,2,2}, \gamma'_{2,2} \in \mathcal{G}^{\text{adm}}(\vec{x}, \vec{\eta})$, hence for the example γ_{ex}, the linear combination (C.10) is given by

$$\gamma_{\text{ex}} = \gamma'_{1,1,1} - \gamma'_{1,2,1} + \gamma'_{1,2,2} - \gamma'_{2,2} \in \mathcal{G}_f^{\text{adm}}(\vec{x}, \vec{\eta}) \tag{C.27}$$

or written out in terms of the actual graphs:



(C.28)

To finish, let us consider the image under $f(x) = 1/x$ of the identity (C.28). It is the following partial fractioning identity in $G_f(\vec{x}, \vec{\eta})$:

$$\frac{1}{x_{41}x_{34}x_{24}} = \frac{1}{x_{12}x_{23}x_{24}} - \frac{1}{x_{12}x_{23}x_{34}} + \frac{1}{x_{12}x_{13}x_{34}} - \frac{1}{x_{12}x_{13}x_{14}} \,. \tag{C.29}$$

Thus, eq. (C.28) is a well-structured representation for the partial fractioning identity (C.29).

C.2.2 Chain graphs

In order to prove eq. (C.9), again, without loss of generality a connected graph $\gamma \in \mathcal{G}(\vec{x}, \vec{\eta})$ is considered and with the first step of the algorithm in the previous subsection rewritten as

$$\gamma = (-1)^m \sigma(\gamma^{\mathrm{adm}}) \,, \tag{C.30}$$

where $\gamma^{\mathrm{adm}} \in \mathcal{G}^{\mathrm{adm}}(\vec{x}, \vec{\eta})$ and $\sigma(i_0(\gamma)) = i_0(\gamma)$, where i_0 is defined in eq. (C.13). Then, the σ-permuted admissible graph $\sigma(\gamma^{\mathrm{adm}})$ can be written as a $\mathbb{Z}$-linear combination of admissible graphs simply by applying the shuffle identity (C.3) to all the branch points and effectively folding back any branches. This iterative application of the shuffle identity can conveniently be described using the following recursive definition of a sum of sequences associated to the vertices of a graph γ:

$$(i, \gamma) = (i, \mathop{\sqcup\!\sqcup}_{w_{ij}(\gamma) \neq 0} (j, \gamma)) \,, \tag{C.31}$$

where the shuffle product runs over all vertices j with an outgoing edge to i and is empty if there is no such vertex j.[1] Using this sum of sequences, the graph $\sigma(\gamma^{\mathrm{adm}})$ can be written as a sum of chains

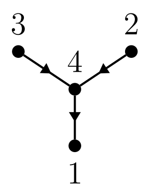
$$\sigma(\gamma^{\mathrm{adm}}) = \varphi(i_0(\gamma), \sigma(\gamma^{\mathrm{adm}})) \,, \tag{C.32}$$

which leads to the linear combination (6.48), i.e.

$$\gamma = (-1)^m \varphi(i_0(\gamma), \sigma(\gamma^{\mathrm{adm}})) = \sum_{\gamma' \in \mathcal{G}^{\mathrm{cha}}(\vec{x}, \vec{\eta})} b^{\mathrm{cha}}_{\gamma'} \gamma' \in \mathcal{G}^{\mathrm{cha}}_f(\vec{x}, \vec{\eta}) \,, \quad b^{\mathrm{cha}}_{\gamma'} \in \mathbb{Z} \,. \tag{C.33}$$

Let us again consider the example $\gamma_{\mathrm{ex}} \in \mathcal{G}(\vec{x}, \vec{\eta}) \backslash \mathcal{G}^{\mathrm{cha}}(\vec{x}, \vec{\eta})$ from eq. (C.11), for which the first step of the algorithm in the previous section leads to eq. (C.16), i.e.

$$\gamma_{\mathrm{ex}} = -\tau_{24}(\gamma^{\mathrm{adm}}_{\mathrm{ex}}) \in \mathcal{G}_f(\vec{x}, \vec{\eta}) \,, \quad \tau_{24}(\gamma^{\mathrm{adm}}_{\mathrm{ex}}) = \tag{C.34}$$



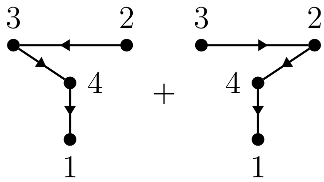
,

[1] This simply generalises the definition (6.58) to non-admissible graphs. Equivalently, we could also go through the second step of the algorithm from subsection C.2.1 and express γ as a linear combination of admissible graphs, then apply eq. (6.58) to recover a linear combination of chain graphs. However, this will not be as efficient since various terms generated by the second step of the above algorithm will again cancel when rewriting the admissible graphs in terms of chain graphs.

with $i_0(\gamma_{\text{ex}}) = 1$. Then,

$$\begin{aligned}(1, \tau_{24}(\gamma_{\text{ex}}^{\text{adm}})) &= (1, (4, \tau_{24}(\gamma_{\text{ex}}^{\text{adm}}))) \\ &= (1, 4, (2, \tau_{24}(\gamma_{\text{ex}}^{\text{adm}})) \sqcup (3, \tau_{24}(\gamma_{\text{ex}}^{\text{adm}}))) \\ &= (1, 4, (2) \sqcup (3)) \\ &= (1, 4, 2, 3) + (1, 4, 3, 2)\,, \end{aligned} \tag{C.35}$$

such that

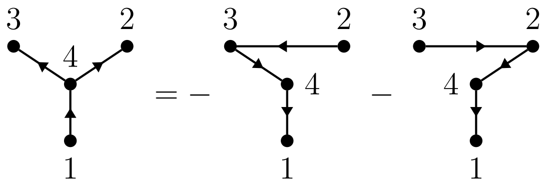
$$\tau_{24}(\gamma_{\text{ex}}^{\text{adm}}) = \varphi(1, \tau_{24}(\gamma_{\text{ex}}^{\text{adm}})) =$$


(C.36)

and, ultimately,

$$\gamma_{\text{ex}} = -\varphi(1, \tau_{24}(\gamma_{\text{ex}}^{\text{adm}})) = -\varphi(1, 4, 3, 2) - \varphi(1, 4, 2, 3) \in \mathcal{G}_f^{\text{cha}}(\vec{x}, \vec{\eta})\,, \tag{C.37}$$

i.e.



$\in \mathcal{G}_f^{\text{cha}}(\vec{x}, \vec{\eta})\,.$ (C.38)

For the example $f(x, \eta) = 1/x$, the image of eq. (C.38) under f is the identity

$$\frac{1}{x_{41}x_{34}x_{24}} = -\frac{1}{x_{14}x_{43}x_{32}} - \frac{1}{x_{14}x_{42}x_{23}}\,. \tag{C.39}$$

C.3 Closed formula

In this last section the closed partial differential equation (6.86) is proven by considering an element of $\boldsymbol{Z}_{n,p}^{f}(B)$,

$$Z_{n,p}^{f}(B; (1, A^1), \ldots, (p, A^p)) = \int_{\Delta_{n,p}} \prod_{a=p+1}^{n} \mathrm{d}x_a\, u_f \prod_{b=1}^{p} \varphi_f\left(b, A^b\right)\,, \tag{C.40}$$

where $A = (p+1, p+2, \ldots, n)$ and $(A^1, \ldots, A^p) = \sigma(A)$ for a permutation $\sigma \in S_{n-p}$. A closed formula for the derivative of the integral in eq. (C.40) with respect to some base point $x_i \in B$ can be calculated as follows:[2] first, using the antisymmetry of

[2]The following derivation generalises the calculation from ref. [3]

$f_{ij}(\eta)$ and integration by parts, any partial derivative may be redirected to only act on the factor u_f, which leads to

$$\begin{aligned}
&\partial_{x_i} Z^f_{n,p}(B;(1,A^1),\dots,(p,A^p)) \\
&= \int_{\Delta_{n,p}} \prod_{a=p+1}^{n} \mathrm{d}x_a \left(\sum_{k\in(i,A^i)} \partial_{x_k} u_f \right) \prod_{b=1}^{p} \varphi_f(b,A^b)\,.
\end{aligned} \tag{C.41}$$

Let us denote for $1 \leq i \leq p$ the label $i = a^i_0$ and the elements of A^i by

$$(i,A^i) = (a^i_0, a^i_1, \dots, a^i_{|A^i|}) \tag{C.42}$$

and define for a sequence $C = (c_1, c_2, \dots, c_m)$ a sum of η-variables by

$$\eta_C = \sum_{i=1}^{m} \eta_{c_i}\,. \tag{C.43}$$

Note that according to the condition (6.8), auxiliary variables

$$\eta_i = -\eta_{A^i} \tag{C.44}$$

are associated to the unintegrated punctures x_i in the product $\prod_{i=1}^{p} \varphi_f(i,A^i)$ in the integrand in eq. (C.40). Moreover, let us define for a sequence C the subsequence C_{ij} by

$$C = (c_1, \dots, c_{i-1}, \underbrace{c_i, c_{i+1} \dots, c_{j-1}}_{C_{i,j}=C_{ij}}, c_j, c_{j+1} \dots, c_m)\,, \tag{C.45}$$

with

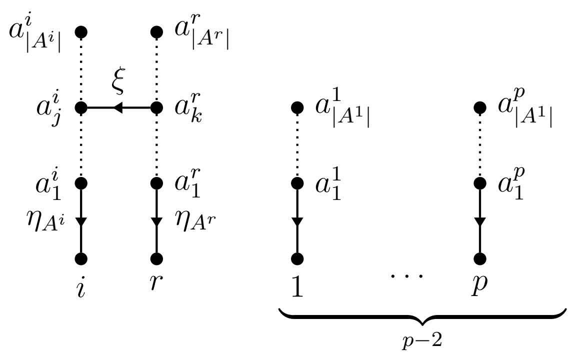
$$\begin{aligned}
C_{ji} &= \varnothing \ \text{ for } j \geq i\,, \\
C_{1,m+1} &= C\,, \\
\tilde{C}_{ij} &= (c_{j-1}, c_{j-2}, \dots, c_i)\,.
\end{aligned} \tag{C.46}$$

Then, the derivative (6.78) of u_f and the antisymmetry of $g_{ij}^{(m)}$ lead to

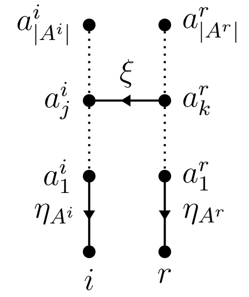
$$
\begin{aligned}
&\partial_{x_i} Z^f_{n,p}(B;(1,A^1),\dots,(p,A^p)) \\
&= -\sum_{\substack{k\in\{i,A^i\}\\ j\notin\{i,A^i\}}} \alpha_{kj} \int_{\Delta_{n,p}} \prod_{a=p+1}^{n} \mathrm{d}x_a\, u_f \left[f_{ij}(\xi) \prod_{b=1}^{p} \varphi_f(b,A^b)\right]_{\xi^0} \\
&= -\sum_{\substack{r\in\{1,\dots,p\}\\ r\neq i}} \left(\alpha_{ir} g_{ir}^{(m)} Z^f_{n,p}(B;A^1,\dots,A^p) \right.\\
&\quad + \sum_{k=1}^{|A^r|} \alpha_{ia^r_k} \int_{\Delta_{n,p}} \prod_{a=p+1}^{n} \mathrm{d}x_a\, u_f \left[f_{ia^r_k}(\xi) \prod_{b=1}^{p} \varphi_f(b,A^b)\right]_{\xi^0} \\
&\quad + \sum_{j=1}^{|A^i|} \alpha_{a^i_j r} \int_{\Delta_{n,p}} \prod_{a=p+1}^{n} \mathrm{d}x_a\, u_f \left[f_{a^i_j r}(\xi) \prod_{b=1}^{p} \varphi_f(b,A^b)\right]_{\xi^0} \\
&\quad \left. + \sum_{k=1}^{|A^r|}\sum_{j=1}^{|A^i|} \alpha_{a^i_j a^r_k} \int_{\Delta_{n,p}} \prod_{a=p+1}^{n} \mathrm{d}x_a\, u_f \left[f_{a^i_j a^r_k}(\xi) \prod_{b=1}^{p} \varphi_f(b,A^b)\right]_{\xi^0} \right), \qquad \text{(C.47)}
\end{aligned}
$$

where $[f_{ij}(\xi)]_{\xi^0}$ is the order zero coefficient $g_{ij}^{(m)}$ in the ξ-expansion of $f_{ij}(\xi)$. Let us consider the product $f_{a^i_j a^r_k}(\xi)\prod_{b=1}^{p}\varphi_f(b,A^b)$ in the last sum, it corresponds to the graph

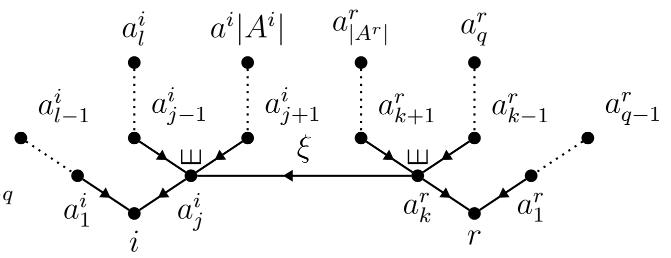
$$f_{a^i_j a^r_k}(\xi) \prod_{b=1}^{p} \varphi_f(b,A^b) \simeq \qquad \text{(C.48)}$$



in $\mathcal{G}_f(\vec{x},\vec{\eta}')$, where the weights $\eta'_l \in \vec{\eta}'$ are the same as $\eta_l \in \vec{\eta}$, except for $l \in \{a^i_j, a^r_k\}$ which acquire a shift $\eta'_{a^i_j} = \eta_{a^i_j} - \xi$ and $\eta'_{a^r_k} = \eta_{a^r_k} + \xi$. The connected subgraph with base points i and r can be expressed in $\mathcal{G}_f(\vec{x},\vec{\eta}')$ such that it has only one base point and an edge between the two vertices i and r: first, the identity (C.5) can be used to shift a^i_j and a^r_k next to i and r, respectively,

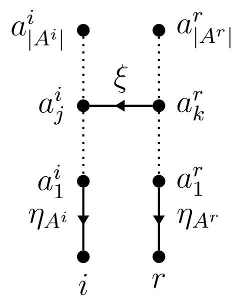


$$= \sum_{l=1}^{j} \sum_{q=1}^{k} (-1)^{j-l+k-q}$$

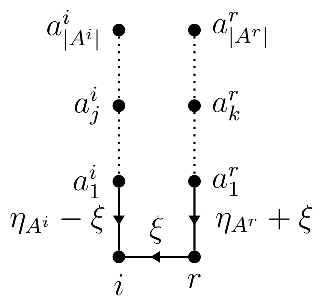


. (C.49)

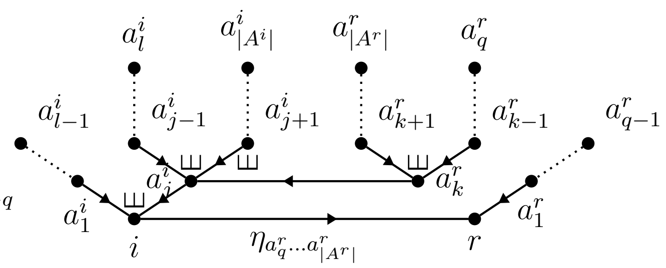
Then, the identity (C.7) can be applied to the vertices i, a^i_j, a^r_k, r, which yields upon applying eq. (C.5) in the reverse direction for the first term and additional shuffle identities at the vertices i and a^i_j for the second and at the vertices r and a^r_k for the third term a sum of chain graphs with an additional edge between the vertices i and r:



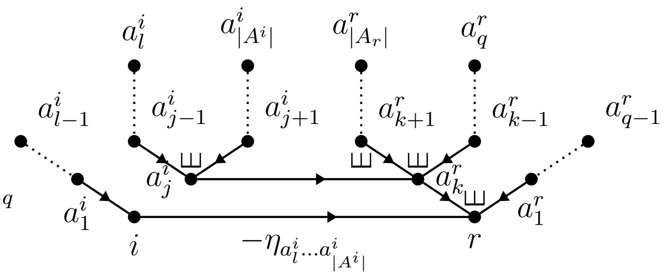
$=$



$$+ \sum_{l=1}^{j} \sum_{q=1}^{k} (-1)^{j-l+k-q}$$



$$+ \sum_{l=1}^{j} \sum_{q=1}^{k} (-1)^{j-l+k-q}$$



. (C.50)

The image of this graphical identity under f in $G_f(\vec{x}, \vec{\eta}')$ is

$$
\begin{aligned}
& f_{a^i_j a^r_k}(\xi)\varphi_f(i, A^i)\varphi_f(r, A^r) \\
& = f_{ir}(\xi)\varphi_f(i, A^i)\varphi_f(r, A^r) \\
& \quad + \sum_{l=1}^{j}\sum_{q=1}^{k}(-1)^{j-l+k-q} f_{ri}(\eta_{A^r_{q,|A^r|+1}})\varphi_f(r, A^r_{1q}) \\
& \quad \varphi_f\left(i, A^i_{1l} ш \left(a^i_j, \tilde{A}^i_{lj} ш A^i_{j+1,|A^i|+1}\right) ш \left(a^r_k, A^r_{k+1,|A^r|+1} ш \tilde{A}^r_{qk}\right)\right) \\
& \quad + \sum_{l=1}^{j}\sum_{q=1}^{k}(-1)^{j-l+k-q} f_{ri}(-\eta_{A^i_{l,|A^i|+1}})\varphi_f(i, A^i_{1l}) \\
& \quad \varphi_f\left(r, A^r_{1q} ш \left(a^r_k, \tilde{A}^r_{qk} ш A^r_{k+1,|A^r|+1}\right) ш \left(a^i_j, A^i_{j+1,|A^i|+1} ш \tilde{A}^i_{lj}\right)\right) . \qquad \text{(C.51)}
\end{aligned}
$$

The zeroth order term $\left[f_{a^i_j a^r_k}(\xi)\varphi_f(i, A^i)\varphi_f(r, A^r)\right]_{\xi^0}$ in ξ can be extracted from the two double sums on the right-hand side above simply by setting the shift $\xi = 0$ in the weights $\vec{\eta}'$, i.e. $\vec{\eta}'|_{\xi=0} = \vec{\eta}$, such that both double sums are elements of $G_f(\vec{x}, \vec{\eta})$. However, due to poles at $\xi = 0$ in $f_{ir}(\xi)$, all the quantities depending on ξ in the first term $f_{ir}(\xi)\varphi_f(i, A^i)\varphi_f(r, A^r) \in G_f(\vec{x}, \vec{\eta}')$ on the right-hand side have to be properly expanded in order to extract the zeroth order term and accountig for the poles. This introduces derivatives due to the Taylor expansion of $\varphi_f(i, A^i)\varphi_f(r, A^r) \in G_f(\vec{x}, \vec{\eta}')$

around $\xi = 0$. Denoting $i = a_0^i$ and $r = a_0^r$, this yields

$$
\begin{aligned}
&\left[f_{ir}(\xi)\varphi_f(i,A^i)\varphi_f(r,A^r)\right]_{\xi^0}\\
&=\Bigg[\left(\sum_{h=0}^{m} g_{ir}^{(h)}\xi^{h-m}\right)\\
&\left(\prod_{l=1}^{j} f_{a_{l-1}^i a_l^i}(\eta_{a_l^i\dots a_{|A^i|}^i}-\xi)\prod_{l=j+1}^{|A^i|} f_{a_{l-1}^i a_l^i}(\eta_{a_l^i\dots a_{|A^i|}^i})\right)\\
&\left(\prod_{q=1}^{k} f_{a_{q-1}^r a_q^r}(\eta_{a_q^r\dots a_{|A^r|}^r}+\xi)\prod_{q=k+1}^{|A^r|} f_{a_{q-1}^r a_q^r}(\eta_{a_q^r\dots a_{|A^r|}^r})\right)\Bigg]_{\xi^0}\\
&=\Bigg[\left(\sum_{h=0}^{m} g_{ir}^{(h)}\xi^{h-m}\right)\\
&\left(\prod_{l=1}^{j}\left(\sum_{m_l=0}^{m}\frac{(-\xi)^{m_l}}{m_l!}\partial_{\eta_{a_j^i}}^{m_l} f_{a_{l-1}^i a_l^i}(\eta_{a_l^i\dots a_{|A^i|}^i})\right)\prod_{l=j+1}^{|A^i|} f_{a_{l-1}^i a_l^i}(\eta_{a_l^i\dots a_{|A^i|}^i})\right)\\
&\left(\prod_{q=1}^{k}\left(\sum_{m_q=0}^{m}\frac{\xi^{m_q}}{m_q!}\partial_{\eta_{a_k^r}}^{m_q} f_{a_{q-1}^r a_q^r}(\eta_{a_q^r\dots a_{|A^r|}^r})\right)\prod_{q=k+1}^{|A^r|} f_{a_{q-1}^r a_q^r}(\eta_{a_q^r\dots a_{|A^r|}^r})\right)\Bigg]_{\xi^0}\\
&=\sum_{h=0}^{m} g_{ir}^{(h)}\sum_{m^j+m^k=m-h}(-1)^{m^j}\sum_{\substack{m_1^j+\dots+m_j^j=m^j\\ m_1^k+\dots+m_k^k=m^k}}\frac{1}{m_1^j!\dots m_j^j!}\frac{1}{m_1^k!\dots m_k^k!}\\
&\left(\prod_{l=1}^{j}\left(\partial_{\eta_{a_j^i}}^{m_l} f_{a_{l-1}^i a_l^i}(\eta_{a_l^i\dots a_{|A^i|}^i})\right)\prod_{l=j+1}^{|A^i|} f_{a_{l-1}^i a_l^i}(\eta_{a_l^i\dots a_{|A^i|}^i})\right)\\
&\left(\prod_{q=1}^{k}\left(\partial_{\eta_{a_k^r}}^{m_q} f_{a_{q-1}^r a_q^r}(\eta_{a_q^r\dots a_{|A^r|}^r})\right)\prod_{q=k+1}^{|A^r|} f_{a_{q-1}^r a_q^r}(\eta_{a_q^r\dots a_{|A^r|}^r})\right),
\end{aligned}
\tag{C.52}
$$

where all the sums run over the non-negative integers. Rewriting the result using the general Leibniz rule and the fact that the partial derivatives $\partial_{\eta_{a_j^i}}^{m_l}$ and $\partial_{\eta_{a_k^r}}^{m_q}$ act trivially on any other factors than the ones they act upon in the last two lines above leads to

$$
\begin{aligned}
&\left[f_{ir}(\xi)\underbrace{\varphi_f(i,A^i)\varphi_f(r,A^r)}_{\in G_f(\vec{x},\vec{\eta}')}\right]_{\xi^0}\\
&=\sum_{h=0}^{m} g_{ir}^{(h)}\sum_{\substack{m^j,m^k\geq 0\\ m^j+m^k=m-h}}\frac{(-1)^{m^j}}{m^j!m^k!}\partial_{\eta_{a_j^i}}^{m^j}\partial_{\eta_{a_k^r}}^{m^k}\underbrace{\varphi_f(i,A^i)\varphi_f(r,A^r)}_{\in G_f(\vec{x},\vec{\eta})},
\end{aligned}
\tag{C.53}
$$

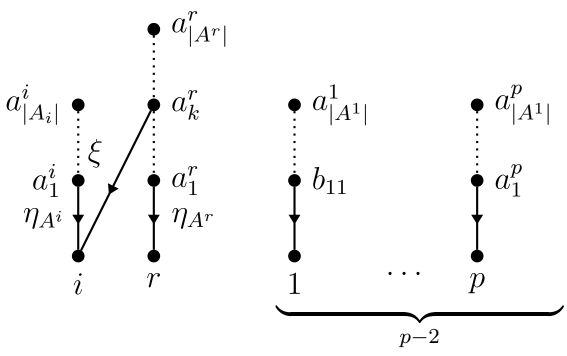
where in the product of chains of functions $\varphi_f(i,A^i)\varphi_f(r,A^r)$ on the right-hand side, the shift in the weights is set to zero $\xi = 0$, while on the left-hand side, there is a sufficiently small non-vanishing shift ξ. Putting all together, the integrals in the double sum on the last line in eq. (C.47) are given by

$$\begin{aligned}
&\int_{\Delta_{n,p}} \prod_{a=p+1}^{n} \mathrm{d}x_a\, u_f \left[f_{a^i_j a^r_k}(\xi) \prod_{b=1}^{p} \varphi_f(b, A^b) \right]_{\xi^0} \\
&= \sum_{h=0}^{m} g^{(h)}_{ir} \sum_{\substack{m^j, m^k \geq 0 \\ m^j + m^k = m-h}} \frac{(-1)^{m^j}}{m^j! m^k!} \partial^{m^j}_{\eta_{a^i_j}} \partial^{m^k}_{\eta_{a^r_k}}\, Z^f_{n,p}(B; (1, A^1), \dots, (p, A^p)) \\
&\quad + \sum_{l=1}^{j} \sum_{q=1}^{k} (-1)^{j-l+k-q} f_{ri}(\eta_{A^r_{q,|A^r|+1}}) \\
&\quad Z^f_{n,p}\Big(B; \dots, \Big(i, A^i_{1l} \sqcup \Big(a^i_j, \tilde{A}^i_{lj} \sqcup A^i_{j+1,|A^i|+1}\Big) \sqcup \Big(a^r_k, A^r_{k+1,|A^r|+1} \sqcup \tilde{A}^r_{qk}\Big)\Big), \\
&\qquad\qquad \dots, (r, A^r_{1q}), \dots\Big) \\
&\quad + \sum_{l=1}^{j} \sum_{q=1}^{k} (-1)^{j-l+k-q} f_{ri}(-\eta_{A^i_{l,|A^i|+1}}) \\
&\quad Z^f_{n,p}\Big(B; \dots, \Big(r, A^r_{1q} \sqcup \Big(a^r_k, \tilde{A}^r_{qk} \sqcup A^r_{k+1,|A^r|+1}\Big) \sqcup \Big(a^i_j, A^i_{j+1,|A^i|+1} \sqcup \tilde{A}^i_{lj}\Big)\Big), \\
&\qquad\qquad \dots, (i, A^i_{1l}), \dots\Big), \qquad \text{(C.54)}
\end{aligned}$$

where any argument not explicitly shown in the integrals $Z^f_{n,p}$ on the right-hand side is the same as on the left-hand side.

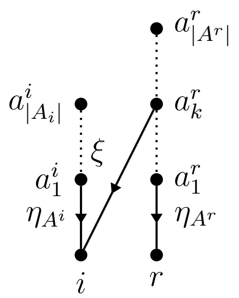
The other two sums in eq. (C.47), which are not yet written in terms of the integrals $Z^f_{n,p}$, can be calculated similarly. The first of these terms corresponds to the graph

$$f_{ia^r_k}(\xi) \prod_{b=1}^{p} \varphi_f(b, A^b) \simeq$$

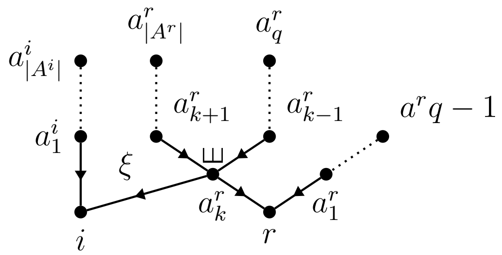


(C.55)

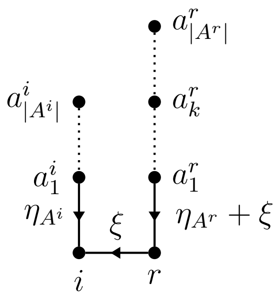
in $\mathcal{G}_f(\vec{x}, \vec{\eta}')$ with the shifted weights $\eta'_k = \eta_k + (\delta_{k,a^r_k} - \delta_{k,i})\xi \in \vec{\eta}'$ and $\eta_k \in \vec{\eta}$. The connected subgraph with an edge from a^r_k to i can be rewritten as follows using the identity (C.5) and the Fay identity (followed by the reverse of eq. (C.5) for one term):



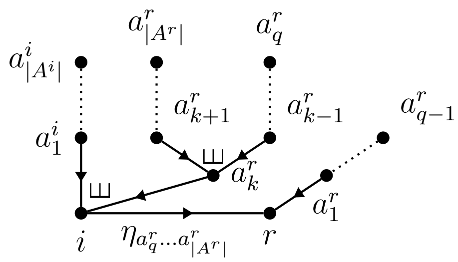
$$= \sum_{q=1}^{k} (-1)^{k-q}$$



$$=$$



$$+ \sum_{q=1}^{k} (-1)^{k-q}$$



$$. \tag{C.56}$$

The image of this graph identity under f in $G_f(\vec{x}, \vec{\eta}')$ is

$$\begin{aligned} f_{ia^r_k}(\xi)\varphi_f(i, B^i)\varphi_f(r, A^r) = {} & f_{ir}(\xi)\varphi_f(i, A^i)\varphi_f(r, A^r) \\ & + \sum_{q=1}^{k} (-1)^{k-q} f_{ri}(\eta_{A^r_{q,|A^r|+1}})\varphi_f(r, A^r_{1q}) \\ & \varphi_f\left(i, A^i \sqcup \left(a^r_k, A^r_{k+1,|A^r|+1} \sqcup \tilde{A}^r_{qk}\right)\right) . \end{aligned} \tag{C.57}$$

The extraction of the zeroth order in ξ in the sum over q can simply be done by setting $\xi = 0$. For the first term, restricting the sum in eq. (C.53) to $m^j = 0$ leads to

$$\left[f_{ir}(\xi) \underbrace{\varphi_f(i, A^i)\varphi_f(r, A^r)}_{\in G_f(\vec{x}, \vec{\eta}')} \right]_{\xi^0} = \sum_{h=0}^{m} g^{(h)}_{ir} \frac{1}{(m-h)!} \partial^{m-h}_{\eta_{a^r_k}} \underbrace{\varphi_f(i, A^i)\varphi_f(r, A^r)}_{\in G_f(\vec{x}, \vec{\eta})} . \tag{C.58}$$

Therefore, the integrals in the second sum of eq. (C.47) are given by

$$\begin{aligned}
&\int_{\Delta_{n,p}} \prod_{a=p+1}^{n} \mathrm{d}x_a\, u_f \left[f_{ia_k^r}(\xi) \prod_{b=1}^{p} \varphi_f(b, A^b) \right]_{\xi^0} \\
&= \sum_{h=0}^{m} g_{ir}^{(h)} \frac{1}{(m-h)!} \partial_{\eta_{a_k^r}}^{m-h} Z_{n,p}^f(B; (1, A^1), \ldots, (p, A^p)) \\
&\quad + \sum_{q=1}^{k} (-1)^{k-q} f_{ri}(\eta_{A^r_{q,|A^r|+1}}) \\
&\qquad Z_{n,p}^f \left(B; \ldots, (r, A^r_{1q}), \ldots, \left(i, A^i \sqcup \left(a_k^r, A^r_{k+1,|A^r|+1} \sqcup \tilde{A}^r_{qk} \right) \right), \ldots \right) . \qquad \text{(C.59)}
\end{aligned}$$

Exchanging the roles of i, j, m^j with r, k, m^k leads to the similar result for the third sum in eq. (C.47)

$$\begin{aligned}
&\int_{\Delta_{n,p}} \prod_{a=p+1}^{n} \mathrm{d}x_a\, u_f \left[f_{a_i^i r}(\xi) \prod_{b=1}^{p} \varphi_f(b, A^b) \right]_{\xi^0} \\
&= \sum_{h=0}^{m} g_{ir}^{(h)} \frac{(-1)^{m-h}}{(m-h)!} \partial_{\eta_{a_j^i}}^{m-h} Z_{n,p}^f(B; (1, A^1), \ldots, (p, A^p)) \\
&\quad + \sum_{l=1}^{j} (-1)^{j-k} f_{ri}(-\eta_{A^i_{l,|A^i|+1}}) \\
&\qquad Z_{n,p}^f \left(B; \ldots, (i, A^i_{1,l}), \ldots, \left(r, A^r \sqcup \left(a_j^i, A^i_{j+1,|A^i|+1} \sqcup \tilde{A}^i_{lj} \right) \right), \ldots \right) . \qquad \text{(C.60)}
\end{aligned}$$

Now, the closed formula of the partial derivative in eq. (C.47) follows from eqns. (C.54), (C.59) and (C.60):

$$
\begin{aligned}
&\partial_{x_i} Z^f_{n,p}(B;(1,A^1),\ldots,(p,A^p))\\
&= -\sum_{\substack{r\in\{1,\ldots,p\}\\ r\neq i}} \Bigg[\alpha_{ir} g_{ir}^{(m)} Z^f_{n,p}(B;(1,A^1),\ldots,(p,A^p))\\
&\quad + \sum_{k=1}^{|A^r|} \alpha_{i a^r_k} \Bigg\{ \sum_{h=0}^{m} g_{ir}^{(h)} \frac{1}{(m-h)!} \partial^{m-h}_{\eta_{a^r_k}} Z^f_{n,p}(B;(1,A^1),\ldots,(p,A^p))\\
&\qquad + \sum_{q=1}^{k} (-1)^{k-q} f_{ri}(\eta_{A^r_{q,|A^r|+1}})\\
&\qquad Z^f_{n,p}\left(B;\ldots,(r,A^r_{1q}),\ldots,\left(i, A^i \sqcup \left(a^r_k, A^r_{k+1,|A^r|+1} \sqcup \tilde{A}^r_{qk}\right)\right),\ldots\right)\Bigg\}\\
&\quad + \sum_{j=1}^{|A^i|} \alpha_{a^i_j r} \Bigg\{ \sum_{h=0}^{m} g_{ir}^{(h)} \frac{(-1)^{m-h}}{(m-h)!} \partial^{m-h}_{\eta_{a^i_j}} Z^f_{n,p}(B;A^1,\ldots,A^p)\\
&\qquad + \sum_{l=1}^{j} (-1)^{j-k} f_{ri}(-\eta_{A^i_{l,|A^i|+1}})\\
&\qquad Z^f_{n,p}\left(B;\ldots,(i,A^i_{1l}),\ldots,\left(r, A^r \sqcup \left(a^i_j, A^i_{j+1,|A^i|+1} \sqcup \tilde{A}^i_{lj}\right)\right),\ldots\right)\Bigg\}\\
&\quad + \sum_{k=1}^{|A^r|}\sum_{j=1}^{|A^i|} \alpha_{a^i_j a^r_k} \Bigg\{ \sum_{h=0}^{m} g_{ir}^{(h)} \sum_{\substack{m^j,m^k\geq 0\\ m^j+m^k=m-h}} \frac{(-1)^{m^j}}{m^j! m^k!} \partial^{m^j}_{\eta_{a^i_j}} \partial^{m^k}_{\eta_{a^r_k}} Z^f_{n,p}(B;A^1,\ldots,A^p)\\
&\qquad + \sum_{l=1}^{j}\sum_{q=1}^{k} (-1)^{j-l+k-q} f_{ri}(\eta_{A^r_{q,|A^r|+1}})\\
&\qquad Z^f_{n,p}\Big(B;\ldots,\left(i, A^i_{1l} \sqcup \left(a^i_j, \tilde{A}^i_{lj} \sqcup A^i_{j+1,|A^i|+1}\right) \sqcup \left(a^r_k, A^r_{k+1,|A^r|+1} \sqcup \tilde{A}^r_{qk}\right)\right),\\
&\qquad\qquad \ldots,(r,A^r_{1q}),\ldots\Big)\\
&\qquad + \sum_{l=1}^{j}\sum_{q=1}^{k} (-1)^{j-l+k-q} f_{ri}(-\eta_{A^i_{l,|A^i|+1}})\\
&\qquad Z^f_{n,p}\Big(B;\ldots,\left(r, A^r_{1q} \sqcup \left(a^r_k, \tilde{A}^r_{qk} \sqcup A^r_{k+1,|A^r|+1}\right) \sqcup \left(a^i_j, A^i_{j+1,|A^i|+1} \sqcup \tilde{A}^i_{lj}\right)\right),\\
&\qquad\qquad \ldots,(i,A^i_{1l}),\ldots\Big)\Bigg\}\Bigg].
\end{aligned} \tag{C.61}
$$

Evaluating the derivative of each component of the vector $\boldsymbol{Z}^f_{n,p}(x_1,\ldots,x_p)$ according to eq. (C.61) and expanding each $f_{ri}(\eta) = \sum_{k\geq 0} g_{ri}^{(k)} \eta^{k-m}$ leads to a partial differential equation of the form

$$\partial_{x_i} \boldsymbol{Z}^f_{n,p}(x_1,\dots,x_p) = \left(\sum_{k\geq 0} \sum_{\substack{r=1\\ r\neq i}}^{p} r^f_{n,p}(\boldsymbol{x}^{ri}_k)\, g^{(k)}_{ri} \right) \boldsymbol{Z}^f_{n,p}(x_1,\dots,x_p)\,. \tag{C.62}$$

The components of the matrices $r^f_{n,p}(\boldsymbol{x}^{ri}_k)$ are given by the coefficients in the linear combination in eq. (C.61), from which certain properties of these matrices may be deduced: they are homogeneous of degree one in the variables α_{ij} and homogeneous of degree $k-m$ in the variables η_i, where a l-th derivative $\partial^l_{\eta_i}$ is counted to be of degree $-l$. The matrices with $k<m$ are differential-operator-valued, while for the matrices $k\geq m$ with pairwise distinct labels r,i,q,j and any k,l, the commutation relation

$$[r^f_{n,p}(\boldsymbol{x}^{ri}_k), r^f_{n,p}(\boldsymbol{x}^{qj}_l)] = 0\,, \quad |\{r,i,q,j\}| = 4 \tag{C.63}$$

holds.

C.3.1 Genus-one example

Let us again consider the genus-one example from subsection 6.2.3 with $\alpha_{ij}=s_{ij}$,

$$f(z,\eta) = F(z,\eta,\tau) = \sum_{k\geq 0} g^{(k)}(z,\tau)\eta^{k-1} \tag{C.64}$$

and the base points $B=\{0,z_2,\dots,z_n\}$, such that

$$\boldsymbol{Z}^\tau_{n,p}(z_2,\dots,z_p) = \boldsymbol{Z}^F_{n,p}(B) \tag{C.65}$$

is the vector from eq. (5.20).

z_i-derivative

The derivative of a component with respect to an unfixed puncture z_i for $2 \leq i \leq p$ is according to eq. (C.61) given by

$$
\begin{aligned}
&\partial_{z_i} Z^\tau_{n,p}((1,A^1),\dots,(p,A^p))\\
&= -\sum_{\substack{r\in\{1,\dots,p\}\\ r\neq i}} \Bigg[\Bigg(s_{(i,A^i),(r,A^r)} g^{(1)}_{ir} + \sum_{k=1}^{|A^r|} s_{(i,A^i),a^r_k} \partial_{\eta_{a^r_k}} - \sum_{j=1}^{|A^i|} s_{a^i_j,(r,A^r)} \partial_{\eta_{a^i_j}} \Bigg)\\
&\qquad\qquad Z^\tau_{n,p}(B;(1,A^1),\dots,(p,A^p))\\
&\quad + \sum_{k=1}^{|A^r|} s_{i a^r_k} \sum_{q=1}^{k} (-1)^{k-q} F_{ri}(\eta_{A^r_{q,|A^r|+1}})\\
&\qquad\qquad Z^\tau_{n,p}\Big(B;\dots,(r,A^r_{1q}),\dots,\Big(i, A^i \sqcup \Big(a^r_k, A^r_{k+1,|A^r|+1} \sqcup \tilde{A}^r_{qk}\Big)\Big),\dots\Big)\\
&\quad + \sum_{j=1}^{|A^i|} s_{a^i_j r} \sum_{l=1}^{j} (-1)^{j-k} F_{ri}(-\eta_{A^i_{l,|A^i|+1}})\\
&\qquad\qquad Z^\tau_{n,p}\Big(B;\dots,(i,A^i_{1l}),\dots,\Big(r, A^r \sqcup \Big(a^i_j, A^i_{j+1,|A^i|+1} \sqcup \tilde{A}^i_{lj}\Big)\Big),\dots\Big)\\
&\quad + \sum_{k=1}^{|A^r|}\sum_{j=1}^{|A^i|} s_{a^i_j a^r_k} \Bigg\{ \sum_{l=1}^{j}\sum_{q=1}^{k} (-1)^{j-l+k-q} F_{ri}(\eta_{A^r_{q,|A^r|+1}})\\
&\qquad\qquad Z^\tau_{n,p}\Big(B;\dots,\Big(i, A^i_{1l} \sqcup \Big(a^i_j, \tilde{A}^i_{lj} \sqcup A^i_{j+1,|A^i|+1}\Big) \sqcup \Big(a^r_k, A^r_{k+1,|A^r|+1} \sqcup \tilde{A}^r_{qk}\Big)\Big),\\
&\qquad\qquad\qquad \dots,(r,A^r_{1q}),\dots\Big)\\
&\qquad\quad + \sum_{l=1}^{j}\sum_{q=1}^{k} (-1)^{j-l+k-q} F_{ri}(-\eta_{A^i_{l,|A^i|+1}})\\
&\qquad\qquad Z^\tau_{n,p}\Big(B;\dots,\Big(r, A^r_{1q} \sqcup \Big(a^r_k, \tilde{A}^r_{qk} \sqcup A^r_{k+1,|A^r|+1}\Big) \sqcup \Big(a^i_j, A^i_{j+1,|A^i|+1} \sqcup \tilde{A}^i_{lj}\Big)\Big),\\
&\qquad\qquad\qquad \dots,(i,A^i_{1l}),\dots\Big)\Bigg\}\Bigg],
\end{aligned}
\tag{C.66}
$$

where for two sequences P, Q the sum of Mandelstam variables $s_{P,Q}$ is defined in eq. (4.81). This yields explicit expressions for the matrices $r^{\text{cha}}_n(\boldsymbol{x}^{ij,k}_p)$ in the elliptic KZB system (5.21), if it is applied to each component of the vector $\boldsymbol{Z}^\tau_{n,p}$ and the

Eisenstein–Kronecker series are expanded. For $p=2$, it reduces to

$$
\begin{aligned}
&\partial_{z_2} Z^\tau_{n,2}((1,A^1),(2,A^2))\\
&= -\left(s_{(1,A^1),(2,A^2)} g^{(1)}_{21} + \sum_{k=1}^{|A^1|} s_{a^1_k,(2,A^2)} \partial_{\eta_{a^1_k}} - \sum_{j=1}^{|A^2|} s_{(1,A^1),a^2_j} \partial_{\eta_{a^2_j}} \right) Z^\tau_{n,2}((1,A^1),(2,A^2))\\
&\quad + \sum_{k=1}^{|A^1|}\sum_{j=1}^{|A^2|} s_{a^1_k,a^2_j} \sum_{i=1}^{k}\sum_{l=1}^{j} (-1)^{k+j-i-l} F_{21}(\eta_{A^2_{l,|A^2|+1}})\\
&\qquad\qquad Z^\tau_{n,2}\left(\left(1, A^1_{1i} \sqcup (a^1_k, (\tilde{A}^1_{ik} \sqcup A^1_{k+1,|A^1|+1}) \sqcup (a^2_j, \tilde{A}^2_{lj} \sqcup A^2_{j+1,|A^2|+1}))\right), \left(2, A^2_{1l}\right)\right)\\
&\quad + \sum_{k=1}^{|A^1|}\sum_{j=1}^{|A^2|} s_{a^1_k,a^2_j} \sum_{i=1}^{k}\sum_{l=1}^{j} (-1)^{k+j-i-l} F_{21}(-\eta_{A^1_{i,|A^1|+1}})\\
&\qquad\qquad Z^\tau_{n,2}\left(\left(1, A^1_{1i}\right), \left(2, A^2_{1l} \sqcup (a^2_j, (\tilde{A}^2_{lj} \sqcup A^2_{j+1,|A^2|+1}) \sqcup (a^1_k, \tilde{A}^1_{ik} \sqcup A^1_{k+1,|A^1|+1}))\right)\right)\\
&\quad + \sum_{j=1}^{|A^2|} s_{1,a^2_j} \sum_{l=1}^{j} (-1)^{j-l} F_{21}(\eta_{A^2_{l,|A^2|+1}})\\
&\qquad\qquad Z^\tau_{n,2}\left(\left(1, A^1 \sqcup (a^2_j, \tilde{A}^2_{lj} \sqcup A^2_{j+1,|A^2|+1})\right), \left(2, A^2_{1l}\right)\right)\\
&\quad + \sum_{k=1}^{|A^1|} s_{a^1_k,2} \sum_{i=1}^{k} (-1)^{k-i} F_{21}(-\eta_{A^1_{i,|A^1|+1}})\\
&\qquad\qquad Z^\tau_{n,2}\left(\left(1, A^1_{1i}\right), \left(2, A^2 \sqcup (a^1_k, \tilde{A}^1_{ik} \sqcup A^1_{k+1,|A^1|+1})\right)\right), \qquad \text{(C.67)}
\end{aligned}
$$

leading to the partial differential equation (5.51) and used for the genus-one recursion, originally derived in ref. [3].

τ-derivative

In this genus-one example, there is another parameter than the unfixed punctures $z_2, \ldots, z_p$: the modular parameter τ. The τ-dependence of $\boldsymbol{Z}^\tau_{n,p}$ is governed by the $2\pi i\tau$-derivative in the elliptic KZB system (5.21). Let us give a brief outline how the closed formula, which leads to the matrices $r^{\text{cha}}_n(\boldsymbol{\epsilon}^k_p)$, can be derived. It essentially follows the same steps as the derivation of eq. (C.66) and is a straightforward generalisation of the derivation for $p=2$ from ref. [3]. In particular, the graphical formulation can be used to carefully structure and prove each step. For the sake of brevity, we simply give the crucial results at each step.

First, integration by parts and the mixed heat equation (3.79) are used such that the derivatives solely act on the Koba–Nielsen factor in $Z^\tau_{n,p}(A^1,\ldots,A^p)$. Then the identity [39]

$$2\pi i\partial_\tau \,\mathrm{KN}^\tau_{12\dots n} = -\sum_{1\leq i<j\leq n} s_{ij}(g^{(2)}_{ij} + 2\zeta_2)\,\mathrm{KN}^\tau_{12\dots n} \tag{C.68}$$

can be applied. The result is the expression

$$\begin{aligned}
&2\pi i\partial_\tau Z^\tau_{n,p}((1,A^1),\dots,(p,A^p))\\
&= -s_{12\dots n}2\zeta_2\, Z^\tau_{n,p}((1,A^1),\dots,(p,A^p))\\
&\quad -\sum_{r=1}^{p}\sum_{k=1}^{|A^r|}\sum_{j=0}^{k-1} s_{a^r_k a^r_j}\int_{\Delta_{n,p}}\prod_{a=p+1}^{n}\mathrm{d}z_a\ \mathrm{KN}^\tau_{12\dots n}\\
&\qquad\qquad \left(g^{(1)}_{a^r_k a^r_j}(\partial_{\eta_{a^r_k}} - \theta_{j\geq 1}\partial_{\eta_{a^r_j}}) + g^{(2)}_{a^r_k a^r_j}\right)\prod_{b=1}^{p}\varphi^\tau(b,A^b)\\
&\quad -\sum_{\substack{r,q=1\\ q<r}}^{p}\sum_{k=0}^{|A^r|}\sum_{j=0}^{|A^q|} s_{a^r_k a^q_j}\int_{\Delta_{n,p}}\prod_{a=2}^{p+1}\mathrm{d}z_a\ \mathrm{KN}^\tau_{12\dots n}\\
&\qquad\qquad \left(g^{(1)}_{a^r_k a^q_j}(\theta_{k\geq 1}\partial_{\eta_{a^r_k}} - \theta_{j\geq 1}\partial_{\eta_{a^q_j}}) + g^{(2)}_{a^r_k a^q_j}\right)\prod_{b=1}^{p}\varphi^\tau(b,A^b)\,,
\end{aligned} \tag{C.69}$$

where $\theta_{j\geq 1}$ is one for $j\geq 1$ and zero for $j=0$.

Second, the antisymmetry, Fay identity and similar relations can again be used to obtain a closed formula by recovering integrals of the form $Z^\tau_{n,p}((1,B^1),\dots,(p,B^p))$. This leads to the following identities [3]:

$$\begin{aligned}
&\sum_{k=1}^{|A^r|}\sum_{j=0}^{k-1} s_{a^r_k a^r_j}\left(g^{(1)}_{a^r_k a^r_j}(\partial_{\eta_{a^r_k}} - \theta_{j\geq 1}\partial_{\eta_{a^r_j}}) + g^{(2)}_{a^r_k a^r_j}\right)\prod_{b=1}^{p}\varphi^\tau(b,A^b)\\
&= \sum_{k=1}^{|A^r|}\sum_{j=0}^{k-1} s_{a^r_k a^r_j}\left[\left(\partial_{\eta_{a^r_k}} - F_{j\geq 1}\partial_{\eta_{a^r_j}} + \partial_\xi\right)\Omega_{a^r_k a^r_j}(\xi)\prod_{b=1}^{p}\varphi^\tau(b,A^b)\right]_{\xi^0}\\
&= -\frac{1}{2}\sum_{k=1}^{|A^r|}\sum_{j=0}^{k-1} s_{a^r_k a^r_j}\left(\partial_{\eta_{a^r_k}} - \theta_{j\geq 1}\partial_{\eta_{a^r_j}}\right)^2\prod_{b=1}^{p}\varphi^\tau(b,A^b)\\
&\quad + \sum_{k=1}^{|A^r|}\sum_{j=0}^{k-1} s_{a^r_k a^r_j}\sum_{l=j+1}^{k}\wp(\eta_{A^r_{l,|A^r|+1}})(-1)^{k-l}\prod_{\substack{b=1\\ b\neq r}}^{p}\varphi^\tau(b,A^b)\\
&\qquad \varphi^\tau(r, A^r_{1,j}, a^r_j, A^r_{j+1,l}\sqcup(a_k,\tilde{A}^r_{l,k}\sqcup A^r_{k+1,|A^r|+1}))
\end{aligned} \tag{C.70}$$

and

$$
\begin{aligned}
&\sum_{k=0}^{|A^r|}\sum_{j=0}^{|A^q|} s_{a_k^r a_j^q}\left(g^{(1)}_{a_k^r a_j^q}(\theta_{k\geq 1}\partial_{\eta_{a_k^r}} - \theta_{j\geq 1}\partial_{\eta_{a_j^q}}) + g^{(2)}_{a_k^r a_j^q}\right)\prod_{b=1}^{p}\varphi^\tau(b,A^b)\\
&= \sum_{k=0}^{|A^r|}\sum_{j=0}^{|A^q|} s_{a_k^r a_j^q}\left[\left(\theta_{k\geq 1}\partial_{\eta_{a_k^r}} - \theta_{j\geq 1}\partial_{\eta_{a_j^q}} + \partial_\xi\right)F_{a_k^r a_j^q}(\xi)\prod_{b=1}^{p}\varphi^\tau(b,A^b)\right]_{\xi^0}\\
&= \left(s_{(r,A^r),(q,A^q)}g^{(2)}_{a_k^r a_j^q} - \frac{1}{2}\sum_{k=0}^{|A^r|}\sum_{j=0}^{|A^q|} s_{a_k^r a_j^q}\left(\theta_{k\geq 1}\partial_{\eta_{a_k^r}} - \theta_{j\geq 1}\partial_{\eta_{a_j^q}}\right)^2\right)\prod_{b=1}^{p}\varphi^\tau(b,A^b)\\
&\quad - \sum_{k=0}^{|A^r|}\sum_{j=0}^{|A^q|} s_{a_k^r a_j^q}\sum_{i=1}^{k}\sum_{l=1}^{j}(-1)^{k+j-i-l}F^+_{qr}(\eta_{A^q_{l,|A^q|+1}})\prod_{\substack{b=1\\ b\neq r,q}}^{p}\varphi^\tau(b,A^b)\\
&\qquad\qquad \varphi^\tau(q,A^q_{1l})\varphi^\tau(r,A^r_{1i}\sqcup(a_k^r,(\tilde{A}^r_{i,k}\sqcup A^r_{k+1,|A^r|+1})\sqcup(a_j^q,\tilde{A}^q_{l,j}\sqcup A^q_{j+1,|A^q|+1})))\\
&\quad - \sum_{k=0}^{|A^r|}\sum_{j=0}^{|A^q|} s_{a_k^r a_j^q}\sum_{i=1}^{k}\sum_{l=1}^{j}(-1)^{k+j-i-l}F^-_{qr}(-\eta_{A^r_{i,|A^r|+1}})\prod_{\substack{b=1\\ b\neq r,q}}^{p}\varphi^\tau(b,A^b)\\
&\qquad\qquad \varphi^\tau(r,A^r_{1i})\varphi^\tau(q,A^q_{1l}\sqcup(a_j^q,(\tilde{A}^q_{l,j}\sqcup A^q_{j+1,|A^q|+1})\sqcup(a_k^r,\tilde{A}^r_{i,k}\sqcup A^r_{k+1,|A^r|+1})))\,,
\end{aligned}
\tag{C.71}
$$

where

$$
\begin{aligned}
F^{\pm}_{ij}(\pm\xi) &= \pm\partial_\xi F_{ij}(\pm\xi)\\
&= \pm\partial_\xi\sum_{k\geq 0} g^{(k)}_{ij}(\pm\xi)^{k-1}\\
&= \sum_{k\geq 0}(k-1)g^{(k)}_{ij}(\pm\xi)^{k-2}\,.
\end{aligned}
\tag{C.72}
$$

The result is the closed form

$$
\begin{aligned}
&2\pi i\partial_\tau Z^\tau_{n,p}((1,A^1),\dots,(p,A^p))\\
&=\Bigg(\frac{1}{2}\sum_{j=p+1}^{n}(s_{(1,\dots,p),j})\partial^2_{\eta_j}+\frac{1}{2}\sum_{p+1\le i<j\le n}s_{ij}(\partial_{\eta_i}-\partial_{\eta_j})^2-2\zeta_2 s_{12\dots n}\\
&\qquad-\sum_{\substack{r,q=1\\q<r}}^{p}s_{(k,A^k),(q,A^q)}g^{(2)}_{kq}\Bigg)Z^\tau_{n,p}(\dots)\\
&\quad-\sum_{r=1}^{p}\sum_{k=1}^{|A^r|}\sum_{j=0}^{k-1}s_{a^r_k,a^r_j}\sum_{l=j+1}^{k}\wp(\eta_{A^r_{l,|A^r|+1}})(-1)^{k-l}\\
&\qquad\qquad Z^\tau_{n,p}\left(\dots,\left(r,A^r_{1j},a^r_j,A^r_{jl}\sqcup(a^r_k,\tilde{A}^r_{lk}\sqcup A^r_{k+1,|A^r|+1})\right),\dots\right)\\
&\quad+\sum_{\substack{r,q=1\\q<r}}^{p}\sum_{k=1}^{|A^r|}\sum_{j=1}^{|A^q|}s_{a^r_k,a^q_j}\sum_{i=1}^{k}\sum_{l=1}^{j}(-1)^{k+j-i-l}F^+_{qr}(\eta_{A^q_{l,|A^q|+1}})\\
&\qquad Z^\tau_{n,p}\Big(\dots,\left(r,A^r_{1i}\sqcup(a^r_k,(\tilde{A}^r_{ik}\sqcup A^r_{k+1,|A^r|+1})\sqcup(a^q_j,\tilde{A}^q_{lj}\sqcup A^q_{j+1,|A^q|+1}))\right),\\
&\qquad\qquad\dots,(q,A^q_{1l}),\dots\Big)\\
&\quad+\sum_{\substack{r,q=1\\q<r}}^{p}\sum_{k=1}^{|A^r|}\sum_{j=1}^{|A^q|}s_{a^r_k,a^q_j}\sum_{i=1}^{k}\sum_{l=1}^{j}(-1)^{k+j-i-l}F^-_{qr}(-\eta_{A^r_{i,|A^r|+1}})\\
&\qquad Z^\tau_{n,p}\Big(\dots,\left(q,A^q_{1l}\sqcup(a^q_j,(\tilde{A}^q_{lj}\sqcup A^q_{j+1,|A^q|+1})\sqcup(a^r_k,\tilde{A}^r_{ik}\sqcup A^r_{k+1,|A^r|+1}))\right),\\
&\qquad\qquad\dots,(r,A^r_{1i}),\dots\Big)\\
&\quad+\sum_{\substack{r,q=1\\q<r}}^{p}\sum_{j=1}^{|A^q|}s_{r,a^q_j}\sum_{l=1}^{j}(-1)^{j-l}F^+_{qr}(\eta_{A^q_{l,|A^q|+1}})\\
&\qquad\qquad Z^\tau_{n,p}\left(\dots,(q,A^q_{1l}),\dots,\left(r,A^r\sqcup(a^q_j,\tilde{A}^q_{lj}\sqcup A^q_{j+1,|A^q|+1})\right),\dots\right)\\
&\quad+\sum_{\substack{r,q=1\\q<r}}^{p}\sum_{k=1}^{|A^r|}s_{a^r_k,q}\sum_{i=1}^{k}(-1)^{k-i}F^-_{qr}(-\eta_{A^r_i,|A^r|+1})\\
&\qquad\qquad Z^\tau_{n,p}\left(\dots,\left(q,A^q\sqcup(a^r_k,\tilde{A}^r_{ik}\sqcup A^r_{k+1,|A^r|+1})\right),\dots,(r,A^r_{1i}),\dots\right).
\end{aligned}
\tag{C.73}
$$

The $2\pi i\tau$-derivative in the elliptic KZB system (5.21) is obtained from this closed formula by applying it to the full vector $\boldsymbol{Z}^\tau_{n,p}$ and expanding the Eisenstein–Kronecker series as well as the Weierstrass $\wp$-functions according to eq. (3.15). This leads to explicit expressions of the matrices $r^{\text{cha}}_n(\boldsymbol{\epsilon}^k_2)$ and their properties can be read off. By specifying to $p=2$, the result relevant for the genus-one recursion is recovered [3]:

$$
\begin{aligned}
&2\pi i\partial_\tau Z^\tau_{n,2}((1,A^1),(2,A^2))\\
&=\Bigg(\frac{1}{2}\sum_{j=3}^{n}(s_{1j}+s_{2j})\partial^2_{\eta_j}+\frac{1}{2}\sum_{3\leq i<j\leq n}s_{ij}(\partial_{\eta_i}-\partial_{\eta_j})^2-2\zeta_2 s_{12\dots n}-s_{(1,A^1),(2,A^2)}g^{(2)}_{21}\Bigg)\\
&\qquad Z^\tau_{n,2}((1,A^1),(2,A^2))\\
&\quad-\sum_{k=1}^{|A^2|}\sum_{j=0}^{k-1}s_{a^2_k,a^2_j}\sum_{l=j+1}^{k}\wp(\eta_{A^2_{l,|A^2|+1}})(-1)^{k-l}\\
&\qquad\quad Z^\tau_{n,2}\Big((1,A^1),\Big(2,A^2_{1j},a^2_j,A^2_{jl}\shuffle(a^2_k,\tilde{A}^2_{lk}\shuffle A^2_{k+1,|A^2|+1})\Big)\Big)\\
&\quad-\sum_{k=1}^{|A^1|}\sum_{j=0}^{k-1}s_{a^1_k,a^1_j}\sum_{l=j+1}^{k}\wp(\eta_{A^1_{l,|A^1|+1}})(-1)^{k-l}\\
&\qquad Z^\tau_{n,2}\Big(\Big(1,A^1_{1j},a^1_j,A^1_{jl}\shuffle(a^1_k,\tilde{A}^1_{lk}\shuffle A^1_{k+1,|A^1|+1})\Big),(2,A^2)\Big)\\
&\quad+\sum_{k=1}^{|A^1|}\sum_{j=1}^{|A^2|}s_{a^1_k,a^2_j}\sum_{i=1}^{k}\sum_{l=1}^{j}(-1)^{k+j-i-l}F^+_{21}(\eta_{A^2_{l,|A^2|+1}})\\
&\qquad Z^\tau_{n,2}\Big(\Big(1,A^1_{1i}\shuffle(a^1_k,(\tilde{A}^1_{i,k}\shuffle A^1_{k+1,|A^1|+1})\shuffle(a^2_j,\tilde{A}^2_{l,j}\shuffle A^2_{j+1,|A^2|+1}))\Big),\big(2,A^2_{1l}\big)\Big)\\
&\quad+\sum_{k=1}^{|A^1|}\sum_{j=1}^{|A^2|}s_{a^1_k,a^2_j}\sum_{i=1}^{k}\sum_{l=1}^{j}(-1)^{k+j-i-l}F^-_{21}(-\eta_{A^1_{i,|A^1|+1}})\\
&\qquad Z^\tau_{n,2}\Big(\big(1,A^1_{1i}\big),\Big(2,A^2_{1l}\shuffle(a^2_j,(\tilde{A}^2_{lj}\shuffle A^2_{j+1,|A^2|+1})\shuffle(a^1_k,\tilde{A}^1_{i,k}\shuffle A^1_{k+1,|A^1|+1}))\Big)\Big)\\
&\quad+\sum_{j=1}^{|A^2|}s_{1,a^2_j}\sum_{l=1}^{j}(-1)^{j-l}F^+_{21}(\eta_{A^2_{l,|A^2|+1}})\\
&\qquad Z^\tau_{n,2}\Big(\Big(1,A^1\shuffle(a^2_j,\tilde{A}^2_{l,j}\shuffle A^2_{j+1,|A^2|+1})\Big),\big(2,A^2_{1l}\big)\Big)\\
&\quad+\sum_{k=1}^{|A^1|}s_{a^1_k,2}\sum_{i=1}^{k}(-1)^{k-i}F^-_{21}(-\eta_{A^1_{i,|A^1|+1}})\\
&\qquad Z^\tau_{n,2}\Big(\big(1,A^1_{1i}\big),\Big(2,A^2\shuffle(a^1_k,\tilde{A}^1_{i,k}\shuffle A^1_{k+1,|A^1|+1})\Big)\Big)\,,
\end{aligned}
\tag{C.74}
$$

which leads to the matrices in the partial differential equation (5.52) relevant for the genus-one recursion.

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