Bounds on Distance Measures in Graphs and Altered Graphs

by

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THESIS

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PREFACE AND DECLARATION

The study described in this thesis was carried out in the Department of Mathematics and Applied Mathematics, University of Johannesburg, from January 2017 to November 2020, under the supervision of Prof. Peter Dankelmann.

This study represents original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it has been duly acknowledged in the text.

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DECLARATION- PUBLICATIONS

Details of contribution to publications that form part and/or include research presented in this thesis (include publications in preparation, submitted, in press and published and give details of the contributions of each author to the experimental work and writing of each publication.)

PUBLICATION 1: A. Alochukwu and P. Dankelmann, Distances in Graphs of Girth 6 and Generalised Cages. *(submitted)*.

PUBLICATION 2: A. Alochukwu and P. Dankelmann, Wiener index in graphs with given minimum degree and maximum degree. *(submitted)*

PUBLICATION 3: A. Alochukwu and P. Dankelmann, Upper Bounds on the Average Eccentricity of Graphs of Girth 6 and (C_4, C_5)-free Graphs. *(ArXiv preprint arXiv:2004.14490)*.

PUBLICATION 4: A. Alochukwu and P. Dankelmann, Bounds on the k-edge-fault-diamter of graphs with no 4-cycles. *(in preparation)*.

PUBLICATION 5: A. Alochukwu and P. Dankelmann, Bounds on the (Edge-)Fault-diameter of Graphs of Girth 6. *(in preparation)*.

Dedication

To

God Almighty: In Whom I live, move and have my being

My treasures: Dimma, Kamsi and Kossy

My mother and sister: Lilian Nwankwo and Chinenye Enemuoh

And in loving memory of my Dad,

Mr Ericmoore Azuka Alochukwu-Nwankwo

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Abstract

The research work presented in this thesis consists of investigation of bounds on the four most important distance measures (radius, diameter/fault-diameter, average eccentricity and average distance) for graphs of girth at least 6 in terms of other graph parameters, namely order, minimum degree and maximum degree.

Let G be a finite, connected graph. The distance between two vertices is defined to be the length of a shortest path between them. The eccentricity of a vertex u is the distance from u to a vertex farthest from u in G, i.e., $e_G(u) = \max_{v \in V(G)} d_G(u, v)$ where $V(G)$ is the vertex set of G. The radius $\text{rad}(G)$ of G is the minimum eccentricity of a vertex, and the diameter $\text{diam}(G)$ of G is the maximum eccentricity of a vertex. The average eccentricity $\text{avec}(G)$ of G is the arithmetic mean of the eccentricities of the vertices of G, and the average distance $\mu(G)$ of G is the average of the distance between all pairs of vertices of G.

Erdös, Pach, Pollack and Tuzá [51] established in terms of order and minimum degree an upper bound on the radius and diameter of connected C_4-free graphs. In Chapter 2, we improve this bound for graphs of girth at least 6 and for (C_4, C_5)-free graphs, i.e., graphs not containing cycles of length 4 or 5. We prove that if G is a graph of girth at least 6 of order n and minimum degree δ, then the diameter is at most $\frac{3n}{\delta^2-\delta+1} - 1$, and the radius is at most $\frac{3n}{2(\delta^2-\delta+1)} + 10$. If $\delta - 1$ is a prime power, then both bounds are sharp apart from an additive constant. This improves the bound given by Erdös et.al in [51] by a factor of approximately 3/5 but under a stronger assumption.

For graphs of large maximum degree Δ, we show in Chapter 3 that these bounds can be improved to $\frac{3n-\Delta\delta}{\delta^2-\delta+1} - \frac{3(\delta-1)\sqrt{\Delta(\delta-2)}}{\delta^2-\delta+1} + 10$ for the diameter, and $\frac{3n-3\Delta\delta}{2(\delta^2-\delta+1)} - \frac{3(\delta-1)\sqrt{\Delta(\delta-2)}}{2(\delta^2-\delta+1)} + \frac{159}{4}$ for the radius. We further show that only slightly weaker bounds hold for (C_4, C_5)-free graphs. As a by-product we obtain a result on a generalisation of cages. For given $\delta, \Delta \in \mathbb{N}$ with $\Delta \geq \delta$ let $n(\delta, \Delta, g)$ be the minimum order of a graph of girth g, minimum degree δ and maximum degree Δ. Then $n(\delta, \Delta, 6) \geq \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}$. If $\delta - 1$ is a prime power, then we construct graphs that show that there exist infinitely many values of Δ such that, for δ constant and Δ large, $n(\delta, \Delta, 6) = \delta\Delta + O(\sqrt{\Delta})$.

In Chapter 4, we show that for a connected graph G of girth at least six, order n and minimum degree δ, $\text{avec}(G) \leq \frac{9}{2}\lceil \frac{n}{2\delta^2-2\delta+2} \rceil + 8$. We construct graphs that show that whenever $\delta - 1$ is a prime power, then this bound is sharp apart from an additive constant. For graphs containing a vertex of large degree we give improved bounds. We further show that if the girth condition on G is relaxed to G having neither a 4-cycle nor a 5-cycle as a subgraph, then similar and only slightly weaker bounds hold.

In Chapter 5, we show that the average distance of a connected graph G of girth at least

six, order n and minimum degree δ is at most $\frac{n}{\delta^2-\delta+1}+11$. Furthermore, we show that if $\delta-1$ is a prime power, then this bound is sharp apart from an additive constant.

To date no upper bound on the average distance of graphs containing a vertex of large degree is known in the literature, except for trees. We prove several such bounds. We show in Section 5.3.3 of Chapter 5 that the average distance of graphs of order n, minimum degree δ and maximum degree Δ is at most $\frac{(n-\Delta+\delta)}{n}\frac{(n-\Delta+\delta-1)}{n-1}\frac{(n+2\Delta+\delta)}{\delta+1}+8$ for connected graphs and $\frac{(n-\Delta+\delta)(n-\Delta+\delta-1)}{n(n-1)}\left[\frac{2}{3}\frac{(n-\Delta+\delta)}{2\delta}+\frac{2\Delta}{\delta}\right]+\frac{35}{3}$ for triangle-free graphs. These bounds are sharp apart from an additive constant and in some sense generalizes the bound given by Dankelmann and Entringer in [30]. Futhermore, we obtain improved bounds for C_4-free graphs and graphs of girth at least 6 in terms of order, minimum degree and maximum degree and prove that these bounds are sharp or close to being sharp apart from the value of the additive constants.

Let G be a $(k+1)$-connected or $(k+1)$-edge-connected graph, where $k \in \mathbb{N}$. The k-fault-diameter and k-edge-fault-diameter of G is the largest diameter of the subgraphs obtained from G by removing up to k vertices and edges, respectively. Dankelmann [28] proved that for a $(k+1)$-connected C_4-free graph G of order n, the k-fault-diameter is at most $\frac{5n}{k^2-k+1}-\frac{5k^2-5k+8}{2}$. This bound is close to being optimal for infinitely many values of k. In Chapter 6, we give a corresponding bound on the k-edge-fault diameter for graphs not containing 4-cycles thus filling a gap in literature. We also establish upper bounds on the k-fault-diameter and k-edge-fault-diameter of graphs of graphs of girth at least 6 and (C_4, C_5)-free graphs in terms of the order of the graph n. These bounds are asymptotically sharp and improve on the bounds by Dankelmann in [28] under a stronger assumption.

We complete our study in Chapter 7 by giving a summary of the previous chapters, as well as, concluding remarks and suggest possible future research arising from our study.

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Chapter 1

Introduction and Preliminaries

In this chapter, we define the most important terms that will be used in this thesis. Most of the definitions and proofs presented herein are known results adapted from graph theory and linear algebra textbooks, see [18] and [79]. We will define other terms that are not defined in this chapter as the need arises. Within this introductory chapter, we present a survey of results that are related to the study, underlying motivation for our study, as well as, relevant background study on distance measures in graph.

1.1 General Terminology

1.1.1 Vertices, Edges, Adjacency and Incidence

Definition 1.1.1. *A **Graph** $G = (V, E)$ consists of a non-empty set of elements V referred to as vertices and a set (possibly empty) E of 2-element subsets of V called edges. We often write $V(G)$ for V, and $E(G)$ for E.*

In this thesis we consider only finite graphs i.e, those graphs with a finite vertex set.

Definition 1.1.2. *The **order** of a graph G, is the number of vertices in G denoted by n or $|V(G)|$, while the **size** of G is the cardinality of the edge set E, denoted by m or $|E(G)|$.*

A graph G is said to be **trivial** if G has order 1, otherwise G is non-trivial.

Definition 1.1.3. *Two vertices u, v of G are said to be **adjacent** if $\{u, v\} \in E(G)$. We usually write uv for the edge $\{u, v\}$. Let $e = uv$. We say that e is incident with u and v or simply e joins u and v.*

1.1.2 Walks, Paths, and Distance

Definition 1.1.4. *A **walk** W in a graph G is an alternating sequence of vertices and edges such that $e_i = v_{i-1}v_i$ for $i = 1, 2, \ldots, k$, denoted as $W : v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$ or simply $W : v_0, v_1, \ldots, v_k$ since the vertices that appear in a walk determine the edges in the walk.*

We say that W is a (v_0, v_k)-walk of length k since W starts at v_0 and ends at v_k. If $v_0 = v_k$, then W is called a **closed walk** in G.

Definition 1.1.5. *Let $n = |V(G)| \geq 3$. If all the vertices in a closed walk are distinct except for v_0 and v_k, then the closed walk is called a **cycle** of length k or a k-cycle.*

Definition 1.1.6. *W is said to be a **path** if all v_i are distinct. The path $P : v_0, v_1, \ldots, v_k$ is often referred to as a (v_0, v_k)-path since it begins at vertex v_0 and ends at v_k.*

Definition 1.1.7. *The **girth** of a graph is the length of a shortest cycle contained in the graph. If a graph does not contain any cycle, then its girth is said to be infinite.*

Definition 1.1.8. *The **distance**, $d_G(u, v)$, between two vertices u, v of a graph G is the length of a shortest (u, v)-path in G.*

1.1.3 Subgraphs, Neighbourhood and Degrees

Definition 1.1.9. *The **degree** of a vertex v of G denoted by $\deg_G(v)$ (or $\deg(v)$) is the number of edges incident with v. The **minimum and maximum degree** of a graph G denoted as $\delta = \delta(G)$ and $\Delta = \Delta(G)$ respectively is the minimum and maximum, of the degrees of vertices in G. A vertex of degree one is called an **end vertex or a leaf**. A graph is said to be k-**regular** if the degree of every vertex in G is k.*

Definition 1.1.10. *A graph H is said to be a **subgraph** of G if the vertex and edge sets H are contained in the vertex and edge sets of G respectively, i.e., $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is a **spanning subgraph** of G if $V(H) = V(G)$.*

Definition 1.1.11. *Let S be a subset of $V(G)$. Then **the subgraph of G induced by** S is the maximal subgraph of G with vertex set S, denoted as $G[S]$.*

For $e \in E(G)$, the subgraph $G - e$ is the graph obtained from G by deleting the edge e. Similarly, for vertex $u \in G$, the subgraph $G - u$ is the graph obtained by deleting the vertex u along with all edges which are incident to u.

Definition 1.1.12. *The **neighbourhood** $N_G(v)$(or simply $N(v)$) of a vertex $v \in V$ is the set of all vertices adjacent to v in G while the closed neighbourhood $N_G[v](N[v])$ is the union of $\{v\}$ and its neighbourhood. Hence, $|N_G(v)| = \deg_G(v)$ and $|N_G[v]| = |N_G(v) \cup \{v\}| = \deg_G(v) + 1$.*

Definition 1.1.13. *Let v be a vertex of G and $k \in N$. The k**-th neighbourhood** of v is the set of vertices of G at distance exactly k from v, denoted by $N_G^k(v)$. Furthermore, the ball of radius k centred at v, $N_G^k[v]$, is the set of vertices of G at distance not more than k from v. For a non empty proper set $A \subseteq V(G)$ and $p \in \mathbb{N}$, the p-th neighbourhood of A, denoted by $N_G^p[A]$, is the set of all vertices v of G of distance at most p to some vertex $a \in A$, i.e., for some $a \in A$ and $p \geq 1$, $N_G^p[A] := \{v \in V(G) \mid d_G(v, a) \leq p \text{ for some } a \in A\}$.*

1.1.4 Specific Graphs

Definition 1.1.14. *A graph G is **connected** if every two of vertices of G are connected.*

Definition 1.1.15. *A graph G of order n is said to be a **complete graph** if all the vertices of G are pairwise adjacent. A complete graph of order n is denoted by K_n.*

Definition 1.1.16. *A **tree** is a connected graph without cycles. A **spanning tree** T of G is a spanning subgraph of G which is a tree. A spanning tree of G with the property that $d_T(v,u) = d_G(v,u)$ for each $u \in V(G)$ is said to be **distance-preserving** from v.*

Definition 1.1.17. *A (not necessarily connected) graph with no cycles is a **forest**.*

Definition 1.1.18. *A **component** of a graph G is a maximal connected subgraph of G. Hence each component of a forest is a tree.*

Definition 1.1.19. *Let $U \subseteq V(G)$. $G - U$ is the graph obtained from G by deleting all vertices in U as well as the edges incident with the vertices in U. U is a **separating set or vertex-cut** if G is connected and $G - U$ has more than one component.*

Definition 1.1.20. *A separating set which consists of only one vertex is a **cut vertex**.*

Definition 1.1.21. *A subset $F \subseteq E(G)$ whose deletion increases the number of components of G is an **edge-cut**. $G - F$ is the the graph obtained from G by deleting all edges in F.*

Definition 1.1.22. *Let G_1 and G_2 be two connected graphs. The **union** $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The union of k disjoint copies of G is denoted by kG. The **join** $G_1 + G_2$ of graphs G_1 and G_2 is the graph consisting of the vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1), v \in V(G_2)\}$.*

Definition 1.1.23. *For $k \geq 3$ vertex disjoint graphs $G_1, G_2, \ldots, G_k$, the **sequential join** $G_1 + G_2 + \ldots + G_k$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup \ldots \cup (G_{k-1} + G_k)$, and the union $G_1 \cup G_2 \cup \ldots \cup G_k$ is the graph with vertex set $V(G_1) \cup V(G_2) \cup \ldots \cup V(G_k)$ and edge set $E(G_1) \cup E(G_2) \cup \ldots \cup E(G_k)$. If $a \in N$, then $[G_1 + G_2 + \ldots + G_k]^a$ stands for $G_1 + G_2 + \ldots + G_k + G_1 + G_2 + \ldots + G_k + \ldots + G_1 + G_2 + \ldots + G_k$, where the pattern $G_1 + G_2 + \ldots + G_k$ appears a times.*

Definition 1.1.24. *A graph G is **bipartite** if $V(G)$ can be partitioned into two non empty subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 to a vertex of V_2. If each vertex of V_1 is joined to every vertex of V_2 , then G is called a **complete bipartite graph**, and is denoted as $K_{n,m}$ where $n = |V_1|$ and $m = |V_2|$, or vice versa.*

Definition 1.1.25. *The star graph S_m sometimes simply known as an m-star is a tree with one internal vertex and m leaves. The star graph S_m is therefore isomorphic to the complete bipartite graph $K_{1,m}$.*

Definition 1.1.26. *The k-th power of G, denoted as G^k, is the graph with the same vertex set as G, in which two vertices $u \neq v \in V(G)$ are adjacent if $d_G(u, v) \leq k$. For a subset $S \subseteq V(G)$, the subgraph of G^k induced by S is denoted by $G^k[S]$.*

Definition 1.1.27. *For any graphs G and H, G is said to be **H-free** if it does not contain a subgraph isomorphic to H (irrespective of whether the subgraph is induced or not).*

The following lemma (see, for example [18]) is a classical result.

Lemma 1.1.28. *Let G be an undirected graph with e edges. Then,*

$$2|E| = \sum_{v \in V} \deg(v).$$

Lemma 1.1.28 is often referred as the handshaking lemma. It implies that every finite undirected graph has an even number of vertices with odd degree.

1.1.5 Matching, Vertex and Edge Connectivity

Definition 1.1.29. *A set M of pairwise non-adjacent edges in a graph G is called a **matching**. In other words, no two edges of M share a common vertex in M. The set of vertices incident with edges of M is denoted by V(M).*

Definition 1.1.30. *The **vertex-connectivity** $\kappa(G)$ of G is defined to be the minimum number of vertices whose deletion from G results in a disconnected or trivial graph. We say that graph G is k-vertex-connected or simply k-connected if $\kappa(G) \geq k$. Similarly, the **edge-connectivity** $\lambda(G)$ is the minimum cardinality of an edge-cut of G. We say that G is k-edge- connected if $k \leq \lambda(G)$.*

Lemma 1.1.31. *[18] Let G be a k-connected graph and let $k \geq 1$. If e is an edge of G, then $G - e$ is at least $(k-1)$-connected.*

Definition 1.1.32. *If A, B are subsets of V, then $(A, B)_G$ denotes the set of edges joining a vertex in A to a vertex in B. If M is a matching of G then we say that M is induced if the only edges of G that join two vertices of V(M), the set of vertices incident with edges in M, are the edges of M.*

In the next section we give definitions to some of the distance concepts and measures in graph theory which will be used in the subsequent chapters.

1.1.6 Distance Concepts and Distance Measures in Graphs

Definition 1.1.33. *The **eccentricity**, $e_G(v)$, of a vertex $v \in V(G)$ is the maximum distance between v and any other vertex in G.*

Definition 1.1.34. *The maximum eccentricity of vertices of G is the* ***diameter*** *of G, denoted by* $\operatorname{diam}(G)$ *and the minimum eccentricity of vertices of G is the* ***radius*** *of G, denoted by* $\operatorname{rad}(G)$*. More precisely,*

$$\operatorname{diam}(G) = \max_{u \in V(G)} e_G(u) = \max_{u,v \in V(G)} d_G(u, v),$$

$$\operatorname{rad}(G) = \min_{u \in V(G)} e_G(u) = \min_{u \in V(G)} \max_{v \in V(G)} d_G(u, v).$$

Definition 1.1.35. *The* ***fault-diameter*** $D_{\kappa(G)-1}$ *of a graph G is the largest diameter obtained by deleting a set of $(\kappa(G) - 1)$ vertices.*

Definition 1.1.36. *The* ***total eccentricity*** $EX(G)$ *is the sum of all eccentricities of vertices in G. The* ***average eccentricity*** $\operatorname{avec}(G)$ *of a connected graph G of order n is the mean eccentricity of the vertices in G, that is* $\operatorname{avec}(G) = \frac{1}{n} EX(G)$.

Definition 1.1.37. *The* ***Wiener index (total distance)*** *is the sum of distances between all unordered pairs of vertices of a connected graph G, that is,*

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v).$$

The ***average distance*** $\mu(G)$ *of a connected graph G of order n is the average of the distances between all pairs of vertices of G, i.e.,*

$$\mu(G) = \binom{n}{2}^{-1} \sum_{\{u,v\} \subseteq V(G)} d_G(u, v).$$

Definition 1.1.38. *Let v be a vertex of a connected graph G. We define the following.*

i) An ***eccentric vertex*** *of a vertex v is a vertex farthest away from v in G.*

ii) Every vertex of G of minimum eccentricity is a ***centre vertex*** *of G.*

iii) The ***centre*** $C(G)$ *of G is the subgraph induced by the set of all centre vertices in G.*

iv) If $\{u, v\} \subseteq V(G)$ is a pair of vertices of G with $d_G(u, v) = \operatorname{diam}(G)$, then $\{u, v\}$ is referred to as a diametral pair of G and any shortest (u, v)-path is called a diametral path.

The first two parts of the following definition is the same as Definition 1.1.13 except that we choose to use a slightly different notation to distinguish the i-th distance layer from i-th neighbourhood.

Definition 1.1.39. *Let u be a vertex of a connected graph G. For any $i \in \mathbb{Z}$, we define the following sets.*

a) *The i-th distance layer of u, denoted by* $N_i(u)$*, is defined as*

$$N_i(u) = \{x \in V(G) : d_G(u, x) = i\}.$$

where $N_i(u) = \emptyset$ *for* $i < 0$ *or* $i > e_G(u)$*.*

b) $N_{\leq j}(u) = \bigcup_{i:\ 0\ \leq\ i\ \leq\ j} N_i(u).$

c) $N_{\geq j}(u) = \bigcup_{i:\ j\ \leq\ i\ \leq\ e_G(u)} N_i(u).$

If it is clear which vertex is meant, then we write N_i *instead of* $N_i(u)$*.*

Definition 1.1.40. *Let G be a connected graph. A* ***packing*** *S of G is a set of vertices such that the distance between any pair of vertices in that set is at least 3, that is* $d_G(u, v) \geq 3$ *for all* $u, v \in S$ *and* $u \neq v$*, where* $S \subset V(G)$*.*

For a positive integer k, a ***k-packing*** *of G is a subset* $A \subset V(G)$ *with* $d_G(u, v) > k$ *for all* $u, v, \in A$ *and* $u \neq v$*. The number of vertices in any k-packing of maximum cardinality is referred to as the k-packing number* $\beta_k(G)$ *of G.*

1.2 Linear Algebra Concepts

In this section we recall some concepts from linear algebra over finite fields which we need for the construction of some graphs in later chapters.

Definition 1.2.1. *A field is a set F together with two operations, addition (+) and multiplication (.) satisfying the following axioms for all* $a, b, c \in F$*:*

- *Associativity of addition and multiplication:* $a + (b + c) = (a + b) + c$ *and* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$*.*
- *Commutativity of addition and multiplication:* $a + b = b + a$ *and* $a \cdot b = b \cdot a$*.*
- *Additive and multiplicative identity: there exist two different elements 0 and 1 in F such that* $a + 0 = a$ *and* $a \cdot 1 = a$*.*
- *Additive inverses: for every a in F, there exists an element in F, denoted* $-a$*, called the additive inverse of a, such that* $a + (-a) = 0$*.*
- *Multiplicative inverses: for every* $a \neq 0$ *in F, there exists an element in F, denoted by* a^{-1}*, or* $1/a$*, called the multiplicative inverse of a, such that* $a \cdot a^{-1} = 1$*.*
- *Distributivity of multiplication over addition:* $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$*.*

Definition 1.2.2. *A **finite field or Galois field** is a field with finite order (number of elements). Let q be a prime power. Then $\mathbb{F} = GF(q)$ is a finite field of order q and $\mathbb{F}^n = GF(q)^n$ denotes the n-dimensional vector space over $GF(q)$ of all n-tuples of elements of $GF(q)$.*

Definition 1.2.3. *Let $\mathbb{F}$ be a field and let $\mathbb{F}^n$ be the vector space over $\mathbb{F}$ of all $n-$tuples of elements of $\mathbb{F}$. If $\underline{u} = (u_1, u_2, \ldots, u_n) \in \mathbb{F}^n$ and $\underline{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{F}^n$, then the **dot product** $\underline{\boldsymbol{u}} \cdot \underline{\boldsymbol{v}}$ of $\underline{u}$ and $\underline{v}$ is defined by*

$$\underline{u} \cdot \underline{v} = u_1v_1 + u_2v_2 + \ldots + u_nv_n$$

Subsequently we use u_1v_1 instead of $u_1 \cdot v_1$ to denote the multiplication in $\mathbb{F}$.

Definition 1.2.4. *If $S = \{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_k}\}$ is a set of vectors in $\mathbb{F}^n$, then the set of all linear combinations of $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_k}$ is called the **span** of $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_k}$, denoted by $\langle S \rangle$. S is called a **spanning set** for $\mathbb{F}^n$ if $\mathrm{span}(S) = \mathbb{F}^n$*

Definition 1.2.5. *A set of vectors $\{\underline{v_1}, \underline{v_2}, \ldots, \underline{v_k}\}$ is **linearly dependent** if there are scalars $c_1, c_2, \ldots, c_k$, at least one of which is not zero, such that*

$$c_1\underline{v_1} + c_2\underline{v_2} + \ldots + c_k\underline{v_k} = \underline{0}$$

*A set of vectors that is not linearly dependent is said to be **linearly independent**.*

Definition 1.2.6. *A **subspace** of $\mathbb{F}^n$ is any collection S of vectors in $\mathbb{F}^n$ such that:*

- *the zero vector, $\underline{0}$, is in S.*
- *S is closed under addition and scalar multiplication.*

Definition 1.2.7. *A **basis** for a subspace S of $\mathbb{F}^n$ is a set of vectors in S that*

- *spans S and*
- *is linearly independent.*

Definition 1.2.8. *If S is a subspace of $\mathbb{F}^n$, then the number of vectors in a basis for S is called the **dimension** of S, denoted $\dim S$.*

Definition 1.2.9. *Two vectors $\underline{u}$ and $\underline{v}$ in $\mathbb{F}^n$ are **orthogonal** to each other if $\underline{u} \cdot \underline{v} = 0$. A vector $\underline{v}$ in $\mathbb{F}^n$ is then said to be self-orthogonal if $\underline{v} \cdot \underline{v} = 0$.*

Definition 1.2.10. *Let W be a subspace of $\mathbb{F}^n$. We say that a vector $\underline{v}$ in $\mathbb{F}^n$ is **orthogonal to** W if $\underline{v}$ is orthogonal to every vector in W. The set of all vectors that are orthogonal to W is called the **orthogonal complement of** W, denoted $W^{\perp}$. That is,*

$$W^{\perp} = \{\underline{v} \ in\ \mathbb{R}^n : \underline{v} \cdot \underline{w} = 0 \ for\ all\ \underline{w}\ in\ W\}$$

Theorem 1.2.11. *[79] If W is a subspace of $\mathbb{F}^n$, then $W^\perp$ is a subspace of $\mathbb{F}^n$ and* $\dim\, W + \dim\, W^\perp = n$.

Theorem 1.2.12. *[79] Let V be a vector space over a field F and let U, W be finite-dimensional subspaces of V. Then* $\dim\,(U + W) = \dim\, U + \dim\, W - \dim\,(U \cap W)$.

Corollary 1.2.13. *Given the n-dimensional vector space $\mathbb{F}^n$, the orthogonal complement $W^\perp$ of a k-dimensional subspace W has dimension $n - k$.*

The following result is a classical result. See, for example [60].

Theorem 1.2.14. *Let $GF(q)$ be a finite field of order q Then $\mathbb{F}^* = GF(q)^* = GF(q)\backslash\{0\}$, the multiplicative group of $GF(q)$ is cyclic.*

The following result is probably known. However, we were unable to find a reference in the exact form as stated below. We observed that [7] has a formulation in a different terminology (for example, see Lemma 3.8.1 in [7]). The proof written below is ours.

Claim 1.2.15. *Let q be a prime power. Then there exists a self-orthogonal vector $\underline{z}$ in $GF(q)^3$.*

Proof. Since q is a prime power, we can write q in the form p^r for some prime p and a positive integer r. Since $GF(p)$ is a subfield of $GF(q)$, and since $GF(p) = \mathbb{Z}_p$ it suffices to show that $GF(p)^3$ contains a non-zero self-orthogonal vector. The multiplicative group $\mathbb{F}^*$ of the field $GF(p) = \mathbb{Z}_p$ contains the elements $\{1, 2, \ldots, p-1\}$ and is cyclic (see Theorem 1.2.14).

Case 1: -1 is a square in $\mathbb{F}^*$.

Then there exists $b \in \mathbb{F}^*$ with $b^2 = -1$. Then $1 + b^2 = 0$, and so the vector $(1, b, 0)^t$ is self-orthogonal.

Case 2: -1 is not a square in $\mathbb{F}^*$.

Let a be a generator of $\mathbb{F}^*$. Then $\mathbb{F}^* = \{a^0, a^1, a^2, \ldots, a^{q-2}\}$. Since $-1 \in \mathbb{F}^*$, there exists a unique $k \in \{0, 1, \ldots, p-2\}$ such that $a^k = -1$. Then k is odd, since otherwise $(a^{k/2})^2 = a^k = -1$, contradicting the fact that -1 is not a square.

Let $b \in \mathbb{F}^*$. We show that

if b is not a square, then $-b$ is a square.

Assume that b is not a square. Then $b = a^\ell$ for some $\ell \in \{0, 1, \ldots, q-2\}$. Clearly, ℓ is odd since otherwise $b = (a^{\ell/2})^2$, a contradiction to b not being square. Then $-b = (-1)b = a^{k+\ell} = (a^{(k+\ell)/2})^2$, so $-b$ is a square since $k + \ell$ is even.

Now 1 is a square in $\mathbb{F}^*$, but $p-1=-1$ is not. Choose b to be the smallest of the numbers $1,2,3,\ldots,p-1$ that is not a square in $\mathbb{F}^*$. Then $b-1$ and $-b$ are squares in $\mathbb{F}^*$, so there exist $c,d\in\mathbb{F}^*$ such that $b-1=c^2$ and $-b=d^2$. Then the vector $(1,c,d)^t$ is self-orthogonal since $(1,c,d)\cdot(1,c,d)^t=1^2+c^2+d^2=1+(b-1)+(-b)=0$ in $\mathbb{F}$. □

1.2.1 Counting in Vector Spaces over Finite Fields

Remark 1.2.16. *Any k-dimensional subspace of $GF(q)^n$ contains q^k-1 non-zero vectors.*

Claim 1.2.17. *Let q be a prime power and $n\in\mathbb{N}$. Every 1-dimensional subspace of $GF(q)^n$ is contained in $(q^{n-2}+q^{n-3}+q^{n-4}+\ldots+q+1)$ distinct 2-dimensional subspaces of $GF(q)^n$. Each 2-dimensional subspace of $GF(q)^n$ contains $q+1$ distinct 1-dimensional subspaces of $GF(q)^n$.*

Proof. Let $\langle\underline{u}\rangle$ be a 1-dimensional subspace of $GF(q)^n$. We want to find the number of 2-dimensional subspaces containing $\langle\underline{u}\rangle$. Every 2-dimensional subspace, say U, containing $\langle\underline{u}\rangle$ is of the form $\langle\underline{u},\underline{v}\rangle$, where $\{\underline{u},\underline{v}\}$ is a basis for U. So $\underline{v}\in GF(q)^n-\langle\underline{u}\rangle$. There are q^n-q vectors in $GF(q)^n-\langle\underline{u}\rangle$. Since every 2-dimensional subspace containing $\langle\underline{u}\rangle$ contains q^2-q different vectors that are linearly independent from vector $\underline{u}$, the q^n-q vectors in $GF(q)^n-\langle\underline{u}\rangle$ come in sets of q^2-q vectors, so that each of these together with $\underline{u}$ generates the same 2-dimensional subspace. Hence there are exactly $(q^n-q)/(q^2-q)=(q^{n-1})/(q-1)=(q^{n-2}+q^{n-3}+\ldots+q+1)$ different 2-dimensional subspaces. Thus, every 1-dimensional subspace of $GF(q)^3$ is contained in $(q^{n-2}+q^{n-3}+\ldots+q+1)$ distinct 2-dimensional subspace of $GF(q)^n$ as desired.

We want to find the number of 1-dimensional subspaces contained in a 2-dimensional subspace of $GF(q)^n$. We know that in a 2-dimensional subspace, there are q^2-1 choices for a non-zero vector $\underline{w}$ and groups of $q-1$ choices for $\underline{w}$ yields the same 1-dimensional subspace. So we have that each 2-dimensional subspace of $GF(q)^n$ contains $(q^2-1)/(q-1)=q+1$ different 1-dimensional subspaces. □

Corollary 1.2.18. *Every 1-dimensional subspace of $GF(q)^3$ is contained in $q+1$ distinct 2-dimensional subspace of $GF(q)^3$. Each 2-dimensional subspace of $GF(q)^3$ contains $q+1$ distinct 1-dimensional subspace of $GF(q)^3$.*

Claim 1.2.19. *[92] Let q be a prime power and $k,n\in\mathbb{N}$ with $k\le n$. Then there are $\frac{(q^n-1)(q^n-q)(q^n-q^2)\ldots(q^n-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)\ldots(q^k-q^{k-1})}$ k-dimensional subspaces of $GF(q)^n$.*

Proof. Let W be a k-dimensional subspace of $GF(q)^n$. Recall from Remark 1.2.16 that any k-dimensional subspace of $GF(q)^n$ contains q^k-1 non-zero vectors. Let $k\in\mathbb{N}$ and let A_n be the set of k-tuples of linearly independent vectors in $GF(q)^n$. We count the number of ways in which we can construct a k-tuple $(v_1,v_2,\ldots,v_k)$ of linearly independent vectors.

Clearly, $\underline{v_1}$ can be selected in $q^n - 1$ ways, $\underline{v_2}$ can be selected in $q^n - q$ ways and so on. Hence,

$$|A_n| = (q^n - 1)(q^n - q)(q^n - q^2) \dots (q^n - q^{k-1}).$$

Next, we want to count the number of k-tuples of linearly independent vectors in W. To select a basis for W, the first member could be selected in $q^k - 1$ ways, second member in $q^k - q$ ways and so on. Hence, each k-dimensional subspace is generated by A_k k-tuples where

$$A_k = \{(q^k - 1)(q^k - q)(q^k - q^2) \dots (q^k - q^{k-1})\}.$$

We know that any member of A in A_n generates W if and only if A lies in A_k. On the other hand, any k-dimensional subspace of $GF(q)^n$ is generated by an element of A_k and since W is a k-dimensional subspace of $GF(q)^n$, in A_n, there are A_k elements which all of them generates the same subspace. Therefore, the number of different k-dimensional subspaces of $GF(q)^n$ is

$$A_n/A_k = \frac{(q^n - 1)(q^n - q)(q^n - q^2) \dots (q^n - q^{k-1})}{(q^k - 1)(q^k - q)(q^k - q^2) \dots (q^k - q^{k-1})}.$$

□

Corollary 1.2.20. *There are* $(q^2 + q + 1)$ *1-dimensional subspaces and* $(q^2 + q + 1)$ *2-dimensional subspaces of* $GF(q)^3$*.*

1.3 Rationale/Motivation for the Study

Herein, we give some motivation for our research and provide background for relevant results.

Graphs can serve as mathematical models for many kinds of real world networks, such as computer networks, the internet, social networks, and transportation networks. In the analysis of graphs, distances play a major role because of its relevance for the efficiency of the network structure and numerous applications, ranging from the construction of more efficient computer networks and transportation networks to modelling the interactions of species in environmental conservation, solving facility location problems and network designs in operation research, to the design and analysis of floor plans in architecture and predicting the properties of chemical compounds in chemistry. For example distances in computer networks indicate through how many intermediate processors information has to be transmitted; or in transportation networks such as the Gautrain it is an indicator for the travel time, while in social networks it is an indicator for closeness of people. The wide application of distance parameters in analysing real-world networks provides strong motivation for studying distance concepts. Buckley and Harary [13], and many others have written extensively on this subject. Hence, research on distance measures in graphs has attracted much attention in the literature.

A distance measure in a graph is a number that describes and quantifies a particular aspect of the distances between the vertices of the graph. Most distance measures of graphs provide important information on the graph. The oldest and best studied distance measures are the diameter and the radius, while the average distance and average eccentricity are more recent distance measures. For example, let a graph describe a transportation network. Then its diameter, defined as the largest of all distances between the vertices of a graph, is an indicator for the worst case travel time between destinations within the network. On the other hand, the radius is an important measure of centrality. It is an indicator for the worst case travel time starting from a vertex that is central within the network. It is often used to identify possible sites for emergency facilities within a network.

The literature on the diameter and radius is vast with [51] been the most influential. [51] presented bounds on the diameter and radius of graphs in terms of order and minimum degree and observed that these bounds can be improved if restricted to certain graph classes. Among others, the paper [51] contains sharp bounds on the diameter and radius of triangle-free graphs and C_4-free graphs. Subsequent to this paper, similar bounds for other graph classes were proved, see for example [23]. In Chapters 2 and 3, we give an upper bound on the radius and diameter of connected graphs of girth at least 6, as well as (C_4, C_5)-free graphs of given order, minimum degree and maximum degree, thus improving on the bound for C_4-free graphs in [51] under the additional assumption that the graph has girth at least 6 and is thus also C_5-free.

The average distance of a graph can loosely be described as a measure for the travel time on average within a transportation network, while the average eccentricity is an indicator for the maximum travel time in a network from a typical vertex. The question how the average distance of a graph can be bounded in terms of its order and minimum degree has been considered in a number of papers, the most important ones being [30, 67]. Since its first systematic investigation in [31], the average eccentricity has attracted much attention in the literature. In particular conjectures by the creators of the computer programme AutoGraphiX spurred much interest in the average eccentricity, see for example [32, 49, 73, 101]. The first bounds on the average eccentricity of graphs in terms of order and minimum degree appeared in [31]. Other bounds on the average eccentricity of graphs in terms of order, minimum degree, independence number, domination number and other graph parameters have also been considered in quite a number of papers in the literature see for example the most recent one [36]. These considerations is an indication that the bounds can be improved for other graph classes if we have additional information on the graph. In Chapters 4 and 5, we slightly modified the techniques developed in [30] and [31] to strengthen these bounds for graphs of girth at least 6 and (C_4, C_5)-free graphs taken into consideration the order, minimum and maximum degree of the graphs.

Many networks are not static but change in time: for example new websites or links are added to the internet or deleted, or processors or links in a computer can fail. These changes in the network can also change distances, in particular the diameter, significantly. For such networks the fault-diameter is an important distance measure since it gives more information on the diameter of a graph after failure of links. It is defined as the largest diameter of the graphs arising from deleting a prescribed number of vertices or edges from the given graph. The fault-diameter has been studied for many very specific graph classes. However, the first bounds on the fault-diameter for all connected graphs appeared in [28]. This paper contains also bounds on the fault-diameter in terms of order and minimum degree. In Chapter 6, we modified techniques developed in [28] to improve these bounds for the classes of graphs considered herein. We also filled the gap in the literature by giving bounds on the edge-fault-diameter for graphs not containing 4-cycles since for C_4-free graphs only bounds on the k-fault diameter are known (see [28].

These findings motivated our research which seek to contribute to a better understanding of distance measures in graphs by finding new bounds in terms of other graph parameters. In the remainder of this chapter, we give short survey of results for each of these distance measures.

1.4 Literature Review

In this section we present a survey of some of the results on the radius and diameter of graphs. We only give a detailed proof of some of the results that will be essential to the proofs of our main results in subsequent chapters.

1.4.1 Survey of Results on Radius and Diameter

Several bounds on the diameter and radius are known in the literature.

The trivial upper bound, on the diameter of a connected graph on n vertices

$$1 \leq \text{diam}(G) \leq n - 1 \tag{1.4.1}$$

is attained only by the path of order n. Furthermore for a nontrivial connected graph G, the inequality

$$\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G) \tag{1.4.2}$$

is a well-known result which follows from the definition of radius and triangle inequality. Thus, establishing a relationship between the radius and diameter of connected graphs. According to Ostrand [81], for any given $r, d \in \mathbb{N}$ such that $r \leq d \leq 2r$, there exists a graph with radius r and diameter d and so (1.4.2) is the only restriction on the diameter in terms of radius. However, Jordan [66] showed that for trees, there exists a stronger relationship between the diameter and radius.

Theorem 1.4.1. *[66] Let T be a tree of order $n \geq 2$, then the centre of T consist of a single vertex or of two adjacent vertices. Furthermore, if the centre of T consists of a single vertex then* $\operatorname{diam}(T) = 2\operatorname{rad}(T)$*, and if the centre of T consists of two adjacent vertices then* $\operatorname{diam}(T) = 2\operatorname{rad}(T) - 1$.

Taking into account also other graph parameters or properties of a graph, bound (1.4.1) can be improved. Ore [80] determined the maximum size of a graph of given order and diameter, and thus gave a bound on the diameter in terms of order and size.

Theorem 1.4.2. *[80] Let G be connected graph of order n and diameter d. Then, the number of edges in G is at most $d + \frac{1}{2}(n-d-1)(n-d+4)$.*

Ali, Mazorodze, Mukwembi and Vetrík [3] improved Ore's bound by taking into account also the edge-connectivity. Fulek, Morić and Pritchard [59] determined the maximum size of planar graphs of given order and diameter. Bounds on the diameter in terms of order and edge-connectivity were given by Caccetta and Smyth [15].

Theorem 1.4.3. *[3] Let G be a graph of order n, size m, diameter d and edge-connectivity λ where $\lambda \geq 8$ is a constant. Then*

$$m \leq \frac{1}{2}\left[n - \frac{d}{3}(\lambda+1)\right]^2 + O(n).$$

Theorem 1.4.4. *[59] For every connected planar graph G,* $\operatorname{diam}(G) \leq \frac{4(n-1)-m}{3}$.

Theorem 1.4.5. *[15] Let G be a K-edge connected graph of order n. If $\alpha = \lceil 2\sqrt{K} \rceil$, then for $n < 2(K+1)+\alpha$, the maximum diameter D^* of G is*

$$D^* = \left\lfloor \frac{n}{K+1} \right\rfloor + \left\lceil \frac{n}{K+1} \right\rceil - 1.$$

The following theorem is a well-known result on the radius of a connected graph

Theorem 1.4.6. *If G is a connected graph of order $n \geq 2$. Then*

$$1 \leq \operatorname{rad}(G) \leq \lfloor \frac{n}{2} \rfloor. \tag{1.4.3}$$

The bound $\lfloor \frac{n}{2} \rfloor$ in (1.4.3) is attained, for example, by the path.

Vizing [93] gave the following bound on the size of a connected graph in terms of order and radius.

Theorem 1.4.7. *[93] For any natural numbers n and r such that $n \geq 2r \geq 2$, the maximum number of edges in a connected graph of order n and radius at least r is $f(n,r)$ where* *(a)* $f(n,1) = \frac{1}{2}n(n-1)$,

(b) $f(n,2) = \frac{1}{2}n(n-1) - \lceil \frac{1}{2}n \rceil = \lfloor \frac{1}{2}n(n-2) \rfloor$,

(c) $f(n,r) = \frac{1}{2}(n^2 - 4rn + 5n + 4r^2 - 6r)$ *for* $n \geq 2r \geq 6$.

On the other hand, Dankelmann et.al. in [41] gave a corresponding bound on the size of a bipartite graph of given order n and radius r.

Theorem 1.4.8. *[41] For any natural numbers n and r such that $n \geq 2r \geq 2$, the maximum number of edges in a bipartite graph of order n and radius at least r is $b(n,r)$ where*

(a) $b(n,1) = n - 1$,

(b) $b(n,2) = \lfloor \frac{n^2}{4} \rfloor$,

(c) $b(n,3) = \lfloor \frac{n^2}{4} \rfloor - \lfloor \frac{n}{2} \rfloor$,

(d) $b(n,r) = \lfloor \frac{n^2}{4} \rfloor - nr + r^2 + 2(n-r)$ *for* $n \geq 2r \geq 8$.

Dankelmann and Volkmann [42] proved that a connected graph G of order n, radius r and minimum degree δ, has at least $\frac{1}{2}\delta n + \frac{(n-1)(\delta-2)}{(\delta-1)^r-1}$ edges for large n. The bound is sharp and they also gave similar bounds for digraphs. Iida and Kobayashi [64] gave upper bounds on the radius in terms of order and connectivity. Ali et al. in [2] showed that for G, a 3-connected planar graph of order n, maximum face length ℓ and radius $\text{rad}(G)$, the bound $\text{rad}(G) \leq \frac{n+5\ell}{6} + \frac{2}{3}$ holds. Harant in [63] also investigated the radius of planar graphs. For results on the radius on neighbourhood graphs see [78].

Bounds on the diameter and radius of a graph of given order and minimum degree that are sharp apart from an additive constant were given by [51]

$$\text{diam}(G) \leq \frac{3n}{\delta+1} - 1. \qquad (1.4.4)$$

Apart from [51], various sets of authors, see for example [6, 10, 61], independently proved the same bound in (1.4.4) or variations thereof. Erdös, Pach, Pollack and Tuzá [51] showed that the radius of a graph is bounded by approximately half the value of the upper bound in (1.4.4).

$$\text{rad}(G) \leq \frac{3(n-3)}{2(\delta+1)} + 5. \qquad (1.4.5)$$

Using a slightly different technique, [29] improved (1.4.5) to a stronger bound.

$$\text{rad}(G) \leq \frac{3n}{2(\delta+1)} + 1. \qquad (1.4.6)$$

In addition, Dankelmann, Mukwembi and Swart [35] determined the maximum radius of a 3-edge-connected graph of given order. Erdös, Pach, Pollack and Tuzá [51] also noticed that their bounds on the radius and diameter can be improved for triangle-free graphs (see details in Theorem 1.4.11). We give the details of the proof in full with more elaboration since the idea of the technique will be useful in subsequent chapters, in particular Chapters 2 and 3.

We start with the following Lemma which will be very useful for the proofs on the bounds on the radius in the subsequent theorem and chapters.

Definition 1.4.9. *Let G be a connected graph and let $v, u \in V(G)$ such that u is a centre vertex. Let T be a spanning tree of G that preserves the distances from u and denote the (u, v)-path in T by $T(u, v)$. Let $\mathrm{rad}(G) = r$ and fix $v' \in N_r(u)$, then a vertex $v'' \in V(G)$ is said to be* ***related*** *to v' if there exists $\overline{v}' \in V(T(u, v')) \cap N_{\geq 9}(u)$ and $\overline{v}'' \in V(T(u, v'')) \cap N_{\geq 9}(u)$ such that*

$$d_G(\overline{v}', \overline{v}'') \leq 4. \tag{1.4.7}$$

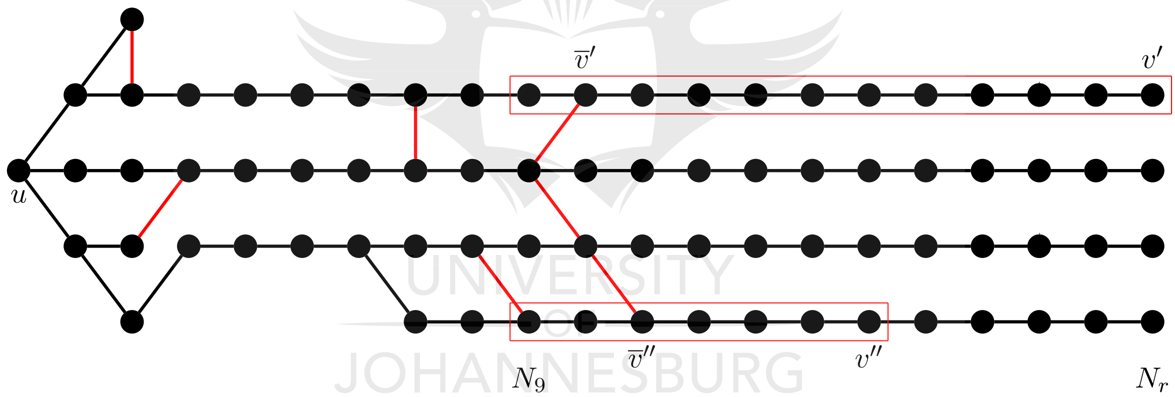

Figure 1.1: Illustration to show that vertices v' and v'' are related.

Lemma 1.4.10. *[51] Let G be a connected graph of radius $r \geq 9$, and let u be a centre vertex of G. Let T be a spanning tree of G that preserves the distances from u, and let $v' \in N_r(u)$. Then there exists a vertex $w \in N_{\geq r-9}(u)$ that is not related to v'.*

Proof. Suppose to the contrary that every vertex $v \in N_{\geq r-9}$ is related to v'. Let u' be the only vertex of $T(u, v')$ belonging to N_9. We show that $d(u', v) \leq r - 1$ for all $v \in V(G)$. To achieve this, we consider the two cases when $v \in N_{\leq r-10}$ or $v \in N_{\geq r-9}$. For any $v \in N_{\leq r-10}$, we have that

$$d_G(u', v) \leq d_G(u', u) + d_G(u, v) \leq 9 + r - 10 = r - 1. \tag{1.4.8}$$

On the other hand, if $v \in N_{\geq r-9}$, then by our assumption, v is related to v' and so there exists $\overline{v}' \in V(T(u,v')) \cap N_{\geq 9}$ and $\overline{v}'' \in V(T(u,v)) \cap N_{\geq 9}$ for which $d_G(\overline{v}',\overline{v}'') \leq 4$. We have that

$$\begin{aligned} d_G(u',v) &\leq d_G(u',\overline{v}') + d_G(\overline{v}',\overline{v}'') + d_G(\overline{v}'',v). \\ &\leq (d_G(u,\overline{v}') - 9) + 4 + [r - d_G(u,\overline{v}'')], \\ &= r - 5 + d_G(u,\overline{v}') - d_G(u,\overline{v}''), \\ &\leq r - 5 + d_G(\overline{v}',\overline{v}''), \\ &= r - 5 + 4, \\ &\leq r - 1. \end{aligned}$$

This implies that $d_G(u',v) \leq r-1$ for all $v \in V(G)$, contradicting our assumption that $r = \operatorname{rad}(G)$. Therefore, we conclude that there is a vertex $w \in N_{\geq r-9}(u)$ that is not related to v'. □

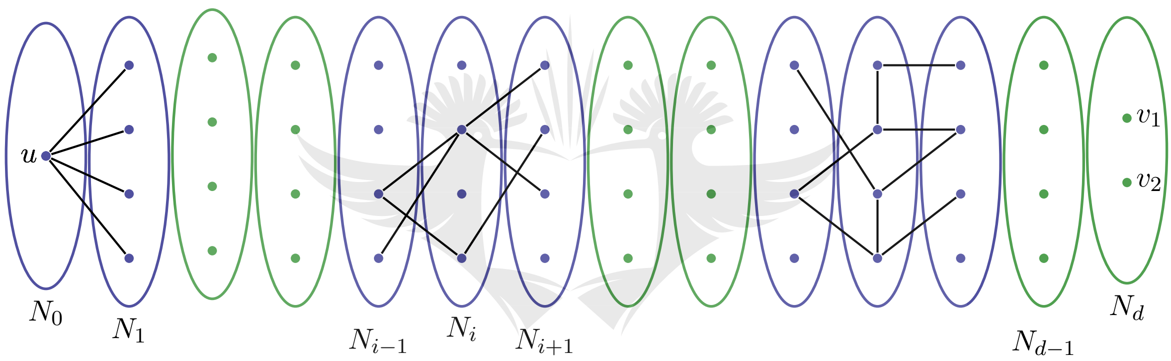

Figure 1.2: The i-th distance layer.

Theorem 1.4.11. *[51] Let G be a connected triangle-free graph of order n and with minimum degree $\delta \geq 2$, then*

$$(i) \quad \operatorname{diam}(G) \leq \frac{4(n-\delta-1)}{2\delta}, \tag{1.4.9}$$

$$(ii) \quad \operatorname{rad}(G) \leq \frac{n}{\delta} + 9. \tag{1.4.10}$$

(i) and (ii) are tight apart from the exact value of the additive constant, and for every $\delta \geq 2$ equality can hold in (i) for infinitely many values of n.

Proof. Recall that $N_i = N_i(u)$ denotes the i-th distance layer of u, i.e., the set of vertices at distance exactly i from u (see figure 1.2 above). Let u and v be two vertices with $d_G(u,v) = d$, the diameter of G. A vertex in N_i can have neighbours only in $N_{i-1} \cup N_i \cup N_{i+1}$ since otherwise there is a shorter path from u to v. If $u \in N_0$, we have that $|N_0| = 1$, $N(u) = N_1$

and so $|N_1| \geq \delta$. Similarly, if $v \in N_d$, we have that $|N_d| \geq 1$ and $N(v) \subseteq N_{d-1} \cup (N_d - \{v\})$. Hence,

$$|N_0| + |N_1| \geq 1 + \deg\ (u) \geq 1 + \delta, \text{ and } |N_{d-1}| + |N_d| \geq 1 + \deg\ (v) \geq 1 + \delta.$$

Considering the i-th distance layers, one of the following possibilities occurs. It is either that vertices of N_i are independent, or N_i contains two adjacent vertices. If the vertices of N_i are independent, then $N(x) \subseteq N_{i-1} \cup N_{i+1}$ for all $x \in N_i$ and so

$$|N_{i-1}| + |N_{i+1}| \geq \delta. \tag{1.4.11}$$

If N_i contains two adjacent vertices, say x and y, then $N(x) \cap N(y) = \emptyset$ since otherwise G contains a triangle. Since the neighbours or x and y are in $N_{i-1} \cup N_i \cup N_{i+1}$, we have

$$|N_{i-1}| + |N_i| + |N_{i+1}| + \geq 2\delta. \tag{1.4.12}$$

It follows from (1.4.11) and (1.4.12) that

$$|N_{i-1}| + |N_i| + |N_{i+1}| + |N_{i+2}| \geq 2\delta \quad \text{for} \quad 0 \leq i \leq d-1. \tag{1.4.13}$$

Indeed equation (1.4.13) follows from (1.4.12) if N_i or N_{i+1} contains an edge. Otherwise, by (1.4.11), $|N_{i-1}| + |N_{i+1}| \geq \delta$ and $|N_i| + |N_{i+2}| \geq \delta$, hence (1.4.13) is true.

Clearly, $n = \sum_{i=0}^{d} |N_i|$ and the distance layers can be grouped into blocks, each containing 4 distance layers. We now consider the distinguishing cases according to the residue class of $d \mod 4$. Let $k = \lfloor \frac{d}{4} \rfloor - 1$,

If $d \equiv 0 \pmod 4$, then we have that

$$\begin{aligned} n &= \sum_{i=0}^{d} |N_i| \\ &= \sum_{i=0}^{k-1} \big(|N_{4i-1}| + |N_{4i}| + |N_{4i+1}| + |N_{4i+2}|\big) + |N_{d-5}| + |N_{d-4}| + |N_{d-3}| \\ &\quad +|N_{d-2}| + |N_{d-1}| + |N_d| \\ &\geq \big(k\big)2\delta + (\delta + 1) + 2\delta, \\ &= (k+1)2\delta + \delta + 1 \\ &= \left\lfloor \frac{d}{4} \right\rfloor 2\delta + \delta + 1. \end{aligned}$$

If $d \equiv 1 \pmod 4$, then

$$\begin{aligned} n &= \sum_{i=0}^{d} |N_i| \\ &= \sum_{i=0}^{k} \big(|N_{4i-1}| + |N_{4i}| + |N_{4i+1}| + |N_{4i+2}|\big) + |N_{d-2}| + |N_{d-1}| + |N_d| \\ &\geq \big(k+1\big)2\delta + 2\delta, \\ &= \left\lfloor \frac{d}{4} \right\rfloor 2\delta + 2\delta. \end{aligned}$$

If $d \equiv 2 \pmod 4$, then we have that

$$\begin{aligned} n &= \sum_{i=0}^{d} |N_i| \\ &= \sum_{i=0}^{k} \big(|N_{4i-1}| + |N_{4i}| + |N_{4i+1}| + |N_{4i+2}|\big) + |N_{d-3}| + |N_{d-2}| + |N_{d-1}| + |N_d| \\ &\geq \big(k+1\big)2\delta + 1 + 2\delta, \\ &= \big(k+2\big)2\delta + 1 \\ &= \left(\left\lfloor \frac{d}{4} \right\rfloor + 1\right)2\delta + 1, \\ &= \left\lfloor \frac{d}{4} \right\rfloor 2\delta + 2\delta + 1. \end{aligned}$$

If $d \equiv 3 \pmod 4$, then

$$\begin{aligned} n &= \sum_{i=0}^{d} |N_i| \\ &= \sum_{i=0}^{k} \big(|N_{4i-1}| + |N_{4i}| + |N_{4i+1}| + |N_{4i+2}|\big) + |N_d| + |N_{d-4}| + |N_{d-3}| + \\ &\quad |N_{d-2}| + |N_{d-1}| + |N_d| \\ &\geq \big(k+1\big)2\delta + 2 + 2\delta, \\ &= \big(k+2\big)2\delta + 2, \\ &= \left(\left\lfloor \frac{d}{4} \right\rfloor + 1\right)2\delta + 2, \\ &= \left\lfloor \frac{d}{4} \right\rfloor 2\delta + 2\delta + 2. \end{aligned}$$

From the above four cases, we have that

$$n = \sum_{i=0}^{d} |N_i| \geq \left\lfloor \frac{d}{4} \right\rfloor 2\delta + \begin{cases} \delta + 1 & \text{if } d \equiv 0 (\text{mod } 4), \\ 2\delta & \text{if } d \equiv 1 (\text{mod } 4), \\ 2\delta + 1 & \text{if } d \equiv 2 (\text{mod } 4), \\ 2\delta + 2 & \text{if } d \equiv 3 (\text{mod } 4). \end{cases} \quad (1.4.14)$$

and thus,

$$n \geq \begin{cases} \left(\frac{d}{4}\right) 2\delta + \delta + 1 & \text{if } d \equiv 0 (\text{mod } 4), \\ \left(\frac{d-1}{4}\right) 2\delta + 2\delta & \text{if } d \equiv 1 (\text{mod } 4), \\ \left(\frac{d-2}{4}\right) 2\delta + 2\delta + 1 & \text{if } d \equiv 2 (\text{mod } 4), \\ \left(\frac{d-3}{4}\right) 2\delta + 2\delta + 2 & \text{if } d \equiv 3 (\text{mod } 4). \end{cases} \quad (1.4.15)$$

which implies that

$$d \leq \begin{cases} \frac{4(n-\delta-1)}{2\delta} & \text{if } d \equiv 0 (\text{mod } 4), \\ \frac{4(n-2\delta)}{2\delta} + 1 & \text{if } d \equiv 1 (\text{mod } 4), \\ \frac{4(n-2\delta-1)}{2\delta} + 2 & \text{if } d \equiv 2 (\text{mod } 4), \\ \frac{4(n-2\delta-2)}{2\delta} + 3 & \text{if } d \equiv 3 (\text{mod } 4). \end{cases} \quad (1.4.16)$$

Therefore

$$d \leq \frac{4(n-\delta-1)}{2\delta}. \quad (1.4.17)$$

And thus inequality (1.4.9) holds.

The extremal graph, $G_{k,\delta}$, described below shows that (1.4.9) is sharp apart from an additive constant.

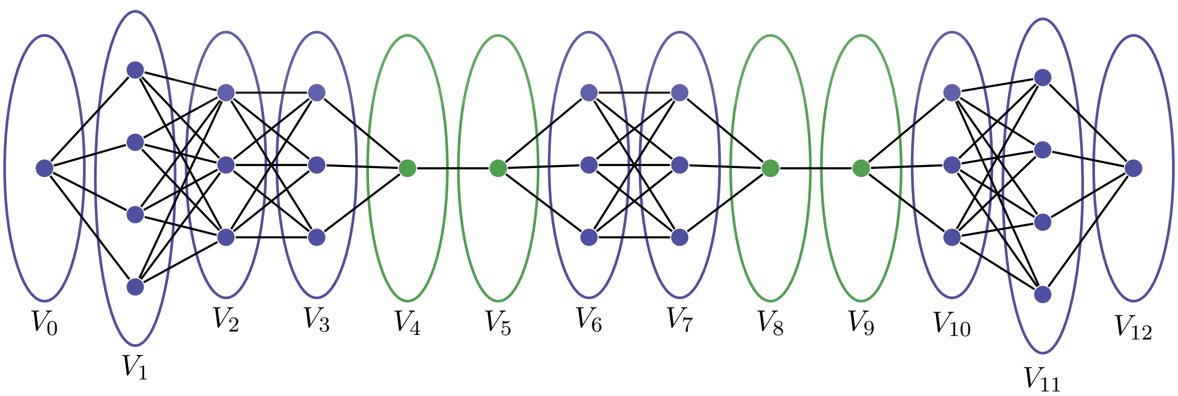

Figure 1.3: The graph $G_{3,\delta}$.

Let $V(G) = V_0 \cup V_1 \cup \ldots \cup V_{4k}$ with

$$|V_i| = \begin{cases} 1 & \text{if } i \equiv 0 \text{ or } 1 (\text{mod } 4) \text{ and } i \neq 1, \\ \delta & \text{if } i = 1 \text{ or } 4k-1, \\ \delta - 1 & \quad \text{otherwise.} \end{cases}$$

and let V_i and V_{i+1} induce a complete bipartite subgraph of G for every i. Observe that $n = |V(G_{k,\delta})| = 2k\delta + \delta + 1$. This implies that $k = (n - \delta - 1)/(2\delta)$. Hence the diameter of the graph, $G_{k,\delta}$, is $4k$. The extremal graph, $G_{3,4}$, is shown in Figure 1.3.

ii) Let u be a centre vertex of G. For any $x \in N_i$, pick a vertex $x' \in N_{i-1}$ such that $xx' \in E(G)$ $(1 \leq i \leq r)$. The collection of the edges of the form $\{xx' : x \in V(G) - \{u\}\}$ defines a distance preserving spanning tree, $T \leq G$, from u, i.e., $d_T(u, v) = d_G(u, v)$ for all $v \in V(G)$.

By Lemma 1.4.10, there is a vertex $w \in N_{\geq r-9}$ that is not related to x'. For any $i \in \mathbb{Z}$, let D'_i and D''_i denote the set of all vertices in N_i whose distance from at least one vertex of $T(u, x') \cap N_{\geq 9}$ $(T(u, w) \cap N_{\geq 9}$ respectively) is at most 2 in G.

By our assumption that x' and w are not related, we have that

$$\Big(\bigcup_{i=7}^{r} D'_i\Big) \bigcap \Big(\bigcup_{i=7}^{r} D''_i\Big) = \emptyset,$$

Let $s = d_G(u, w)$. For $0 \leq i \leq r - 1$, we have by (1.4.13) that

$$|N_{i-1}| + |N_i| + |N_{i+1}| + |N_{i+2}| \geq 2\delta,$$

and following similar arguments, we obtain that

$$|D'_{i-1}| + |D'_i| + |D'_{i+1}| + |D'_{i+2}| \geq 2\delta \quad \forall i \in \{8, 9, \ldots r-1\}, \tag{1.4.18}$$

$$|D''_{i-1}| + |D''_i| + |D''_{i+1}| + |D''_{i+2}| \geq 2\delta \quad \forall i \in \{8, 9, \ldots s-1\}, \tag{1.4.19}$$

where $s \geq r - 9$. Based on this fact, we have that

$$n = \sum_{i=0}^{r} |N_i| \;\geq\; |N_{\leq 6}| + \sum_{i=7}^{r} |D'_i| + \sum_{i=7}^{s+1} |D''_i|, \tag{1.4.20}$$

and so,

$$n \geq \sum_{i=0}^{6} |N_i| + \sum_{i=7}^{r} |D'_i| + \sum_{i=7}^{r-8} |D''_i| \;\; \text{since } s \geq r - 9.$$

Let k be the largest integer with $k \leq r$ and $k \equiv 2 \pmod 4$. Then $k \geq r-3$ and

$$\begin{aligned}
n &\geq \sum_{i=0}^{6}|N_i| + \sum_{i=7}^{k}|D'_i| + \sum_{i=7}^{k-8}|D''_i|,\\
&\geq \sum_{i=0}^{6}|N_i| + \sum_{i=2}^{\frac{k-2}{4}}\left(|D'_{4i-1}|+|D'_{4i}|+|D'_{4i+1}|+|D'_{4i+2}|\right)\\
&\quad + \sum_{i=2}^{\frac{k-10}{4}}\left(|D''_{4i-1}|+|D''_{4i}|+|D''_{4i+1}|+|D''_{4i+2}|\right), \qquad \text{by (1.4.18) and (1.4.19)}\\
&\geq 4\delta + \left(\frac{k-2}{4}\right)2\delta - 2\delta + \left(\frac{k-10}{4}\right)2\delta - 2\delta,\\
&= \left(\frac{2k-12}{4}\right)2\delta,\\
&= (k-6)\delta\\
&\geq (r-9)\delta \qquad \text{since } k \geq r-3.
\end{aligned}$$

Thus, $n \geq (r-9)\delta$ and it follows that $r \leq n/\delta + 9$. Hence, the inequality (1.4.10) holds. □

In the same paper [51], the authors conjectured an improvement of the bound (1.4.4) for graphs not containing a complete subgraph of order k, $k \geq 4$.

Conjecture 1.4.12. *[51] Let $r, \delta \in \mathbb{N}$ and let G be a connected graph on n vertices with minimum degree δ.*

(i) If G does not contain a complete subgraph on $2r$ vertices, and if δ is a multiple of $(r-1)(3r+2)$, then, for large n,

$$\operatorname{diam}(G) \leq \frac{2(r-1)(3r+2)}{(2r^2-1)\delta}n + O(1).$$

(ii) If G does not contain a complete subgraph on $2r+1$ vertices, and if δ is a multiple of $3r-1$, then, for large n,

$$\operatorname{diam}(G) \leq \frac{3r-1}{r\delta}n + O(1).$$

This conjecture remains open, however Czabarka, Dankelmann and Székely [23] gave a bound on the diameter of 4-colourable graphs, and thus proved a weaker form of this conjecture for $k = 5$. Recently, Czabarka, Singgih and Székely [24] disproved part (i) of the conjecture. They further proved the bound $\operatorname{diam}(G) \leq (3 - \frac{1}{k-1})n + O(1)$ for k-colourable graphs of minimum degree at least δ and order n. Further bounds on radius or diameter in terms of vertex degrees can be found in [75, 76, 77]. By slightly modifying the technique in [30, 31], Mazorodze and Mukwembi in [74] proved an upper bound on the radius and diameter of connected graphs in terms of order, minimum degree and maximum degree.

$$\text{diam}(G) \le \frac{3(n-\Delta)}{\delta+1} + O(1). \tag{1.4.21}$$

$$\text{rad}(G) \le \frac{3(n-\Delta)}{2(\delta+1)} + O(1). \tag{1.4.22}$$

The authors showed that the bounds in (1.4.21) and (1.4.22) are asymptotically sharp and can be improved for triangle-free and C_4-free graphs. They also showed that given the irregularity index t such that $\Delta < \frac{3}{2}t$, then their results on the radius and diameter of triangle-free graphs can be also be improved.

For graphs not containing 4-cycles, Erdös, Pach, Pollack and Tuzá [51] showed that

$$\text{diam}(G) \le \frac{5n}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1}, \tag{1.4.23}$$

and they proved that the radius of such a graph is bounded by approximately half the value in the above bound (1.4.23). Similar bounds was obtained for graphs not containing a complete bipartite graph $K_{3,3}$, $\ell \ge 2$ in [29]. We present in full (see Section 2.2 of Chapter 2) the proof of the bound in (1.4.23), including the sharpness construction since some of the properties of the graph will be useful in subsequent chapters.

1.4.2 Survey of Results on Average Distance and Average Eccentricity

The average distance, also known as mean distance or transmission delay, originally introduced in graph theory in 1977 by [45] has proven to be one of the tools used for performance evaluation or measuring the efficiency of an interconnection network modelled by a graph. One fact taken into account when investigating any communication network is the diameter of a graph, which is the maximum distance between any two nodes of the network. Nonetheless, those pairs of nodes that give the diameter may represent only a small fraction of the total number of pairs. The average distance can therefore be a more effective estimate of the network efficiency on average above its diameter since it is a measure of the estimated travel time between two randomly selected points of the network.

The average distance (named in honour of Wiener) has been introduced in 1977, but it received significant attention only after the classical paper by Plesnik [82], through conjectures of the computer programme GRAFFITI and recently in chemical graph theory [95]. The Wiener index $W(G)$ of a graph G as defined by [95] is the sum of distances between all unordered pairs of vertices. Thus,

$$W(G) = \binom{n}{2}\mu(G).$$

The following result is a trivial bound on the average distance of a connected graph

$$1 \leq \mu(G) \leq \text{diam}(G).$$

Plesnik in [82] showed that aside from the above well known bound, the average distance of a graph is independent of its radius and diameter. He proved that given the radius and diameter of a graph and any $t, \epsilon > 0 \in \mathbb{R}$ such that (1.4.2) holds and $1 \leq t \leq \text{diam}(G)$, then there exists a graph G with $|\mu(G) - t| < \epsilon$.

Given that G is a connected graph of α independence number, [55] conjectured that

$$\mu(G) \leq \alpha(G). \tag{1.4.24}$$

This has generated considerable interest as several authors have obtained different bounds for the average distance taken into account the order, minimum degree, size, independence number, k-packing number, domination number and k-domination number, see for example [30, 25, 26, 27].

Chung in [21] gave a proof to the above conjecture and showed that equality holds only if $\alpha = 1$, that is, if the graph is complete. This was later improved in [25] by Dankelmann. In addition, [56] showed that a weaker version $\mu(G) \leq \alpha(G) + 1$ of the conjecture also holds. Two other GRAFFITI conjectures in [55] that involve the average distance are

$$\text{rad}(G) \leq \mu(G) + R(G), \tag{1.4.25}$$

where $R(G)$ is the Randic index of the graph, and

$$\mu(G) \leq \frac{n}{\delta}, \tag{1.4.26}$$

for every δ-regular connected graph G of order n. The inequality (1.4.25) involving two distance parameters was disproved by Dankelmann, Oellermann and Swart [39] while that of (1.4.26) has generated tremendous interest as several authors attempted to improve on the bound.

Kouider and Winkler in [67] proved a slightly asymptotically stronger bound on the above conjecture (1.4.26).

Theorem 1.4.13. *Let G be a connected graph with minimum degree δ and order n, then*

$$\mu(G) \leq \frac{n}{\delta + 1} + 2. \tag{1.4.27}$$

Dankelmann and Entriger in [30] showed that for every connected graph G with n vertices and minimum degree δ, there exists a spanning tree of G satisfying the bound in (1.4.27), that is,

$$\mu(T) \le \frac{n}{\delta+1} + 5.$$

In the same paper [30], the authors using the same technique showed that their bound can be improved further for triangle-free graphs and graphs not containing a (not necessarily induced) 4-cycle. The technique used in the proof of Theorem 1.4.15 will be useful in Chapter 5, hence we omit the proof here and present it later.

Theorem 1.4.14. *[30] Let G be a connected triangle-free graph of order n and minimum degree $\delta \ge 2$. Then G has a spanning tree T with*

$$\mu(T) \le \frac{2n}{3\delta} + \frac{25}{3}.$$

This inequality is best possible apart from the additive constant.

Theorem 1.4.15. *[30] (i) Let G be a connected C_4-free graph of order n and minimum degree δ. Then G has a spanning tree T with*

$$\mu(T) \le \frac{5}{3}\frac{n}{\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1} + \frac{29}{3}.$$

(ii) There exists an infinite number of C_4-free graphs with n vertices and minimum degree δ for which $\delta - 1$ is a prime power, such that for every spanning tree T of G,

$$\mu(T) \ge \frac{5}{3}\frac{n}{\delta^2 + 3\delta + 2} + O(1).$$

Theorem 1.4.16. *Let G be a connected graph of order n, then*

$$1 \le \mu(G) \le \frac{n+1}{3}.$$

This bound is maximised by a path of order n, see [50, 45, 72].

A similar technique used in [30] will be used in Chapters 4 and 5 to obtain upper bounds on the average eccentricity and average distance of graphs of girth at least 6 taken into account the order and minimum degree. In that same chapter, we will now present in full the proof to Theorem 1.4.15 with some elaborations on the original proof.

Since the radius, diameter and average eccentricity is defined to be the smallest, the largest and average of all eccentricities in a graph respectively, we have for a connected graph G of order n, the following inequality

$$rad(G) \le \text{avec}(G) \le diam(G),$$

which shows the relationship between the average eccentricity, radius and diameter of a graph. The bound is attained by a self-centred graph, i.e. $\text{rad}(G) = \text{diam}(G)$.

Subsequently, we state some useful results on eccentricity of a vertex and average eccentricity of a graph. For some of the results that will be useful in subsequent chapters, we present their proof in full and intentionally omit the proof of others making reference to the original proof.

The average eccentricity was introduced under the name *eccentric mean* by Buckley and Harary [13], but it attracted major attention only after its first systematic study in [31]. One of the basic results in this paper determined the maximum average eccentricity of a connected graph of given order:

Theorem 1.4.17. [31] *If G is a connected graph of order n, then*

$$\operatorname{avec}(G) \leq \frac{1}{n}\left\lfloor \frac{3n^2}{4} - \frac{n}{2} \right\rfloor,$$

with equality if and only if G is a path.

In the same paper the authors established by direct calculations the following basic results for the average eccentricity of a complete graph, the cycle and the complete bipartite graph.

Theorem 1.4.18. *[31] Let G be a graph of order n, then*

a) $\operatorname{avec}(K_n) = 1$,

b) $\operatorname{avec}(C_n) = \lfloor \frac{1}{2}n \rfloor$,

c) $\operatorname{avec}(K_{n_1,n_2}) = 2$, *if* $n_1, n_2 \geq 2$.

Furthermore, Dankelmann et al. [31] established a relation between average eccentricity and average distance, and proved that in trees one can determine the average eccentricity given information on the eccentricity and distance of the central vertex.

Theorem 1.4.19. *[31] For any graph G,*

- $\operatorname{avec}(G) \geq \mu(G)$,
- $\operatorname{avec}(G) \leq \frac{1}{n}\sigma(C(G)) + \operatorname{rad}(G)$, *where $\sigma(C(G))$ is the distance of $C(G)$.*

The following corollary is a consequence of the the previous theorem since the path and its centre has maximum radius and maximum distance respectively.

Corollary 1.4.20. *[31] The connected graph with maximum average eccentricity for given order is the path.*

The following lemma due to [32] is a well-known basic result on the eccentricity of a vertex.

Lemma 1.4.21. *[32] Let T be a tree and let u, v be two vertices at distance* $\operatorname{diam}(T)$*. Then,*

$$e_G(x) = \max\{d(x,u), d(x,v)\} \quad \textit{for all} \;\; x \in V(T).$$

The natural question if the bound in Theorem 1.4.17 can be improved for graphs whose minimum degree is greater than 1 was answered in the affirmative in [31], where it was shown that if G is a graph of order n and minimum degree δ, then

$$\operatorname{avec}(G) \le \frac{9n}{4(\delta+1)} + \frac{15}{4}, \tag{1.4.28}$$

and this inequality is best possible apart from a small additive constant. However, by a slight modification, we show in the preliminary results in Chapter 4 that this bound is at most $\frac{9n}{4(\delta+1)} + \frac{5}{2}$. In addition to the bounds already established in [31], the authors in the same paper examined the change in the average eccentricity when a graph is replaced either by a spanning tree or removing an edge. It was observed in [38] that the upper bound (1.4.28) can be improved for triangle-free graphs and for graphs not containing four-cycles. Using similar methods as in [30, 31], the authors proved the following results.

Theorem 1.4.22. *[38] Let G be a connected triangle-free graph of order n and minimum degree $\delta \ge 2$. Then G has a spanning tree T with*

$$\operatorname{avec}(T) \le 3\left\lceil \frac{n}{2\delta} \right\rceil + 5.$$

This inequality is best possible apart from the additive constant.

Theorem 1.4.23. *[38] (i) Let G be a connected C_4-free graph of order n and minimum degree δ. Then G has a spanning tree T with*

$$\operatorname{avec}(T) \le \frac{15}{4}\left\lceil \frac{n}{\delta^2 - 2\lfloor \frac{\delta}{2} \rfloor + 1} \right\rceil + \frac{11}{2}.$$

(ii) If $\delta \in \mathbb{N}$ such that $\delta + 1$ is a prime power, then there exists an infinite number of C_4-free graphs with n vertices and minimum degree δ such that for every spanning tree T of G,

$$\operatorname{avec}(T) \ge \frac{15}{4} \frac{n}{\delta^2 + 3\delta + 2} + O(1).$$

In Chapter 4, we aim to further pursue the idea of improving (1.4.28) for graphs not containing certain subgraphs.

Recently [37] gave upper bounds on the average eccentricity of connected graphs, triangle-free graphs and connected C_4-free graphs of given order, minimum and maximum degree.

Theorem 1.4.24. *[37] Let G be a connected graph of order n and minimum degree δ and* *maximum degree Δ. Then,*

$$\operatorname{avec}(G) \leq \frac{9}{4}\frac{n-\Delta-1}{\delta+1}\Big(1+\frac{\Delta-\delta}{3n}\Big)+7. \tag{1.4.29}$$

Theorem 1.4.25. *[37] Let G be a connected triangle-free graph of order n, minimum degree δ and maximun degree Δ. Then,*

$$\operatorname{avec}(G) \leq \frac{3}{2}\frac{n-\Delta}{\delta}\Big(1+\frac{\Delta-\delta}{3n}\Big)+\frac{19}{2}. \tag{1.4.30}$$

Theorem 1.4.26. *[37] (i) Let G be a connected C_4-free graph of order n, minimum degree δ and maximum degree Δ. Then*

$$\operatorname{avec}(G) \leq \frac{15}{4}\frac{n-\varphi_\Delta+\varphi_\delta}{\varphi_\delta}\Big[1+\frac{\varphi_\Delta-\varphi_\delta}{3n}\Big]+\frac{37}{4}, \tag{1.4.31}$$

where $\varphi_\Delta := \Delta\delta - 2\lfloor\frac{\Delta}{2}\rfloor + 1$, $\varphi_\delta := \delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1$.

(ii) If $\delta \geq 3 \in \mathbb{Z}$ such that $\delta+1$ is a prime power, then for $n, \Delta \in \mathbb{N}$ with $2\delta - 3 \leq \Delta < n$ and $n \equiv 0 \pmod{(\delta+1)(\delta+2)}$ and $\Delta \equiv \delta + 1 (\operatorname{mod} \delta+2)$ there exists a C_4-free graph G with n vertices, minimum degree δ and maximum degree Δ whose average eccentricity satisfies,

$$\operatorname{avec}(G) \geq \frac{3}{4}\frac{n-\varphi_\Delta}{\varphi'_\delta}(1+\frac{\Delta(\delta+1)}{3n})+O(1),$$

where $\varphi'_\delta := (\delta+1)(\delta+2)$.

By substituting δ for Δ yields Theorem 1.4.24, except for a slightly weaker additive constant. Hence, Theorem 1.4.24 is in some sense a generalisation of (1.4.28) (see also Theorem 4.2.3). Moreover, the bound for C_4-free graphs above is close to being best possible and not far from being sharp if $\delta+1$ is a prime power.

Several other bounds on the average eccentricity also exist in literature. For example for graphs of given order and size [1, 91], and for maximal planar graphs [1]. Furthermore, several relations between the average eccentricity and other graph parameters, for example independence number [32, 36, 65], domination number [32, 36, 49, 48, 47], clique number [44, 65], chromatic number [90], proximity [73] and Wiener index [43] have also been explored in the literature. Bounds on the average eccentricity of the strong product of graphs were given in [17]. Further results relating the average eccentricity of a graph to its vertex degrees are known. Bounds on the average eccentricity of trees of given order and maximum degree were given in [65]. Trees with given degree sequence that minimise or maximise the average eccentricity were determined in [88]. For relations between average eccentricity and Randić index see [71]. An upper bound on the average eccentricity in terms of order, size and first Zagreb index was also given in [44].

1.4.3 Survey of Results on (Edge)-Fault-Diameter

Often graphs are not static, but change over time. For example, in a transportation or communication network links may fail, which can change the diameter. Hence, it becomes significant to consider faulty networks and investigate the fault-diameter since it is an important measure of the network efficiency and reliability.

A number of researchers have investigated the diameter of graphs after an edge or a vertex have been removed prior to the introduction of fault-diameter in [68].

Chung and Garey [22], and independently Plesník [83], showed that if an edge is removed that is not a bridge from a connected graph, then the diameter of the new graph is at most double of the diameter of the original graph. Schoone, Bodlaender, and van Leeuwen [87] proved that deleting two or three edges from a graph G of diameter d, either leaves the resulting graph disconnected, or yields a graph whose diameter is at most $3d-1$ or $4d-1$.

The graph $K_1 + P_{n-1}$ has diameter 2, but removing the universal vertex leaves a graph with diameter $n-2$. Thus, after a vertex or an edge has been removed from a graph, the diameter cannot be bounded in terms of the diameter of the original graph. Consequently, the diameter of a graph provides little or no information on the diameter after vertices or edges are removed. This problem of determining what the diameter of resulting graph after vertices or edges have been removed was addressed in [8] and [28] under the name k-(edge-)fault diameter. Diameter vulnerability and fault-tolerant diameter are other names used by different authors to mean fault-diameters.

Krishnamoorthy and Krishnamurthy [68] gave an upper bound of the fault-tolerant diameter of the Cartesian product graph $G_1 \times G_2$ to be

$$D_{k_1+k_2}(G_1 \times G_2) \leq D_{k_1}(G_1) + D_{k_2}(G_2),$$

where $k_1 + k_2$ is the diameter of $G_1 \times G_2$. This bound however happened to be false as Xu et al. [99] showed that the above bound ought to be

$$D_{k_1+k_2}(G_1 \times G_2) \leq D_{k_1}(G_1) + D_{k_2}(G_2) + 1.$$

Banič and Žerovnik in [9] considered a generalisation, the Cartesian graph bundles.

Upper bounds on the diameter of a κ-connected and λ-edge-connected graphs of order n also exist in literature. The first two results that follows are bounds on the diameter of a graph and not the fault-diameter but since they are also related to removing vertices and edges, we decided to include it under the survey for the fault-diameter.

Theorem 1.4.27. *[94] Let G be a κ-connected graph of order n, then*

$$\mathrm{diam}(G) \leq \lfloor \frac{n-2}{\kappa} \rfloor + 1,$$

and this bound is sharp.

In addition, Plesník [82] also gave upper bound on the diameter of 2-edge-connected graphs in terms of order while Caccetta and Smyth [15] proved same for $3 \leq \lambda \leq 7$. They showed that $\text{diam}(G) \leq 4$ or $\text{diam}(G) \leq \lfloor \frac{2n-2}{3} \rfloor$, if $\lambda \in \{2\}$; $\text{diam}(G) \leq \lfloor \frac{n-1}{2} \rfloor$, if $\lambda \in \{3,4\}$; $\text{diam}(G) \leq \lfloor \frac{2(n-3)}{5} \rfloor$ if $\lambda \in \{5,6\}$; $\text{diam}(G) \leq \lfloor \frac{n-5}{3} \rfloor$, if $\lambda = 7$. The above bounds are asymptotically sharp.

Since the fault-tolerant diameter is not the only parameter used in measuring reliability and efficiency of interconnection networks, studies on the wide-diameter also abound in the literature. The wide diameter d_k of a graph G is a natural generalisation of the diameter that takes into account the connectivity of the graph. More precisely, it is defined to be the minimum integer d' for which there exists at least k internally disjoint paths of length at most d' between any two distinct vertices in G. Most authors are interested in determining how large the difference between the wide diameter d_k and the fault-tolerant diameter D_k can be. For example, Flandrin and Li [58] established the following result for any 2-connected graph G with diameter d,

$$d_2 \leq \begin{cases} D_2 + 1 & \text{if } d = 2, \\ (d-1)(D_2 - 1) & \text{if } d \geq 3. \end{cases}$$

Yin et al. [100] improved on the bound and the result to

$$d_2 \leq \begin{cases} \max\{D_2 + 1, (d-1)(D_2 - d) + 2\} & \text{if } d \leq \lceil (D_2 - 1)2 \rceil, \\ \max\{(D_2 + 1), \lfloor (D_2 - 1)^2/4 \rfloor + 2\} & \text{if } d \geq \lceil (D_2 - 1)/2 \rceil + 1. \end{cases}$$

The fault-diameter and edge-fault-diameter have been studied for many very specific graph classes , see for example, [19, 86, 70] for results on the fault-diameters of the 2-dimensional mesh of trees and star graphs; [20, 16, 89, 57] for results on the fault-diameters of directed double-loop, pyramid networks, folded Petersen graphs and generalised cycles respectively. Moreover, results on the fault-diameter and the edge-fault-diameter prior 2001 were documented in [97].

Guowen and Zhang [62] determined the maximum size for k-connected graphs of order n and with a given $(k-1)$-fault diameter or k-diameter. Banič, Erveš and Žerovnik [8] showed that in a $(k+1)$-connected graph the k-fault-diameter can exceed the k-edge-fault-diameter by at most one. For results on the fault-tolerant diameter of product, see, for example, [98]. The fault-tolerant diameter on hypercubes, undirected de Bruijn networks, directed Kautz networks and undirected Kautz networks were given in [68, 53, 46, 69] respectively. Bounds on the maximum value of the k-fault-diameter and the k-edge-fault-diameter of graphs of given order appeared first in Dankelmann [28]. The author showed that if G is a $(k+1)$-connected graph G of order n then the k-fault-diameter of G is bounded from above by $n-k+1$ while on the other hand the k-edge-fault-diameter is bounded by $n-1$ if $k=1$, by $\lfloor\frac{2n-1}{3}\rfloor$ if $k=2$, and by approximately $\frac{3}{k+2}n$ if $k\geq 3$. Dankelmann went further to show that the above bound can be improved for triangle-free graphs.

Theorem 1.4.28. *[28] Let G be a triangle-free $(k+1)$-connected graph of order n, where $k\geq 2$. Then*

$$D_k(G)\leq \frac{4}{k+2}n-\frac{4k}{k+2},$$

and for $k\geq 2$ this bound is sharp apart from an additive constant.

Theorem 1.4.29. *[28] Let G be a triangle-free graph of order n*

(i) If G is a 2-edge-connected and $n\geq 4$, then

$$D_1'(G)\leq n-1.$$

Equality holds, if and only if $G=C_n$.

(ii) If G is a 3-edge-connected graph of order $n\geq 6$, then

$$D_2'(G)\leq \lfloor\frac{3n-1}{5}\rfloor.$$

The bound is sharp for all $n\geq 11$.

(iii) If $k\geq 3$ and G is $(k+1)$-edge-connected, then

$$D_k'(G)\leq \frac{2n}{k+1}+\frac{4k}{k+1},$$

and this bound is sharp apart except for an additive constant.

For $k=1,2$, the bounds in Theorem 1.4.29 are sharp for all n and for $k\geq 3$, the bounds are best possible. In particular, the bound strengthened the well-known bounds in [51] in the sense that their bounds on the diameter of graphs with minimum degree δ is asymptotically sharp even after removal of at most $\delta-1$ edges, given that the graph is δ-edge-connected.

In the same paper, Dankelmann showed that the bound on the k-fault diameter can be further improved to approximately $\frac{5n}{(k-1)^2}$ if G does not contain 4-cycles. However he did not give the corresponding result for the k-edge-fault-diameter. We intentionally omit the proof of this result here and present it in detail in Chapter 6 since the proof of our original result follows closely follows the proof given in [28].

In Chapter 6, we improve Dankelmann's bound in Theorem 1.4.29 for graphs not containing 4-cycles thereby filling the gap in the literature. Using a similar technique, we also give bounds on the (edge-)fault-diameter of graphs with girth at least 6 and (C_4, C_5)-free graphs.

1.5 Thesis Outline

The remaining chapters of this thesis is arranged as follows: In Chapter 2, we give upper bounds on the diameter and radius of graphs of girth at least 6, as well as, (C_4, C_5)-free graphs taken into account the minimum degree and the order of the graphs. We first present lower bounds on the cardinality of the second neighbourhood, i.e., the set of vertices at distance at most 2, of a vertex or a pair of adjacent vertices for both, graphs of girth at least 6 and (C_4, C_5)-free graphs. This result will be used in the proofs in the chapters that follow. In Chapter 3, we present lower bounds on the cardinality of the third neighbourhood, i.e., the set of vertices at distance at most 3, of a vertex. These results imply lower bounds on the order of graphs of girth at least 6 in terms of minimum degree and maximum degree. This leads us to the generalisation of a cage, in which not regular graphs of given girth but graphs of given minimum degree and maximum degree and prescribed girth are considered. We also present a similar lower bound on the order of (C_4, C_5)-free graphs in terms of minimum degree and maximum degree, and we discuss our results in relation to a different, but related generalisation of cages introduced in [11]. Making use of these results, we give improved bounds on diameter and radius for graphs of girth at least 6 and also for (C_4, C_5)-free graphs in terms of order and minimum degree and maximum degree.

By slightly modifying the widely used technique for constructing spanning trees developed by Dankelmann and Entringer [30], we obtain bounds on the average eccentricities and average distance of graphs of girth at least 6 and (C_4, C_5)-free graphs in Chapters 4 and Chapter 5 respectively. Bounds on the (edge)-fault-diameter of graphs of girth at least 6, as well as, (C_4, C_5)-free graphs were obtained in Chapter 6.

Chapter 2

Upper Bounds on the Radius and Diameter of Graphs of Girth at least 6 and (C_4, C_5)-free Graphs.

2.1 Introduction

In this chapter, we give upper bounds on the diameter and radius of graphs of girth at least 6, as well as, (C_4, C_5)-free graphs taking into account the minimum degree and the order of the graphs. More precisely, we want to develop further a particular aspect of an idea that appeared in [51] where the authors observed that bounds on distance measures can be improved if information on the presence or absence of certain substructures is given. Our results show that the bounds in [51] can be improved further by a factor of about 3/5 for graphs of girth at least 6 and that is best possible. The techniques used in [51], [30] and [31], were very useful in obtaining these bounds. In addition, we construct graphs to show that these upper bounds on the distance measures are asymptotically sharp.

2.2 Preliminary Results

The following bound is an extension of a bound on the number of vertices in the second neighbourhood of a vertex given in [51] which takes into account only the minimum degree, but not the degree of v.

Lemma 2.2.1. *Let G be a C_4-free graph of minimum degree at least δ and v a vertex of G. Then*

$$|N_{\leq 2}(v)| \geq \deg(v)\delta - \deg(v) + \varepsilon_{\deg(v)} + 1,$$

where

$$\varepsilon_{\deg(v)} = \begin{cases} 0 & \text{if } \deg(v) \text{ is even,} \\ 1 & \text{if } \deg(v) \text{ is odd.} \end{cases}$$

Proof. Let $v \in V(G)$. Clearly, $|N_{\leq 2}(v)| = |N_0(v)| + |N_1(v)| + |N_2(v)|$ and since G is a graph containing no 4-cycle, each vertex at distance 1 from v can only have one neighbour in $N_1(v)$, otherwise G would contain a 4-cycle. So each vertex in $N_1(v)$ has at least $\delta - 2$ neighbours in $N_2(v)$. Moreover, no two vertices in $N_1(v)$ have a common neighbour in

$N_2(v)$, otherwise v, together with its two neighbours in $N_1(v)$ and their common neighbour in $N_2(v)$ will then form a 4-cycle contradicting the fact that G is C_4-free.

Based on the above established fact, we now give a bound on $|N_{\leq 2}(v)|$ by considering different cases for the degree of v.

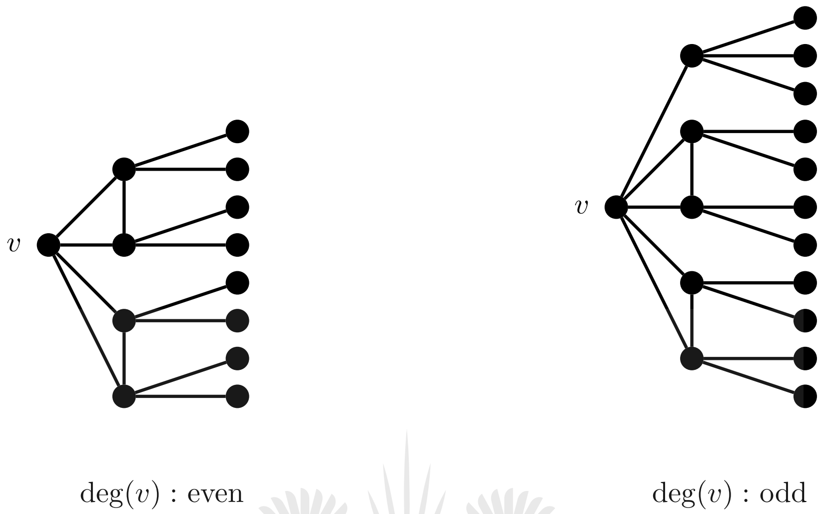

Figure 2.1: Illustration to show vertices within distance two from a vertex.

If $\deg(v)$ is even, we have that

$$\begin{aligned} |N_{\leq 2}(v)| &\geq 1 + \deg(v) + \deg(v)(\delta - 2) \\ &= \deg(v)\delta - \deg(v) + 1. \end{aligned} \tag{2.2.1}$$

Note that if $\deg(v)$ is odd, then it follows from Lemma 1.1.28 that there is a vertex in $N_1(v)$ that is not adjacent to any other vertex in $N_1(v)$, so this vertex has at least $\delta - 1$ neighbours in $N_2(v)$. Hence, we have that

$$\begin{aligned} |N_{\leq 2}(v)| &\geq 1 + \deg(v) + (\deg(v) - 1)(\delta - 2) + (\delta - 1) \\ &= \deg(v)\delta - \deg(v) + 2. \end{aligned} \tag{2.2.2}$$

Therefore,

$$\begin{aligned} |N_{\leq 2}(v)| &\geq \deg(v)\delta - 2\lfloor \deg(v)/2 \rfloor + 1 \\ &= \deg(v)\delta - \deg(v) + \varepsilon_{\deg(v)} + 1. \end{aligned} \tag{2.2.3}$$

□

We now give a bound by [51] on the diameter of a C_4-free graph. Since the idea of the proof of our improvement for graphs of girth at least 6 is based on the proof of this result, we present their proof in full with some elaboration.

Theorem 2.2.2. *[51] Let G be a connected C_4-free graph of order n and minimum degree $\delta \geq 2$, then*

$$(i)\ \mathrm{diam}(G) \leq \frac{5(n-2\delta)}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1} - 1. \tag{2.2.4}$$

$$(ii)\ \mathrm{rad}(G) \leq \frac{5n}{2(\delta^2 - 2\lfloor \delta/2 \rfloor + 1)} + \frac{11}{2}. \tag{2.2.5}$$

Proof. (i) Let u_0 and u_d be two vertices at distance $d = \mathrm{diam}(G)$ and $u_0u_1u_2\ldots u_d$ be a shortest (u_0, u_d)-path of length in G.

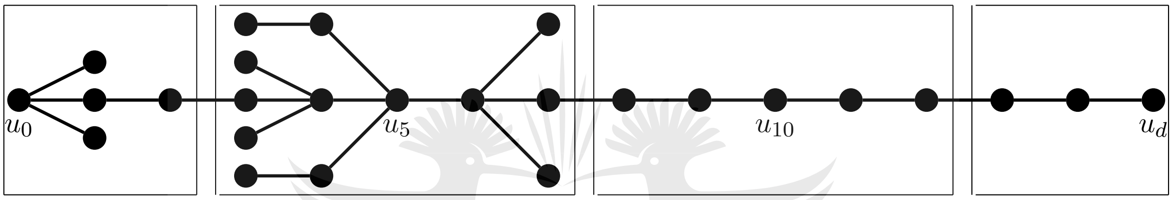

Figure 2.2: Illustration to show the (u_0, u_d)-path of length d in a graph.

By Lemma 2.2.1 we have $|N_{\leq 2}(u)| \geq \deg(u)\delta - \deg(u) + \varepsilon_{\deg(u)} + 1$. Since $\deg(u) \geq \delta$, this implies

$$|N_{\leq 2}(u)| \geq \delta^2 - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1 \quad \text{for every} \quad u \in V(G). \tag{2.2.6}$$

For $i \in \{0, 1, \ldots, d\}$, we have that $N_{\leq 2}(u_i) \subseteq N_{i-2} \cup N_{i-1} \cup N_i \cup N_{i+1} \cup N_{i+2}$. Hence by (2.2.6) we have that

$$|N_{i-2}| + |N_{i-1}| + |N_i| + |N_{i+1}| + |N_{i+2}| \geq \delta^2 - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1. \tag{2.2.7}$$

In view of the fact that $N_{\leq 2}(u_{5i}) \cap N_{\leq 2}(u_{5j}) = \emptyset$ for all $0 \leq i \neq j \leq d/5$, we let $k = \lfloor \frac{d}{5} \rfloor - 1$ and consider the distinguishing cases according to the residue class of $d \mod 5$. In our bounds below, we also make use of the fact that $|N_{i-1}| + |N_i| + |N_{i+1}| \geq \delta + 1$ for all $i \in \{0, \ldots, d\}$. Clearly $n = \sum_{i=0}^{d} |N_i|$.

If $d \equiv 0 \pmod 5$, we have

$$\begin{aligned} n &= \sum_{i=0}^{k} \big(|N_{5i-2}| + |N_{5i-1}| + |N_{5i}| + |N_{5i+1}| + |N_{5i+2}|\big) + |N_{d-2}| + |N_{d-1}| + |N_d| \\ &\geq (k+1)\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + \big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big), \\ &= (k+2)\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big), \\ &= \Big(\Big\lfloor\frac{d}{5}\Big\rfloor + 1\Big)\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big). \end{aligned}$$

If $d \equiv 1 \pmod 5$, we have

$$\begin{aligned} n &= \sum_{i=0}^{k} \big(|N_{5i-2}| + |N_{5i-1}| + |N_{5i}| + |N_{5i+1}| + |N_{5i+2}|\big) + |N_{d-3}| + |N_{d-2}| + |N_{d-1}| + |N_d| \\ &\geq (k+1).\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + 1 + \big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + 1, \\ &= (k+2)\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + 1, \\ &= \Big(\Big\lfloor\frac{d}{5}\Big\rfloor + 1\Big)\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + 1. \end{aligned}$$

If $d \equiv 2 \pmod 5$, we have

$$\begin{aligned} n &= \sum_{i=0}^{k} \big(|N_{5i-2}| + |N_{5i-1}| + |N_{5i}| + |N_{5i+1}| + |N_{5i+2}|\big) + |N_{d-4}| + |N_{d-3}| + \\ &\quad |N_{d-2}| + |N_{d-1}| + |N_d|, \\ &\geq (k+1).\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + 2 + \big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big), \\ &= (k+2).\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + 2, \\ &= \Big(\Big\lfloor\frac{d}{5}\Big\rfloor + 1\Big)\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + 2. \end{aligned}$$

If $d \equiv 3 \pmod 5$, we have

$$\begin{aligned} n &= \sum_{i=0}^{k} \big(|N_{5i-2}| + |N_{5i-1}| + |N_{5i}| + |N_{5i+1}| + |N_{5i+2}|\big) + |N_{d-5}| + |N_{d-4}| + |N_{d-3}| \\ &\quad + |N_{d-2}| + |N_{d-1}| + |N_d|, \\ &\geq (k+1).\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + (\delta + 1) + \big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big), \\ &= (k+2).\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + \delta + 1, \\ &= \Big(\Big\lfloor\frac{d}{5}\Big\rfloor + 1\Big)\big(\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1\big) + \delta + 1. \end{aligned}$$

If $d \equiv 4 \pmod 5$, we have

$$\begin{aligned} n &= \sum_{i=0}^{k} \big(|N_{5i-2}| + |N_{5i-1}| + |N_{5i}| + |N_{5i+1}| + |N_{5i+2}|\big) + |N_{d-6}| + |N_{d-5}| + |N_{d-4}| + \\ &\quad |N_{d-3}| + |N_{d-2}| + |N_{d-1}| + |N_d|, \\ &\geq (k+1).\big(\delta^2 - 2\big\lfloor\frac{\delta}{2}\big\rfloor + 1\big) + 2\delta + \big(\delta^2 - 2\big\lfloor\frac{\delta}{2}\big\rfloor + 1\big), \\ &= (k+2).\big(\delta^2 - 2\big\lfloor\frac{\delta}{2}\big\rfloor + 1\big) + 2\delta, \\ &= \Big(\Big\lfloor\frac{d}{5}\Big\rfloor + 1\Big)\big(\delta^2 - 2\big\lfloor\frac{\delta}{2}\big\rfloor + 1\big) + 2\delta. \end{aligned}$$

From the above five cases, we conclude that

$$n = \sum_{i=0}^{d} |N_i| \geq \Big(\Big\lfloor\frac{d}{5}\Big\rfloor + 1\Big)\big(\delta^2 - 2\big\lfloor\frac{\delta}{2}\big\rfloor + 1\big) + \begin{cases} 0 & \text{if } d \equiv 0 (\text{mod } 5), \\ 1 & \text{if } d \equiv 1 (\text{mod } 5), \\ 2 & \text{if } d \equiv 2 (\text{mod } 5), \\ \delta + 1 & \text{if } d \equiv 3 (\text{mod } 5), \\ 2\delta & \text{if } d \equiv 4 (\text{mod } 5). \end{cases} \tag{2.2.8}$$

This implies that

$$n \geq \begin{cases} \big(\frac{d+5}{5}\big)\big(\delta^2 - 2\big\lfloor\frac{\delta}{2}\big\rfloor + 1\big) & \text{if } d \equiv 0 (\text{mod } 5), \\ \big(\frac{d+4}{5}\big)\big(\delta^2 - 2\big\lfloor\frac{\delta}{2}\big\rfloor + 1\big) + 1 & \text{if } d \equiv 1 (\text{mod } 5), \\ \big(\frac{d+3}{5}\big)\big(\delta^2 - 2\big\lfloor\frac{\delta}{2}\big\rfloor + 1\big) + 2 & \text{if } d \equiv 2 (\text{mod } 5), \\ \big(\frac{d+2}{5}\big)\big(\delta^2 - 2\big\lfloor\frac{\delta}{2}\big\rfloor + 1\big) + \delta + 1 & \text{if } d \equiv 3 (\text{mod } 5), \\ \big(\frac{d+1}{5}\big)\big(\delta^2 - 2\big\lfloor\frac{\delta}{2}\big\rfloor + 1\big) + 2\delta & \text{if } d \equiv 4 (\text{mod } 5). \end{cases} \tag{2.2.9}$$

and thus

$$d \leq \begin{cases} \frac{5n}{\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1} - 5 & \text{if } d \equiv 0 (\text{mod } 5), \\ \frac{5(n-1)}{\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1} - 4 & \text{if } d \equiv 1 (\text{mod } 5), \\ \frac{5(n-2)}{\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1} - 3 & \text{if } d \equiv 2 (\text{mod } 5), \\ \frac{5(n-\delta-1)}{\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1} - 2 & \text{if } d \equiv 3 (\text{mod } 5), \\ \frac{5(n-2\delta)}{\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1} - 1 & \text{if } d \equiv 4 (\text{mod } 5). \end{cases} \tag{2.2.10}$$

Therefore

$$d \leq \frac{5(n - 2\delta)}{\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1} - 1, \tag{2.2.11}$$

and this yields inequality (2.2.4).

The proof of (ii) follows essentially the same way as that of part (ii) of Theorem 1.4.11. (ii) Let u be a centre vertex of G and let $N_i = N_i(u)$. For any $x \in N_i$, pick a vertex $x' \in N_{i-1}$ such that $xx' \in E(G)$ $(1 \le i \le r)$. The collection of the edges of the form $\{xx' : x \in V(G) - \{u\}\}$ defines a distance preserving spanning tree, $T \le G$, from u, i.e., $d_T(u,v) = d_G(u,v)$ for all $v \in V(G)$.

For $v \in V(G)$, denote the (u,v)-path in T by $T(u,v)$. Fix a vertex $v' \in N_r$. We say that a vertex $v'' \in V(G)$ is related to v' if there exists $\overline{v}' \in V(T(u,v')) \cap N_{\ge 9}$ and $\overline{v}'' \in V(T(u,v'')) \cap N_{\ge 9}$ such that

$$d_G(\overline{v}', \overline{v}'') \le 4. \tag{2.2.12}$$

By Lemma 1.4.10, there is a vertex $w \in N_{\ge r-9}$ that is not related to v'.

For any $i \in \mathbb{Z}$, let D_i' and D_i'' denote the set of all vertices in N_i whose distance from at least one vertex of $T(u,v') \cap N_{\ge 9}$ $(T(u,w) \cap N_{\ge 9}$ respectively) is at most 2 in G. By our assumption that v' and w are not related, we have that

$$\Big(\bigcup_{i=7}^{r} D_i'\Big) \bigcap \Big(\bigcup_{i=7}^{r} D_i''\Big) = \emptyset.$$

Let $s = d_G(u,w)$. Let $u_0, u_1, u_2, \ldots, u_r$ be a shortest (u,v')-path and let $w_0, w_1, w_2, \ldots, w_s$ be a shortest (u,w)-path in T. Since $N_{\le 2}(u_i) \subseteq D_{i-2}' \cup D_{i-1}' \cup D_i' \cup D_{i+1}' \cup D_{i+2}'$, and $N_{\le 2}(w_i) \subseteq D_{i-2}'' \cup D_{i-1}'' \cup D_i'' \cup D_{i+1}'' \cup D_{i+2}''$, we have by the condition on $|N_{\le 2}(u)|$ in 2.2.6 that

$$|D_{i-2}'| + |D_{i-1}'| + |D_i'| + |D_{i+1}'| + |D_{i+2}'| \ge \delta^2 - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1 \quad \forall i \in \{9, 10 \ldots r, \} \tag{2.2.13}$$

$$|D_{i-2}''| + |D_{i-1}''| + |D_i''| + |D_{i+1}''| + |D_{i+2}''| \ge \delta^2 - 2\left\lfloor \frac{\delta}{2} \right\rfloor + 1 \quad \forall i \in \{9, 10, \ldots s, \} \tag{2.2.14}$$

where $s \ge r - 9$.

Similarly to (1.4.20), we now obtain that

$$n = \sum_{i=0}^{r} |N_i| = \sum_{i=0}^{7} |N_i| + \sum_{i=8}^{r} |N_i| \ge \ |N_{\le 7}| + \sum_{i=8}^{r} |D_i'| + \sum_{i=8}^{s+2} |D_i''|. \tag{2.2.15}$$

Since $s \ge r - 9$, we have that

$$\begin{aligned} n &\ge \sum_{i=0}^{7} |N_i| + \sum_{i=8}^{r} |D_i'| + \sum_{i=8}^{r-7} |D_i''|, \\ &\ge \sum_{i=0}^{2} |N_i| + \sum_{i=3}^{7} |N_i| + \sum_{i=8}^{r-3} |D_i'| + \sum_{r-2}^{r} |D_i'| + \sum_{i=8}^{r-10} |D_i''| + \sum_{r-9}^{r-7} |D_i''|. \end{aligned}$$

Let k be the largest integer with $k \leq r$ and $k \equiv 2 \pmod 5$. Then $k \geq r-4$ and

$$\begin{aligned} n &\geq \sum_{i=0}^{2}|N_i| + \sum_{i=3}^{7}|N_i| + \sum_{i=8}^{k-3}|D'_i| + \sum_{k-2}^{k}|D'_i| + \sum_{i=8}^{k-10}|D''_i| + \sum_{k-9}^{k-7}|D''_i|, \\ &\geq \sum_{i=0}^{2}|N_i| + \sum_{i=3}^{7}|N_i| + \sum_{i=2}^{\frac{k-5}{5}}\left(|D'_{5i-2}| + |D'_{5i-1}| + |D'_{5i}| + |D'_{5i+1}| + |D'_{5i+2}|\right) + \sum_{i=k-2}^{k}|D'_i| \\ &\quad + \sum_{i=2}^{\frac{k-12}{5}}\left(|D''_{5i-2}| + |D''_{5i-1}| + |D''_{5i}| + |D''_{5i+1}| + |D''_{5i+2}|\right) + \sum_{i=k-9}^{k-7}|D''_i|, \end{aligned}$$

$$\begin{aligned} n &\geq 2\left(\delta^2 - 2\left\lfloor\frac{\delta}{2}\right\rfloor + 1\right) + \left(\frac{k-5}{5}\right)\left(\delta^2 - 2\left\lfloor\frac{\delta}{2}\right\rfloor + 1\right) - \left(\delta^2 - 2\left\lfloor\frac{\delta}{2}\right\rfloor + 1\right), \\ &\quad + \left(\delta^2 - 2\left\lfloor\frac{\delta}{2}\right\rfloor + 1\right) + \left(\frac{k-12}{5}\right)\left(\delta^2 - 2\left\lfloor\frac{\delta}{2}\right\rfloor + 1\right) - \left(\delta^2 - 2\left\lfloor\frac{\delta}{2}\right\rfloor + 1\right) + \left(\delta^2 - 2\left\lfloor\frac{\delta}{2}\right\rfloor + 1\right), \\ &= \left(2 + \frac{2k-17}{5}\right)\left(\delta^2 - 2\left\lfloor\frac{\delta}{2}\right\rfloor + 1\right), \\ &= \left(\frac{2k-7}{5}\right)\left(\delta^2 - 2\left\lfloor\frac{\delta}{2}\right\rfloor + 1\right), \\ &\geq \left(\frac{2r-11}{5}\right)\left(\delta^2 - 2\left\lfloor\frac{\delta}{2}\right\rfloor + 1\right), \end{aligned}$$

with the last inequality holding since $k \geq r-4$, and so

$$\frac{n}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1} \geq \frac{2r-11}{5}.$$

Therefore,

$$r \leq \frac{5n}{2(\delta^2 - 2\lfloor \delta/2 \rfloor + 1)} + \frac{11}{2}.$$

Hence, the inequality (2.2.5) holds. □

We now describe the following graph H_q constructed by [12] and [52] since it will be useful later.

Example 2.2.3. *Let q be a prime power and let $GF(q)^3$ be a 3-dimensional vector space over the finite field $GF(q)$ of order q. Let H_q be the graph whose vertices are the 1-dimensional subspaces of $GF(q)^3$. Denote the subspace generated by $\underline{x}$, as $\langle \underline{x} \rangle$. Two vertices, $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$, are said to be adjacent in H_q if they are orthogonal, i.e., $\underline{x} \cdot \underline{y} = 0$ where $\underline{x} \cdot \underline{y}$ denotes the dot product.*

Claim 2.2.4. *H_q is C_4-free.*

Proof. Let $\langle \underline{a} \rangle, \langle \underline{b} \rangle$ be distinct 1-dimensional subspaces of $GF(q)^3$ representing vertices in H_q. The following subclaims hold for any two vertices of H_q.

(a) If $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ are adjacent in H_q, then at most one of $\underline{a}$ and $\underline{b}$ is self-orthogonal.

(b) If $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ are adjacent in H_q, and one of $\underline{a}$ and $\underline{b}$ is self orthogonal, then $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ have no common neighbour in H_q.

(c) If $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ are adjacent in H_q and both $\underline{a}$ and $\underline{b}$ are not self-orthogonal, then $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ have exactly one common neighbour in H_q.

(d) If $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ are not adjacent in H_q, then they have exactly one common neighbour.

We have previously shown in Theorem 1.2.11 that if X is a subset of $GF(q)^n$, then $X^\perp$ is a subspace of $GF(q)^n$.

Let $X = \{\underline{a}, \underline{b}\}$ and $X^\perp = \langle \underline{a}, \underline{b} \rangle^\perp = \langle \underline{c} \rangle$ for some $\underline{c} \in GF(q)^3$ such that $\underline{c}$ is not the zero vector. Recall from Theorem 1.2.11 that the orthogonal complement of a k-dimensional subspace of a vector space of dimension n has dimension $n - k$. Therefore, since X has dimension 2, we have that $X^\perp$ has dimension $3 - 2 = 1$. The following two observations will be used in the proving the above subclaims.

i) If $\langle \underline{c} \rangle = \langle \underline{a} \rangle$ or $\langle \underline{c} \rangle = \langle \underline{b} \rangle$, then one of $\underline{a}$ or $\underline{b}$ is self-orthogonal. Hence, $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ have no common neighbour in H_q. If the definition of H_q allowed loops, then the self-orthogonal vertex would be the common neighbour of $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$.

ii) If $\langle \underline{c} \rangle \neq \langle \underline{a} \rangle$ and $\langle \underline{c} \rangle \neq \langle \underline{b} \rangle$, then $\underline{c}$ is orthogonal to both $\underline{a}$ and $\underline{b}$. Hence $\langle \underline{c} \rangle$ is the unique neighbour of $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ in H_q. Note that $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ can be adjacent or not adjacent in H_q.

Next, we give a justification to the above **subclaims (a) to (d)**.

subclaim (a): Since $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ are adjacent vertices in H_q, we have $\underline{a} \cdot \underline{b} = 0$. Suppose to the contrary that both are self-orthogonal, then $\underline{a} \cdot \underline{a} = 0$ and $\underline{b} \cdot \underline{b} = 0$. Thus $\langle \underline{a} \rangle, \langle \underline{b} \rangle \subseteq X^\perp$. This implies that $X^\perp$ has dimension at least 2, a contradiction since $X^\perp$ is 1-dimensional subspace of $GF(q)^3$. Hence at most one of $\underline{a}$ and $\underline{b}$ is self orthogonal if both are adjacent in H_q.

subclaim (b): $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ are adjacent in H_q. Without loss of generality, let one of them, say $\underline{a}$, be self-orthogonal. Then $\langle \underline{a} \rangle \subseteq X^\perp = \langle \underline{a}, \underline{b} \rangle^\perp$. Therefore $\langle \underline{a} \rangle = \langle \underline{c} \rangle$ since $X^\perp = \langle \underline{c} \rangle$. Hence $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ have no common neighbour in H_q since loops are not defined in H_q.

subclaim (c): Since $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ are adjacent vertices in H_q but are not self orthogonal, we have that $\langle \underline{a} \rangle, \langle \underline{b} \rangle \nsubseteq X^\perp$. This implies that $\langle \underline{c} \rangle \neq \langle \underline{a} \rangle$ and $\langle \underline{c} \rangle \neq \langle \underline{b} \rangle$. Hence, $\langle \underline{c} \rangle$ is the unique neighbour common to both $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ in H_q.

subclaim (d): Since $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ are not adjacent in H_q, $\underline{a} \cdot \underline{b} \neq 0$ and $\langle \underline{a} \rangle, \langle \underline{b} \rangle \nsubseteq X^\perp$ since they are non-adjacent in H_q. Thus $\langle \underline{c} \rangle \neq \langle \underline{a} \rangle$ and $\langle \underline{c} \rangle \neq \langle \underline{b} \rangle$. Hence, $\langle \underline{a} \rangle$ and $\langle \underline{b} \rangle$ have exactly one common neighbour, $\langle \underline{c} \rangle$, in H_q.

It follows from the above subclaims a) to d) that any two vertices of H_q have at most one common neighbour. Hence H_q is C_4-free and so Claim 2.2.4 holds. $\square$

Claim 2.2.5. *(a) H_q has q^2+q+1 vertices.*

(b) For each vertex $\langle \underline{v} \rangle$ in $V(H)$, the degree of $\langle \underline{v} \rangle$, $\deg_{H_q}(\langle \underline{v} \rangle) = q$ if $\underline{v}$ is self-orthogonal and $\deg_{H_q}(\langle \underline{v} \rangle) = q+1$ otherwise.

(c) There exists a self-orthogonal vertex, say $\langle \underline{z} \rangle$ in H_q.

(d) No two neighbours of a self-orthogonal vertex in H_q are adjacent.

Proof. (a) Since H_q is the graph whose vertices are the 1-dimensional subspaces of $GF(q)^3$, we have by Theorem 1.2.20, that H_q has q^2+q+1 vertices.

(b) Since $\langle \underline{v} \rangle$, a 1-dimensional subspace of $GF(q)^3$, is a vertex of H_q, we have by Theorem 1.2.11, that $\langle \underline{v} \rangle^{\perp}$, its orthogonal complement, is a 2-dimensional subspace of $GF(q)^3$. Hence $\langle \underline{v} \rangle^{\perp}$ has q^2 vectors and q^2-1 non-zero vectors. Thus the number of 1-dimensional subspaces of $\langle \underline{v} \rangle^{\perp}$ is $\frac{q^2-1}{q-1} = (q+1)$. And so, if $\langle \underline{v} \rangle$ is self orthogonal, then $\deg_{H_q}(\langle \underline{v} \rangle) = q$. Otherwise $\deg_{H_q}(\langle \underline{v} \rangle) = q+1$.

(c) Recall from Claim 1.2.15 that there exists a non-zero self-orthogonal vector $\underline{z}$ in $GF(q)^3$ and since H_q is a graph whose vertices are the 1-dimensional subspaces of $GF(q)^3$ generated by a non-zero vector. Then, we have that there is one of the 1-dimensional subspaces of $GF(q)^3$ generated by $\underline{z}$. Hence H_q contains a self orthogonal vertex.

(d) Let $\langle \underline{z} \rangle$ be the self orthogonal vertex in H_q and let $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ be neighbours of $\langle \underline{z} \rangle$. Clearly, (d) follows directly from **subclaim (b)** of Claim 2.2.4. If $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ were adjacent, then $\langle \underline{z} \rangle$ and $\langle \underline{x} \rangle$ would have a common neighbour, which is a contradiction to **subclaim (b)** of Claim 2.2.4. $\square$

From here onwards, we fix a self-orthogonal vertex in H_q, $\langle \underline{z} \rangle$, and two of its neighbours $\langle \underline{x} \rangle, \langle \underline{y} \rangle$.

Let $\langle \underline{x}_0 \rangle = \langle \underline{z} \rangle, \langle \underline{x}_1 \rangle, \langle \underline{x}_2 \rangle, \ldots, \langle \underline{x}_q \rangle$ and $\langle \underline{y}_0 \rangle = \langle \underline{z} \rangle, \langle \underline{y}_1 \rangle, \langle \underline{y}_2 \rangle, \ldots, \langle \underline{y}_q \rangle$ denote the neighbours of $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ in H_q respectively.

Since $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ are neighbours of a self-orthogonal vertex, we have by Claim 2.2.5(b) that $\deg(\langle \underline{x} \rangle) = \deg(\langle \underline{y} \rangle) = q+1$ since by **subclaim (b)** of Claim 2.2.4, $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ cannot be self-orthogonal. Moreover, by **subclaim (c)** of Claim 2.2.4, $\langle \underline{x} \rangle$ and $\langle \underline{z} \rangle$ have no common neighbour in H_q and similarly $\langle \underline{y} \rangle$ and $\langle \underline{z} \rangle$ have no common neighbour in H_q.

We now prove the following claim.

Claim 2.2.6. *For $1 \leq i \leq q$,*

(a) $\langle \underline{z} \rangle$ is not adjacent to any $\langle \underline{x}_i \rangle$ or $\langle \underline{y}_i \rangle$.

(b) There is a uniquely determined j_i $(1 \leq j_i \leq q)$ *such that* $\langle \underline{x}_i \rangle \langle \underline{y}_{j_i} \rangle \in E(H)$ *for every* i *with* $1 \leq i \leq q$.

(c) No $\langle \underline{x}_i \rangle$ *is adjacent to* $\langle \underline{y} \rangle$ *and no* $\langle \underline{y}_i \rangle$ *is adjacent to* $\langle \underline{x} \rangle$ *in* H_q.

Proof. (a) Suppose that $\langle \underline{x}_i \rangle$ is adjacent to $\langle \underline{z} \rangle$ in H_q. By our assumption, $\langle \underline{z} \rangle$ is a self-orthogonal vertex and is also adjacent to $\langle \underline{x} \rangle$ in H_q. We now have that both $\langle \underline{x} \rangle$ and $\langle \underline{x}_i \rangle$ are neighbours of a self-orthogonal vertex. But $\langle \underline{x} \rangle$ and $\langle \underline{x}_i \rangle$ are adjacent in H_q, a contradiction to Claim 2.2.5(d). Hence, no $\langle \underline{x}_i \rangle$ is adjacent to $\langle \underline{z} \rangle$ in H_q for $1 \leq i \leq q$. Similarly no $\langle \underline{y}_i \rangle$ is adjacent to $\langle \underline{z} \rangle$ in H_q since the proof is analogous.

(b) Since each $\langle \underline{x}_i \rangle$ and $\langle \underline{y} \rangle$ are not adjacent in H_q, it follows from **subclaim (d)** of Claim 2.2.4 that each $\langle \underline{x}_i \rangle$ and $\langle \underline{y} \rangle$ have exactly one common neighbour. We show that the common neighbour is not $\langle \underline{y}_0 \rangle = \langle \underline{z} \rangle$. By a), $\langle z \rangle$ is not adjacent to $\langle x_i \rangle$. Hence, $\langle \underline{x}_i \rangle$ and $\langle \underline{y} \rangle$ cannot have $\langle \underline{y}_0 \rangle$ as a common neighbour. It immediately follows from above that there exists j_i with $1 \leq j_i \leq q$ such that $\langle \underline{y}_{j_i} \rangle$ is the common neighbour of $\langle \underline{x}_i \rangle$ and $\langle \underline{y} \rangle$.

(c) Suppose $\langle \underline{x}_i \rangle$ is adjacent to $\langle \underline{y} \rangle$ in H_q. Then, H_q contains a C_4, vis $\langle \underline{z} \rangle, \langle \underline{x} \rangle, \langle \underline{x}_i \rangle, \langle \underline{y} \rangle$, a contradiction to Claim 2.2.4. Hence, no $\langle \underline{x}_i \rangle$ is a adjacent to $\langle \underline{y} \rangle$ in H_q. Similarly no $\langle \underline{y}_i \rangle$ is a adjacent to $\langle \underline{x} \rangle$ in H_q. This proves Claim 2.2.6. □

Let H_0 be the graph obtained from H_q by removing the self-orthogonal vertex $\langle \underline{z} \rangle$ and all edges of the form $\langle \underline{x}_i \rangle \langle \underline{y}_{j_i} \rangle$ for $1 \leq i \leq q$. We claim that $d_{H_0}(\langle \underline{x} \rangle, \langle \underline{y} \rangle) \geq 4$.

Claim 2.2.7. *If* $\langle \underline{z} \rangle$*, a self-orthogonal vertex, and two of its neighbours,* $\langle \underline{x} \rangle$ *and* $\langle \underline{y} \rangle$ *are fixed in* H_q*, then* $d_{H_0}(\langle \underline{x} \rangle, \langle \underline{y} \rangle) \geq 4$.

Proof. Recall that $d_{H_q}(\langle \underline{x} \rangle, \langle \underline{y} \rangle) = 2$ since both $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ share a common neighbour, $\langle \underline{z} \rangle$. Since $\langle \underline{z} \rangle$ and and two of its neighbours $\langle \underline{x} \rangle, \langle \underline{y} \rangle$ were fixed in H_q, removal of $\langle \underline{z} \rangle$ in H_0 implies that there is no $(\langle \underline{x} \rangle, \langle \underline{y} \rangle)$-path of length 2 in H_0, thus $d_{H_0}(\langle \underline{x} \rangle, \langle \underline{y} \rangle) \geq 3$. Moreover, removal of all edges of the form $\langle \underline{x}_i \rangle \langle \underline{y}_{j_i} \rangle$ for $1 \leq i \leq q$ destroys all $(\langle \underline{x} \rangle, \langle \underline{y} \rangle)$-path of length 3 in H_0. Hence, $d_{H_0}(\langle \underline{u} \rangle, \langle \underline{v} \rangle) \geq 4$. □

Since the degree of each vertex in H_q is either q or $q+1$, H_0 has minimum degree $\delta = q-1$ since the removal of $\langle \underline{z} \rangle$ and all edges of the form $\langle \underline{x}_i \rangle \langle \underline{y}_{j_i} \rangle$ for $1 \leq i \leq q$ will reduce the degree of all neighbours of $\langle \underline{z} \rangle$ and the degree of all vertices associated with the removed edges, $\langle \underline{x}_i \rangle \langle \underline{y}_{j_i} \rangle$ by 1. We note that the degree of these vertices that were affected by the edge-removal cannot be reduced by 2 since they are not adjacent to $\langle \underline{z} \rangle$, the self orthogonal vertex. Thus, since some of these vertices that their degrees were affected are either a self-orthogonal vertex or not, we have that the degree of vertices of H_0 is either $q-1$ or q. The number of vertices in H_0 is q^2+q and the number of edges in H_0 will be reduced by $2q$.

Theorem 2.2.8. *[51] If $\delta+1$ is a prime power, then there exists an infinite family of C_4-free graphs G of order n and minimum degree δ such that*

$$(i)\ \operatorname{diam}(G) \geq \frac{5n}{\delta^2+3\delta+2} - 1. \tag{2.2.16}$$

$$(ii)\ \operatorname{rad}(G) \geq \frac{5n}{2(\delta^2+3\delta+2)} - \frac{1}{2}. \tag{2.2.17}$$

Proof. Let $q = \delta + 1$ be prime power and H_0 be the graph constructed above. Let $G_{k,\delta}$ be the graph obtained from the union of k disjoint copies, $H_0^1, H_0^2, \ldots, H_0^k$, of H_0 by adding the following edges $\langle \underline{y}^t\rangle\langle \underline{x}^{t+1}\rangle$ for every $1 \leq t < k$, where $\langle \underline{x}^t\rangle, \langle \underline{y}^t\rangle \in V(H_0^t)$ are the corresponding vertices to $\langle \underline{x}\rangle, \langle \underline{y}\rangle \in V(H_0)$. Clearly, $\langle \underline{y}^t\rangle \in N_{G_{k,\delta}}[\langle \underline{x}^{t+1}\rangle]$ for every $1 \leq t < k$ and the addition of the edges $\langle \underline{y}^t\rangle\langle \underline{x}^{t+1}\rangle$ does not create any 4-cycle since no two neighbours of $\langle \underline{x}^{t+1}\rangle$ share two common neighbours. Since H_0 is obtained from H_q, we have by Claim 2.2.4 that $G_{k,\delta}$ is C_4-free. Moreover, since H_0 has q^2+q vertices, there are $k(q^2+q)$ vertices in $G_{k,\delta}$. The minimum degree of $G_{k,\delta}$ is $\delta = q-1$ since the minimum degree of vertices of H_0 is $\delta = q-1$. The order of $G_{k,\delta}$, is given by

$$\begin{aligned} |V(G_{k,\delta})| = n &= k(q^2+q) \\ &= k[(\delta+1)^2 + (\delta+1)] \\ &= k(\delta^2+3\delta+2). \end{aligned}$$

By Claim 2.2.7, $d_{H_0}(\langle \underline{x}\rangle, \langle \underline{y}\rangle) \geq 4$, and so we have that $\operatorname{diam}(H_0) \geq 4$. Thus,

$$\begin{aligned} \operatorname{diam}(G_{k,\delta}) &\geq k.\operatorname{diam}(H_0) + k - 1 \\ &\geq 4k + k - 1 \\ &= 5k - 1 \\ &= \frac{5n}{\delta^2+3\delta+2} - 1. \end{aligned}$$

Part (ii) follows from the fact that

$$\operatorname{rad}(G_{k,\delta}) \geq \frac{1}{2}\operatorname{diam}(G_{k,\delta}) \geq \frac{5n}{2(\delta^2+3\delta+2)} - \frac{1}{2}$$

□

The graph $G_{k,\delta}$ demonstrates that the bound in Theorem 2.2.2 is not far from, best possible. Indeed, as δ gets large, then the ratio of the coefficient of n in the bound in Theorem 2.2.2, and the coefficient of n in the diameter of $G_{k,\delta}$ equals $\{\frac{5}{\delta^2-2\lfloor\delta/2\rfloor+1} / \frac{5}{\delta^2+3\delta+2}\}$ which tends to 1 as δ gets large.

2.3 Main Results

2.3.1 Bounds on Diameter and Radius of Graphs of Girth at least 6

In this subsection we improve the bounds on the diameter and radius of a connected C_4-free graph in Theorem 2.2.2 under the additional assumption that the graph also has girth at least 6.

Lemma 2.3.1 and 2.3.2 are essentially the well-known Moore bounds on the order of a graph of given minimum degree δ and girth at least 5 or 6, respectively. Since Lemma 2.3.1 takes into account the degree of a given vertex and is thus very slightly more general than the Moore bound, we give a proof. The proof of Lemma 2.3.2 is well-known, and it is almost identical to Case 1 in the proof of Lemma 2.3.9.

Lemma 2.3.1. *Let G be a graph of girth at least 5 and $v \in V(G)$. If every vertex in $V(G) - \{v\}$ has degree at least $\delta \geq 3$, then*

$$|N_{\leq 2}(v)| \geq 1 + \deg(v)\delta.$$

Proof. There are $\deg(v)$ vertices at distance exactly 1 from v. Each vertex in $N(v)$ has only one neighbour in $N_{\leq 1}(v)$ since otherwise G would contain C_3. Hence each vertex in $N(v)$ has at least $\delta - 1$ neighbours in $N_2(v)$. Moreover, the sets $N(x) \cap N_2(v)$, $x \in N(v)$, are pairwise disjoint, otherwise G would contain C_4. Hence $|N_2(v)| \geq \deg(v)(\delta - 1)$. Since $|N_{\leq 2}(v)| = 1 + \deg(v) + |N_2(v)|$, the proposition follows. □

Lemma 2.3.2. *Let G be a graph of minimum degree $\delta \geq 3$ and girth at least 6. If u and v are adjacent vertices of G, then*

$$|N_{\leq 2}(u) \cup N_{\leq 2}(v)| \geq 2(\delta^2 - \delta + 1).$$

Proof. We have by Lemma 2.3.1 that $|N_{\leq 2}(u)| \geq \deg_G(u)\delta + 1$ and $|N_{\leq 2}(v)| \geq \deg_G(v)\delta + 1$ and by the inclusion-exclusion principle, that

$$|N_{\leq 2}(u) \cup N_{\leq 2}(v)| = |N_{\leq 2}(u)| + |N_{\leq 2}(v)| - |N_{\leq 2}(u) \cap N_{\leq 2}(v)|.$$

We claim that $N_{\leq 2}(u) \cap N_{\leq 2}(v) = N(u) \cup N(v)$. Clearly, $N(u) \cup N(v) \subseteq N_{\leq 2}(u) \cap N_{\leq 2}(v)$ as $N(u) = N_1(u)$. Conversely, let $y \in N_{\leq 2}(u) \cap N_{\leq 2}(v)$. Thus we have that $y \in N_0(u) \cap N_1(v)$, or $y \in N_1(u) \cap N_2(v)$, or $y \in N_0(v) \cap N_1(u)$, or $y \in N_1(v) \cap N_2(u)$ since G has girth at least 6. In either case, we get $y \in N(u) \cup N(v)$. By Lemma 2.3.1 and using the fact that

$N(u) \cap N(v) = \emptyset$, it follows that

$$\begin{aligned} |N_{\leq 2}(u) \cap N_{\leq 2}(v)| &= |N_{\leq 2}(u)| + |N_{\leq 2}(v)| - |N_{\leq 2}(u) \cap N_{\leq 2}(v)|, \\ &\geq 1 + \delta \cdot \deg_G(u) + 1 + \delta \cdot \deg_G(v) - (\deg_G(u) + \deg_G(v)), \\ &= (\delta - 1)\big(\deg_G(u) + \deg_G(v)\big) + 2, \\ &\geq (\delta - 1)(2\delta) + 2. \end{aligned}$$

Hence,

$$|N_{\leq 2}(u) \cup N_{\leq 2}(v)| \geq 2\delta^2 - 2\delta + 2.$$

□

Lemma 2.3.3. *Let G be a connected graph of girth at least 6 and minimum degree $\delta \geq 3$. If $P : x_0, x_1, \ldots, x_s$ is a shortest (x_0, x_s)-path in G, then*

$$|N_{\leq 2}(V(P))| \geq \frac{s}{3}(\delta^2 - \delta + 1) + \frac{1}{3}(\delta^2 + 2\delta + 2).$$

Proof. Let $P : x = x_0, x_1, x_2, \ldots, x_{s-1}, x_s = y$ be a shortest (x, y)-path. Clearly, we have that $N_{\leq 2}(x) = N_{\leq 2}(P)$ if P is a path of length 0 containing only x. It follows immediately from Lemmas 2.3.1 and 2.3.2 that $|N_{\leq 2}(x)| \geq \delta^2 + 1$ and $|N_{\leq 2}(x_0) \cup N_{\leq 2}(x_1)| \geq 2\delta^2 - 2\delta + 2$ for any $x_0x_1 \in E(G)$. We now consider the set $N_{\leq 2}[V(P)]$.

In view of the fact that $[N_{\leq 2}(x_i) \cup N_{\leq 2}(x_{i+1})] \cap \big[N_{\leq 2}(x_j) \cup N_{\leq 2}(x_{j+1})\big] = \emptyset$ if $|i - j| \geq 6$, we have that

$$\begin{aligned} N_{\leq 2}[V(P)] &= \bigcup_{i=0}^{s} N_{\leq 2}(x_i), \\ &\supseteq \bigcup_{i=0}^{\lfloor \frac{s-6}{6} \rfloor} [N_{\leq 2}(x_{6i}) \cup N_{\leq 2}(x_{6i+1})] \cup \bigcup_{j=6+6(\lfloor \frac{s-6}{6} \rfloor)}^{s} N_{\leq 2}(x_j). \end{aligned}$$

Considering the distinguishing cases according to the residue class of $s (\bmod\ 6)$ with $s \geq 6$, we have that

$$N_{\leq 2}[V(P)] \supseteq \bigcup_{i=0}^{\lfloor \frac{s-6}{6} \rfloor} [N_{\leq 2}(x_{6i}) \cup N_{\leq 2}(x_{6i+1})] \cup \begin{cases} N_{\leq 2}(x_s) & \text{if } s \equiv 0 (\bmod\ 6), \\ N_{\leq 2}(x_{s-1}) \cup N_{\leq 2}(x_s) & \text{if } s \equiv 1 (\bmod\ 6), \\ \{x_{s-4}\} \cup N_{\leq 2}(x_{s-1}) \cup N_{\leq 2}(x_s) & \text{if } s \equiv 2 (\bmod\ 6), \\ \{x_{s-5}, x_{s-4}\} \cup N_{\leq 2}(x_{s-1}) \cup N_{\leq 2}(x_s) & \text{if } s \equiv 3 (\bmod\ 6), \\ N_{\leq 1}(x_{s-5}) \cup N_{\leq 2}(x_{s-1}) \cup N_{\leq 2}(x_s) & \text{if } s \equiv 4 (\bmod\ 6), \\ N_{\leq 1}(x_{s-6}) \cup N_{\leq 1}(x_{s-5}) \cup N_{\leq 2}(x_{s-1}) \cup N_{\leq 2}(x_s) & \text{if } s \equiv 5 (\bmod\ 6), \end{cases}$$

This implies that

$$|N_{\leq 2}[V(P)]| \geq \sum_{i=0}^{\lfloor \frac{s-6}{6} \rfloor} |N_{\leq 2}(x_{6i}) \cup N_{\leq 2}(x_{6i+1})| + \begin{cases} \delta^2+1 & \text{if } s \equiv 0 \pmod 6, \\ 2\delta^2-2\delta+2 & \text{if } s \equiv 1 \pmod 6, \\ 2\delta^2-2\delta+3 & \text{if } s \equiv 2 \pmod 6, \\ 2\delta^2-2\delta+4 & \text{if } s \equiv 3 \pmod 6, \\ 2\delta^2-\delta+3 & \text{if } s \equiv 4 \pmod 6, \\ 2\delta^2+2 & \text{if } s \equiv 5 \pmod 6. \end{cases}$$

Hence by Lemma 2.3.2,

$$|N_{\leq 2}[V(P)]| \geq \frac{s}{3}(\delta^2-\delta+1) + \begin{cases} \delta^2+1 & \text{if } s \equiv 0 \pmod 6, \\ (\frac{5}{3})(\delta^2-\delta+1) & \text{if } s \equiv 1 \pmod 6, \\ (\frac{1}{3})(4\delta^2-4\delta+7) & \text{if } s \equiv 2 \pmod 6, \\ \delta^2-\delta+3 & \text{if } s \equiv 3 \pmod 6, \\ (\frac{1}{3})(2\delta^2+\delta+5) & \text{if } s \equiv 4 \pmod 6, \\ (\frac{1}{3})(\delta^2+5\delta+1) & \text{if } s \equiv 5 \pmod 6. \end{cases}$$

It is easy to verify that $|N_{\leq 2}[V(P)]| \geq \frac{s}{3}(\delta^2-\delta+1) + \frac{1}{3}(\delta^2+2\delta+2)$ in all cases, so the lemma holds if $s \geq 6$.

To check the remaining cases $s < 6$, we note that for $s = 0$ we have $|N_{\leq 2}(v_0)| \geq \delta^2+1$, for $s \in \{1,2,3\}$ we have $|N_{\leq 2}(v_0) \cup N_{\leq 2}(v_1)| \geq 2\delta^2-2\delta+2$, for $s = 4$ we have $|N_{\leq 2}(v_0) \cup N_{\leq 2}(v_1) \cup \{v_4\}| \geq 2\delta^2-2\delta+3$, and for $s = 5$ we have $|N_{\leq 2}(v_0) \cup N_{\leq 2}(v_1) \cup N[v_5]| \geq 2\delta^2-\delta+3$. All of these terms are not less than $\frac{s}{3}(\delta^2-\delta+1)+\frac{1}{3}(\delta^2+2\delta+2)$, and so the lemma holds. $\square$

For fixed δ we define a real function g by

$$g(x) := \frac{x}{3}(\delta^2-\delta+1) + \frac{1}{3}(\delta^2+2\delta+2). \tag{2.3.1}$$

So the inequality in Lemma 2.3.3 becomes $|N_{\leq 2}(V(P))| \geq g(s)$. This inequality is used in the proofs of all bounds on diameter and radius of graphs of girth at least 6 both in the following section and next chapter.

Theorem 2.3.4. *Let G be a connected graph of girth at least 6, order n and with minimum degree $\delta \geq 3$, then*

$$(i) \quad \operatorname{diam}(G) \leq \frac{3n}{\delta^2-\delta+1} - 1. \tag{2.3.2}$$

$$(ii)\ \mathrm{rad}(G) \quad \leq \quad \max\Big\{18,\ \frac{3n}{2(\delta^2-\delta+1)}+10\Big\}. \tag{2.3.3}$$

Proof. Let v and w be two vertices at distance $d := \mathrm{diam}(G)$ and $P : v_0, v_1, v_2 \dots v_d$ be a shortest (v,w)-path where $v_0 = v, v_d = w$. It follows from Lemma 2.3.3 that

$$n \geq |N_{\leq 2}(V(P)| \geq g(d) = \frac{d}{3}(\delta^2-\delta+1)+\frac{1}{3}(\delta^2+2\delta+2).$$

Solving for d yields

$$d \leq \frac{3n}{\delta^2-\delta+1} - \frac{\delta^2+2\delta+2}{\delta^2-\delta+1} < \frac{3n}{\delta^2-\delta+1} - 1,$$

as desired. This yields inequality (2.3.2). We now give a proof for (ii).

Let G be a graph of radius r, let u be a centre vertex of G and let v' be a vertex at distance r from v. For any $w \in N_i(u)$, let $w' \in N_{i-1}(u)$ such that $ww' \in E(G)$ $(1 \leq i \leq r)$. The collection of the edges of the form $\{ww' : w \in V(G) - \{u\}\}$ defines a distance preserving spanning tree, $T \leq G$, from u. Recall from Definition 1.4.9 that a vertex $v'' \in V(G)$ is related to $v' \in N_r(u)$ if there exists $\overline{v}' \in V(T(u,v')) \cap N_{\geq 9}$ and $\overline{v}'' \in V(T(u,v'')) \cap N_{\geq 9}$ such that $d_G(\overline{v}', \overline{v}'') \leq 4$. For $v \in V(G)$, denote the (u,v)-path in T by $T(u,v)$.

If $\mathrm{rad}(G) \leq 18$, then there is nothing to prove, hence we may assume that $\mathrm{rad}(G) \geq 19$. Let $u = v_0', v_1', v_2', \dots, v_r'$ be the vertices of the path $T(u,v')$, and denote the segment $v_a', v_{a+1}', \dots, v_b'$ by $P'_{a,b}$. Let $u = v_0'', v_1'', v_2'', \dots, v_s''$ be the vertices of the path $T(u,v'')$, where $s = d(u,v'')$, and denote the segment $v_a'', v_{a+1}'', \dots, v_b''$ by $P''_{a,b}$. Consider the three paths $P'_{0,4}$, $P'_{9,r}$ and $P''_{9,s}$. Since v'' is not related to v', the sets $N_{\leq 2}(V(P'_{9,r}))$ and $N_{\leq 2}(V(P''_{9,s}))$ are disjoint. Both sets clearly do not share any vertices with $N_{\leq 2}(V(P'_{0,4}))$. Hence we have

$$n \geq |N_{\leq 2}(V(P'_{0,4}))| + |N_{\leq 2}(V(P'_{9,r}))| + |N_{\leq 2}(V(P''_{9,s}))|.$$

Applying Lemma 2.3.3 to the three paths, in conjunction with the inequality $s \geq r-9$, we obtain

$$\begin{aligned} n &\geq g(4)+g(r-9)+g(s-9) \\ &\geq g(4)+g(r-9)+g(r-18) \\ &= \frac{2}{3}r(\delta^2-\delta+1) - \frac{20}{3}\delta^2 + \frac{29}{3}\delta - \frac{17}{3}. \end{aligned}$$

Solving for r now yields

$$r \leq \frac{3n}{2(\delta^2-\delta+1)} + \frac{20\delta^2-29\delta+17}{2(\delta^2-\delta+1)} < \frac{3n}{2(\delta^2-\delta+1)} + 10.$$

as desired. This together with the assumption that $r \geq 18$ yields inequality (2.3.3). $\square$

The above bounds are sharp apart from the additive constant. We now describe the construction of a graph due to [85] that we will make use of to demonstrate the sharpness of our bound in Theorem 2.3.4.

Example 2.3.5. *Let q be a prime power. Recall that $GF(q)^3$ is the vector space of triples of elements of the finite field $GF(q)$. Let H^*_{q+1} be the graph whose vertices are the 1-dimensional and 2-dimensional subspaces of $GF(q)^3$ generated by $\underline{w}$ and $\{\underline{u}, \underline{v}\}$ respectively where $\underline{w}, \underline{u}, \underline{v} \neq \underline{0}$. Let U be the set of all 1-dimensional subspaces of $GF(q)^3$ and W be the set of all 2-dimensional subspaces of $GF(q)^3$. Two vertices, $\langle \underline{w} \rangle \in U$ and $\langle \underline{u}, \underline{v} \rangle \in W$ are said to be adjacent in H^*_{q+1} if and only if $\langle \underline{w} \rangle$ is contained in $\langle \underline{u}, \underline{v} \rangle$.*

Claim 2.3.6. *The following are the properties of H^*_{q+1}.*

a) H^*_{q+1} is bipartite.

b) Each partite set has $q^2 + q + 1$ vertices, and so H^*_{q+1} has $2(q^2 + q + 1)$ vertices.

c) H^*_{q+1} contains no 4-cycle.

d) Every vertex of H^*_{q+1} has degree $q + 1$.

e) For any two vertices, u and v of H^*_{q+1}, $d_{H^*_{q+1}}(u, v) \leq 3$.

f) If uv is an edge of H^*_{q+1} then, $d_{H^*_{q+1}-uv}(u, v) \geq 5$.

Proof. **a)** From the description of H^*_{q+1}, we have that the vertex sets of H^*_{q+1} can be partitioned into two different sets, U and W, representing the 1-dimensional and 2-dimensional subspaces of $GF(q)^3$. Moreover, each vertex in U can only be adjacent to a vertex in W. Hence, we conclude that H^*_{q+1} is bipartite.

b) Recall from Corollary 1.2.20 that there are $(q^2 + q + 1)$ 1-dimensional subspaces of $GF(q)^3$ and $(q^2 + q + 1)$ 2-dimensional subspaces of $GF(q)^3$. Thus, each partite set (U, W) of H^*_{q+1} has (q^2+q+1) vertices since U and W are the 1-dimensional and 2-dimensional subspaces of $GF(q)^3$ generated by $\underline{w}$ and $\{\underline{u}, \underline{v}\}$ respectively. Hence, H^*_{q+1} has $2(q^2+q+1)$ vertices.

c) Let $\langle \underline{x} \rangle, \langle \underline{y} \rangle$ be any two vertices of H^*_{q+1} belonging to U. $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ are not adjacent in H^*_{q+1} since they belong to the same partite set. $\langle \underline{x} \rangle$ and $\langle \underline{y} \rangle$ have exactly one common neighbour. Since any two vertices in U have only one common neighbour. H^*_{q+1} contains no 4-cycle.

d) Recall from Corollary 1.2.18(a) that every 1-dimensional subspace of $GF(q)^3$ is contained in $q + 1$ distinct 2-dimensional subspace of $GF(q)^3$ and since the partite set, U, is the set of all 1-dimensional subspaces of $GF(q)^3$, we have that $\deg_{H^*_{q+1}}(\langle \underline{u} \rangle) = q + 1$ for every $\langle \underline{u} \rangle \in U$. By Corollary 1.2.18(b), each 2-dimensional subspace of $GF(q)^3$ contains $q+1$ distinct 1-dimensional subspaces of $GF(q)^3$ and since the partite set, V, is the set of all 2-dimensional subspaces of $GF(q)^3$, it follows immediately that $\deg_{H^*_{q+1}}(\langle \underline{u}, \underline{v}\rangle) = q+1$ for every $\langle \underline{u}, \underline{v}\rangle \in V$. Hence each vertex of H^*_{q+1} has degree $q+1$.

e) Let u and v be any two vertices of H^*_{q+1} and let $u_1, u_2, \ldots, u_{q+1}$ be the neighbours of u. If u and v are adjacent in H^*_{q+1}, then $d_{H^*_{q+1}}(u,v) = 1$. If u and v belong to the same partite set, then u and v are not adjacent in H^*_{q+1} and so they have a common neighbour in H^*_{q+1}. Hence, we can find (u,v)-path of length 2 in H^*_{q+1}, that is, $d_{H^*_{q+1}}(u,v) = 2$. If u and v belong to different partite sets and are not adjacent in H^*_{q+1}, then v and u_i have a common neighbour, w, since they belong to the same partite set and are not adjacent in H^*_{q+1}. Thus, there exists a (u,v)-path of length 3 in H^*_{q+1} and so $d_{H^*_{q+1}}(u,v) = 3$. Hence we conclude from the above cases that $d_{H^*_{q+1}}(u,v) \geq 3$ and $\mathrm{diam}(H^*_{q+1}) = 3$.

f) Clearly, the girth of H^*_{q+1} is at least 6 since H^*_{q+1} is a bipartite C_4-free graph. If x and y are adjacent vertices in H^*_{q+1}, then since $\mathrm{girth}(H^*_{q+1}) \geq 6$, we have that any (x,y)-path together with the edge xy is a cycle of length at least 6 in H^*_{q+1}. Thus, we have that any (x,y)-path in $H^*_{q+1} - uv$ is a path of length at least 5. Hence $d_{H^*_{q+1}-uv}(x,y) \geq 5$. $\square$

From here onwards, we let $u, v \in V(H^*_{q+1})$ be two fixed adjacent vertices. Let H_e be the graph $H^*_{q+1} - uv$. By Claim 2.3.6 (f), $d_{H_e}(x,y) \geq 5$. Moreover, H_e has minimum degree, $\delta = q$ and $|V(H_e)| = |V(H^*_{q+1})| = 2(q^2+q+1)$.

Theorem 2.3.7. *If $\delta - 1$ is a prime power, then there exists an infinite family of graphs G of girth at least 6, order n and minimum degree δ such that*

$$(i)\ \mathrm{diam}(G) \geq \frac{3n}{\delta^2 - \delta + 1} - 5, \tag{2.3.4}$$

$$(ii)\ \mathrm{rad}(G) \geq \frac{3n}{2(\delta^2 - \delta + 1)} - \frac{5}{2}. \tag{2.3.5}$$

Proof. Let $q = \delta - 1$ be a prime power. Let H_1 and H_k be disjoint copies of H^*_{q+1} and let $H_2, H_3, \ldots, H_{k-1}$ disjoint isomorphic copies of H_e. From the disjoint union of $H_1, H_2, \ldots H_k$, we obtain the graph $G^*_{k,\delta}$ by adding the edges $v^{(t)}u^{(t+1)}$ for every $(1 \leq t < k)$ where u^t and v^t are the vertices of H_t corresponding to the vertices u and v, respectively, of H^*_{q+1} and H_e. Since $G^*_{k,\delta}$ is the graph obtained from the union of both H_e and H^*_{q+1}, we have by Claim 2.3.6 (a and c) that $G^*_{k,\delta}$ has girth 6. Moreover, since both H^*_{q+1} and H_e has $2(q^2+q+1)$ vertices, we have that there are $2k(q^2+q+1)$ vertices in $G^*_{k,\delta}$. The degree of vertices of $G^*_{k,\delta}$ is either $q+1$ or $q+2$, hence the minimum degree of $G^*_{k,\delta}$ is

$\delta = q + 1$. The order of $G^*_{k,\delta}$ is

$$\begin{aligned} |V(G^*_{k,\delta})| &= 2k(q^2 + q + 1) \\ &= 2k[(\delta - 1)^2 + (\delta - 1) + 1] \\ &= 2k(\delta^2 - \delta + 1) \end{aligned}$$

By Claim 2.3.6(e), $d_{H^*_{q+1}}(x, y) \leq 3$ for any two vertices, x and y, of H^*_{q+1}, and so we have that $\mathrm{diam}(H_1) = \mathrm{diam}(H_k) = 3$. By Claim 2.3.6(f) that $d_{H_e} \geq 5$. Letting u^1 be a vertex of H_1 with $d_{H_1}(u^1, v^1) = 3$, and v^k a vertex of H_k with $d_{H_k}(v^k, u^k) = 3$, we have that the diameter of $G^*_{k,\delta}$ is

$$\begin{aligned} \mathrm{diam}(G^*_{k,\delta}) &\geq d(u^1, v^k) \\ &\geq \mathrm{diam}(H_1) + (k-2).\mathrm{diam}(H_e) + \mathrm{diam}(H_k) + k - 1 \\ &\geq 3 + 5(k-2) + 3 + k - 1 \\ &= 6k - 5 \\ &= \frac{3n}{\delta^2 - \delta + 1} - 5. \end{aligned}$$

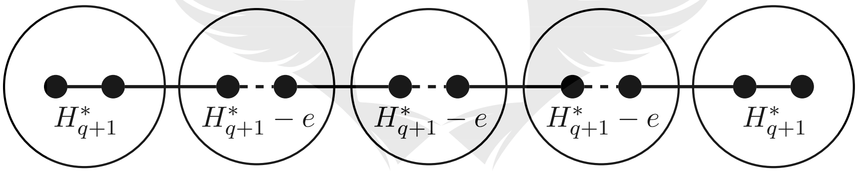

Figure 2.3: The graph $G^*_{5,\delta}$.

This concludes the proof of part (i) of Theorem (2.3.7). The proof for the part (*ii*) follows from the fact that

$$\mathrm{rad}(G^*_{k,\delta}) \geq \frac{1}{2}\left(\mathrm{diam}(G^*_{k,\delta})\right) \geq \frac{3n}{2(\delta^2 - \delta + 1)} - \frac{5}{2}.$$

□

2.3.2 Bounds on Diameter and Radius of Connected (C_4, C_5)-Free Graphs.

In this subsection we show that very slightly weaker versions of the bound on the diameter and radius of graphs of girth at least 6 in Theorem 2.3.4 hold. We note that if we relax the conditions on the graph in Theorem 2.3.4 by allowing triangles, we obtain similar bounds on the order of C_4 free graphs in Lemma 2.3.8 and (C_4, C_5)-free graphs in Lemma 2.3.9. The former is a very slightly more general version of the well-known lower bound on the order of C_4-free graphs of given minimum degree.

Lemma 2.3.8. *Let G be a C_4-free graph and v a vertex of G. If every vertex in $V(G)-\{v\}$ has degree at least $\delta \geq 3$, then*

$$|N_{\leq 2}(v)| \geq 1 + \deg(v)(\delta - 1) + \varepsilon_{\deg(v)}.$$

where

$$\varepsilon_{\deg(v)} = \begin{cases} 0 & \text{if } \varepsilon_{\deg(v)} \text{ is even,} \\ 1 & \text{if } \varepsilon_{\deg(v)} \text{ is odd.} \end{cases}$$

Proof. There are $\deg(v)$ vertices at distance exactly 1 from v. Each vertex in $N(v)$ is adjacent to v and at most one vertex in $N(v)$ since otherwise G would contain C_4. If $\deg(v)$ is odd, then at least one of the vertices in $N(v)$ has no neighbour in $N(v)$. Hence all but $\varepsilon_{\deg(v)}$ vertices in $N(v)$ have at least $\delta - 2$ neighbours in $N_2(v)$, and $\varepsilon_{\deg(v)}$ vertices in $N(v)$ have at least $\delta - 1$ neighbours in $N_2(v)$. As above, the sets $N(x) \cap N_2(v)$, $x \in N(v)$, are pairwise disjoint. Hence $|N_2(v)| \geq \deg(v)(\delta - 2) + \varepsilon_{\deg(v)}$, and the lemma follows. □

Lemma 2.3.9. *Let G be a (C_4, C_5)-free graph of minimum degree $\delta \geq 3$. If u and v are adjacent vertices of G, then*

$$|N_{\leq 2}(v) \cup N_{\leq 2}(v)| \geq 2\delta^2 - 5\delta + 2 + 2\varepsilon_\delta.$$

where

$$\varepsilon_\delta = \begin{cases} 0 & \text{if } \delta \text{ is even,} \\ 1 & \text{if } \delta \text{ is odd.} \end{cases}$$

Proof. We consider two cases.

CASE 1: $N(u) \cap N(v) = \emptyset$.

Consider $G' = G - uv$. The sets $N^2_{G'}(u)$ and $N^2_{G'}(v)$ are disjoint, otherwise G would contain a cycle C_4 or C_5 through uv. Clearly G' is C_4-free, and u and v have degree at least $\delta - 1$. By Lemma 2.3.8 we thus have $|N^2_{G'}(u)| \geq 1 + (\delta - 1)^2 + \varepsilon_{\delta-1}$ and $|N^2_{G'}(v)| \geq 1 + (\delta - 1)^2 + \varepsilon_{\delta-1}$.

Since $N^2_G(u) \cup N^2_G(v) = N^2_{G'}(u) \cup N^2_{G'}(u)$ we obtain

$$|N^2_G(u) \cup N^2_G(v)| \geq 2[(\delta - 1)^2 + 1 + \varepsilon_{\delta-1}] \geq 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta,$$

and the lemma follows.

CASE 2: $N(u) \cap N(v) \neq \emptyset$.

Let w be a common neighbour of u and v. Then w is the only common neighbour since otherwise G would contain C_4. We first consider the second neighbourhood of u and v, respectively, in $G' - w$. As in Case 1, the sets $N^2_{G'-w}(u)$ and $N^2_{G'-w}(v)$ are disjoint, and each has at least $1 + (\delta - 2)(\delta - 1) + \varepsilon_\delta$ vertices. The set $N_G[w] - \{u, v\}$ is also contained in $N_G^2(u) \cup N_G^2(v)$, and it does not share any vertex with $N_{G'-w}^2(u) \cup N_{G'-w}^2(v)$, otherwise G would contain a C_4 or a C_5. Hence

$$\begin{aligned} |N_G^2(u) \cup N_G^2(v)| &\geq 2[1 + (\delta - 2)(\delta - 1) + \varepsilon_\delta] + (\deg_G(w) - 1) \\ &\geq 2\delta^2 - 5\delta + 2 + 2\varepsilon_\delta, \end{aligned}$$

as desired. □

If we now relax the condition in Theorem 2.3.3 that G has girth at least 6 to G being (C_4, C_5)-free, then only marginally weaker bounds on the diameter and radius holds. We omit the details of the proofs, which are very similar to the proofs of Lemma 2.3.3 and Theorem 2.3.3.

Lemma 2.3.10. *Let G be a connected (C_4, C_5)-free graph of order n and minimum degree $\delta \geq 3$. If $P : x_0, x_1, \ldots, x_s$ is a shortest (x_0, x_s)-path in G, then*

$$|N_{\leq 2}[V(P)]| \geq \frac{s}{6}(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + \frac{1}{3}\delta^2 - \frac{7}{6}\delta + \frac{5}{6} + \frac{1}{3}\varepsilon_\delta. \tag{2.3.6}$$

Proof. The proof is analogous to that of Lemma 2.3.3. Recall that $N_{\leq 2}[V(P)]$, is the set of those vertices of G that are within distance at most 2 to a vertex of P.

Let $P := x_0, x_1, x_2, \ldots, x_{s-1}, x_s$ and consider the set $N_{\leq 2}[V(P)]$. Recall from Lemma 2.3.3 that for $s \geq 6$

$$\begin{aligned} |N_{\leq 2}[V(P)]| \geq & \sum_{i=0}^{\lfloor \frac{s-6}{6} \rfloor} |N_{\leq 2}(x_{6i}) \cup N_{\leq 2}(x_{6i+1})| \\ & + \begin{cases} |N_{\leq 2}(x_s)| & \text{if } s \equiv 0 (\text{mod } 6), \\ |N_{\leq 2}(x_{s-1}) \cup N_{\leq 2}(x_s)| & \text{if } s \equiv 1 (\text{mod } 6), \\ |\{x_{s-4}\} \cup N_{\leq 2}(x_{s-1}) \cup N_{\leq 2}(x_s)| & \text{if } s \equiv 2 (\text{mod } 6), \\ |\{x_{s-5}, x_{s-4}\} \cup N_{\leq 2}(x_{s-1}) \cup N_{\leq 2}(x_s)| & \text{if } s \equiv 3 (\text{mod } 6), \\ |N_{\leq 1}(x_{s-5}) \cup N_{\leq 2}(x_{s-1}) \cup N_{\leq 2}(x_s)| & \text{if } s \equiv 4 (\text{mod } 6), \\ |N_{\leq 1}(x_{s-6}) \cup N_{\leq 1}(x_{s-5}) \cup N_{\leq 2}(x_{s-1}) \cup N_{\leq 2}(x_s)| & \text{if } s \equiv 5 (\text{mod } 6). \end{cases} \end{aligned}$$

Furthermore, we have by Lemma 2.3.8 that $|N_{\leq 2}(v)| \geq \delta^2 - \delta + 1 + \varepsilon_\delta$ for any $u \in V(G)$ since $\deg(u) \geq \delta$, and by Lemma 2.3.9 that $|N_{\leq 2}(u) \cup N_{\leq 2}(v)| \geq 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta$ for any $uv \in E(G)$.

Hence,

$$|N_{\leq 2}[V(P)]| \geq (1+\lfloor \frac{s-6}{6} \rfloor)(2\delta^2-5\delta+5+2\varepsilon_\delta)+\begin{cases} \delta^2-\delta+1+\varepsilon_\delta & \text{if } s \equiv 0 \pmod 6, \\ 2\delta^2-5\delta+5+2\varepsilon_\delta & \text{if } s \equiv 1 \pmod 6, \\ 2\delta^2-5\delta+6+2\varepsilon_\delta & \text{if } s \equiv 2 \pmod 6, \\ 2\delta^2-5\delta+7+2\varepsilon_\delta & \text{if } s \equiv 3 \pmod 6, \\ 2\delta^2-4\delta+6+2\varepsilon_\delta & \text{if } s \equiv 4 \pmod 6, \\ 2\delta^2-3\delta+5+2\varepsilon_\delta & \text{if } s \equiv 5 \pmod 6. \end{cases}$$

This implies that

$$|N_{\leq 2}[V(P)]| \geq (\frac{s}{6})(2\delta^2-5\delta+5+2\varepsilon_\delta)+\begin{cases} \delta^2-\delta+1+\varepsilon_\delta & \text{if } s \equiv 0 \pmod 6, \\ (\frac{1}{6})(10\delta^2-25\delta+25+10\varepsilon_\delta) & \text{if } s \equiv 1 \pmod 6, \\ (\frac{1}{3})(4\delta^2-10\delta+13+4\varepsilon_\delta) & \text{if } s \equiv 2 \pmod 6, \\ (\frac{1}{2})(2\delta^2-5\delta+9+2\varepsilon_\delta) & \text{if } s \equiv 3 \pmod 6, \\ (\frac{1}{3})(2\delta^2-2\delta+8+2\varepsilon_\delta) & \text{if } s \equiv 4 \pmod 6, \\ (\frac{1}{6})(2\delta^2+7\delta+5+2\varepsilon_\delta) & \text{if } s \equiv 5 \pmod 6. \end{cases}$$

It is easy to verify that $|N_{\leq 2}[V(P)]| \geq \frac{s}{6}(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + \frac{1}{3}\delta^2 - \frac{7}{6}\delta + \frac{5}{6} + \frac{1}{3}\varepsilon_\delta$in all cases, so the lemma holds if $s \geq 6$.

To check the remaining cases $s < 6$, we note that for $s = 0$ we have $|N_{\leq 2}(v_0)| \geq \delta^2 - \delta + 1 + \varepsilon_\delta$, for $s \in \{1, 2, 3\}$ we have $|N_{\leq 2}(v_0) \cup N_{\leq 2}(v_1)| \geq 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta$, for $s = 4$ we have $|N_{\leq 2}(v_0) \cup N_{\leq 2}(v_1) \cup \{v_4\}| \geq 2\delta^2 - 5\delta + 6 + 2\varepsilon_\delta$, and for $s = 5$ we have $|N_{\leq 2}(v_0) \cup N_{\leq 2}(v_1) \cup N[v_5]| \geq 2\delta^2 - 4\delta + 6 + 2\varepsilon_\delta$. All of these terms are not less than $\frac{s}{6}(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + \frac{1}{3}\delta^2 - \frac{7}{6}\delta + \frac{5}{6} + \frac{1}{3}\varepsilon_\delta$, and so the lemma holds. □

From now on, we will define a function $h(x)$, as

$$h(x) := \frac{s}{6}(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + \frac{1}{3}\delta^2 - \frac{7}{6}\delta + \frac{5}{6} + \frac{1}{3}\varepsilon_\delta. \tag{2.3.7}$$

Theorem 2.3.11. *Let G be a connected (C_4, C_5)-free graph with n vertices and with minimum degree $\delta \geq 3$. Then*

$$\text{diam}(G) \leq \frac{6n}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} - 1, \tag{2.3.8}$$

$$\text{rad}(G) \leq \max\{18, \frac{3n}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} + 10\}, \tag{2.3.9}$$

where ε_δ takes on the value of 0 or 1 depending on whether δ is even or odd.

Proof. Let v and w be two vertices at distance $d := \text{diam}(G)$ and $P : v_0, v_1, v_2 \ldots v_d$ be a shortest (v,w)-path where $v_0 = v, v_d = w$. It follows from Lemma 2.3.10 that

$$n \geq |N_{\leq 2}(V(P))| \geq h(d) = \frac{d}{6}(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + \frac{1}{3}\delta^2 - \frac{7}{6}\delta + \frac{5}{6} + \frac{1}{3}\varepsilon_\delta$$

Solving for d,we have

$$d \leq \frac{6n}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} - \frac{2\delta^2 - 7\delta + 5 + 2\varepsilon_\delta}{2(\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta)} < \frac{6n}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} - 1, \tag{2.3.10}$$

which yields inequality (2.3.8) as desired.

We now give a proof for the radius, which is analogous to the proof of Theorem 2.3.4(ii).

If $\mathrm{rad}(G) \leq 18$, then there is nothing to prove, hence we may assume that $\mathrm{rad}(G) \geq 19$. Let u, r, T, v' and v'' be as defined in Theorem 2.3.4(ii). Let $u = v'_0, v'_1, v'_2, \ldots, v'_r$ be the vertices of the path $T(u, v')$, and denote the segment $v'_a, v'_{a+1}, \ldots, v'_b$ by $P'_{a,b}$. Let $u = v''_0, v''_1, v''_2, \ldots, v''_s$ be the vertices of the path $T(u, v'')$, where $s = (d(u, v'')$, and denote the segment $v''_a, v''_{a+1}, \ldots, v''_b$ by $P''_{a,b}$. Consider the three paths $P'_{0,4}$, $P'_{9,r}$ and $P''_{9,s}$. Since v'' is not related to v', the sets $N_{\leq 2}(V(P'_{9,r}))$ and $N_{\leq 2}(V(P''_{9,s}))$ are disjoint. Both sets clearly do not share any vertices with $N_{\leq 2}(V(P'_{0,4}))$. Hence we have

$$n \geq |N_{\leq 2}(V(P'_{0,4}))| + |N_{\leq 2}(V(P'_{9,r}))| + |N_{\leq 2}(V(P''_{9,s}))|.$$

Applying Lemma 2.3.10 to the three paths, in conjunction with the inequality $s \geq r - 9$, we obtain

$$\begin{aligned} n &\geq h(4) + h(r-9) + h(s-9) \\ &\geq h(4) + h(r-9) + h(r-18). \end{aligned}$$

Solving for r, we have

$$n \geq \frac{r}{3}(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) - \frac{23}{6}(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + \delta^2 - \frac{7}{2}\delta + \frac{5}{2} + \varepsilon_\delta,$$

which now yields

$$r \leq \frac{3n}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} + \frac{20\delta^2 - 47\delta + 50 + 20\varepsilon_\delta}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} < \frac{3n}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} + 10, \tag{2.3.11}$$

as desired. □

We do not know if the bound in Theorem 2.3.11 is sharp. The graph $G^*_{\delta,k}$ shows that the coefficient $\frac{3}{\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta}$ of n is at least close to being best possible if δ is large. Indeed, the ratio between the coefficients of n in the bound in Theorem 2.3.11 and in Theorem 2.3.7 tends to 1 as δ tends to infinity.

Chapter 3

Diameter, Radius, Maximum Degree and Minimum Degree

3.1 Introduction

In the previous chapter, we gave bounds on the diameter and radius of graph of girth at least 6 and (C_4, C_5)-free graphs of given order and minimum degree. In our sharpness example, we saw that the degree of each vertex is close to the minimum degree, this suggests that our bounds can be improved a little further if we have a vertex of large degree, say $\Delta(G) = cn$ for some $c \in \mathbb{R}$ with $0 < c < 1$. Herein, we present upper bounds on the diameter and radius of graphs of girth at least 6 and (C_4, C_5)-free graphs taken into consideration the order of the graph n, minimum degree δ and maximum degree Δ. We also present a construction to show that the bound is best possible in a sense specified later.

3.2 Main Results

3.2.1 Generalised Cages

In this section we obtain a bound on the cardinality of the third neighbourhood of a vertex of large degree in a graph of girth at least 6. As a corollary, we obtain a lower bound on the order of a graph of girth at least 6 whose minimum degree and maximum degree are prescribed.

This leads us to a natural generalisation of the classical problem in the theory of cages. Given positive integers $\delta \geq 2$ and $g \geq 3$, a (δ, g)-cage is a δ-regular graph of girth g that has minimum order among such graphs. The classical cage problem is to find, for given δ and g, to determine a (δ, g)-cage and its order. See [54, 96] for a survey on this topic.

Our problem is closely related to a problem introduced by Boben, Jajcay and Pisanski [11]: Given a positive integer N, a nonempty set $A = \{k_1, k_2, \ldots, k_t\} \subseteq \mathbb{N}$ with and a possibly empty set $B = \{g_1, g_2, \ldots, g_s\} \subseteq \mathbb{N}$ with $3 \leq g_i < N$ for $i = 1, 2, \ldots, s$. A graph G is an (A, B, N)-graph if its degree set equals A and the set of cycle lengths less than N occurring in G equals B. (Note that for this definition it is irrelevant if G has cycles of lengths N or more.) In [11] it was shown that for every choice of N, A and B such graphs exist if $A \neq \{1\}$. The minimum order of an (A, B, N)-graph is denoted by $n(A, B, N)$, and an (A, B, N)-graph of order $n(A, B, N)$ is an (A, B, N)-cage. The generalisation of the cage problem introduced in [11] is to determine, for given A, B and N, the value $n(A, B, N)$ and an (A, B, N)-cage.

In our generalisation of the cage problem, only the minimum degree and maximum degree are prescribed, but not the degree set, so it is not a special case of the problem in [11]. However, it follows from the above-mentioned existence result in [11], for example by choosing $A = \{\delta, \Delta\}$, $B = \emptyset$ and $N = 6$, that a graph of minimum degree δ, maximum degree Δ and girth at least 6 exists for every given values of δ and Δ with $\Delta \geq \delta \geq 2$. Clearly, the minimum order of such a graph is $\min_{A \subseteq [\delta+1, \Delta-1]} n(\{\delta, \Delta\} \cup A, \emptyset, 6)$, where $[\delta + 1, \Delta - 1]$ is the set $\{\delta + 1, \delta + 2, \ldots, \Delta - 1\}$.

Lemma 3.2.1. *Let G be a graph of girth at least 6, minimum degree δ and maximum degree Δ. If y is a vertex of degree Δ, then*

$$|N_{\leq 3}(y)| \geq \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}. \tag{3.2.1}$$

Proof. Fix $y \in N_i$. We have by Definition 1.1.39 that $|N_{\leq 3}(y)| = |N_{\leq 2}(y)| + |N_3(y)|$. Since by Lemma 2.3.1, $|N_{\leq 2}(y)| \geq 1 + \deg(y)\delta = 1 + \Delta\delta$, it suffices to show that

$$|N_3(y)| \geq \frac{1}{2} + (\delta - 1)\sqrt{\Delta(\delta - 2)}.$$

To achieve this, we now consider the sets, $N_2(y)$ and $N_3(y)$. Since G has no cycles of length less than 6, every vertex in $N(y)$ has all its neighbours except one in $N_2(y)$. Hence, the sets $N(v)/\{y\}$, $v \in N(y)$ are disjoint, and so we have that

$$|N_2(u)| = \sum_{v \in N(y)} (\deg_G(v) - 1) \geq (\delta - 1)\deg(y) = \Delta(\delta - 1). \tag{3.2.2}$$

For every $u \in N_2(y)$, the number of 2-stars centered at u with leaves in $N_3(y)$ is $\binom{\deg_G(u)-1}{2}$. Hence, denoting by S_2 the number of 2-stars with centre in N_2 and leaves in N_3, we have,

$$S_2 = \sum_{u \in N_2(y)} \binom{\deg_G(u) - 1}{2} \geq \sum_{u \in N_2(y)} \binom{\delta - 1}{2}, \tag{3.2.3}$$

Using the result in (3.2.2), we conclude that

$$S_2 \geq \Delta(\delta - 1)\binom{\delta - 1}{2}. \tag{3.2.4}$$

Furthermore, no two 2−stars with leaves in $N_3(y)$ have the same two leaves. Otherwise, G would contain a C_4. Hence, for every set of 2 vertices in $N_3(y)$, there is at most one

2−star having this set as its set of leaves. Thus,

$$S_2 \leq \binom{|N_3(y)|}{2}. \tag{3.2.5}$$

Comparing (3.2.4) and (3.2.5), we have that

$$\Delta(\delta-1)\binom{\delta-1}{2} \leq S_2 \leq \binom{|N_3(y)|}{2},$$

Thus,

$$\binom{|N_3(y)|}{2} \geq \Delta(\delta-1)\binom{\delta-1}{2}. \tag{3.2.6}$$

By letting $t := |N_3(y)|$, we have from (3.2.6) that $t^2 - t \geq \Delta(\delta-1)^2(\delta-2)$. And so, $t^2 - t - \Delta(\delta-1)^2(\delta-2) \geq 0$. Solving the inequality, we obtain that $t \geq \frac{1}{2} + \sqrt{\frac{1}{4} + \Delta(\delta-1)^2(\delta-2)} > \frac{1}{2} + (\delta-1)\sqrt{\Delta(\delta-2)}$. Thus,

$$|N_3(y)| \geq \frac{1}{2} + (\delta-1)\sqrt{\Delta(\delta-2)}. \tag{3.2.7}$$

Hence, we conclude that

$$|N_{\leq 3}(y)| = |N_{\leq 2}(x)| + |N_3(y)| \geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}.$$

□

Let $n(\delta, \Delta, g)$ be the minimum order of a connected graph of girth at least 6 with minimum degree δ and maximum degree Δ and girth g. Adding the bounds on $|N_{\leq 2}(v)|$ in Lemma 2.3.1 and $|N_3(v)|$ in Lemma 3.2.1, where y is a vertex of maximum degree, we obtain the following corollary on $n(\delta, \Delta, 6)$.

Corollary 3.2.2. *Given $\delta, \Delta \in \mathbb{N}$ with $3 \leq \delta \leq \Delta$. Then*

$$n(\delta, \Delta, 6) \geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}.$$

Rephrasing this result in the terminology of [11] we obtain the following corollary.

Corollary 3.2.3. *Let $A \subseteq \mathbb{N}$ be nonempty and finite. If $\min(A) \geq 3$, then*

$$n(A, \emptyset, 6) \geq \max(A)\min(A) + (\min(A)-1)\sqrt{\max(A)(\min(A)-2)} + \frac{3}{2}.$$

We also have a similar, only slightly weaker lower bound on the number of vertices within distance three of a vertex of maximum degree in a (C_4, C_5)-free graph.

Lemma 3.2.4. *Let G be a (C_4, C_5)-free graph of minimum degree δ and maximum degree*

Δ. *If v is a vertex of degree Δ, then*

$$|N_{\leq 3}(v)| \geq \Delta(\delta-1) + (\delta-2)\sqrt{\Delta(\delta-3)} + \frac{3}{2}.$$

Proof. Let y be a vertex of degree Δ. By Lemma 2.2.1 we have

$$|N_{\leq 2}(y)| \geq \Delta(\delta-1)+1. \tag{3.2.8}$$

In order to bound $|N_3(y)|$ from below we count the number of unordered pairs of vertices in $N_3(y)$. Clearly, there are exactly $\binom{|N_3(y)|}{2}$ such pairs. On the other hand, each vertex $w \in N_2(y)$ has at least $\deg(w)-2$ neighbours in $N_3(y)$. Indeed, if $w_1 \in N_1(y)$ is the unique common neighbour of w and v, then the only vertices in $N_{\leq 2}(y)$ to which w can be adjacent are w_1 and one other neighbour of w_1, otherwise it is easy to see that w and v lie on a 4-cycle or a 5-cycle, a contradiction. Hence there are at least $\binom{\deg(w)-2}{2}$ pairs of vertices of $N_3(y)$ that are both adjacent to w. Since G is C_4-free, no two vertices of $N_2(y)$ have a common pair of neighbours in $N_3(y)$. Hence the number of pairs of vertices in $N_3(y)$ is at least $\sum_{w\in N_2(y)} \binom{\deg(w)-2}{2}$. Since each vertex in $N_2(y)$ has degree at least δ, this implies

$$|N_2(y)|\binom{\delta-2}{2} \leq \binom{|N_3(y)|}{2}.$$

Since $|N_2(y)| \geq \Delta(\delta-2)$ by Lemma 2.2.1, we thus have

$$\frac{1}{2}\Delta(\delta-2)^2(\delta-3) \leq \frac{1}{2}|N_3(y)|(|N_3(y)|-1).$$

Solving for $N_3(y)$ yields $|N_3(y)| \geq \frac{1}{2} + \sqrt{\frac{1}{4}+\Delta(\delta-2)^2(\delta-3)} > \frac{1}{2} + (\delta-2)\sqrt{\Delta(\delta-3)}$. This, in conjunction with (3.2.8), yields the lemma. □

Lemma 3.2.4 yields a lower bound on the order of a (C_4, C_5)-free graph with given minimum degree and maximum degree, which we state using the notation of [11].

Corollary 3.2.5. *Let $A \subseteq \mathbb{N}$ be nonempty and finite. If $\min(A) \geq 3$, then*

$$n(A,\{3\},6) \geq \max(A)(\min(A)-1) + (\min(A)-2)\sqrt{\max(A)(\min(A)-3)} + \frac{3}{2}.$$

3.2.2 Graph of Girth 6 with Minimum Degree and Maximum Degree

In this section we construct a graph to show that the bound in Corollary 3.2.2 is close to best possible.

We start by showing that for all δ for which $\delta-1$ is a prime power, there exist infinitely many values of Δ for which

$$\Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2} \leq n(\delta, \Delta, 6) \leq \delta\Delta + (\delta + 1)\sqrt{\Delta(\delta - 2)} + 2,$$

so the bound in Corollary 3.2.2 is close to being best possible in the sense that the second term, $(\delta - 1)\sqrt{\Delta(\delta - 2)}$, is of the right order of magnitude.

Our construction is based on the following graph F^*, constructed first by [85].

Example 3.2.6. *Let q be a prime power. Recall that $GF(q)^n$ is an n-dimensional vector space over the finite field $GF(q)$. Let X be the set of all 1-dimensional subspaces of $GF(q)^n$ and Y be the set of all 2-dimensional subspaces of $GF(q)^n$. Let F^* be the graph with vertex set $X \cup Y$, where two vertices, $\langle \underline{w} \rangle \in X$ and $\langle \underline{u}, \underline{v} \rangle \in Y$ are adjacent if and only if $\langle \underline{w} \rangle$ is contained in $\langle \underline{u}, \underline{v} \rangle$.*

Claim 3.2.7. *The following are the properties of F^*.*

a) *F^* is bipartite.*

b) *F^* has $\frac{(q^n-1)(q^n+q^3-2q)}{(q^2-1)(q^2-q)}$ vertices.*

c) *F^* contains no 4-cycle.*

d) *Each vertex in X has degree $q^{n-2} + q^{n-3} + \ldots + q + 1$. Each vertex in Y has degree $q + 1$.*

e) $\mathrm{diam}(F^*) \geq 4$ *if* $|F^*| > 3$.

Proof. **a)** From the definition of F^*, we have that the vertex set of F^* can be partitioned into two different sets, X and Y, representing the 1-dimensional and 2-dimensional subspaces of $GF(q)^n$. Moreover, each vertex in X can only be adjacent to a vertex in Y. Hence, we conclude that F^* is bipartite.

b) Recall from Claim 1.2.19 that there are $\frac{(q^n-1)(q^n-q)(q^n-q^2)\ldots(q^n-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)\ldots(q^k-q^{k-1})}$ k-dimensional subspaces of $GF(q)^n$. Hence, we have $\frac{(q^n-1)}{(q-1)}$ 1-dimensional subspaces of $GF(q)^n$ and $\frac{(q^n-1)(q^n-q)}{(q^2-1)(q^2-q)}$ 2-dimensional subspaces of $GF(q)^n$. Since F^* consists of both 1-dimensional and 2-dimensional subspaces of $GF(q)^n$, we have that

$$|F^*| = |X| + |Y| = \frac{(q^n - 1)}{(q - 1)} + \frac{(q^n - 1)(q^n - q)}{(q^2 - 1)(q^2 - q)} = \frac{(q^n - 1)(q^n + q^3 - 2q)}{(q^2 - 1)(q^2 - q)}.$$

c) Let $\langle \underline{u} \rangle, \langle \underline{v} \rangle$ be any two vertices of F^* belonging to X. $\langle \underline{u} \rangle$ and $\langle \underline{v} \rangle$ are not adjacent in F^* since they belong to the same partite set. $\langle \underline{u} \rangle$ and $\langle \underline{v} \rangle$ have exactly one common neighbour. Since any two vertices in X have only one common neighbour. F^* contains no 4-cycle.

d) Recall from Claim 1.2.17(a) that every 1-dimensional subspace of $GF(q)^n$ is contained in $q^{n-2} + q^{n-3} + \ldots + q + 1$ distinct 2-dimensional subspaces of $GF(q)^n$ and since the partite set X is the set of all 1-dimensional subspaces of $GF(q)^n$, we have that for every $\langle \underline{w} \rangle \in X$,

$$\deg_{F^*}(\langle \underline{w} \rangle) = q^{n-2} + q^{n-3} + \ldots + q + 1.$$

By Claim 1.2.17(b), each 2-dimensional subspace of $GF(q)^n$ contains $q+1$ distinct 1-dimensional subspace of $GF(q)^n$ and since the partite set, Y, is the set of all 2-dimensional subspaces of $GF(q)^n$, it follows immediately that for every $\langle \underline{u}, \underline{v} \rangle \in Y$, $\deg_{F^*}(\langle \underline{u}, \underline{v} \rangle) = q+1$.

e) Let u and v be any two vertices of F^* and let $u_1, u_2, \ldots, u_{q+1}$ be the neighbours of u.. If u and v are adjacent in F^*, then $d_{F^*}(u,v) = 1$. If u and v belong to different partite sets and are not adjacent in F^*, then v and u_i have a common neighbour, w, since they belong to the same partite set and are not adjacent in F^*. Thus, there exists a (u,v)-path of length 3 in F^* and so $d_{F^*}(u,v) = 3$. If u and v belong to the same partite set, then u and v are not adjacent in F^*. Clearly $u, v \in X$ or $u, v \in Y$ have a common neighbour in F^* if u, v have a nontrivial intersection. Thus, we can find a (u,v)-path of length 2 in F^*, that is, $d_{F^*}(u,v) = 2$. For $u, v \in Y$, if their intersection is trivial then u and v have no common neighbour and thus, $d_{F^*}(u,v) \neq 2$. It therefore follows immediately that $d_{F^*}(u,v) \geq 4$ since the distance between two vertices that represent 2-dimensional subspaces that intersect only trivially must be even. Hence we conclude from the above cases that $d_{F^*}(u,v) \geq 4$ and so $\text{diam}(F^*) \geq 4$. □

Lemma 3.2.8. *Let $q, m \in \mathbb{N}$ with q a prime power and $m \geq 7$. Then the set of 2-dimensional subspaces of $GF(q)^m$ can be partitioned into parts $U_1, U_2, \ldots, U_t$, where $t = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)q} - \frac{1}{q}$, $|U_i| = q$ for all $i \in \{1, 2, \ldots, t-1\}$, and $|U_1| = q+1$, such that any two 2-dimensional subspaces contained in the same part U_i intersect trivially.*

Proof. We construct an auxiliary graph H whose vertices are the 2-dimensional subspaces of $GF(q)^m$, and in which two vertices are adjacent if, as subspaces, their intersection is non-trivial. Define $t = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)q} - \frac{1}{q}$. In order to prove the lemma it suffices to prove that $V(H)$ can be partitioned into independent sets $U_1, U_2, \ldots, U_t$ of the desired cardinalities.

We first give an expression for the order of H. Clearly, $n(H) = |V_2| = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)}$. Simple calculations show that this equals

$$\begin{array}{ll} (q^{m-2} + q^{m-4} + q^{m-6} + \cdots + q^2 + 1)(q^{m-2} + q^{m-3} + q^{m-4} + \cdots + 1) & \text{if } m \text{ is even,} \\ (q^{m-3} + q^{m-5} + q^{m-7} + \cdots + q^2 + 1)(q^{m-1} + q^{m-2} + q^{m-3} + \cdots + 1) & \text{if } m \text{ is odd.} \end{array}$$

We now determine the degrees of the vertices in H. Fix a 2-dimensional subspace U. Choosing a non-zero vector $a_1 \in U$ and a non-zero vector $b_1 \in GF(q)^m - U$, we obtain the 2-dimensional subspace $W = \langle a_1, b_1 \rangle$ that has a nontrivial intersection with U. There are $(q^2-1)(q^m-q^2)$ ways to choose (a_1, b_1), and clearly every 2-dimensional subspace that intersects U nontrivially can be obtained in this way. Since W shares $q-1$ non-zero vectors with U, and q^2-q non-zero vectors with $GF(q)^m - U$, there are $(q-1)(q^2-q)$ choices for the pair (a_1, b_1) that yield the same subspace W. Hence there are $\frac{(q^2-1)(q^m-q^2)}{(q-1)(q^2-q)}$ distinct 2-dimensional subspaces that have a non-trivial intersection with U. Since H is clearly regular, every vertex has degree $\frac{(q^2-1)(q^m-q^2)}{(q-1)(q^2-q)}$.

We next prove that $V(H)$ can be partitioned into $t-1$ sets, $U_1, U_2, \ldots, U_{t-1}$ with q vertices each, and one set U_t with $q+1$ vertices. Multiplying out the terms in the above expression for $n(H)$ we see that $n(H) \equiv 1 \pmod{q}$. Hence there exists a partition of $V(H)$ into sets where all but one set have cardinality q and the remaining set has cardinality $q+1$. Clearly the number of sets in such a partition is $\frac{|V_2|-1}{q}$, which equals t.

Among all such partitions, choose one for which $\sum_{j=1}^{t} m(G[U_j])$ is minimum. We claim that $\sum_{j=1}^{t} m(G[U_j]) = 0$. Suppose not. Then there exists a set U_i containing two adjacent vertices v_i and w_i. Since each vertex has degree $\frac{(q^2-1)(q^m-q^2)}{(q-1)(q^2-q)}$, we have $|N(U_i)| \leq |U_i|\frac{(q^2-1)(q^m-q^2)}{(q-1)(q^2-q)} \leq (q+1)\frac{(q^2-1)(q^m-q^2)}{(q-1)(q^2-q)}$. It is easy to check that this is less than t for $m \geq 7$. Hence there exists a set U_ℓ not containing any neighbour of a vertex in U_i. Choose a vertex $v_\ell \in U_\ell$, and replace in U_i vertex v_i by v_ℓ, and replace in U_ℓ vertex v_ℓ by v_i. Then clearly $\sum_{j=1}^{t} m(G[U_j])$ has decreased, a contradiction to our choice of the sets $U_1, \ldots, U_t$. Hence $U_1, U_2, \ldots, U_t$ are independent sets of H, and the lemma follows. □

Claim 3.2.9. *Let $q, m \in \mathbb{N}$ with q a prime power and $m \geq 4$. Let Y be the set of all 2-dimensional subspaces of the vector space $GF(q)^m$. If $\delta = q+1$ and $\Delta = \frac{(q^m-1)(q^m-q)}{q(q^2-1)(q^2-q)} - \frac{1}{q}$, then*

$$|Y| \leq (\delta+1)\sqrt{\Delta(\delta-2)}. \tag{3.2.9}$$

Proof. By Claim 1.2.19, $\frac{|Y|}{q\sqrt{\Delta(q-1)}} = \frac{\frac{q^m-1}{q-1}}{q\sqrt{\Delta(q-1)}}$ and since $\Delta = \frac{(q^m-1)(q^m-q)}{q(q^2-1)(q^2-q)} - \frac{1}{q}$, we have that

$$\begin{aligned}
\frac{\frac{q^m-1}{q-1}}{q\sqrt{\Delta(q-1)}} &= \frac{q^m-1}{q-1}\cdot\frac{1}{q}\cdot\sqrt{\frac{q(q-1)(q+1)}{(q^m-1)(q^{m-1}-1)-(q^2-1)(q-1)}},\\
&= \sqrt{\frac{(q^m-1)^2}{q^2(q-1)^2}\cdot\frac{q(q-1)(q+1)}{(q^m-1)(q^{m-1}-1)-(q^2-1)(q-1)}},\\
&= \sqrt{\frac{(q^{2m}-2q^m+1)(q+1)}{(q^2-q)(q^{2m-1}-q^m-q^{m-1}-q^3+q^2+q)}},\\
&= \sqrt{1+\frac{2q^{2m}+q^{m+2}-2q^{m+1}-3q^m+q^5-2q^4+q^2+q+1}{q^{2m+1}-q^{2m}-q^{m+2}+q^m-q^5+2q^4-q^2}},
\end{aligned}$$

Since $-2q^{m+1}-3q^m+q^5-2q^4+q^2+q+1 < 0$ for $m \geq 5$ and $q^m-q^5+2q^4-q^2 > 0$ for

$m \geq 5$, we have that

$$\frac{\frac{q^m-1}{q-1}}{q\sqrt{\Delta(q-1)}} \leq \sqrt{1+\frac{2q^{2m}+q^{m+2}}{q^{2m+1}-q^{2m}-q^{m+2}}}, = \sqrt{1+\frac{2q^{m-2}+1}{q^{m-1}-q^{m-2}-1}},$$

subclaim:

1. $\frac{2q^{m-2}+1}{q^{m-1}-q^{m-2}-1} < \frac{4}{q}$.
   This is true if and only if $(2q^{m-2}+1)q < 4(q^{m-1}-q^{m-2}-1) \Leftrightarrow 0 < 2q^{m-1}-4q^{m-2}-q-4 \Leftrightarrow 0 \leq (2q^{m-2}-1)(q-2)-6$.
2. $\sqrt{1+x} \leq 1+\frac{x}{2}$ for $x \in \mathbb{R}$.
   This is true since $\sqrt{1+x}-1 = \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{\sqrt{1+x}+1} = \frac{x}{\sqrt{1+x}+1} \leq \frac{x}{2}$.

Using the above facts established above, we have that

$$\frac{\frac{q^m-1}{q-1}}{q\sqrt{\Delta(q-1)}} < \sqrt{1+\frac{2q^{m-2}+1}{q^{m-1}-q^{m-2}-1}}, \leq \sqrt{1+\frac{4}{q}}, \leq 1+\frac{2}{q},$$

Hence,

$$|Y| = \frac{q^m-1}{q-1} \leq q\sqrt{\Delta(q-1)}\left(1+\frac{2}{q}\right). \tag{3.2.10}$$

Substituting $q = \delta - 1$ in (3.2.10) yields inequality (3.2.9) as desired. □

Theorem 3.2.10. *Let $\delta, m \in \mathbb{N}$ such that $q := \delta - 1$ is a prime power and $m \geq 7$. Then there exist a graph $F_{q,m}$ of girth 6, minimum degree at least δ, maximum degree $\Delta = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q^2-q)} - \frac{1}{q}$ and order $n(F_{q,m})$, where*

$$n(F_{q,m}) \leq 2 + \delta\Delta + (\delta+1)\sqrt{\Delta(\delta-2)}. \tag{3.2.11}$$

Proof. Let δ be fixed such that $q := \delta - 1$ is a prime power. For $m \in \mathbb{N}$ with $m \geq 7$ consider the graph F^* with partite sets X and Y defined above. By Lemma 3.2.8 there exists a partition of Y into t sets $U_1, U_2, \ldots, U_t$ of cardinality q or $q+1$, where $t = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)q} - \frac{1}{q}$, such that two vertices in Y belonging to the same set U_i, as subspaces, intersect trivially and thus have no common neighbour in X. Add vertices $u_1, u_2, \ldots, u_t$

to F^* and join u_i to all vertices in U_i for all $i \in \{1, 2, \ldots, t\}$. Finally, add a vertex z and join it to $u_1, u_2, \ldots, u_t$. Denote the obtained graph by $F_{q,m}$.

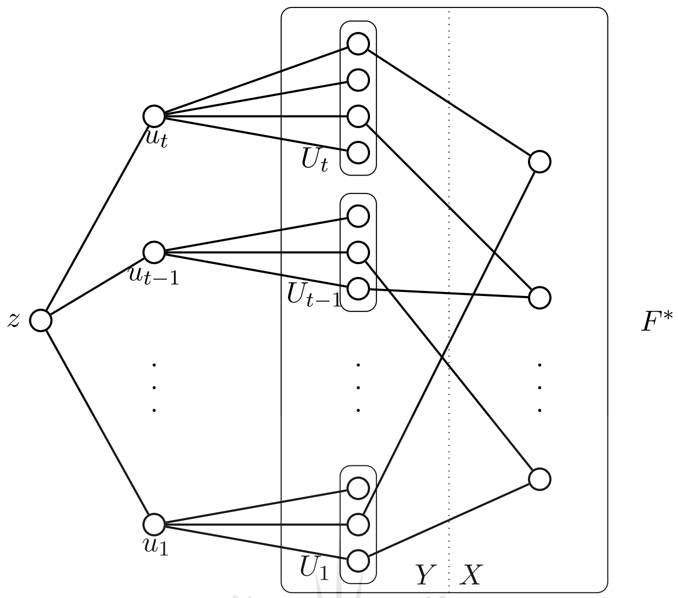

Figure 3.1: The graph $F_{q,m}$.

Clearly, $F_{q,m}$ is bipartite with partite sets $X \cup \{u_1, u_2, \ldots, u_t\}$ and $Y \cup \{z\}$. We have $|Y| = tq + 1$. The degrees of the vertices in X and Y in $F_{q,m}$ equal $\frac{q^{m-1}-1}{q-1}$ and $q + 2$, respectively, while $\deg(u_i) = q + 1$ for $i = 1, 2, \ldots, t - 1$ and $q + 2$ for $i = t$. Finally, $\deg(z) = t = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)q} - \frac{1}{q}$. Hence $\delta(G_{q,m}) = q+1$ and $\Delta(G_{q,m}) = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)q} - \frac{1}{q}$.

We now show that $F_{q,m}$ has girth 6. Clearly, $F_{q,m}$ contains a cycle of length 6 through z and a vertex in X. Hence we need to show that $F_{q,m}$ does not contain a shorter cycle. Since $F_{q,m}$ is bipartite, it suffices to show that it is C_4-free. Suppose to the contrary that $F_{q,m}$ contains a cycle C of length 4.

Since $F_{q,m}[X \cup Y] = F^*$, and since F^* is C_4-free, it follows that $V(C)$ is not contained in $X \cup Y$, and so C contains a vertex in $\{u_1, u_2, \ldots, u_t\}$. On the other hand, C cannot have two vertices in $\{u_1, \ldots, u_t\}$ since by the construction of $F_{q,m}$, any two vertices of $\{u_1, \ldots, u_t\}$ have only z as a common neighbour. Hence C contains exactly one vertex, u_i say, of $\{u_1, \ldots, u_t\}$, and z is not on C. Hence C contains u_i, two neighbours of u_i in U_i, and a fourth vertex, which is in X. But by Lemma 3.2.8 no two vertices in U_i have, as subspaces of $GF(q)^m$, a nontrivial intersection, which means that no two vertices in U_i have a common neighbour in X. This contradiction shows that no such cycle C exists, and $F_{q,m}$ is C_4-free.

We now determine the order of $F_{q,m}$. Let $\Delta = \Delta(F_{q,m})$. Since $\Delta(F_{q,m}) = t = \frac{|Y|-1}{q} =$

$\frac{|Y|-1}{\delta-1}$, we have $t + |Y| = \delta\Delta + 1$, and thus

$$n(F_{q,m}) = 1 + t + |Y| + |X| = 2 + \delta\Delta + |X|.$$

Applying Claim 3.2.9, we have that

$$n(F_{q,m}) \leq 2 + \delta\Delta + (\delta + 1)\sqrt{\Delta(\delta - 2)},$$

Thus the inequality (3.2.11) holds, completing the proof of the theorem. □

3.2.3 Bounds on Diameter and Radius of Graphs of Girth at least 6 and (C_4, C_5)-Free Graphs

In this section we give improved bounds on the diameter of graphs of girth at least 6 and (C_4, C_5)-free graphs in terms of order, minimum degree and maximum degree.

Theorem 3.2.11. *Let G be a connected graph of girth at least 6, order n, minimum degree $\delta \geq 3$ and maximum degree Δ. Then*

$$(i)\ \mathrm{diam}(G) \leq \frac{3n - 3\Delta\delta}{\delta^2 - \delta + 1} - \frac{3(\delta - 1)\sqrt{\Delta(\delta - 2)}}{\delta^2 - \delta + 1} + 11, \tag{3.2.12}$$

$$(ii)\ \mathrm{rad}(G) \leq \max\Big\{28, \frac{3n - 3\Delta\delta}{2(\delta^2 - \delta + 1)} - \frac{3(\delta - 1)\sqrt{\Delta(\delta - 2)}}{2(\delta^2 - \delta + 1)} + 22\Big\}. \tag{3.2.13}$$

Proof. Let d, v, w, P be as in the proof of Theorem 2.3.4. Let y be a vertex of maximum degree in G. We have by Lemma 3.2.1, that $|N_{\leq 3}(y)| \geq \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}$.

Let $P : v_0, v_1, \ldots, v_d$, where $v_0 = v$ and $v_d = w$. Let z be a vertex of degree Δ and let $j = d_G(v, z)$. Then the only vertices x of P for which possibly $N_{\leq 2}(x) \cap N_{\leq 3}(z) \neq \emptyset$ are the vertices in $\{v_i \in V(P) \mid |j - i| \leq 5\}$. We consider two cases, depending on the value of j.

Case 1: $6 \leq j \leq d - 6$.

Define the subpaths P_1 and P_2 of P by

$$P_1 : v_0, v_1, \ldots, v_{j-6}, \quad P_2 : v_{j+6}, v_{j+7}, \ldots, v_d.$$

The sets $N_{\leq 2}(V(P_1))$, $N_{\leq 2}(V(P_2))$ and $N_{\leq 3}(z)$ are disjoint. Applying Lemmas 2.3.3 and

3.2.1 we obtain

$$\begin{aligned} n &\geq |N_{\leq 2}(V(P_1))| + |N_{\leq 2}(V(P_2))| + |N_{\leq 3}(z)|, \\ &\geq g(j-6) + g(d-j-6) + \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}, \\ &= \frac{d}{3}(\delta^2-\delta+1) - \frac{10}{3}\delta^2 + \frac{16}{3}\delta - \frac{7}{6} + \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)}. \end{aligned} \tag{3.2.14}$$

Solving for d now yields

$$\begin{aligned} d &\leq \frac{3n-3\Delta\delta}{\delta^2-\delta+1} - \frac{3(\delta-1)\sqrt{\Delta(\delta-2)}}{\delta^2-\delta+1} + \frac{10\delta^2-16\delta+7/2}{\delta^2-\delta+1} \\ &< \frac{3n-3\Delta\delta}{\delta^2-\delta+1} - \frac{3(\delta-1)\sqrt{\Delta(\delta-2)}}{\delta^2-\delta+1} + 10, \end{aligned}$$

as desired.

Case 2: $0 \leq j \leq 5$ or $d-5 \leq j \leq d$.

If $0 \leq j \leq 5$, then define the subpath P_1 of P by $P_1 : v_{12}, v_{13}, \ldots, v_d$. In the other case, $d-5 \leq j \leq d$, we choose $P_1 : v_0, v_1, \ldots, v_{d-12}$. In both cases $N_{\leq 3}(z)$ and $N_{\leq 2}(V(P_1))$ are disjoint. Now the same calculations as in Case 1 yields

$$\begin{aligned} n &\geq |N_{\leq 2}(V(P_1))| + |N_{\leq 3}(z)|, \\ &\geq g(d-12) + \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}, \\ &= \frac{d}{3}(\delta^2-\delta+1) - \frac{11}{3}\delta^2 + \frac{14}{3}\delta - \frac{11}{6} + \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)}. \end{aligned} \tag{3.2.15}$$

and solving for d now yields the desired bound. Hence, (3.2.12) holds.

We now give a proof for the radius

If $\text{rad}(G) \leq 28$, then there is nothing to prove, hence we may assume that $\text{rad}(G) \geq 29$. Let u, r, T, v', v'', $P'_{a,b}$ and $P''_{a,b}$ be as in the proof of Theorem 2.3.4.

Since v' and v'' are not related, the sets $N_{\leq 2}(V(P'_{a,b}))$ and $N_{\leq 2}(V(P''_{c,d}))$ are disjoint for all a, b, c, d with $9 \leq a \leq b \leq r$ and $9 \leq c \leq d \leq r-9$.

Let z be a vertex of degree Δ and let $j = d_G(u,z)$. Then the only vertices x of $T(u,v')$ and $T(u,v'')$ for which possibly $N_{\leq 2}(x) \cap N_{\leq 3}(z) \neq \emptyset$ are the vertices in $\{v'_i, v''_i \mid |j-i| \leq 5\}$. We consider five cases, depending on the value of j.

Case 1: $j \leq 5$.

Then the sets $N_{\leq 3}(z)$, $N_{\leq 2}(V(P'_{11,r}))$ and $N_{\leq 2}(V(P''_{11,r-9}))$ are disjoint. Applying Lemmas

2.3.3 and 3.2.1 we obtain

$$\begin{aligned} n &\geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2} + g(r-11) + g(r-20) \\ &\geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{2r}{3}(\delta^2-\delta+1) - \frac{29}{3}\delta^2 + \frac{35}{3}\delta - \frac{45}{6}. \end{aligned}$$

Case 2: $6 \leq j \leq 14$.

Then the sets $N_{\leq 3}(z)$, $N_{\leq 2}(V(P'_{0,j-6}))$, $N_{\leq 2}(V(P'_{j+6,r}))$ and $N_{\leq 2}(V(P''_{j+6,r-9}))$ are disjoint. Applying Lemmas 2.3.3 and 3.2.1 we obtain

$$\begin{aligned} n &\geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2} + g(j-6) + g(r-j-6) + g(r-j-15) \\ &\geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{2r}{3}(\delta^2-\delta+1) - \frac{38}{3}\delta^2 + \frac{47}{3}\delta - \frac{61}{6}. \end{aligned}$$

Case 3: $15 \leq j \leq r-15$.

Then the sets $N_{\leq 3}(z)$, $N_{\leq 2}(V(P'_{0,4}))$, $N_{\leq 2}(V(P'_{9,j-6}))$, $N_{\leq 2}(V(P'_{j+6,r}))$, $N_{\leq 2}(V(P''_{9,j-6}))$ and $N_{\leq 2}(V(P''_{j+6,r-9}))$ are disjoint. Applying Lemmas 2.3.3 and 3.2.1 we obtain

$$\begin{aligned} n &\geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2} + g(4) + g(j-15) + g(r-j-6) + g(j-15) + g(r-j-15) \\ &\geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{2r}{3}(\delta^2-\delta+1) - \frac{42}{3}\delta^2 + \frac{57}{3}\delta - \frac{65}{6}. \end{aligned}$$

Case 4: $r-14 \leq j \leq r-6$.

Then the sets $N_{\leq 3}(z)$, $N_{\leq 2}(V(P'_{0,4}))$, $N_{\leq 2}(V(P'_{9,j-6}))$, $N_{\leq 2}(V(P'_{j+6,r}))$, and $N_{\leq 2}(V(P''_{9,j-6}))$ are disjoint. Applying Lemmas 2.3.3 and 3.2.1 we obtain

$$\begin{aligned} n &\geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2} + g(4) + g(j-15) + g(r-j-6) + g(j-15) \\ &\geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{2r}{3}(\delta^2-\delta+1) - \frac{42}{3}\delta^2 + \frac{54}{3}\delta - \frac{67}{6}. \end{aligned}$$

Case 5: $r-5 \leq j$.

Then the sets $N_{\leq 3}(z)$, $N_{\leq 2}(V(P'_{0,4}))$, $N_{\leq 2}(V(P'_{9,r-11}))$, and $N_{\leq 2}(V(P''_{9,r-11}))$ are disjoint. Applying Lemmas 2.3.3 and 3.2.1 we obtain

$$\begin{aligned} n &\geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2} + g(4) + g(r-20) + g(r-20) \\ &\geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{2r}{3}(\delta^2-\delta+1) - \frac{33}{3}\delta^2 + \frac{42}{3}\delta - \frac{21}{6}. \end{aligned}$$

It is easy to see that the smallest lower bound for n is always the expression in Case 4, so we have, irrespective of the value of j,

$$n \geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{2r}{3}(\delta^2-\delta+1) - \frac{42}{3}\delta^2 + \frac{54}{3}\delta - \frac{67}{6}.$$

Solving for r and using the estimate $\frac{42\delta^2-54\delta+67/2}{2(\delta^2-\delta+1)} \leq 22$ now yields inequality (3.2.13), thus completing the proof of the theorem. □

Theorem 3.2.12. *If $\delta-1$ is a prime power, then there exists an infinite family of graphs $F^*_{l,m,\delta}$ of girth at least 6 with n vertices, minimum degree δ, maximum degree Δ such that*

$$(i)\ \mathrm{diam}(G) \geq \frac{3(n-\Delta\delta)}{\delta^2-\delta+1} - \frac{3(\delta+1)\sqrt{\Delta(\delta-2)}}{\delta^2-\delta+1} + 3, \quad (3.2.16)$$

$$(ii)\ \mathrm{rad}(G) \geq \frac{3(n-\Delta\delta)}{2(\delta^2-\delta+1)} - \frac{3(\delta+1)\sqrt{\Delta(\delta-2)}}{2(\delta^2-\delta+1)} + \frac{3}{2}. \quad (3.2.17)$$

Proof. Let $q=\delta-1$ be a prime power and $m \in \mathbb{N}$ with $m \geq 4$. By Theorem 3.2.10, there exists a connected graph $F_{q,m}$ of girth at least 6 with minimum degree $q+1$, maximum degree $\Delta = \frac{(q^m-1)(q^m-q)}{q(q^2-1)(q^2-q)} - \frac{1}{q}$ whose order n satisfies $n(F_{q,m}) \leq 2+\delta\Delta+(\delta+1)\sqrt{\Delta(\delta-2)}$. Let $u_1 \in F_{q,m}$ be a vertex of maximum degree, viz z, and let $v_1 \in Y$ where Y is as defined in Theorem 3.2.10 (Example 3.2.6).

Now let $l \in \mathbb{N}$ with $l \geq 2$ and l sufficiently large. Consider the graph $G^*_{k,\delta}$ constructed in the proof of Theorem 2.3.7, and let H_1 be a copy of $F_{q,m}$. Denote the resulting graph by $F^*_{l,m,\delta}$. Figure 3.2 is an illustration of the graph $F^*_{5,m,\delta}$.

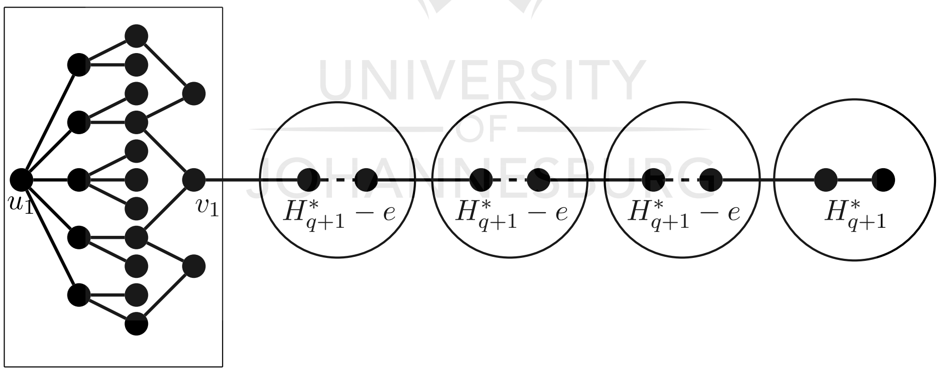

Figure 3.2: The graph $F^*_{5,m,\delta}$.

We have by Theorem 3.2.10, Claims 2.3.6 and 3.2.7 that $F^*_{l,m,\delta}$ is a connected graph of girth at least 6 with minimum degree $\delta = q+1$ and maximum degree is $\Delta = \frac{(q^m-1)(q^m-q)}{q(q^2-1)(q^2-q)} - \frac{1}{q}$.

By Claim 2.3.6, graph H_i (for $2 \leq i \leq l$) has $2(q^2+q+1)$ vertices, and since $n(F_{q,m}) \leq 2+\delta\Delta+(\delta+1)\sqrt{\Delta(\delta-2)}$, we have that the order of $F^*_{l,m,\delta}$ is

$$\begin{aligned} n(F^*_{l,m,\delta}) = |V(F^*_{l,m,\delta},| &\leq \Delta\delta+(\delta+1)\sqrt{\Delta(\delta-2)}+2+2(l-1)(q^2+q+1) \\ &= \Delta\delta+(\delta+1)\sqrt{\Delta(\delta-2)}+2+(l-1)2(\delta^2-\delta+1) \end{aligned}$$

By Claim 2.3.6(e), $d_{H^*_{q+1}}(x,y) \leq 3$ for any two vertices x and y of H^*_{q+1}, and so we have that $\mathrm{diam}(H_l) = 3$. Similarly, we have by Claim 2.3.6(f) that $d_{H_e}(x,y) \geq 5$. Thus, the diameter of $F^*_{l,m,\delta,}$ is obtained as follows

$$\begin{aligned}\mathrm{diam}(F^*_{l,m,\delta}) = d &\geq d_{H_1}(u_1,v_1) + (l-2)\cdot d_{H_e}(u,v) + \mathrm{diam}(H_l) + l - 1,\\ &= 6l-3.\end{aligned}$$

This implies that $l \leq \frac{d+3}{6}$ and so substituting this value in $n(F^*_{l,m,\delta})$ and making use of the fact that $\delta \geq 3$ we have that

$$\frac{3(n-\Delta\delta)}{\delta^2-\delta+1} - \frac{3(\delta+1)\sqrt{\Delta(\delta-2)}}{\delta^2-\delta+1} + \frac{3(\delta^2-\delta-1)}{\delta^2-\delta+1} \leq d.$$

Now using the estimate that $\frac{3(\delta^2-\delta-1)}{\delta^2-\delta+1} < 3$ yields the desired bound in Theorem 3.2.12.

The proof for part (ii) follows from the fact that

$$\mathrm{rad}(F^*_{l,m,\delta}) \geq \frac{1}{2}\,\mathrm{diam}(F^*_{l,m,\delta}) \geq \frac{3(n-\Delta\delta)}{2(\delta^2-\delta+1)} - \frac{3(\delta+1)\sqrt{\Delta(\delta-2)}}{2(\delta^2-\delta+1)} + \frac{3}{2}.$$

□

The graphs constructed in Theorem 3.2.12 show that the bound on the diameter in Theorem 2.3.7 is best possible if $\delta-1$ is a prime power in the following sense. For $\delta-1$ a fixed prime power and n and δ large, the maximum diameter of a graph of girth at least 6 with n vertices, minimum degree δ and maximum degree Δ is $\frac{3(n-\Delta\delta)}{\delta^2-\delta+1} - (1+f(\delta))\sqrt{\Delta(\delta-2)}$ where $|f(\delta)| \leq \frac{2}{\delta-1}$.

Theorem 3.2.13. *Let G be a connected (C_4, C_5)-free graph of order n, minimum degree $\delta \geq 3$ and maximum degree Δ. Then*

$$(i)\ \mathrm{diam}(G) \leq \frac{3n-3\Delta(\delta-1)}{\delta^2-\frac{5}{2}\delta+\frac{5}{2}+\varepsilon_\delta} - \frac{3(\delta-2)\sqrt{\Delta(\delta-3)}}{\delta^2-\frac{5}{2}\delta+\frac{5}{2}+\varepsilon_\delta} + 11. \tag{3.2.18}$$

$$(ii)\ \mathrm{rad}(G) \leq \max\Big\{28, \frac{3n-3\Delta(\delta-1)}{2\delta^2-5\delta+5+2\varepsilon_\delta} - \frac{3(\delta-2)\sqrt{\Delta(\delta-3)}}{2\delta^2-5\delta+5+2\varepsilon_\delta} + 20\Big\}. \tag{3.2.19}$$

Proof. Let d, v, w, P be as in the proof of Theorem 2.3.11.

Let $P : v_0, v_1, \ldots, v_d$, where $v_0 = v$ and $v_d = w$. Let z be a vertex of degree $\Delta$and let $j = d_G(v,z)$. Then the only vertices x of P for which $N_{\leq 2}(x) \cap N_{\leq 3}(z) \neq \emptyset$ are the vertices in $\{v_i \in V(P) \mid |j-i| \leq 5\}$.

We consider two cases, depending on the value of j.

Case 1: $6 \leq j \leq d-6$.

Define the subpaths P_1 and P_2 of P by

$$P_1 : v_0, v_1, \ldots, v_{j-6}, \quad P_2 : v_{j+6}, v_{j+7}, \ldots, v_d.$$

The sets $N_{\leq 2}(V(P_1))$, $N_{\leq 2}(V(P_2))$ and $N_{\leq 3}(z)$ are disjoint.

Applying Lemmas 2.3.10 and 3.2.4 we obtain

$$\begin{aligned} n &\geq |N_{\leq 2}(V(P_1))| + |N_{\leq 2}(V(P_2))| + |N_{\leq 3}(z)| \\ &\geq h(j-6) + h(d-j-6) + \Delta(\delta-1) + (\delta-2)\sqrt{\Delta(\delta-3)} + \frac{3}{2}. \end{aligned}$$

Recall from Lemma 2.3.10 that

$$|N_{\leq 2}(V(P)| \geq h(d) = \frac{d}{6}(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + \frac{1}{3}\delta^2 - \frac{7}{6}\delta + \frac{5}{6} + \frac{1}{3}\varepsilon_\delta,$$

and so we have that

$$n \geq \frac{d-12}{6}(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + \frac{2}{3}\delta^2 - \frac{7}{3}\delta + \frac{19}{6} + \frac{2}{3}\varepsilon_\delta + \Delta(\delta-1) + (\delta-2)\sqrt{\Delta(\delta-3)},$$

Solving for d now yields

$$\begin{aligned} d &\leq \frac{3n - 3\Delta(\delta-1)}{\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta} - \frac{3(\delta-2)\sqrt{\Delta(\delta-3)}}{\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta} + \frac{20\delta^2 - 46\delta + 41 + 20\varepsilon_\delta}{2(\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta)} \\ &< \frac{3n - 3\Delta(\delta-1)}{\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta} - \frac{3(\delta-2)\sqrt{\Delta(\delta-3)}}{\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta} + 10, \end{aligned}$$

as desired

Case 2: $0 \leq j \leq 5$ or $d - 5 \leq j \leq d$.

If $0 \leq j \leq 5$, then define the subpath P_1 of P by $P_1 : v_{12}, v_{13}, \ldots, v_d$. In the other case, $d - 5 \leq j \leq d$, we choose $P_1 : v_0, v_1, \ldots, v_{d-12}$. In both cases $N_{\leq 3}(z)$ and $N_{\leq 2}(V(P_1)$ are disjoint. Now the same calculations as in Case 1

$$\begin{aligned} n &\geq |N_{\leq 2}(V(P_1))| + |N_{\leq 3}(z)| \\ &\geq h(d-12) + \Delta(\delta-1) + (\delta-2)\sqrt{\Delta(\delta-3)} + \frac{3}{2} \\ &= \frac{d-12}{6}(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + \frac{1}{3}\delta^2 - \frac{7}{6}\delta + \frac{14}{6} + \frac{1}{3}\varepsilon_\delta + \Delta(\delta-1) + (\delta-2)\sqrt{\Delta(\delta-3)}. \end{aligned}$$

Solving for d, we have that

$$\begin{aligned} d &\leq \frac{3n - 3\Delta(\delta-1)}{\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta} - \frac{3(\delta-2)\sqrt{\Delta(\delta-3)}}{\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta} + \frac{22\delta^2 - 53\delta + 46 + 22\varepsilon_\delta}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta}, \\ &< \frac{3n - 3\Delta(\delta-1)}{\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta} - \frac{3(\delta-2)\sqrt{\Delta(\delta-3)}}{\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta} + 11. \end{aligned}$$

This yields inequality (3.2.18) and so the theorem holds.

We now give a proof for the radius.

If $\mathrm{rad}(G) \leq 28$, then there is nothing to prove, hence we may assume that $\mathrm{rad}(G) \geq 29$. Let u, r, T, v', v'', $P'_{a,b}$ and $P''_{a,b}$ be as in the proof of Theorem 2.3.11.

Since v' and v'' are not related, the sets $N_{\leq 2}(V(P'_{a,b}))$ and $N_{\leq 2}(V(P''_{c,d}))$ are disjoint for all a, b, c, d with $9 \leq a \leq b \leq r$ and $9 \leq c \leq d \leq r-9$.

Let z be a vertex of degree Δ and let $j = d_G(v, z)$. Then the only vertices x of $T(u, v')$ and $T(u, v'')$ for which $N_{\leq 2}(x) \cap N_{\leq 3}(z) \neq \emptyset$ are the vertices in $\{v'_i, v''_i \mid |j - i| \leq 5\}$. We consider five cases, depending on the value of j.

Case 1: $j \leq 5$.

Then the sets $N_{\leq 3}(z)$, $N_{\leq 2}(V(P'_{11,r}))$ and $N_{\leq 2}(V(P''_{11,r-9}))$ are disjoint. Applying Lemmas 2.3.10 and 3.2.4 we obtain

$$n \geq \Delta(\delta - 1) + (\delta - 2)\sqrt{\Delta(\delta - 3)} + \frac{3}{2} + h(r - 11) + h(r - 20).$$

Thus,

$$n \geq \Delta(\delta-1)+(\delta-2)\sqrt{\Delta(\delta-3)} + \left(\frac{2r-31}{6}\right)(2\delta^2-5\delta+5+2\varepsilon_\delta)+\frac{2}{3}\delta^2-\frac{7}{3}\delta+\frac{19}{6}+\frac{2}{3}\varepsilon_\delta.$$

Case 2: $6 \leq j \leq 14$.

Then the sets $N_{\leq 3}(z)$, $N_{\leq 2}(V(P'_{0,j-6}))$, $N_{\leq 2}(V(P'_{j+6,r}))$ and $N_{\leq 2}(V(P''_{j+6,r-9}))$ are disjoint. Applying Lemmas 2.3.10 and 3.2.4 we obtain

$$\begin{aligned} n \geq\ & \Delta(\delta - 1) + (\delta - 2)\sqrt{\Delta(\delta - 3)} + \frac{3}{2} + h(j - 6) + h(r - j - 6) \\ & + h(r - j - 15), \end{aligned}$$

and so,

$$n \geq \Delta(\delta - 1) + (\delta - 2)\sqrt{\Delta(\delta - 3)} + \left(\frac{2r - 41}{6}\right)(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + \delta^2 - \frac{7}{2}\delta + 4 + \varepsilon_\delta.$$

Case 3: $15 \leq j \leq r - 15$.

Then the sets $N_{\leq 3}(z)$, $N_{\leq 2}(V(P'_{0,4}))$, $N_{\leq 2}(V(P'_{9,j-6}))$, $N_{\leq 2}(V(P'_{j+6,r}))$, $N_{\leq 2}(V(P''_{9,j-6}))$ and $N_{\leq 2}(V(P''_{j+6,r-9}))$ are disjoint. Applying Lemmas 2.3.10 and 3.2.4 we obtain

$$\begin{aligned} n \geq\ & \Delta(\delta - 1) + (\delta - 2)\sqrt{\Delta(\delta - 3)} + \frac{3}{2} + h(4) + h(j - 15) + h(r - j - 6) + h(j - 15) \\ & + h(r - j - 15). \end{aligned}$$

Thus,

$$n \geq \Delta(\delta-1)+(\delta-2)\sqrt{\Delta(\delta-3)} + \left(\frac{2r-45}{6}\right)(2\delta^2-5\delta+5+2\varepsilon_\delta)+\frac{5}{3}\delta^2-\frac{35}{6}\delta+\frac{44}{6}+\frac{5}{3}\varepsilon_\delta.$$

Case 4: $r-14 \leq j \leq r-6$.

Then the sets $N_{\leq 3}(z)$, $N_{\leq 2}(V(P'_{0,4}))$, $N_{\leq 2}(V(P'_{9,j-6}))$, $N_{\leq 2}(V(P'_{j+6,r}))$, and $N_{\leq 2}(V(P''_{9,j-6}))$ are disjoint. Applying Lemmas 2.3.10 and 3.2.4 we obtain

$$n \geq \Delta(\delta-1)+(\delta-2)\sqrt{\Delta(\delta-3)}+\frac{3}{2}+h(4)+h(j-15)+h(r-j-6)+h(j-15),$$

and so,

$$n \geq \Delta(\delta-1)+(\delta-2)\sqrt{\Delta(\delta-3)} + \left(\frac{2r-38}{6}\right)(2\delta^2-5\delta+5+2\varepsilon_\delta)+\frac{4}{3}\delta^2-\frac{14}{3}\delta+\frac{29}{6}+\frac{4}{3}\varepsilon_\delta$$

Case 5: $r-5 \leq j$.

Then the sets $N_{\leq 3}(z)$, $N_{\leq 2}(V(P'_{0,4}))$, $N_{\leq 2}(V(P'_{9,r-11}))$, and $N_{\leq 2}(V(P''_{9,r-11}))$ are disjoint. Applying Lemmas 2.3.10 and 3.2.4 we obtain

$$n \geq \Delta(\delta-1)+(\delta-2)\sqrt{\Delta(\delta-3)}+\frac{3}{2}+h(4)+h(r-20)+h(r-20).$$

Thus,

$$n \geq \Delta(\delta-1)+(\delta-2)\sqrt{\Delta(\delta-3)} + \left(\frac{2r-36}{6}\right)(2\delta^2-5\delta+5+2\varepsilon_\delta)+\delta^2-\frac{7}{2}\delta+4+\varepsilon_\delta.$$

Clearly, from the different cases considered, we can see that the smallest lower bound for n is always the expression in Case 3, so we have, irrespective of the value of j that

$$n \geq \Delta(\delta-1)+(\delta-2)\sqrt{\Delta(\delta-3)} + \left(\frac{2r-45}{6}\right)(2\delta^2-5\delta+5+2\varepsilon_\delta)+\frac{5}{3}\delta^2-\frac{35}{6}\delta+\frac{44}{6}+\frac{5}{3}\varepsilon_\delta.$$

Solving for r now yields

$$\begin{aligned} r &\leq \frac{3n-3\Delta(\delta-1)}{2\delta^2-5\delta+5+2\varepsilon_\delta}-\frac{3(\delta-2)\sqrt{\Delta(\delta-3)}}{2\delta^2-5\delta+5+2\varepsilon_\delta}+\frac{40\delta^2-95\delta+181/2+40\varepsilon_\delta}{2\delta^2-5\delta+5+2\varepsilon_\delta}, \\ &< \frac{3n-3\Delta(\delta-1)}{2\delta^2-5\delta+5+2\varepsilon_\delta}-\frac{3(\delta-2)\sqrt{\Delta(\delta-3)}}{2\delta^2-5\delta+5+2\varepsilon_\delta}+20. \end{aligned}$$

This yields inequality (3.2.19), thus completing the proof of the Theorem 3.2.13. □

Chapter 4

Upper Bounds on the Average Eccentricity of Graphs of Girth at least 6 and (C_4, C_5)-free Graphs.

4.1 Introduction

In this Chapter, we give bounds on the average eccentricity of graphs of girth at least 6, as well as connected (C_4, C_5)-free graphs taken into account the minimum degree and the order of the graphs. To achieve this goal, we adapt the approach given in [30] and [31] wherein the average eccentricity of graphs is bounded. Moreover, we show that for certain values of δ the bounds obtained for graphs of girth at least 6 are sharp apart from an additive constant. We also prove upper bounds on the average eccentricity that take into account not only order and minimum degree, but also maximum degree. Our bound is best possible in a sense specified later.

4.2 Preliminary Results

We first present a definition of the weighted eccentricity and weighted average eccentricity and a bound on the weighted average eccentricity. Both play an important role in the proof of our main results in this chapter.

Definition 4.2.1. *[31] Let G be a connected graph and $c : V(G) \to \mathbb{R}$ be a nonnegative weight function on the vertices of G. Then the eccentricity of G with respect to c is defined by*

$$EX_c(G) = \sum_{v \in V(G)} c(v)e_G(v).$$

Let $N = \sum_{v \in V(G)} c(v)$ be the total weight of the vertices of G. If $N > 0$, then we define the weighted average eccentricity of G with respect to c as

$$\text{avec}_c(G) = \frac{EX_c(G)}{\sum_{v \in V(G)} c(v)}.$$

Lemma 4.2.2. *[31] Let G be a weighted graph with a weight function $c : V(G) \to \mathbb{R}$ such* *that* $c(v) \geq 1$ *for every vertex* v*. Then for* $N = \sum_{v \in V(G)} c(v)$,

$$\operatorname{avec}_c(G) \leq \operatorname{avec}(P_{\lceil N \rceil}).$$

Proof. Let $k = \frac{\lceil \sum_{v \in V(G)} c(v) \rceil}{\sum_{v \in V(G)} c(v)}$ and $\overline{c}(v) = k\, c(v)$ be a new weight function defined on G. We now prove the following.

i) $\overline{c}(v) \geq 1$,

ii) $\sum\limits_{v \in V(G)} \overline{c}(v) = \lceil N \rceil$,

iii) $\operatorname{avec}_{\overline{c}}(G) = \operatorname{avec}_c(G)$.

Clearly $\overline{c}(v) \geq 1$ since $c(v)$ is a nonnegative weight function and $k \geq 1$. The total weight of vertices of G with respect to the new function is given by $\sum\limits_{v \in V(G)} \overline{c}(v)$, so the second statement follows from

$$\begin{aligned} \sum_{v \in V(G)} \overline{c}(v) &= \sum_{v \in V(G)} kc(v) \\ &= k \sum_{v \in V(G)} c(v) \\ &= \frac{\lceil \sum_{v \in V(G)} c(v) \rceil}{\sum_{v \in V(G)} c(v)} \sum_{v \in V(G)} c(v) \\ &= \lceil N \rceil. \end{aligned} \tag{4.2.1}$$

By Definition 4.2.1, we have that

$$\begin{aligned} \operatorname{avec}_{\overline{c}}(G) &= \frac{\sum_{v \in V(G)} \overline{c}(v) e_G(v)}{\sum_{v \in V(G)} \overline{c}(v)} \\ &= \frac{\sum_{v \in V(G)} kc(v) e_G(v)}{\sum_{v \in V(G)} kc(v)} \\ &= \frac{k \sum_{v \in V(G)} c(v) e_G(v)}{k \sum_{v \in V(G)} c(v)} \\ &= \operatorname{avec}_c(G), \end{aligned} \tag{4.2.2}$$

and the third statement follows.

Hence, it suffices to show that $\operatorname{avec}_{\overline{c}}(G) \leq \operatorname{avec}(P_{\lceil N \rceil})$. We prove the equivalent statement $EX_{\overline{c}}(G) \leq EX(P_{\lceil N \rceil})$. Since deleting an edge does not decrease the eccentricity of any vertex, we have $\operatorname{avec}(G) \leq \operatorname{avec}(T)$ for every spanning tree T of G. Hence it suffices to prove the bounds for trees.

Given $N \in \mathbb{R}^+$, let T be a tree with weight function $\overline{c}$ such that $\overline{c} \geq 1$ and $\lceil N \rceil =$

$\sum_{v \in V(T)} \overline{c}(v)$ satisfying the hypothesis of the lemma for which $EX_{\overline{c}}(T)$ is maximum. We show that $EX_{\overline{c}}(T) \leq EX(P_{\lceil N \rceil})$ holds. To do this, we first show that T is a path, then we show that $T = P_{\lceil N \rceil}$.

CLAIM 1: T is a path.

Suppose T is not a path. Then T has a vertex of degree at least 3.

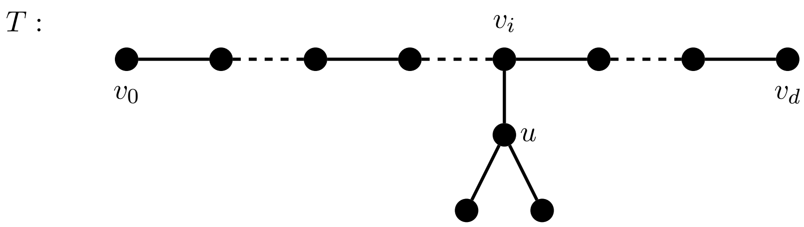

Figure 4.1: The spanning tree T with a vertex of degree 3.

Let $P = v_0, v_1, v_2, v_3, \cdots, v_d$ be a diametral (longest) path of T of length d and let v_i be a vertex on P of degree at least three. Since P is a diametral path of T, we have that for each vertex $w \in V(T)$, at least one of the vertices v_1 and v_d is an eccentric vertex, hence by Lemma 1.4.21

$$e_T(w) = \max\{d_T(w, v_0), d_T(w, v_d)\}. \tag{4.2.3}$$

Let u be a neighbour of v_i in T which is not on P and let T_u be the component of $T - V(P)$ containing u. Without loss of generality, we let v_1 to be an eccentric vertex of u and thus of every vertex of T_u.

Now we consider the tree $\overline{\mathcal{T}}$ with weight function $\overline{c}$ defined by $\overline{\mathcal{T}} = T - uv_i + uv_d$. Then for each vertex in T_u, the distance to v_1 increased by $d - i$, hence its eccentricity also has increased by $d - i$. On the other hand, for every vertex, x, not in T_u, the distances to v_1 and v_d remains unchanged, hence its eccentricity has not decreased. Thus for all $x \in V(T) - V(T_u)$, we have by Lemma 1.4.21 that

$e_{\overline{\mathcal{T}}}(x) \geq \max\{d_{\overline{\mathcal{T}}}(x, v_0), d_{\overline{\mathcal{T}}}(x, v_d)\} = \max\{d_T(x, v_0), d_T(x, v_d)\} = e_T(x)$. Therefore,

$$EX_{\overline{c}}(\overline{\mathcal{T}}) \geq EX_{\overline{c}}(T) + \overline{c}(d - i)|V(T_u)| > EX_{\overline{c}}(T),$$

a contradiction to our choice of T since $EX_{\overline{c}}(T)$ is maximum. Hence T is a path $v_0, v_1, v_2, \cdots, v_d$ and so Claim 1 holds.

CLAIM 2: all internal vertices of T have weight exactly 1.

Suppose not, then there is an internal vertex, v_j, of $V(T)$ with weight greater than 1. That is, there exists $j \in \{1, 2, 3, \cdots, d\}$ with $\overline{c}(v_j) = 1 + k$ where k is a positive real number.

Next, we define a new weight function, c', on $V(T)$ for which $c'(v_j) = 1$ and move the extra weight on v_j to one of the end vertices of T, say v_0. Thus, for all $v_i \in V(T)$, the new weight function $c'(v_i)$ is defined by

$$c'(v_i) = \begin{cases} 1 & \text{if } i = j, \\ \overline{c}(v_i) & \text{if } i \neq j \text{ or } i \neq 0, \\ \overline{c}(v_i) + k & \text{if } i = 0. \end{cases}$$

Then,

$$\begin{aligned} EX_{c'}(T) - EX_{\overline{c}}(T) &= \sum_{i=0}^{d} c'(v_i)e_T(v_i) - \sum_{i=0}^{d} \overline{c}(v_i)e_T(v_i), \\ &= \sum_{i=0}^{d} e_T(v_i)[c'(v_i) - \overline{c}(v_i)], \\ &= e_T(v_0)[c'(v_0) - \overline{c}(v_0)] + e_T(v_j)[c'(v_j) - \overline{c}(v_j)] \\ &= e_T(v_0)[\overline{c}(v_0) + k - \overline{c}(v_0)] + e_T(v_j)[1 - (1 + k)], \\ &= k[e_T(v_0) - e_T(v_j)], \end{aligned}$$

and so, $EX_{c'}(T) = EX_{\overline{c}}(T) + k[e_T(v_0) - e_T(v_i)]$. We have that $e_T(v_j) < e_T(v_0)$ since v_j is an internal vertex of T. Hence, $EX_{c'}(T) > EX_{\overline{c}}(T)$, is a contradiction to our choice of T and assumption that $EX_{\overline{c}}(T)$ is maximum. Therefore, we conclude that all internal vertices of T must have weight exactly 1.

CLAIM 3: the end vertices of T also have weight exactly 1.

Suppose not, then at least one of v_0 or v_d has weight greater than 1. If one of them, say $c(v_0) > 1$, then similarly as in the proof for the internal vertices, we can define a new weight function, $c''(v_k)$, for which $c''(v_0) = 1$ and transfer the extra weight to v_d. Since we have shown that all internal vertices of T has weight exactly 1, $c''(v_k)$ is then defined by,

$$c''(v_k) = \begin{cases} 1 & \text{if } i = 0, 1, 2, \ldots d - 1, \\ \overline{c}(v_d) + \overline{c}(v_0) - 1 & \text{if } i = d. \end{cases}$$

Clearly $c''(v_k) \geq 1$ for all $v_k \in V(T)$ and $\sum_{v_k \in V(T)} c''(v_k) = \lceil N \rceil$.

Observe that $e_T(v_0) = e_T(v_n) = d$ and so we have that

$$EX_{c''}(T) = c''(v_0)e_T(v_0) + \sum_{k=1}^{d-1} c''(v_k)e_T(v_k) + c''(v_d)e_T(v_d),$$

$$
\begin{aligned}
&= e_T(v_0) + \sum_{k=1}^{d-1} \overline{c}(v_k)e_T(v_k) + [\overline{c}(v_d) + \overline{c}(v_0) - 1]e_T(v_d) \\
&= \sum_{k=0}^{d-1} \overline{c}(v_k)e_T(v_k) + \overline{c}(v_d)e_T(v_d) + \overline{c}(v_0)e_T(v_d), \\
&= \sum_{k=1}^{d-1} \overline{c}(v_k)e_T(v_k) + \overline{c}(v_d)e_T(v_d) + \overline{c}(v_0)e_T(v_0), \\
&= \sum_{k=0}^{d} \overline{c}(v_k)e_T(v_k), \\
&= EX_{\overline{c}}(T).
\end{aligned}
$$

Hence $EX_{c''}(T)$ is also maximal. Next we show that $\overline{c}(v_d) = 1$.

Suppose to the contrary that $\overline{c}(v_d) > 1$. Since $v_0, v_1, \ldots, v_{d-1}$ have weight 1, and since the sum of all weights is an integer, we have $\overline{c}(v_d) \geq 2$ then $\overline{c}(v_d) \geq 2$.

Thus, we can define a new graph $T^* = T + v_{n+1}$ and transfer one weight unit from v_n to v_{d+1}. In other words, T^* is the path $v_0, v_1, v_2, \ldots v_d, v_{d+1}$ with weight function $c^*(v_k)$ where

$$
c^*(v_k) = \begin{cases} 1 & \text{if } i = 0, 1, 2, \ldots d-1 \text{ and } i = v_{d+1} \\ \overline{c}(v_d) - 1 & \text{if } i = d. \end{cases}
$$

Clearly T^* is a weighted tree with higher eccentricity than T, that is, $EX_{\overline{c}'}(T) > EX_{\overline{c}}(T)$, a contradiction to the assumption that $EX_{\overline{c}}(T)$ is maximum. Therefore, we conclude that $\overline{c}(v_d) = 1$ and $T^* = T = v_0, v_1, v_2, \ldots, v_d$.

CLAIM 4: $T = P_{\lceil N \rceil}$.

Let $d + 1 = |V(T)|$. Then it follows from **Claim 1** that $T = v_0, v_1, v_2, \cdots, v_d$ is a path of order $d + 1$. We have by **Claim 2** and **Claim 3** that all vertices of T have weight exactly 1. Since $\overline{c} = 1$ for all vertices of T, it follows that the total weight of vertices of T, $\sum_{v \in V(T)} \overline{c}(v) = d + 1$. But from (4.2.1), $\sum_{v \in V(T)} \overline{c}(v) = \lceil N \rceil$, and so $d + 1 = \lceil N \rceil$. Therefore, $T = P_{d+1} = P_{\lceil N \rceil}$. This proves Claim 4.

Since $T = P_{\lceil N \rceil}$, we then have that $EX_{\overline{c}}(T) = EX_{\overline{c}}(P_{\lceil N \rceil})$ and so $\text{avec}_{\overline{c}}(T) = \text{avec}_{\overline{c}}(P_{\lceil N \rceil})$. By (4.2.2), $\text{avec}_{\overline{c}}(P_{\lceil N \rceil}) = \text{avec}(P_{\lceil N \rceil})$, so we have that $\text{avec}_{\overline{c}}(T) = \text{avec}(P_{\lceil N \rceil})$. Hence we conclude that $\text{avec}(T) \leq \text{avec}(P_{\lceil N \rceil})$ and thus $\text{avec}_{\overline{c}}(G) \leq \text{avec}(P_{\lceil N \rceil})$. And so, Lemma 4.2.2 holds. □

The following result appeared in [31]. The proof follows the proof in [31], but we have added some elaboration.

Theorem 4.2.3. *[31] Let G be a connected graph of order n and minimum degree δ. Then*

$$\operatorname{avec}(G) \leq \frac{9}{4}\left\lceil \frac{n}{\delta+1} \right\rceil + \frac{5}{2}.$$

This inequality is best possible apart from the additive constant.

Proof. Recall that for $k \in \mathbb{N}$, a k-packing is a set of vertices whose pairwise distance is greater than k. To prove this, we start first by finding a maximal 2-packing, A, of G as follows. Choose an arbitrary vertex u_1 of G and let $A = \{u_1\}$. If there exists a vertex u_2 in G with $d_G(u_2, A) = 3$, add u_2 to A. Add vertices u_j with $d_G(u_j, A) = 3$ to A until, after k steps, say, every vertex not in A is within distance two of A. Thus $A = \{u_1, u_2, \ldots u_k\}$ and $|A| = k$.

Let $N[A]$ denote the vertex set consisting of A and any vertex adjacent to A. Let $T_1 \leq G$ be the subforest of G with vertex set $N[A]$, whose edge set consists of all edges incident with a vertex in A, such that each component of T_1 is a star centered at a vertex in A. By our construction of A, there exist $|A| - 1$ edges in G, each one joining two neighbours of distinct vertices of A, whose addition to T_1 yields a subtree T_2 of G. Now, each vertex $v \in V(G) - V(T_2)$ is adjacent to some vertex $v' \in V(T_2)$. Let T be a spanning tree of G with edge set $E(T) = E(T_2) \cup \{vv' : v \in V(G) - V(T_2)\}$. Since taking a spanning tree or deleting edges does not decrease the eccentricity or average eccentricity and we have $\operatorname{avec}(G) \leq \operatorname{avec}(T)$, so it suffices to prove that

$$\operatorname{avec}(T) \leq \frac{9}{4}\left\lceil \frac{n}{\delta+1} \right\rceil + \frac{5}{2}. \tag{4.2.4}$$

For every vertex $u \in V(T)$, let u_A be a vertex in A closest to u in T. We define a weight function $c : V(T) \to \mathbb{R}^+$ by

$$c(u) = |\{x \in V(T) \mid x_A = u\}| \text{ for } u \in V(T),$$

where $c(u) = 0$ if $u \notin A$, $c(u) \geq \deg(u) + 1 \geq \delta + 1$ for each $u \in A$, and $\sum_{u \in V(T)} c(u) = n$. For each vertex u of T, the weight of u is moved to u_A and since each vertex of T is within distance two of u_A, each weight was moved over a distance not exceeding two, hence

$$\operatorname{avec}(T) \leq \operatorname{avec}_c(T) + 2. \tag{4.2.5}$$

Observe that the weight c is concentrated only on the vertices of A. Next, let U be the induced subgraph $T^3[A]$ of T^3 with A as the vertex set.

Claim 1: U is connected.

To verify that U is connected, it suffices to prove using induction on i that for any $u_i \in$

$A, i \in \{1, 2, \cdots, k\}$, there exists a path from u_i to u_1 in U. For $i = 1$, we have u_1 and hence there is a walk from u_1 to u_1 of zero length. For $i > 1$, we have by our construction of A, that there is an j for which $j < i$, and $d_T(u_j, u_i) = 3$. Therefore, by the induction hypothesis, there is a path from u_j to u_1 in $U = T^3[A]$. This path together with the edge $u_j u_i$ in U yields a path from u_i to u_1 in U. Hence U is connected.

Since A is a maximal 2-packing, we have that $d_T(u_i, u_j) \geq 3$ for all pairs $u_i, u_j \in A$. It follows immediately that $d_T(u_i, u_j) \leq 3d_U(u_i, u_j)$ for all pairs of vertices $u_i, u_j \in A$. Since every vertex of T that is not in A is within distance two of A, we have that $e_T(u_i) \leq 3e_U(u_i) + 2$ for each vertex $u_i \in A$. Hence

$$\mathrm{avec}_c(T) \leq 3\mathrm{avec}_c(U) + 2. \tag{4.2.6}$$

Recall that for all vertices $u \in A$, $c(u) \geq \delta + 1$ and since $n = \sum_{u \in A} c(u)$, we have that $|A|(\delta + 1) \leq n$ and so $|A| \leq n/(\delta + 1)$.

To normalize the weight, $c(u)$, on the vertices of A, we now define a new weight $c'(u)$ on $V(U)$ by $c'(u) = c(u)/(\delta + 1)$. Observe that $c'(u) \geq 1$ for all $u \in V(U)$ and $\sum_{u \in A} c'(u) = 1/(\delta + 1) \sum_{u \in A} c(u) = n/(\delta + 1)$. Letting $N = \sum_{u \in A} c'(u)$, we have that $N = \frac{n}{\delta+1}$ and $|A| \leq N$.

We have that

$$\mathrm{avec}_{c'}(U) = \frac{EX_{c'}(U)}{\sum_{u \in A} c'(u)} = \frac{1/(\delta + 1) \sum_{u \in A} c(u) e_U(u)}{1/(\delta + 1) \sum_{u \in A} c(u)} = \mathrm{avec}_c(U). \tag{4.2.7}$$

Now, $c'(u_i) \geq 1$ for all u_i in A. Applying (4.2.7), Lemma 4.2.2 and Theorem 1.4.18(c), we have that

$$\mathrm{avec}_c(U) = \mathrm{avec}_{c'}(U) \leq \mathrm{avec}(P_{\lceil N \rceil}) \leq \frac{3\lceil N \rceil}{4} - \frac{1}{2}$$

In order to bound $\mathrm{avec}(T)$, we apply the inequalities in (4.2.5) and (4.2.6) together with the fact that $N = \frac{n}{\delta+1}$. Thus

$$\begin{aligned} \mathrm{avec}(T) &\leq \mathrm{avec}_c(T) + 2, \\ &\leq 3\mathrm{avec}_c(U) + 2 + 2, \\ &\leq 3\Big(\frac{3\lceil N \rceil}{4} - \frac{1}{2}\Big) + 4, \\ &= \frac{9\lceil N \rceil}{4} + \frac{5}{2}, \\ &= \frac{9}{4}\left\lceil \frac{n}{\delta + 1} \right\rceil + \frac{5}{2}. \end{aligned}$$

Thus, we have proved (4.2.4). Therefore the theorem holds since $\mathrm{avec}(G) \leq \mathrm{avec}(T)$.

The graph, $G_{k,\delta}$, described below shows that the bound in (4.2.4) is sharp apart from an additive constant.

For positive integers, k, n, δ, let $n = k(\delta + 1)$ and let $x_i y_i \in E(G_i)$ where $G_i = K_{\delta+1}$ for $i = 1, 2, \ldots, k$. Let $G_{k,\delta}$ be the graph obtained from $G_1 \cup G_2 \cup \ldots \cup G_k$ by deleting the edges $x_i y_i$ for $i = 2, 3, \ldots, k-1$ and adding the edges $x_{i+1} y_i$ for $i = 1, 2, \ldots, k-1$. Then for large n,

$$\operatorname{avec}(G_{k,\delta}) = \frac{9n}{4\delta + 4} + O(1).$$

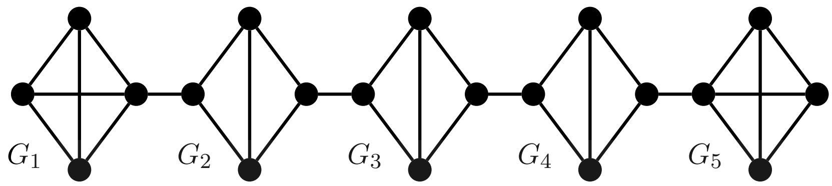

Figure 4.2: The graph $G_{k,\delta}$ with $\delta = 3$ and $k = 5$.

□

4.3 Main Results

4.3.1 Bounds on Average Eccentricity of Graphs of Girth at least 6

If G is a graph of girth at least 6, then the following theorem shows that the bound in Theorem 1.4.23 can be improved by a factor of about 5/3.

Theorem 4.3.1. *Let G be a graph of girth at least 6 with n vertices and minimum degree $\delta \geq 2$. Then G has a spanning tree T with*

$$\operatorname{avec}(T) \leq \frac{9}{2}\left\lceil \frac{n}{2(\delta^2 - \delta + 1)} \right\rceil + 8. \tag{4.3.1}$$

Proof. To prove this, we start by finding a matching, M, of G as follows: Choose an arbitrary edge $e_1 = uv \in E(G)$ and let $M = \{e_1\}$. Let $V(M)$ be the set of vertices incident with some edge of M. Recall that for an edge e, $d_G(e, V(M))$ is the smallest distance between a vertex incident with e and a vertex in $V(M)$. If there exists an edge e_2 in G with $d_G(e_2, V(M)) = 5$, add e_2 to M. Add edges e_i with $d_G(e_i, V(M)) = 5$ to M until each of the edges not in M is within distance four of M. Thus $M = \{e_1, e_2, \cdots e_k\}$ where $|M| = k$.

Let $N_{\leq 2}(u)$ denote the set of vertices at distance at most 2 from u. For $i \in \{1, 2, \ldots, k\}$ let $e_i = u_i v_i$. Let $T^*_{e_i}$ be a subtree of G with vertex set $N_{\leq 2}(u_i) \cup N_{\leq 2}(v_i)$ that preserves the distance to $\{u_i v_i\}$.

Let $T_1 = \cup_{e_i \in M} T^*_{e_i}$. Then, $T_1 \leq G$ is a subforest of G with vertex set $N_{\leq 2}[V(M)]$. By our construction of M, there exist $|M|-1$ edges in G, each joining two distinct components of T_1, whose addition to T_1 yields a tree $T_2 \leq G$, which contains T_1 and has the same vertex set as T_1.

Now, each vertex $v \in V(G) - V(T_2)$ is within distance five of some vertex w in $V(T_2)$.

Let $T \geq T_2$ be a spanning tree of G in which $d_T(x, V(M)) = d_G(x, V(M))$ for each $x \in V(G)$.

Since taking a spanning tree or removal of an edge, which is not a bridge, does not decrease the eccentricity or average eccentricity and we have $\text{avec}(G) \leq \text{avec}(T)$, so it suffices to show that

$$\text{avec}(T) \leq \frac{9}{2}\left\lceil \frac{n}{2(\delta^2 - \delta + 1)} \right\rceil + 8. \tag{4.3.2}$$

For every vertex $u \in V(T)$, let u_M be a vertex in $V(M)$ closest to u in T. The tree, T, can be viewed as a weighted tree where each vertex has weight exactly 1. We now move the weight of every vertex to the closest vertex in $V(M)$, that is, we define a weight function $c : V(T) \to \mathbb{R}^+$ by:

$$c(u) = |\{x \in V(M) \mid x_M = u\}| \quad \text{for } u \in V(T).$$

Since each vertex of T is within distance five from some vertex in M, we have that $d_T(x, x_M) \leq 5$ and each weight was moved over a distance not exceeding five, hence

$$\text{avec}(T) \leq \text{avec}_c(T) + 5. \tag{4.3.3}$$

Observe that the weight c is concentrated only on the vertices of $V(M)$. Our interest is to ensure that the weights are concentrated on the edges of T. We consider the line graph $L = L(T)$ and define a new weight function $\overline{c}$ on $V(L) = E(T)$ by

$$\overline{c}(wz) = \begin{cases} c(w) + c(z) & \text{if } wz \in M, \\ 0 & \text{if } wz \notin M. \end{cases}$$

Let $wz \in M$. For each vertex $x \in N_{\leq 2}(w) \cup N_{\leq 2}(z)$, we have $x_M \in \{w, z\}$. Hence

$$c(w) + c(z) \geq |N_{\leq 2}(w) \cup N_{\leq 2}(z)|.$$

By Lemma 2.3.2, we have

$$|N_{\leq 2}(w) \cup N_{\leq 2}(z)| \geq 2(\delta^2 - \delta + 1), \tag{4.3.4}$$

and so we have that

$$\bar{c}(wz) \geq 2\delta^2 - 2\delta + 2 \quad \text{for } wz \in M. \tag{4.3.5}$$

Note that $\bar{c}(wz) = 0$ if $wz \notin M$ and $\sum_{e\in M} \bar{c}(wz) = \sum_{u\in V(T)} c(u) = n$.

Claim 1: Let $x, y \in V(T)$ and let $e_x, e_y \in E(T)$ be edges of T incident with x and y respectively. Then,

$$d_T(x, y) \leq d_L(e_x, e_y) + 1. \tag{4.3.6}$$

Let $k = d_L(e_x, e_y)$. Let the vertices of L on a shortest (e_x, e_y)-path be $e_x, f_1, f_2, \ldots, f_{k-1}, e_y$. So $\{e_x, f_1, f_2, \ldots, f_{k-1}, e_y\}$ induces a connected subgraph of T with $k+1$ edges, and it contains vertices x and y. Hence, $d_T(x, y) \leq k + 1 = d_L(e_x, e_y) + 1$.

Assume that $u, v \in V(T)$, and that v is an eccentric vertex of u. Then $e_T(u) = d_T(u, v)$. Let $e_u \in E(T)$ be an edge incident with u and e_v. We have by Claim 1 that $d_T(u, v) \leq d_L(e_u, e_v) + 1 \leq e_L(e_u) + 1$. Hence, $e_T(u) \leq e_L(e_u) + 1$. Consequently, we have that

$$\begin{aligned} \sum_{v\in V(T)} e_T(v)c(v) &= \sum_{uv\in M} [e_T(u)c(u) + e_T(v)c(v)], \\ &\leq \sum_{uv\in M} \bar{c}(uv)\big(e_L(uv) + 1\big), \\ &= \sum_{e\in M} \bar{c}(e)e_L(e) + \sum_{e\in M} \bar{c}(e). \end{aligned}$$

It follows immediately from above that $EX_c(T) \leq EX_{\bar{c}}(L) + \sum_{e\in M} \bar{c}(e)$ and since $\sum_{e\in M} \bar{c}(e) = \sum_{v\in V(T)} c(v)$ and $\text{avec}(G) = EX_c(G)/\sum_{v\in V(T)} c(v)$, we have that

$$\text{avec}_c(T) \leq \text{avec}_{\bar{c}}(L) + 1. \tag{4.3.7}$$

Now, if the distance $d_T(e_1, e_2)$ between two matching edges e_1, e_2 in M equals five, then $d_L(e_1, e_2) \leq 6$. Let $\mathcal{U} = L^6[M]$. Following a similar argument as in the proof of Claim 1 of Theorem 4.2.3, we have that $L^6[M]$ is connected, and for all pairs $e, f \in M$,

$$d_L(e, f) \leq 6d_{L^6[M]}(e, f).$$

Now, for every $e' \in V(L) = E(T)$, there exists an edge $f' \in M$ such that $d_L(e', f') \leq 5$, and so for every $f \in M$, we have that $e_L(f) \leq 6e_{L^6[M]}(f) + 5$.

$$\text{avec}_{\bar{c}}(L) \leq 6\text{avec}_{\bar{c}}(L^6[M]) + 5. \tag{4.3.8}$$

Recall that for all edges in M, $\bar{c}(e) \geq 2\delta^2 - 2\delta + 2$ and since $n = \sum_{e\in M} \bar{c}(e) = \sum_{v\in V(T)} c(v)$, we have that $|M|(2\delta^2 - 2\delta + 2) \leq n$ and so $|M| \leq n/(2\delta^2 - 2\delta + 2)$.

To normalise the weight, $c(u)$, on the vertices of $L^6[M]$, we now define a new weight, $\bar{c}'$, on $V(L^6[M])$ by $\bar{c}'(v) = \bar{c}(v)/(2\delta^2 - 2\delta + 2)$. Observe that $c'(u) \geq 1$ for all $u \in V(L^6[M])$

and

$$\sum_{v\in V(L^6[M])}\overline{c}'(v)=\frac{1}{2\delta^2-2\delta+2}\sum_{v\in L^6[M]}c(v)=\frac{n}{2\delta^2-2\delta+2}.$$

Letting $N^*=\sum_{v\in V(L^6[M])}\overline{c}'(v)$, we have that $N^*=\frac{n}{2\delta^2-2\delta+2}$ and $|M|\leq N^*$.

We have that $\text{avec}_{\overline{c}'}(L^6[M])=\text{avec}_{\overline{c}}(L^6[M])$, since

$$\begin{aligned}\text{avec}_{\overline{c}'}(L^6[M]) &= \frac{EX_{\overline{c}'}(L^6[M])}{\sum_{v\in V(L^6[M])}\overline{c}'(v)},\\ &= \frac{\frac{1}{2\delta^2-2\delta+2}\sum_{v\in A}c(v)e_U(v)}{\frac{1}{2\delta^2-2\delta+2}\sum_{v\in V(L^6[M])}\overline{c}(v)},\\ &= \text{avec}_{\overline{c}}(L^6[M]).\end{aligned}\tag{4.3.9}$$

Now, $\overline{c}'(e_i)\geq 1$ for all e_i in M. Applying (4.3.9), Lemma 4.2.2 and Theorem 1.4.18(c), we have that

$$\text{avec}_{\overline{c}}(L^6[M])=\text{avec}_{\overline{c}'}(L^6[M])\leq\text{avec}(P_{\lceil N^*\rceil})\leq\frac{3\lceil N^*\rceil}{4}-\frac{1}{2}.$$

To bound avec(T), we apply the inequalities in (4.3.3), (4.3.7) and (4.3.8) in conjunction with the fact that $N^*=\frac{n}{2\delta^2-2\delta+2}$.

$$\begin{aligned}\text{avec}(T) &\leq \text{avec}_c(T)+5,\\ &\leq \text{avec}_{\overline{c}}(L)+6,\\ &\leq 6\text{avec}_{\overline{c}}(L^6[M])+11,\\ &\leq 6\Big(\frac{3\lceil N^*\rceil}{4}-\frac{1}{2}\Big)+11,\\ &= \frac{9}{2}\lceil N^*\rceil+8\\ &= \frac{9}{2}\Big\lceil\frac{n}{2\delta^2-2\delta+2}\Big\rceil+8..\end{aligned}$$

This completes the proof of Theorem 4.3.1.

We now prove that for $\delta-1$ a prime power. the above bound in Theorem 4.3.1 is sharp apart from the additive constant. This is shown in the next theorem.

Theorem 4.3.2. *If $\delta-1$ is a prime power, then there exists an infinite family of graphs of girth at least 6, G, with n vertices and minimum degree δ such that*

$$\text{avec}(G)\geq\frac{9n}{2\kappa_{\delta}.}-5,\tag{4.3.10}$$

where $\kappa_\delta:=2(\delta^2-\delta+1)$.

To prove this theorem, we let H^*_{q+1} be the graph described in Example 2.3.5.

Recall that H^*_{q+1} be the graph whose vertices are the 1-dimensional and 2-dimensional subspaces of $GF(q)^3$. Let U be the set of all 1-dimensional subspaces of $GF(q)^3$ and V be the set of all 2-dimensional subspaces of $GF(q)^3$. Two vertices, $\langle \underline{w} \rangle \in U$ and $\langle \underline{u}, \underline{v} \rangle \in V$ are said to be adjacent in H^*_{q+1} if and only if $\langle \underline{w} \rangle$ is contained in $\langle \underline{u}, \underline{v} \rangle$.

By Claim 2.3.6, H^*_{q+1} has girth 6 and contains no 4-cycle. H^*_{q+1} has $2(q^2+q+1)$ vertices, each vertex is of degree $q+1$. Moreover for any two vertices of u and v of H^*_{q+1}, it is easy to verify that $d_{H^*_{q+1}}(u,v) \leq 3$ and so $\text{diam}(H^*_{q+1}) = 3$.

Let $u \in V(H^*_{q+1})$ be a fixed vertex of H^*_{q+1} and let v be a neighbour of u. Let H_e be the graph $H^*_{q+1} - uv$.

By Claim 2.3.6(f), we have that for any two vertices, k and l, of H_e, $d_{H_e}(k,l) \geq 5$ and so $\text{diam}(H_e) \geq 5$. H_e has $2(q^2+q+1)$ vertices and minimum degree δ.

Let $G^*_{k,\delta}$ be the graph obtained from the union of $k-2$ copies of $\mathcal{H}_2, \mathcal{H}_3, \ldots, \mathcal{H}_{k-1}$ of H_e and two copies $\mathcal{H}_1$ and $\mathcal{H}_k$ by adding the edges $v^{(i)}u^{(i+1)}$ for every $(1 \leq i < k)$ where u^i and v^i are the vertices of $\mathcal{H}_i$ corresponding to the vertices u and v, respectively of H^*_{q+1}.

Since $G^*_{k,\delta}$ is the graph obtained from the union of both H_e and H^*_{q+1}, we have by Claim 2.3.6 (a and c) that $G^*_{k,\delta}$ is bipartite, contains no 4-cycle and has of girth at least 6. Moreover, since both H^*_{q+1} and H_e have $2(q^2+q+1)$ vertices, we have that there are $2k(q^2+q+1)$ vertices in $G^*_{k,\delta}$. The degree of vertices of $G^*_{k,\delta}$ is either $q+1$ or $q+2$, hence the minimum degree of $G^*_{k,\delta}$ is $\delta = q+1$. Therefore, $q = \delta - 1$ is a prime power and $n = |V(G^*_{k,\delta})| = 2k(q^2+q+1) = 2k(\delta^2 - \delta + 1)$.

In order to bound the average eccentricity of $G^*_{k,\delta}$ from below choose vertices u^* of $\mathcal{H}_1$ and v^* of $\mathcal{H}_k$ with $d(u^*, v^1) = d(u^k, v^*) = 3$. Since H^*_{q+1} has girth at least 6, the distance between $u^{(i)}$ and $v^{(i)}$ in $\mathcal{H}_i$ is at least 5 for $i = 2, 3, \ldots, k-1$. Clearly $\text{diam}(G^*_{k,\delta}) = d(u^*, v^*) \geq 6k - 5 = \frac{3n}{\delta^2 - \delta + 1} - 5$.

If $w \in V(\mathcal{H}_i)$, then $e(w) = d(w, v^*) \geq d(v^i, v^*) = 6(k-1-i) + 4$ if $i \leq \frac{k}{2}$, and $e(w) \geq d(w, u^*) \geq d(u^i, v^*) = 6(i-2) + 4$ if $i > \frac{k}{2}$. Hence

$$e_{\mathcal{H}_i}(w) \geq \begin{cases} 6(k-i) - 2 & \text{if } 1 \leq i \leq \frac{k}{2}, \\ 6(i-1) - 2 & \text{if } \frac{k}{2} + 1 \leq i \leq k, \end{cases}$$

and so,

$$\begin{aligned} EX(G^*_{k,\delta}) &= \sum_{i=1}^{k/2} \sum_{w\in V(\mathcal{H}_i)} e(w) + \sum_{i=k/2+1}^{k} \sum_{w\in V(\mathcal{H}_i)} e(w) \\ &\geq \sum_{i=1}^{k/2} 2(\delta^2-\delta+1)\big[6(k-i)-2\big] + \sum_{i=k/2+1}^{k} 2(\delta^2-\delta+1)\big[6(i-1)-2\big] \\ &= 2(\delta^2-\delta+1)(\frac{9}{2}k^2-5k). \end{aligned}$$

Since $n = 2k(\delta^2-\delta+1)$, division by n yields that

$$\text{avec}(G^*_{k,\delta}) \geq \frac{2(\delta^2-\delta+1)(\frac{9}{2}k^2-5k)}{2k(\delta^2-\delta+1)} = \frac{9}{2}k - 5 = \frac{9n}{4(\delta^2-\delta+1)} - 5,$$

as desired. □

4.3.2 Bounds on Average Eccentricity of (C_4, C_5)-Free Graphs.

If we relax the condition of G having girth at least 6 to the weaker condition that G contains no 4-cycle and 5-cycle as subgraphs, we show in the next theorem that a very slightly weaker version of the bound on average eccentricity of graphs of girth at least 6 holds.

Theorem 4.3.3. *Let G be a connected (C_4, C_5)-free graph of order n and with minimum degree $\delta \geq 2$. Then, G has a spanning tree T with*

$$\text{avec}(T) \leq \frac{9}{2}\left\lceil \frac{n}{2\delta^2-5\delta+5+2\varepsilon_\delta} \right\rceil + 8, \tag{4.3.11}$$

where

$$\varepsilon_\delta = \begin{cases} 1 & \text{if } \delta \text{ is odd,} \\ 0 & \text{if } \delta \text{ is even.} \end{cases}$$

Proof. The proof of this theorem follows essentially the same way as that of Theorem 4.3.1 except for little modification.

We start by finding a maximal matching, M', of G using the procedure described below: Choose an arbitrary edge $f_1 = v_iv_{i+1} \in E(G)$ and let $M' = \{f_1\}$. Let $V(M')$ be the set of vertices incident with some edge of M'. Let $d_G(e, V(M'))$ and $N_{\leq 2}(v)$ be as defined in the proof of Theorem 4.3.1. If there exists an edge f_2 in G with $d_G(f_2, V(M')) = 5$, add f_2 to M'. Subsequently, add edges f_i with $d_G(f_i, V(M')) = 5$ to M' until each of the edges not in M' is within distance four of M'. Thus $M' = \{f_1, f_2, \cdots f_t\}$ where $|M'| = t$.

For $i \in \{1, 2, \ldots, t\}$ let $f_i = v_iu_i$. Let $T^*_{f_i}$ be a subtree of G with vertex set $N_{\leq 2}(v_i) \cup$

$N_{\leq 2}(u_i)$ that preserves the distance to $\{v_iu_i\}$.

Let $\mathcal{T} = \cup_{f_i \in M'} T^*_{f_i}$. Then, $\mathcal{T} \leq G$ is a subforest of G with a vertex set $N_{\leq 2}[V(M')]$. By our construction of M', there exist $|M'| - 1$ edges in G, each one joining two distinct components of $\mathcal{T}$, whose addition to $\mathcal{T}$ yields a tree $\mathcal{T}^* \leq G$, which contains $\mathcal{T}$ and has the same vertex set as $\mathcal{T}$.

Now, each vertex $x \in V(G) - V(\mathcal{T}^*)$ is within distance five of some vertex y in $V(\mathcal{T}^*)$. Let $T \geq \mathcal{T}^*$ be a spanning tree of G in which $d_T(u, V(M')) = d_G(u, V(M'))$ for each $u \in V(G)$. Since taking a spanning tree or removal of an edge, which is not a bridge, does not decrease the eccentricity or average eccentricity, and we have $\mathrm{avec}(G) \leq \mathrm{avec}(T)$, it suffices to show that

$$\mathrm{avec}(T) \leq \frac{9}{2}\left\lceil \frac{n}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} \right\rceil + 8. \tag{4.3.12}$$

For every vertex $v \in V(T)$, let $v_{M'}$ be a vertex in $V(M')$ closest to v in T. The tree, T, can be viewed as a weighted tree where each vertex has weight exactly 1. We now move the weight of every vertex to the closest vertex in $V(M')$, that is, we define a weight function $c' : V(T) \to \mathbb{N} \cup \{0\}$ by:

$$c'(v) = |\{w \in V(M') \mid w_{M'} = v\}| \ \text{ for } v \in V(T).$$

Since the weight of each vertex was moved over a distance not exceeding four, we have that $d_T(w, w_{M'}) \leq 5$ and so

$$\mathrm{avec}(T) \leq \mathrm{avec}_{c'}(T) + 5. \tag{4.3.13}$$

Clearly, the weight c' is concentrated only on the vertices of $V(M')$. To ensure that the weights are now concentrated on the edges of T, we consider the line graph $L = L(T)$ and define a new weight function $\overline{c}'$ on $V(L) = E(T)$ by

$$\overline{c}'(v_iv_{i+1}) = \begin{cases} c'(v_i) + c'(v_{i+1}) & \text{if } v_iv_{i+1} \in M', \\ 0 & \text{if } v_iv_{i+1} \notin M'. \end{cases}$$

Let $v_iv_{i+1} \in M'$. For each vertex $w \in N_{\leq 2}(v_i) \cup N_{\leq 2}(v_{i+1})$, we have that $w_{M'} \in \{v_i, v_{i+1}\}$. Hence,

$$c'(v_i) + c'(v_{i+1}) \geq |N_{\leq 2}(v_i) \cup N_{\leq 2}(v_{i+1})|.$$

Since G is a (C_4, C_5)-free graph, we have by Lemma 2.3.9 that

$$|N_{\leq 2}(v_i) \cup N_{\leq 2}(v_{i+1})| \geq 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta,$$

and so we have that

$$\overline{c}'(v_iv_{i+1}) \geq 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta \quad \text{for } v_iv_{i+1} \in M'. \tag{4.3.14}$$

Note that $\overline{c}'(v_iv_{i+1}) = 0$ if $v_iv_{i+1} \notin M'$ and $\sum_{f\in M'} \overline{c}'(v_iv_{i+1}) = \sum_{v\in V(T)} c(v) = n$.

Let $y, z \in V(T)$ and let P_T be the (y, z)-path in T. If e_y, e_z, are edges of T incident with y and z respectively, then

$$d_T(y, z) \leq d_L(e_y, e_z) + 1, \tag{4.3.15}$$

following a similar argument as in the proof of Claim 1 of Theorem 4.3.1.

By letting v to be an eccentric vertex of w for any $v, w \in V(T)$, we have that $e_T(w) = d_T(v, w)$. If $e_w \in E(T)$ is an edge incident with w, then we have by (4.3.15) that $d_T(v, w) \leq d_L(e_v, e_w) + 1$. Hence, $e_T(w) \leq e_L(e_w) + 1$. Consequently, we have that

$$\begin{aligned}
\sum_{v\in V(T)} e_T(v)c'(v) &= \sum_{v_iv_{i+1}\in M'} [e_T(v_i)c'(v_{i+1}) + e_T(v_i)c'(v_{i+1})], \\
&\leq \sum_{v_iv_{i+1}\in M'} \overline{c}'(v_iv_{i+1})(e_{L'}(v_iv_{i+1}) + 1), \\
&= \sum_{f\in M'} \overline{c}'(f)e_L(f) + \sum_{f\in M'} \overline{c}'(f).
\end{aligned}$$

It follows immediately from above that $EX_{c'}(T) \leq EX_{\overline{c}'}(L) + \sum_{f\in M'} \overline{c}'(f)$.

Since $\sum_{f\in M'} \overline{c}'(f) = \sum_{v\in V(T)} c'(v)$ and $\text{avec}(G) = EX_{c'}(G)/\sum_{v\in V(T)} c'(v)$, we have that

$$\text{avec}_{c'}(T) \leq \text{avec}_{\overline{c}'}(L) + 1. \tag{4.3.16}$$

Consider two matching edges f_1, f_2. If the distance $d_T(f_1, f_2)$ between f_1, f_2 in M' equals five, then $d_L(f_1, f_2) \leq 6$. Let U' be the subgraph of L^6 induced by M', that is $U' = L^6[M']$. Clearly U' is connected and for all pairs $f, g \in M'$,

$$d_L(f, g) \leq 6d_{U'}(f, g).$$

Now, for every $f' \in V(L) = E(T)$, there exists an edge g' such that $d_L(f', g') \leq 5$, and so for every $f \in M'$, we have that $e_L(f) \leq 6e_{U'}(f) + 5$. Hence,

$$avec_{\overline{c}'}(L) \leq 6avec_{\overline{c}'}(U') + 5. \tag{4.3.17}$$

Recall that for all edges in M', $\overline{c}'(f) \geq 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta$ and since $n = \sum_{f\in M'} \overline{c}'(f) = \sum_{v\in V(T)} c'(v)$, we have that $|M'|(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) \leq n$ and so $|M'| \leq n/(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta)$.

We normalise the weight on the vertices of U', by defining a new weight, $\overline{c}$, by

$$\overline{c}(v) = \overline{c}'(v)/(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta).$$

Clearly $\overline{c}' \geq 1$ for all $u \in V(U')$ and

$$\sum_{v \in V(U')} \overline{c}(v) = \frac{1}{2\delta^2 + 5\delta + 5 + 2\varepsilon_\delta} \sum_{v \in U'} \overline{c}'(v) = \frac{n}{2\delta^2 + 5\delta + 5 + 2\varepsilon_\delta}.$$

Letting $N^* = \sum_{v \in V(U')} \overline{c}(v)$, we have that $N^* = \frac{n}{2\delta^2+5\delta+5+\varepsilon_\delta}$ and $|M^{'}| \leq N^*$. We have that $\text{avec}_{\overline{c}}(U') = \text{avec}_{\overline{c}'}(U')$, since

$$\begin{aligned} \text{avec}_{\overline{c}}(U') &= \frac{EX_{\overline{c}}(U')}{\sum_{v \in V(U')} \overline{c}(v)}, \\ &= \frac{1}{2\delta^2 + 5\delta + 5 + 2\varepsilon_\delta} \sum_{v \in M'} \overline{c}'(v) e_{U'}(v) \Big/ \frac{1}{2\delta^2 + 5\delta + 5 + 2\varepsilon_\delta} \sum_{v \in V(U')} \overline{c}'(v), \\ &= \text{avec}_{\overline{c}'}(U'). \end{aligned} \tag{4.3.18}$$

Clearly, $\overline{c}(f_i) \geq 1$ for all f_i in $M^{'}$. Applying (4.3.18), Lemma 4.2.2, and Theorem 1.4.18(c), we have that

$$\text{avec}_{\overline{c}}(U') = \text{avec}_{\overline{c}'}(U') \leq \text{avec}(P_{\lceil N^* \rceil}) \leq \frac{3\lceil N^* \rceil}{4} - \frac{1}{2}.$$

Next, we bound $\text{avec}(T)$ by applying the inequalities in (4.3.13), (4.3.16) and (4.3.17) in conjuction with the fact that $N^* = \frac{n}{2\delta^2+5\delta+5+2\varepsilon_\delta}$.

$$\begin{aligned} \text{avec}(T) &\leq \text{avec}_{c'}(T) + 5, \\ &\leq \text{avec}_{\overline{c}'}(L) + 6, \\ &\leq 6\text{avec}_{\overline{c}'}(L^6[M^{'}]) + 11, \\ &\leq 6\Big(\frac{3\lceil N^* \rceil}{4} - \frac{1}{2}\Big) + 11, \\ &= \frac{9}{2}\lceil N^* \rceil + 8 \\ &= \frac{9}{2}\Big\lceil \frac{n}{2\delta^2 + 5\delta + 5 + 2\varepsilon_\delta} \Big\rceil + 8. \end{aligned}$$

Therefore, Theorem (4.3.3) holds since $\text{avec}(G) \leq \text{avec}(T)$. □

4.3.3 Average Eccentricity, Maximum and Minimum Degree

In the previous section, we gave bounds on the average eccentricity of graphs of girth at least 6 and connected (C_4, C_5)-free graphs of given order and minimum degree. We saw from the sharpness example of the bound that the degree of each vertex is close to the minimum degree. That is an indication that the bounds can be improved if the graph under consideration contains a vertex of large degree. Herein, we show that the bounds for connected C_4-free graphs can be improved by a factor of about 3/5 if the graph has girth at least 6. We also give corresponding bound for graphs containing neither 4-cycle or 5-cycle. Moreover, we construct graphs to show that our bound for connected graphs of girth at least 6 is best possible in the sense that the coefficient of n in the bound is best possible, and the minor order term $\sqrt{n}$ is of the right order of magnitude.

4.3.3.1 Bounds on Average Eccentricity, Minimum Degree and Maximum Degree of Graphs of girth at least 6

We begin by presenting our result on the average eccentricity of connected graphs of girth at least 6 of given order, minimum degree and maximum degree. The technique used throughout this section is a modification of that used in previous sections, it follows the approach taken in [37].

Theorem 4.3.4. *Let G be a graph of girth at least 6 with n vertices, minimum degree $\delta \geq 2$, and maximum degree Δ. Then,*

$$\operatorname{avec}(G) \leq \frac{n-\kappa_\Delta}{2\kappa_\delta}\left\lceil \frac{9n+3\kappa_\Delta}{n} \right\rceil + 21 \tag{4.3.19}$$

where $\kappa_\Delta := \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}$ and $\kappa_\delta := 2(\delta^2 - \delta + 1)$. This bound is sharp apart from the value of the additive constant.

Proof. Let v_1 be a vertex of degree Δ and let e_1 be an edge incident with v_1. We obtain a maximal matching M of G as follows. Let $M = \{e_1\}$. Let $V(M)$ be the set of vertices incident with an edge of M. Recall that for an edge e, $d_G(e, V(M))$ is the minimum of the distances between a vertex incident with e and a vertex in $V(M)$. If there exists an edge e_2 with $d_G(e_2, e_1) = 6$, add e_2 and let $M_0 = \{e_2\}$. If there exists an edge e_3 with

(i) $d_G(e_3, e_1) \geq 6$

(ii) $d_G(e_3, e_2) \geq 5$ and

(iii) we have equality in (i) or (ii) or both,

then we add e_3 to M_0. Repeat this process: Let $M_0 = \{e_2, e_3, \ldots, e_{i-1}\}$. If there exists an edge e_i satisfying

(a) $d_G(e_i, e_1) \geq 6$

(b) $\min\{d_G(e_i, e_j) \mid j = 2, 3, \ldots, i-1\} \geq 5$, and

(c) we have equality in (a) or (b) or both,

then add e_i to M_0. We repeat this process until, after k steps say, no further edge can be added to M_0. Let $M = \{e_1\} \cup M_0$, so $M = \{e_1, \ldots, e_k\}$ and $|M| = k$. Then every edge not in M is within distance 5 of an edge in M.

Let $T^*_{v_1}$ be a tree with vertex set $N_{\leq 3}(v_1)$ which is distance preserving from v_1. For $i \in \{1, 2, \ldots, k\}$ let $e_i = u_i v_i$. Let $T^*_{e_i}$ be a subtree of G with vertex set $N_{\leq 2}(u_i) \cup N_{\leq 2}(v_i)$ that preserves the distance to $\{u_i v_i\}$.

Let $T_1 = T^*_{v_1} \cup \bigcup_{e_i \in M} T^*_{e_i}$. Then, $T_1 \leq G$ is a subforest of G with vertex set, $N_{\leq 3}(v_1) \cup N_{\leq 2}\big(V(M) - \{v_1\}\big)$. By our construction of M, there exists $|M| - 1$ edges in G, each joining two distinct components of T_1, whose addition to T_1 yields a tree $T_2 \leq G$, so that T_2 contains T_1 and has the same vertex set as T_1.

Now, each vertex $v \in V(G) - V(T_2)$ is within distance five of some vertex w in $V(M)$ closest to it. Let $T \geq T_2$ be a spanning tree of G containing T_2 and distance preserving from $V(M)$, i.e. $d_T(x, V(M)) = d_G(x, V(M))$ for each $x \in V(G)$. Clearly, tree T has the same maximum degree as G since $\deg_T(v_1) = \deg_G(v_1)$. Furthermore, since $\mathrm{avec}(G) \leq \mathrm{avec}(T)$, it suffices to prove the bound for T.

For every vertex $u \in V(T)$, let u_M be a vertex in $V(M)$ closest to u in T. We can view T as a weighted tree where each vertex has weight exactly 1. We now move the weight of every vertex to the closest vertex in $V(M)$, by defining a new weight function $c : V(T) \to \mathbb{N} \cup \{0\}$ by:

$$c(u) = |\{x \in V(M) \mid x_M = u\}| \quad \text{for } u \in V(T).$$

Note that $c(u) = 0$ if $u \notin V(M)$ and $\sum_{u \in V(M)} c(u) = n$, where n is the order of G.

Since the weight of each vertex was moved over a distance not exceeding six, we have that $d_T(x, x_M) \leq 6$ and $|e_T(x) - e_T(x_M)| \leq 6$. Hence,

$$\begin{aligned} \mathrm{avec}(T) &= \frac{1}{n} \sum_{x \in V(T)} e_T(x), \\ &\leq \frac{1}{n} \sum_{x \in V(T)} (e_T(x_M) + 6), \\ &\leq \Big(\frac{1}{n} \sum_{u \in V(M)} c(u) e_T(u)\Big) + 6, \\ &\leq \mathrm{avec}_c(T) + 6. \end{aligned} \tag{4.3.20}$$

Clearly the weight of c is concentrated on the vertices incident with an edge of M. To ensure that the weights are concentrated on the edges of T, we consider the line graph $L = L(T)$ and define a new weight function $\bar{c}$ on $V(L) = E(T)$ by

$$\bar{c}(wz) = \begin{cases} c(w) + c(z) & \text{if } wz \in M, \\ 0 & \text{if } wz \notin M. \end{cases}$$

Let $wz \in M - \{e_1\}$. For each vertex $x \in N_{\leq 2}(w) \cup N_{\leq 2}(z)$, we have $x_M \in \{w, z\}$. Hence

$$c(w) + c(z) \geq |N_{\leq 2}(w) \cup N_{\leq 2}(z)|.$$

By Lemma 2.3.2, we have

$$|N_{\leq 2}(w) \cup N_{\leq 2}(z)| \geq (\delta - 1)\big(\deg(w) + \deg(z)\big) + 2 \geq 2(\delta^2 - \delta + 1) \tag{4.3.21}$$

and so we have that

$$\bar{c}(wz) \geq 2\delta^2 - 2\delta + 2 \text{ for } wz \in M - \{e_1\}. \tag{4.3.22}$$

On the other hand, let $e_1 := v_1 w$. For each vertex $x \in N_{\leq 3}(v_1)$, we have $x_M \in \{v_1, w\}$. Hence,

$$c(v_1) + c(w) \geq |N_{\leq 3}(v_1)|.$$

By Lemma 3.2.1, we have

$$|N_{\leq 3}(v_1)| \geq \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}. \tag{4.3.23}$$

and so we have that

$$\bar{c}(e_1) = \bar{c}(v_1 w) \geq \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}. \tag{4.3.24}$$

Note that $\bar{c}(wz) = 0$ if $wz \notin M$ and $\sum_{e \in M} \bar{c}(e) = \sum_{u \in V(T)} c(u) = n$. It follows that

$$\begin{aligned} n &\geq \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2} + \sum_{x \in M - \{e_1\}} (2\delta^2 - 2\delta + 2), \\ &= \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2} + (2\delta^2 - 2\delta + 2)(|M| - 1), \end{aligned}$$

and rearranging yields

$$|M| \leq \frac{n - \left[\Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}\right]}{2\delta^2 - 2\delta + 2} + 1. \tag{4.3.25}$$

Following a similar argument as in the proof of Theorem 4.3.1 (See Claim 1), we have that $|d_T(x,y) - d_L(e_x,e_y)| \leq 1$ where L is the line graph of T, $e_x, e_y \in E(T)$ are edges of T incident with x and y respectively. Hence, if u is a vertex of G, v an eccentric vertex of u, and e_u and e_v are edges incident with u and v, respectively, then

$$e_T(u) = d_T(u,v) \leq d_L(e_u,e_v) + 1 \leq e_L(e_u) + 1.$$

Summation over all the vertices of T yields that

$$\begin{aligned}\sum_{v\in V(T)} e_T(v)c(v) &= \sum_{uv\in M} [e_T(u)c(u) + e_T(v)c(v)],\\ &\leq \sum_{uv\in M} \overline{c}(uv)\big(e_L(uv)+1\big),\\ &= \sum_{e\in M} \overline{c}(e)e_L(e) + \sum_{e\in M} \overline{c}(e),\end{aligned}$$

It follows immediately from the above that

$$EX_c(T) \leq EX_{\overline{c}}(L) + \sum_{e\in M} \overline{c}(e),$$

and since $\sum_{e\in M} \overline{c}(e) = \sum_{v\in V(T)} c(v) = n$ and $\operatorname{avec}(G) = EX_c(G)/\sum_{v\in V(T)} c(v)$, we have that

$$\operatorname{avec}_c(T) \leq \operatorname{avec}_{\overline{c}}(L) + 1. \tag{4.3.26}$$

If f_1, f_2 are two matching edges in M with $d_T(f_1,f_2) = 5$, then $d_L(f_1,f_2) \leq 6$. Now the weights lie solely on M. Let H be the graph obtained from $L^6[M]$ by joining e_1 to every e_i in M for which $d_L(e_1,e_i) \leq 7$. Such edges exist since by construction of M we have that $d_T(e_1,e_2) = 6$ and thus $d_L(e_1,e_2) \leq 7$. Essentially the same argument as in the proof of Theorem 4.2.3) shows that H is connected.

Let $e, f \in M$ and let P be a shortest path from e to f in H of length ℓ say. First assume that P does not pass through e_1. Then each edge of P yields a path in L of length 6, so P yields a path from e to f of length at most 6ℓ. Now assume that P passes through e_1. Then each edge on P not incident with e_1 yields a path of of length at most 6 in L, while each edge of P incident with e_1 yields a path of length at most 7 in L. Since P has at most two edges incident with e_1, P yields a path of length at most $6\ell + 2$. Hence

$$d_L(e,f) \leq 6d_{L^6[M]}(e,f) + 2 \quad \text{for every} \quad e,f \in M.$$

Now, for every $e' \in V(L) = E(T)$, there exists an edge $f' \in M$ such that $d_L(e',f') \leq 6$, so that $e_L(f) \leq 6e_{L^6[M]}(f) + 6$ for every $f \in M$. Hence,

$$\text{avec}_{\overline{c}}(L) \leq 6\text{avec}_{\overline{c}}(L^6[M]) + 8. \tag{4.3.27}$$

We now modify the weight function $\overline{c}$ to obtain a new weight function $\overline{c}'$ on M for which $\overline{c}'(e) \geq 1$ for all $e \in M$.

For $e \in M - \{e_1\}$, we define

$$\overline{c}'(e) = \frac{\overline{c}(e)}{2\delta^2 - 2\delta + 2},$$

and

$$\overline{c}'(e_1) = \frac{\overline{c}(e_1) - \left[\Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}\right] + (2\delta^2 - 2\delta^2 + 2)}{2\delta^2 - 2\delta^2 + 2}.$$

Since $\overline{c}(e_1) \geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}$ and $\overline{c}(e) \geq 2\delta^2 - 2\delta^2 + 2$ for $e \in M - \{e_1\}$, we have that $\overline{c}'(e_i) \geq 1$ for all $e_i \in M$. Furthermore,

$$\begin{aligned}
\sum_{v \in V(L^6[M])} \overline{c}'(v) &= \left[\frac{1}{2\delta^2 - 2\delta + 2} \sum_{e \in M - \{e_1\}} \overline{c}(e)\right] + \overline{c}'(e_1) \\
&= \frac{1}{2\delta^2 - 2\delta + 2}\left[\sum_{e \in M} \overline{c}(e) - \left(\Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}\right)\right] + 1 \\
&= \frac{\sum_{v \in V(T)} c(v) - \left(\Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}\right) + (2\delta^2 - 2\delta^2 + 2)}{2\delta^2 - 2\delta + 2}.
\end{aligned}$$

Letting $N^* = \sum_{v \in V(L^6[M])} \overline{c}'(v)$, we have that $N^* = \frac{n - \left(\Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}\right) + (2\delta^2 - 2\delta^2 + 2)}{2\delta^2 - 2\delta + 2}$ and $|M| \leq N^*$. We now express $\text{avec}_{\overline{c}}(L^6[M])$ in terms of $\text{avec}_{\overline{c}'}(L^6[M])$.

Let $\kappa_\Delta := \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}$ and $\kappa_\delta := 2\delta^2 - 2\delta^2 + 2$, then

$$\begin{aligned}
\text{avec}_{\overline{c}'}(L^6[M]) &= \frac{EX_{\overline{c}'}(L^6[M])}{\sum_{v \in V(L^6[M])} \overline{c}'(v)}, \\
&= \frac{\frac{1}{\kappa_\delta}\left[\sum_{u \in M - \{e_1\}} \overline{c}(u) e_{L^6[M]}(u) + (\overline{c}(e_1) - \kappa_\Delta + \kappa_\delta) e_{L^6[M]}(e_1)\right]}{N^*}, \\
&= \frac{EX_{\overline{c}}(L^6[M]) + (\kappa_\delta - \kappa_\Delta) e_{L^6[M]}(e_1)}{n - \kappa_\Delta + \kappa_\delta}, \\
&= \frac{\sum_{v \in V(T)} \overline{c}(v)}{n - \kappa_\Delta + \kappa_\delta} \text{avec}_{\overline{c}}(L^6[M]) + \frac{(\kappa_\delta - \kappa_\Delta) e_{L^6[M]}(e_1)}{n - \kappa_\Delta + \kappa_\delta}, \\
&= \frac{n}{n - \kappa_\Delta + \kappa_\delta} \text{avec}_{\overline{c}}(L^6[M]) + \frac{(\kappa_\delta - \kappa_\Delta) e_{L^6[M]}(e_1)}{n - \kappa_\Delta + \kappa_\delta},
\end{aligned}$$

and rearranging, we have that

$$\text{avec}_{\overline{c}}(L^6[M]) = \frac{n - \kappa_\Delta + \kappa_\delta}{n} \text{avec}_{\overline{c}'}(L^6[M]) + \frac{(\kappa_\Delta - \kappa_\delta)}{n} e_{L^6[M]}(e_1) \tag{4.3.28}$$

We now bound the two terms on the right hand side of (4.3.28) separately. Since $L^6[M]$

has order $|M|$ and since $|M| \leq \frac{n-\kappa_\Delta+\kappa_\delta}{\kappa_\delta}$, we have

$$e_{L^6[M]}(e_1) \leq \operatorname{diam}(L^6[M]) \leq |M| - 1 \leq \frac{n-\kappa_\Delta}{\kappa_\delta}. \quad (4.3.29)$$

Now, $\overline{c}'(e_i) \geq 1$ for all e_i in M. Applying (4.3.9), Lemma 4.2.2 and Theorem 1.4.18(c), we have that

$$\operatorname{avec}_{\overline{c}'}(L^6[M]) \leq \operatorname{avec}(P_{\lceil N^* \rceil}) \leq \frac{3\lceil N^* \rceil}{4} - \frac{1}{2}.$$

Since $\lceil N^* \rceil = \lceil \frac{n-\kappa_\Delta+\kappa_\delta}{\kappa_\delta} \rceil < \frac{n-\kappa_\Delta}{\kappa_\delta} + 2$, we have that

$$\operatorname{avec}_{\overline{c}'}(L^6[M]) \leq \operatorname{avec}(P_{\lceil N^* \rceil}) < \frac{3}{4}\Big[\frac{n-\kappa_\Delta}{\kappa_\delta} + 2\Big] - \frac{1}{2} = \frac{3(n-\kappa_\Delta)}{4\kappa_\delta} + 1. \quad (4.3.30)$$

Substituting (4.3.29) and (4.3.30) in (4.3.28), we have that

$$\begin{aligned}
\operatorname{avec}_{\overline{c}}(L^6[M]) &< \frac{n-\kappa_\Delta+\kappa_\delta}{n}\Big(\frac{3(n-\kappa_\Delta)}{4\kappa_\delta} + 1\Big) + \frac{\kappa_\Delta-\kappa_\delta}{n}\frac{n-\kappa_\Delta}{\kappa_\delta}, \\
&= \Big(\frac{n-\kappa_\Delta}{n} + \frac{\kappa_\delta}{n}\Big)\Big(\frac{3n-3\kappa_\Delta+4\kappa_\delta}{4\kappa_\delta}\Big) + \frac{\kappa_\Delta-\kappa_\delta}{n}\frac{n-\kappa_\Delta}{\kappa_\delta}, \\
&= \Big(\frac{n-\kappa_\Delta}{n}\Big)\Big(\frac{3n-3\kappa_\Delta+4\kappa_\delta}{4\kappa_\delta}\Big) + \frac{3n-3\kappa_\Delta+4\kappa_\delta}{4n} + \frac{\kappa_\Delta-\kappa_\delta}{n}\frac{n-\kappa_\Delta}{\kappa_\delta}, \\
&= \frac{n-\kappa_\Delta}{n}\Big(\frac{3n-3\kappa_\Delta+4\kappa_\delta}{4\kappa_\delta} + \frac{\kappa_\Delta-\kappa_\delta}{\kappa_\delta}\Big) + \frac{3n-3\kappa_\Delta+4\kappa_\delta}{4n}, \\
&\leq \frac{n-\kappa_\Delta}{n}\Big(\frac{3n+\kappa_\Delta}{4\kappa_\delta}\Big) + 1. \quad (4.3.31)
\end{aligned}$$

Applying the inequalities in (4.3.20), (4.3.26), (4.3.27) and (4.3.31), we obtain a bound on $avec(T)$, as follows

$$\begin{aligned}
\operatorname{avec}(T) &\leq \operatorname{avec}_c(T) + 6, \\
&\leq \operatorname{avec}_{\overline{c}}(L) + 7, \\
&\leq 6\operatorname{avec}_{\overline{c}}(L^6[M]) + 15, \\
&< 6\Big[\frac{n-\kappa_\Delta}{n}\Big(\frac{3n+\kappa_\Delta}{4\kappa_\delta}\Big) + 1\Big] + 15, \\
&= \frac{n-\kappa_\Delta}{2\kappa_\delta}\frac{9n+3\kappa_\Delta}{n} + 21.
\end{aligned}$$

Thus, we have proved Theorem (4.3.4). □

Theorem 4.3.5. *Let $\delta \geq 3$ be an integer such that $\delta - 1$ is a prime power. Then for $n, \Delta \in N$, there exists infinitely many values of Δ for which there exists infinitely many values of n such that there exists a graph of girth at least 6 $F^*_{\ell,\delta,\Delta}$ with n vertices, minimum degree δ and maximum degree Δ whose average eccentricity satisfies*

$$\operatorname{avec}(F^*_{\ell,\delta,\Delta}) \geq \frac{n-\kappa_\Delta}{2\kappa_\delta}\,\frac{9n+3\kappa_\Delta}{n} - O(\sqrt{\Delta}), \tag{4.3.32}$$

where $\kappa_\delta := 2\delta^2 - 2\delta + 2$ *and* $\kappa_\Delta := \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}$.

To prove this theorem, we make use of the graph described in Example 3.2.6. For the readers convenience, we briefly recall its construction here.

Proof. Let $q = \delta - 1$ be a prime power and $m \in \mathbb{N}$ with $m \geq 4$. Recall from Section 3.2.2 (Theorem 3.2.10), there exists a connected graph of girth at least 6 $F_{m,\delta,\Delta}$ whose order n satisfies $\Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2} \leq n(F_{m,\delta,\Delta}) \leq 2 + \delta\Delta + (\delta+1)\sqrt{\Delta(\delta-2)}$. $F_{m,\delta,\Delta}$ has of girth at least 6, minimum degree δ and maximum degree $\Delta = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q^2-q)} - \frac{1}{q}$.

Let $u_1 \in F_{m,\delta,\Delta}$ be a vertex of maximum degree, viz z, and let v_1 be any vertex of $F_{m,\delta,\Delta}$ that is not of degree Δ. Without loss of generality, we let $v_1 \in Y$ as defined there.

Now let $\ell \in \mathbb{N}$ with $\ell \geq 2$. Consider the graph $G^*_{k,\delta}$ constructed in the proof of Theorem 4.3.2, and let H_1 be a copy of $F_{m,\delta,\Delta}$ and $H_2, \ldots, H_l$ as defined there. Denote the resulting graph by $F^*_{\ell,\delta,\Delta}$. The sketch of the graph $F^*_{\ell,\delta,\Delta}$ in Figure 3.2 is reproduced in Figure 4.3 below.

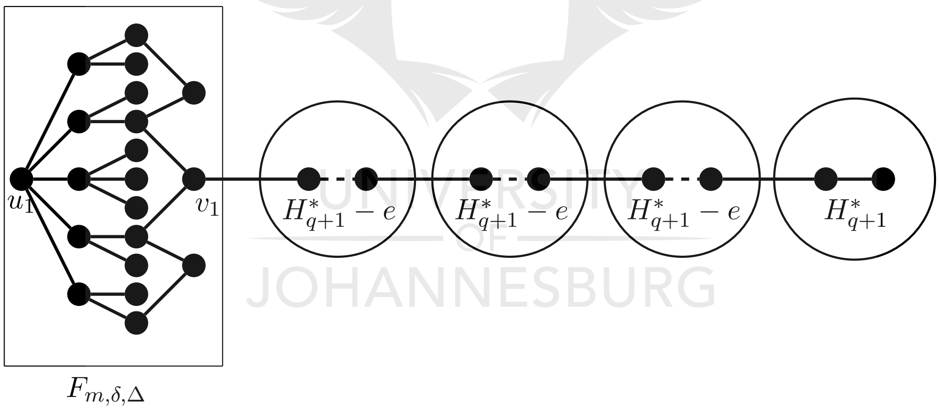

Figure 4.3: The graph $F^*_{5,\delta,\Delta}$.

By Claim 2.3.6, graph H_i (for $2 \leq i \leq l$) has $\kappa_\delta := 2(q^2+q+1)$ vertices and so for the order n of $F^*_{\ell,\delta,\Delta}$, we have that

$$n = |V(F^*_{\ell,\delta,\Delta}| = n(H_1) + (\ell-1)\kappa_\delta. \tag{4.3.33}$$

In subsequent calculations, we denote the order of H_1 of $F^*_{\ell,\delta,\Delta}$ as ω_Δ. $F^*_{\ell,\delta,\Delta}$ is bipartite, C_4-free, has girth at least 6, minimum degree δ, maximum degree Δ. As shown in Theorem 3.2.12, the diameter of $F^*_{\ell,\delta,\Delta}$ is $d(u_1, v_\ell) = 6\ell - 3$. Moreover, by Section 3.2.2 (Theorem

3.2.10) we have that

$$\kappa_\Delta \leq \omega_\Delta \leq \kappa_\Delta + 2\sqrt{\Delta(\delta-2)} + \frac{1}{2}. \tag{4.3.34}$$

We now bound the average eccentricity of $F^*_{\ell,\delta,\Delta}$. Let ℓ be even since for odd ℓ the proof is similar. If $w \in V(H_i)$, $i \leq \frac{\ell}{2}$, then $e(w) \geq d(w, v^\ell) \geq d(v^i, v^\ell) = 6(\ell-1-i)+4$. If $w \in V(H_i)$, $i > \frac{\ell}{2}$, then we have $e(w) = d(w, u^1) \geq d(u^i, u^1) = 6(i-2)+4$. Hence,

$$e_{H_i}(w) \geq \begin{cases} 6(\ell-i)-2 & \text{if } 1 \leq i \leq \frac{\ell}{2}, \\ 6(i-1)-2 & \text{if } \frac{\ell}{2}+1 \leq i \leq \ell, \end{cases}$$

and so we have,

$$\begin{aligned} EX(F^*_{\ell,\delta,\Delta}) &= \sum_{i=1}^{\ell/2} \sum_{w \in V(H_i)} e(w) + \sum_{i=\ell/2+1}^{\ell} \sum_{v \in V(H_i)} e(v), \\ &\geq e_{H_1}(u)\omega_\Delta + \sum_{i=2}^{\ell/2} \sum_{w \in V(H_i)} e(w) + \sum_{i=\ell/2+1}^{\ell} \sum_{v \in V(H_i)} e(v), \\ &\geq (6\ell-8)\omega_\Delta + \Big(\sum_{i=1}^{\ell/2} \kappa_\delta[6(\ell-i)-2]\Big) + \Big(\sum_{i=\ell/2+1}^{\ell} \kappa_\delta[6i-8]\Big), \\ &= (6\ell-8)\omega_\Delta + \kappa_\delta\Big(\frac{9\ell^2}{2} - 11\ell + 8\Big), \\ &= (6\ell-8)(n-(\ell-1)\kappa_\delta) + \kappa_\delta\Big(\frac{9\ell^2}{2} - 11\ell + 8\Big), \\ &= (6\ell-8)n - \kappa_\delta\Big(\frac{3\ell^2}{2} - 3\ell\Big). \end{aligned}$$

Now $\ell = \frac{n-\omega_\Delta}{\kappa_\delta} + 1$ by (4.3.33). Substituting this and dividing by n yields, after simplification,

$$\begin{aligned} \text{avec}(F^*_{\ell.\delta,\Delta}) &\geq \frac{n-\omega_\Delta}{2\kappa_\delta} \frac{9n+3\omega_\Delta}{n} - 2 + \frac{3\kappa_\delta}{2n} \\ &> \frac{n-\omega_\Delta}{2\kappa_\delta} \frac{9n+3\omega_\Delta}{n} - 2. \end{aligned}$$

Now let $\varepsilon = \omega_\Delta - \kappa_\Delta$. Replacing ω_Δ by $\kappa_\Delta + \varepsilon$ in the above lower bound, we obtain

$$\begin{aligned} \text{avec}(F^*_{\ell,\delta,\Delta}) &> \frac{n-\kappa_\Delta-\varepsilon}{2\kappa_\delta} \frac{9n+3\kappa_\Delta+3\varepsilon}{n} - 2 \\ &= \frac{n-\kappa_\Delta}{2\kappa_\delta} \frac{9n+3\kappa_\Delta}{n} - \frac{\varepsilon}{2\kappa_\delta n}(6n+6\kappa_\Delta+3\varepsilon) - 2. \end{aligned}$$

Since $6n+6\kappa_\Delta+3\varepsilon \leq 12n$, and since $0 \leq \varepsilon \leq 2\sqrt{\Delta(\delta-2)} + \frac{1}{2}$ by (4.3.34) we have, for constant δ and large n and Δ,

$$\begin{aligned}\text{avec}(F^*_{\ell,\delta,\Delta}) &> \frac{n-\kappa_\Delta}{2\kappa_\delta}\frac{9n+3\kappa_\Delta}{n} - O(\sqrt{\Delta}),\\ &= \frac{9(n-\kappa_\Delta)}{2\kappa_\delta} + \frac{n-\kappa_\Delta}{2\kappa_\delta}\frac{3\kappa_\Delta}{n} - O(\sqrt{\Delta}),\end{aligned}$$

as desired in Theorem 4.3.5. □

Theorem 4.3.4 generalises Theorem 4.3.1 in the sense that it implies (by setting $\Delta = \delta$) a bound that differs from Theorem 4.3.1 only by having a weaker additive constant.

The next theorem shows that slightly weaker bounds hold for graphs containing no cycles of length 4 or 5-cycle as subgraphs. We omit the proof, since it is very similar to the proof of Theorem 4.3.4 except for little modification. We do not know if this bound is sharp

Theorem 4.3.6. *Let G be a connected (C_4, C_5)-free graph with n vertices, minimum degree $\delta \geq 2$, maximum degree Δ. Then,*

$$\text{avec}(G) \leq \frac{n-\tau_\Delta}{2\tau_\delta}\frac{9n+3\tau_\Delta}{n} + 21. \tag{4.3.35}$$

where $\tau_\Delta := \Delta(\delta-1) + (\delta-2)\sqrt{\Delta(\delta-3)} + \frac{3}{2}$, $\tau_\delta := 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta$ *and*

$$\varepsilon_\delta = \begin{cases} 1 & \text{if } \delta \text{ is odd,} \\ 0 & \text{if } \delta \text{ is even.} \end{cases}$$

Chapter 5

Upper Bounds on the Average Distance of Graphs of Girth at least 6 and Connected (C_4, C_5)-Free Graphs.

5.1 Introduction

In this Chapter, we give bounds on the average distance of graphs with girth at least 6 as well as connected (C_4, C_5)-free graphs, taken into account the minimum degree and the order of the graphs. To achieve this goal, we adapt the approach given in [30]. Moreover, we show that the bounds are asymptotically sharp apart from additive constants. In addition, we give upper bounds on the average distance of connected graphs, triangle-free, C_4-free graphs, graphs of girth at least 6 and (C_4, C_5)-free graphs that take into account not only order and minimum degree, but also maximum degree.

5.2 Preliminary Results

The following definition and lemma will be very useful in proving our results.

Definition 5.2.1. *[30] Let G be a weighted graph with weight function $c : V(G) \to \mathbb{R}^+$, then the distance of G with respect to c and average distance of G with respect to c is defined by*

$$\sigma_c(G) = \sum_{\{x,y\}\subset V(G)} c(x)c(y)d_G(x, y),$$

and

$$\mu_c(G) = \binom{N}{2}^{-1} \sigma_c(G),$$

respectively, where $N = \sum_{x\in V(G)} c(x)$ is the total weight of the vertices in G.

It was proved in [50], [45] and [72] that the average distance of a connected graph of order n is maximised by the path. The following lemma generalises this fact. The proof given follows the proof in [30], but is simplified slightly.

Lemma 5.2.2. *[30] Let G be a weighted graph with a weight function $c : V(G) \to \mathbb{R}^{\geq 0}$ and let k, N be positive integers, N a multiple of k such that $c(v) \geq k$ for every vertex v*

of G and $\sum_{v\in V(G)} c(v) \leq N$. Then

$$\mu_c(G) \leq \frac{N-k}{N-1}\frac{N+k}{3k}.$$

Equality holds if and only if G is a path and $c(v) = k$ for every $v \in V(G)$.

Proof. We prove the equivalent statement

$$\sigma_c(G) \leq \frac{N(N-k)(N+k)}{6k}.$$

The proof is by induction on N/k. If $N/k = 1$, then $N = k$ and $c(v) = k$. Thus, G has only one vertex of weight k and the average distance of G equals zero. Hence let $N > k$. Let G be a graph satisfying the hypothesis of the lemma such that $\sigma_c(G)$ is maximum. Then G is a tree. We note that the order of G is not specified in the statement of the lemma, hence we denote the order of G by n. We now show that G is a path and one of the end vertices has weight exactly k.

Claim 1: G is a path.

Suppose G is not a path, then there is a vertex of G whose degree is at least 3. Let u be the vertex on G of degree at least three and let $u_1, u_2, \cdots, u_d$ denote be the neighbours of u. Let G_i and C_i denote the component of $G - u$ containing u_i and the total weight of G_i respectively. Without loss of generality, we assume that $C_1 \geq C_2 \geq \cdots \geq C_d$. Now consider the graph H with the same vertex weight function c defined by $H = G - uu_d + u_d u_{d-1}$. By our definition of H, the distance from vertices in G_d to each vertex in $\{u\} \cup V(G_1) \cup V(G_2) \cup \cdots V(G_{d-2})$, have increased by one. On the other hand, the distances between the vertices of G_{d-1} and G_d have decreased by one. Hence the distance of H with respect to c is given by

$$\begin{aligned}
\sigma_c(H) &= \sigma_c(G) + c(u)C_d + C_1C_d + C_2C_d + \cdots + C_{d-2}C_d - C_dC_{d-1}, \\
&= \sigma_c(G) + c(u)C_d + C_d(C_1 + C_2 + \cdots + C_{d-2} - C_{d-1}), \\
&\geq \sigma_c(G) + c(u)C_d, \\
&> \sigma_c(G),
\end{aligned}$$

contradicting our choice of G since $\sigma_c(G)$ is maximum. Hence G is a path, say, $v_1, v_2, \cdots, v_n$.

Claim 2: $c(v_1) = k$ or $c(v_n) = k$.

By the statement of the Lemma, $c(v) \geq k$ for every vertex v of G. We consider two cases, $c(v_1) \geq 2k$ and $c(v_1) \leq 2k - 1$. First assume that $c(v_1) \geq 2k$, then we create a new vertex v_0 and join it to v_1 and assign to v_0 the weight k, and reduce the weight of v_1 by k. Let G' be the graph obtained from G by adding v_0 and the edge v_0v_1. Clearly G' has the same total weights as G and $c(v_0), c(v_1), c(v_2), \cdots, c(v_n) \geq k$. Thus,

$$\begin{aligned}\sigma_c(G') &\geq \sigma_c(G) + k\Big[c(v_1) - k + c(v_2) + c(v_3) + \cdots + c(v_n)\Big],\\ &= \sigma_c(G) + k(N' - k),\\ &= \sigma_c(G) + k(nk),\\ &> \sigma_c(G).\end{aligned}$$

The addition of the new vertex v_0 and splitting the weight of v_1 between v_1 and v_0 yielded a graph with larger average distance, contradicting our choice of G. Hence, $c(v_1) \leq 2k-1$. Similarly, we have that $v_n \leq 2k - 1$. Now suppose that $c(v_1)$, $c(v_n) \geq k+1$. Let $v_r \in V(G) - \{v_1\}$ be the vertex closest to v_1 with $c(v_r) > k$. Define the weight function $\overline{c}$ by

$$\overline{c}(v_i) = \begin{cases} c(v_1) + 1 & \text{if } i = 1,\\ c(v_r) - 1 & \text{if } i = r,\\ c(v_i) & \text{otherwise.}\end{cases}$$

By our definition, we have that

$$\begin{aligned}\sigma_{\overline{c}}(G) &= \sigma_c(G) + (r-1)\big[c(v_r) - 1 + c(v_{r+1}) + \cdots + c(v_n)\big] - (r-1)c(v_1)\big],\\ &= \sigma_c(G) + (r-1)\big[-c(v_1) - 1 + c(v_r) + c(v_{r+1}) + \cdots + c(v_n)\big],\\ &= \sigma_c(G) + (r-1)\big[-c(v_1) - 1 + N - (c(v_1) + c(v_2) + \cdots + c(v_{r-1}))\big],\\ &= \sigma_c(G) + (r-1)\big[-c(v_1) - 1 + N - c(v_1) - k(r-2)\big],\\ &= \sigma_c(G) + (r-1)\big[N - (r-2)k - 2c(v_1) - 1\big].\end{aligned}$$

Since $\sigma_c(G)$ is maximum, we have that

$$\begin{aligned} N - (r-2)k - 2c(v_1) - 1 \leq 0 \quad &\Leftrightarrow \quad N \leq (r-2)k + 2c(v_1) + 1,\\ \Longrightarrow N \leq (r-2)k + 1 + 4k - 2 \quad &\Leftrightarrow \quad N \leq (r+2)k - 1\\ \Longrightarrow N \leq (r+1)k. &\end{aligned}$$

But $N > nk \Longrightarrow N \geq (n+1)k$. Thus, $(n+1)k \leq N \leq (r+1)k$. Therefore $r = n$. This means that the end vertex v_n is the closest vertex to v_1 with weight greater than k. Hence each vertex except v_1 and v_n has weight exactly k.

Recall that $c(v_1) \leq 2k-1$ and $c(v_n) \leq 2k-1$. Thus $c(v_1)+c(v_n) \leq 4k-2$ and $c(v_1)+c(v_n) = 3k$. Let $c(v_1) = x$ and $c(v_n) = 3k - x$. We introduce a new vertex v_0 to G and reduce the weight of v_1 and v_n to k by shifting the difference to v_0. Let G'' be the graph obtained from G by adding v_0 and the edge v_0v_1. It suffices to show that G'' is a graph with larger average distance than G. Since $c(v_1) = x$ and $c(v_n) = 3k - x$, then we are moving $(x - k)$

and $(2k-x)$ respectively from v_1 and v_n to v_0. Now, $c(v)=k$ for all vertices v of G'' and

$$\begin{aligned}\sigma_c(G'') &= \sigma_c(G)+(x-k)\big[nk\big]+(2k-x)\big[nk\big]-(x-k)(2k-x)(n-1),\\ &= \sigma_c(G)+nk^2-(n-1)(x-k)(2k-x).\end{aligned}$$

Observe that $n-1<n$ and $(x-k)+(2k-x)=k$. If $(x-k)+(2k-x)=k$, then $(x-k)(2k-x)\leq\left(\frac{k}{2}\right)^2$. Thus,

$$\begin{aligned}\sigma_c(G'') &\geq \sigma_c(G)+nk^2-\frac{nk^2}{4},\\ &= \sigma_c(G)+\frac{n}{4}\big(3k^2\big),\\ &= \sigma_c(G)+\frac{nk}{4}\Big[c(v_1)+c(v_n)\Big] \qquad (\text{since} \quad c(v_1)+c(v_n)=3k),\\ &> \sigma_c(G)+\frac{N}{4}\Big[c(v_1)+c(v_n)\Big] \quad (\text{since } N \text{ is a multiple of } k \text{ and } N>nk),\\ &> \sigma_c(G).\end{aligned}$$

Hence, G'' is a graph with a larger average distance than G, contradicting our choice of G since $\sigma_c(G)$ is maximum. Therefore, our assumption that both $c(v_1)$ and $c(v_n)$ has weight greater than k is false. Hence, G has at least one end vertex of weight k, that is, $c(v_1)=k$ or $c(v_n)=k$. This proves our claim.

Without loss of generality, let v_1 be the end vertex with weight k. Let $\overline{c}$ be the vertex weight function restricted to $V(G)-\{v_1\}$. The total weight on $V(G)-\{v_1\}$ is $N-k$. By the induction hypothesis,

$$\sigma_{\overline{c}}(G-v_1)\leq\frac{(N-k)(N-k-k)(N-k+k)}{6k}=\frac{(N-k)(N-2k)N}{6k}.$$

Since $\sigma_{\overline{c}}(v_1,G)$ is maximized subject to $\sum_{i\geq 2}c(v_i)=n-k$, if $c(v_2),c(v_3),\cdots,c(v_n)=k$, we have that

$$\begin{aligned}\sigma_c(v_1,G) &= \sum_{v_i\in V(G),\ i=2}^{n}c(v_i)d(v_1,v_i),\\ &= k[1+2+3+\cdots+n-1],\\ &< k\Big[1+2+3+\cdots+\frac{N-k}{k}\Big] \quad (\text{since } N>nk),\\ &= \frac{N}{2}\frac{N-k}{k}.\end{aligned}$$

This yields in total

$$\begin{aligned}\sigma_c(G) &= \sigma_{\overline{c}}(G - v_1) + k\sigma_c(v_1, G), \\ &\leq \frac{(N-k)(N-2k)N}{6k} + \frac{1}{2}N(N-k), \\ &= \frac{N(N-k)(N+k)}{6k},\end{aligned}$$

as desired. □

We now present a bound on the average distance of a connected graphs in terms of order and minimum degree. The proof we give is an elaboration of the original proof.

Theorem 5.2.3. *[30] Let G be a connected graph of order n and minimum degree δ. Then (i) G has a spanning tree T with*

$$\mu(T) \leq \frac{n}{\delta+1} + 5.$$

(ii)

$$\mu(G) \leq \frac{n}{\delta+1} + 5.$$

(iii) The bounds in (i) and (ii) are sharp apart from an additive constant.

Proof. The proof is similar to the proof of Theorem (4.2.3).

(i) We find a maximal 2-packing, B, of G as follows. Choose an arbitrary vertex v_1 of G and let $B = v_1$. If there exists a vertex v_i in G with $d_G(v_i, B) = 3$, add v_i to B. Add vertices v_i with $d_G(v_i, B) = 3$ to B until every vertex not in B is within distance two of B. Thus $B = \{v_1, v_2, \cdots v_k\}$.

Let $N_G[B]$ denote the vertex set consisting of B and any vertex adjacent to B. Also, let T_1 be the subforest of G with vertex set $N_G[B]$ and whose edge set consists of all edges incident with a vertex in B. Each component of T_1 is a star with a vertex in B as the center. By our construction of B, there exist $|B| - 1$ edges in G, each one joining two neighbours of distinct vertices of B, whose addition to T_1 yields a subtree T_2 of G. Now, each vertex $u \in V(G) - V(T_2)$ is adjacent to some vertex $u' \in V(T_2)$.

Let T be a spanning tree of G with edge set $E(T) = E(T_2) \cup \{uu' : u \in V(G) - V(T_2)\}$. We now prove that

$$\mu(T) \leq \frac{n}{\delta+1} + 5. \tag{5.2.1}$$

For every vertex $x \in V(T)$, let x_B be a vertex in B closest to u in T. We now move the weight of every vertex to the closest vertex in B by defining the weight function as follows,

$c : V(T) \to \mathbb{R}^+$ by

$$c(u) = |\{x \in V(T) \mid x_B = u\}| \;\text{ for } u \in V(T),$$

where $c(u) = 0$ if $u \notin B$, $c(u) \geq \delta + 1$ for each $u \in B$ and $\sum_{u \in B} c(u) = n$.

Since each vertex of T is within distance two of the nearest vertex in B, each weight was moved over a distance not exceeding two and no distance between two weights have changed by more than 4. Hence,

$$\mu(T) \leq \mu_c(T) + 4.$$

Observe that the weight c is concentrated only on the vertices of B. Let $T' = T^3[B]$. Following a similar argument as in the proof of Claim 1 in Theorem 4.2.3, T' is connected. Since B is a maximal 2-packing, we have that $d_T(u, v) \geq 3$ for all pairs $u, v \in B$. We also have that $d_T(u, v) \leq 3d_{T'}(u, v)$ for all pair of vertices $u, v \in B$ and since $V(B) = V(T')$, we have that $\sigma_c(T) \leq 3\sigma_c(T')$. Hence,

$$\mu_c(T) \leq 3\mu_c(T').$$

Recall that $c(v) \geq \delta + 1$ for all vertices $v \in B$. Let N be the least multiple of $\delta + 1$ such that $\sum_{v \in B} c(v) = n \leq N$. By Lemma (5.2.2) we have

$$\mu_c(T') \leq \frac{N - \delta - 1}{N - 1} \frac{N + \delta + 1}{3(\delta + 1)} \leq \frac{N + 1}{3(\delta + 1)}.$$

Combining these inequalities, in conjunction with $N \leq n + \delta$, yields

$$\begin{aligned} \mu(T) &\leq \mu_c(T) + 4, \\ &\leq 3\mu_c(T') + 4, \\ &\leq 3\Big(\frac{N + 1}{3(\delta + 1)}\Big) + 4, \\ &\leq \frac{n + \delta + 1}{\delta + 1} + 4, \\ &= \frac{n}{\delta + 1} + 5, \end{aligned}$$

as desired.

(ii) Since the average distance of a spanning tree of G is not more than that the average distance of G itself, the statement of (ii) holds.

(iii) Next we show that the bound is best possible apart from the value of the additive constant. To show this, we consider the graph illustrated in [30]. For given integers, n, δ, k with $n = k(\delta+1)$, let $G_1, G_2, \cdots, G_k$ be disjoint copies of the complete graph $K_{\delta+1}$ and let

$u_i v_i \in E(G_i)$. Let $G_{n,\delta}$ be the graph obtained from the union of $G_1, G_2, \cdots, G_k$ by deleting the edges $u_i v_i$ for $i = 2, 3, \cdots, k-1$ and adding the edges $u_{i+1} v_i$ for $i = 1, 2, \cdots, k-1$. $G_{n,\delta}$ has order n and minimum degree δ.

For each $v \in V(G_i)$, if $w \in V(G_j)$ where $i < j$, we have that $d(v, w) \geq 1 + 3(j - i - 1)$. Thus

$$\begin{aligned}
\sum_{\{v,w\} \subset V(G_{n,\delta})} d(v,w) &\geq \sum_{1 \leq i < j \leq k} \Big(1 + 3(j-i-1)\Big)(\delta+1)^2 + \sum_{i=1}^{k} \binom{(\delta+1)}{2}, \\
&= \big(\delta+1\big)^2 \sum_{1 \leq i < j \leq k} \Big[3(j-i) - 2\Big] + k\frac{\delta(\delta+1)}{2}, \\
&= \big(\delta+1\big)^2 \sum_{1 \leq i < j \leq k} \Big[3(j-i) - 2\Big] + k\frac{\delta(\delta+1)}{2}, \\
&= \big(\delta+1\big)^2 \Big[1/2(k^3 - k) - k(k-1)\Big] + \frac{k}{2}\delta(\delta+1), \\
&= \frac{1}{2}(\delta+1)^2 \big[k^3 - 2k^2 + k\big] + \frac{k}{2}\delta(\delta+1), \\
&> \frac{1}{2}(\delta+1)^2 \big[k^3 - 2k^2\big] + \frac{k}{2}\delta(\delta+1), \\
&= \frac{1}{2}k(\delta+1)\big[k(\delta+1)(k-2)\big] + \delta, \\
&> \frac{1}{2}k(\delta+1)\big[k(\delta+1)(k-2)\big].
\end{aligned}$$

Since $n = k(\delta+1)$, dividing by $\binom{n}{2}$ yields

$$\begin{aligned}
\mu(G_{n,\delta}) &\geq \frac{n^2(k-2)}{n(n-1)}, \\
&\geq \frac{n^2(k-2)}{n^2}, \\
&= k - 2, \\
&= \frac{n}{\delta+1} - 2.
\end{aligned}$$

Therefore,

$$\mu(G_{n,\delta}) > \frac{n}{\delta+1} - 2.$$

Hence every spanning tree of $G_{n,\delta}$ has average distance greater than $n/(\delta+1)$. □

The bound in Theorem 5.2.3 can be improved further for graphs not containing a (not necessarily induced) 4-cycle. The proof is a slight variation of the proof in Theorem 5.2.3. Since the proof of our main result follows the same idea, we present the proof in full, with some elaborations of the original proof.

Theorem 5.2.4. *[30] (i) Let G be a connected C_4-free graph of order n and minimum* *degree δ. Then G has a spanning tree T with*

$$\mu(T) \leq \frac{5}{3}\frac{n}{\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1} + \frac{29}{3}.$$

(ii) There exists an infinite number of C_4-free graphs with n vertices and minimum degree δ such that for every spanning tree T of G

$$\mu(T) \geq \frac{5}{3}\frac{n}{\delta^2 + 3\delta + 2} + O(1).$$

Proof. We first construct a spanning tree in exactly the same way as in the proof of the upper bound on the average eccentricity of connected graphs and C_4-free graphs(Theorem 4.2.3 and Theorem 1.4.23). For the reader's convenience we repeat the construction. We start first by finding a maximal 4-packing, C, of G using the following procedure. Choose an arbitrary vertex x_1 of G and let $C = x_1$. If there exists a vertex x_i in G with $d_G(x_i, C) = 5$, add x_i to C. Add vertices x_i with $d_G(x_i, C) = 5$ to C until, after k steps say, every vertex not in C is within distance four of C. Thus $C = \{x_1, x_2, \cdots x_k\}$.

Let $N_G[u]$ and $N_{\leq 2}(u)$ denote the closed neighbourhood and set of vertices within distance two of u, respectively. For each $u \in V(C)$ let $T_1(u)$ be a tree with vertex set $N_{\leq 2}(u)$ satisfying $d_T(u, v) = d_G(u, v)$ for each $v \in N_{\leq 2}(u)$. $T_1(u)$ is distance preserving to u. Then $T_1 = \bigcup_{u\in V(C)} T_1(u)$ is a subforest of G. By our construction of C, there exist $|C|-1$ edges in G, each one joining two distinct components of T_1, whose addition to T_1 yields a tree $T_2 \leq G$. Now, each vertex $v \in V(G) - V(T_1)$ is within distance five of some vertex in T_2.

Let $T \geq T_2$ be a spanning tree of G in which $d_T(x, V(C)) = d_G(x, V(C))$ for each $x \in V(G)$. We now prove that

$$\mu(T) \leq \frac{5}{3}\frac{n}{\delta^2 - 2\lfloor\delta/2\rfloor + 1} + \frac{29}{3}.$$

For every vertex $u \in V(T)$, let u_C be a vertex in C closest to u in T. We move the weight of every vertex to the closest vertex in C by defining a weight function $c : V(T) \to \mathbb{R}^+$ by

$$c(u) = |\{x \in V(T) \mid x_C = u\}| \text{ for } u \in V(T),$$

where $c(u) = 0$ if $u \notin C$.

By Lemma 2.2.1, $\left|N_{\leq 2}[x_i]\right| \geq \delta^2 - 2\lfloor\delta/2\rfloor + 1$ for all $x_i \in C$. Therefore,

$$c(x_i) \geq \delta^2 - 2\lfloor\delta/2\rfloor + 1 \text{ for all } x_i \in C.$$

Since each weight was moved over a distance not exceeding four and no distance between two weights has changed by more than 8 and thus

$$\mu(T) \leq \mu_c(T) + 8.$$

We now construct an induced subgraph, $T^5[C] = T''$, of T^5. Following a similar argument as in the proof of Claim 1 in Theorem 4.2.3, T'' is connected. Clearly $d_T(u,v) \leq 5d_{T''}(u,v)$ for all pair of vertices $u, v \in C$ and since $V(C) = V(T'')$, we have that $\sigma_c(T) \leq 5\sigma_c(T'')$. Hence,

$$\mu_c(T) \leq 5\mu_c(T'').$$

Recall that $c(v) \geq \delta^2 - 2\lfloor \delta/2 \rfloor + 1$ for all vertices $v \in C$. Let N be the least multiple of $\delta^2 - 2\lfloor \delta/2 \rfloor + 1$ such that $\sum_{v \in C} c(v) \leq N$ and $N \geq n$. By Lemma 5.2.2 we have that

$$\mu_c(T'') \leq \frac{N - \delta^2 + 2\lfloor \delta/2 \rfloor - 1}{N-1} \frac{N + \delta^2 - 2\lfloor \delta/2 \rfloor + 1}{3(\delta^2 - 2\lfloor \delta/2 \rfloor + 1)} \leq \frac{N+1}{3(\delta^2 - 2\lfloor \delta/2 \rfloor + 1)}.$$

Combining these inequalities, in conjunction with $N \leq n + \delta^2 - 2\lfloor \delta/2 \rfloor$, yields

$$\begin{aligned}
\mu(T) &\leq \mu_c(T) + 8, \\
&\leq 5\mu_c(T'') + 8, \\
&\leq 5\left[\frac{N+1}{3(\delta^2 - 2\lfloor \delta/2 \rfloor + 1)}\right] + 8, \\
&\leq 5\left[\frac{n + \delta^2 - 2\lfloor \delta/2 \rfloor + 1}{3(\delta^2 - 2\lfloor \delta/2 \rfloor + 1)}\right] + 8, \\
&= \frac{5}{3} \frac{n}{\delta^2 - 2\lfloor \delta/2 \rfloor + 1} + \frac{29}{3},
\end{aligned}$$

as desired.

To prove the second part of the theorem, we consider the following graph $G''_{n,\delta}$ that was first described in [51] and the detailed proof shown in (2.2.16) of Theorem (2.2.2).

Let $q = \delta + 1$ be a prime power. Let H_q be the graph described in Example 2.2.3 whose vertices are the 1-dimensional subspaces of $GF(q)^3$. Two vertices are adjacent in H_q if, as subspaces of $GF(q)^3$, they are orthogonal. Clearly H_q is C_4-free, has $q^2 + q + 1$ vertices, each of degree q or $q + 1$ and $\text{diam}(H_q) = 2$. By Claim 2.2.5, there exists a self orthogonal vertex z in H_q. Let u and v be two neighbours of z. Let H_0 denote the graph obtained from H_q by deleting the vertex z and all edges of joining a neighbour of u to a neighbour of v. Then $n(H_0) = q^2 + q$, $d_{H_0}(u,v) = \text{diam}(H_0) = 4$ and $\delta(H_0) \geq q - 1$ in H_0.

For n a multiple of $q^2 + q = \delta^2 + 3\delta + 2$, let $G''_{n,\delta}$ be the graph obtained from the union of $k = n/(\delta^2 + 3\delta + 2)$ disjoint copies $H_0^1, H_0^2, \cdots, H_0^k$ of H_0 by adding the edges $u^t v^{t+1}$ for $1 \leq t \leq k - 1$, where u^t and v^t are the vertices in H_0^t corresponding to $\langle \underline{u} \rangle$ and $\langle \underline{v} \rangle$ in H_0. There are $q^2 + q$ vertices in each H_0^i, hence $|V(G''_{n,\delta})| = n = k(q^2 + q) = k(\delta^2 + 3\delta + 2)$.

For each $v \in V(H_0^i)$, if $w \in V(H_0^j)$ where $i < j$, we have that $d(v,w) \geq 1 + 5(j-i-1)$. Thus

$$\begin{aligned}
\sum_{\{v,w\}\subset V(G''_{n,\delta})} d(v,w) &\geq \sum_{1\leq i<j\leq k} \Big(1+5(j-i-1)\Big)\Big[(q^2+q)\Big]^2 + \sum_{i=1}^{k} \binom{(q^2+q)}{2}, \\
&= (q^2+q)^2 \sum_{i<j} \Big[5(j-i)-4\Big] + \frac{k}{2}(q^2+q)\Big[(q^2+q)-1\Big], \\
&= (q^2+q)^2 \sum_{i<j} \Big[5(j-i)-4\Big] + \frac{k}{2}(q^2+q)\Big[(q^2+q)-1\Big], \\
&= (q^2+q)^2 \Big[\frac{5}{6}(k^3-k) - \frac{1}{2}k(k-1)\Big] + \frac{k}{2}(q^2+q)\Big[(q^2+q)-1\Big], \\
&= \frac{1}{6}k(q^2+q)^2\big[5k^2-3k-2\big] + \frac{k}{2}(q^2+q)\Big[(q^2+q)-1\Big], \\
&> \frac{1}{6}k(q^2+q)^2\big[5k^2-3k-2\big].
\end{aligned}$$

Since $n = k(q^2+q)$, $q^2+q = \delta^2+3\delta+2$, dividing by $\binom{n}{2}$ yields and so

$$\begin{aligned}
\mu(G''_{n,\delta}) &\geq \frac{k(q^2+q)\Big[(q^2+q)(5k^2-3k-2)\Big]}{3n(n-1)} \\
&\geq \frac{k(q^2+q)\Big[(q^2+q)(5k^2-3k-2)\Big]}{3n^2} \\
&= \frac{5k}{3} - 1 - \frac{2}{3k}
\end{aligned}$$

Hence,

$$\mu(G''_{n,\delta}) \geq \frac{5}{3}\frac{n}{\delta^2+3\delta+2} - \frac{2\delta^2+6\delta+4}{3n} - 1.$$

□

Corollary 5.2.5. *Let G be a connected C_4-free graph of order n and minimum degree δ. Then*

$$\mu(G) \leq \frac{5}{3}\frac{n}{\delta^2 - 2\lfloor\frac{\delta}{2}\rfloor + 1} + \frac{29}{3}.$$

While we don't know if the bounds in Theorems 5.2.3 and 5.2.5 are sharp, the graph constructed in part (ii) show that for $\delta+1$ a prime power the coefficient of n is close to best possible.

5.3 Main Results

We now present upper bounds on the average distance of graphs of girth at least 6 and for graphs not containing a C_4 or C_5 as a subgraph. These bounds improve on the bounds on the average distance of C_4-free graphs in the previous sections. Unlike the bounds for C_4-free graphs, the bounds for graphs of girth at least 6 are sharp apart from an additive constant, provided $\delta - 1$ is a prime power.

5.3.1 Bounds on Average Distance of Graphs of Girth at least 6.

Theorem 5.3.1. *Let G be a graph of girth at least 6 with order n and with minimum degree $\delta \geq 2$. Then,*

i) G has a spanning tree T with

$$\mu(T) \leq \frac{n}{\delta^2 - \delta + 1} + 11. \tag{5.3.1}$$

ii)

$$\mu(G) \leq \frac{n}{\delta^2 - \delta + 1} + 11. \tag{5.3.2}$$

iii) If $\delta - 1$ is a prime power, then there exists an infinite number of graphs with girth at least 6, G^*, with n vertices and minimum degree $\delta \geq 2$ such that (ii) above holds and

$$\mu(G^*) > \frac{n}{\delta^2 - \delta + 1} - 5. \tag{5.3.3}$$

Proof. The proof is similar to that of the previous theorem with some minor modification.

i) We start by first finding a matching, M', of G as follows: Choose an arbitrary edge $f_1 \in E(G)$ and let $M' = f_1$. If there exists an edge f_2 in G with $d_G(f_2, M') = 5$, then add f_2 to M'. Repeat and add edges at distance 5 to M' until each of the edges not in M' is within distance four of M'. Thus $M' = \{f_1, f_2, \cdots f_k\}$ and $|M'| = k$. Recall that $V(M')$ is the vertex set consisting of vertices incident with an edge in M'.

   Let $N_G[u]$ and $N_{\leq 2}(u)$ denote the closed neighbourhood and set of vertices within distance two of u respectively. For each $u \in V(M')$, let $T_1(u)$ be a tree with vertex set $N_{\leq 2}(u)$ satisfying $d_T(u, v) = d_G(u, v)$ for each $v \in N_{\leq 2}(u)$. $T_1(u)$ is distance preserving to u. Then $T_1 = \bigcup_{u \in V(M')} T_1(u)$ is a subforest of G. By our construction of M', there exist $|M'| - 1$ edges in G, each one joining two distinct components of T_1, whose addition to T_1 yields a tree $T_2 \leq G$. Now, each vertex $v \in V(G) - V(T_1)$ is within distance five of some vertex in T_2. Let $T \geq T_2$ be a spanning tree of G in which $d_T(x, V(M')) = d_G(x, V(M'))$ for each $x \in V(G)$. We now prove that

$$\mu(T) \leq \frac{n}{\delta^2 - \delta + 1} + 11.$$

   For every vertex $u \in V(T)$, let $u_{M'}$ be a vertex in $V(M')$ closest to u in T. We now move the weight of every vertex to a closest vertex in $V(M')$ by defining a weight function $c : V(T) \to \mathbb{R}^+$ by

$$c(u) = |\{x \in V(M') \mid x_{M'} = u\}| \text{ for } u \in V(T),$$

where $c(u) = 0$ if $u \notin V(M')$.

The weight of each vertex was moved over a distance not exceeding four, and so we have that no distance between two weights has changed by more than 8. Thus

$$\mu(T) \leq \mu_c(T) + 8. \tag{5.3.4}$$

Observe that the weight c is concentrated exclusively on the vertices belonging to $V(M')$. We consider the line graph $L = L(T)$ and define a new weight function c' on $V(L) = E(T)$ by

$$c'(e) = c'(uv) = \begin{cases} c(u) + c(v) & \text{if } uv \in M', \\ 0 & \text{if } uv \notin M'. \end{cases}$$

The weight, c', on e is the sum of the weights on u and v. We have by Lemma 2.3.2 that

$$c'(uv) \geq 2\delta^2 - 2\delta + 2 \text{ for } uv \in M',$$

and $c'(uv) = 0$ if $uv \notin M'$. We have shown in the previous proof of Claim 1 in Theorem 4.3.1 that if $e_1, e_2 \in E(T)$ are edges incident with vertices $v_1, v_2 \in V(T)$ respectively, then $d_T(v_1, v_2) \leq d_L(e_1, e_2) + 1$. Hence, no distance between weights has increased by more than one and thus

$$\mu_c(T) \leq \mu_{c'}(L) + 1. \tag{5.3.5}$$

We now construct an induced subgraph, $L^6[M']$, of L^6. $L^6[M']$ is connected following a similar argument as in the proof of Claim 1 in Theorem 4.2.3. Furthermore, $d_L(e, f) \leq 6d_{L^6[M']}(e, f)$ for all pairs $e, f \in M'$ and since $V(M') = V(L^6[M'])$, we have that $\sigma_{c'}(L) \leq 6\sigma_{c'}(L^6[M'])$. Hence,

$$\mu_{c'}(L) \leq 6\mu_{c'}(L^6[M']).$$

Recall that $c'(v) \geq 2\delta^2 - 2\delta + 2$ for all vertices $v \in M'$ and that $\sum_{v \in M'} c'(v) = n$. Let N be the least multiple of $2\delta^2 - 2\delta + 2$ such that $N \geq n$. By Lemma (5.2.2) we have

$$\begin{aligned}
\mu_{c'}(L^6[M']) &\leq \frac{N-(2\delta^2-2\delta+2)}{N-1}\frac{N+(2\delta^2-2\delta+2)}{3(2\delta^2-2\delta+2)},\\
&= \frac{N^2-(2\delta^2-2\delta+2)^2}{6(N-1)(\delta^2-\delta+1)},\\
&< \frac{N^2-1^2}{6(N-1)(\delta^2-\delta+1)},\\
&= \frac{N+1}{6(\delta^2-\delta+1)}.
\end{aligned}$$

Combining these inequalities, in conjunction with $N \leq n+(2\delta^2-2\delta+1)$, yields

$$\begin{aligned}
\mu(T) &\leq \mu_c(T)+8,\\
&\leq \mu_{c'}(L)+9,\\
&\leq 6\mu_{c'}(L^6[M'])+9,\\
&\leq 6\Big[\frac{N+1}{6(\delta^2-\delta+1)}\Big]+9,\\
&\leq \frac{n+2(\delta^2-\delta+1)}{\delta^2-\delta+1}+9,\\
&= \frac{n}{\delta^2-\delta+1}+11,
\end{aligned}$$

as desired. Thus, we have proved the first part of Theorem 5.3.1.

ii) Since the average distance of a spanning tree of a graph is not more than the average distance of the graph itself, (5.3.2) holds.

iii) To prove the last part of the theorem, we consider the graph, G^* described in Example 2.3.5. For the reader's convenience we briefly recall its definition.

Let H^*_{q+1} be graph whose vertices consists of the 1-dimensional and 2-dimensional subspaces of $GF(q)^3$.

Two vertices, $\langle \underline{u}\rangle \in U$ and $\langle \underline{v}\rangle \in V$ are adjacent in H^*_{q+1} if and only if $\langle \underline{u}\rangle$ is contained in $\langle \underline{v}\rangle$. Clearly H^*_{q+1} is bipartite, C_4 free and has girth at least 6. Moreover, H^*_{q+1} has $2(q^2+q+1)$ vertices, each of degree $q+1$ and $\mathrm{diam}(H^*_{q+1})=3$. Let $u \in V(H^*_{q+1})$ be fixed. Let H_0 be the graph obtained from H^*_{q+1} after the removal of one of the edges, uv, incident with u. Then, $d_{H_0}(u,v)=5$ and the minimum degree of H_0 is q.

Let G^* be the graph obtained from the union of $H'_k (k \geq 2)$ disjoint copies of H^*_{q+1} with $H'_2, \cdots, H'_{k-2}$ being disjoint isomorphic copies of H_0 by adding the edges $u^{(t)}v^{(t+1)}$ for every $(1 \leq t < k)$, where u^t and v^t are vertices in H'_t corresponding to $\underline{u}$ and $\underline{v}$ in H^*_{q+1}. A simple calculation shows that G^* has $2k(q^2+q+1) = 2k(\delta^2-\delta+1)$ vertices since $\delta = q+1$. Thus $n = 2k(q^2+q+1)$.

For each $v \in V(H'_i)$, if $w \in V(H'_j)$ where $i < j$, we have that $d(v,w) \geq 1+6(j-i-1)$. Thus

$$\begin{aligned}
\sum_{\{v,w\}\subset V(G^*)} d(v,w) &\geq \sum_{1\leq i<j\leq k} \Big(1+6(j-i-1)\Big)\Big[2(q^2+q+1)\Big]^2 + \sum_{i=1}^{k} \binom{2(q^2+q+1)}{2}, \\
&= 4\big(q^2+q+1\big)^2 \sum_{i<j} \Big[6(j-i)-5\Big] + k(q^2+q+1)\Big[2(q^2+q+1)-1\Big], \\
&= 4\big(q^2+q+1\big)^2 \Big[(k^3-k) - 5/2(k^2-k)\Big] + k(q^2+q+1)\Big[2(q^2+q+1)-1\Big], \\
&= 2k\big(q^2+q+1\big)^2 \Big[2k^2-5k+3\Big] + k(q^2+q+1)\Big[2(q^2+q+1)-1\Big], \\
&> 2k\big(q^2+q+1\big)^2 \big[2k^2-5k\big].
\end{aligned}$$

Since $n = 2k\big(q^2+q+1\big)$ and $q^2+q+1 = \delta^2-\delta+1$, dividing by $\binom{n}{2}$ yields

$$\begin{aligned}
\mu(G^*) &\geq \frac{2k\big(q^2+q+1\big)\big[k(q^2+q+1)(2k-5)}{\binom{n}{2}}\big] \\
&\geq \frac{2n\big[nk-\frac{5n}{2}\big]}{n(n-1)} \\
&\geq \frac{2n\big[nk-\frac{5n}{2}\big]}{n^2} \\
&= 2k-5
\end{aligned}$$

Hence,

$$\mu(G^*) \geq \frac{n}{\delta^2-\delta+1} - 5.$$

□

5.3.2 Bounds on Average Distance of Connected (C_4, C_5)-Free Graphs

Theorem 5.3.2. *Let G be a connected (C_4, C_5)-free graph of order n and with minimum degree $\delta \geq 2$. Then,*

i) G has a spanning tree T with

$$\mu(T) \leq \frac{n+1}{\delta^2-\frac{5}{2}\delta+\frac{5}{2}+\varepsilon_\delta} + 11. \tag{5.3.6}$$

ii)

$$\mu(G) \leq \frac{n+1}{\delta^2-\frac{5}{2}\delta+\frac{5}{2}+\varepsilon_\delta} + 11. \tag{5.3.7}$$

Proof. The proof is essentially similar to that of the previous theorem except for some minor modification.

We start by first finding a matching, M', of G as follows: Choose an arbitrary edge $f_1 \in E(G)$ and let $M' = f_1$. If there exists an edge f_2 in $E(G)$ with $d_G(f_2, M') = 5$, then add f_2 to M'. Repeat and add edges at distance 5 to M' until each of the edges not in M'

is within distance four of M'. Thus $M' = \{f_1, f_2, \cdots f_k\}$ and $|M'| = k$. Denote by $V(M')$ the vertex set consisting of vertices incident with an edge in M'.

For each $u \in V(M')$, let $N_G[u]$, $N_{\leq 2}(u)$ and $T_1(u)$ be as defined in previous section. Recall that $T_1 = \bigcup_{u \in V(M')} T_1(u)$ is a subforest of G and by our construction of M', there exist $|M'| - 1$ edges in G, each one joining two distinct components of T_1, whose addition to T_1 yields a tree $T_2 \leq G$. Now, each vertex $v \in V(G) - V(T_1)$ is within distance five of some vertex in T_2.

By letting $T \geq T_2$ to be a spanning tree of G in which $d_T(x, V(M')) = d_G(x, V(M'))$ for each $x \in V(G)$, we now prove that

$$\mu(T) \leq \frac{2(n+1)}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} + 11.$$

For every vertex $u \in V(T)$, let $u_{M'}$ be a vertex in $V(M')$ closest to u in T. We now move the weight of every vertex to the closest vertex in $V(M')$ by defining a weight function $c : V(T) \to \mathbb{R}^+$ by

$$c(u) = |\{x \in V(M') \mid x_{M'} = u\}| \text{ for } u \in V(T),$$

where $c(u) = 0$ if $u \notin V(M')$.

The weight of each vertex was moved over a distance not exceeding four, and so we have that no two distance between two weights has changed by more than 8. Thus,

$$\mu(T) \leq \mu_c(T) + 8. \tag{5.3.8}$$

Clearly the weight c is concentrated exclusively on the vertices belonging to $V(M')$. We now consider the line graph $L = L(T)$ and define a new weight function c' on $V(L) = E(T)$ by

$$c'(e) = c'(uv) = \begin{cases} c(u) + c(v) & \text{if } uv \in M', \\ 0 & \text{if } uv \notin M'. \end{cases}$$

The weight, c', on e is the sum of the weights on u and v. We have by Lemma 2.3.9 that

$$c'(uv) \geq 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta \text{ for } uv \in M',$$

and $c'(uv) = 0$ if $uv \notin M'$. We have shown previously in Theorem 4.3.1 (see Claim 1) that if $e_1, e_2 \in E(T)$ are edges incident with vertices $v_1, v_2 \in V(T)$ respectively, then $d_T(v_1, v_2) \leq d_L(e_1, e_2) + 1$. Hence, no distance between weights has increased by more than one and thus

$$\mu_c(T) \leq \mu_{c'}(L) + 1. \tag{5.3.9}$$

Next, we construct an induced subgraph, $L^6[M']$, of L^6. Let $L'' = L^6[M']$. Following a similar argument as in the proof of Claim 1 in Theorem 4.3.1, L'' is connected.

Furthermore, $d_L(e,f) \le 6d_{L''}(e,f)$ for all pairs $e, f \in M'$ and since $V(M') = V(L'')$, we have that $\sigma_{c'}(L) \le 6\sigma_{c'}(L'')$. Hence,

$$\mu_{c'}(L) \le 6\mu_{c'}(L'').$$

Recall that $c'(v) \ge 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta$ for all vertices in M'. Let N be the least multiple of $2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta$ such that $\sum_{v \in L''} c'(v) \le N$ and $N \ge n$.

By Lemma (5.2.2) we have

$$\begin{aligned} \mu_{c'}(L'') &\le \frac{N - (2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta)}{N-1} \frac{N + (2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta)}{3(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta)}, \\ &= \frac{N^2 - (2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta)^2}{3(N-1)(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta)}, \\ &< \frac{N^2 - 1^2}{3(N-1)(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta)}, \\ &= \frac{N+1}{3(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta)}. \end{aligned}$$

Combining these inequalities, in conjunction with $N \le n + (2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta)$, yields

$$\begin{aligned} \mu(T) &\le \mu_c(T) + 8, \\ &\le \mu_{c'}(L) + 9, \\ &\le 6\mu_{c'}(L'') + 9, \\ &\le 6\Big[\frac{N+1}{3(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta)}\Big] + 9, \\ &\le \frac{2n + 2(2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta) + 2}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} + 9, \\ &= \frac{2(n+1)}{2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta} + 11. \end{aligned}$$

Hence

$$\mu(T) \le \frac{n+1}{\delta^2 - \frac{5}{2}\delta + \frac{5}{2} + \varepsilon_\delta} + 11. \tag{5.3.10}$$

Therefore, Theorem 5.3.2 (*ii*) holds since $\mu(G) \le \mu(T)$. □

5.3.3 Average Distance, Maximum and Minimum Degree

In the previous section, bounds on the average distance of connected graphs, triangle free graphs, connected C_4-free graphs, connected graphs of girth at least 6 and connected

(C_4, C_5)-free graphs of given order and minimum degree were presented. We observed from the sharpness example of the bounds that the degree of each vertex is close to the minimum degree for each class of graph considered. This is an indication that these bounds can be improved if the graph under consideration contains a vertex of large degree. Herein, we present our results on the average distance of given order, minimum and maximum degree for each of the graph considered in the previous section. Furthermore, we construct graphs to show that our bounds are sharp apart from an additive constants for connected graphs, triangle-free graphs. For connected C_4-free graphs and graphs of girth at least 6, we show that the bounds are best possible in a sense specified later. The technique used here is based on that used in previous sections, but had to be modified. Since there are no upper bounds in the literature on the average distance of graphs with given minimum degree and maximum degree (except for trees), we also present such bounds for triangle-free graphs and for graphs with no forbidden subgraphs.

5.3.3.1 Bounds on Average Distance in terms of Order, Minimum Degree and Maximum Degree

We now present bounds on the average distance of connected graphs in terms of order, minimum degree and maximum degree.

Theorem 5.3.3. *Let G be a connected graph of order n, minimum degree δ and maximum degree Δ. Then*

(i) G has a spanning tree T with

$$\mu(T) \leq \frac{(n-\Delta+\delta)}{n}\frac{(n-\Delta+\delta-1)}{(n-1)}\frac{n+2\Delta}{\delta+1}+4.$$

(ii)

$$\mu(G) \leq \frac{(n-\Delta+\delta)}{n}\frac{(n-\Delta+\delta-1)}{(n-1)}\frac{n+2\Delta}{\delta+1}+4.$$

(iii) The bounds in (i) and (ii) are sharp apart from an additive constant.

Proof. (i) Let w_1 be a vertex of maximum degree Δ. We find a maximal 2-packing, A, of G as follows: Let $A = \{w_1\}$. If there exists a vertex w_2 in G with $d_G(w_2, A) = 3$, add w_2 to A. Repeat and add vertices at distance 3 to A until every vertex not in A is within distance two of A. Then $A = \{w_1, w_2, w_3, \cdots w_r\}$ and $|A| = r$.

Let $N_G[A]$ denote the vertex set consisting of A and any vertex adjacent to some vertex in A. Furthermore, let T_1 be the subforest of G with vertex set $N_G[A]$ and whose edge set consists of all edges incident with a vertex in A. Clearly, each component of T_1 is a star with a vertex in A as the center. By our construction of A, there exist $|A|-1$ edges in G, each one joining two neighbours of distinct vertices of

A, whose addition to T_1 yields a subtree T_2 of G. Now, each vertex $u \in V(G)-V(T_2)$ is adjacent to some vertex $u' \in V(T_2)$.

Let T be the spanning tree of G with edge set $E(T) = E(T_2) \cup \{uu' : u \in V(G) - V(T_2)\}$. We now prove that

$$\mu(T) \leq \frac{(n-\Delta+\delta)}{n} \frac{(n-\Delta+\delta-1)}{(n-1)} \frac{n+2\Delta}{\delta+1} + 4. \tag{5.3.11}$$

We think of T as a weighted tree in which each vertex has weight 1. We now obtain a new weight function by moving the weight of every vertex to a a nearest vertex in A. More specifically, for every vertex $x \in V(T)$, let x_A be a vertex in A closest to u in T. Let $c : V(T) \to \mathbb{R}^+$ be the new weight function defined by

$$c(u) = |\{x \in V(T) \mid x_A = u\}| \quad \text{for } x \in V(T).$$

Then $c(u) = 0$ if $u \notin A$, $c(u) \geq \delta+1$ for each $u \in A - \{w_1\}$ and $c(w_1) \geq \Delta+1$. Note that $\sum_{u\in A} c(u) = n$ where n is the order of G. It follows that $n = \sum_{u\in V(T)} c(u) = \sum_{u\in A} c(u) \geq |A|(\delta+1) + \Delta - \delta$ and so

$$|A| \leq \frac{n-\Delta+\delta}{\delta+1}. \tag{5.3.12}$$

Since each vertex of T is within distance two of the nearest vertex in A, each weight was moved over a distance not exceeding two and no distance between two weights have changed by more than 4. Hence,

$$\mu(T) \leq \mu_c(T) + 4. \tag{5.3.13}$$

Note that the weight c is concentrated only on the vertices of A. Thus we construct an induced subgraph, $T^3[A]$, of A which by our construction of A is connected. Let $T' = T^3[A]$. Since A is a maximal 2-packing, we have that $d_T(u,v) \geq 3$ for all pairs $u, v \in A$. We also have that $d_T(u,v) \leq 3d_{T'}(u,v)$ for all pairs of vertices $u, v \in A$ and since $V(A) = V(T')$, we have that $\sigma_c(T) \leq 3\sigma_c(T')$. Hence,

$$\mu_c(T) \leq 3\mu_c(T'). \tag{5.3.14}$$

We now modify the weight function c to obtain a new weight function c' which satisfies $c'(a) \geq \delta+1$ for all $a \in A$. Define the new weight c' by

$$c'(u) = \begin{cases} c(u) & \text{if } u \in A - \{w_1\}, \\ c(u) - \Delta + \delta & \text{if } u = w_1. \end{cases}$$

Since $\deg_G(w_1) = \Delta$ while $\deg_G(w_i) \geq \delta$ for all $w_i \in A - \{w_1\}$, we have that

$c'(u) \geq \delta + 1$ for all $u \in A$.

Furthermore,

$$\sum_{u \in A} c'(u) = \sum_{u \in A} c(u) - \Delta + \delta = n - \Delta + \delta.$$

By letting $N = \sum_{u \in A} c'(u)$, we have that $N = n - \Delta + \delta$. We now express $\mu_c(T')$ in terms of $\mu_{c'}(T')$.

$$\begin{aligned}
\mu_{c'}(T') &= \binom{N}{2}^{-1} \sigma_{c'}(T'), \\
&= \binom{N}{2}^{-1} \Big[\sum_{i=2}^{r} (c(w_1) - \Delta + \delta) c(w_i) d_{T'}(w_1, w_i) \Big] \\
&\quad + \binom{N}{2}^{-1} \Big[\sum_{(u,v) \subseteq A - \{w_1\}} c(u)c(v) d_{T'}(u,v) \Big], \\
&= \binom{N}{2}^{-1} \Big[\sum_{(u,v) \subseteq A} c(u)c(v) d_{T'}(u,v) - \sum_{i=2}^{r} (\Delta - \delta) c(w_i) d_{T'}(w_1, w_i) \Big], \\
&= \binom{N}{2}^{-1} \Big[\binom{n}{2} \mu_c(T') - (\Delta - \delta) \sum_{i=2}^{r} c(w_i) d_{T'}(w_1, w_i) \Big],
\end{aligned}$$

and thus, by rearranging

$$\mu_c(T') = \frac{N(N-1)}{n(n-1)} \mu_{c'}(T') + \frac{2(\Delta - \delta)}{n(n-1)} \sum_{i=2}^{r} c(w_i) d_{T'}(w_1, w_i). \tag{5.3.15}$$

Clearly $\sum\limits_{i=2}^{r} c(w_i) d_{T'}(w_1, w_i) = \sigma_c(w_1, T')$. We now bound the two terms of the right hand side of (5.3.15) separately.

Let the vertices $w_2, w_3, \ldots, w_r$ be relabelled $u_1, u_2, \ldots, u_{r-1}$ respectively such that $d_{T'}(w_1, u_1) \leq d_{T'}(w_1, u_2) \leq \ldots d_{T'}(w_1, u_{r-1})$. Since T' is connected, we have

$$d_{T'}(w_1, u_1) \leq 1, d_{T'}(w_1, u_2) \leq 2, d_{T'}(w_1, u_3) \leq 3, \ldots, d_{T'}(w_1, u_{r-1}) \leq r - 1.$$

Hence we have

$$\sum_{i=2}^{r} c(u_i) d_{T'}(w_1, u_i) \leq \sum_{i=1}^{r-1} i\, c(u_i). \tag{5.3.16}$$

Now $c(u_i) \geq \delta + 1$ for $i = 1, 2, \ldots, r-2$ and $\sum_{i=1}^{r-1} c(u_i) = n - c(w_1) \leq n - \Delta - 1$, Subject to these conditions, the right hand side of (5.3.16) is maximised if $c(u_1) = c(u_2) = \ldots = c(u_{r-2}) = \delta + 1$ and $c(u_{r-1}) = n - \Delta - 1 - (r-2)(\delta + 1)$. Substituting these values yields, after simplification,

$$\sum_{i=1}^{r-1} i\,c(u_i) \leq (n-\Delta-1)(r-1) - \frac{1}{2}(\delta+1)(r-1)(r-2). \tag{5.3.17}$$

By (5.3.12), we have

$$r \leq \frac{n-\Delta+\delta}{\delta+1}, \tag{5.3.18}$$

and it is easy to verify that the right hand side of the inequality (5.3.17) is increasing in r for $r \leq \frac{n-\Delta+\delta}{\delta+1}$. Substituting this value for r yields thus

$$\sum_{i=1}^{r-1} i\,c(u_i) \leq \frac{(n-\Delta-1)^2}{2(\delta+1)} + \frac{1}{2}(n-\Delta-1). \tag{5.3.19}$$

Since $\frac{(n-\Delta-1)^2}{2(\delta+1)} + \frac{n-\Delta-1}{2} = \frac{(n-\Delta-1)(n-\Delta+\delta)}{2(\delta+1)} \leq \frac{N(N-1)}{2(\delta+1)}$, this implies that

$$\sum_{i=1}^{r-1} ic(u_i) \leq \frac{(N-1)N}{2(\delta+1)}. \tag{5.3.20}$$

To bound $\mu_{c'}(T')$, we recall that $c'(u) \geq \delta+1$ for all vertices $u \in T'$. Let M be the least multiple of $\delta+1$ such that $\sum_{u\in A} c'(u) = n-\Delta+\delta \leq M$. Since $n-\Delta+\delta = N$, we have that $M \leq N+\delta$ and so by Lemma 5.2.2

$$\mu_{c'}(T') \leq \frac{M-\delta-1}{M-1}\frac{M+\delta+1}{3(\delta+1)} \leq \frac{M+1}{3(\delta+1)} \leq \frac{N+\delta+1}{3(\delta+1)}. \tag{5.3.21}$$

Substituting (5.3.19) and (5.3.21) into (5.3.15) yields

$$\begin{aligned} \mu_c(T') &\leq \frac{N(N-1)}{n(n-1)}\frac{N+\delta+1}{3(\delta+1)} + \frac{2(\Delta-\delta)}{n(n-1)}\frac{N(N-1)}{2(\delta+1)} \\ &\leq \frac{N(N-1)}{n(n-1)}\frac{n+2\Delta}{3(\delta+1)}. \end{aligned} \tag{5.3.22}$$

Combining the inequalities (5.3.13), (5.3.14) and (5.3.22), we obtain

$$\begin{aligned} \mu(T) &\leq \mu_c(T)+4, \\ &\leq 3\mu_c(T')+4, \\ &\leq \frac{N(N-1)}{n(n-1)}\frac{n+2\Delta}{(\delta+1)}+4. \end{aligned} \tag{5.3.23}$$

Thus (5.3.23) yields the desired result.

(ii) The bound on $\mu(G)$ follows immediately from the above proof since the average distance of a spanning tree of G is not more than that the average distance of G itself.

(iii) To see that the bound above is sharp apart from the value of the additive constant, let $n, \delta, \Delta, \ell \in \mathbb{N}$ for which $\Delta \geq \delta$ and $n \geq \Delta + \delta + 1$.

Consider the following graph. Let $G_1, G_2, \cdots, G_{\ell-1}$ be disjoint copies of the complete graph $K_{\delta+1}$ and G_ℓ, a copy of the complete graph K_Δ. Let $u_i v_i \in E(G_i)$. Let $G_{\ell,\delta,\Delta}$ be the graph obtained from the union of $G_1, G_2, \cdots, G_\ell$ by deleting the edges $u_i v_i$ for $i = 2, 3, \cdots, \ell - 1$ and adding the edges $u_{i+1} v_i$ for $i = 1, 2, \cdots, \ell - 1$. The graph $G_{6,3,8}$ is illustrated in Figure 5.1

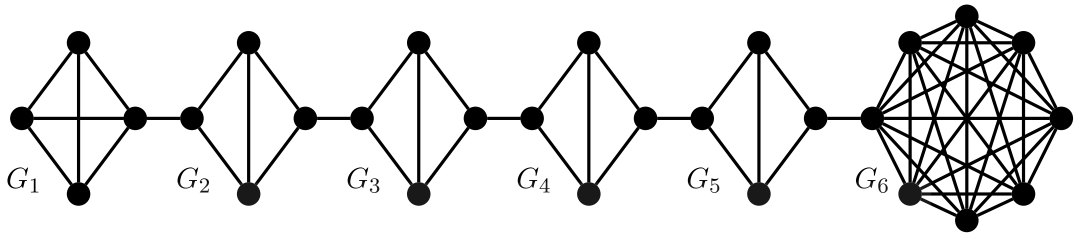

Figure 5.1: The graph $G_{6,3,8}$

We now bound the average distance of $G_{\ell,\delta,\Delta}$ from below. The graph G_i has $\delta + 1$ vertices for $i = 1, 2, \ldots, \ell - 1$, and Δ vertices for $i = \ell$. If $v \in V(G_i)$ and $w \in V(G_j)$, where $1 \leq i < j \leq \ell$, then $d(v, w) \geq 1 + 3(j - i - 1) = 3(j - i) - 2$. Hence

$$\begin{aligned} W(G_{\ell,\delta,\Delta}) &= \sum_{\{x,y\} \subseteq V(G_{\ell,\delta,\Delta})} d(x, y) \\ &> \sum_{1 \leq i < j \leq \ell-1} \sum_{x \in V(G_i), y \in V(G_j)} d(x, y) + \sum_{i=1}^{\ell-1} \sum_{x \in V(G_i), y \in V(G_\ell)} d(x, y) \\ &\geq (\delta + 1)^2 \sum_{1 \leq i < j \leq \ell-1} \big(3(j - i) - 2\big) + (\delta + 1)\Delta \sum_{i=1}^{\ell-1} \big(3(\ell - i) - 2\big). \end{aligned}$$

Straightforward calculations shows that $\sum_{1 \leq i < j \leq \ell-1} \big(3(j-i)-2\big) = \frac{1}{2}(\ell-1)(\ell-2)^2 = \frac{1}{2}(\ell^3 - 5\ell^2 + 8\ell - 4) > \frac{1}{2}(\ell^3 - 5\ell^2)$ and $\sum_{i=1}^{\ell-1} \big(3(\ell - i) - 2\big) = \frac{1}{2}(\ell - 1)(3\ell - 4) = \frac{1}{2}(3\ell^2 - 7\ell + 4) > \frac{1}{2}(3\ell^2 - 7\ell)$. Substituting these values, we obtain

$$\begin{aligned} W(G_{\ell,\delta,\Delta}) &> \frac{1}{2}(\delta + 1)^2(\ell^3 - 5\ell^2) + \frac{1}{2}(\delta + 1)\Delta(3\ell^3 - 7\ell) \\ &= \frac{1}{2}\big((\delta + 1)^2\ell^3 + 3(\delta + 1)\Delta\ell^2\big) - \frac{5}{2}(\delta + 1)^2\ell^2 - \frac{7}{2}(\delta + 1)\Delta\ell. \end{aligned}$$

Since $N = n - \Delta + \delta$ and $(\ell - 1)(\delta + 1) = n - \Delta = N - \delta$, we have that $\ell(\delta + 1) = N + 1$. Therefore, $(\delta + 1)^2\ell^3 + 3(\delta + 1)\Delta\ell^2 = \frac{(N+1)^3}{\delta+1} + \frac{3\Delta(N+1)^2}{\delta+1} > \frac{N(N-1)(n+2\Delta)}{\delta+1}$. Also $\frac{5}{2}(\delta+1)^2\ell^2 + \frac{7}{2}(\delta+1)\Delta\ell < \frac{7}{2}(N+1)((\delta+1)\ell+\Delta) = \frac{7}{2}(N+1)(n+\delta+1) < 7n(n-1)$. Applying these inequalities to the bound on $W(G_{\ell,\delta,\Delta})$ and dividing by $\binom{n}{2}$ thus yields

$$\mu(G_{\ell,\delta,\Delta}) > \frac{N(N-1)}{n(n-1)}\frac{n+2\Delta}{\delta+1} - 14,$$

which differs from the upper bound by at most 18.

□

Observe that the bound in Theorem 5.3.3 in some sense is a generalization of the bound in Theorem 5.2.3 given by [30]. This is true since $\Delta \geq \delta$ and replacing Δ by δ yields a bound on the average distance in terms of order and minimum degree that has an only slightly weaker additive constant than Theorem 5.2.3. Now define $\mu(\ell, \delta, \Delta)$ and $W(\ell, \delta, \Delta)$ to be the maximum average distance and maximum Wiener index, respectively, among all connected graphs of order n, minimum degree δ and maximum degree Δ. Theorem 5.3.3 shows that

$$\mu(\ell, \delta, \Delta) = \frac{(n-\Delta+\delta)(n-\Delta+\delta-1)}{n(n-1)}\frac{n+2\Delta}{\delta+1} + O(1)$$

and

$$W(\ell, \delta, \Delta) = \binom{n-\Delta+\delta}{2}\frac{n+2\Delta}{\delta+1} + O(n^2).$$

We show in the next theorem that the bound in Theorem 5.3.3 can be improved by a factor of 2/3 for triangle-free graphs. The proof techniques follow essentially the previous one with slight modification.

Theorem 5.3.4. *Let G be a connected triangle-free graph of order n, minimum degree δ and maximum degree Δ. Then*

(i) G has a spanning tree T with

$$\mu(T) \leq \frac{2}{3}\frac{(n-\Delta+\delta)}{n}\frac{(n-\Delta+\delta-1)}{n(n-1)}\frac{n+2\Delta}{\delta} + 7.$$

(ii)

$$\mu(G) \leq \frac{2}{3}\frac{(n-\Delta+\delta)}{n}\frac{(n-\Delta+\delta-1)}{n(n-1)}\frac{n+2\Delta}{\delta} + 7.$$

(iii) The bounds in (i) and (ii) are sharp apart from an additive constant.

Proof. (i) Let w_1 be a vertex of degree Δ in G. We start by finding a matching, M, of G as follows: Choose an arbitrary edge e_1 incident with w_1 in G and let $M = \{e_1\}$. Let $V(M)$ be the set of vertices incident with some edges of M. Recall that for an edge e, $d_G(e, V(M))$ is the smallest distance between a vertex incident with e and a vertex in $V(M)$. If there exists an edge e_2 in G with $d_G(e_2, V(M)) = 3$, add e_2 to M. Add edges e_i with $d_G(e_i, V(M)) = 3$ to M until each of the edges not in M is within distance two of M. Thus $M = \{e_1, e_2, \cdots e_r\}$ where $|M| = r$.

Let T_1 be the subforest of G with vertex set $N_G[V(M)]$ and whose edge set consists of all edges incident with a vertex in $V(M)$. By our construction of M, there exist $|M|-1$ edges in G, each one joining two components of T_1, whose addition to T_1 yields a subtree T_2 of G such that T_2 which contains T_1 and has the same vertex set as T_1. Now, each vertex $u \in V(G) - V(T_2)$ is within distance two of some vertex $u' \in V(T_2)$ closest to it, i.e., $d_G(u, u') \leq 2$.

Let T be a spanning tree of G that is distance preserving from $V(M)$ and with edge set $E(T) = E(T_2) \cup \{uu' : u \in V(G) - V(T_2)\}$. Hence it suffices to prove the bound for T, that is

$$\mu(T) \leq \frac{2}{3} \frac{(n-\Delta+\delta)}{n} \frac{(n-\Delta+\delta-1)}{n(n-1)} \frac{n+2\Delta}{\delta} + 7. \tag{5.3.24}$$

For every vertex $u \in V(T)$, let u_M be a vertex in $V(M)$ closest to u in T. The tree, T, can be viewed as a weighted tree where each vertex has weight exactly 1. We now move the weight of every vertex to the closest vertex in $V(M)$, that is, we define a weight function $c : V(T) \to \mathbb{R}^+$ by:

$$c(u) = |\{x \in V(M) \mid x_M = u\}| \quad \text{for } u \in V(T).$$

Then $c(u) = 0$ if $u \notin V(M)$ and $\sum_{u \in V(M)} c(u) = n$ where n is the order of G. Since G is triangle-free, no two incident vertices of an edge in M have a common neighbour. Hence $\deg(u) \geq \delta$ and so we have that $c(u) \geq \delta$ for $u \in V(M) - \{w_1\}$ and $\deg(w_1) = \Delta$.

Since the weight of each vertex was moved over a distance not exceeding three and no distance between two weights has changed by more than 6. Thus,

$$\mu(T) \leq \mu_c(T) + 6. \tag{5.3.25}$$

Clearly, the weight c is concentrated only on the vertices of $V(M)$. We now consider the line graph $L = L(T)$ and define a new weight function $\overline{c}$ on $V(L) = E(T)$ by

$$\overline{c}(uv) = \begin{cases} c(u) + c(v) & \text{if } uv \in M, \\ 0 & \text{if } uv \notin M. \end{cases}$$

Since e_1 is an edge incident with w_1 in T, we have that $\overline{c}(e_1) \geq \Delta + \delta$ and $\overline{c}(e) \geq 2\delta$ for $e \in M - \{e_1\}$. Observe that $\sum\limits_{e \in M} \overline{c}(e) = \sum\limits_{u \in V(T)} c(u) = n$ and so we have that

$$n \geq \Delta + \delta + \sum_{e \in M - \{e_1\}} 2\delta = \Delta + \delta + 2\delta(|M| - 1).$$

This implies that

$$|M| \leq \frac{n-\Delta+\delta}{2\delta}. \tag{5.3.26}$$

Following a similar argument as in the proof of Theorem 4.3.1 (See Claim 1), we have that $|d_T(u,v)-d_L(e_u,e_v)| \leq 1$ where L is the line graph of T, $e_u, e_v \in E(T)$ are edges of T incident with u and v respectively. Hence, no distance between weights has increased by more than one and thus

$$\mu_c(T) \leq \mu_{\overline{c}}(L) + 1. \tag{5.3.27}$$

We now construct an induced subgraph, $L^4[M]$, of L^4. $L^4[M]$ is connected following a similar argument as in the proof of Claim 1 in Theorem 4.2.3. Furthermore, $d_{L^4[M]}(e_i,e_j) \leq 4d_{L^4[M]}(e_i,e_j)$ for all pairs $e_i, e_j \in M$ and since $V(M) = V(L^4[M])$, we have that $\sigma_{\overline{c}}(L) \leq 4\sigma_{\overline{c}}(L^4[M])$. Hence,

$$\mu_{\overline{c}}(L^4[M]) \leq 4\mu_{\overline{c}}(L^4[M]). \tag{5.3.28}$$

We now modify the weight function c to obtain a new weight function c' which satisfies $c'(e) \geq 2\delta$ for all $e \in M$. Define the new weight c' by

$$c'(e) = \begin{cases} \overline{c}(e) & \text{if } e \in M - \{e_1\}, \\ \overline{c}(e_1) - \Delta + \delta & \text{if } e = e_1. \end{cases}$$

Since $\overline{c}(e) \geq 2\delta$ for all $e \in M$ and $\overline{c}(e_1) \geq \Delta + \delta$, we have that $c'(e) \geq 2\delta$ for all $e \in M$.

By letting $N = \sum_{v \in V(M)} c'(u)$, we have that $N = n - \Delta + \delta$. We now express $\mu_{\overline{c}}(L^4[M])$ in terms of $\mu_{c'}(L^4[M])$.

$$\begin{aligned}
\mu_{c'}(L^4[M]) &= \binom{N}{2}^{-1} \sigma_{c'}(L^4[M]), \\
&= \binom{N}{2}^{-1} \Big[\sum_{i=2}^{r} (\overline{c}(e_1) - \Delta + \delta)\overline{c}(e_i) d_{L^4[M]}(e_1, e_i) \Big] \\
&\quad + \binom{N}{2}^{-1} \Big[\sum_{(e,f) \subseteq M - \{e_1\}} \overline{c}(e)\overline{c}(f) d_{L^4[M]}(e,f) \Big], \\
&= \binom{N}{2}^{-1} \Big[\sum_{(e,f) \subseteq M} c(e)c(f) d_{L^4[M]}(e,f) - \sum_{i=2}^{r} (\Delta - \delta)\overline{c}(e_i) d_{L^4[M]}(e_1, e_i) \Big], \\
&= \binom{N}{2}^{-1} \Big[\binom{n}{2} \mu_{\overline{c}}(L^4[M]) - (\Delta - \delta) \sum_{i=2}^{r} \overline{c}(e_i) d_{L^4[M]}(e_1, e_i) \Big],
\end{aligned}$$

and thus, by rearranging

$$\mu_{\overline{c}}(L^4[M]) \quad = \quad \frac{N(N-1)}{n(n-1)}\mu_{c'}(L^4[M]) + \frac{2(\Delta-\delta)}{n(n-1)}\sum_{i=2}^{r}\overline{c}(e_i)d_{L^4[M]}(e_1,e_i). \tag{5.3.29}$$

Clearly $\sum_{i=2}^{r}\overline{c}(e_i)d_{L^4[M]}(e_1,e_i) = \sigma_{\overline{c}}(e_1, L^4[M])$. We now bound the two terms of the right hand side of (5.3.29) separately.

Following an argument similar to the proof of Theorem 5.3.3, we obtain

$$\sigma_{\overline{c}}(e_1, L^4[M]) \leq (n-\Delta-\delta)(r-1) - \delta(r-1)(r-2). \tag{5.3.30}$$

Since $r \leq \frac{n-\Delta+\delta}{2\delta}$, and since the right hand side of the above inequality is increasing in r for $r \leq \frac{n-\Delta+\delta}{2\delta}$, we obtain by substituting this value for r that

$$\sigma_{\overline{c}}(e_1, L^4[M]) \leq \frac{(n-\Delta-\delta)^2}{4\delta} + \frac{1}{2}(n-\Delta-\delta). \tag{5.3.31}$$

Since $N = n - \Delta + \delta$, the right hand side equals $\frac{(N-2\delta)^2}{4\delta} + \frac{1}{2}(N-2\delta) = \frac{N(N-2\delta)}{4\delta}$, and so we obtain

$$\sigma_{\overline{c}}(e_1, L^4[M]) \leq \frac{N(N-2\delta)}{4\delta} < \frac{N(N-1)}{4\delta}. \tag{5.3.32}$$

To bound $\mu_{c'}(L^4[M])$, we recall that $c'(u) \geq 2\delta$ for all $u \in M$. Let C be the least multiple of 2δ such that $\sum_{u\in M} c'(u) \leq C$. Then by Lemma 5.2.2 we have

$$\mu_{c'}(L^4[M]) \leq \frac{C-2\delta}{C-1}\frac{C+2\delta}{6\delta} \leq \frac{C+1}{6\delta}.$$

Now $\sum_{u\in M} c'(u) = N$ so $C \leq N + 2\delta - 1$. Hence we obtain

$$\mu_{c'}(L^4[M]) \leq \frac{N+2\delta}{6\delta}. \tag{5.3.33}$$

Substituting (5.3.32) and (5.3.33) into (5.3.29) yields

$$\mu_{\overline{c}}([L^4[M]) \leq \frac{N(N-1)}{n(n-1)}\frac{N+2\delta}{6\delta} + \frac{\Delta-\delta}{n(n-1)}\frac{N(N-1)}{2\delta} = \frac{N(N-1)}{n(n-1)}\frac{N+3\Delta-\delta}{6\delta},$$

and thus

$$\mu_{\overline{c}}[L^4[M]) \leq \frac{N(N-1)}{n(n-1)}\frac{n+2\Delta}{6\delta}. \tag{5.3.34}$$

Combining the inequalities (5.3.25), (5.3.27) and (5.3.28) yields

$$\begin{aligned}\mu(T) &\leq \mu_c(T)+6\\ &\leq \mu_{\overline{c}}(L)+7\\ &\leq 4\mu_{\overline{c}}L^4[M])+7\\ &\leq \frac{2}{3}\frac{N(N-1)}{n(n-1)}\frac{n+2\Delta}{\delta}+7,\end{aligned}$$

as desired.

(ii) The bound on $\mu(G)$ follows immediately from (i) since the average distance of a spanning tree of G is not more than that the average distance of G itself.

(iii) To see that the bound above is sharp apart from the value of the additive constant, let δ, Δ, ℓ, be positive with $\Delta \geq 2\delta$.

Consider the following graph described below. For $1 \leq i \leq \ell$; $i \neq 2, \ell-1$, let H'_i be a disjoint copy of the empty graph $\lfloor \delta/2 \rfloor K_1$ if $i \equiv 1$ or $2 (\bmod\ 4)$ and $\lceil \delta/2 \rceil K_1$ if $i \equiv 0$ or $3 (\bmod\ 4)$. Furthermore, let H'_2 be a copy of $(\Delta - \lceil \delta/2 \rceil)K_1$ and $H'_{\ell-1}$ be a copy of the empty graph δK_1. Let $H'_{\ell,\delta,\Delta}$ be the graph obtained from the union of $H'_1, H'_2, \cdots, H'_\ell$ by joining each vertex in H'_i to each vertex in H'_{i+1} for $1 \leq i \leq \ell-1$. A sketch of $H'_{12,5,10}$ is shown in Figure 5.2.

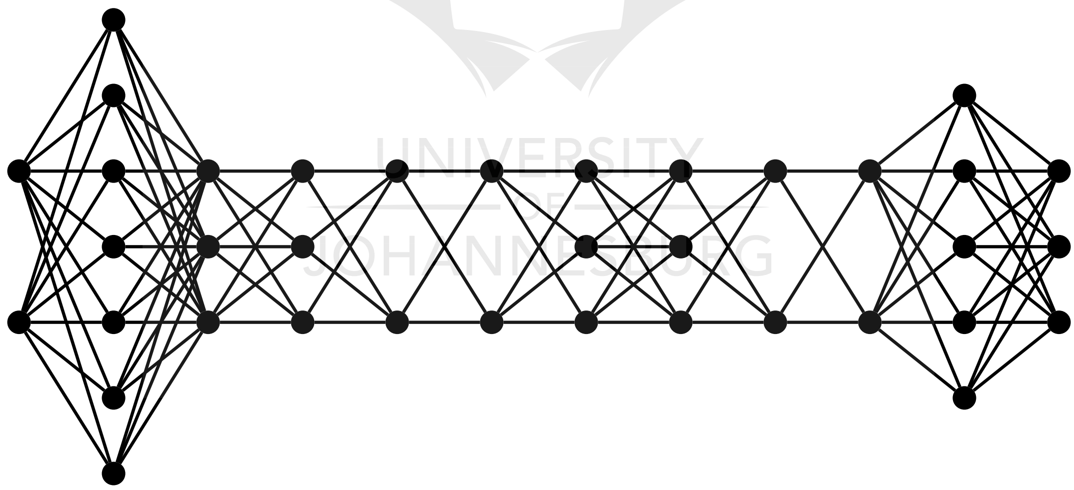

Figure 5.2: The graph $H'_{12,5,10}$

Clearly $H'_{\ell,\delta,\Delta}$ is the graph defined by the sequential sum

$$\overline{K}_{\lfloor\frac{\delta}{2}\rfloor}+\overline{K}_{\Delta-\lceil\frac{\delta}{2}\rceil}+\overline{K}_{\lceil\frac{\delta}{2}\rceil}+\overline{K}_{\lceil\frac{\delta}{2}\rceil}+\overline{K}_{\lfloor\frac{\delta}{2}\rfloor}+\overline{K}_{\lfloor\frac{\delta}{2}\rfloor}+\ldots+\overline{K}_{\lceil\frac{\delta}{2}\rceil}+\overline{K}_{\lceil\frac{\delta}{2}\rceil}+\overline{K}_{\lfloor\frac{\delta}{2}\rfloor}+\overline{K}_{\lfloor\frac{\delta}{2}\rfloor}+\overline{K}_{\delta}+\overline{K}_{\lceil\frac{\delta}{2}\rceil}$$

For the calculation that follows, we consider only the case that δ is even. For odd δ a similar calculation yields the same result.

Fix $\delta, \Delta \in \mathbb{N}$. For $\ell \in \mathbb{N}$ and δ even, it is easy to see that $H'_{\ell,\delta,\Delta}$ is triangle-free, has maximum degree Δ, minimum degree δ and $n(H'_{\ell,\delta,\Delta}) = \Delta + \delta\binom{\ell}{2} - \frac{\delta}{2}$.

Now, the Wiener index of $H'_{\ell,\delta,\Delta}$, $W(H'_{\ell,\delta,\Delta})$ becomes

$$W(H'_{\ell,\delta,\Delta}) = \sum_{1\le i<j<\ell} \sum_{\substack{v\in V(H'_i)\\ w\in V(H'_j)}} d(v,w) + \sum_{i=1}^{\ell} \sum_{v,w\in V(H'_i)} d(v,w). \qquad (5.3.35)$$

For $i \in \{1,2,\dots,\ell\} - \{2, \ell-1\}$ let $V_i = V(H'_i)$, and for $i \in \{2, \ell-1\}$ let V_i be a set of $\frac{\delta}{2}$ vertices of $V(H'_i)$, and let $W_i = V(H'_i) - V_i$. So $|V_i| = \frac{\delta}{2}$ for $1 \le i \le \ell$, $|W_2| = \Delta - \delta$ and $|W_{\ell-1}| = \frac{\delta}{2}$. Now the distance between two vertices $x \in V(H'_i)$ and $y \in V(H'_j)$, where $i < j$, equals $j - i$. Counting only the distances between pairs (x, y) of vertices that are either in distinct V_i, or $x \in W_2$ and $y \in V_j$ for $j \in \{3, 4, \dots, \ell\}$, we obtain

$$\begin{aligned} W(H'_{\ell,\delta,\Delta}) &> \sum_{1\le i<j\le\ell} \sum_{x\in V_i} \sum_{y\in V_j} d(x,y) + \sum_{x\in W_2} \sum_{j=3}^{\ell} \sum_{y\in V_j} d(x,y) \\ &= \sum_{1\le i<j\le \ell} |V_i|\cdot|V_j|\cdot(j-i) + \sum_{j=3}^{\ell} |W_2|\cdot|V_j|\cdot(j-2) \\ &= \frac{\delta^2}{4}\sum_{1\le i<j\le\ell}(j-i) + (\Delta-\delta)\frac{\delta}{2}\sum_{i=1}^{\ell-2} i. \end{aligned}$$

Now $N = n - \Delta + \delta$, $\sum_{1\le i<j\le\ell}(j-i) = \frac{\ell^3-\ell}{6}$ and $\sum_{i=1}^{\ell-2} i = \frac{\ell^2-3\ell+2}{2}$. Hence

$$W(H'_{\ell,\delta,\Delta}) > \frac{1}{24}\delta^2(\ell^3-\ell) + \frac{1}{4}(\Delta-\delta)\delta(\ell^2-3\ell+2).$$

We now make use of the fact that $(\ell+1)\frac{\delta}{2} = N$ and so $\ell + 1 = \frac{2N}{\delta}$. Clearly, $\ell^3 - \ell > (\ell+1)^3 - 3(\ell+1)^2$ and $\ell^2 - 3\ell + 2 > (\ell+1)^2 - 5(\ell+1)$. Hence

$$\begin{aligned} W(H'_{\ell,\delta,\Delta}) &> \frac{1}{24}\delta^2(\ell+1)^3 - \frac{3}{24}\delta^2(\ell+1)^2 + \frac{1}{4}(\Delta-\delta)\delta(\ell+1)^2 - \frac{5}{4}(\Delta-\delta)\delta(\ell+1) \\ &= \frac{N^3}{3\delta} - \frac{N^2}{2} + \frac{(\Delta-\delta)N^2}{\delta} - \frac{5(\Delta-\delta)N}{2} \\ &= \frac{N^2(N+3\Delta)}{3\delta} - \frac{1}{2}N(3N - 5(\Delta-\delta)) \\ &> \frac{N(N-1)(N+3\Delta)}{3\delta} - 4n(n-1). \end{aligned}$$

Now $N + 3\Delta = n - 2\Delta + \delta > n + 2\Delta$. Division by $\binom{n}{2}$ now yields

$$\mu(H'_{\ell,\delta,\Delta}) > \frac{2}{3}\frac{N(N-1)}{n(n-1)}\frac{n+2\Delta}{\delta} - 8.$$

This implies that the bound on the average distance in Theorem 5.3.4 is sharp apart from an additive constant.

□

Define $\mu'(\ell, \delta, \Delta)$ and $W'(\ell, \delta, \Delta)$ to be the maximum average distance and maximum Wiener index, respectively, among all connected, triangle-free graphs of order n, minimum degree δ and maximum degree Δ. Theorem 5.3.4 shows that

$$\mu'(\ell, \delta, \Delta) = \frac{2}{3}\frac{(n-\Delta+\delta)(n-\Delta+\delta-1)}{n(n-1)}\frac{n+2\Delta}{\delta+1} + O(1)$$

and

$$W'(\ell, \delta, \Delta) = \frac{2}{3}\binom{n-\Delta+\delta}{2}\frac{n+2\Delta}{\delta+1} + O(n^2).$$

In the next theorem we show that the bound in Theorem 5.3.3 can be improved significantly for graphs not containing a 4-cycle. If $\delta+1$ is a prime power, then our bound is best possible in a sense specified later.

Theorem 5.3.5. *Let G be a connected C_4-free graph of order n, minimum degree δ and maximum degree Δ. Then*

i) G has a spanning tree T with

$$\mu(T) \le \frac{5}{3}\frac{n-\xi_\Delta+\xi_\delta}{n}\frac{n-\xi_\Delta+\xi_\delta-1}{n-1}\frac{n+2\xi_\Delta}{\xi_\delta} + 8.$$

ii)

$$\mu(G) \le \frac{5}{3}\frac{n-\xi_\Delta+\xi_\delta}{n}\frac{n-\xi_\Delta+\xi_\delta-1}{n-1}\frac{n+2\xi_\Delta}{\xi_\delta} + 8.$$

where $\xi_\Delta = \Delta\delta - 2\lfloor \Delta/2 \rfloor + 1$ and $\xi_\delta = \delta^2 - 2\lfloor \delta/2 \rfloor + 1$.

Proof. Let w_1 be as defined previously. We first construct a spanning tree in exactly the same way as in the previous proofs. For the reader's convenience we repeat the construction. We start first by finding a maximal 4-packing, B, of G using the following procedure. Let $B = w_1$. If there exists a vertex w_i in G with $d_G(w_i, V(B)) = 5$, add w_i to B. Add vertices w_i satisfying $d_G(w_i, V(B)) = 5$ to B until, after r steps say, every vertex not in B is within distance four of B. Thus $B = \{w_1, w_2, \cdots w_r\}$.

Let $N_G[u]$ and $N_{\le 2}(u)$ denote the closed neighbourhood and set of vertices within distance two of u, respectively. For each $u \in B$ let $T_1(u)$ be a tree with vertex set $N_{\le 2}(u)$ satisfying $d_T(u, v) = d_G(u, v)$ for each $v \in N_{\le 2}(u)$. $T_1(u)$ is distance preserving to u. Then $T_1 = \bigcup_{u \in V(B)} T_1(u)$ is a subforest of G. By our construction of B, there exist $|B| - 1$ edges in G, each one joining two distinct components of T_1, whose addition to T_1 yields a tree $T_2 \leq G$. Now, each vertex $v \in V(G) - V(T_1)$ is within distance five of some vertex in T_2.

Let $T \geq T_2$ be a spanning tree of G in which $d_T(x, V(B)) = d_G(x, V(B))$ for each $x \in V(G)$. Hence it suffices to prove the bound for T, that is

$$\mu(T) \leq \frac{5}{3}\frac{n-\xi_\Delta+\xi_\delta}{n}\frac{n-\xi_\Delta+\xi_\delta-1}{n-1}\frac{n+2\xi_\Delta}{\xi_\delta}+8.$$

For every vertex $u \in V(T)$, let u_B be a vertex in B closest to u in T. We move the weight of every vertex to the closest vertex in B by defining a weight function $c : V(T) \to \mathbb{R}^+$ by

$$c(u) = |\{u \in V(T) \mid u_B = v\}| \quad \text{for } v \in V(T),$$

where $c(u) = 0$ if $u \notin B$.

By Lemma 2.2.1,

$$\left|N_{\leq 2}[w_i]\right| \geq \delta^2 - 2\lfloor \delta/2 \rfloor + 1 \quad \text{for all} \quad w_i \in B.$$

Therefore, $c(w_i) \geq \delta^2 - 2\lfloor \delta/2 \rfloor + 1$ for all $w_i \in B - \{w_1\}$. It follows immediately from the same argument that $c(w_1) \geq \Delta\delta - 2\lfloor \Delta/2 \rfloor + 1$. Subsequently, we let $\xi_\Delta = \Delta\delta - 2\lfloor \Delta/2 \rfloor + 1$. and $\xi_\delta = \delta^2 - 2\lfloor \delta/2 \rfloor + 1$. This implies that $c(w_1) \geq \xi_\Delta$ and $c(w_i) \geq \xi_\delta$ for all $w_i \in B - \{w_i\}$. We also note that $\sum_{v \in V(T)} c(u) = n$. This yields $n = \sum_{u \in B} c(u) \geq \xi_\Delta + (r-1)\xi_\delta$, and so

$$r \leq \frac{n - \xi_\Delta + \xi_\delta}{\xi_\delta}. \tag{5.3.36}$$

Since each weight was moved over a distance not exceeding four and no distance between two weights has changed by more than 8 and thus

$$\mu(T) \leq \mu_c(T) + 8. \tag{5.3.37}$$

As in the proof of previous theorem, we construct an induced subgraph, $T^5[B] = T''$, of T^5. T'' is connected. Clearly $d_T(u,v) \leq 5d_{T''}(u,v)$ for all pair of vertices $u, v \in B$ and since $V(B) = V(T'')$, we have that $\sigma_c(T) \leq 5\sigma_c(T'')$. Hence,

$$\mu_c(T) \leq 5\mu_c(T''). \tag{5.3.38}$$

We now modify the weight function c to obtain a new weight function c' satisfying $c'(u) \geq \xi_\delta$ for all $u \in B$. Define the new weight c' by

$$c'(u) = \begin{cases} c(u) & \text{if } u \in B - \{w_1\}, \\ c(u) - \xi_\Delta + \xi_\delta & \text{if } u = w_1. \end{cases}$$

Since $c(u) \geq \xi_\delta$ for all $u \in B$ and $c(w_1) \geq \xi_\Delta$, we have that $c'(u) \geq \xi_\delta$ for all $u \in B$.

Furthermore,

$$\sum_{u \in B} c'(u) = \sum_{u \in A} c(u) - \xi_\Delta + \xi_\delta = n - \xi_\Delta + \xi_\delta.$$

By letting $N'' = \sum_{u \in A} c'(u)$, we have that $N'' = n - \xi_\Delta + \xi_\delta$. We now express $\mu_c(T'')$ in terms of $\mu_{c'}(T'')$.

$$\begin{aligned}
\mu_{c'}(T'') &= \binom{N''}{2}^{-1} \sigma_{c'}(T''), \\
&= \binom{N''}{2}^{-1} \Big[\sum_{i=2}^{r} (c(w_1) - \xi_\Delta + \xi_\delta) c(w_i) d_{T''}(w_1, w_i) + \sum_{(u,v) \subseteq B - \{w_1\}} c(u)c(v) d_{T''}(u,v) \Big], \\
&= \binom{N''}{2}^{-1} \Big[\sum_{(u,v) \subseteq B} c(u)c(v) d_{T''}(u,v) - \sum_{i=2}^{r} (\xi_\Delta - \xi_\delta) c(w_i) d_{T''}(w_1, w_i) \Big], \\
&= \binom{N''}{2}^{-1} \Big[\binom{n}{2} \mu_c(T'') - (\xi_\Delta - \xi_\delta) \sum_{i=2}^{r} c(w_i) d_{T''}(w_1, w_i) \Big],
\end{aligned}$$

and thus, by rearranging

$$\mu_c(T'') = \frac{N''(N''-1)}{n(n-1)} \mu_{c'}(T'') + \frac{2(\xi_\Delta - \xi_\delta)}{n(n-1)} \sum_{i=2}^{r} c(w_i) d_{T''}(w_1, w_i) \qquad (5.3.39)$$

Clearly $\sum_{i=2}^{r} c'(e_i) d_{T''}(w_1, w_i) = \sigma_{c'}(w_1, T'')$. We now bound the two terms of the right hand side of (5.3.39) separately. Following an argument similar to the proof of Theorem 5.3.3, we obtain

$$\sigma_{c'}(w_1, T'') \leq (n - \xi_\Delta)(r-1) - \frac{1}{2}\xi_\delta (r-1)(r-2). \qquad (5.3.40)$$

Since $r \leq \frac{n - \xi_\Delta + \xi_\delta}{\xi_\delta}$, and since the right hand side of the above inequality is increasing in r for $r \leq \frac{n - \xi_\Delta + \xi_\delta}{\xi_\delta}$, we obtain by substituting this value for r that

$$\sigma_{c'}(w_1, T'') \leq \frac{(n - \xi_\Delta)^2}{2\xi_\delta} + \frac{1}{2}(n - \xi_\Delta). \qquad (5.3.41)$$

Since $N'' = n - \xi_\Delta + \xi_\delta$, the right hand side of (5.3.41) equals $\frac{(N'' - \xi_\delta)^2}{2\xi_\delta} + \frac{N'' - \xi_\delta}{2} = \frac{N''(N'' - \xi_\delta)}{2\xi_\delta}$, and so we obtain

$$\sigma_{c'}(w_1, T'') \leq \frac{N''(N'' - \xi_\delta)}{2\xi_\delta} < \frac{N''(N''-1)}{2\xi_\delta}. \qquad (5.3.42)$$

To bound $\mu_{c'}(T'')$, we recall that $c'(u) \geq \xi_\delta$ for all $u \in B$. Let C' be the least multiple of ξ_δ such that $\sum_{u \in B} c'(u) \leq C'$. Then by Lemma 5.2.2 we have

$$\mu_{c'}(T'') \leq \frac{C' - \xi_\delta}{C' - 1} \frac{C' + \xi_\delta}{3\xi_\delta} \leq \frac{C' + 1}{3\xi_\delta}.$$

Now $\sum_{u\in B} c'(u) = N''$ so $C' \le N'' + \xi_\delta - 1$. Hence we obtain

$$\mu_{c'}(T'') \le \frac{N'' + \xi_\delta}{3\xi_\delta}. \tag{5.3.43}$$

Substituting (5.3.42) and (5.3.43) into (5.3.39) yields

$$\mu_{c'}(T'') \le \frac{N''(N''-1)}{n(n-1)} \frac{N''+\xi_\delta}{3\xi_\delta} + \frac{\xi_\Delta - \xi_\delta}{n(n-1)} \frac{N''(N''-1)}{\xi_\delta} = \frac{N''(N''-1)}{n(n-1)} \frac{N''+3\xi_\Delta - 2\xi_\delta}{3\xi_\delta},$$

and thus

$$\mu_{c'}(T'') \le \frac{N''(N''-1)}{n(n-1)} \frac{n+2\xi_\Delta - \xi_\delta}{3\xi_\delta} < \frac{N''(N''-1)}{n(n-1)} \frac{n+2\xi_\Delta}{3\xi_\delta}. \tag{5.3.44}$$

Combining the inequalities (5.3.37), (5.3.38) and (5.3.44) yields

$$\begin{aligned} \mu(T) &\le \mu_c(T) + 8 \\ &\le 5\mu_{c'}(T'') + 8 \\ &\le \frac{5}{3} \frac{N''(N''-1)}{n(n-1)} \frac{n+2\xi_\Delta}{\xi_\delta} + 8, \end{aligned}$$

as desired. Hence, Theorem 5.3.5 holds. $\square$

(ii) The proof of part (ii) follows from part (i).

The following theorem shows that the bound in Theorem 5.3.5 is not far from being best possible in the sense described later if $\delta + 1$ is prime power. Our construction is based on the graph H_q constructed independently by Erdös and Rényi [52] and Brown [12], and a modification H_0, first described in [51] (see Example 2.3.5). For the readers convenience, we recall below the description of the graph H_q and its modification H_0.

Let q be a prime power. Let $GF(q)$ be the field of order q and let $GF(q)^3$ be the 3-dimensional vector space over $GF(q)$ whose vectors are the triples of elements of $GF(q)$. We define H_q to be the graph whose vertices are the 1-dimensional subspaces of $GF(q)^3$, where two vertices are adjacent if, as subspaces, they are orthogonal. It is easy to verify that H_q has $q^2 + q + 1$ vertices, that its vertices have degree either q or $q + 1$, and that H_q is C_4-free and connected.

Erdös, Pach, Pollack and Tuza [51] described a modification H_0 of the graph H_q, obtained as follows. Choose a vertex z of degree q in H_q and let u, v be two distinct neighbours of z. Delete z and all edges joining a neighbour of u to a neighbour of v. The resulting graph H_0 is connected, C_4-free and has $q^2 + q$ vertices, its minimum degree is at least $q - 1$, and $d_{H_0}(u, v) \ge 4$.

Theorem 5.3.6. *Let $\delta \in \mathbb{N}$ such that $\delta \ge 3$ and $\delta + 1$ is a prime power. Let $\Delta \in \mathbb{N}$*

such that $\Delta - 1$ is a positive multiple of $\delta + 2$. Let $n \in \mathbb{N}$ such that $n - (\Delta - 1)(\delta + 1)$ is a positive multiple of $(\delta + 1)(\delta + 2)$. Then there exists a C_4-free graph G of order n, maximum degree Δ and minimum degree at least δ which satisfies

$$\mu(G) > \frac{5}{3} \frac{(n - \theta_\Delta + \theta_\delta)(n - \theta_\Delta + \theta_\delta - 1)}{n(n-1)} \frac{n + 2\theta_\Delta}{\theta_\delta} - 13,$$

where $\theta_\Delta = (\Delta - 1)(\delta + 1) + 1$ and $\theta_\delta = (\delta + 2)(\delta + 1)$.

Proof. Let $q = \delta + 1$. By the assumptions on Δ and n we can find $k, \ell \in \mathbb{N}$ with $\ell \geq 2$ such that $\Delta = k(q + 1) + 1$ and $n = (k + \ell - 1)(q^2 + q)$.

We construct a graph G_1 by taking k disjoint copies of H_q, choosing a vertex of degree $q + 1$ in each copy, and then identifying these k vertices to a new vertex v_1. Clearly, $n(G_1) = k(q^2 + q) + 1$ and $\deg_{G_1}(v_1) = k(q + 1)$. For $i = 2, 3 \ldots, \ell$ let G_i be a copy of the graph H'_q. Let u_i and v_i denote the vertices of G_i corresponding to u and v, respectively, of H'_q. Let $G_{k,\ell,q}$ be the graph obtained from the disjoint union $\bigcup_{i=1}^{\ell} G_i$ by adding the edges $v_i u_{i+1}$ for $i = 1, 2, \ldots, \ell - 1$. A sketch of the graph $G_{2,5,q}$ is shown in Figure 5.3. Then $G_{k,\ell,q}$ has order n, maximum degree Δ and minimum degree at least δ. We define $\theta_\Delta = (\Delta - 1)(\delta + 1) + 1$, $\theta_\delta = (\delta + 2)(\delta + 1)$, and $M = n - \theta_\Delta + \theta_\delta$.

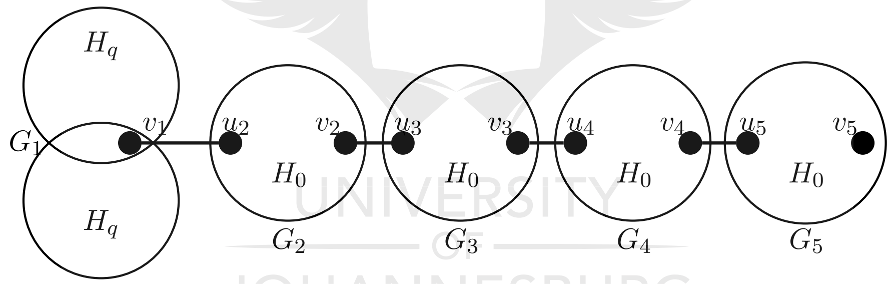

Figure 5.3: The graph $G_{k,\ell,q}$ for $k = 2$ and $\ell = 5$.

We now bound the average distance of $G_{n,\Delta,\delta}$ from below in terms of n, θ_Δ and θ_δ. Let $V_i := V(G_i)$ for $i = 1, 2, \ldots, \ell$. Then $|V_1| = \theta_\Delta$ and $|V_i| = \theta_\delta$ for $i = 2, 3, \ldots, \ell$.

For our lower bound we only count the distances between pairs x, y with either $x \in V_i$ and $y \in V_j$ where $2 \leq i < j \leq \ell$, or $x \in V_1$ and $y \in V_j$ for $j = 2, 3, \ldots, \ell$, ignoring all other pairs of vertices. Clearly, if $x \in V_i$, $y \in V_j$ and $i < j$, then $d(x, y) \geq 5(j - i) - 4$. If $x \in V_1$

and $y \in V_j$, then $d(x,y) \geq 5j-9$. Hence

$$\begin{aligned} W(G_{k,\ell,q}) &> \sum_{2\leq i<j\leq \ell} \sum_{x\in V_i, y\in V_j} d(x,y) + \sum_{j=2}^{\ell} \sum_{x\in V_1, y\in V_j} d(x,y) \\ &\geq \sum_{2\leq i<j\leq \ell} |V_i|\,|V_j|\big(5(j-i)-4\big) + \sum_{j=2}^{\ell} |V_1|\,|V_j|\big(5j-9\big) \\ &= \theta_\delta^2 \sum_{2\leq i<j\leq \ell} \big(5(j-i)-4\big) + \theta_\delta\theta_\Delta \sum_{j=2}^{\ell} \big(5j-9\big). \end{aligned}$$

Straightforward calculations show that $\sum_{2\leq i<j\leq \ell} \big(5(j-i)-4\big) = \frac{1}{6}(5\ell^3-27\ell^2+46\ell-24) \geq \frac{1}{6}(5\ell^3-27\ell^2)$ and $\sum_{j=2}^{\ell}\big(5j-9\big) = \frac{1}{2}(5\ell^2-13\ell+8) \geq \frac{1}{2}(5\ell^2-13\ell)$. Substituting these values we obtain

$$\begin{aligned} W(G_{k,\ell,q}) &> \frac{1}{6}\theta_\delta^2(5\ell^3-27\ell^2) + \frac{1}{2}\theta_\delta\theta_\Delta(5\ell^2-13\ell) \\ &= \frac{5}{6}\big(\theta_\delta^2\ell^3 + 3\theta_\delta\theta_\Delta\ell^2\big) - \frac{9}{2}\theta_\delta^2\ell^2 - \frac{13}{2}\theta_\Delta\theta_\delta\ell. \end{aligned}$$

Since $\ell\theta_\delta = M$, we have $\theta_\delta^2\ell^3 + 3\theta_\delta\theta_\Delta\ell^2 = \frac{M^3}{\theta_\delta} + 3\frac{\theta_\Delta M^2}{\theta_\delta} \geq \frac{M(M-1)(n+2\theta_\Delta)}{\theta_\delta}$. Also $\frac{9}{2}\theta_\delta^2\ell^2 + \frac{13}{2}\theta_\Delta\theta_\delta\ell = \frac{13}{2}\theta_\delta\ell(\frac{9}{13}\theta_\delta\ell + \theta_\Delta) < \frac{13}{2}(M-1)M \leq \frac{13}{2}(n-1)n$. Dividing the above lower bound on $W(G_{k,\ell,q})$ by $\binom{n}{2}$ thus yields

$$\mu(G_{k,\ell,q}) > \frac{5}{3}\frac{M(M-1)}{n(n-1)}\frac{n+2\theta_\Delta}{\theta_\delta} - 13,$$

as desired. □

To see that Theorem 5.3.5 is not far from best possible, even for very large maximum degree, assume that $\delta, \Delta \in \mathbb{N}$ satisfy the hypothesis of Theorem 5.3.6, and that moreover $\Delta = cn$ for some $c \in \mathbb{R}$ with $0 < c < \frac{1}{\delta+1}$ (note that by Lemma 2.2.1 c cannot be greater than $\frac{1}{\delta+1}$). Then the leading term in the upper bound in Theorem 5.3.5 is $\frac{5}{3}\frac{(1-c\delta)^2(1+2c\delta)}{\delta^2-2\lfloor\delta/2\rfloor+1}n$ while the leading term in the lower bound in Theorem 5.3.6 is $\frac{5}{3}\frac{(1-c(\delta+1)^2(1+2c(\delta+1))}{\delta^2+\delta-2}n$. It is easy to see that the ratio of the two coefficients of n approaches 1 as δ gets large. So the larger δ, the closer the bound in Theorem 5.3.5 to being sharp.

The next theorem shows that the bound on the average distance can be improved if in addition, the graph described in Theorem 5.3.5 has girth at least 6. The proof technique follows essentially from Theorem 4.3.4 with a little modification. For the readers convenience, we repeat the proof here.

Theorem 5.3.7. *Let G be a graph of girth at least 6 with n vertices, minimum degree $\delta \geq 2$, and maximum degree Δ. Then,* *i) G has a spanning tree T with*

$$\mu(T) \leq \frac{2(n-\kappa_\Delta+\kappa_\delta)}{n}\frac{n-\kappa_\Delta+\kappa_\delta-1}{n-1}\frac{n+2\kappa_\Delta}{\kappa_\delta}+13. \tag{5.3.45}$$

ii)

$$\mu(G) \leq \frac{2(n-\kappa_\Delta+\kappa_\delta)}{n}\frac{n-\kappa_\Delta+\kappa_\delta-1}{n-1}\frac{n+2\kappa_\Delta}{\kappa_\delta}+13. \tag{5.3.46}$$

where $\kappa_\Delta := \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)}+\frac{3}{2}$, *and* $\kappa_\delta := 2(\delta^2-\delta+1)$.

Proof. Let v_1 be a vertex of degree Δ and let e_1 be an edge incident with v_1. We obtain a maximal matching M of G as follows. Let $M = \{e_1\}$. Let $V(M)$ be the set of vertices incident with an edge of M. Recall that for an edge e, $d_G(e, V(M))$ is the minimum of the distances between a vertex incident with e and a vertex in $V(M)$. If there exists an edge e_2 with $d_G(e_2, e_1) = 6$, add e_2 and let $M_0 = \{e_2\}$. If there exists an edge e_3 with

(i) $d_G(e_3, e_1) \geq 6$

(ii) $d_G(e_3, e_2) \geq 5$ and

(iii) we have equality in (i) or (ii) or both,

then we add e_3 to M_0. Repeat this process: Let $M_0 = \{e_2, e_3, \ldots, e_{i-1}\}$. If there exists an edge e_i satisfying

(a) $d_G(e_i, e_1) \geq 6$

(b) $\min\{d_G(e_i, e_j) \| \; j = 2, 3, \ldots, i-1\} \geq 5$, and

(c) we have equality in (a) or (b) or both,

then add e_i to M_0. We repeat this process until, after k steps say, no further edge can be added to M_0. Let $M = \{e_1\} \cup M_0$, so $M = \{e_1, \ldots, e_k\}$ and $|M| = k$. Then every edge not in M is within distance 5 of an edge in M.

Let $T^*_{v_1}$ be a tree with vertex set $N_{\leq 3}(v_1)$ which is distance preserving from v_1. For $i \in \{1, 2, \ldots, k\}$ let $e_i = u_i v_i$. Let $T^*_{e_i}$ be a subtree of G with vertex set $N_{\leq 2}(u_i) \cup N_{\leq 2}(v_i)$ that preserves the distance to $\{u_i v_i\}$.

Let $T_1 = T^*_{v_1} \cup \bigcup_{e_i \in M} T^*_{e_i}$. Then, $T_1 \leq G$ is a subforest of G with vertex set, $N_{\leq 3}(v_1) \cup N_{\leq 2}\big(V(M) - \{v_1\}\big)$. By our construction of M, there exists $|M| - 1$ edges in G, each joining two distinct components of T_1, whose addition to T_1 yields a tree $T_2 \leq G$, so that T_2 contains T_1 and has the same vertex set as T_1.

Now, each vertex $v \in V(G) - V(T_2)$ is within distance five of some vertex w in $V(M)$ closest to it. Let $T \geq T_2$ be a spanning tree of G containing T_2 and distance preserving from $V(M)$, i.e. $d_T(x, V(M)) = d_G(x, V(M))$ for each $x \in V(G)$. Clearly, tree T has the same maximum degree as G since $\deg_T(v_1) = \deg_G(v_1)$. It suffices to prove the bound for T since it directly implies part *(ii)* of the theorem.

For every vertex $u \in V(T)$, let u_M be a vertex in $V(M)$ closest to u in T. We can view T as a weighted tree where each vertex has weight exactly 1. We now move the weight of every vertex to the closest vertex in $V(M)$, by defining a new weight function $c : V(T) \to \mathbb{N} \cup \{0\}$ by:

$$c(u) = |\{x \in V(M) \mid x_M = u\}| \text{ for } u \in V(T).$$

Note that $c(u) = 0$ if $u \notin V(M)$ and $\sum_{u \in V(M)} c(u) = n$, where n is the order of G.

Since the weight of each vertex was moved over a distance not exceeding five and no distance between two weights have changed more than 10, we have that

$$\mu(T) \leq \mu_c(T) + 10 \tag{5.3.47}$$

Now the weight of c is concentrated exclusively on the vertices incident with an edge of M. Consider the line graph $L = L(T)$ and define a new weight function $\bar{c}$ on $V(L) = E(T)$ by

$$\bar{c}(wz) = \begin{cases} c(w) + c(z) & \text{if } wz \in M, \\ 0 & \text{if } wz \notin M. \end{cases}$$

Let $wz \in M - \{e_1\}$. For each vertex $x \in N_{\leq 2}(w) \cup N_{\leq 2}(z)$, we have $x_M \in \{w, z\}$. Hence

$$c(w) + c(z) \geq |N_{\leq 2}(w) \cup N_{\leq 2}(z)|.$$

By Lemma 2.3.2, we have

$$|N_{\leq 2}(w) \cup N_{\leq 2}(z)| \geq (\delta - 1)\big(\deg(w) + \deg(z)\big) + 2 \geq 2(\delta^2 - \delta + 1) \tag{5.3.48}$$

and so we have that

$$\bar{c}(wz) \geq 2\delta^2 - 2\delta + 2 \text{ for } wz \in M - \{e_1\}. \tag{5.3.49}$$

On the other hand, let $e_1 := v_1 w$. For each vertex $x \in N_{\leq 3}(v_1)$, we have $x_M \in \{v_1, w\}$. Hence,

$$c(v_1) + c(w) \geq |N_{\leq 3}(v_1)|.$$

By Lemma 3.2.1, we have

$$|N_{\leq 3}(v_1)| \geq \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}. \tag{5.3.50}$$

and so we have that

$$\overline{c}(e_1) = \overline{c}(v_1 w) \geq \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}. \tag{5.3.51}$$

Note that $\overline{c}(wz) = 0$ if $wz \notin M$ and $\sum_{e \in M} \overline{c}(e) = \sum_{u \in V(T)} c(u) = n$. It follows that

$$\begin{aligned} n &\geq \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2} + \sum_{x \in M - \{e_1\}} (2\delta^2 - 2\delta + 2), \\ &= \Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2} + (2\delta^2 - 2\delta + 2)(|M| - 1), \end{aligned}$$

and rearranging yields

$$|M| \leq \frac{n - \left[\Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}\right]}{2\delta^2 - 2\delta + 2} + 1. \tag{5.3.52}$$

Following a similar argument as in the proof of Theorem 4.3.1 (See Claim 1), we have that $|d_T(x, y) - d_L(e_x, e_y)| \leq 1$ where L is the line graph of T, $e_x, e_y \in E(T)$ are edges of T incident with x and y respectively. Hence no distance between weights has increased by more than 1 and thus

$$\mu_c(T) \leq \mu_{\overline{c}}(L) + 1. \tag{5.3.53}$$

If f_1, f_2 are two matching edges in M with $d_T(f_1, f_2) = 5$, then $d_L(f_1, f_2) \leq 6$. Now the weights lie solely on M. Let H be the graph obtained from $L^6[M]$ by joining e_1 to every e_i in M for which $d_L(e_1, e_i) \leq 7$. Such edges exist since by construction of M we have that $d_T(e_1, e_2) = 6$ and thus $d_L(e_1, e_2) \leq 7$. Essentially the same argument as in the proof of Theorem 4.2.3) shows that H is connected.

Let $e, f \in M$ and let P be a shortest path from e to f in H of length ℓ say. A similar argument as in the proof of Theorem 4.3.4 shows that P yields a path of length at most $6\ell + 2$. Hence

$$d_L(e, f) \leq 6 d_H(e, f) + 2 \quad \text{for every} \quad e, f \in M,$$

and so we have that

$$\mu_{\overline{c}}(L) \leq 6\mu_{\overline{c}}(H) + 2. \tag{5.3.54}$$

We now modify the weight function $\overline{c}$ to obtain a new weight function $\overline{c}'$ on M for which $\overline{c}'(e) \geq 2\delta^2 - 2\delta + 2$ for all $e \in M$.

Let $\overline{c}'(e_1) = \overline{c}(e_1) - \left[\Delta\delta + (\delta - 1)\sqrt{\Delta(\delta - 2)} + \frac{3}{2}\right] + (2\delta^2 - 2\delta + 2)$ and let $\overline{c}'(e) = \overline{c}(e)$ for

$e \in M - \{e_1\}$.

Clearly $\overline{c}'(e_i) \geq 2\delta^2 - 2\delta + 2$ for all $e_i \in M$ since $\overline{c}(e_1) \geq \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}$ and $\overline{c}(e) \geq 2\delta^2 - 2\delta + 2$ for $e \in M - \{e_1\}$.

Furthermore,

$$\begin{aligned}\sum_{v\in V(L^6[M])} \overline{c}'(v) &= \sum_{e\in M-\{e_1\}} \overline{c}(e) + \overline{c}'(e_1)\\ &= \sum_{e\in M} \overline{c}(e) - \Big(\Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}\Big) + (2\delta^2 - 2\delta + 2)\Big]\\ &= \sum_{v\in V(T)} c(v) - \big(\Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}\big) + (2\delta^2 - 2\delta + 2)\end{aligned}$$

Let $\kappa_\Delta := \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}$ and $\kappa_\delta := 2\delta^2 - 2\delta + 2$. By letting $N^* = \sum_{v\in V(L^6[M])} \overline{c}'(v)$, we have that $N^* = n - \kappa_\Delta + \kappa_\delta$.

We now express $\mu_{\overline{c}}(H)$ in terms of $\mu_{\overline{c}'}(H)$.

$$\begin{aligned}\mu_{\overline{c}'}(H) &= \binom{N^*}{2}^{-1} \sigma_{\overline{c}'}(H),\\ &= \binom{N^*}{2}^{-1}\Big[\sum_{i=2}^{r}(\overline{c}(e_1) - \kappa_\Delta + \kappa_\delta)\overline{c}(e_i)d_H(e_1,e_i)\Big]\\ &\quad + \binom{N^*}{2}^{-1}\Big[\sum_{(e,f)\subseteq M-\{e_1\}} \overline{c}(e)\overline{c}(f)d_H(e,f)\Big],\\ &= \binom{N^*}{2}^{-1}\Big[\sum_{(e,f)\subseteq M} \overline{c}(e)\overline{c}(f)d_H(e,f) - \sum_{i=2}^{r}(\kappa_\Delta - \kappa_\delta)\overline{c}(e_i)d_H(e_1,e_i)\Big],\\ &= \binom{N^*}{2}^{-1}\Big[\binom{n}{2}\mu_{\overline{c}}(H) - (\kappa_\Delta - \kappa_\delta)\sum_{i=2}^{r}\overline{c}(e_i)d_H(e_1,e_i)\Big],\end{aligned}$$

and thus, by rearranging

$$\mu_{\overline{c}}(H) = \frac{N^*(N^*-1)}{n(n-1)}\mu_{\overline{c}'}(H) + \frac{2(\kappa_\Delta - \kappa_\delta)}{n(n-1)}\sum_{i=2}^{r}\overline{c}(e_i)d_H(e_1,e_i) \tag{5.3.55}$$

Clearly $\sum_{i=2}^{r}\overline{c}(e_i)d_H(e_1,e_i) = \sigma_{\overline{c}}(e_1,H)$. We now bound the two terms of the right hand side of (5.3.55) separately. Following an argument similar to the proof of Theorem 5.3.3, we obtain

$$\sigma_{\overline{c}}(e_1,H) \leq (n-\kappa_\Delta)(r-1) - \frac{1}{2}\kappa_\delta(r-1)(r-2). \tag{5.3.56}$$

Since $r \leq \frac{n-\kappa_\Delta+\kappa_\delta}{\kappa_\delta}$, and since the right hand side of the above inequality is increasing in r for $r \leq \frac{n-\kappa_\Delta+\kappa_\delta}{\kappa_\delta}$, we obtain by substituting this value for r that

$$\sigma_{\overline{c}}(e_1, H) \leq \frac{(n-\kappa_\Delta)^2}{2\kappa_\delta} + \frac{1}{2}(n-\kappa_\Delta). \tag{5.3.57}$$

Since $N^* = n - \kappa_\Delta + \kappa_\delta$, the right hand side equals $\frac{(N^*-\kappa_\delta)^2}{2\kappa_\delta} + \frac{N^*-\kappa_\delta}{2} = \frac{N^*(N^*-\kappa_\delta)}{2\kappa_\delta}$, and so we obtain

$$\sigma_{\overline{c}}(e_1, H) \leq \frac{N^*(N^*-\kappa_\delta)}{2\kappa_\delta} < \frac{N^*(N^*-1)}{2\kappa_\delta}. \tag{5.3.58}$$

To bound $\mu_{\overline{c}'}(H)$, we recall that $\overline{c}'(u) \geq \kappa_\delta$ for all $u \in M$. Let C'' be the least multiple of κ_δ such that $\sum_{u\in M} \overline{c}'(u) \leq C''$. Then by Lemma 5.2.2 we have

$$\mu_{\overline{c}'}(H) \leq \frac{C''-\kappa_\delta}{C''-1}\frac{C''+\kappa_\delta}{3\kappa_\delta} \leq \frac{C''+1}{3\kappa_\delta}.$$

Now $\sum_{u\in M} \overline{c}'(u) = N^*$ so $C'' \leq N^* + \kappa_\delta - 1$. Hence we obtain

$$\mu_{\overline{c}'}(H) \leq \frac{N^*+\kappa_\delta}{3\kappa_\delta}. \tag{5.3.59}$$

Substituting (5.3.58) and (5.3.59) into (5.3.55) yields

$$\mu_{\overline{c}}(H) \leq \frac{N^*(N^*-1)}{n(n-1)}\frac{N^*+\kappa_\delta}{3\kappa_\delta} + \frac{\kappa_\Delta-\kappa_\delta}{n(n-1)}\frac{N^*(N^*-1)}{\kappa_\delta} = \frac{N^*(N^*-1)}{n(n-1)}\frac{N^*+3\kappa_\Delta-2\kappa_\delta}{3\kappa_\delta},$$

and thus

$$\mu_{\overline{c}}(H) \leq \frac{N^*(N^*-1)}{n(n-1)}\frac{n+2\kappa_\Delta-\kappa_\delta}{3\kappa_\delta} < \frac{N^*(N^*-1)}{n(n-1)}\frac{n+2\kappa_\Delta}{3\kappa_\delta}. \tag{5.3.60}$$

Combining the inequalities (5.3.47), (5.3.53), (5.3.54) and (5.3.60) yields

$$\begin{aligned} \mu(T) &\leq \mu_c(T) + 10 \\ &\leq \mu_{\overline{c}}(L) + 11 \\ &\leq 6\mu_{\overline{c}}(H) + 13 \\ &\leq \frac{2N^*(N^*-1)}{n(n-1)}\frac{n+2\kappa_\Delta}{\kappa_\delta} + 13, \end{aligned}$$

as desired. Hence, Theorem 5.3.7 holds. □

The following theorem shows that the bound in Theorem 5.3.7 is sharp apart from an additive constant. To prove the theorem, we make use of the graph constructed in Example 3.2.6 and briefly recall its construction here.

Theorem 5.3.8. *Let $\delta \geq 3$ be an integer such that $\delta - 1$ is a prime power. Then for*

$n, \Delta \in \mathbb{N}$, there exists infinitely many values of Δ for which there exists infinitely many values of n such that there exists a graph of girth at least 6, $F^*_{\ell,\delta,\Delta}$ with n vertices, minimum degree δ and maximum degree Δ whose average distance satisfies

$$\mu(F^*_{\ell,\delta,\Delta}) \geq \frac{2(n-\kappa_\Delta)(n-\kappa_\Delta-1)}{n(n-1)}\frac{n+2\kappa_\Delta}{\kappa_\delta} + O(\sqrt{\Delta}), \tag{5.3.61}$$

where $\kappa_\delta := 2\delta^2 - 2\delta + 2$ and $\kappa_\Delta := \Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2}$.

Proof. Let $q = \delta - 1$ be a prime power and $m \in \mathbb{N}$ with $m \geq 4$. Recall from Section 3.2.2 (Theorem 3.2.10), there exists a connected graph of girth at least 6, $F_{q,m}$ whose order n satisfies

$$\Delta\delta + (\delta-1)\sqrt{\Delta(\delta-2)} + \frac{3}{2} \leq n(F_{q,m}) \leq 2 + \delta\Delta + (\delta+1)\sqrt{\Delta(\delta-2)}.$$

$F_{q,m}$ has minimum degree δ and maximum degree $\Delta = \frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q^2-q)} - \frac{1}{q}$.

Let $u_0 \in F_{q,m}$ be a vertex of maximum degree, viz z, and let v_0 be any vertex of $F_{q,m}$ that is not of degree Δ. Without loss of generality, we let $v_0 \in Y$ as defined there.

Let $\ell \in \mathbb{N}$ with $\ell \geq 2$ and ℓ sufficiently large. Now consider the graph H^*_{q+1} constructed in Example 2.3.5. Recall that if $e = uv$ is an edge of H^*_{q+1}, then we have by Claim 2.3.6 that $d_{H^*_{q+1}-uv}(u,v) \geq 5$. Let F_0 be a copy of $F_{q,m}$ and let $F_1, \dots, F_\ell$ be disjoint isomorphic copies of H_e. From the disjoint copies of $F_1, F_2, \dots, F_\ell$, we obtain a graph of girth at least 6, $F^*_{\ell,\delta,\Delta}$ by adding the edges $v_i u_{i+1}$ for every $(0 \leq i \leq \ell-1)$. For $1 \leq i \leq \ell$, vertices u_i and v_i of F_i correspond to vertices u and v, respectively, of H_e. The sketch of the graph $F^*_{\ell,\delta,\Delta}$ in Figure 3.2 is reproduced in Figure 5.4 below.

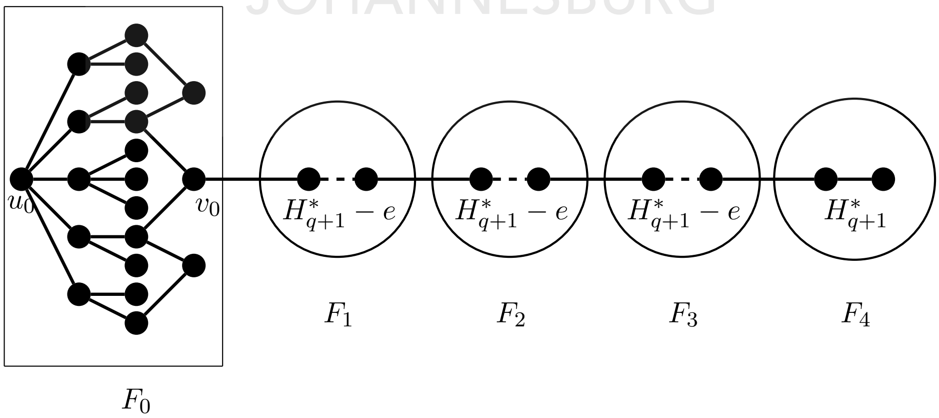

Figure 5.4: The graph $F^*_{4,\delta,\Delta}$.

It is easy to verify that $F^*_{\ell,\delta,\Delta}$ has minimum degree δ and maximum degree Δ. By Claim 2.3.6, graph F_i (for $1 \leq i \leq \ell$) has $\kappa_\delta := 2(q^2+q+1)$ vertices and so if we denote the order of F_0 by ω_Δ, then we have that the order of the graph $F^*_{\ell,\delta,\Delta}$ is $n = |V(F^*_{\ell,\delta,\Delta})| = \omega_\Delta + \ell\kappa_\delta$, and by Section 3.2.2 (Theorem 3.2.10),

$$\kappa_\Delta \leq \omega_\Delta \leq \kappa_\Delta + 2\sqrt{\Delta(\delta-2)} + \frac{1}{2}. \tag{5.3.62}$$

We now bound the average distance of $F^*_{\ell,\delta,\Delta}$ from below.

For $i \in \{1, 2, \dots, \ell\}$, let $U_i = V(F_i)$ and let $U' = V(F_0)$. So $|U_i| = \kappa_\delta$ and $|U'| = \omega_\Delta$. Now the distance between two vertices $v \in U_i$, if $w \in U_j$ where $i < j$ is at least $6(j-i)-5$. Counting only the distances between pairs of vertices (v, w) for $v \in U'$ and $w \in U_i$ and the pairs in U_i and U_j, we obtain that

$$\begin{aligned} W(F^*_{\ell,\delta,\Delta}) &> \sum_{1\leq i<j\leq \ell}\sum_{v\in U_i}\sum_{w\in U_j} d(v,w) + \sum_{v\in U'}\sum_{j=1}^{\ell}\sum_{w\in U_j} d(v,w) \\ &= \sum_{1\leq i<j\leq \ell} |U_i|\cdot|U_j|\cdot(6(j-i)-5) \ + \ \sum_{j=1}^{\ell}|U'|\cdot|U_j|\cdot(6j-5) \\ &= (\kappa_\delta)^2 \sum_{1\leq i<j\leq \ell}(6(j-i)-5) \ + \ \omega_\Delta\kappa_\delta\sum_{j=1}^{\ell}(6j-5). \end{aligned}$$

Simple calculation shows that $\sum_{1\leq i<j\leq \ell}(6(j-i)-5) = \frac{1}{2}(\ell^2-\ell)(2\ell-3)$ and $\sum_{j=1}^{\ell}(6j-5) = 3\ell^2 - 2\ell$. Substituting these values and using the fact that $n = \omega_\Delta + \ell\kappa_\delta$, we obtain

$$\begin{aligned} W(F^*_{\ell,\delta,\Delta}) &> \frac{1}{2}(\kappa_\delta)^2(2\ell^3 - 5\ell^2 + 3\ell) + \omega_\Delta\kappa_\delta(3\ell^2 - 2\ell) \\ &= \frac{1}{\kappa_\delta}(n-\omega_\Delta)^2(n+2\omega_\Delta) - \frac{1}{2}(n-\omega_\Delta)(5n-\omega_\Delta-3\kappa_\delta) \end{aligned}$$

Clearly, $(n-\omega_\Delta)^2 > (n-\omega_\Delta)(n-\omega_\Delta-1)$, $n-\omega_\Delta \leq n-1$ and, since $5n-\omega_\Delta-3\kappa_\delta \leq 5n$, we have that $\frac{1}{2}(n-\omega_\Delta)(5n-\omega_\Delta-3\kappa_\delta) \leq 5\binom{n}{2}$. Now, division by $\binom{n}{2}$ yields

$$\mu(F^*_{\ell,\delta,\Delta}) > \frac{n+2\omega_\Delta}{\kappa_\delta}\frac{2(n-\omega_\Delta)(n-\omega_\Delta-1)}{n(n-1)} - 5.$$

Now let $\varepsilon = \omega_\Delta - \kappa_\Delta$. Replacing ω_Δ by $\kappa_\Delta + \varepsilon$ in the above lower bound, we obtain

$$\begin{aligned} \mu(F^*_{\ell,\delta,\Delta}) &> \frac{n+2\kappa_\Delta+2\varepsilon}{\kappa_\delta}\frac{2(n-\kappa_\Delta-\varepsilon)(n-\kappa_\Delta-\varepsilon-1)}{n(n-1)} - 5 \\ &= \frac{2}{\kappa_\delta n(n-1)}\Big[(n+2\kappa_\Delta)(n-\kappa_\Delta)(n-\kappa_\Delta-1) + \epsilon[6\kappa_\Delta^2 - 6n\kappa_\Delta + 4\kappa_\Delta - n] \\ &\quad +\epsilon^2[-3n+6\kappa_\Delta+2] + 2\epsilon^3\Big] \end{aligned}$$

Since $(\kappa_\Delta^2 - 6n\kappa_\Delta + 4\kappa_\Delta - n \geq -7n\kappa_\Delta \geq -7n(n-1)$ and $-3n+6\kappa_\Delta+2 \geq -3n \geq -n(n-1)$, and also since $0 \leq \varepsilon \leq 2\sqrt{\Delta(\delta-2)} + \frac{1}{2} = O(\sqrt{\Delta})$ by (5.3.62), we have for constant δ and large n and Δ,

$$\mu(F^*_{\ell,\delta,\Delta}) > \frac{2(n-\kappa_\Delta)(n-\kappa_\Delta-1)}{n(n-1)} \frac{n+2\kappa_\Delta}{\kappa_\delta} + O(\sqrt{\Delta}),$$

as desired in Theorem 5.3.7. □

The graph $F^*_{\ell,\delta,\Delta}$ constructed above demonstrates that for $\delta-1$ a prime power, the bound on the average distance in Theorem 5.3.7 is sharp apart from a term$O(\sqrt{\Delta})$, and the second term is of the right order of magnitude.

The next theorem shows that slightly weaker bounds hold for graphs containing no cycles of length 4 or 5-cycle as subgraphs. We omit the proof, since it is very similar to the proof of Theorem 5.3.7 except for little modification. We do not know if this bound is sharp

Theorem 5.3.9. *Let G be a connected (C_4, C_5)-free graph with n vertices, minimum degree $\delta \geq 2$, maximum degree Δ. Then,*

i) G has a spanning tree T with

$$\mu(T) \leq \frac{2(n-\tau_\Delta+\tau_\delta)(n-\tau_\Delta+\tau_\delta-1)}{n(n-1)} \frac{n+2\tau_\Delta}{\tau_\delta} + 13. \tag{5.3.63}$$

ii)

$$\mu(G) \leq \frac{2(n-\tau_\Delta+\tau_\delta)(n-\tau_\Delta+\tau_\delta-1)}{n(n-1)} \frac{n+2\tau_\Delta}{\tau_\delta} + 13. \tag{5.3.64}$$

where $\tau_\delta := 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta$, $\tau_\Delta := \Delta(\delta-1) + (\delta-2)\sqrt{\Delta(\delta-3)} + \frac{3}{2}$ and

$$\varepsilon_\delta = \begin{cases} 1 & \text{if } \delta \text{ is odd,} \\ 0 & \text{if } \delta \text{ is even.} \end{cases}$$

Chapter 6

Bounds on the (Edge-)Fault-Diameter of Graphs of Girth at least 6 and (C_4, C_5)-free graphs.

6.1 Introduction

In Chapter 2, we gave bounds on the diameter of graphs of girth at 6 and connected (C_4, C_5)-free graphs of given order and minimum degree. Since graphs are not always static, for example graphs modelling communication networks can change if communication links or relays fail, it is also desirable to have information on the diameter increase or decrease in some of the vertices or edges of the network fail. Hence, we consider the concept of the k-fault-diameter and k-edge-fault-diameter first introduced in [68] and give upper bounds on the k-fault-diameter and k-edge-fault-diameter of graphs at least girth 6 and (C_4, C_5)-free graphs in terms of the order of the graph n. Our results show that the bounds in [28] can be improved further for graphs of girth at least 6. The techniques used in [28] were very useful in obtaining these bounds. We further present a construction to show that the bound is best possible in a sense specified later.

6.2 Preliminary Results

Herein, we recall some recent results by [28] on the k-fault-diameter and k-edge-fault-diameter of connected graphs, triangle free graphs and connected C_4-free graphs of given order. From now onwards, all graphs are considered to be $(k+1)$-connected since the k-fault-diameter of a graph that is not $(k+1)$-connected is infinite. We will make use of some of the definitions in Chapter 1 and some of the notation used in [28].

Definition 6.2.1. *Let G be a $(k+1)$-connected-graph or $(k+1)$-edge-connected-graph where $k \in \mathbb{N}$. The k-fault diameter $D_k(G)$ and the k-edge-fault-diameter $D'_k(G)$ of G is the largest diameter of the subgraphs obtained from G by removing at most k vertices and edges, respectively.*

In the proofs of the subsequent theorems and bounds, we make use of the following notation. G is a $(k+1)$-(edge-)connected graph, d denotes the k-(edge-)fault-diameter of G. The set $S \subseteq V(G)$ is a set of k vertices of G such that $\mathrm{diam}(G-S) = d$. If we consider the edge-fault-diameter, then S is a set of edges. We denote $G-S$ by H. Then H contains

v_0, v_d with $d_H(v_0, v_d) = d$. Let $P : v_0v_1 \dots v_d$ be a (v_0, v_d)-path of length d in H. For $i \in \mathbb{Z}$, we define the set N_i to contain the vertices whose distance to v_0 (in H) equals i, and we let $n_i = |N_i|$. So $N_i \neq \emptyset$ only for $i = 0, 1, \dots, d$.

The following upper bound on $D_k(G)$ follows from the fact that removing a set of k vertices from a $(k+1)$-connected graph of order n yields a connected graph of order $n-k$.

Proposition 6.2.2. *[28] Let G be a $(k+1)$-connected graph of order n. Then*

$$D_k(G) \leq n - k - 1.$$

Equality holds, for example, if $G = K_k + P_{n-k}$.

We present the following result and its proof since it is closely related to the original results on the fault-diameter presented in the next section, in particular the bound on the fault-diameter of graphs of girth 6 in Theorem 6.3.2. The proof closely follows the proof given in [28].

Theorem 6.2.3. *[28] Let G be a $(k+1)$-connected C_4-free graph of order n, where $k \geq 2$. Then*

$$D_k(G) \leq \frac{5}{k^2 - k + 1} n - \frac{5k^2 - 5k + 8}{2}. \tag{6.2.1}$$

Proof. Since G is $(k+1)$-connected, we have that $\delta(G) \geq k+1$. For $i \in \{0, 1, 2, \dots, d\}$ and $v_i \in N_i$, we have that that $N_H^2[v_i] \subseteq N_{i-2} \cup N_{i-1} \cup N_i \cup N_{i+1} \cup N_{i+2}$. Hence, we now consider the set $N_H^2[v_{5i}]$ for $i = 0, 1, 2, \dots, \lfloor \frac{d}{5} \rfloor$ and show that $N_H^2[v_{5i}]$ are disjoint. Suppose there exists $i \neq j$ with $N_H^2[v_{5i}] \cap N_H^2[v_{5j}] \neq \emptyset$, then $d_H(v_{5i}, v_{5j}) \leq 4$ and replacing the (v_{5i}, v_{5j})-section of P with a shortest (v_{5i}, v_{5j})-path would yield a shorter (v_0, v_d)-path than P, a contradiction.

We now bound the number of vertices in $N_H^2[v_{5i}]$.

Since $H = G - S$, it is possible that for $i = 0, 1, \dots, \lfloor \frac{d}{5} \rfloor$, $N_H^2[v_{5i}]$ has a neighbour in S. If it happens that a vertex $v \in N_H[v_{5i}]$ has only one common neighbour in S, then, $\deg_H(v)$ drops by 1 and this won't significantly affect the bound in $N_H^2[v_{5i}]$. On the other hand, if $v \in N_H[v_{5i}]$ has more than one neighbour in S, say ℓ neighbours, then $\deg_H(v)$ drops by ℓ and if ℓ is almost close to $\deg(v)$, then this would adversely affect the bound in $N_H^2[v_{5i}]$. Thus, we partition the set $\{0, 1, 2, \dots, \lfloor \frac{d}{5} \rfloor\}$ into two disjoint sets I_1 and I_2 such that for each $i \in \{0, 1, 2, \dots, \lfloor \frac{d}{5} \rfloor\}$, we have that $i \in I_1$ if $|N_G(v) \cap S| \leq 1$ for each $v \in N_H[v_{5i}]$ and $i \in I_2$ if $|N_G(v) \cap S| \geq 2$ for some $v \in N_H[v_{5i}]$.

CLAIM 1: $|N_H^2[v_{5i}]| \geq k^2 - k + 1$ if $i \in I_1$.

Fix $i \in I_1$. For all $v \in N_H[v_{5i}]$, $\deg_H(v) = \deg_G(v) - |N_G(v) \cap S| \geq (k+1) - 1 = k$. Since H is C_4-free, any two vertices in $N_H(v_{5i})$ have no common neighbour other than v_{5i}, and each vertex in $N_H(v_{5i})$ has at most one neighbour in $N_H(v_{5i})$. It follows that,

$$\begin{aligned}
|N_H^2[v_{5i}]| &\geq 1 + \deg_H(v_{5i}) + \sum_{x \in N_H(v_{5i})} \big[\deg_H(x) - 2\big] \\
&\geq 1 + k + \sum_{j=1}^{\deg_H(v_{5i})} \big[\deg_H(x_j) - 2\big] \\
&\geq 1 + k + k(k-2) \\
&= k^2 - k + 1, \qquad (6.2.2)
\end{aligned}$$

as desired in Claim 1.

CLAIM 2: $|I_2| \leq \binom{k}{2}$

Recall that for $i \in \{0, 1, 2, \ldots \lfloor \frac{d}{5} \rfloor\}$, $i \in I_2$ if $|N_G(v) \cap S| \geq 2$ for some $v \in N_H[v_{5i}]$. If there are too many i for which $N_H[v_{5i}]$ contains a vertex with two or more neighbours in S, then G contains two vertices that share two neighbours in S, a contradiction to G being C_4-free. Thus we want to show that there are not many such sets $N_H[v_{5i}]$ containing a vertex with more than 2 neighbours in S.

For each set $N_H^2[v_{5i}]$, $i \in I_2$, choose a vertex w_i with $|N_G(w_i) \cap S| \geq 2$ and a subset $S_i \subset N_G(w_i) \cap S$ of order two. Observe that the sets S_i, $i \in I_2$ are distinct, since otherwise two vertices of H share two common neighbours in S and G contains a C_4, a contradiction. Hence $|I_2| \leq \binom{k}{2}$, which is Claim 2.

CLAIM 3: $n \geq \left(\frac{d-4}{5} - \binom{k}{2}\right)(k^2 - k + 1)$.

It follows from Claim 2 that

$$|I_1| \geq \Big\lfloor \frac{d}{5} \Big\rfloor - \binom{k}{2} \geq \frac{d-4}{5} - \binom{k}{2}.$$

Clearly,

$$\begin{aligned}
n &\geq \sum_{i=0}^{\lfloor \frac{d}{5} \rfloor} |N_H^2[v_{5i}]| \\
&\geq \sum_{i \in I_1} |N_H^2[v_{5i}]| \\
&\geq \left[\frac{d-4}{5} - \binom{k}{2}\right](k^2 - k + 1).
\end{aligned}$$

Solving for d, we have that

$$\begin{aligned} d &\leq 5\Big(\frac{n}{k^2-k+1}\Big) + 5\binom{k}{2} + 4 \\ &= \frac{5n}{k^2-k+1} + 5\Big(\frac{k^2-k}{2}\Big) + 4 \\ &= \frac{5n}{k^2-k+1} + \frac{5k^2-5k+8}{2}, \end{aligned}$$

which yields the desired bound on d as stated in inequality (6.2.1). □

The subsequent theorem by Dankelmann [28] shows that the order of magnitude of the bound in Theorem 6.2.3 is close to being optimal for infinitely many values of k. The construction is based on a modification of the graph H_q described in Example 2.2.3, first constructed by Erdös and Rényi [52] and independently Brown [12].

Recall from Example 2.2.3 that two vertices x and y of H_q are said to be adjacent if x and y, as 1-dimensional subspaces of $GF(q)^3$, are orthogonal. Furthermore by Claim 2.2.5, we have that H_q is C_4-free and has q^2+q+1 vertices. The degree of each vertex of H_q is q if the vertex is self-orthogonal and $q+1$ otherwise. Subsequently, we denote by V_{q+1} and V_q the set of vertices of H_q of degree $q+1$ and q, respectively.

The following properties of V_q and V_{q+1} in H_q which follow from Claim 2.2.4 (see subclaims a-d) will be useful in the proofs of subsequent lemmas and theorems.

(i) the vertices in V_q are pairwise non-adjacent,

(ii) if $u \in V_q$ and $v \in V_{q+1}$ are adjacent, then $N(u) \cap N(v) = \emptyset$,

(iii) if $u \in V_q$ and $v \in V_{q+1}$ are non-adjacent, then $|N(u) \cap N(v)| = 1$,

(iv) if $u, v \in V_q$ or $u, v \in V_{q+1}$ then $|N(u) \cap N(v)| = 1$.

Lemma 6.2.4. *[28] Let q be a prime power and let $GF(q)^3$ be the 3-dimensional vector space over the finite field $GF(q)$ of order q. Let H_q be the graph whose vertices are the 1-dimensional subspaces of $GF(q)^3$, where two vertices are adjacent if, as subspaces, they are orthogonal.*

$$\kappa(H_q) \geq q. \tag{6.2.3}$$

Lemma 6.2.5. *[28] Choose $z \in V_q$ and two neighbours u and v.*

Let $N(u) = \{z, u_1, u_2, \ldots, u_{q-1}\}$ and $N(v) = \{z, v_1, v_2, \ldots, v_{q-1}\}$. Then there exists a perfect matching M between $N(u) - \{z\}$ and $N(v) - \{z\}$. Then,

i) $V(M) \subseteq V_{q+1}$.

ii) M is an induced matching of H_q

Lemma 6.2.6. *[28] Let H_q and M be as defined above. If $H'_q = H_q - M$, then*

$$\kappa(H'_q) = q. \tag{6.2.4}$$

Theorem 6.2.7. *[28] Let $k \in \mathbb{N}$ be such that $k+1$ is a prime power. Then for infinitely many values of n there exists a $(k+1)$-connected, C_4-free graph G_n of order n with*

$$D_k(G_n) \geq \frac{5n}{k^2+3k+2} - 2.$$

Proof. Let $q = k+1$, so q is a prime power. Let the graph H_q be as defined above. By Lemma 6.2.4, $\kappa(H_q) \geq q$. Let M and H'_q be as defined above. By Lemma 6.2.6, $\kappa(H'_q) = q$.

For $i = 1, 2, \ldots, t$ let F_i be a copy of H'_q. We now let $u^i, u^i_1, u^i_2, \ldots, u^i_{q-2}, v^i$, and z^i be the vertices of F_i corresponding to $u, u_1, u_2, \ldots, u_{q-2}, v$ and z in H'_q. For $i = 1, 2, \ldots, t-1$, let $v^i u^{i+1}$ be an edge of F_i. For $j = 1, 2, \ldots, q-2$ and $i = 1, 2, \ldots, t$, let $w_1, w_2, \ldots, w_{q-2}$ be new vertices of F_i such $u^i_j w_j$ is an edge of F_i. Furthermore we identify the vertices $z^1, z^2, \ldots, z^t$ to a single vertex z_0.

Let G_n be the graph with vertex set $V(G_n) = V(F_1) \cup V(F_2) \cup \ldots V(F_t)$, and edge set $E(G_n) = E(F_1) \cup E(F_2) \cup \ldots E(F_t) \cup \{v^1u^2, v^2u^3, \ldots, v^{t-1}u^t\} \cup \{u^1_1w_1, u^2_1w_1, \ldots u^t_1w_1\}$ $\cup\{u^1_2w_2, u^2_2w_2, \ldots u^t_2w_2\} \ldots \cup\{u^1_{q-2}w_{q-2}, u^2_{q-2}w_{q-2}, \ldots u^t_{q-2}w_{q-2}\}$. A sketch of the graph G_n for $t = 3$ is shown in Figure 6.1. The order of G_n is $t(q^2+q+1) - (t-1) + q - 2 = t(k^2+3k+2) + k$.

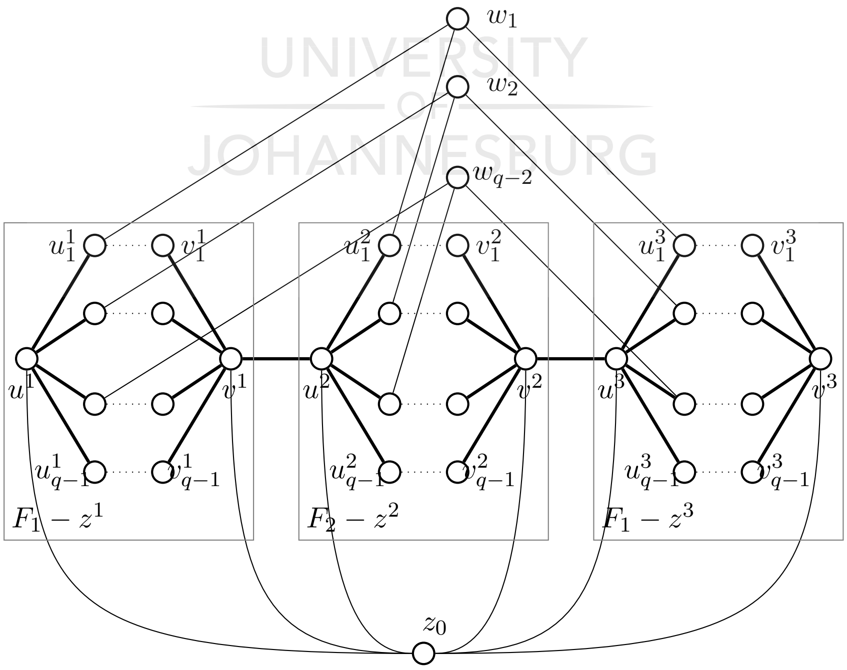

Figure 6.1: The graph G_n for $t = 3$.

subclaim: G_n is a C_4-free, $(k+1)$-connected graph.

We first show that G_n is q-connected. Let S be a set of $q-1$ vertices of G_n. Then each subgraph $F_i - (S \cap V(F_i))$ is connected since F_i is q-connected. Since each subgraph F_i is connected to other subgraphs F_j through at least q vertices, viz $z_0, w_1, w_2, \ldots, w_{q-2}$ and at least one of u^i, v^i, $G_n - S$ is connected and G_n is q-connected. Clearly H'_q is C_4-free since H_q is C_4-free. Moreover, since G_n is obtained from the disjoint union of F_i's with each F_i a copy of H'_q, we have that G_n is also C_4-free.

To bound the k-fault-diameter from below, we choose the set S as $\{z_0, w_1, w_2, \ldots, w_{q-2}\}$. Since $d(u^i, v^i) = 4$ in $F_i - z^i$, we have

$$D_{q-1}(G_n) \geq d_{G_n - S}(u^1, v^t) = 5t - 1 = \frac{5}{k^2+3k+2}n - \frac{k^2+8k+2}{k^2+3k+2} \geq \frac{5}{k^2+3k+2}n - 2.$$

Hence there exist infinitely many values of n for which there is a C_4-free graph with the desired properties. □

6.3 Main Results

In this section we present our original results on the fault-diameter and the edge-fault-diameter.

6.3.1 Bounds on the Fault-Diameter of $(k+1)$-connected Graphs of Girth at least 6

We begin by presenting our result on the fault-diameter of $(k+1)$-connected graphs of girth at least 6. The technique used throughout this section is a slight modification of that used in the previous section. We omit the proof of Lemma 6.3.1 since it is identical to that of Lemma 2.3.2.

Lemma 6.3.1. *Let G be a connected graph of girth at least 6. If u and v are adjacent vertices of G such that all vertices in $N_G(u) \cup N_G(v)$ has degree at least δ, then*

$$|N_{\leq 2}(u) \cup N_{\leq 2}(v)| \geq 2(\delta^2 - \delta + 1).$$

Theorem 6.3.2. *Let $k \geq 2 \in \mathbb{N}$ and let G be a $(k+1)$-connected graph of girth at least 6 graph of order n. Then*

$$D_k(G) \leq \frac{3n}{k^2-k+1} + 3(k^2-k+2). \tag{6.3.1}$$

Proof. Let P, d, S, H, N_i, $N^2_G(v)$, v_i be as defined in the proof of Theorem 6.2.3. For

$i \in \{0, 1, 2, \dots, d-1\}$, we have that,

$$N_H^2[v_i] \cup N_H^2[v_{i+1}] \subseteq N_{i-2} \cup N_{i-1} \cup N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3},$$

and so for $i = 0, 1, 2, \dots, \lfloor \frac{d-1}{6} \rfloor$, we consider the sets $N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]$ which are clearly disjoint. Clearly, the sets $\left(N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]\right)$ and $\left(N_H^2[v_{6j}] \cup N_H^2[v_{6j+1}]\right)$ are disjoint for all distinct $i, j \in \{0, 1, 2 \dots \lfloor \frac{d-1}{6} \rfloor\}$. Indeed if there exists $i \neq j$ with $\left(N_H^2[v_{6i}] \cup N_H^2[v_{6j+1}]\right) \cap \left(N_H^2[v_{6j}] \cup N_H^2[v_{6j+1}]\right) \neq \emptyset$, then $d_H(v_{6i}, v_{6j}) \leq 5$ or $d_H(v_{6i+1}, v_{6j+1}) \leq 5$. Replacing the (v_{6i}, v_{6j})-section of P with a shortest (v_{6i}, v_{6j})-path or replacing the (v_{6i+1}, v_{6j+1})-section of P with a shortest (v_{6i+1}, v_{6j+1})-path or replacing the (v_{6i}, v_{6j+1})-section of P with a shortest (v_{6i}, v_{6j+1})-path or replacing the (v_{6i+1}, v_{6j})-section of P with a shortest (v_{6i+1}, v_{6j})-path would yield a shorter (v_0, v_d)-path than P, a contradiction.

Following a similar argument as in the proof of Theorem 6.2.3, we partition the set $\{0, 1, 2, \dots, \lfloor \frac{d-1}{6} \rfloor\}$ into two disjoint sets I_1' and I_2' such that for each $i \in \{0, 1, 2, \dots, \lfloor \frac{d}{6} \rfloor\}$, we have that $i \in I_1'$ if $|N_G(v) \cap S| \leq 1$ for each $v \in N_H[v_{6i}] \cup N_H[v_{6i+1}]$ and $i \in I_2'$ if $|N_G(v) \cap S| \geq 2$ for some $v \in N_H[v_{6i}] \cup N_H[v_{6i+1}]$. We now bound the number of vertices in $\left[N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]\right]$ for $i \in I_1'$.

CLAIM 1: $|N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]| \geq 2(k^2 - k + 1)$ if $i \in I_1'$.

Fix $i \in I_1'$. Since G is $(k+1)$-connected, $\delta(G) \geq k+1$. Moreover for all $v \in N_H[v_{6i}] \cup N_H[v_{6i+1}]$, $\deg_H(v) = \deg_H(v) - |N_G(v) \cap S| \geq (k+1) - 1 = k$ and since H has girth at least 6, we have by Lemma 6.3.1 that

$$|N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]| = 2k^2 - 2k + 2,$$

as desired in Claim 1.

Following a similar argument as in the proof of Claim 2 (Theorem 6.2.3), $|I_2'| \leq \binom{k}{2}$. It follows immediately that

$$|I_1'| \geq \left\lfloor \frac{d-1}{6} \right\rfloor - \binom{k}{2} \geq \frac{d-6}{6} - \binom{k}{2}. \tag{6.3.2}$$

CLAIM 2: $n \geq \left(\frac{d-6}{6} - \binom{k}{2}\right)(2k^2 - 2k + 2)$.

Applying Claim 1 and (6.3.2) we obtain,

$$\begin{aligned} n &\geq \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} |N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]| \\ &\geq \sum_{i \in I_1'} |N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]| \\ &\geq \left[\frac{d-6}{6} - \binom{k}{2}\right](2k^2 - 2k + 2). \end{aligned}$$

Solving for d, we have that

$$\begin{aligned} d &\leq 6\Big(\frac{n}{2(k^2-k+1)}\Big)+6\binom{k}{2}+6 \\ &= \frac{3n}{k^2-k+1}+6\Big(\frac{k^2-k}{2}\Big)+6 \\ &= \frac{3n}{k^2-k+1}+3k^2-3k+6, \end{aligned}$$

which yields the desired bound on d as stated in inequality (6.3.1). □

Next, we show in the following theorem that the bound in Theorem 6.3.2 is best possible for infinitely many values of k. The construction makes use of the graph H^* described in Example 2.3.5 however with some modifications. We will make use of some of the notation and definitions from the Linear Algebra Section used in Example 2.3.5.

Let H^*_{q+1} be the graph described in Example 2.3.5, whose vertices correspond to the one-dimensional subspaces and two-dimensional subspaces of the vector space $GF(q)^3$ with two vertices x and y being adjacent if x is a subspace of y or y is a subspace of x. Denote by U_{q+1} and W_{q+1} the set of 1-dimensional and 2-dimensional subspaces, respectively. By Claim 2.3.6, we have the following properties of H^*_{q+1}.

a) H^*_{q+1} is bipartite with partite sets U_{q+1} and W_{q+1}, contains no 4-cycle.

b) U_{q+1} and W_{q+1} have q^2+q+1 vertices each, and so H^*_{q+1} has $2(q^2+q+1)$ vertices.

c) Every vertex of H^*_{q+1} has degree $q+1$.

d) If $x, y \in U_{q+1}$ or $x, y \in W_{q+1}$, then $|N(x) \cap N(y)| = 1$.

Claim 6.3.3. *H^*_{q+1} is $(q+1)$-connected.*

Proof. For $u, w \in V(H^*_{q+1})$, let $\kappa(u, w)$ be the number of internally disjoint (u, w)-path in H^*_{q+1}. It suffices to show that $\kappa(u, w) \geq q+1$ for all pairs u, w of non-adjacent vertices of H^*_{q+1}. For any $u, w \in V(H^*_{q+1})$ that are non-adjacent, we have the following cases to consider.

CASE A: u, w belong to the same partite set

We assume that $u, w \in U_{q+1}$; the case $u, w \in W_{q+1}$ is analogous. By property (d) above u and w have a common neighbour z in W_{q+1}. Let

$$N[u] - \{z\} = \{u_1, u_2, \ldots, u_q\} \text{ and } N[w] - \{z\} = \{w_1, w_2, \ldots, w_q\}.$$

Observe that for $i \in \{1, 2, \ldots, q\}$. the vertices u_i and w_i are elements of W_{q+1} and by the same property (d), any two elements of W_{q+1} have a common neighbour x_i in U_{q+1}.

Let P_i be the path $uu_ix_iw_iw$ and P_0 be the path uzw. We claim the following

i) $x_i \neq w, u$ for all $i \in \{1, 2, \dots q\}$. Suppose that $x_i = w$ for some $i \in \{1, 2, \dots q\}$, then uu_ix_iz or $x_iu_iw_iz$ forms a 4-cycle contradicting the hypothesis of the theorem. Thus, $x_i \neq w, u$ for all $i \in \{1, 2, \dots q\}$.

ii) $x_i \neq x_j$ for $i \neq j$. Suppose to the contrary that $x_i = x_j$, then $uu_ix_iu_j$ now forms a 4-cycle, a contradiction to the fact that H^*_{q+1} is C_4-free. Hence $x_i \neq x_j$ for $i \neq j$

Therefore P_i for $i \in \{1, 2, \dots, q\}$ are internally disjoint. Hence, $P_0, P_1, P_2, \dots, P_q$ are internally disjoint (u, w)-paths, and so we have that $\kappa(u, w) = q + 1$.

CASE B: u and w belong to different partite sets.

Let $u \in U_{q+1}$ and let $w \in W_{q+1}$. Since H^*_{q+1} is bipartite by property (a), u and w have no common neighbour since they belong to different partite sets. Let

$$N[u] = \{u_0, u_1, u_2, \dots, u_q\} \text{ and } N[w] = \{w_0, w_1, w_2, \dots, w_q\}.$$

Observe that for $i \in \{0, 1, 2, \dots, q\}$, $u_i \in W_{q+1}$ and $w_i \in U_{q+1}$. Furthermore, $w \neq u_i$ and $u \neq w_i$ for some $i \in 0, 1, 2, \dots, q$ else u and w are adjacent, a contradiction to the assumption that u and w are non adjacent vertices of H_q.

Let $U'_{q+1} = \{u\} \cup N[w]$ and $W'_{q+1} = \{w\} \cup N[u]$. Thus, $U'_{q+1} \subset U_{q+1}$ and $W'_{q+1} \subset W_{q+1}$. By property (d), any two vertices, say $u, v \in U'_{q+1}$ have a common neighbour. Clearly, $N(w_i) \cap N(w_j) = \{u\}$ for $i \neq j$. For $i \in \{0, 1, 2, \dots, q\}$, let v_i be the unique common neighbour of u and w_i. Clearly, the v_i are distinct, otherwise H^*_{q+1} would contain a 4-cycle. Hence $\{v_0, v_1, \dots, v_q\} = \{u_0, u_1, \dots, u_q\}$. Renumbering the u_i, we may assume that $u_i = v_i$ for all $i \in \{0, 1, \dots, q\}$.

It follows immediately that there is an edge $e_i = u_iw_i$ such that $P_i : u, u_iw_iw$ is a (u, w)-path. For $i \in \{0, 1, 2, \dots, q\}$, P_i are internally disjoint paths. Hence we conclude that for $u \in U_{q+1}$, $w \in W_{q+1}$, there are $(q + 1)$-internally disjoint (u, w)-paths and so $\kappa(u, w) \geq k + 1$. □

Theorem 6.3.4. *Let $k \in \mathbb{N}$ be such that k is a prime power. Then for infinitely many values of n, there exists a $(k + 1)$-connected graph G of girth at least 6 with*

$$D_k(G) \geq \frac{3n}{k^2 + k + 1} - 5,$$

where n is the order of G.

Proof. Let $q = k$, so q is a prime power. Fix two adjacent vertices $u \in U_{q+1}$, $w \in W_{q+1}$ of H^*_{q+1} and let $H'_{q+1} = H^*_{q+1} - uw$. By Claim 6.3.3, $\kappa(H^*_{q+1}) \geq q + 1$ and so we have by Lemma 1.1.31 that $\kappa(H'_{q+1}) \geq q$.

Let $n \in \mathbb{N}$ be such that $n = 2t(k^2 + k + 1) + k$ for some $t \in \mathbb{N}$ with $t \geq 3$. Let F_1 and let F_t be disjoint copies of H^*_{q+1} and let $F_2, F_3, \ldots, F_{t-1}$ be disjoint copies of H'_{q+1}. Let u^i and w^i be the vertices of F_i corresponding to the vertices u and w, respectively, of H^*_{q+1} and H'_{q+1}.

For $i \in \{2, \ldots, t-1\}$, let $N(w^i) = \{w^i_1, w^i_2, \ldots, w^i_q\}$ and for $i \in \{1, t\}$, let $N(w^i) = \{w^i_0, w^i_1, \ldots, w^i_q\}$ where $w^i_0 = u^i$. From the disjoint union of $F_1, F_2, \ldots F_t$, we obtain the graph G by adding the edges $w^i u^{i+1}$ for every $(1 \leq i < t)$ and q new vertices $\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_q\}$ joining α_j to w^i_j for $j = 1, 2, \ldots, q$ and for $i = 1, 2, \ldots, t$. A sketch of the graph is shown below in Figure 6.2.

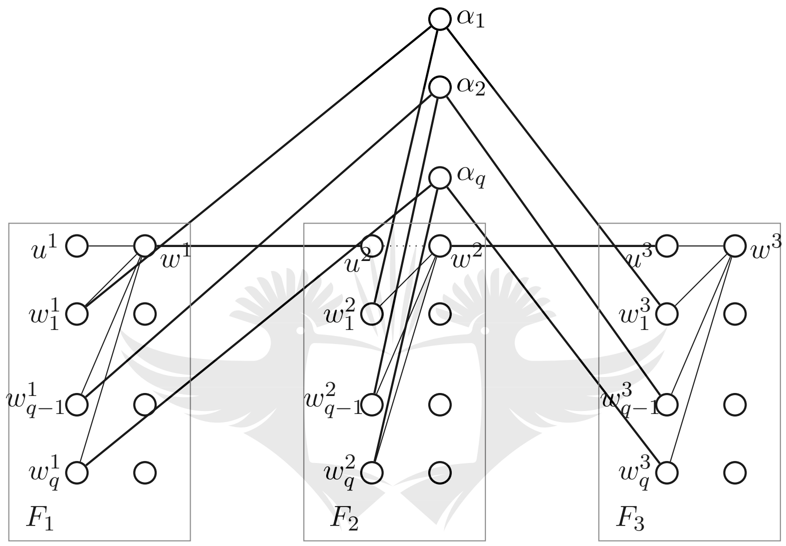

Figure 6.2: The graph G for $t = 3$.

The order of G is n. Let $X_q = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_q\}$. Let U_i and W_i be the partite set corresponding to U_{q+1} and W_{q+1} in F_i. If we denote by $\mathcal{U} = \bigcup_{i=1}^{t} U_i$ and $\mathcal{W} = \bigcup_{i=1}^{t} W_i \cup X_q$, then G is bipartite and contains no 4-cycle.

We now show that G is $(q+1)$-connected.

Suppose to the contrary that $\kappa(G) \leq q$. Then there is a set $S \subseteq V(G)$ with $|S| \leq q$ such that $G - S$ is disconnected.

Claim 1: Let $i \in \{1, 2, \ldots, t\}$. Then the vertices in $V(F_i) - S$ belong to the same component of $G - S$.

It suffices to show that between every two vertices $x, y \in V(F_i)$. There exist $q+1$ internally disjoint paths in G. This is immediate for $i = 1$ or $i = t$ since then F_i is isomorphic to H^*_{q+1} and $\kappa(H^*_{q+1}) = q + 1$, so assume $i \in \{2, 3, \ldots, t-1\}$. Since $F_i + u^i w^i$ is isomorphic to H^*_{q+1}, we have $q + 1$ internally disjoint (x, y)-paths that are contained in $F_i + u^i w^i$. If none of these paths uses the edge $u^i w^i$, then we are done, and if one of these paths uses the edge u^iw^i, then replacing this edge with a path through $u^i, w^{i-1}, w_1^{i-1}, \alpha_1, w^{i+1}$ and then in F_{i+1} to u^{i+1} and w^i. Hence any two vertices in $V(F_i)$ are joined in G by $q+1$ internally disjoint paths, and so they belong to the same component of $G-S$.

Claim 2: If $i \in \{1, 2, \ldots, t-1\}$, then the vertices in $(V(F_i) \cup V(F_{i+1})) - S$ belong to the same component of $G-S$.

It suffices to show that there exist $q+1$ disjoint paths in G from $V(F_i)$ to $V(F_{i+1})$. The $q+1$ paths w^i, α_i, w^{i+1} for $i = 1, 2, \ldots, q$ and the path w^iu^{i+1} are such paths, so Claim 2 follows.

It follows from Claims 1 and 2 that the vertices of $\bigcup_{i=1}^{t}(V(F_i) - S)$ all belong to the same component of $G-S$. Since each vertex in $X_q - S$ has at least one neighbour in $\bigcup_{i=1}^{t}(V(F_i) - S)$, it follows that also all vertices of $X_q - S$ belong to this component. This proves that $G-S$ is connected.

In order to bound the k-fault diameter from below, choose the set S as X_q. By Claim 2.3.6(f) that $d_{H'_{q+1}}(u^i, w^i) \geq 5$. Choosing a vertex x_1 of F_1 with $d(x_1, w^1) \geq 3$, and a vertex x_t, we have that

$$\begin{aligned} D_q(G) &\geq \mathrm{diam}(G-S) \geq d(x_1, x_t) = d(x_1, w^1) + \sum_{i=2}^{t-1} d(u^i, w^i) + d(u^t, x_t) + t - 1 \\ &\geq 3 + 5(t-2) + 3 + (t-1) \\ &= 6t - 5 \\ &= \frac{3n}{k^2+k+1} - \frac{5k^2+8k+5}{k^2+k+1}, \end{aligned}$$

which yields the desired result.

□

As in the previous chapters, we show that a bound slightly weaker than that in Theorem 6.3.2 holds if we relax the condition to (C_4, C_5)-free graphs.

Lemma 6.3.5. *Let G be a connected (C_4, C_5)-free graph. If u and v are adjacent vertices of G such that all vertices in $N_G(u) \cup N_G(v)$ have degree at least δ, then*

$$|N_{\leq 2}(u) \cup N_{\leq 2}(v)| \geq 2\delta^2 - 5\delta + 5 + 2\varepsilon_\delta.$$

where

$$\varepsilon_\delta = \begin{cases} 0 & \text{if } \delta \text{ is even,} \\ 1 & \text{if } \delta \text{ is odd.} \end{cases}$$

The proof is identical to that of Lemma 2.3.9, hence we omit it.

Theorem 6.3.6. *Let $k \geq 2 \in \mathbb{N}$ and let G be a $(k+1)$-connected (C_4, C_5)-free graph of order n. Then*

$$D_k(G) \leq \frac{3n}{\chi_k} + 3(k^2 - k + 2). \tag{6.3.3}$$

where $\chi_k = k^2 - \frac{5}{2}k + \frac{5}{2}$ if $k+1$ is even and $\chi_k = k^2 - \frac{5}{2}k + \frac{7}{2}$ if $k+1$ is odd.

Proof. Let P, d, S, H, N_i, v_i, $N_G^2(v)$ be as defined in the proof of Theorem 6.2.3. By the hypothesis of Theorem 6.3.6, G may contain triangles. Since G is $(k+1)$-connected, we have that $\delta(G) \geq k+1$. For $i \in \{0, 1, 2, \ldots, d-1\}$, we have that,

$$N_H^2[v_i] \cup N_H^2[v_{i+1}] \subseteq N_{i-2} \cup N_{i-1} \cup N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3},$$

and so for $i = 0, 1, 2, \ldots, \lfloor \frac{d-1}{6} \rfloor$, the set $\left(N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]\right) \cap \left(N_H^2[v_{6j}] \cup N_H^2[v_{6j+1}]\right)$ are disjoint for $i \neq j$. We now bound the number of vertices in $\left[N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]\right]$ for $i = 0, 1, 2, \ldots, \lfloor \frac{d-1}{6} \rfloor$.

Following a similar argument as in the proof of Theorem 6.2.3, we partition the set $\{0, 1, 2, \ldots, \lfloor \frac{d-1}{6} \rfloor\}$ into two disjoint sets I_1' and I_2' such that for each $i \in \{0, 1, 2, \ldots, \lfloor \frac{d-1}{6} \rfloor\}$, we have that $i \in I_1'$ if $|N_G(v) \cap S| \leq 1$ for each $v \in N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]$ and $i \in I_2'$ if $|N_G(v) \cap S| \geq 2$ for some $v \in N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]$.

CLAIM 1: $|N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]| \geq 2(k^2 - k + 1)$ if $i \in I_1'$.

Fix $i \in I_1'$. For all $v \in N_H[v_{6i}] \cup N_H^2[v_{6i+1}]$, $\deg_H(v) = \deg_H(v) - |N_G(v) \cap S| \geq (k+1) - 1 = k$. Since H is (C_4, C_5)-free and $i \in I_1'$, we have by Lemma 6.3.5 that

$$|N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]| \geq 2k^2 - 5k + 5 + 2\varepsilon_{\delta_G},$$

as desired in Claim 1.

Following a similar argument as in the proof of Claim 2 (Theorem 6.2.3), $|I_2'| \leq \binom{k}{2}$. It follows immediately that

$$|I_1'| \geq \left\lfloor \frac{d-1}{6} \right\rfloor - \binom{k}{2} \geq \frac{d-6}{6} - \binom{k}{2}. \tag{6.3.4}$$

Applying Claim 1 above and (6.3.4), we obtain

$$\begin{aligned} n &\geq \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} |N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]| \\ &\geq \sum_{i \in I_1'} |N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]| \\ &\geq \left[\frac{d-6}{6} - \binom{k}{2}\right](2k^2 - 5k + 5 + 2\varepsilon_{\delta_G}). \end{aligned}$$

Solving for d, we have that

$$\begin{aligned} d &\leq 6\Big(\frac{n}{2k^2-5k+5+2\varepsilon_{\delta_G}}\Big)+6\binom{k}{2}+6 \\ &= \frac{3n}{k^2-\frac{5}{2}(k-1)+\varepsilon_{\delta_G}}+6\Big(\frac{k^2-k}{2}\Big)+6 \\ &= \frac{3n}{k^2-\frac{5}{2}k+\frac{5}{2}+\varepsilon_{\delta_G}}+3k^2-3k+6, \end{aligned}$$

which yields the desired bound on d as stated in inequality (6.3.3). □

6.3.2 Bounds on the (Edge-)Fault-Diameter of $(k+1)$-connected C_4-free Graphs

In this section we give upper bounds on the k-edge-fault-diameter in terms of their order. We aim to improve the bound in Theorem 1.4.29 for graphs not containing 4-cycles. This fills a gap in the literature since for C_4-free graphs only bounds on the k-fault diameter are known (see [28]). We also give bounds on the edge-fault-diameter of graphs with girth at least 6 and (C_4, C_5)-free graphs. We present an upper bound on the k-edge-fault-diameter of C_4-free graphs. Since our bound is close to being sharp for large values of k, but not for small values, we first present a sharp bound on the 2-edge-fault diameter of 3-connected C_4-free graphs.

We make use of the following notation in the proofs of the subsequent lemmas and theorems where P, d, S, H, N_i, v_i, $N_G^2(v)$ is as before. Let $\{i_0, i_1, \ldots, i_s\}$ be the set of all indices $i \in \{0, 1, \ldots, d\}$ for which $n_i = 1$, where $0 = i_0 < i_1 < \cdots < i_s$. For convenience we also define $i_{s+1} = d+1$. For $j \in \{0, 1, \ldots, s\}$ we denote the set $\{i_j, i_j+1, \ldots, i_{j+1}-1\}$ by I_j. and refer to it as the jth segment of H. For $V', U' \subseteq V(G)$ and $E' \subseteq E(G)$ we further define $(V', U')_G$ to denote the set of edges joining a vertex in V' to a vertex in U' and $\text{inc}(E', V')$ to be the total number of incidences of E' with vertices in V'. By $\text{inc}(S, I_j)$, we mean $\text{inc}(S, \bigcup_{p\in I_j} N_p)$. We make repeated use of the fact that $\text{inc}(E', V(G)) = 2|E'|$ for every set E' of edges in G.

We begin with a sharp bound on the 2-edge fault-diameter of 3-connected graphs. The subsequent lemmas will be useful in the proof of our main results.

Lemma 6.3.7. *Let G be 3-connected and $S \subseteq E(G)$ with $|S| = 2$. Then H has at most one bridge.*

Proof. Suppose to the contrary that H contains two bridges e_1 and e_2. Then $H - e_1$ contains two components. One of these components contains e_2, and e_2 is a bridge of this component. Hence $H - \{e_1, e_2\}$ contains three components. Let their vertex sets be V_1, V_2 and V_3. Since G is 3-edge-connected, there are at least three edges in $(V_i, V(G) - V_i)_G$ for each $i \in \{1,2,3\}$. These edges are necessarily in $S \cup \{e_1, e_2\}$. Hence $\mathrm{inc}(S \cup \{e_1, e_2\}, V_i) \geq 3$ for every $i \in \{1,2,3\}$. Since the V_i form a partition of $V(G)$ we have

$$8 = 2|S \cup \{e_1, e_2\}| = \mathrm{inc}(S \cup \{e_1, e_2\}, V(G)) \geq \sum_{i=1}^{3} \mathrm{inc}(S \cup \{e_1, e_2\}, V_i) \geq 9,$$

and this contradiction to $|S| = 2$ proves the lemma. □

Lemma 6.3.8. *Let G be 3-edge-connected and let S, H, d, n, N_i, n_i, s, i_j and I_j be as above. Assume that $s \geq 2$. For $j \in \{0, 1, \dots, s-1\}$ define β_j to be 1 if the set $E_H(N_{i_{j+1}-1}, N_{i_{j+1}})$ contains a bridge, and 0 otherwise.*

(a) If $j > 0$ and $i_{j+1} \geq i_j + 2$, then

$$\sum_{p \in I_j} n_p \geq 2|I_j| - \frac{1}{2}\mathrm{inc}(S, I_j) - \frac{1}{2}\beta_j. \tag{6.3.5}$$

(b) For the segment I_0 we have

$$\sum_{p \in I_0} n_p \geq 2|I_0| - \frac{1}{2}\mathrm{inc}(S, I_0) - \frac{1}{2}\beta_0 + \frac{1}{2}. \tag{6.3.6}$$

(c) If $n_d = 1$, then we have $I_s = \{d\}$ and

$$\sum_{p \in I_{s-1} \cup I_s} n_p \geq 2|I_{s-1} \cup I_s| - \frac{1}{2}\mathrm{inc}(S, I_{s-1} \cup I_s) - \frac{1}{2}\beta_{s-1} - \frac{1}{2}. \tag{6.3.7}$$

Proof. Let I_j be the set $\{i_j, i_j + 1, \dots, i_{j+1} - 1\}$. By definition of a segment we have $n_{i_j} = 1$ and $n_{i_j+1}, n_{i_j+2} \dots, n_{i_{j+1}-1} \geq 2$. Hence

$$\sum_{p \in I_j} n_p \geq 2|I_j| - 1. \tag{6.3.8}$$

If now $\bigcup_{p \in I_j} N_p$ contains a vertex of degree 1 or two vertices of degree 2 in H, then it follows from the fact that these vertices have degree at least three in G, that there are at least two incidences of edges in S with these vertices, so $\mathrm{inc}(S, \bigcup_{p \in I_j} N_p) \geq 2$, which implies $2|I_j| - \frac{1}{2}\mathrm{inc}(S, I_j) \leq 2|I_j| - 1$, and (6.3.5) holds in this case. Hence we may assume that $\bigcup_{p \in I_j} N_p$ contains at most one vertex of degree less than 3 in H, and if there is such a vertex it has degree 2. Moreover we may assume that equality holds in (6.3.8) since otherwise, if $\sum_{p \in I_j} n_p \geq 2|I_j|$, (6.3.5) clearly holds.

Let $a := i_j$ and $b := i_{j+1}$. Then $n_a = n_b = 1$ and $n_{a+1} = n_{a+2} =, \dots = n_{b-1} = 2$. For $p \in \{a+1, a+2, \dots, b-1\}$ denote the two vertices of N_p by v_p and w_p.

We prove that

$$v_{a+1} w_{a+1} \in E(H). \tag{6.3.9}$$

Suppose to the contrary that $v_{a+1}w_{a+1} \notin E(H)$. We may assume that $\deg_H(v_{a+1}) \geq 3$ and $\deg_H(w_{a+1}) \geq 2$. Then v_{a+1} has two neighbours in N_{a+2}, and w_{a+1} has a neighbour in N_{a+2}. This neighbour, together with v_{a+1}, w_{a+1} and v_a form a 4-cycle, a contradiction to G being C_4-free. Hence (6.3.9) follows.

Case 1: $\deg_H(w_{a+1}) \geq 3$.

We first observe that v_{a+1} and w_{a+1} cannot have a common neighbour in N_{a+2} since otherwise H would contain a 4-cycle. Hence the set N_{a+2} contains two vertices, v_{a+2} and w_{a+2}. We may assume that $v_{a+1}v_{a+2}, w_{a+1}w_{a+2}$ are in $E(H)$. Since H is C_4-free, we have $v_{a+1}w_{a+2}, w_{a+1}v_{a+2}, v_{a+2}w_{a+2} \notin E(H)$.

At least one of the vertices in N_{a+2} has degree at least three, without loss of generality $\deg_H(v_{a+2}) \geq 3$. Since v_{a+2} has only one neighbour in $N_{a+1} \cup N_{a+2}$, v_{a+2} is adjacent to both vertices of N_{a+3}. Since w_{a+2} shares at most one neigbour in N_{a+3} with v_{a+2}, it follows, in conjunction with the above, that

$$\deg_H(w_{a+2}) = 2. \tag{6.3.10}$$

We furthermore claim that

$$v_{a+3}w_{a+3} \in E(H). \tag{6.3.11}$$

Suppose to the contrary that $v_{a+3}w_{a+3} \notin E(H)$. Since $\deg_H(v_{a+3}) \geq 3$ and $\deg_H(w_{a+3}) \geq 3$, v_{a+3} has two neighbours in N_{a+4}, and w_{a+3} has one neighbour in N_{a+4}. Hence v_{a+3} and w_{a+3} have a common neighbour in N_{a+4}, in addition to v_{a+2}, and so H contains a 4-cycle, a contradiction. This proves (6.3.11).

Since the two vertices in N_{a+3} have v_{a+2} as a common neighbour, it follows that they cannot have a common neighbour in N_{a+4}, otherwise H would contain C_4. Hence, if now $n_{a+4} = 1$, so $i_{j+1} = a+4$, then v_{a+4} has only one edge joining it to N_{a+3}, so this edge would be a bridge. Since by $\deg_H(w_{a+2}) = 2$ implies $\mathrm{inc}(S, I_j) \geq 1$, it follows that (6.3.5) holds in this case. Hence we may assume that $n_{a+4} = 2$ and that H contains the edges $v_{a+3}v_{a+4}$ and $w_{a+3}w_{a+4}$. None of the edges $v_{a+3}w_{a+4}$, $w_{a+3}v_{a+4}$ and $v_{a+4}w_{a+4}$ are present in H since H is C_4-free. Since both, v_{a+4} and w_{a+4} have degree at least three, and since they are non-adjacent and have only one neighbour each in N_{a+3}, it follows that both, v_{a+4} and w_{a+4} have two (common) neighbours in N_{a+5}, and so H contains a 4-cycle, a contradiction.

Case 2: $\deg_H(w_{a+1}) = 2$.

We first show that

$$n_{a+2} = 1. \tag{6.3.12}$$

Suppose to the contrary that $n_{a+2} = 2$. Since $\deg_H(w_{a+1}) = 2$, both vertices of N_{a+2} are adjacent to v_{a+1}, but non-adjacent to w_{a+1}. The same considerations as in Case 1, with

a replaced by $a+1$, now prove that $\deg_H(w_{a+3}) = 2$ as in (6.3.10). This contradiction to our assumption that we have only one vertex of degree two proves (6.3.12).

It follows from $n_{a+2} = 1$ that v_{a+2} is adjacent to only one vertex in N_{a+1}, say v_{a+1}, otherwise v_{a+1} and w_{a+1} have two common neighbours and so H contains a 4-cycle, a contradiction. Hence $E_H(N_{a+1}, N_{a+2})$ contains only one edge, $v_{a+1}v_{a+2}$, which is a bridge. We have $\mathrm{inc}(S, I_j) \geq 1$ since $\deg_H(w_2) = 2$. Therefore $2|I_j| - \frac{1}{2}\mathrm{inc}(S, I_j) - \frac{1}{2} \leq 2|I_j| - 1$, and by (6.3.8) we have that (6.3.5) holds. This completes the proof of (a).

(b) First assume that $n_1 \geq 3$. Since $n_0 = 1$, $n_1 \geq 3$ and $n_p \geq 2$ for $p = 2, 3, \ldots, i_1 - 1$, we have $\sum_{p \in I_0} n_p \geq 2|I_0|$. If the inequality is strict, then (6.3.6) clearly holds. Hence we may assume that $\sum_{p \in I_0} n_p = 2|I_0|$. Then we have $n_1 = 3$ and $n_2 \leq 2$. We bound the sum of the degrees in H of the vertices in N_1. There are exactly three edges joining a vertex in N_1 to the vertex in N_0. Since H contains no C_4, there is most one edge joining two vertices in N_1, and each vertex in N_2 has at most one edge joining it to vertices in N_1. Hence $\sum_{x \in N_1} \deg_H(x) \leq 5 + n_2 \leq 7$. Since every vertex of G has degree at least 3, it follows that $\mathrm{inc}(S, I_1) \geq \mathrm{inc}(S, N_2) \geq 2$, and so (6.3.6) holds if $n_1 \geq 3$.

Now assume that $n_1 = 2$. Then the proof of part (a) in conjunction with the fact that there is an additional incidence between v_0 and S, yields (6.3.6). If $n_1 = 1$, then $I_0 = \{0\}$. Since $\deg_H(v_0) = 1$, we have $\mathrm{inc}(S, I_0) \geq 2$, so (6.3.6) holds.

(c) Let $n_d = 1$. Then clearly $I_s = \{d\}$. First assume that $\deg_H(v_d) \geq 3$. Then $n_{d-1} \geq \deg_H(v_d)$. Each of the $\deg_H(v_d)$ neighbours of v_d has a neighbour in N_{d-2}, and these $\deg_H(v_d)$ neighbours in N_{d-2} are distinct since H is C_4-free.
It follows that $n_{d-2}, n_{d-1} \geq \deg_H(v_d) \geq 3$, and since $n_{i_{s-1}+1}, n_{i_{s-1}+2}, \ldots, n_{d-3} \geq 2$, we have $\sum_{p \in I_{s-1} \cup I_s} n_p \geq 2|I_{s-1} \cup I_s|$ and (6.3.7) follows in this case. Now assume that $\deg_H(v_d) = 2$. Then $E_H(N_{d-1}, N_d)$ does not contain a bridge. Hence by (a) and the fact that $\mathrm{inc}(S, N_d) \geq 1$, we have that

$$\sum_{p \in I_{s-1} \cup I_s} n_p \geq 2|I_{s-1}| - \frac{1}{2}\mathrm{inc}(S, I_{s-1}) + 1 \geq 2|I_{s-1} \cup I_s| - \frac{1}{2}\mathrm{inc}(S, I_{s-1} \cup I_s) - \frac{1}{2},$$

as desired. The proof for the case that $\deg_H(v_d) = 1$ is similar and thus omitted. This completes the proof of the lemma. □

Theorem 6.3.9. *Let G be a 3-edge-connected C_4-free graph of order n, then*

$$D_2'(G) \leq \frac{n}{2}. \tag{6.3.13}$$

The bound is sharp for all $n \geq 22$ with $n \equiv 2 \pmod{10}$.

Proof. Let G, S, H, d, i_j for $j = 0, 1, \ldots, s+1$ and I_j for $j = 0, 1, \ldots, s$ be as above. We clearly have

$$n = \sum_{j=0}^{s} \sum_{p \in I_j} n_p.$$

Since $\sum_{p \in I_j} n_p \geq 2|I_j| - 1$ for all $j \in \{0, 1, \ldots, d\}$, we have $n \geq \sum_{j=0}^{s}(2|I_j| - 1) = 2d + 2 - (s + 1) = 2d + 1 - s$. Hence the theorem holds if $s \leq 1$, and so we may assume that $s \geq 2$.

Case 1: $n_d > 1$ and there is no i with $n_i = n_{i+1} = 1$.

Applying (6.3.6) and (6.3.5) we obtain

$$\begin{aligned} n &= \sum_{p \in I_0} n_p + \sum_{j=1}^{s} \sum_{p \in I_j} n_p \\ &\geq 2|I_0| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p \in I_0} N_p) - \frac{1}{2}\beta_0 + \frac{1}{2} + \sum_{j=1}^{s} \Big(2|I_j| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p \in I_j} N_p) - \frac{1}{2}\beta_j\Big) \\ &= 2\sum_{j=0}^{s} |I_j| - \frac{1}{2}\sum_{j=0}^{s} \mathrm{inc}(S, \bigcup_{p \in I_j} N_p) - \frac{1}{2}\sum_{j=0}^{s-1} \beta_j + \frac{1}{2}. \end{aligned}$$

Now $\sum_{j=0}^{s} |I_j| = d+1$, $\sum_{j=0}^{s} \mathrm{inc}(S, \bigcup_{p \in I_j} N_p) = \mathrm{inc}(S, V(G)) = 2|S| = 4$, and $\sum_{j=0}^{s-1} \beta_j \leq 1$ by Lemma 6.3.7. Hence $n \geq 2d$, as desired.

Case 2: $n_d = 1$ and there is no i with $n_i = n_{i+1} = 1$.

Applying (6.3.6), (6.3.5) and (6.3.7) we obtain

$$\begin{aligned} n &= \sum_{p \in I_0} n_p + \sum_{j=1}^{s-2} \sum_{p \in I_j} n_p + \sum_{j=s-1}^{s} \sum_{p \in I_j} n_p \\ &\geq 2|I_0| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p \in I_0} N_p) - \frac{1}{2}\beta_0 + \frac{1}{2} + \sum_{j=1}^{s-2} \Big(2|I_j| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p \in I_j} N_p) - \frac{1}{2}\beta_j\Big) \\ &\quad + 2|I_{s-1} \cup I_s| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p \in I_{s-1} \cup I_s} N_p) - \frac{1}{2}\beta_{s-1} - \frac{1}{2} \\ &= 2\sum_{j=0}^{s} |I_j| - \frac{1}{2}\sum_{j=0}^{s} \mathrm{inc}(S, \bigcup_{p \in I_j} N_p) - \frac{1}{2}\sum_{j=0}^{s-1} \beta_j. \end{aligned}$$

As above, $\sum_{j=0}^{s} |I_j| = d + 1$, $\sum_{j=0}^{s} \mathrm{inc}(S, \bigcup_{p \in I_j} N_p) = \mathrm{inc}(S, V(G)) = 2|S| = 4$, and $\sum_{j=0}^{s-1} \beta_j \leq 1$ by Lemma 6.3.7. Hence $n \geq 2d - \frac{1}{2}$. Since n and d are integers, this implies $n \geq 2d$ as desired.

Case 3: $n_d > 1$ and there exists i with $n_i = n_{i+1} = 1$.

We note that there is only one $i \in \{0, 1, \ldots, s - 1\}$ for which $n_i = n_{i+1} = 1$, otherwise

H would have more than one bridge in contradiction to Lemma 6.3.7. Then $\{i\}$ forms a segment of H, I_k say. We have that $k \neq s-1$ since $n_d > 1$.

Applying (6.3.6) and (6.3.5) we obtain

$$\begin{aligned}
n &= \sum_{p\in I_0} n_p + \Big(\sum_{j\in\{1,2,\dots,s\}-\{k\}} \sum_{p\in I_j} n_p \Big) + 1 \\
&\geq 2|I_0| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p\in I_0} N_p) - \frac{1}{2}\beta_0 + \frac{1}{2} \\
&\quad + \sum_{j\in\{1,2,\dots,s\}-\{k\}} \Big(2|I_j| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p\in I_j} N_p) - \frac{1}{2}\beta_j \Big) + 1 \\
&= 2\sum_{j\in\{0,1,\dots,s\}-\{k\}} |I_j| - \frac{1}{2}\sum_{j\in\{0,1,\dots,s\}-\{k\}} \mathrm{inc}(S, \bigcup_{p\in I_j} N_p) - \frac{1}{2}\sum_{j\in\{0,1,\dots,s-1\}-\{k\}} \beta_j + \frac{3}{2}.
\end{aligned}$$

Now $\sum_{j\in\{0,1,\dots,s\}-\{k\}} |I_j| = d$, $\sum_{j\in\{0,1,\dots,s\}-\{k\}} \mathrm{inc}(S, \bigcup_{p\in I_j} N_p) \leq \mathrm{inc}(S, V(G)) = 2|S| = 4$, and $\beta_j = 0$ for all $j \in \{0,1,\dots,s\} - \{k\}$ since by Lemma 6.3.7 there exists no bridge in H besides the bridge joining the vertices in N_i and N_{i+1}. Hence $n \geq 2d - \frac{1}{2}$. Since n and d are integers, this implies $n \geq 2d$, as desired.

Case 4: $n_d = 1$ and there exists i with $n_i = n_{i+1} = 1$.

As in Case 3 let $\{i\} = I_k$. We may assume that $k \geq 1$ since otherwise, if $k = 0$, inequality (6.3.6) holds for I_0 and the proof of Case 1 applies. We may further assume that $k < s-1$ since otherwise, if $k = s-1$, inequality (6.3.7) holds for I_0 and the proof of Case 2 applies.

Applying (6.3.6), (6.3.7) and (6.3.5) we obtain

$$\begin{aligned}
n &= \sum_{p\in I_0} n_p + \Big(\sum_{j\in\{1,2,\dots,s\}-\{k,s-1,s\}} + \sum_{j\in\{s-1,s\}} \Big) \sum_{p\in I_j} n_p + 1 \\
&\geq 2|I_0| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p\in I_0} N_p) - \frac{1}{2}\beta_0 + \frac{1}{2} \\
&\quad + \sum_{j\in\{1,2,\dots,s\}-\{k,s-1,s\}} \Big(2|I_j| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p\in I_j} N_p) - \frac{1}{2}\beta_j \Big) \\
&\quad + 2|I_{s-1}\cup I_s| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p\in I_{s-1}\cup I_s} N_p) - \frac{1}{2}\beta_{s-1} - \frac{1}{2} + 1 \qquad (6.3.14) \\
&= 2\sum_{j\in\{0,1,\dots,s\}-\{k\}} |I_j| - \frac{1}{2}\sum_{j\in\{0,1,\dots,s\}-\{k\}} \mathrm{inc}(S, \bigcup_{p\in I_j} N_p) - \frac{1}{2}\sum_{j\in\{0,1,\dots,s-1\}-\{k\}} \beta_j + 1.
\end{aligned}$$

As above, $\sum_{j\in\{0,1,\dots,s\}-\{k\}} |I_j| = d$, $\sum_{j\in\{0,1,\dots,s\}-\{k\}} \mathrm{inc}(S, \bigcup_{p\in I_j} N_p) \leq \mathrm{inc}(S, V(G)) = 2|S| = 4$, and $\beta_j = 0$ for all $j \in \{0,1,\dots,s\} - \{k\}$ since by Lemma 6.3.7 there exists no bridge in H besides the bridge joining the vertices in N_i and N_{i+1}. Hence $n \geq 2d - 1$.

To complete the proof, it suffices to show that this inequality is strict. Suppose that $n = 2d - 1$. Therefore, we have equality in (6.3.14). This implies that $\sum_{p\in I_0} n_p = 2|I_0| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p\in I_0} N_p) + \frac{1}{2}$, so $\mathrm{inc}(S, \bigcup_{p\in I_0} N_p)$ is odd, otherwise the right hand side of the equation would not be an integer. For all $j \in \{1, 2, \ldots, s\} - \{k, s-1, s\}$ we have $\sum_{p\in I_j} n_p = 2|I_j| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p\in I_j} N_p)$, implying that $\mathrm{inc}(S, \bigcup_{p\in I_j} N_p)$ is even. Finally, we have $\sum_{j\in\{s-1,s\}} \sum_{p\in I_j} n_p = 2|I_{s-1} \cup I_s| - \frac{1}{2}\mathrm{inc}(S, \bigcup_{p\in I_{s-1}\cup I_s} N_p) - \frac{1}{2}$ implying that $\mathrm{inc}(S, \bigcup_{p\in I_{s-1}\cup I_s} N_p)$ is odd. We also have that $\sum_{j\in\{0,1,\ldots,s\}-\{k\}} \mathrm{inc}(S, \bigcup_{p\in I_j} N_p) = 4$. It follows that one of the following occurs. A: $\mathrm{inc}(S, \bigcup_{p\in I_0} N_p) = \mathrm{inc}(S, \bigcup_{p\in I_{s-1}\cup I_s} N_p) = 1$ and $\mathrm{inc}(S, \bigcup_{p\in I_j} N_p) = 2$ for some $j \in \{1, 2, \ldots, s-2\} - \{k\}$, or B: one of $\mathrm{inc}(S, \bigcup_{p\in I_0} N_p)$ and $\mathrm{inc}(S, \bigcup_{p\in I_{s-1}\cup I_s} N_p)$ equals 1 while the other one equals 3.

In both cases, A and B, we have that three of the incidences of S with $V(G)$ occur with vertices in $\bigcup_{j=0}^{k-1} \bigcup_{p\in I_j} N_p$ and one occurs in $\bigcup_{j=k+1}^{s} \bigcup_{\in I_j} N_p$, or vice versa. In either case only one one edge in S joins a vertex in $\bigcup_{p=0}^{i_k} N_p$ to a vertex in $\bigcup_{p=i_k+1}^{d} N_p$. Now H contains only one edge joining these two sets, viz $v_{i_k}v_{i_k+1}$. It follows that G contains only two edges joining these two sets. This contradiction to G being 3-edge-connected proves that our assumption $n = 2d - 1$ is false, hence we have $n \geq 2d$, and so (6.3.13) follows completing the proof of the theorem. □

We now construct graphs to show the sharpness of the bound in (6.3.13).

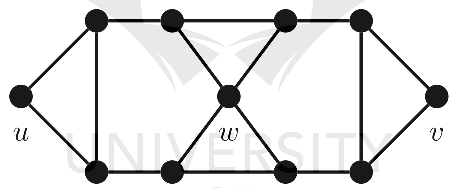

Figure 6.3: The graph H_5.

Let H_5 be the graph depicted in Figure 6.3 above. Let u, v, and w be the vertices described above in H_5. Clearly $\mathrm{diam}(H_5) = 5$ and $V(H_5) = 11$. For $t \in \mathbb{N}$, let $F_0, F_1, \ldots, F_t$ be disjoint copies of H_5. Let u_i, v_i, w_i be vertices of F_i corresponding to u, v and w respectively of H_5. Consider the disjoint union of $F_0, F_1, \ldots, F_t$. Let z be a new vertex and $e_1 = zu_0$, $e_2 = zu_{\lfloor t/2\rfloor}, e_3 = zv_t$ be new edges respectively. Let G_n be the graph with vertex set $V(G_n) = V(F_1) \cup V(F_2) \ldots \cup V(F_t) \cup \{z\}$ and edge set $E(G_n) = E(F_1) \cup E(F_2) \ldots \cup E(F_t) \cup \{e_1, e_2, e_3\}$. By identifying v_i and u_{i+1} as u_{i+1} (for $0 \leq i \leq t-1$), we obtain a 3-edge-connected C_4-free graph. Figure 6.4 is an illustration of G_n when $t = 2$.

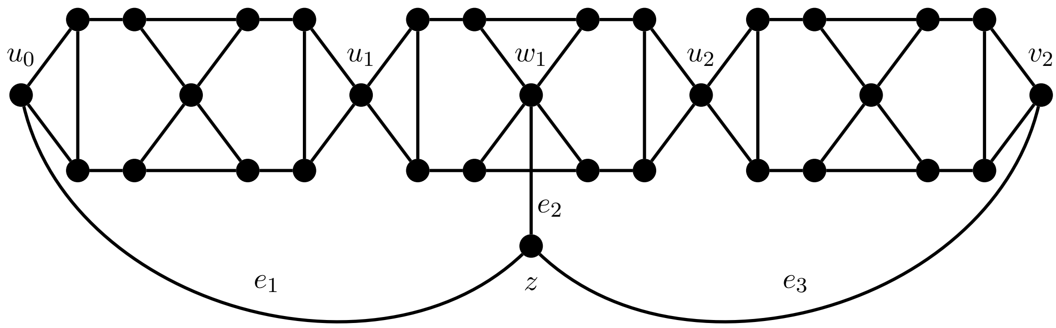

Figure 6.4: The graph G_n for $t = 2$.

The order of G_n is

$$n = |G_n| = 10(t+1) + 2 \tag{6.3.15}$$

Then, letting $S = \{e_1, e_2\}$,

$$\begin{aligned} D'_2(G_n) &\geq \operatorname{diam}(G_n - S) \\ &= d_{G_n - S}(z, u_0) \\ &= 1 + \frac{1}{2}(n-2) \\ &= \frac{n}{2}. \end{aligned}$$

This shows that for all n with $n \geq 22$ and $n \equiv 2 \pmod{10}$ the bound in Theorem 6.3.9 is sharp.

Theorem 6.3.10. *Let G be a C_4-free graph of order n. If $k \geq 3$ and G is $(k+1)$-edge-connected, then*

$$D'_k(G) \leq \frac{5n}{k^2+k+1} + \frac{9k^2+9k-1}{k^2+k+1}. \tag{6.3.16}$$

Proof. Let P, d, S, H, N_i, v_i, $N_G^2(v)$ be as defined previously. Since G is $(k+1)$-edge connected, we have that $\delta(G) \geq k+1$. For $i \in \{0, 1, 2, \ldots, d\}$, we have that,

$$N_H^2[v_i] \subseteq N_{i-2} \cup N_{i-1} \cup N_i \cup N_{i+1} \cup N_{i+2},$$

and so for $0 \leq i \neq j \leq \lfloor \frac{d}{5} \rfloor$, the sets $N_H^2[v_{5i}]$ and $N_H^2[v_{5j}]$ are disjoint. It follows that

$$n \geq \sum_{i=0}^{\lfloor d/5 \rfloor} |N_H^2[v_{5i}]|. \tag{6.3.17}$$

Since H is C_4-free, any two vertices in $N_H(v_{5i})$ have no common neighbour other than v_{5i} and each vertex in $N_H(v_{5i})$ has at most one neighbour in $N_H(v_{5i})$. Moreover since

$H = G - S$, it is possible that an edge in S is incident with a vertex (vertices) in $N_H[v_{5i}]$. We now bound the number of vertices in $N_H^2[v_{5i}]$.

Fix $i \in \{0, 1, \ldots, \lfloor \frac{d}{5} \rfloor\}$ and consider v_{5i}. Let $w_1, w_2, \ldots, w_{\deg_H(v_{5i})}$ be the neighbours of v_{5i} in H. Let a_i be the number of incidences of edges in S with v_{5i}, and let c_i be the number of incidences of edges in S with vertices in $N_H[v_{5i}]$. Then $\deg_H(v_{5i}) = \deg_G(v_{5i}) - a_i$ and $\sum_{j=1}^{\deg_H(v_{5i})} \deg_H(w_j) = \sum_{j=1}^{\deg_H(v_{5i})} \deg_G(w_j) - c_i + a_i$. Since H is C_4-free, each w_j is adjacent to at most one other vertex in $N_H(v_{5i})$ and has thus at least $\deg_H(w_j) - 2$ neighbours in $N_H^2(v_{5i})$. Moreover, no two neighbours of v_{5i} have a common neighbour in $N_H^2(v_{5i})$. Hence

$$\begin{aligned}
|N_H^2[v_{5i}]| &\geq 1 + \deg_H(v_{5i}) + \sum_{j=1}^{\deg_H(v_{5i})} (\deg_H(w_j) - 2) \\
&\geq 1 + \deg_G(v_{5i}) - a_i + (\deg_G(v_{5i}) - a_i)(k-1) - (c_i - a_i) \\
&\geq k + 2 + (k + 1 - a_i)(k-1) - c_i \\
&\geq k^2 + k + 1 - c_i k,
\end{aligned}$$

with the last inequality holding since $a_i \leq c_i$.

Since removing the k edges of S reduces the total degree sum of G by $2k$ and $\sum_{i=0}^{\lfloor d/5 \rfloor} c_i \leq 2k$, equation 6.3.17 yields

$$\begin{aligned}
n &\geq \sum_{i=0}^{\lfloor d/5 \rfloor} (k^2 + k + 1 - c_i k) - 2k \\
&\geq \sum_{i=0}^{\lfloor d/5 \rfloor} (k^2 + k + 1) - k \sum_{i=1}^{\lfloor d/5 \rfloor} c_i - 2k \\
&\geq (\lfloor d/5 \rfloor + 1)(k^2 + k + 1) - 2k^2 - 2k \qquad (6.3.18)
\end{aligned}$$

Since $\lfloor d/5 \rfloor \geq \frac{d-4}{5}$, we have that $n \geq (\frac{d+1}{5})(k^2 + k + 1) - 2k^2 - 2k$. Solving for d yields the desired result in (6.3.16). □

We now construct graphs to show that the bound in (6.3.16) is close to the being optimal.

Let q be an odd prime power. Let H_q be the graph described in Example 2.2.3 whose vertices are the 1-dimensional subspaces of the field $GF(q)^3$, where two vertices are adjacent if, as subspaces, they are orthogonal.By Claim 2.2.5 we have $V_q \neq \emptyset$. Vertices in H_q have degree q or $q + 1$ and so we denote by V_q and V_{q+1} the set of vertices of H_q of degree q and $q + 1$ respectively. The following results will be useful in the our construction.

Proposition 6.3.11. *Let q be a prime power. Then H_q contains a vertex z such that $N(z)$ can be partitioned into two sets U and W such that:*

(i) $|U| \geq \frac{q}{2}$ *and* $|W| \geq \frac{q}{2}$*, and*

(ii) there is no edge in H_q *joining a vertex in* W *to a vertex in* U*.*

Proof. CASE 1: q is even.

Let z be a vertex of degree q. Then $N(z)$ is an independent set by Claim 2.2.4. Partition $N(z)$ into two sets U and W of cardinality $q/2$.

CASE 2: q is odd.

Since every vertex of H_q has degree either q or $q+1$, and since there exists a vertex of degree q, it follows from the handshake lemma that H_q has at least two vertices, x and y say, of degree q. Let z be the common neighbour of x and y. Then $\deg(z) = q+1$. Let $\mathcal{I}_z$ be the subgraph of H_q induced by $N(z)$.

CLAIM 1: $\mathcal{I}_z$ has at least two vertices of degree 0, and its maximum degree is at most 1.

Clearly, every vertex in $N_{H_q}(z) \cap V_q$ is non-adjacent to every other vertex in $N_{H_q}(z)$ by Claim 2.2.4(b), so it has degree 0. Since $x, y \in N_{H_q}(z) \cap V_q$, $\mathcal{I}_z$ has at least two vertices of degree 0. Every vertex in $N_{H_q}(z) \cup V_{q+1}$ is adjacent to exactly one other vertex in $N_{H_q}(z)$ by Claim 2.2.4(c). This proves the claim.

Let $e_1, e_2, \ldots, e_s$ be the set of edges of $\mathcal{I}_z$. Then there are $q+1-2s$ vertices of degree 0 in $\mathcal{I}_z$, and $q+1-2s \geq 2$ by CLAIM 1. If s is even, let the vertices incident with one of the edges $e_1, e_2, \ldots, e_{s/2}$ be in U, let the vertices incident with one of the edges $e_{s/2+1}, e_{s/2+2}, \ldots, e_s$ be in W, and distribute the remaining vertices, which have degree 0, evenly between U and W. If s is odd, let the vertices incident with one of the edges $e_1, e_2, \ldots, e_{(s+1)/2}$ be in U, let the vertices incident with one of the edges $e_{(s+3)/2}, e_{(s+5)s/2}, \ldots, e_s$ as well as x and y be in W, and distribute the remaining vertices, which have degree 0, evenly between U and W. □

Now, replace vertex z by two vertices, u and w, where u is adjacent to all vertices in U, and w is adjacent to all vertices in W. Denote the new graph by H_q''. Clearly, H_q'' is C_4-free, has q^2+q+2 vertices, and all vertices other than u and w have degree q or $q+1$, while u and w have degree at least $\frac{q}{2}$.

We note some properties which will be used later.

CLAIM 1: $d_{H_q''}(u, w) \geq 5$.

Indeed u and w are neither adjacent nor have a common neighbour, Hence $d_{H_q''}(u, w) \geq 3$ and every (u, w)-path in H_q'' is of the form $u, u_1, \ldots, w_1, w$, where $u_1 \in U$ and $w_1 \in W$. Since there is no edge joining a vertex in U to a vertex in W since any two vertices of $N(z)$ have no common neighbour other then z, we have $d_{H_q''}(u_1, w_1) \geq 3$, and thus $d_{H_q''}(u, w) \geq 5$, as desired.

Claim 2: Let $S \subseteq E(H_q'')$ with $|S| < q$ and $v \in V(H_q'')$. Then $H_q'' - S$ contains a (v,u)-path or a (v,w)-path.

If $v \in \{u,w\}$, then the statement holds trivially, so assume that $v \notin \{u,w\}$. We make use of the fact that H_q is obtained from H_q'' by identifying u and w to obtain vertex z. Since H_q is q-edge-connected, there exists a (v,z)-path in H_q. This path yields a (v,u)-path or a (w,u)-path in $H_q'' - S$. (Here and below we denote by S a set of edges in H_q'', and we denote the set of corresponding edges in H_q also by S.)

Claim 3: Let $S \subseteq E(H_q'')$ with $|S| < \frac{q}{2}$ and $v \in V(H_q'')$. Then $H_q'' - S$ is connected.

By Lemma 6.2.4, graph H_q is q-connected, so $H_q - z$ is $(q-1)$-connected and thus $\frac{q}{2}$-connected. Since H_q'' is obtained from $H_q - z$ by attaching two vertices of degree at least $\frac{q}{2}$, it follows that H_q'' is also $\frac{q}{2}$-connected. Claim 3 follows.

Theorem 6.3.12. *Let $k \in \mathbb{N}$ such that $k+1$ is a prime power. Then for infinitely many values of n, there exists a $(k+1)$-edge-connected, C_4-free graph $G_{q,\ell}^*$ of order n and with k-fault diameter*

$$D_k'(G_{q,\ell}^*) \geq \frac{5n}{k^2+3k+3} - \frac{6k^2+18k+13}{k^2+3k+3}. \tag{6.3.19}$$

Proof. For a given prime power q and an integer ℓ with $\ell \geq 2$ we define the graph $G_{q,\ell}^*$ as follows. Let $\mathcal{G}_1$ and $\mathcal{G}_\ell$ be disjoint copies of H_q, and let $\mathcal{G}_2, \mathcal{G}_3, \ldots, \mathcal{G}_{\ell-1}$ be disjoint copies of H_q''. Let w^1 and u^ℓ be vertices of $\mathcal{G}_1$ and $\mathcal{G}_\ell$, respectively, and for $i = 2,3,\ldots,\ell-1$ let u^i and w^i be the vertices of $\mathcal{G}_i$ corresponding to u and w. Now $G_{q,\ell}^*$ is obtained from the union of $\mathcal{G}_1, \ldots, \mathcal{G}_\ell$ by identifying w^i with u^{i+1} for $i = 1,2,\ldots,\ell-1$ by adding $q-1$ edges, each joining a vertex of $\mathcal{G}_1$ to a vertex of $\mathcal{G}_\ell$. A sketch of the graph $G_{q,\ell}^*$ for q and $\ell = 3$ is shown in Figure 6.5.

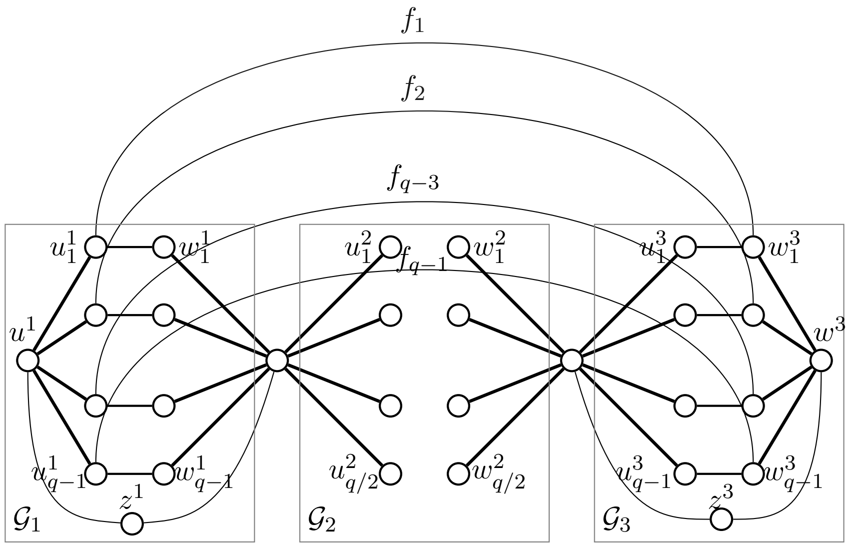

Figure 6.5: The graph $G_{q,\ell}^*$ for $\ell = 3$

We first show that $G^*_{q,\ell}$ is q-edge-connected. Assume that S is a set of $q-1$ edges of $G^*_{q,\ell}$. We show that $G^*_{q,\ell} - S$ is connected.

Case 1: There exists $i \in \{2, 3, \ldots, \ell - 1\}$ with $|S \cap V(\mathcal{G}_i)| \geq \frac{q}{2}$.

Then for each $j \in \{0, 1, \ldots, \ell\} - \{i\}$ the subgraph $\mathcal{G}_j - S$ is connected by Claim 3 since S has less than $\frac{q}{2}$ vertices in $\mathcal{G}_j$. Hence $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{i-1}$ belong to the same component of $G^*_{q,\ell} - S$, and $\mathcal{G}_{i+1}, \mathcal{G}_{i+2}, \ldots, \mathcal{G}_\ell$ belong to the same component of $G^*_{q,\ell} - S$. Since not all $q-1$ edges between $\mathcal{G}_1$ and $\mathcal{G}_\ell$ are in S, it follows that all $\mathcal{G}_j$ with $j \neq i$ are in the same component of $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{i-1}$ belong to the same component of $G^*_{q,\ell} - S$. Since for every vertex x of $\mathcal{G}_i$ there exists a path from x to u^i or to w^i in $\mathcal{G}_i - S$ by Claim 2, it follows that $G^*_{q,\ell} - S$ is connected, as desired.

Case 2: For all $i \in \{2, 3, \ldots, \ell - 1\}$ we have $|S \cap V(\mathcal{G}_i)| < \frac{q}{2}$.

Then it follows that $\mathcal{G}_i - S$ is connected for all $i \in \{1, 2, \ldots, \ell - 1\}$ by Claim 3. Since H_q is q-edge-connected, the graphs $\mathcal{G}_1 - S$ and $\mathcal{G}_\ell - S$ are also connected. Hence $G^*_{q,\ell} - S$ is connected, as desired.

We now bound the k-edge-fault-diameter from below. Choose the set S as the $q-1$ edges joining a vertex of $\mathcal{G}_1$ to a vertex of $\mathcal{G}_\ell$. Since $d_{H''_q}(u^i, w^i) = 5$ by Claim 1. Choosing vertices u^1 and w^ℓ of $\mathcal{G}$ with $d(u^1, w^1) = 2 = d(u^\ell, w^\ell)$, we have

$$\begin{aligned} D'_{q-1}(G^*_{q,\ell}) &\geq d(u^1, w^1) + d_{G^*_{q,\ell} - S}(w^1, u^\ell) + d(uu^\ell, w^\ell) \\ &= 5(\ell - 2) + 4 \\ &= 5\ell - 6. \end{aligned}$$

Hence

$$D'_k(G^*_{q,\ell}) \geq \frac{5n}{k^2 + 3k + 3} - \frac{6k^2 + 18k + 13}{k^2 + 3k + 3},$$

as desired in (6.3.19). □

6.3.3 Bounds on the (Edge-)Fault-Diameter of $(k+1)$-connected Graphs of Girth at least 6

In this section we give upper bounds on the k-edge-fault diameter of $(k+1)$-edge connected graphs of girth 6 in terms of order. We make use of the notation described in the previous section.

Theorem 6.3.13. *Let G be a connected graph with girth at least 6 and order n.*

(i) If G is 2-edge-connected and $n \geq 6$, then

$$D'_1(G) \leq n - 1. \tag{6.3.20}$$

Equality holds, if and only if $G = C_n$.

(ii) If $k \geq 2$ and G is $(k+1)$-edge-connected, then

$$D'_k(G) \leq \frac{3n}{k^2+k+1} + \frac{6k^2+6k}{k^2+k+1}, \tag{6.3.21}$$

and this bound is sharp apart from an additive constant.

Proof. (i) Since H is connected and has n vertices, we have $\operatorname{diam}(H) \leq n-1$. On the other hand, equality holds if and only if G is 2-connected and G contains an edge e such that $G - e$ is a path which only holds if G is a cycle.

(ii) Let H, S, P, d, v_i be as defined earlier. Since G is $(k+1)$-edge connected, we have that $\delta(G) \geq k+1$. For $i \in \{0, 1, 2, \ldots, d-1\}$, we have that,

$$N_H^2(v_i) \cup N_H^2(v_{i+1}) \subseteq N_{i-2} \cup N_{i-1} \cup N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3},$$

and so for $0 \leq i \neq j \leq \lfloor \frac{d-1}{6} \rfloor$, the sets $\big(N_H^2[v_{6i}] \cup N_H^2(v_{6i+1})\big)$ and $\big(N_H^2(v_{6j}) \cup N_H^2(v_{6j+1})\big)$ are disjoint. It follows that

$$n \geq \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} |N_H^2(v_{6i}) \cup N_H^2(v_{6i+1})| \tag{6.3.22}$$

Since H is a graph of girth at least 6, any two vertices in $N_H(v_{6i})$ have no common neighbour other than v_{6i} and each vertex in $N_H(v_{6i})$ can only be adjacent to v_{6i} and to vertices in $N_H^2(v_{6i})$ since otherwise each H has a cycle of length less than 6.

We now bound the number of vertices in $N_H^2(v_{6i}) \cup N_H^2(v_{6i+1})$. Note that the minimum degree of H is not necessarily $k+1$ or more.

Fix $i \in \{0, 1, \ldots \lfloor \frac{d-1}{6} \rfloor\}$ and consider v_{6i} and v_{6i+1}. Let $x_1, x_2, \ldots, x_{\deg_H(v_{6i})}$ and $y_1, y_2, \ldots, y_{\deg_H(v_{6i+1})}$ be the neighbours of v_{6i} and v_{6i+1} respectively in H. Let $\tau_i(\rho_i$ resp.) be the number of incidences of edges in S with $v_{6i}(v_{6i+1}$ resp.). Let $\alpha_i(\beta_i$ resp.) be the number of incidences of edges in S with vertices in $N_H(v_{6i})$ $(N_H(v_{6i+1})$ resp.). Then $\deg_H(v_{6i}) = \deg_G(v_{6i}) - \tau_i$, $\deg_H(v_{6i+1}) = \deg_G(v_{6i+1}) - \rho_i$, $\sum_{j=1}^{\deg_H(v_{6i})} \deg_H(x_j) = \sum_{j=1}^{\deg_H(v_{6i})} \deg_G(x_j) - \alpha_i + \tau_i$ and $\sum_{j=1}^{\deg_H(v_{6i+1})} \deg_H(y_j) = \sum_{j=1}^{\deg_H(v_{6i+1})} \deg_G(y_j) - \beta_i + \rho_i$.

Since H is a graph with girth at least 6, each x_j can only be adjacent to v_{6i} and to a vertex in $N_H^2(v_{6i})$ and has thus $\deg_H(x_j) - 1$ neighbours in $N_H^2(v_{6i})$. Similarly each y_j has $\deg_H(y_j) - 1$ neighbours in $N_H^2(v_{6i+1})$. Moreover, no two neighbours of $v_{6i}(v_{6i+1}$ resp.) have a common neighbour in $N_H^2(v_{6i})(N_H^2(v_{6i+1})$ resp.) since otherwise H has a cycle of length less than 6.

Claim: $N_H(v_{6i}) \cap N_H(v_{6i+1}) = \emptyset$.

Consider $H - v_{6i}v_{6i+1}$ and let $H' = H - v_{6i}v_{6i+1}$. The sets $N^2_{H'}(v_{6i})$ and $N^2_{H'}(v_{6i+1})$ are disjoint, otherwise H would contain a cycle of length less than 6. Clearly H' is a graph of girth at least 6 and degree of v_{6i} and v_{6i+1} in H' is at least $\deg_H(v_{6i}) - 1$ and $\deg_H(v_{6i+1}) - 1$ respectively. It follows from Lemma 2.3.1 that

$$\begin{aligned} |N^2_{H'}(v_{6i})| &\geq 1 + \deg_{H'}(v_{6i}) + \sum_{j=1}^{\deg_{H'}(v_{6i})} (\deg_{H'}(x_j) - 1) \\ &\geq \deg_G(v_{6i}) - \tau_i + (\deg_G(v_{6i}) - \tau_i - 1)(k) - (\alpha_i - \tau_i) \\ &\geq k + 1 - \tau_i + (k - \tau_i)k - (\alpha_i - \tau_i) \\ &\geq k^2 + k + 1 - \alpha_i(k+1), \end{aligned} \tag{6.3.23}$$

with the last inequality holding since $\tau_i \leq \alpha_i$. A similar argument yields

$$|N^2_{H'}(v_{6i+1})| \geq k^2 + k + 1 - \beta_i(k+1). \tag{6.3.24}$$

Since $N^2_H(v_{6i}) \cup N^2_H(v_{6i+1}) = N^2_{H'}(v_{6i}) \cup N^2_{H'}(v_{6i+1})$ we obtain

$$\left|N^2_H(v_{6i}) \cup N^2_H(v_{6i+1})\right| \geq 2k^2 + 2k + 2 - (\alpha_i + \beta_i)k. \tag{6.3.25}$$

Since removing the k edges of S reduces the total degree sum of G by $2k$, equation (6.3.22) yields

$$\begin{aligned} n &\geq \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} [2k^2 + 2k + 2 - k(\alpha_i + \beta_i)] - 2k \\ &\geq \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} (2k^2 + 2k + 2) - k \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} (\alpha_i + \beta_i) - 2k \\ &\geq (\lfloor \frac{d-1}{6} \rfloor + 1)(2k^2 + 2k + 2) - 2k - k \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} (\alpha_i + \beta_i) \\ &\geq (\lfloor \frac{d-1}{6} \rfloor + 1)(2k^2 + 2k + 2) - 2k - 2k^2, \end{aligned} \tag{6.3.26}$$

with the last inequality holding since $\sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} (\alpha_i + \beta_i) \geq 2k$. Since $\lfloor \frac{d-1}{6} \rfloor > \frac{d-6}{6}$, we have that $n \geq (\frac{d}{3})(k^2 + k + 1) - 2k^2 - 2k$. Solving for d yields the desired result in (6.3.21).

□

We now construct graphs to show that the bound in (6.3.21) is sharp apart from an additive constant for infinitely many values of k.

Consider the graph H^*_{q+1} described in Example 2.3.5, whose vertices correspond to the 1-dimensional and 2-dimensional subspaces of $GF(q)^3$. Clearly H^* is C_4-free, has girth 6 and $(k+1)$-connected (see Claim 2.3.6 and Claim 6.3.3). By Claim 2.3.6, H^* has $2(q^2+q+1)$ vertices and is $(q+1)$-regular.

Fix a vertex z of H^*_{q+1}. Partition $N_{H^*_{q+1}}(z)$ into two sets U and W of order $\frac{q+1}{2}$. Replace vertex z by two vertices, u and w, where u is adjacent to all vertices in U, and w is adjacent to all vertices in W. Denote the new graph by $\mathcal{F}^*_{q+1}$. Clearly, $\mathcal{F}^*_{q+1}$ is C_4-free, has $2q^2+2q+3$ vertices, and all vertices other than u and w have degree $q+1$, while u and w have degree $\frac{q+1}{2}$.

The following properties will be useful later in our construction.

Claim 1: $d_{\mathcal{F}^*_{q+1}}(u,w) \geq 6$.

Indeed u and w are neither adjacent nor have a common neighbour, Hence $d_{\mathcal{F}^*_{q+1}}(u,w) \geq 3$. By identifying u and w, we get the original graph H^*_{q+1}. This implies that the shortest path between these two vertices will form a cycle and since H^*_{q+1} has girth at least 6, this cycle must have length at least 6. Thus, $d_{\mathcal{F}^*_{q+1}}(u,w) \geq 6$, as desired.

Claim 2: Let $S \subseteq E(\mathcal{F}^*_{q+1})$ with $|S| < q+1$ and $v \in V(\mathcal{F}^*_{q+1})$. Then $\mathcal{F}^*_{q+1} - S$ contains a (v,u)-path or a (v,w)-path.

If $v \in \{u,w\}$, then the statement holds trivially, so assume that $v \notin \{u,w\}$. We make use of the fact that H^*_{q+1} is obtained from $\mathcal{F}^*_{q+1}$ by identifying u and w to obtain vertex z. Since H^*_{q+1} is $q+1$-edge-connected, there exists a (v,z)-path in H^*_{q+1}. This path yields a (v,u)-path or a (w,u)-path in $\mathcal{F}^*_{q+1} - S$. (Here and below we denote by S a set of edges in $\mathcal{F}^*_{q+1}$, and we denote the set of corresponding edges in H^*_{q+1} also by S.)

Claim 3: Let $S \subseteq E(\mathcal{F}^*_{q+1})$ with $|S| < \frac{q+1}{2}$ and $v \in V(\mathcal{F}^*_{q+1})$. Then $\mathcal{F}^*_{q+1} - S$ is connected.

By Claim 6.3.3, H^*_{q+1} is $q+1$-connected, so $H^*_{q+1} - z$ is q-connected and thus $\big((q+1)/2\big)$-connected. Since $\mathcal{F}^*_{q+1}$ is obtained from $H^*_{q+1} - z$ by attaching two vertices of degree at least $\frac{q+1}{2}$, it follows that $\mathcal{F}^*_{q+1}$ is also $\big((q+1)/2\big)$-connected. Claim 3 follows.

Theorem 6.3.14. *Let $k \in \mathbb{N}$ such that k is an odd prime power. Then, there exists a $(k+1)$-edge-connected graph with girth at least 6, $G_{q,\ell}$, of order n and k-edge-fault diameter*

$$D'_k(G_{q,\ell}) \geq \frac{3n}{k^2+k+1} - \frac{6k^2+6k+3}{k^2+k+1} \tag{6.3.27}$$

Proof. For a given odd prime power q and an integer ℓ with $\ell \geq 2$ we define the graph $G_{q,\ell}$ as follows. Let $\mathcal{G}_1$ and $\mathcal{G}_\ell$ be disjoint copies of H^*_{q+1}, and let $\mathcal{G}_2, \mathcal{G}_3, \ldots, \mathcal{G}_{\ell-1}$ be disjoint copies of $\mathcal{F}^*_{q+1}$. Let w^1 and u^ℓ be vertices of $\mathcal{G}_1$ and $\mathcal{G}_\ell$, respectively, and for $i = 2,3,\ldots,\ell-1$ let u^i and w^i be the vertices of $\mathcal{G}_i$ corresponding to u and w. Now $G_{q,\ell}$ is obtained from the union of $\mathcal{G}_1, \ldots, \mathcal{G}_\ell$ by identifying w^i with u^{i+1} for $i = 1,2,\ldots,\ell-1$

by adding q edges, each joining a vertex of $\mathcal{G}_1$ to a vertex of $\mathcal{G}_\ell$. A sketch of the graph $G_{q,\ell}$ for $\ell = 3$ is shown in Figure 6.6.

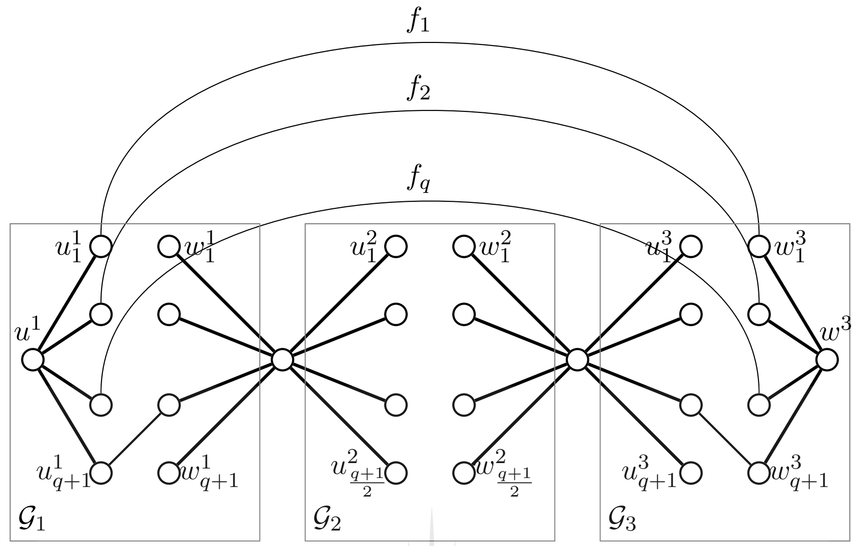

Figure 6.6: The graph $G_{q,\ell}$ for $\ell = 3$

We first show that $G_{q,\ell}$ is $(q+1)$-edge-connected. Assume that S is a set of q edges of $G_{q,\ell}$. We show that $G_{q,\ell} - S$ is connected.

Case 1: There exists $i \in \{2, 3, \ldots, \ell - 1\}$ with $|S \cap V(\mathcal{G}_i)| \geq \frac{q+1}{2}$.

Then for each $j \in \{0, 1, \ldots, \ell\} - \{i\}$ the subgraph $\mathcal{G}_j - S$ is connected by Claim 3 since S has less than $\frac{q+1}{2}$ vertices in $\mathcal{G}_j$. Hence $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{i-1}$ belong to the same component of $G_{q,\ell} - S$, and $\mathcal{G}_{i+1}, \mathcal{G}_{i+2}, \ldots, \mathcal{G}_\ell$ belong to the same component of $G_{q,\ell} - S$. Since not all q edges between $\mathcal{G}_1$ and $\mathcal{G}_\ell$ are in S, it follows that all $\mathcal{G}_j$ with $j \neq i$ are in the same component of $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{i-1}$ belong to the same component of $G_{q,\ell} - S$. Since for every vertex x of $\mathcal{G}_i$ there exists a path from x to u^i or to w^i in $\mathcal{G}_i - S$ by Claim 2, it follows that $G_{q,\ell} - S$ is connected, as desired.

Case 2: For all $i \in \{2, 3, \ldots, \ell - 1\}$ we have $|S \cap V(\mathcal{G}_i)| < \frac{q+1}{2}$.

Then it follows that $\mathcal{G}_i - S$ is connected for all $i \in \{1, 2, \ldots, \ell - 1\}$ by Claim 3. Since H_{q+1}^* is $(q+1)$-edge-connected, the graphs $\mathcal{G}_1 - S$ and $\mathcal{G}_\ell - S$ are also connected. Hence $G_{q,\ell} - S$ is connected, as desired.

We now bound the k-edge-fault-diameter from below. Choose the set S as the q edges joining a vertex of $\mathcal{G}_1$ to a vertex of $\mathcal{G}_\ell$. Since $d_{\mathcal{F}_{q+1}^*}(u^i, w^i) = 6$ and by choosing vertices

u^1 and w^ℓ of $\mathcal{G}$ with $d(u^1, w^1) = 3 = d(u^\ell, w^\ell)$,

$$\begin{aligned} D'_{q-1}(G_{q,\ell}) &\geq d(u^1, w^1) + d_{G_{q,\ell}-S}(w^1, u^\ell) + d(u^\ell, w^\ell) \\ &= 6(\ell - 2) + 6 \\ &= 6(\ell - 1). \end{aligned}$$

Hence

$$D'_k(G_{q,\ell}) \geq \frac{3n}{k^2 + k + 1} - \frac{6k^2 + 6k + 3}{k^2 + k + 1},$$

as desired in (6.3.27). □

Next we show that a slightly weaker bound than the one given in Theorem 6.3.2 holds if we relax the condition to (C_4, C_5)-free (k+1)-edge connected graphs.

Theorem 6.3.15. *Let $k \geq 2 \in \mathbb{N}$ and let G be a $(k+1)$-edge-connected (C_4, C_5)-free graph of order n. Then the k-edge-fault-diameter is given by*

$$D'_k(G) \leq \frac{3n}{k^2 - \frac{1}{2}k + 2 + \varepsilon_{k+1}} + \frac{6k^2 + 6k}{k^2 - \frac{1}{2}k + 2 + \varepsilon_{k+1}}, \tag{6.3.28}$$

where $\varepsilon_{k+1} = 0$ if $k+1$ is even and $\varepsilon_{k+1} = 1$ if $k+1$ is odd.

Proof. Let $P, d, S, H, N_i, v_i, N_G^2$ be as defined earlier. By the hypothesis of Theorem 6.3.15, G may contain triangles. Since G is $(k+1)$-connected, we have that $\delta(G) \geq k + 1$. For $i \in \{0, 1, 2, \ldots, d-1\}$ and $v_i \in N_i, v_{i+1} \in N_{i+1}$, we have that,

$$N_H^2(v_i) \cup N_H^2(v_{i+1}) \subseteq N_{i-2} \cup N_{i-1} \cup N_i \cup N_{i+1} \cup N_{i+2} \cup N_{i+3},$$

and so for $i = 0, 1, 2, \ldots, \lfloor \frac{d-1}{6} \rfloor$, the sets $\big(N_H^2(v_{6i}) \cup N_H^2(v_{6i+1})\big)$ and $\big(N_H^2(v_{6j}) \cup N_H^2(v_{6j+1})\big)$ are disjoint for $i \neq j$. It follows that

$$n \geq \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} |N_H^2[v_{6i}] \cup N_H^2[v_{6i+1}]|. \tag{6.3.29}$$

We now bound the number of vertices in $N_H^2(v_{6i}) \cup N_H^2(v_{6i+1})$.

Fix $i \in \{0, 1, \ldots \lfloor \frac{d-1}{6} \rfloor\}$ and consider v_{6i} and v_{6i+1}.

Let $x_1, x_2, \ldots, x_{\deg_H(v_{6i})}$ and $y_1, y_2, \ldots, y_{\deg_H(v_{6i+1})}$ be the neighbours of v_{6i} and v_{6i+1} respectively in H. Let $\tau_i (\rho_i$ resp.) be the number of incidences of edges in S with $v_{6i} (v_{6i+1}$ resp.). Furthermore, let $\alpha_i (\beta_i$ resp.) be the number of incidences of edges in S with vertices in $N_H(v_{6i})$ $(N_H(v_{6i+1})$ resp.). Then $\deg_H(v_{6i}) = \deg_G(v_{6i}) - \tau_i$, $\deg_H(v_{6i+1}) = \deg_G(v_{6i}) - \rho_i$, $\sum_{j=1}^{\deg_H(v_{6i})} \deg_H(x_j) = \sum_{j=1}^{\deg_H(v_{6i})} (\deg_G(x_j) - 2) - \alpha_i + \tau_i$ and $\sum_{j=1}^{\deg_H(v_{6i+1})} \deg_H(y_j) = \sum_{j=1}^{\deg_H(v_{6i+1})} (\deg_G(y_j) - 2) - \beta_i + \rho_i$.

We consider the following cases since H is (C_4, C_5)-free and v_{6i} can either have (or not have) a common neighbour with v_{6i+1}.

Case 1 : $N_H(v_{6i}) \cap N_H(v_{6i+1}) = \emptyset$.

Consider $H - v_{6i}v_{6i+1}$ and let $H' = H - v_{6i}v_{6i+1}$. The sets $N^2_{H'}(v_{6i})$ and $N^2_{H'}(v_{6i+1})$ are disjoint, otherwise H would contain a cycle of C_4 or C_5 through $v_{6i}v_{6i+1}$. Clearly H' is C_4-free and degree of v_{6i} and v_{6i+1} in H' is at least $\deg_H(v_{6i}) - 1$ and $\deg_H(v_{6i+1}) - 1$ respectively. It follows from Lemma 2.3.8 that

$$\begin{aligned}
|N^2_{H'}(v_{6i})| &\geq 1 + \deg_{H'}(v_{6i}) + \sum_{j=1}^{\deg_{H'}(v_{6i})} \deg_{H'}(x_j) + \varepsilon_{\deg_{H'}(v_{6i})} \\
&\geq 1 + \deg_{H'}(v_{6i}) + \deg_{H'}(v_{6i})(\deg_G(x_j) - 2) - (\alpha_i - \tau_i) + \varepsilon_{\deg_{H'}(v_{6i})} \\
&\geq k + 1 - \tau_i + (k - \tau_i)(k-1) - (\alpha_i - \tau_i) + \varepsilon_{\deg_{H'}(v_{6i})} \\
&\geq 1 + k^2 - k\alpha_i + \varepsilon_{\deg_{H'}(v_{6i})},
\end{aligned}$$

with the last inequality holding since $\tau_i \leq \alpha_i$. A similar argument yields

$$|N^2_{H'}(v_{6i+1})| \geq 1 + k^2 - k\beta_i + \varepsilon_{\deg_{H'}(v_{6i})}.$$

Since $N^2_H(v_{6i}) \cup N^2_H(v_{6i+1}) = N^2_{H'}(v_{6i}) \cup N^2_{H'}(v_{6i+1})$ we obtain

$$\left|N^2_H(v_{6i}) \cup N^2_H(v_{6i+1})\right| \geq 2k^2 + 2 - k(\alpha_i + \beta_i) + 2\varepsilon_{k+1}. \tag{6.3.30}$$

Case 2: $N_H(v_{6i}) \cap N_H(v_{6i+1}) \neq \emptyset$.

Let w be a common neighbour of v_{6i} and v_{6i+1}. Then w is the only common neighbour since otherwise H would contain C_4. We first consider the second neighbourhood of v_{6i} and v_{6i+1}, respectively, in $H' - w$. As in Case 1, the sets $N^2_{H'-w}(v_{6i})$ and $N^2_{H'-w}(v_{6i+1})$ are disjoint, and each has at least $k^2 - k + 2 - k\alpha_i + 2\varepsilon_{k+1}$ vertices. The set $N_H[w] - \{v_{6i}, v_{6i+1}\}$ is also contained in $N^2_H(v_{6i}) \cup N^2_H(v_{6i+1})$, and it does not share any vertex with $N^2_{H'-w}(v_{6i}) \cup N^2_{H'-w}(v_{6i+1})$, otherwise H would contain a C_4 or a C_5. Hence

$$\begin{aligned}
|N^2_H(v_{6i+1}) \cup N^2_H(v_{6i+1})| &\geq 2[k^2 - k + 2 - k\alpha_i + 2\varepsilon_{k+1}] + (\deg_H(w) - 1) \\
&\geq 2k^2 - k + 4 - k(\alpha_i + \beta_i) + 2\varepsilon_{k+1}, \qquad (6.3.31)
\end{aligned}$$

Comparing equations (6.3.30) and (6.3.31), we conclude that

$$|N^2_H(v_{6i}) \cup N^2_H(v_{6i+1})| \geq 2k^2 - k + 4 + 2\varepsilon_{k+1} - k(\alpha_i + \beta_i).$$

Since removing the k edges of S reduces the total degree sum of G by $2k$, equation (6.3.29) yields

$$
\begin{aligned}
n &\geq \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} [2k^2 - k + 4 + 2\varepsilon_{k+1} - k(\alpha_i + \beta_i)] - 2k \\
&\geq \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} (2k^2 - k + 4 + 2\varepsilon_{k+1}) - k \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} (\alpha_i + \beta_i) - 2k \\
&\geq (\lfloor \frac{d-1}{6} \rfloor + 1)(2k^2 - k + 4 + 2\varepsilon_{k+1}) - 2k - k \sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} (\alpha_i + \beta_i) \\
&\geq (\lfloor \frac{d-1}{6} \rfloor + 1)(2k^2 - k + 4 + 2\varepsilon_{k+1}) - 2k - 2k^2, \qquad (6.3.32)
\end{aligned}
$$

with the last inequality holding since $\sum_{i=0}^{\lfloor \frac{d-1}{6} \rfloor} (\alpha_i + \beta_i) \geq 2k$. Furthermore, we have that $\lfloor \frac{d-1}{6} \rfloor > \frac{d-6}{6}$ and so $n \geq (\frac{d}{6})(2k^2 - k + 4 + 2\varepsilon_{k+1}) - 2k - 2k^2$. Solving for d yields the desired result in (6.3.28). □

We do not know if the bound in Theorem 6.3.15 is sharp. The graph $G_{q,\ell}$ constructed in Theorem 6.3.14 shows that the coefficient $\frac{3}{k^2 - \frac{1}{2}k + 2 + \varepsilon_{k+1}}$ of n is at least close to being best possible if $k+1$ is large.

Chapter 7

Conclusion

In this thesis, we considered the four most important distance measures in graph theory: diameter, radius, average distance and average eccentricity, and determined upper bounds on each of these measures for certain graph classes, in particular graphs of girth at least 6, as well as, (C_4, C_5)-free graphs in terms of order and minimum degree. We further obtained similar bounds for the radius and diameter that take into account also the maximum degree. For this purpose we introduced a very natural generalisation of the classical cage problem, the determination of the minimum order of graphs of given minimum and maximum degree and given girth. We also proved upper bounds on the average distance of graphs with given minimum degree and maximum degree. This problem has not been considered in the literature to date. We also considered altered graphs which arises as a result of deleting an edge or vertex from the original graph and determined what the diameter will be for the graphs considered above.

A natural question for further research that arises from the results in this thesis is if similar results can be obtained for graphs of larger girth than 6. While it seems possible to prove similar bounds for larger girth, it is not clear if these bounds will be sharp. This remains an interesting challenge for future research.

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