Bivariate Sarmanov Phase-Type Distributions for Joint Lifetimes Modeling

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Abstract

In this paper, we are interested in the dependence between lifetimes based on a joint survival model. This model is built using the bivariate Sarmanov distribution with Phase-Type marginal distributions. Capitalizing on these two classes of distributions' mathematical properties, we drive some useful closed-form expressions of distributions and quantities of interest in the context of multiple-life insurance contracts. The dependence structure that we consider in this paper is based on a general form of kernel function for the Bivariate Sarmanov distribution. The introduction of this new kernel function allows us to improve the attainable correlation range.

Keywords : *Dependence*; *multiple life insurance*; *Sarmanov distribution*; *Phase-Type distribution.*

1 Introduction

Dependence modeling is one of the most important topics in financial and actuarial modeling. While the independence assumption seems to be very simple to implement and makes most of the models very tractable, this assumption is not realistic in some cases and either underestimate or overestimate the level of risks. To fairly evaluate and price risks, one should consider any possible dependence between risks. In life contingencies, modeling dependence is not required when individual risks are considered unless they are part of a group life insurance or subject to multiple life insurance products. Multiple life products are contracts issued on two or more lives, as opposed to a single individual. The payoff associated with such insurance policies is contingent on the lifetimes of these individuals. It is usually assumed that the remaining lifetimes of the lives involved are mutually independent, mainly because of this assumption's computational feasibility. In this paper, we relax this assumption and address the problem of evaluating multiple insurance products for dependent risks. One way to introduce this dependence is using common shock models, which allow positive dependence between risks. For this kind of model, it is generally assumed that individual risks are following exponential distributions, limiting the practicability of such a model. Alternatively, the dependence is modeled using copula functions. Copulas allow separating the marginal distributions from the dependence structure of a given multivariate distribution, which introduces some flexibility in building dependent multivariate models. Dufresne et al. (2018) used copula theory to model the dependence between lifetimes within a married couple. Frees et al. (1996) developed a model based on copulas to show positive dependence between couples' lifetimes. Luciano et al. (2008) presented a bivariate stochastic mortality model where the joint mortality intensities are modeled via an Archimedean copula. Gobbi et al. (2019) suggested an extended Marshall-Olkin model and copulas to analyze the impact of dependence between risks in joint life insurance products pricing. This paper adds to the literature on joint life modeling with dependence by introducing a joint model based on the well-known Sarmanov multivariate distribution. For simplicity and without loss of generality, we only consider policies with two individuals. The generalization to three or more individuals follows easily. Sarmanov distribution was first introduced in bivariate form in Sarmanov (1966) then extended to the multivariate case in Lee (1996). Over the past few years, Sarmanov distribution has been used in many actuarial and financial applications due to its flexibility and mathematical features, which allow the derivation of closed-form expressions for some interesting actuarial and financial quantities. The applications of Sarmanov distribution in insurance are mainly in risk theory, ruin theory, claim reserving, and capital allocation, see e.g., Yang and Hashorva (2013), Hashorva and Ratovomirija (2015), Abdallah et al. (2016a), Abdallah et al. (2016b), Ratovomirija et al. (2017), Vernic (2017), Vernic (2018), and Bolancé and Vernic (2019). To our knowledge, no work has been done on applying Sarmanov distribution to model multiple life insurance contracts, and this paper is unique in this context.

To build a suitable bivariate lifetime model, we need an accurate marginal distribution. Various parametric distributions are used to model mortality, including the exponential distribution, Gompertz distribution, Weibull distribution. Stochastic mortality models have also been proposed in the literature; see Pitacco et al. (2009) for more details. However, these laws are normally focused on the underlying mortality itself. Simultaneously, actuarial quantities such as actuarial present values, premium, and other important quantities are not readily available in analytic form. This paper considers a large class of marginal distribution and works with the Phase-Type (PH) distribution. This specific distribution has been highly-used in risk theory; first introduced by NEUTS (1975), Neuts (1994). The PH distribution has many applications in queuing systems, engineering, and other applications due to its simple analytic expression and easy interpretation. It is also a generalization for many well-known and extensively used distributions such as the exponential distribution, Erlang distribution, and hyper-exponential distribution. X. Sheldon Lin PhD and Liu (2007) have proposed the use of the PH distribution as a mortality law. They calibrate the PH distribution to human mortalities of various countries and cohorts, based on the weighted least squares method. They derived analytical forms for the actuarial present value of some of the standard insurance products. Kim et al. (2017) extended the results of X. Sheldon Lin PhD and Liu (2007) by finding closed-form life contingencies expressions for a wide range of common life insurance products. They supported the efficiency of using the PH mortality model in risk management by referring to many papers where the distribution was used in many actuarial applications. For example, Govorun and Latouche (2014) used the PH mortality model to estimate the financial impact of health on profits and losses of a pension fund. Another example is Govorun et al. (2015) who studied the distribution of the net present value of a health care contract based on the PH model. Kim et al. (2017) provided analytic derivations for actuarial present values, premiums, reserves, higher moments of benefit, loaded premiums, risk measures, and many other applications for various standard life products. In the last section of their paper, they used the fact that the PH distribution is closed under convolution to price actuarial present values of multiple life products, assuming independence of the underlying mortalities. Zadeh et al. (2014) applied PH distributions in actuarial calculations for disability insurance. Recently, Asmussen et al. (2019) fit PH distribution to human mortality then price equity-linked life insurance contracts.

Motivated by the tractability and mathematical features of both Sarmanov and PH distributions, we build a bivariate survival model to evaluate multiple life insurance contracts. Capitalizing on these two classes of distributions' properties, we derive some closed-form expressions for relevant distributions, actuarial present values, and risk measures. Using our model, we can calculate premiums and reserves for life insurance and life annuities contingent on two dependent lives.

The rest of the paper will be organized as follows. We give a brief review of the PH distribution class with its main properties, and we also define the dependence structure using Sarmanov distribution. In Section 3, we derive our main results, mainly arriving to the actuarial present values of multiple life insurance and annuities contracts. To illustrate our findings, we provide some numerical examples in Section 4. Then, a summary and conclusion are given in Section 5.

2 The model

In this section, our joint lifetime model's components are presented. First, the marginal risks are specified to be following Phase-Type distribution, and the main definitions and characteristics of this family of distributions are given. Then, the dependence structure is defined using a multivariate Sarmanov distribution with a general kernel function.

2.1 Phase-Type Distribution

Phase-Type (PH) distributions have attracted a lot of interest in the actuarial and financial literature due to their useful properties and tractability. It is also possible to approximate any distribution on the non-negative real numbers by a PH distribution, which justifies its uses both in practice and theory. The PH distribution is defined as the distribution of the time to absorption for a given Markov chain. Consider a continuous-time Markov chain (CTMC) with a state space $\{1, 2, ..., n, 0\}$ where 0 is an absorbing state while all other states are transient. Let the initial distribution of the CTMC be $(\alpha, 0)$ and the infinitesimal generator be

$$\begin{pmatrix} \Lambda & \lambda \\ 0_{1,n} & 0 \end{pmatrix}, \lambda = -\Lambda e_n,$$

where e_n is an n-dimensional column vector of 1's and $0_{1,n}$ is an n-dimensional row vector of 0's. Note that α is an n dimensional row vector and Λ is an $n \times n$ matrix. Note also that Λ is a sub-stochastic matrix. Thus it has non-negative off-diagonal elements and strictly negative diagonal elements. The PH distribution is defined as the distribution of the time, T, till absorption in the CTMC, and we write, $T \sim PH(\alpha, \Lambda)$.

The cumulative distribution function (cdf) of T is

$$F_T(t) = 1 - \alpha e^{\Lambda t} e_n, \tag{2.1}$$

and the probability density function (pdf) is given by

$$f_T(t) = \alpha e^{\Lambda t} \lambda, \tag{2.2}$$

for $t \geq 0$ where $\lambda = -\Lambda e_n$. Many well-known and important distributions, such as the exponential, Erlang, and hyper-exponential distributions, are special cases of the PH distribution, as shown in the following examples.

Example 2.1. *If $T \sim \exp(\lambda)$ where $\lambda > 0$, and density function $f_T(t) = \lambda e^{-\lambda t}$, for $t \geq 0$, then it is a PH with*

$$\alpha = 1, \qquad \Lambda = -\lambda.$$

□

Example 2.2. *If T follows hyper-exponential distribution with density function $f_T(t) = \sum_{i=1}^{n} \alpha_i \lambda_i e^{-\lambda_i t}$, for $t \geq 0$, then it is a PH with*

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \qquad \Lambda = \begin{pmatrix} -\lambda_1 & 0 & \dots & 0 \\ 0 & -\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -\lambda_n \end{pmatrix}.$$

□

Example 2.3. *If T has an Erlang (n) distribution with density*

$$f_T(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \quad t \geq 0, \quad n \in \mathbb{N}^+, \quad \lambda > 0$$

then it is also PH with

$$\alpha = (1, 0, ..., 0), \qquad \Lambda = \begin{pmatrix} -\lambda & \lambda & 0 & \dots & 0 & 0 \\ 0 & -\lambda & \lambda & \dots & 0 & 0 \\ 0 & 0 & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda & \lambda \\ 0 & 0 & 0 & \dots & 0 & -\lambda \end{pmatrix}.$$

□

It is well-known that for any substochastic matrix, Λ, the real parts of its eigenvalues are non positive since substochastic matrices are diagonally dominant and all their diagonals are negative (see Definition 5.1 in the Appendix). Thus, we have the following proposition

Proposition 2.1. *Let Λ be a non-singular, square substochastic matrix. Then*

$$\lim_{t\to\infty} e^{\Lambda t} = 0.$$

Proof. This easily follows by the diagonalization of Λ, and using the definition of matrix exponential (see Definition 5.2 in the appendix). □

From the definition of the matrix exponential, we have

$$\Lambda \int_0^\tau e^{\Lambda t}\, dt = e^{\Lambda \tau} - I_n.$$

where I_n is the identity matrix. Using Proposition 2.1, and the fact that Λ is non-singular, we get

$$\int_0^\infty e^{\Lambda t}\, dt = \lim_{\tau\to\infty} \int_0^\tau e^{\Lambda t}\, dt = \lim_{\tau\to\infty} \Lambda^{-1}(e^{\Lambda\tau} - I_n) = -\Lambda^{-1}. \tag{2.3}$$

The expectation of T, it is given by

$$\begin{aligned} \mathbb{E}[T] &= \int_0^\infty [1 - F_T(t)]\, dt \\ &= \alpha \int_0^\infty e^{\Lambda t}\, dt\, e_n \\ &= -\alpha\Lambda^{-1} e_n. \end{aligned}$$

From Eq. 2.3, and using the fact that the identity matrix I_n commutes with any other matrix (see Proposition 5.1. in the appendix), one can derive the moment generating function (mgf) and Laplace transform to T

$$M_T(\delta) = \int_0^\infty e^{\delta t} f_T(t)\, dt = \alpha \int_0^\infty e^{(\delta I_n + \Lambda)t}\, dt \lambda = -\alpha(\delta I_n + \Lambda)^{-1}\lambda, \tag{2.4}$$

and

$$\Phi(\delta) = \int_0^\infty e^{-\delta t} f_T(t)\, dt = \alpha \int_0^\infty e^{(-\delta I_n + \Lambda)t}\, dt \lambda = \alpha(\delta I_n - \Lambda)^{-1}\lambda. \tag{2.5}$$

The kth moment is obtained by taking the kth derivative of the mgf using matrix-to-scalar identities of matrix calculus (see the appendix)

$$\begin{aligned}\mathbb{E}[T^k] &= \frac{d^k}{d\delta^k} M_T(\delta)|_{\delta=0} = -\frac{d^k}{d\delta^k}\alpha(\delta I_n - \Lambda)^{-1}\lambda|_{\delta=0} \\ &= -\alpha(-1)^k k!(\delta I_n - \Lambda)^{-(k+1)}\lambda|_{\delta=0} = \alpha(-1)^k k!(\delta I_n - \Lambda)^{-k} e_n|_{\delta=0} \\ &= \alpha(-1)^k k!\Lambda^{-k} e_n.\end{aligned} \tag{2.6}$$

One of the most useful properties of a PH distribution is that its excess life $E_x = T - x|T > x$ distribution is also a PH where

$$Pr(E_x > t) = \frac{\alpha e^{\Lambda(t+x)} e_n}{\alpha e^{\Lambda x} e_n},$$

i.e. E_x is a PH distributed with parameters (α_x, Λ), with

$$\alpha_x = \frac{\alpha e^{\Lambda x}}{\alpha e^{\Lambda x} e_n}.$$

In a context of life insurance, this property offers a convenient way to construct various life contingencies quantities because the future lifetime of an individual of age (x) is again $PH(\alpha_x, \Lambda_x)$ random variable. We refer readers to Bladt and Nielsen (2017) for more details on PH distributions and its properties. We end this section by giving some closed form expressions for some Actuarial Present Values (APV). Let δ be the continuous constant force of interest and consider an insured of age (x), where T_x is $PH(\alpha_x, \Lambda_x)$ random variable. The APV for a whole life insurance of a one unit future benefit is

$$\overline{A}_x = \int_0^\infty e^{-\delta T_x} f_{T_x}(t)\, dt = \int_0^\infty e^{-\delta T_x} \alpha_x e^{\Lambda_x t} \lambda_x \, dt = \alpha_x(\delta I_n - \Lambda_x)^{-1}\lambda_x. \tag{2.7}$$

The APV for a whole life annuity of \$1 is

$$\overline{a}_x = \int_0^\infty e^{-\delta T_x} \overline{F}_{T_x}(t)\, dt = \int_0^\infty e^{-\delta T_x} \alpha_x e^{\Lambda_x t} e_n \, dt = \alpha_x(\delta I_n - \Lambda_x)^{-1} e_n. \tag{2.8}$$

The APV of the $t-$term life insurance is given by

$$\overline{A}^{1}_{x:\overline{t}|} = \int_0^t e^{-\delta T_x} f_{T_x}(t)\, dt = \int_0^t e^{-\delta T_x} \alpha_x e^{\Lambda_x t} \lambda_x \, dt = \alpha_x(\delta I_n - \Lambda_x)^{-1}(I_n - e^{-(\delta I_n - \Lambda_x)t})\lambda_x. \tag{2.9}$$

2.2 The dependence structure

Consider two random variables T_x and T_y which are assumed to be dependent. These two rv's represent the future lifetimes for two lives ages x and y, respectively. The dependence could be introduced using copulas or a common shock model. In this paper, we use the bivariate Sarmanov distribution that is given by

$$h(s,t) = f_x(s) f_y(t) \left(1 + \omega \psi_x(s) \psi_y(t)\right), \tag{2.10}$$

where f_x and f_y are the marginal pdfs for T_x and T_y, respectively.

The kernel functions ψ_i, for $i = x, y$, are assumed to be bounded and non-constant such that $E[\psi_i(T_i)] = 0$. The parameter ω is a real number such that

$$1 + \omega\psi_x(s)\psi_y(t) \geq 0, \tag{2.11}$$

$\forall s, t \in \mathbb{R}^+$. Note that the independence is reached when $\omega = 0$. Define $\nu_i = \int_0^\infty s\psi_i(s)f_i(s)ds$, for $i = x, y$, then the covariance and correlation coefficient are given by

$$Cov(T_x, T_y) = \omega\nu_x\nu_y, \tag{2.12}$$

and

$$Corr(T_x, T_y) = \frac{\omega\nu_x\nu_y}{\sqrt{Var[T_x]Var[T_y]}}. \tag{2.13}$$

The maximum attainable correlation for a bivariate Sarmanov distribution is discussed in Shubina and Lee (2004) for different marginal distributions. In our numerical applications, we show how our model produces correlations that are very close to the bounds provided by Shubina and Lee (2004) in the case of exponential marginal distributions.

In this paper, it is assumed that both T_x and T_y are following PH distribution with orders n_x and n_y, i.e.

$$T_x \sim PH(\alpha_x, \Lambda_x) \text{ and } T_x \sim PH(\alpha_y, \Lambda_y).$$

The resulting joint distribution is referred to as Sarmanov Phase-Type (SPH) distribution. The choice of a suitable kernel function is very important in the definition of our SPH distribution. In the literature, the most used kernel functions are (See Lee (1996) for details)

(i) Farlie-Gumbel-Morgenstern (FGM) copula case: $\psi_i(t) = 1 - 2F_i(t)$ where F_i is the cdf associated to T_i;

(ii) Exponential Kernel: $\psi_i(t) = e^{-\gamma_i t} - \mathbb{E}[e^{-\gamma_i T_i}]$;

(iii) The marginal Kernel: $\psi_i(t) = f_i(t) - \mathbb{E}[f_i(T_i)]$.

Yang and Hashorva (2013) considered the case where ψ depends on some function g as follows

$$\psi_i(t) = g_i(t) - \mathbb{E}[g_i(T_i)], \tag{2.14}$$

where $\mathbb{E}[g_i(T_i)] < \infty$. In order to satisfy the condition 2.11, the dependence parameter ω is set to be in the following range

$$\frac{-1}{\max\{C_xC_y, (M_x - C_x)(M_y - C_y)\}} \leqslant \omega \leqslant \frac{1}{\max\{C_x(M_y - C_y), (M_x - C_x)C_y\}}, \tag{2.15}$$

where $C_i = E(g_i(T_i))$ and $M_i = \max_{s\in\mathbb{R}} g_i(s), i = x, y$.

Given that the marginal distributions are PH distributed, we consider a general form for the functions g_i and it is assumed that

$$g_i(t) = \beta_i e^{B_i t} b_i, \tag{2.16}$$

where β_i, B_i, and b_i are $1 \times m_i$ vector, $m_i \times m_i$ matrix, and $m_i \times 1$ vector, respectively.

This general kernel function includes all the well-known kernel functions and we have

(I) For the FGM copula, we have $\psi_i(t) = 1 - 2F_i(t)$, then

$$g_i(t) = 2\bar{F}_i(t).$$

The survival function for a PH has the following form

$$\bar{F}_i(t) = \alpha_i e^{\Lambda_i t} e_{n_i}.$$

It follows that

$$\beta_i = 2\alpha_i, \quad B_i = \Lambda_i, \text{and} \quad b_i = e_{n_i}.$$

(II) For the exponential kernel case, it is assumed that $\psi_i(t) = e^{-\gamma_i t} - \mathbb{E}[e^{-\gamma_i T_i}]$. Then, it is straightforward to identify

$$\beta_i = 1, \ \ B_i = -\gamma_i, \text{and} \ \ b_i = 1.$$

(III) For the marginal kernel function, we have $\psi_i(t) = f_i(t) - \mathbb{E}[f_i(T_i)]$, then

$$\beta_i = \alpha_i, \ \ B_i = \Lambda_i, \text{and} \ \ b_i = \lambda_i = -\Lambda_i e_{n_i}.$$

It is also possible to consider new forms of kernel functions. For example, we suggest the following Erlang-type kernel function with

$$g_i(s) = \sum_{r=0}^{k-1} \frac{(\gamma s)^r}{r!} e^{-\gamma s} \tag{2.17}$$

where $\gamma \geq 0$ and k is an integer. From Example 2.3, it is clear that g_i is the survival function for an Erlang distribution and it could be written as in Eq. 2.16. It is obvious that when $k = 1$ the Erlang-kernel function is reduced to the exponential kernel function.

Remark 2.1. *Note that it is possible to set some parameters in the kernel function g_x and g_y to have the same values which could simplify the model. Although, we keep the general form in our results.*

In order to have a valid distribution, we need some conditions on the general kernel function in Eq. 2.16. The joint distribution is defined as long as $\mathbb{E}\left[g_i\left(T_i\right)\right]$ is finite. Thus, we assume that this expectation exists and the conditions for its existence are given in the following subsection.

2.3 The joint distribution of T_x and T_y

The joint pdf of T_x and T_y in Eq. 2.10 could be written as follows

$$\begin{aligned} h(s,t) &= f_x(s)f_y(t)\left(1 + \omega C_x C_y\right) - \omega C_y f_x(s) g_x(s) f_y(t) \\ &- \omega C_x f_x(s) f_y(t) g_y(t) + \omega f_x(s) g_x(s) f_y(t) g_y(t), \end{aligned} \tag{2.18}$$

or in a compact form

$$h(s,t) = f_x(s)f_y(t) + \omega \sum_{k,l=0}^{1} c_{k,l} f_x(s) g_x^k(s) f_y(t) g_y^l(t), \tag{2.19}$$

with $g_i^0(s) = 1$, for $i = x, y$ and for all s with

$$c_{k,l} = (-1)^{k+l} C_1^{1-k} C_2^{1-l},$$

for l and k in $\{0, 1\}$. The following lemma is used to compute the products $f_i(.)g_i(.)$ for $i = x$ and y.

Lemma 2.1. *Assume that*

$$K(t) = \alpha e^{At} a,$$

and

$$L(t) = \beta e^{Bt} b,$$

where α, β, A, B, c, and b are $1 \times n$ vector, $1 \times m$ vector, $n \times n$ matrix, $m \times m$ matrix, $n \times 1$ vector and $m \times 1$ vector, respectively. Let $M(t) = K(t)L(t)$, then

$$M(t) = (\alpha \otimes \beta) e^{(A \oplus B)t} (a \otimes b).$$

Proof. First, we note that

$$M(t) = \left[\alpha e^{At} a\right] \otimes \left[\beta e^{Bt} b\right].$$

By the mixed product property of the Kronecker product (see Proposition 5.3), we find

$$M(t) = (\alpha \otimes \beta) e^{At} \otimes e^{Bt} (a \otimes b).$$

Then, the desired result follows by using the following property of the matrix exponential

$$e^{At} \otimes e^{Bt} = e^{(A \oplus B)t}.$$

□

Using Lemma 2.1, one can write

$$f_i(s)g(s) = (\alpha_i \otimes \beta_i) e^{\Lambda_i \oplus B_i s} (\lambda_i \otimes b_i). \tag{2.20}$$

In the rest of the paper, we use the notations

$$\alpha_i^{(k)} = \begin{cases} \alpha_i, & \text{if } k = 0 \\ \alpha_i \otimes \beta_i, & \text{if } k = 1 \end{cases},$$

$$\Lambda_i^{(k)} = \begin{cases} \Lambda_i, & \text{if } k = 0 \\ \Lambda_i \oplus B_i, & \text{if } k = 1 \end{cases},$$

and

$$\lambda_i^{(k)} = \begin{cases} \lambda_i, & \text{if } k = 0 \\ \lambda_i \otimes b_i, & \text{if } k = 1 \end{cases},$$

As stated above, the validity of our SPH model holds only if $\mathbb{E}\left[g_i\left(T_i\right)\right] < \infty$ where

$$\begin{aligned} \mathbb{E}\left[g_i\left(T_i\right)\right] &= \int_0^\infty f_i(s)g_i(s)ds \\ &= \alpha_i^{(1)} \int_0^\infty e^{\Lambda_i \oplus B_i s} ds \lambda_i^{(1)}. \end{aligned}$$

Then, a sufficient condition to have $\mathbb{E}\left[g_i\left(T_i\right)\right] < \infty$ is to satisfy the condition $\int_0^\infty e^{\Lambda_i \oplus B_i s} ds < \infty$. Following a similar reasoning as in Proposition 2.1, we can state that our model is defined if and only if the real parts of the eigenvalues of $\Lambda_i \oplus B_i$ are negative. Using Proposition 5.4, we can find a concrete condition on the matrix B_i.

Proposition 2.2. *Let v be the largest real part of the eigenvalues of B_i and let z be the smallest real part of the eigenvalues of Λ_i. Then $\int_0^\infty e^{\Lambda_i \oplus B_i s} ds < \infty$ if and only if $v + z < 0$.*

Proof. This result is a simple application of Proposition 5.4. □

Since all eigenvalues of Λ_i are supposed to be negative, a sufficient condition on B_i follows

Proposition 2.3. *If all the eigenvalues of B_i are nonpositive, then $\int_0^\infty e^{\Lambda_i \oplus B_i s} ds < \infty$.*

For all examples that we consider in this paper the condition in Proposition 2.3 holds.

Using our notation, the joint pdf in Eq. 2.19 becomes

$$h(s,t) \quad = \quad \left(\alpha_x e^{\Lambda_x s} \lambda_x\right)\left(\alpha_y e^{\Lambda_y t} \lambda_y\right) + \omega \sum_{k,l=0}^{1} c_{k,l} \left(\alpha_x^{(k)} e^{\Lambda_x^{(k)} s} \lambda_x^{(k)}\right)\left(\alpha_y^{(l)} e^{\Lambda_y^{(l)} t} \lambda_y^{(l)}\right). \tag{2.21}$$

While, in general, the product $f_i g_i$ is not a pdf and does not have the form of a Phase-type distribution, it is still possible to see the joint pdf in 2.21 as a combination of joint pdf's for independent bivariate Exponential Matrix (EM) distributions. More details on the EM distributions are available in Bladt and Nielsen (2017). Alternatively, one can write the joint pdf h as a combination of joint pdf's of independent PH distributions as we are going to show in the Subsection 2.4.

Remark 2.2. *It is interesting to notice that the expressions for $\alpha_i^{(1)}$, $\Lambda_i^{(1)}$, and $\lambda_i^{(1)}$ are not unique. For example, consider the case of FGM, we have $f_i(s)g_i(s) = 2f_i(s)\bar{F}_i(s)$ which is the pdf of the minimum of two independent rvs following $PH(\alpha_i, \Lambda_i)$. It is well known that the PH distributions are stable with respect to the order statistics and the distribution of the minimum is also following PH with parameters $\alpha_i \otimes \alpha_i$ and $\Lambda_i \oplus \Lambda_i$ (See Bladt and Nielsen (2017)). It follows that*

$$f_i(s)g_i(s) = -(\alpha_i \otimes \alpha_i) e^{(\Lambda_i \oplus \Lambda_i)s} (\Lambda_i \oplus \Lambda_i) e_{n_i^2}. \tag{2.22}$$

From the joint pdf in Eq. 2.21, expressions for the cdf and survival functions follow

$$\bar{H}(s,t) \quad = \quad \left(\alpha_x e^{\Lambda_x s} e_{n_x}\right)\left(\alpha_y e^{\Lambda_y t} e_{n_y}\right) + \omega \sum_{k,l=0}^{1} c_{k,l} \left(\alpha_x^{(k)} e^{\Lambda_x^{(k)} s} \hat{\lambda}_x^{(k)}\right)\left(\alpha_y^{(l)} e^{\Lambda_y^{(l)} t} \hat{\lambda}_y^{(l)}\right), \tag{2.23}$$

where

$$\hat{\lambda}_i^{(k)} = \left[\Lambda_i^{(k)}\right]^{-1} \lambda_i^{(k)},$$

and

$$\begin{aligned} H(s,t) \quad &= \quad \left(1 - \alpha_x e^{\Lambda_x s} e_{n_x}\right)\left(1 - \alpha_y e^{\Lambda_y t} e_{n_y}\right) \qquad (2.24) \\ &+ \quad \omega \sum_{k,l=0}^{1} c_{k,l} \left(1 - \alpha_x^{(k)} e^{\Lambda_x^{(k)} s} \hat{\lambda}_x^{(k)}\right)\left(1 - \alpha_y^{(l)} e^{\Lambda_y^{(l)} t} \hat{\lambda}_y^{(l)}\right). \end{aligned}$$

An expression for the joint Laplace transform is derived and we have

$$\begin{aligned} L(p,q) &= \mathbb{E}\left[e^{-(pT_x+qT_y)}\right] \\ &= \left(\alpha_x\left[pI_{n_x}-\Lambda_x\right]^{-1}\lambda_x\right)\left(\alpha_y\left[qI_{n_y}-\Lambda_y\right]^{-1}\lambda_y\right) \\ &+ \omega\sum_{k,l=0}^{1} c_{k,l}\left(\alpha_x^{(k)}\left[pI_{n_x m_x^k}-\Lambda_x^{(k)}\right]^{-1}\lambda_x^{(k)}\right)\left(\alpha_y^{(l)}\left[qI_{n_y m_y^l}-\Lambda_y^{(l)}\right]^{-1}\lambda_y^{(l)}\right). \end{aligned} \tag{2.25}$$

Differentiating the LT, we get an expression for joint moments

$$\begin{aligned} \mathbb{E}\left[T_x^iT_y^j\right] = (-1)^{i+j}i!j!\Big\{ &\left(\alpha_x\Lambda_x^{-i}e_{n_x}\right)\left(\alpha_y\Lambda_y^{-j}e_{n_y}\right) \\ &+\omega\sum_{k,l=0}^{1} c_{k,l}\left(\alpha_x^{(k)}\left[\Lambda_x^{(k)}\right]^{-i}\hat{\lambda}_x^{(k)}\right)\left(\alpha_y^{(l)}\left[\Lambda_y^{(l)}\right]^{-j}\hat{\lambda}_y^{(l)}\right)\Big\}. \end{aligned} \tag{2.26}$$

2.4 The joint distribution as a combination of PH joint distributions

In this subsection, we show that the joint pdf in Eq. 2.21 could be written in terms of PH pdfs. This result is based on the following lemma.

Lemma 2.2. *Consider a function $M(t)=\alpha e^{At}a$ such that $0<\int_0^\infty M(t)dt<\infty$, where α, A, and a are $1\times n$ vector, $n\times n$ matrix, and $n\times 1$ vector, respectively. Then,*

$$M(t)=cN(t),$$

where N is a pdf of an n order $PH(\tilde{\alpha},\tilde{A})$ with

$$c=-\alpha A^{-1}a,\quad \tilde{\alpha}=\frac{\alpha\Delta}{c},\quad \tilde{A}=\Delta^{-1}A\Delta,$$

and Δ is a diagonal matrix with $diag(\Delta)=-A^{-1}a$.

Proof. Define

$$c=\int_0^\infty M(t)dt.$$

We have $c=-\alpha A^{-1}a$ and $\Delta e_n=-A^{-1}a$. Then, $c=\alpha\Delta e_n$. The function M could be written as follows

$$M(t)=c\left[\tilde{\alpha}\Delta^{-1}e^{At}a\right].$$

Using the definition of exponential matrix, we get

$$\begin{aligned} \Delta^{-1}e^{At} &= \Delta^{-1}\left[\sum_{k=0}^{\infty}\frac{A^k}{k!}t^k\right]\Delta\Delta^{-1} \\ &= \sum_{k=0}^{\infty}\frac{\tilde{A}^k}{k!}t^k\Delta^{-1} \\ &= e^{\tilde{A}t}\Delta^{-1}. \end{aligned}$$

Thus,

$$M(t) = c\left[\tilde{\alpha} e^{\tilde{A}t} \Delta^{-1} a\right].$$

The proof is concluded by showing that

$$\Delta^{-1} a = \Delta^{-1} A A^{-1} a = -\Delta^{-1} A \Delta e_n = -\tilde{A} e_n,$$

i.e. $\tilde{\alpha} e^{\tilde{A}t} \Delta^{-1} a = -\tilde{\alpha} e^{\tilde{A}t} \tilde{A} e_n$ is a pdf of $PH(\tilde{\alpha}, \tilde{A})$. □

Using the previous lemma, one can write the joint pdf in Eq. 2.21 as follows

$$h(s,t) = \left(\alpha_x e^{\Lambda_x s} \lambda_x\right)\left(\alpha_y e^{\Lambda_y t} \lambda_y\right) + \omega \sum_{k,l=0}^{1} \tilde{c}_{k,l} f_{x,k}(s) f_{y,l}(t), \tag{2.27}$$

where $f_{u,i}$ is a pdf of a $PH(\tilde{\alpha}_u^{(i)}, \tilde{\Lambda}_u^{(i)})$ with

$$\tilde{\alpha}_u^{(i)} = \frac{\alpha_u^{(i)} \Delta}{c}, \quad \tilde{\Lambda}_u^{(i)} = \Delta^{-1} \Lambda_u^{(i)} \Delta,$$

and Δ is a diagonal matrix with $diag(\Delta) = -[\Lambda_u^{(i)}]^{-1} \lambda_u^{(i)}$, for $u = x, y$ and $i = k, l$. The constant $\tilde{c}_{k,l}$ is given by

$$\tilde{c}_{k,l} = c_{k,l}\left[\alpha_x^{(k)} (\Lambda_x^{(k)})^{-1} \lambda_x^{(k)}\right]\left[\alpha_y^{(l)} (\Lambda_y^{(l)})^{-1} \lambda_y^{(l)}\right].$$

This result is very important to analyze the statistical characteristics for the SPH distribution but for the sake of simplicity we work in the rest of the paper with joint pdf defined in Eq. 2.21. Although, similar results could be derived using the joint pdf in Eq. 2.27.

3 Multiple-life insurance model

In this section, we apply the SPH distributions in the context of joint-life insurance modeling. This family of distributions allow us to derive some closed-form expression for many useful actuarial quantities. The survival of the two lives is referred to as the status of interest or simply the status. There are two common types of status: the joint life status and the last survival status.

3.1 Joint life status

The joint-life status is one that requires the survival of both lives. Accordingly, the status terminates upon the first death of one of the two lives. The joint-life status of two lives (x) and (y) will be denoted by (xy), and the moment of death random variable is given by $T_{(xy)} = \min(T_x, T_y)$.

Theorem 3.1. *The survival function for $T_{(xy)}$ could be written as*

$$\overline{F}_{T_{(xy)}}(t) \quad = \quad \alpha_{xy} e^{\Lambda_{xy} t} e_{n_{xy}} + \omega \sum_{k,l=0}^{1} c_{k,l} \alpha_{xy}^{(kl)} e^{\Lambda_{xy}^{(kl)} t} \hat{\lambda}_{xy}^{(kl)}, \tag{3.1}$$

where

$$n_{xy} = n_x n_y, \quad \alpha_{xy} = \alpha_x \otimes \alpha_y, \quad \Lambda_{xy} = \Lambda_x \oplus \Lambda_y, \tag{3.2}$$

$$\alpha_{xy}^{(kl)} = \alpha_x^{(k)} \otimes \alpha_y^{(l)}, \quad \Lambda_{xy}^{(kl)} = \Lambda_x^{(k)} \oplus \Lambda_y^{(l)}, \textit{and} \quad \hat{\lambda}_{xy}^{(kl)} = \hat{\lambda}_x^{(k)} \otimes \hat{\lambda}_y^{(l)}. \tag{3.3}$$

Proof. The survival function of $T_{(xy)}$ is given by

$$\overline{F}_{T_{(xy)}}(t) = \overline{H}_{T_x, T_y}(t, t). \tag{3.4}$$

It follows that

$$\overline{F}_{T_{(xy)}}(t) \quad = \quad \left(\alpha_x e^{\Lambda_x t} e_{n_x}\right)\left(\alpha_y e^{\Lambda_y t} e_{n_y}\right) + \omega \sum_{k,l=0}^{1} c_{k,l} \left(\alpha_x^{(k)} e^{\Lambda_x^{(k)} t} \hat{\lambda}_x^{(k)}\right)\left(\alpha_y^{(l)} e^{\Lambda_y^{(l)} t} \hat{\lambda}_y^{(l)}\right). \tag{3.5}$$

Applying Lemma 2.1, we can state that

$$\left(\alpha_x e^{\Lambda_x t} e_{n_x}\right)\left(\alpha_y e^{\Lambda_y t} e_{n_y}\right) = \left(\alpha_x \otimes \alpha_y\right) e^{\Lambda_x \oplus \Lambda_y t} e_{n_x n_y}, \tag{3.6}$$

and

$$\left(\alpha_x^{(k)} e^{\Lambda_x^{(k)} t} \hat{\lambda}_x^{(k)}\right)\left(\alpha_y^{(l)} e^{\Lambda_y^{(l)} t} \hat{\lambda}_y^{(l)}\right) = \left(\alpha_x^{(k)} \otimes \alpha_y^{(l)}\right) e^{\Lambda_x^{(k)} \oplus \Lambda_y^{(l)} t} \left(\hat{\lambda}_x^{(k)} \otimes \hat{\lambda}_y^{(l)}\right). \tag{3.7}$$

Which leads to the desired result. □

Using the survival function, we derive the following pdf for $T_{(xy)}$ is given by

$$f_{T_{(xy)}}(t) \quad = \quad \alpha_{xy} e^{\Lambda_{xy} t} \lambda_{xy} + \omega \sum_{k,l=0}^{1} c_{k,l} \alpha_{xy}^{(kl)} e^{\Lambda_{xy}^{(kl)} t} \lambda_{xy}^{(kl)}, \tag{3.8}$$

where

$$\lambda_{xy} = -\Lambda_{xy} e_{n_{xy}},$$

and

$$\lambda_{xy}^{(kl)} = -\Lambda_{xy}^{(kl)} \hat{\lambda}_{xy}^{(kl)}.$$

Example 3.1. *For the case of FGM and using the result in Remark 2.2, we get*

$$\alpha_i^{(k)} = \alpha_i^{\otimes k+1}, \quad \Lambda_i^{(k)} = \Lambda_i^{\oplus k+1},$$

$$\lambda_i^{(k)} = \Lambda_i^{(k)} e_{n_i^{k+1}}, \textit{and} \quad \hat{\lambda}_i^{(k)} = e_{n_i^{k+1}}$$

It follows that

$$\alpha_{xy}^{(kl)} = \alpha_x^{\otimes_{k+1}} \otimes \alpha_y^{\otimes_{l+1}},$$

$$\Lambda_{xy}^{(kl)} = \Lambda_x^{\oplus_{k+1}} \oplus \Lambda_y^{\oplus_{l+1}}$$

and

$$\hat{\lambda}_{xy}^{(kl)} = e_{n_x^{k+1} n_y^{l+1}},$$

where for any matrix A the kth Kronecker power, $A^{\otimes_{k+1}}$, and the kth Kronecker sum, $A^{\oplus_{k+1}}$, are defined in the appendix. Then, the expression for $\overline{F}_{T_{(xy)}}$ is as follows

$$\overline{F}_{T_{(xy)}}(t) = \alpha_{xy} e^{\Lambda_{xy} t} e_{n_{xy}} + \omega \sum_{k,l=0}^{1} c_{k,l} (\otimes_{k+1}\alpha_x) \otimes (\otimes_{k+1}\alpha_y) e^{(\oplus_{k+1}\Lambda_x) \oplus (\oplus_{l+1}\Lambda_y) t} e_{n_x^{k+1} n_y^{l+1}}.$$

Example 3.2. *Assume that the kernel functions are exponential with the same parameter γ. We find*

$$\alpha_i^{(k)} = \alpha_i, \quad \Lambda_i^{(k)} = \Lambda_i - k\gamma I_{n_i},$$

$$\lambda_i^{(k)} = \lambda_i, and \quad \hat{\lambda}_i^{(k)} = (\Lambda_i - k\gamma I_{n_i})^{-1} \lambda_i.$$

It follows that

$$\alpha_{xy}^{(kl)} = \alpha_x \otimes \alpha_y,$$

$$\begin{aligned}
\Lambda_{xy}^{(kl)} &= [\Lambda_x - k\gamma I_{n_x}] \oplus [\Lambda_y - l\gamma I_{n_y}] \\
&= [\Lambda_x - k\gamma I_{n_x}] \otimes I_{n_y} + I_{n_x} \otimes [\Lambda_y - l\gamma I_{n_y}] \\
&= \Lambda_x \otimes I_{n_y} - k\gamma I_{n_x n_y} - l\gamma I_{n_x n_y} + I_{n_x} \otimes \Lambda_y \\
&= \Lambda_x \oplus \Lambda_y - (k+l)\gamma I_{n_x n_y},
\end{aligned}$$

and

$$\lambda_{xy}^{(kl)} = \left(\Lambda_x^{(k)} \otimes \Lambda_y^{(l)}\right)^{-1} (\lambda_x \otimes \lambda_y).$$

The sf of T_{xy} becomes

$$\overline{F}_{T_{(xy)}}(t) = (\alpha_x \otimes \alpha_y) \left[e^{\Lambda_x \oplus \Lambda_y t} + \omega \sum_{k,l=0}^{1} c_{k,l} e^{[\Lambda_x \oplus \Lambda_y - (k+l)\gamma I_{n_x n_y}] t} \left(\Lambda_x^{(k)} \otimes \Lambda_y^{(l)}\right)^{-1} \right] (\lambda_x \otimes \lambda_y).$$

3.2 The last survivor status

The other common status is the last-survivor status. The last-survivor status is one that ends upon the death of both lives. That is, the status survives as long as at least one of the component members remains alive. The last-survivor status of two lives (x) and (y) will be denoted by $(\overline{xy})$, and the moment of death random variable is given by

$$T_{(\overline{xy})} = \max(T_x, T_y).$$

The cdf for $T_{(\overline{xy})}$ is given by

$$F_{T_{(\overline{xy})}}(t) \quad = \quad H(t,t),$$

where the joint cdf H is given by 2.24. The expression for the cdf $F_{T_{(\overline{xy})}}$ is given in the following theorem.

Theorem 3.2. *The cdf for $T_{(\overline{xy})}$ could be written as*

$$F_{T_{(\overline{xy})}}(t) \quad = \quad \left(1 - \alpha_{\overline{xy}} e^{\Lambda_{\overline{xy}} t} e_{n_{\overline{xy}}}\right) + \omega \sum_{k,l=0}^{1} c_{k,l} \left(1 - \alpha_{\overline{xy}}^{(kl)} e^{\Lambda_{\overline{xy}}^{(kl)} t} \hat{\lambda}_{\overline{xy}}^{(kl)}\right),$$

where

$$n_{\overline{xy}} = n_x n_y + n_x + n_y, \quad \alpha_{\overline{xy}} = (\alpha_x, \alpha_y, -\alpha_x \otimes \alpha_y), \quad A_{\overline{xy}} = diag(\Lambda_x, \Lambda_y, \Lambda_x \oplus \Lambda_y), \tag{3.9}$$

$$\alpha_{\overline{xy}}^{(kl)} = (\alpha_x^{(k)}, \alpha_y^{(l)}, -\alpha_x^{(k)} \otimes \alpha_y^{(l)}), \;\; \Lambda_{\overline{xy}}^{(kl)} = Blocdiag(\Lambda_x^{(k)}, \Lambda_y^{(l)}, \Lambda_x^{(k)} \oplus \Lambda_y^{(l)}), \;\; \hat{\lambda}_{\overline{xy}}^{(kl)} = diag(\hat{\lambda}_x^{(k)}, \hat{\lambda}_y^{(l)}, \hat{\lambda}_x^{(k)} \otimes \hat{\lambda}_y^{(l)}). \tag{3.10}$$

Proof. The result follows by applying Lemma 1 to expressions with the following forms

$$M(t) = K(t)L(t),$$

with

$$K(t) = 1 - \alpha e^{At} C,$$

and

$$L(t) = 1 - \beta e^{Bt} D.$$

We note that

$$M(t) = 1 - \alpha e^{A1} C - \beta e^{Bt} D + \left(\alpha e^{At} C\right) \left(\beta e^{Bt} D\right).$$

By Lemma 1, we can state that

$$M(t) = 1 - \mu e^{Et} F,$$

where

$$\mu = (\alpha, \beta, -\alpha \otimes \beta),$$

$$E = diag(A, B, A \oplus B),$$

and

$$F = diag(C, D, C \otimes D).$$

The result in the theorem follows easily. □

The survival function and density function for $T_{(\overline{xy})}$ are as follows

$$\overline{F}_{T_{(\overline{xy})}}(t) \quad = \quad \alpha_{\overline{xy}} e^{\Lambda_{\overline{xy}} t} e_{n_{\overline{xy}}} + \omega \sum_{k,l=0}^{1} c_{k,l} \alpha_{\overline{xy}}^{(kl)} e^{\Lambda_{\overline{xy}}^{(kl)} t} \hat{\lambda}_{\overline{xy}}^{(kl)},$$

and

$$f_{T_{(\overline{xy})}}(t) \;=\; \alpha_{\overline{xy}} e^{\Lambda_{\overline{xy}} t} \lambda_{\overline{xy}} + \omega \sum_{k,l=0}^{1} c_{k,l} \alpha_{\overline{xy}}^{(kl)} e^{\Lambda_{\overline{xy}}^{(kl)} t} \lambda_{\overline{xy}}^{(kl)},$$

where

$$\lambda_{\overline{xy}} = -\Lambda_{\overline{xy}} e_{n_{\overline{xy}}},$$

and

$$\lambda_{\overline{xy}}^{(kl)} = -\Lambda_{\overline{xy}}^{(kl)} \hat{\lambda}_{\overline{xy}}^{(kl)}.$$

3.3 Actuarial Present Values

From Theorems 3.1 and 3.2, we notice that the distributions for T_{xy} and $T_{(\overline{xy})}$ have the same form with different parameters. In the rest of the paper, let u be the status of interest that could be either xy or $\overline{xy}$. Given the results in the previous subsections, we derive expressions for some well-known actuarial present values. First, we give the APV for an n-year term insurance that pays \$1 at the moment of the decrement of the status u. This means that the insurance pays \$1 at the moment of the first death if $u = xy$ or at the moment of the last death if $u = \overline{xy}$. Let Z be the present value for this life insurance, we have

$$Z = e^{-\delta \min(T_u, n)},$$

where δ is the constant force of interest. The expected value for Z is as follows

$$\begin{aligned}
\overline{A}^{1}_{u:\overline{n}|} &= \int_0^n e^{-\delta t} f_{T_u}(t) dt \\
&= \alpha_u (\delta I_{n_u} - \Lambda_u)^{-1} (I_{n_u} - e^{-(\delta I_{n_u} - \Lambda_u) n}) \lambda_u \\
&+ \omega \sum_{k,l=0}^{1} c_{k,l} \alpha_u^{(kl)} (\delta I_{n_u^{(kl)}} - \Lambda_u^{(kl)})^{-1} (I_{n_u^{(kl)}} - e^{-(\delta I_{n_u^{(kl)}} - \Lambda_u^{(kl)}) n}) \lambda_u^{(kl)},
\end{aligned}$$

Letting $n \to \infty$ leads to the APV for whole life insurance and we obtain

$$\overline{A}_u \;=\; \alpha_u (\delta I_{n_u} - \Lambda_u)^{-1} \lambda_u + \omega \sum_{k,l=0}^{1} c_{k,l} \alpha_u^{(kl)} (\delta I_{n_u^{(kl)}} - \Lambda_u^{(kl)})^{-1} \lambda_u^{(kl)}.$$

Define W to be the present of a continuous life annuity that pays \$1 per year as long as the status u exists and let $\overline{a}_{u:\overline{n}|}$ be its expected value. Then, we have

$$\begin{aligned}
\overline{a}_{u:\overline{n}|} &= \int_0^n e^{-\delta t} \overline{F}_{T_u}(t) dt \\
&= \alpha_u (\delta I_{n_u} - \Lambda_u)^{-1} (I_{n_u} - e^{-(\delta I_{n_u} - \Lambda_u) n}) e_{n_u} \\
&+ \omega \sum_{k,l=0}^{1} c_{k,l} \alpha_u^{(kl)} (\delta I_{n_u^{(kl)}} - \Lambda_u^{(kl)})^{-1} (I_{n_u^{(kl)}} - e^{-(\delta I_{n_u^{(kl)}} - \Lambda_u^{(kl)}) n}) \hat{\lambda}_u^{(kl)}.
\end{aligned}$$

For a whole life annuity, we get

$$
\begin{aligned}
\overline{a}_u &= \int_0^\infty e^{-\delta t}\overline{F}_{T_u}(t)dt \\
&= \alpha_u(\delta I_{n_u} - \Lambda_u)^{-1}e_{n_u} + \omega \sum_{k,l=0}^{1} c_{k,l}\alpha_u^{(kl)}(\delta I_{n_u^{(kl)}} - \Lambda_u^{(kl)})^{-1}\hat{\lambda}_u^{(kl)}.
\end{aligned}
$$

3.4 Other properties of T_u

Laplace transform for T_u is given by

$$
\begin{aligned}
L(r) &= \mathbb{E}\left[e^{-rT_u}\right] \\
&= \alpha_u\left[rI_{n_u} - \Lambda_u\right]^{-1}\lambda_u + \omega \sum_{k,l=0}^{1} c_{k,l}\alpha_u^{(kl)}\left[rI_{n_u^{(kl)}} - \Lambda_u^{(kl)}\right]^{-1}\lambda_u^{(kl)}.
\end{aligned}
$$

The higher moment for T_u are as follows

$$
\mathbb{E}\left[T_u^r\right] = (-1)^r r!\left\{\alpha_u\Lambda_u^{-r}e_{n_u} + \omega \sum_{k,l=0}^{1} c_{k,l}\alpha_u^{(kl)}\left[\Lambda_u^{(kl)}\right]^{-r}\hat{\lambda}_u^{(kl)}\right\}.
$$

Another Way to measure uncertainty is the Conditional Tail Expectation (also known as Tail Value at Risk, TVaR). the Conditional Tail Expectation (CTE) has received much attention in insurance risk management applications, such as the solvency or risk capital measurement. Let the underlying random variable be T_u, the CTE of T_u at a confidence level $0 < p < 1$ is defined as

$$
CTE_p(T_u) = E(T_u|T_u > Q_p),
$$

where Q_p is the p quantile of the distribution of the distribution of T_u. Given the distribution of T_u, the CTE is analytically obtained as follows

$$
\begin{aligned}
CTE_p(T_u) &= Q_p + \mathbb{E}[T_u - Q_p|T_u > Q_p] \\
&= Q_p - \frac{e^{Q_p\Lambda_u}\alpha_u(\Lambda_u)^{-1}e_u + \omega\sum_{k,l=0}^{1} c_{k,l}e^{Q_p\Lambda_u^{(kl)}}\alpha_u^{(kl)}(\Lambda_u^{(kl)})^{-1}\hat{\lambda}_u^{(kl)}}{1-p}.
\end{aligned} \tag{3.11}
$$

4 Applications of SPH distribution in multiple life insurance

This section gives some numerical examples to illustrate our model's features and its applications in life insurance. Some of the applications are a simple adaptation of the work done in Kim et al. (2017) to the context of insurance policies contingent on two dependent lives.

4.1 The correlation structure

Our first numerical illustration aims to analyze the correlation generated by joint SPH distribution. We mainly compute the upper and lower bounds for the correlation coefficient based on the condition in Eq. 2.15. We consider three examples with a Gamma kernel function.

Example 4.1. ***The Erlang kernel function with exponential marginal distributions*** *Assume that*

$$f_{T_x}(t) = \mu_x e^{-\mu_x t},$$

and

$$f_{T_y}(t) = \mu_y e^{-\mu_y t}.$$

We present our results for the following two cases

(i) $\mu_x = \mu_y = 1$

(ii) $\mu_x = 0.02$ *and* $\mu_y = 0.021$.

The first case aims the comparison of our attainable correlation bounds and those obtained by Shubina and Lee (2004). The second case assumes realistic values for μ_x *and* μ_y *in the context of life insurance. For both cases, we consider the Erlang-type kernel functions with* $k = 1, 2, 3, 4, 5, 10, 30$ *and* 70*. Then, for each value of* k*, we compute the upper and the lower bounds for the correlation between* T_x *and* T_y*. Tables 1 and 2 give the obtained range of correlation for different values in each case. It worths noting that the correlation structure improves from* $k = 1$ *(exponential kernel) to the case with large values of* k*. This shows that the introduction of our PH-type form of kernel functions provide a flexible tool to fit a wide range of dependence structure compared to the exponential kernel of the FGM dependence structure. It is important to note that the obtained values of the upper and lower correlations are very close to those given by Shubina and Lee (2004). In the case of* $\mu_x = \mu_y = 1$*, Shubina and Lee (2004) states that the maximal attainable correlation range is* $[-0.4804, 0.6476]$ *and the values displayed in Table 1 show how our Erlang-type kernel functions allow us to cover most of this maximal attainable interval.*

k	1	2	3	4	5	10	30	50	70
ρ_{upper}	0.2551	0.3608	0.4231	0.4633	0.4912	0.5586	0.6150	0.6276	0.6332
γ_{upper}	1.0207	1.6191	2.2406	2.8651	3.4905	6.6232	19.1694	31.7190	44.2677
ρ_{lower}	-0.2551	-0.3433	-0.3830	-0.4050	-0.4189	-0.4485	-0.4695	-0.4738	-0.4757
γ_{lower}	1.0205	2.4154	3.8474	5.2852	6.7250	13.9327	42.7828	71.6359	100.4895

Table 1: The values of the upper and lower bounds for the correlation coefficient when $\mu_x = \mu_y = 1$

k	1	2	3	4	5	10	30	50	70
ρ_{upper}	0.2440	0.3499	0.4090	0.4467	0.4728	0.5354	0.5871	0.5986	0.6037
γ_{upper}	0.0205	0.0334	0.0464	0.0595	0.0726	0.1383	0.4017	0.6653	0.9288
ρ_{lower}	-0.2501	-0.3431	-0.3830	-0.4049	-0.4189	-0.4484	-0.4694	-0.4738	-0.4756
γ_{lower}	0.0205	0.0495	0.0789	0.1083	0.1378	0.2856	0.8769	1.4682	2.0596

Table 2: The values of the upper and lower bounds for the correlation coefficient when $\mu_x = 0.02$ and $\mu_y = 0.021$

The previous example gives a good illustration that our model could generate a better dependence structure in the case of bi-variate exponential distribution. The next example considers the case of mixture exponential marginal distribution (i.e. Hyper-exponential).

Example 4.2. ***The Erlang kernel function with Hyper-exponential marginal distributions*** *We assume the same kernel functions as in the previous example but with marginal risks following Hyper-exponential distributions, i.e.*

$$f_{T_x}(t) = 0.8(0.025e^{-0.025t}) + 0.2(0.02e^{-0.02t}).$$

Similarly,

$$f_{T_y}(t) = 0.3(0.03e^{-0.03t}) + 0.7(0.02e^{-0.02t}).$$

For each value of k, we compute the upper and the lower bounds for the correlation between T_x and T_y. The obtained values are given in Table 3. As in the previous example, this illustration shows the usefulness of the Phase-Type kernel function and how it improves the attainable range of correlation compared to the well-known kernel functions (FGM, exponential,...,etc).

k	1	2	3	4	5	7	9	15	30
ρ_{upper}	0.2413	0.3456	0.4035	0.4403	0.4656	0.4982	0.5183	0.5492	0.5748
γ_{upper}	0.0225	0.0368	0.0514	0.0660	0.0808	0.1103	0.1399	0.2289	0.4517
ρ_{lower}	-0.2482	-0.3382	-0.3765	-0.3975	-0.4108	-0.4266	-0.4357	-0.4488	-0.4590
γ_{lower}	0.0232	0.0562	0.0896	0.1231	0.1566	0.2238	0.2911	0.4928	0.9973

Table 3: The values of the upper and lower bounds for the correlation coefficient

4.2 Joint life insurance premiums

For simplicity, we consider the insurance policy where the unit benefit paid upon the death of both lives, and the premium payment terminates upon the first death too. The loss of this contract is given by

$$Z_{(xy)} - P\frac{1 - Z_{(xy)}}{\delta}.$$

The net premium for this insurance policy is found by setting the expected value of the above expression to zero. Thus, the net premium is

$$P = \frac{\overline{A}_{(xy)}}{\overline{a}_{(xy)}}$$

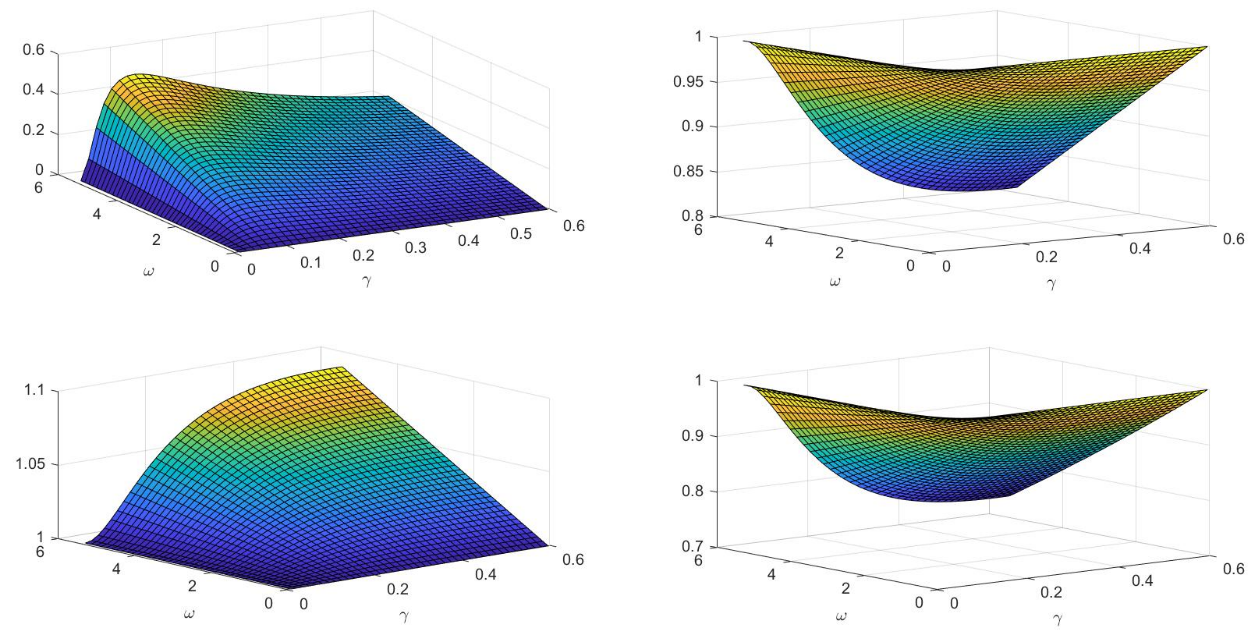

Figure 1: The impact of the dependence parameters on ρ (top left), $R(\overline{A}_{(xy)})$ (top right), $R(\overline{a}_{(xy)})$ (bottom left), and $R(P)$ (bottom right).

where $\overline{A}_{(xy)}$ and $\overline{a}_{(xy)}$ are the expressions obtained in Section 3. For this policy, we would like to measure the impact of the dependence on the level of the APV as well as on the premiums. In our example, we consider The Erlang kernel function with the Hyper-exponential marginal distributions as in the previous example. Given each set of dependence parameters, we can compute the values for correlation coefficient ρ, $\overline{A}_{(xy)}$, $\overline{a}_{(xy)}$, and P. Let $\overline{A}^{ind}_{(xy)}$, $\overline{a}^{ind}_{(xy)}$, and P^{ind} be the values of the APVs and the net premium under the independence assumption (i.e. when $\omega = 0$). Note that the independence could be reached as a limiting case when $\gamma \to 0$. The following ratios are computed

$$R(\overline{A}_{(xy)}) = \frac{\overline{A}_{(xy)}}{\overline{A}^{ind}_{(xy)}}, R(\overline{a}_{(xy)}) = \frac{\overline{a}_{(xy)}}{\overline{a}^{ind}_{(xy)}}, R(P) = \frac{P}{P^{ind}}.$$

These ratios allow us to quantify the impact of modeling dependence between lives in joint life contingencies contract.

Example 4.3. *We set $\delta = 10\%$ and $k = 7$. First, we find*

$$\overline{A}^{ind}_{(xy)} = 0.3190, \quad \overline{a}^{ind}_{(xy)} = 6.8105, \textit{and} \quad P^{ind} = 0.0468.$$

Figure 1 shows the values of the correlation ρ as well as these three ratios. From this figure, we can see the correlation coefficient is not linear in terms of the parameter γ. Regarding the impact of the dependence on the APVs, the positive dependence between the two lifetimes reduces the value for $\overline{A}_{(xy)}$ by up to 20% and increases the value of $\overline{A}_{(xy)}$ by up to 10%. This leads to a reduction in the net premium up to 25%.

4.3 Percentile and CTE premiums

Based on the result derived in Section 4, we have an expression for the survival function of T_u which would allow us to obtain the percentiles for T_u,Q_p, at a given level $0 < p < 1$. Then, we compute the CTE_p risk measure for different values of the dependence parameters. Let us now consider the whole life insurance random variable for the status u. Then,

$$\begin{aligned}\mathbb{E}\left[Z_{(u)}|Z_{(u)} > Q_p(Z_{(u)})\right] &= \mathbb{E}\left[e^{-\delta T_{(u)}}|e^{-\delta T_{(u)}} > Q_P(e^{-\delta T_{(u)}})\right] \\ &= \mathbb{E}\left[e^{-\delta T_{(u)}}|T_{(u)} < \tau_p\right] \\ &= \frac{\int_0^{\tau_p} e^{-\delta T_{(u)}} f_{T_{(u)}}(t)\,dt}{1-p} \\ &= \frac{\overline{A}^{\,1}_{u:\overline{\tau_p}|}}{1-p},\end{aligned} \tag{4.1}$$

where $\tau_p = Q_{(1-p)}(T_{(u)})$ and the expression for $\overline{A}^{\,1}_{u:\overline{\tau_p}|}$ is given in Subsection 3.3. The CTE's derivation for the present value of other standard joint-life insurance policies is quite similar, following the same steps as in the univariate case in Kim et al. (2017).

The obtained actuarial present values in Section 4 facilitate the calculations for the premiums and reserves of difference standard multiple-life insurance contracts using the equivalence principle; for more details, see Bowers et al. (1997) and Dickson et al. (2019). Alternatively, one can consider two other pricing principles: the percentile principle and the CTE-loaded premium (e.g., Kim et al. (2017)). Consider a life insurance contingent on the status u with premium paid continuously at a rate P as long as the status u holds. We define the loss function at time 0, L, as follows

$$L = Z_{(u)} - P\frac{1-Z_{(u)}}{\delta} = \left(1+\frac{P}{\delta}\right) Z_{(u)} - \frac{P}{\delta}. \tag{4.2}$$

Finding a risk-loaded premium can be done in different ways, and it depends on which risk measure is adopted as a loading factor. First, we compute the percentile or the value at risk premium, P_{var}, which is defined as the solution for

$$P(L > 0) = 1-p, \tag{4.3}$$

for a certain confidence level p. This means that the premium P_{var} is set such that the probability to generate an actual loss is $1-p$. From Eq. 4.3 and using Eq. 4.2, the condition on P_{var} could be written

$$\overline{F}_{Z_{(xy)}}\left(\frac{P_{var}}{P_{var}+\delta}\right) = 1-p,$$

i.e.,

$$\overline{F}_{T_{(xy)}}\left(-\frac{1}{\delta}\ln\left(\frac{P_{var}}{P_{var}+\delta}\right)\right) = p,$$

where $\overline{F}_{T_{(xy)}}$ is giving in Eq. 3.1. The premium is as follows

$$P_{var} = \frac{\delta exp\left(-\delta\overline{F}^{-1}_{T_{(xy)}}(p)\right)}{1-exp\left(-\delta\overline{F}^{-1}_{T_{(xy)}}(p)\right)},$$

i.e.,

$$P_{var} = \frac{\delta exp\left(-\delta\tau_p\right)}{1 - exp\left(-\delta\tau_p\right)}. \tag{4.4}$$

Another possible premium is computed based on the CTE risk measure, which we call the CTE-premium. Under this principle, the premium rate P_{cte} is such that

$$CTE_p(L) = 0, \tag{4.5}$$

for a confidence level $0 < p < 1$. It follows that

$$\begin{aligned} CTE_p[L] &= CTE_p\left[-\frac{P_{cte}}{\delta} + \left(1 + \frac{P_{cte}}{\delta}\right) Z_{(xy)}\right] \\ &= -\frac{P_{cte}}{\delta} + \left(1 + \frac{P_{cte}}{\delta}\right) CTE_p(Z_{(xy)}) = 0, \end{aligned}$$

where the second line holds because the CTE is a coherent risk measures (e.g. Artzner et al. (1999)). Thus, the P^{CTE} loaded-premium is given by

$$P_{cte} = \frac{\delta CTE_p(Z_{xy})}{1 - CTE_p(Z_{xy})}, \tag{4.6}$$

where $CTE_p(Z_{(xy)})$ is given by Eq. 4.1.

Example 4.4. *Consider the same model and parameters as in the previous example. For different value of γ and ω, the values for the net premium P are given in Table 4 and risk loaded premiums P_{var} and P_{cte} are displayed in Table 5. In our numerical application , the level of confidence that we consider are $p = 60\%$.*

ω	$\gamma = 0.01$	$\gamma = 0.11$	$\gamma = 0.21$	$\gamma = 0.31$	$\gamma = 0.41$	$\gamma = 0.51$
0	0.0468	0.0468	0.0468	0.0468	0.0468	0.0468
0.9886	0.0468	0.0460	0.0451	0.0446	0.0443	0.0442
1.9772	0.0468	0.0452	0.0434	0.0424	0.0419	0.0417
2.9658	0.0468	0.0444	0.0417	0.0402	0.0395	0.0393
3.9544	0.0468	0.0435	0.0401	0.0382	0.0373	0.0369
4.9430	0.0468	0.0427	0.0385	0.0361	0.0351	0.0347

Table 4: The net premiums, P, for different values for γ and ω

P_{var}	ω	$\gamma = 0.01$	$\gamma = 0.11$	$\gamma = 0.21$	$\gamma = 0.31$	$\gamma = 0.41$	$\gamma = 0.51$
	0	0.0507	0.0507	0.0507	0.0507	0.0507	0.0507
	0.9886	0.0507	0.0497	0.0481	0.0466	0.0453	0.0446
	1.9772	0.0507	0.0487	0.0455	0.0422	0.0396	0.0385
	2.9658	0.0506	0.0478	0.0427	0.0373	0.0338	0.0328
	3.9544	0.0506	0.0467	0.0396	0.0320	0.0281	0.0278
	4.9430	0.0506	0.0457	0.0363	0.0263	0.0230	0.0236
P_{cte}	ω	$\gamma = 0.01$	$\gamma = 0.11$	$\gamma = 0.21$	$\gamma = 0.31$	$\gamma = 0.41$	$\gamma = 0.51$
	0	0.1756	0.1756	0.1756	0.1756	0.1756	0.1756
	0.9886	0.1756	0.1743	0.1724	0.1704	0.1684	0.1668
	1.9772	0.1756	0.1729	0.1690	0.1648	0.1607	0.1573
	2.9658	0.1756	0.1716	0.1655	0.1587	0.1523	0.1474
	3.9544	0.1756	0.1702	0.1618	0.1522	0.1432	0.1370
	4.9430	0.1756	0.1688	0.1579	0.1449	0.1336	0.1264

Table 5: Impact of the dependence parameters on risk loaded premiums P_{var} and P_{cte}

In general, the VaR-based and CTE-based premiums are more conservative than the equivalence premium (i.e., net premium). From the obtained premiums, we can see that our results reflect this feature of risk-loaded premiums for some values for the dependence parameters. But for other values of these parameters, the value of P_{var} could be less than the value of P, which means that the tail of T_{xy} is very light. This is very important to keep in mind when our model is implemented using real data. It is also obvious that the positive dependence has the same impact on the three premiums, and taking into consideration this dependence reduces the level of the premium. But the severity of this impact is different for each premium.

5 Conclusion

This paper presented a joint lifetime model based on Sarmanov bivariate distribution and PH marginal distributions. This work extends the fields of applications of Sarmanov distribution to life insurance. We also introduced a general form of Kernel distributions inspired by the PH distributions, which allows us to extend the range of the correlation that our bivariate model can generate. The suggested model has many tractable and mathematical properties that help to derive closed-form expressions for distributions and actuarial quantities in the multiple life insurance contexts. We believe that this paper's main contribution is to introduce and study the properties of SPH distribution and its applications in modeling dependent lifetimes. Although the paper is limited to life insurance modeling, the SPH model can easily be applied in risk theory, capital allocation, claim reserving, and joint default modeling.

Appendix

Substochastic matrix

Definition 5.1. *A substochastic matrix is a square matrix with nonnegative entries so that every row adds up to at most 1*

Matrix exponential

Definition 5.2. *Let A be a square matrix of order n, then we call matrix exponential, denoted by e^A, the matrix*

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k,$$

with $e^{0_n} = I_n$, where 0_n is the zero matrix of order n.

It is well known that $f(x) = e^x$ is the only function to have the property that $f(x+y) = f(x)f(y)$, i.e. $e^x e^y = e^{x+y}$, where $x, y \in \mathbb{R}$. However, this is not true for matrix exponential.

Proposition 5.1. *Let A and B be square matrices of order n. Then,*

$$e^{tA} e^{tB} = e^{t(A+B)}, \quad t \in \mathbb{R},$$

if $ABB = BA$.

The derivative of matrix exponential function is given in the following proposition.

Proposition 5.2. *The derivative of a matrix exponential function is given by*

$$\frac{d}{dx} e^{Ax} = Ae^{Ax} = e^{Ax} A.$$

Kronecker product and Kronecker sum

The Kronecker product and the Kronecker sum are defined as follows

Definition 5.3. *Let $A = (a_{ij})$ and B be $(n \times m)$ and $(l \times k)$ real matrices, respectively. Then the* Kronecker product $A \otimes B \in \mathbb{R}^{nl*mk}$ *is the partitioned matrix*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1m}B \\ a_{21}B & a_{22}B & \dots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \dots & a_{nm}B \end{pmatrix}.$$

Definition 5.4. *Let A and B be square matrices of orders n and m respectively. Then the* Kronecker sum $A \oplus B$ *is a square matrix of order nm and is given by*

$$A \oplus B = A \otimes I_m + I_n \otimes B.$$

Definition 5.5. *Consider a matrix A. For $k \geq 1$, the kth Kronecker power, $A^{\otimes_{k+1}}$, and the kth Kronecker sum, $A^{\oplus_{k+1}}$, are defined inductively by*

- $A^{\otimes_1} = A$ *and* $A^{\otimes_k} = A \otimes A^{\otimes_{(k-1)}}$ *for* $k = 2, 3, \ldots$.
- $A^{\oplus_1} = A$ *and* $A^{\oplus_k} = A \oplus A^{\oplus_{(k-1)}}$ *for* $k = 2, 3, \ldots$.

Some properties of Kronecker product and Kronecker sum

Proposition 5.3. *Define Let $M_{l,j}$ denote the space of $i \times j$ real (or complex) matrices. Let $A \in M_{m,n}, B \in M_{p,q}, C \in M_{n,k}$, and $D \in M_{q,r}$. Then*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

Proposition 5.4. *Define Let M_l the space of square real (or complex) matrices. Consider two matrices $A \in M_p$ and $B \in M_q$*

(i) *Assume that μ is an eigenvalue for A with corresponding eigenvector x, and ξ is an eigenvalue for B with corresponding eigenvector y. Then $\mu + \xi$ is an eigenvalue of $A \oplus B$ with corresponding eigenvector $y \otimes x$.*

(ii) *Any eigenvalue of $A \oplus B$ arises as such a sum of eigenvalues of A and B*

We refer the readers to Horn and Johnson (1991) for more details on Matrix Mathematics and proofs of the previous propositions.

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