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On holomorphic matrices on bordered Riemann surfaces

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Abstract

Let $\mathbb{D}$ be the unit disk. Kutzschebauch and Studer (*Bull. Lond. Math. Soc.* 51 (2019) 995–1004) recently proved that, for each continuous map $A : \overline{\mathbb{D}} \to \mathrm{SL}(2, \mathbb{C})$, which is holomorphic in $\mathbb{D}$, there exist continuous maps $E, F : \overline{\mathbb{D}} \to \mathfrak{sl}(2, \mathbb{C})$, which are holomorphic in $\mathbb{D}$, such that $A = e^E e^F$. Also they asked if this extends to arbitrary compact bordered Riemann surfaces. We prove that this is possible.

1. *Introduction*

Let $\overline{X}$ be a compact bordered Riemann surface[†], and let X be the interior of $\overline{X}$. Denote by $\mathrm{SL}(2, \mathbb{C})$ the group of complex 2×2 matrices with determinant 1, and by $\mathfrak{sl}(2, \mathbb{C})$ its Lie algebra of complex 2×2 matrices with trace zero. We prove the following.

Theorem 1.1. *Let $A : \overline{X} \to \mathrm{SL}(2, \mathbb{C})$ be a continuous map, which is holomorphic in X. Then there exist continuous maps $E, F : \overline{X} \to \mathfrak{sl}(2, \mathbb{C})$, which are holomorphic in X, such that $A = e^E e^F$ on $\overline{X}$.*

Let $\overline{\mathbb{D}}$ be the closed unit disk in $\mathbb{C}$. For $\overline{X} = \overline{\mathbb{D}}$, Theorem 1.1 was recently proved by Kutzschebauch and Studer [**11**, Theorem 2]. In [**11**] also, the question is asked if Theorem 1.1 is true in general, and it is noted that there is some problem to adapt in a straightforward way the proof of [**11**] to the general case. The problem is that $\overline{X}$ need not be simply connected. Our proof of Theorem 1.1 is nevertheless some adaption of the proof given in [**11**] for the case $\overline{X} = \overline{\mathbb{D}}$.

Let $\mathcal{A}(\overline{X})$ be the algebra of complex-valued functions which are continuous on $\overline{X}$ and holomorphic in X. The first step in our proof of Theorem 1.1 is the following.

Lemma 1.2. *Let $a, b \in \mathcal{A}(\overline{X})$ with $\{a = 0\} \cap \{b = 0\} = \emptyset$ and, moreover, $\{a = 0\} \neq \overline{X}$. Then there exist $g, h \in \mathcal{A}(\overline{X})$ such that $b + ga = e^h$.*

Recall that (by definition) the Bass stable rank of a commutative unital ring R is equal to 1, if, for all $a, b \in R$ with $aR + bR = R$, there exists $g \in R$ such that $b + ga$ is invertible. Although not used in the present paper, let us note the following immediate corollary of Lemma 1.2.

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[Correction added on 17 May 2021, after first online publication: The spelling of the author name in reference 2 was corrected.]

[†]In the sense of [**1**, II.3A], which includes that $\overline{X}$ is connected. For example, $\overline{X}$ can be the closure of a bounded smooth domain X in the complex plane.

COROLLARY 1.3. *The Bass stable rank of* $\mathcal{A}(\overline{X})$ *is equal to* 1.[†]

That the Bass stable rank of $\mathcal{A}(\overline{\mathbb{D}})$ is one is an important ingredient of the proof of Theorem 1.1 given in [**11**] for $\overline{X} = \overline{\mathbb{D}}$. As pointed out there, this makes it possible to limit to matrices of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\{a = 0\} = \emptyset$. In the same way, Lemma 1.2 makes it possible to limit to matrices of the form $\left(\begin{smallmatrix} e^h & b \\ c & d \end{smallmatrix}\right)$, and, for matrices of this form, it is possible to adapt the proof from [**11**] to the case of non-simply connected $\overline{X}$.

Let $\mathrm{M}(2, \mathbb{C})$ be the algebra of all complex 2×2 matrices, and $\mathrm{GL}(2, \mathbb{C})$ the group of its invertible elements. Then, in the same way as in [**11**, Corollary 1], the following corollary can be deduced from Theorem 1.1.

COROLLARY 1.4. *Let* $A : \overline{X} \to \mathrm{GL}(2, \mathbb{C})$ *be continuous on* $\overline{X}$*, holomorphic in* X*, and null-homotopic. Then there exist continuous maps* $E, F : \overline{X} \to \mathrm{M}(2, \mathbb{C})$*, which are holomorphic in* X*, such that* $A = e^E e^F$ *on* $\overline{X}$*.*

The study of the question 'how many exponentials factors are necessary to represent a given holomorphic matrix' was started by Mortini and Rupp [**14**]. In the case of an invertible 2×2 matrix with entries from $\mathcal{A}(\overline{\mathbb{D}})$, they proved that four exponentials are sufficient [**14**, Theorem 7.1]. Then Doubtsov and Kutzschebauch [**6**, Proposition 3] improved this to three exponentials. Eventually Kutzschebauch and Studer obtained that two exponentials are sufficient, which cannot be further improved, by an example Mortini and Rupp [**14**, Example 6.4]. This example shows that, under the hypotheses of Theorem 1.1 or Corollary 1.4, in general there does not exist a continuous $B : \overline{X} \to \mathrm{M}(2, \mathbb{C})$ with $A = e^B$. As noted in [**6**], to find such B with values in $\mathfrak{sl}(2, \mathbb{C})$ is impossible already by the fact that not every matrix in $\mathrm{SL}(2, \mathbb{C})$ has a logarithm in $\mathfrak{sl}(2, \mathbb{C})$.

NOTE: After this paper was written and the preprint was posted in the arXiv [**12**], I got to know the preprint [**2**, Theorem 1.3] with a substantial generalization of Theorem 1.1. This generalization, in particular, contains Theorem 1.1 with $\mathrm{SL}(n, \mathbb{C})$ in place of $\mathrm{SL}(2, \mathbb{C})$, for arbitrary $n \geqslant 2$ (see [**2**, Example 1.4 (1)]).

2. *A sufficient criterion for the existence of a logarithm*

A matrix $\Phi \in \mathrm{M}(2, \mathbb{C})$ will be often considered as the linear operator in $\mathbb{C}^2$ defined by multiplication from the left by Φ (considering the vectors in $\mathbb{C}^2$ as column vectors). The kernel and the image of this operator will be denoted by $\operatorname{Ker} \Phi$ and $\operatorname{Im} \Phi$, respectively. For $\Phi \in \mathrm{M}(2, \mathbb{C})$ and $\lambda \in \mathbb{C}$, we often write $\lambda - \Phi$ instead of $\lambda I - \Phi$. A matrix $\Phi \in \mathrm{M}(2, \mathbb{C})$ will be called a *projection*, if it is a linear projection as an operator, that is, if $\Phi^2 = \Phi$.

LEMMA 2.1. *Let* X *be a topological space and let* $B : X \to \mathrm{SL}(2, \mathbb{C})$ *be continuous. Suppose there exists a continuous complex-valued function* λ *on* X *such that, for all* $\zeta \in X$*:*

(a) $e^{\lambda(\zeta)}$ *is an eigenvalue of* $B(\zeta)$*;*
(b) $e^{\lambda(\zeta)} \neq e^{-\lambda(\zeta)}$.

[†]If $\overline{X}$ is the closure of a bounded smooth domain in $\mathbb{C}$, this was proved by Corach and Suaréz [**5**, Theorem 2.3]. Actually they proved the stronger result that, if K is an arbitrary compact subset of $\mathbb{C}$, then the Bass stable rank of the algebra of functions which are continuous on K and holomorphic in the inner points of K is equal to 1. That the Bass stable rank of $\mathcal{A}(\overline{\mathbb{D}})$ is equal to 1 was obtained before in [**4, 10**]. I do not know if there already exists a published proof of Corollary 1.3 for arbitrary compact bordered Riemann surfaces.

Then there exists a uniquely determined map $F : X \to \mathfrak{sl}(2, \mathbb{C})$ such that $B = e^F$ on X and, for all $\zeta \in X$, $\lambda(\zeta)$ is an eigenvalue of $F(\zeta)$. This map is continuous. If X is a complex space[†] *and B, λ are holomorphic, then F is even holomorphic.*

Proof. *Existence:* Since $e^{\lambda(\zeta)}$ is an eigenvalue of $B(\zeta)$ and $\det B(\zeta) = 1$, $e^{-\lambda(\zeta)}$ is the other eigenvalue of $B(\zeta)$, which is distinct from $e^{\lambda(\zeta)}$, by condition (b). Therefore

$$\mathbb{C}^2 = \operatorname{Ker}\left(e^{\lambda(\zeta)} - B(\zeta)\right) \oplus \operatorname{Ker}\left(e^{-\lambda(\zeta)} - B(\zeta)\right) \quad \text{for all} \quad \zeta \in X,$$

where '$\oplus$' means 'direct sum' (not necessarily orthogonal). Let $P : X \to \mathrm{M}(2, \mathbb{C})$ be the map which assigns to each $\zeta \in X$ the linear projection from $\mathbb{C}^2$ onto $\operatorname{Ker}(e^{\lambda(\zeta)} - B(\zeta))$ along $\operatorname{Ker}(e^{-\lambda(\zeta)} - B(\zeta))$. Then

$$B = e^{\lambda} P + e^{-\lambda}(I - P), \tag{2.1}$$

which implies

$$P = \frac{1}{e^{\lambda} - e^{-\lambda}} B - \frac{e^{-\lambda}}{e^{\lambda} - e^{-\lambda}} I. \tag{2.2}$$

This shows that P is continuous on X and, if X is a complex space and B, λ are holomorphic, then P is even holomorphic on X. Now

$$F := \lambda P - \lambda(I - P) \tag{2.3}$$

has the desired properties.

Uniqueness: Let $\zeta \in X$ and $\Theta \in \mathfrak{sl}(2, \mathbb{C})$ such that $e^{\Theta} = B(\zeta)$, and $\lambda(\zeta)$ is an eigenvalue of Θ. Then Θ and $B(\zeta)$ commute. By (2.2), also Θ and $P(\zeta)$ commute. Therefore $\Theta = \alpha P(\zeta) + \beta(I - P(\zeta))$ for some numbers $\alpha, \beta \in \mathbb{C}$, which then are the eigenvalues of Θ, that is, either $\alpha = \lambda(\zeta)$ and $\beta = -\lambda(\zeta)$, or $\alpha = -\lambda(\zeta)$ and $\beta = \lambda(\zeta)$. $\alpha = -\lambda(\zeta)$ and $\beta = \lambda(\zeta)$ is not possible, since otherwise, by condition (b) and by (2.1), we would have

$$e^{\Theta} = e^{-\lambda(\zeta)} P + e^{\lambda(\zeta)}(I - P(\zeta)) \neq e^{\lambda(\zeta)} P(\zeta) + e^{-\lambda(\zeta)}(I - P(\zeta)) = B(\zeta).$$

Therefore $\alpha = \lambda(\zeta)$ and $\beta = -\lambda(\zeta)$. Hence, by (2.3),

$$\Theta = \lambda(\zeta) P(\zeta) - \lambda\zeta(I - P(\zeta)) = F(\zeta).$$

□

3. *Proof of Lemma* 1.2 *and Theorem* 1.1

In this section, $\overline{X}$ is a compact bordered Riemann surface, where we assume (as always possible[‡]) that X is a bounded smooth domain in some larger open Riemann surface $\widetilde{X}$, and $\overline{X}$ is the closure of X in $\widetilde{X}$. The boundary of $\overline{X}$ will be denoted by ∂X. If we speak about an *open subset* U *of* $\overline{X}$, then we always mean that U is a subset of $\overline{X}$ which is open in the topology of $\overline{X}$ (and in general not open in $\widetilde{X}$). For $K \subseteq \overline{X}$, let $\overline{K}$ be the closure of K (in $\overline{X}$ or in $\widetilde{X}$).

If U is an open subset of $\overline{X}$, then we denote by $\mathcal{A}(U)$ the algebra of continuous complex valued functions on U which are holomorphic in $U \cap X$.

To prove Theorem 1.1, we begin with the observation that

$$\Theta e^{\Phi} \Theta^{-1} = e^{\Theta \Phi \Theta^{-1}} \qquad \text{for all } \Theta, \Phi \in \mathrm{GL}(2, \mathbb{C}). \tag{3.1}$$

[†] By a complex space we mean a *reduced* complex space in the terminology of [**8**], which is the same as an analytic space in the terminology of [**3, 13**]. For example, each Riemann surface is a complex space.

[‡] One can take for $\widetilde{X}$ a non-compact open neighborhood of $\overline{X}$ in the double of $\overline{X}$ (for the definition of the double of $\overline{X}$, see, for example, [**1**, II. 3E]).

This shows that conjugation does not change the number of exponential factors needed to represent a given matrix. As in [**11**], we will use this observation several times.

Next we recall some known facts (Lemma 3.1, its Corollary 3.2 and Lemma 3.3), for completeness with proofs.

LEMMA 3.1. *Let α be a continuous* (0,1)*-form on $\overline{X}$ (that is, a continuous section over $\overline{X}$ of the holomorphic cotangential bundle of $\widetilde{X}$) which is $\mathcal{C}^\infty$ in X. Then there exists a continuous function $u : \overline{X} \to \mathbb{C}$ which is $\mathcal{C}^\infty$ in X such that $\overline{\partial} u = \alpha$ in X.*

Proof. As observed by Forstneric, Fornæss and Wold in [**7**, Section 2, formula (8)] (together with corresponding references), to solve the $\overline{\partial}$-equation on Riemann surfaces, one can use the following know fact: There exists a 1-form, ω, defined and holomorphic on $(\widetilde{X} \times \widetilde{X}) \setminus \Delta$, where Δ is the diagonal in $\widetilde{X} \times \widetilde{X}$, such that, if $h : U \to \mathbb{C}$ is a holomorphic coordinate on some open set $U \subseteq \widetilde{X}$, then, on $(U \times U) \setminus \Delta$, ω is of the form

$$\omega(\zeta, \eta) = \left(\frac{1}{h(\zeta) - h(\eta)} + \theta_h(\zeta, \eta) \right) dh(\zeta), \quad (\zeta, \eta) \in (U \times U) \setminus \Delta, \tag{3.2}$$

where θ_h is a holomorphic function on $U \times U$. Since $\overline{X}$ is compact, and α is continuous on $\overline{X}$, then it is clear that the function $u : \widetilde{X} \to \mathbb{C}$ defined by

$$u(\eta) = \frac{1}{2\pi i} \int_{\zeta \in X} \omega(\zeta, \eta) \wedge \alpha(\zeta), \quad \eta \in \widetilde{X},$$

is continuous on $\widetilde{X}$. To prove that, in X, u is $\mathcal{C}^\infty$ and solves the equation $\overline{\partial} u = \alpha$, we consider a point $\xi \in X$ and take an open neighborhoods V and U of ξ such that $\overline{V} \subseteq U$, $U \subseteq X$ and there exists a holomorphic coordinate $h : U \to \mathbb{C}$ of $\widetilde{X}$. Further choose a $\mathcal{C}^\infty$-function $\chi : \widetilde{X} \to [0,1]$ such that $\chi = 1$ in a neighborhood $\overline{V}$. Then $u = u_1 + u_2 + u_3$, where

$$u_1(\eta) = \frac{1}{2\pi i} \int_{\zeta \in V} \omega(\zeta, \eta) \wedge \alpha(\zeta),$$

$$u_2(\eta) = \frac{1}{2\pi i} \int_{\zeta \in X \setminus V} \chi(\zeta) \omega(\zeta, \eta) \wedge \alpha(\zeta),$$

$$u_3(\eta) = \frac{1}{2\pi i} \int_{\zeta \in X \setminus V} (1 - \chi(\zeta)) \omega(\zeta, \eta) \wedge \alpha(\zeta).$$

Then u_2 and u_3 are holomorphic in V. Therefore it remains to prove that u_1 is $\mathcal{C}^\infty$ and $\overline{\partial} u_1 = \alpha$, in V. By (3.2), $u_1 = u_1' + u_1''$, where

$$u_1'(\eta) = \frac{1}{2\pi i} \int_{\zeta \in V} \frac{dh(\zeta) \wedge \alpha(\zeta)}{h(\zeta) - h(\eta)} \quad \text{and} \quad u_1''(\eta) = \int_{\zeta \in V} \theta_h(\zeta, \eta) dh(\zeta) \wedge \alpha(\zeta).$$

Since θ_h is holomorphic, u_1'' is holomorphic. Further

$$(u_1' \circ h^{-1})(w) = \frac{1}{2\pi i} \int_{z \in h(V)} \frac{dz \wedge \big((h^{-1})^* \alpha\big)(z)}{w - z} \quad \text{for } w \in h(V).$$

Therefore, as is well known (see, for example, [**9**, Theorem 1.2.2]), $u_1' \circ h^{-1}$ is $\mathcal{C}^\infty$ and $\overline{\partial}(u_1' \circ h^{-1}) = (h^{-1})^* \alpha$, in $h(V)$, which implies that u_1' is $\mathcal{C}^\infty$ and $\overline{\partial} u_1' = \alpha$, in V. □

COROLLARY 3.2. *Let U_1, U_2 be non-empty open subsets of $\overline{X}$ with $U_1 \cup U_2 = \overline{X}$, and let $f \in \mathcal{A}(U_1 \cap U_2)$. Then there exist $f_1 \in \mathcal{A}(U_1)$ and $f_2 \in \mathcal{A}(U_2)$ with $f = f_1 - f_2$ on $U_1 \cap U_2$.*

Proof. For $K \subseteq \overline{X}$, we denote by $\partial_{\overline{X}} K$ the boundary of K with respect to the topology of $\overline{X}$ (which is, in general, smaller than the boundary in $\widetilde{X}$). Since U_1 and U_2 are open subsets of $\overline{X}$ and $U_1 \cup U_2 = \overline{X}$, we have

$$\overline{U_1 \setminus U_2} \cap \overline{U_2 \setminus U_1} = \emptyset.$$

Therefore we can find a $\mathcal{C}^\infty$ function $\chi : \widetilde{X} \to [0,1]$ with $\chi = 1$ in an $\widetilde{X}$-neighborhood of $\overline{U_1 \setminus U_2}$, and $\chi = 0$ in an $\widetilde{X}$-neighborhood of $\overline{U_2 \setminus U_1}$. Then we have well-defined continuous functions $c_1 : U_1 \to \mathbb{C}$ and $c_2 : U_2 \to \mathbb{C}$ which are $\mathcal{C}^\infty$ in $X \cap U_1$ and $X \cap U_2$, respectively, such that

$$c_1 = \begin{cases} (1-\chi)f & \text{on } U_1 \cap U_2, \\ 0 & \text{on } U_1 \setminus U_2, \end{cases} \quad \text{and} \quad c_2 = \begin{cases} -\chi f & \text{on } U_1 \cap U_2, \\ 0 & \text{on } U_2 \setminus U_1. \end{cases}$$

Then

$$f = c_1 - c_2 \quad \text{on} \quad U_1 \cap U_2, \tag{3.3}$$

$$\overline{\partial} c_1 = -\overline{\partial}\chi f = \overline{\partial} c_2 \quad \text{on} \quad X \cap U_1 \cap U_2. \tag{3.4}$$

Relation (3.4) shows that there is a well-defined continuous (0,1)-form on $\overline{X}$, α, which is $\mathcal{C}^\infty$ in X, such that

$$\alpha = \overline{\partial} c_j \text{ on } X \cap U_j, \quad \text{for} \quad j = 1, 2. \tag{3.5}$$

By the preceding lemma, we can find a continuous function $u : \overline{X} \to \mathbb{C}$ which is $\mathcal{C}^\infty$ in X such that $\overline{\partial} u = \alpha$ in X. Set $f_j = c_j - u$, $j = 1, 2$. Then, by (3.5), $f_j \in \mathcal{A}(U_j)$ and, by (3.3), $f = f_1 - f_2$ on $U_1 \cap U_2$. □

Lemma 3.3. *For each $a \in \mathcal{A}(\overline{X})$, either $\{a = 0\} = \overline{X}$ or $\partial X \cap \{a = 0\}$ is nowhere dense in ∂X.*

Proof. Assume $\partial X \cap \{a = 0\}$ is not nowhere dense in ∂X. Then there exist $\xi \in \partial X$ and an open subset U of $\overline{X}$ with $\xi \in U$ and $a \equiv 0$ on $U \cap \partial X$. Then (by definition of a bordered Riemann surface), we have an open subset V of $\overline{X}$ with $\xi \in V$, and a homeomorphism $\varphi : V \to \{z \in \mathbb{C} \,|\, |z| < 1, \operatorname{Im} z \geqslant 0\}$, which is biholomorphic from $V \setminus \partial X$ onto $\{z \in \mathbb{C} \,|\, |z| < 1, \operatorname{Im} z > 0\}$ and such that $\varphi(V \cap \partial X) =]-1, 1[$. Then the continuous function $a \circ \varphi^{-1}$ is holomorphic in $\{z \in \mathbb{C} \,|\, |z| < 1, \operatorname{Im} z > 0\}$ and has the real value 0 on $]-1, 1[$. Therefore, by the Schwarz reflection principle, there is a holomorhic function $\widetilde{a}$ on $\{z \in \mathbb{C} \,|\, |z| < 1\}$ with

$$\widetilde{a} = a \circ \varphi^{-1} \quad \text{on} \quad \big\{z \in \mathbb{C} \,\big|\, |z| < 1, \operatorname{Im} z \geqslant 0\big\}. \tag{3.6}$$

Since $a = 0$ on $\varphi^{-1}(]-1, 1[) = V \cap \partial X$, from (3.6) we get $\widetilde{a} = 0$ on $]-1, 1[$. Therefore $\widetilde{a} = 0$ on $\{z \in \mathbb{C} \,|\, |z| < 1\}$. Again by (3.6) this implies that $a = 0$ on $V \setminus \partial X$. Hence ($\overline{X}$ is connected) $\{a = 0\} = \overline{X}$. □

The first step in the proof of Lemma 1.2 is the following lemma.

Lemma 3.4. *Let $a, b \in \mathcal{A}(\overline{X})$ such that $\{a = 0\} \cap \{b = 0\} = \emptyset$. Then there exist finitely many closed subsets $K_1, \dots, K_\ell$ of $\overline{X}$ such that*

$$K_j \cap K_k = \emptyset \quad \textit{for all} \quad 1 \leqslant j, k \leqslant \ell \textit{ with } j \neq k, \tag{3.7}$$

$$\{a = 0\} \subseteq K_1 \cup \ldots \cup K_\ell, \tag{3.8}$$

and, for some open disks $\mathbb{D}_1, \dots, \mathbb{D}_\ell$ contained in $\mathbb{C} \setminus \{0\}$,

$$b(K_j) \subseteq \mathbb{D}_j \quad \textit{for } j = 1, \dots, \ell. \tag{3.9}$$

Proof. If $\{a=0\}=\emptyset$, the claim of the lemma is trivial. Therefore we may assume that $\{a=0\}\neq\emptyset$.

First let $\partial X\cap\{a=0\}=\emptyset$. Since $\overline{X}$ is compact and $\{a=0\}$ has no accumulation points in X, and since $\{a=0\}\neq\emptyset$, then $\{a=0\}$ consists of a finite number of points $\xi_1,\dots,\xi_\ell\in X$. Then $b(\xi_1)\neq 0$, …, $b(\xi_\ell)\neq 0$, and $K_1:=\{\xi_1\}$, …, $K_\ell:=\{\xi_\ell\}$ have the desired properties.

Now let $\partial X\cap\{a=0\}\neq\emptyset$. Fix a metric $\rho(\cdot,\cdot)$ on $\widetilde{X}$. For a subset K of $\widetilde{X}$ we denote by $\operatorname{diam}K$ the diameter of K with respect to this metric. Since $\overline{X}$ is compact, a,b are continuous and $\{a=0\}\cap\{b=0\}=\emptyset$, we have

$$\theta:=\min_{\zeta\in\overline{X}}\left(|a(\zeta)|+|b(\zeta)|\right)>0,$$

and we can find $\varepsilon>0$ such that

$$\left|b(\zeta)-b(\eta)\right|<\theta\quad\text{for all }\zeta,\eta\in\overline{X}\text{ with }\rho(\zeta,\eta)<\varepsilon. \tag{3.10}$$

We call a set $\Lambda\subseteq\partial X$ a *closed Interval* in ∂X if there is a homeomorphic map ψ from [0,1] onto Λ.

Since $\overline{X}$ is compact, ∂X is the union of a finite number of pairwise disjoint Jordan curves.

STATEMENT 1. Let Γ be one of these Jordan curves. Then there exists a finite number of closed intervals $\Lambda_1,\dots,\Lambda_q$ in Γ such that

$$\Lambda_j\cap\Lambda_k=\emptyset\quad\text{for}\quad 1\leqslant j,k\leqslant q\text{ with }j\neq k, \tag{3.11}$$

$$\Gamma\cap\{a=0\}\subseteq\Lambda_1\cup\ldots\cup\Lambda_q, \tag{3.12}$$

$$\Lambda_j\cap\{a=0\}\neq\emptyset\quad\text{for }j=1,\dots,q, \tag{3.13}$$

$$\operatorname{diam}(\Lambda_j)<\varepsilon\quad\text{for}\quad 1\leqslant j\leqslant q, \tag{3.14}$$

Proof of Statement 1. If $\Gamma\cap\{a=0\}=\emptyset$, the claim of the statement is trivial. Therefore we may assume that $\Gamma\cap\{a=0\}\neq\emptyset$.

Since Γ is a Jordan curve, we have a homeomorphism ϕ from $\mathbb{T}:=\{z\in\mathbb{C}\,|\,|z|=1\}$ onto Γ. Since $\{a=0\}\neq\overline{X}$, $\{a=0\}\cap\Gamma$ is nowhere dense in Γ (Lemma 3.3). Therefore we can find $0<t_1<t_2<\ldots<t_p<2\pi$ such that

$$a\big(\phi(e^{it_\kappa})\big)\neq 0\quad\text{for}\quad\kappa=1,\dots,p, \tag{3.15}$$

and

$$\begin{gathered}\operatorname{diam}\phi\Big(e^{i[t_\kappa,t_{\kappa+1}]}\Big)<\varepsilon\text{ for }\kappa=1,\dots,p-1,\text{ and}\\ \operatorname{diam}\Big(\phi\Big(e^{i[t_p,2\pi]}\Big)\cup\phi\Big(e^{i[0,t_1]}\Big)\Big)<\varepsilon.\end{gathered} \tag{3.16}$$

By (3.15), we can find $\sigma>0$ such that $t_\kappa+\sigma<t_{\kappa+1}$ for $\kappa=1,\dots,p-1$, $t_p+\sigma<2\pi$, and

$$a\big(\phi(e^{it})\big)\neq 0\text{ for }t_j\leqslant t\leqslant t_j+\sigma\text{ and }\kappa=1,\dots,p. \tag{3.17}$$

Define closed intervals in Γ, $\Delta_1,\dots,\Delta_p$, by

$$\Delta_\kappa=\phi\Big(e^{i[t_\kappa+\sigma,t_{\kappa+1}]}\Big)\text{ for }\kappa=1,\dots,p-1,\text{ and }\Delta_p=\phi\Big(e^{i[t_p+\sigma,2\pi]}\Big)\cup\phi\Big(e^{i[0,t_1]}\Big).$$

Then it is clear that

$$\Delta_\kappa\cap\Delta_\lambda=\emptyset\quad\text{for all }\kappa,\lambda\in\{1,\dots,p\}\text{ with }\kappa\neq\lambda, \tag{3.18}$$

from (3.16) it follows that

$$\operatorname{diam}\Delta_\kappa<\varepsilon\quad\text{for}\quad\kappa=1,\dots,p, \tag{3.19}$$

and from (3.17) it follows that

$$\Gamma \cap \{a = 0\} \subseteq \Delta_1 \cup \ldots \cup \Delta_p. \tag{3.20}$$

Let $\{\kappa_1, \ldots, \kappa_q\}$ be the set of all $\kappa \in \{1, \ldots, p\}$ with $\Delta_\kappa \cap \{a = 0\} \neq \emptyset$ (such κ exist, as $\Gamma \cap \{a = 0\} \neq \emptyset\}$), and define $\Lambda_j = \Delta_{\kappa_j}$ for $j = 1, \ldots, q$. Then (3.11) is clear by (3.18). (3.12) and (3.13) hold by (3.20) and the definition of the set $\{\kappa_1, \ldots, \kappa_q\}$. (3.14) is clear by (3.19). Statement 1 is proved.

From Statement 1, we obtain a finite number of closed intervals $\Lambda_1, \ldots, \Lambda_r$ in ∂X such that

$$\Lambda_j \cap \Lambda_k = \emptyset \quad \text{for} \quad 1 \leqslant j, k \leqslant r \text{ with } j \neq k, \tag{3.21}$$

$$\partial X \cap \{a = 0\} \subseteq \Lambda_1 \cup \ldots \cup \Lambda_r, \tag{3.22}$$

$$\Lambda_j \cap \{a = 0\} \neq \emptyset \quad \text{for } j = 1, \ldots, r, \tag{3.23}$$

$$\operatorname{diam}(\Lambda_j) < \varepsilon \quad \text{for} \quad j = 1, \ldots, r. \tag{3.24}$$

By (3.21) and (3.24), we can find open subsets U_j of $\overline{X}$, $j = 1, \ldots, r$, with

$$\Lambda_j \subseteq U_j \quad \text{for} \quad 1 \leqslant j \leqslant r, \tag{3.25}$$

$$\overline{U}_j \cap \overline{U}_k = \emptyset \quad \text{for all } 1 \leqslant j, k \leqslant r \text{ with } j \neq k, \tag{3.26}$$

$$\operatorname{diam}\left(\overline{U}_j\right) < \varepsilon \quad \text{for } j = 1, \ldots, r. \tag{3.27}$$

Note that then, by (3.23),

$$U_j \cap \{a = 0\} \neq \emptyset \quad \text{for } j = 1, \ldots, r. \tag{3.28}$$

Set $K_j = \overline{U}_j$ for $j = 1, \ldots, r$. Then, by (3.22) and (3.25),

$$\{a = 0\} \cap (\partial X \cup K_1 \cup \ldots \cup K_r) = \{a = 0\} \cap (K_1 \cup \ldots \cup K_r). \tag{3.29}$$

Statement 2. $N := \{a = 0\} \cap (\overline{X} \setminus (\partial X \cup K_1 \cup \ldots \cup K_r))$ is finite.

Proof of Statement 2. Assume N is infinite. Since $\overline{X}$ is compact, then N has an accumulation point $\xi \in \overline{X}$. Since $\{a = 0\}$ is closed, $\xi \in \{a = 0\}$. As $\{a = 0\} \cap X$ is discrete in X, this implies that $\xi \in \partial X \cap \{a = 0\}$ and further, by (3.22) and (3.25), that $\xi \in U_1 \cup \ldots \cup U_r$. In particular, with respect to the topology of $\overline{X}$, ξ is an inner point of $\partial X \cup K_1 \cup \ldots \cup K_r$, which is not possible, for ξ is an accumulation point of N and therefore, in particular, an accumulation point of $\overline{X} \setminus (\partial X \cup K_1 \cup \ldots \cup K_r)$. Statement 2 is proved.

Let $\xi_{r+1}, \ldots, \xi_\ell$ the distinct points of N, and define $K_j = \{\xi_j\}$ for $j = r+1, \ldots, \ell$. We claim that $K_1, \ldots, K_\ell$ have the desired properties (3.7)–(3.9).

Indeed, (3.7) follows from (3.26) and the fact that $\xi_{r+1}, \ldots, \xi_\ell$ are pairwise distinct and lie in N and, hence, outside $K_1 \cup \ldots \cup K_r$. By (3.29),

$$\{a = 0\} \cap (\partial X \cup K_1 \cup \ldots \cup K_r) \subseteq K_1 \cup \ldots \cup K_r,$$

and, by definition of $K_{r+1}, \ldots, K_\ell$,

$$\{a = 0\} \cap \left(\overline{X} \setminus (\partial X \cup K_1 \cup \ldots \cup K_r)\right) = N = K_{r+1} \cup \ldots \cup K_\ell.$$

Together implies (3.8). To prove (3.9), we first note that by (3.28) and the definition of $K_{r+1}, \ldots, K_\ell$, for each $j \in \{1, \ldots, \ell\}$, we have a point $\xi_j \in K_j$ with $a(\xi_j) = 0$. Since, by definition of θ, $|b(\xi_j)| \geqslant \theta$, setting $\mathbb{D}_j = \{z \in \mathbb{C} \mid |z - b(\xi_j)| < \theta\}$, we obtain open disks $\mathbb{D}_1, \ldots, \mathbb{D}_\ell \subseteq \mathbb{C} \setminus \{0\}$. Since $\operatorname{diam} K_j < \varepsilon$ for $j = 1 \ldots, \ell$ (for $1 \leqslant j \leqslant r$ this holds by (3.27), and for $r + 1 \leqslant j \leqslant \ell$, we have $\operatorname{diam} K_j = 0$), now (3.9) follows from (3.10). $\square$

Proof of Lemma 1.2. If $\{a = 0\} = \emptyset$, we set $g = (1-b)/a$. Then $b + ga = 1 = e^0$ on X, and the claim of the lemma is proved.

Now let $\{a = 0\} \neq \emptyset$. By Lemma 3.4, we can find finitely many closed subsets $K_1, \dots, K_\ell$ of $\overline{X}$ and open disks $\mathbb{D}_1, \dots, \mathbb{D}_\ell$ in $\mathbb{C} \setminus \{0\}$ satisfying (3.7)–(3.9). Choose open subsets $W_1, \dots, W_\ell$ of $\overline{X}$ such that

$$K_j \subseteq W_j \quad \text{for} \quad 1 \leqslant j \leqslant \ell, \tag{3.30}$$

$$W_j \cap W_k = \emptyset \quad \text{for all} \quad 1 \leqslant j, k \leqslant \ell \text{ with } j \neq k, \tag{3.31}$$

$$b(W_j) \subseteq \mathbb{D}_j \quad \text{for } j = 1, \dots, \ell. \tag{3.32}$$

Since $D_j \subseteq \mathbb{C} \setminus \{0\}$, we can find holomorphic functions $\log_j : \mathbb{D}_j \to \mathbb{C}$ with $e^{\log_j z} = z$ for $z \in \mathbb{D}_j$. Set $W = W_1 \cup \ldots \cup W_\ell$ and $V = \overline{X} \setminus \{a = 0\}$. Then, by (3.30) and (3.8), $V \cup W = \overline{X}$, and, by (3.31) and (3.32), we can define $f \in \mathcal{A}(W)$ setting $f = \log_j \circ b$ on W_j. Then

$$b = e^f \quad \text{on } W. \tag{3.33}$$

Since $a \neq 0$ on V and $f \in \mathcal{A}(W)$, we have $f/a \in \mathcal{A}(V \cap W)$. Therefore, by Corollary 3.2, we can find $v \in \mathcal{A}(V)$ and $w \in \mathcal{A}(W)$ with $f/a = v - w$, that is,

$$f + aw = av \quad \text{on} \quad V \cap W.$$

Therefore, we have a function $h \in \mathcal{A}(\overline{X})$ with

$$h = f + aw \quad \text{on} \quad W. \tag{3.34}$$

The series $\sum_{\mu=0}^{\infty} \frac{a^\mu w^\mu}{\mu!} \frac{bw}{\mu+1}$ converges uniformly on the compact subsets of W to some $s \in \mathcal{A}(W)$, and, by (3.34) and (3.33), we have

$$e^h - b = e^{f+aw} - b = be^{aw} - b = b(e^{aw} - 1) \quad \text{on} \quad W.$$

Together this implies that, on $V \cap W = W \setminus \{a = 0\}$,

$$\frac{e^h - b}{a} = \frac{b}{a} \sum_{\mu=1}^{\infty} \frac{a^\mu w^\mu}{\mu!} = \frac{b}{a} \sum_{\mu=0}^{\infty} \frac{a^{\mu+1} w^{\mu+1}}{(\mu+1)!} = \sum_{\mu=0}^{\infty} \frac{a^\mu w^\mu}{\mu!} \frac{bw}{\mu+1} = s.$$

Therefore, we have a function $g \in \mathcal{A}(\overline{X})$ with $g = \frac{e^h - b}{a}$ on V and $g = s$ on W. Then, on $V = \overline{X} \setminus \{a = 0\}$, it is clear that

$$b + ga = b + \frac{e^h - b}{a} a = e^h.$$

Since $\{a = 0\}$ is nowhere dense in $\overline{X}$, it follows by continuity that $b + ga = e^h$ on all of $\overline{X}$. $\square$

Proof of Theorem 1.1. For $f \in \mathcal{A}(\overline{X})$, we denote by $\operatorname{Re} f$ and $|f|$ the functions $\overline{X} \ni \zeta \to \operatorname{Re} f(\zeta)$, and $\overline{X} \ni \zeta \to |f(\zeta)|$, respectively. By $\mathcal{A}^{\mathrm{SL}(2,\mathbb{C})}(\overline{X})$ and $\mathcal{A}^{\mathfrak{sl}(2,\mathbb{C})}(\overline{X})$, we denote the sets of continuous maps from $\overline{X}$ to $\mathrm{SL}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})$, respectively, which are holomorphic in X.

Now let $A \in \mathcal{A}^{\mathrm{SL}(2,\mathbb{C})}(\overline{X})$ be given.

If $A \equiv I$ or $A \equiv -I$, the claim of Theorem 1.1 is trivial. Therefore it is sufficient to consider the following three cases.

(I) A is of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\{c = 0\} \neq \overline{X}$.

(II) A is of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $\{b = 0\} \neq \overline{X}$.

(III) A is of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ where neither $\{a = 1\} = \{d = 1\} = \overline{X}$ nor $\{a = -1\} = \{d = -1\} = \overline{X}$.

By observation (3.1), Case (II) can be reduced to Case (I), since

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}^{-1} = \begin{pmatrix}d & 0\\ b & a\end{pmatrix}.$$

Consider Case (III). Since $\det A \equiv 1$, then $a \neq 0$ and $d = a^{-1}$ on $\overline{X}$. Moreover, then $\{a - a^{-1} = 0\} \neq \overline{X}$, for otherwise we would have $\{a^2 = 1\} = \overline{X}$, that is, either $\{a = 1\} = \{d = 1\} = \overline{X}$ or $\{a = -1\} = \{d = -1\} = \overline{X}$. As

$$\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^{-1} = \begin{pmatrix}a & 0\\ a - a^{-1} & a^{-1}\end{pmatrix},$$

this shows, again by (3.1), that also Case (III) can be reduced to Case (I).

So, we may assume that $A = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ where $\{c = 0\} \neq \overline{X}$. Since also $\{c = 0\} \cap \{a = 0\} = \emptyset$ (the values of A are invertible), then we can apply Lemma 1.2, which gives $g, h \in \mathcal{A}(\overline{X})$ with $a + gc = e^h$ on $\overline{X}$. Then

$$\begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix} A \begin{pmatrix}1 & g\\ 0 & 1\end{pmatrix}^{-1} = \begin{pmatrix}e^h & *\\ * & *\end{pmatrix}.$$

Therefore, again by observation (3.1), finally we see that $A = \left(\begin{smallmatrix}e^h & b\\ c & d\end{smallmatrix}\right)$ with $h, b, c, d \in \mathcal{A}(\overline{X})$ can be assumed.

The remaining part of the proof is an adaption of the proof given in [**11**] for $\overline{X} = \overline{\mathbb{D}}$. Chose $\delta > 0$ so large that, on $\overline{X}$,

$$\operatorname{Re}\left(e^{\delta} + e^{h-\delta} d\right) > 0, \tag{3.35}$$

$$\left|(1 + e^{h-2\delta} d)^2 - 4e^{-2\delta} - 1\right| < \frac{1}{2}, \tag{3.36}$$

and define

$$E = \begin{pmatrix}h - \delta & 0\\ 0 & \delta - h\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}e^{\delta} & e^{\delta - h} b\\ e^{h-\delta} c & e^{h-\delta} d\end{pmatrix}.$$

Then

$$E \in \mathcal{A}^{\mathfrak{sl}(2,\mathbb{C})}(\overline{X}), \quad B \in \mathcal{A}^{\mathrm{SL}(2,\mathbb{C})}(\overline{X}), \quad \text{and} \quad A = e^E B \text{ on } \overline{X}. \tag{3.37}$$

It follows from (3.36) that $\log((1 + e^{h-2\delta} d)^2 - 4e^{-2\delta})$ is well defined, where, since $|\log z| < 1$ if $|z - 1| < 1/2$,

$$\left|\log\left((1 + e^{h-2\delta} d)^2 - 4e^{-2\delta}\right)\right| < 1 \quad \text{on} \quad \overline{X}. \tag{3.38}$$

Since

$$\frac{(\operatorname{tr} B)^2}{4} - 1 = \frac{e^{2\delta}}{4}\left(\left(1 + e^{h-2\delta} d\right)^2 - 4e^{-2\delta}\right),$$

this implies that also $\log(\frac{(\operatorname{tr} B)^2}{4} - 1)$ is well defined, where

$$\log\left(\frac{(\operatorname{tr} B)^2}{4} - 1\right) = 2\delta - \log 4 + \log\left(\left(1 + e^{h-2\delta} d\right)^2 - 4e^{-2\delta}\right) \quad \text{on} \quad \overline{X}. \tag{3.39}$$

Set

$$\varphi = \exp\left(\frac{1}{2}\log\left(\frac{(\operatorname{tr} B)^2}{4} - 1\right)\right) \quad \text{on} \quad \overline{X}.$$

Then, by (3.39),

$$\varphi = \exp\left(\delta - \frac{\log 4}{2}\right) \exp\left(\frac{1}{2}\log\left(\left(1 + e^{h-2\delta}d\right)^2 - 4e^{-2\delta}\right)\right).$$

Since $|e^z - 1| < 1$ if $|z| < 1/2$ and therefore, by (3.38),

$$\left|\exp\left(\frac{1}{2}\log\left(\left(1 + e^{h-2\delta}d\right)^2 - 4e^{-2\delta}\right)\right) - 1\right| < 1,$$

this shows that

$$\operatorname{Re}\varphi > 0 \quad \text{on} \quad \overline{X}. \tag{3.40}$$

Since $\varphi^2 = \frac{(\operatorname{tr} B)^2}{4} - 1$, we see that, for each $\zeta \in \overline{X}$,

$$\theta_+(\zeta) := \frac{\operatorname{tr} B(\zeta)}{2} + \varphi(\zeta) \quad \text{and} \quad \theta_-(\zeta) := \frac{\operatorname{tr} B(\zeta)}{2} - \varphi(\zeta)$$

are the eigenvalues of $B(\zeta)$, where $\theta_+(\zeta) \neq \theta_-(\zeta)$ (as $\varphi(\zeta) \neq 0$). Since $\det B(\zeta) = 1$ and therefore $\theta_-(\zeta) = \theta_+(\zeta)^{-1}$, it follows that $\theta_+(\zeta) \neq \theta_+(\zeta)^{-1}$ for all $\zeta \in \overline{X}$. Since, by (3.35), also $\operatorname{Re}(\operatorname{tr} B) > 0$, it follows from (3.40) that $\operatorname{Re}\theta_+ > 0$ on $\overline{X}$. Therefore $\lambda = \log\theta_+$ is well defined. So, we have found a function $\lambda \in \mathcal{A}(\overline{X})$ with the property that, for all $\zeta \in \overline{X}$, $e^{\lambda(\zeta)}$ $(= \theta_+(\zeta))$ is an eigenvalue of $B(\zeta)$ and $\lambda(\zeta) \neq -\lambda(\zeta)$ (as $\theta_+(\zeta) \neq \theta_+(\zeta)^{-1}$). This implies by Lemma 2.1 that there exists $F \in \mathcal{A}^{\mathfrak{sl}(2,\mathbb{C})}(\overline{X})$ with $B = e^F$. By (3.37) this completes the proof of Theorem 1.1.

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