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DYNAMICS ON THE BOUNDARY OF FATOU COMPONENTS

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Abstract

The aim of this project is to compile the known results about the dynamics on the boundary of invariant simply-connected Fatou components, as well as the questions which are still open concerning the topic. We focus on ergodicity and recurrence. One of the main tools to deal with this kind of questions is to study the boundary behaviour of the associate inner functions. Therefore, the project is divided in two parts. Firstly, ergodicity and recurrence are studied for inner functions. Secondly, these results are applied to study the dynamics on the boundary of invariant simply-connected Fatou components.

Moreover, we study the concrete example $f(z) = z + e^{-z}$, which presents infinitely many invariant doubly-parabolic Baker domains U_k. Making use of the associate inner function, which can be computed explicitly, we give a complete characterization of the periodic points in ∂U_k and prove the existence of uncountably many curves of non-accessible escaping points.

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Contents

Introduction

Holomorphic dynamics studies the iteration of analytic functions on the Riemann sphere and on the complex plane. Central objects include the *Fatou set*, where iterates behave in a stable manner; and its complement, the *Julia set*, where iterates behave chaotically. In the case of transcendental entire functions, it is also relevant the *escaping set*, where iterates converge to infinity, the essential singularity. The Fatou set is open and typically consists of infinitely many components known as *Fatou components.* Due to the invariance of the Fatou and Julia sets, Fatou components are periodic, preperiodic or wandering.

In 1920, P. Fatou classified periodic Fatou components according to its internal dynamics (Thm. 1.3.10). Although doubly-connected periodic Fatou components appear for rational maps, the so-called *Herman rings*, we focus our study on the simply-connected ones: basins of attraction, parabolic basins, Siegel disks and Baker domains. Under this assumption, the dynamics on the Fatou component is conjugate to the dynamics of some self-map of the unit disk $\mathbb{D}$. Indeed, if U is an invariant simply-connected Fatou component of a holomorphic map f, then the map

$$g\colon \mathbb{D} \to \mathbb{D}, \qquad g = \varphi^{-1} \circ f \circ \varphi,$$

where $\varphi\colon \mathbb{D} \to U$ is a Riemann map, is a holomorphic self-map of $\mathbb{D}$ whose dynamics is conjugate to the one of $f_{|U}$. The advantage of this construction is that the dynamics of holomorphic maps of the unit disk are completely understood with the Denjoy-Wolff theorem and Cowen's classification (Thms. 1.3.1 and 1.3.7, respectively). The Denjoy-Wolff theorem ensures the existence of a point, the *Denjoy-Wolff point*, either in the unit disk or in the boundary towards which all orbits converge, except for the case when the map is conjugate to a rotation. Apart from distinguishing if the Denjoy-Wolff point is inside the disk (*elliptic*) or in the boundary, this second case can be divided in three subcases: *doubly-parabolic*, *hyperbolic* and *simply-parabolic*, according to the classification given by C. Cowen (Thm. 1.3.7). By extension, we refer to the type of invariant simply-connected Fatou component according to the type of the associated self-map of $\mathbb{D}$. Therefore, one deduces that basins of attraction are elliptic and parabolic basins are doubly-parabolic. The three cases of convergence to the boundary are realizable for Baker domains (components on which all iterates converge to the essential singularity), thus providing a classification (Thm. 1.3.11).

Although the dynamics inside invariant simply-connected Fatou are completely understood, the dynamical behaviour on their boundary is far more complicated and still unexplored. For instance, given a basin of attraction, it is known that no point in the boundary can converge to the attracting fixed point. But how about a parabolic basin? Can points in the boundary converge to the parabolic fixed point? If so, how many? Also, how many points have a dense orbit? In the concrete case of Baker domains, in view of examples of their different types (see [BF01]), one may ask if always there are points in their boundaries which escape to infinity and if so, how large is the set of such points. In this project we try to answer these questions and related ones.

A naive approach to solve this problem is to study the dynamics of g on $\partial\mathbb{D}$ to deduce properties of f on ∂U. Although this is the strategy we follow, there are several problems that need to be solved in the process. In the first place, g may not be defined in $\partial\mathbb{D}$. Secondly, since f maps ∂U to itself, we also need that g preserves $\partial\mathbb{D}$, at least for the points where it is defined. Finally, the Riemann map φ may not extend continuously to the boundary, so it is no longer a conjugacy.

A given holomorphic self-map g of the unit disk $\mathbb{D}$ may not be defined at any point in the boundary, but according to the Fatou, Riez and Riez theorem, its radial limits g^* exist for Lebesgue almost every point (Thm. 2.1.2). Since we want the boundary to be mapped to itself, it is reasonable to ask that almost every radial limit has modulus 1. Those maps are known as *inner functions*.

Definition. A holomorphic function $g\colon \mathbb{D} \to \mathbb{D}$ is an **inner function** if $\left|g^*(e^{i\theta})\right| = 1$, for λ-almost all θ, where λ denotes the Lebesgue measure.

Blaschke products (products of Möbius transformations which preserve $\mathbb{D}$) are examples of inner functions. Moreover, O. Frostman proved that any inner function is conjugate to a Blaschke product (Thm. 2.1.5).

Since almost every point in the boundary has an image by g^*, forward orbits are defined for almost every point, so it makes sense to consider the dynamical system induced by g^* on $\partial\mathbb{D}$. Note that this dynamical system is only defined almost everywhere and no regularity can be assumed, so standard techniques cannot be applied. This is precisely the area of knowledge of ergodic theory.

In a general setting, *ergodic theory* studies the dynamics of a measurable map defined at almost every point in a measure space. The aim of ergodic theory is to study dynamical systems from a metric point of view, i.e. describing the typical orbit according to the measure. In our context, the measure space is always $\partial\mathbb{D}$ endowed with the Borel σ-algebra and the Lebesgue measure. Although other ergodic properties such as exactness or conservativeness are studied in [DM91] and [BFJK19], we focus on ergodicity and recurrence, which are defined as follows.

Definition. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $T\colon X \to X$ be a measurable map. We say that T is **ergodic**, if for every $A \in \mathcal{A}$ such that $T^{-1}(A) = A$, there holds $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. We say that T is **recurrent**, if for every $A \in \mathcal{A}$ and μ-almost every $x \in A$, there exists an increasing infinite sequence of positive integers $\{n_k\}_k$ such that $T^{n_k}(x) \in A$.

In some sense, ergodicity can be seen as the impossibility of decomposing the dynamical system into smaller parts. It plays the same role as topological transitivity when one looks at the dynamical system from a topological point of view. However, observe that ergodicity is a stronger notion. On the other hand, recurrence captures the idea that the phase space remains the same although time passes: every point returns to any Borel set containing it infinitely many times.

First J. Aaronson ([Aar78; Aar81]) and later C. Doering and R. Mañé ([DM91]) give an almost complete characterization of the ergodic properties of inner functions, summarized in Table 1.

INNER FUNCTION		Ergodicity	Recurrence	Dense orbits a.e.	Convergence to p a.e.
Rational rotation		✗	✓	✗	✗
Irrational rotation		✓	✓	✓	✗
Elliptic		✓	✓	✓	✗
Doubly-parabolic	Finite deg.	✓	✓	✓	✗
	Infinite deg.	✓	?	?	?
Hyperbolic		✗	✗	✗	✓
Simply-parabolic		✗	✗	✗	✓

Table 1: Ergodic properties at $\partial\mathbb{D}$ for the different types of inner functions.

It is left to determine recurrence for the case of doubly-parabolic maps of infinite degree: there are examples which are recurrent and examples which are not, and there is no criterion to distinguish them.

Although in general the study of ergodic properties of a measurable map can be extremely difficult, our case is slightly easier. Since the considered maps come from inner functions, one wants to take advantage of the high degree of regularity of the function inside the unit disk. Indeed, the preferred tool to prove ergodic properties, the *harmonic measure* of $\partial\mathbb{D}$, precisely connects the indicator function of a Borel set on $\partial\mathbb{D}$ with a harmonic function in $\mathbb{D}$, where the inner function is holomorphic (Section 2.4). Apart from ergodicity and recurrence, we study the measure of the set of points having dense orbit and the set of points converging towards the Denjoy-Wolff point p. Dense orbits almost everywhere is an easy consequence of ergodicity and recurrence (Prop. 1.6.4). The convergence towards the Denjoy-Wolff point almost everywhere is in fact the alternative behaviour when recurrence does not occur, by Aaronson's dichotomy (Thm. 3.4.4).

From the ergodic properties of the different types of inner functions described above, one shall deduce ergodic (and dynamical) properties on the boundary of Fatou components. Recall that the Riemann map may not extend continuously to the boundary. This problem is solved by considering an appropriate measure on the boundary of the Fatou component. Indeed, we consider the harmonic measure, defined as follows.

Definition. Let $U \subset \widehat{\mathbb{C}}$ be simply-connected and let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, such that $\varphi(0) = z \in U$. The **harmonic measure** of ∂U with base point $z \in U$ is the image under φ of the normalized Lebesgue measure of $\partial\mathbb{D}$.

Ergodicity and recurrence of the inner function imply directly ergodicity and recurrence in the boundary of the Fatou component with respect to the harmonic measure (Prop. 2.5.4). Results are summarized in Table 2. As before, apart from ergodicity and recurrence, we also look for dense orbits. In the case of Baker domains, the escaping set $\mathcal{I}(f)$ is especially relevant and the ergodic properties we proved give us information about the set's measure.

FATOU COMPONENT		Ergodicity	Recurrence	Dense orbits a.e.	
Attracting basin		✓	✓	✓	
Parabolic basin		✓	✓	✓	
Siegel disk		✓	✓	✓	
BAKER DOMAIN		Ergodicity	Recurrence	Dense orbits a.e.	Measure of $\mathcal{I}(f)$
Doubly-parabolic	Finite degree	✓	✓	✓	Zero
	Infinite degree	✓	?	?	?
Hyperbolic	Finite degree	✗	✗	✗	Full
	Infinite degree	✗	✗	✗	?
Simply-parabolic	Finite degree	✗	✗	✗	Full
	Infinite degree	✗	✗	✗	?

Table 2: Ergodic properties at the boundary of the different Fatou components.

It is worth mentioning that this is an active field of research, in our setting of invariant Fatou components ([RS18; BFJK19; ERS20]) as well as in the setting of wandering domains ([RS11; Ben+19]). Many questions are still not answered, and they are also collected along the project.

Finally, with the aim of answering some of the open questions about the dynamics on the boundary of Baker domains, we study the concrete example $f(z) = z + e^{-z}$. This function was previously studied in [BD99; FH06; BFJK19]. It possesses infinitely many Baker domains of doubly-parabolic type and degree two. Moreover, the inner function can be computed explicitly. In [BD99; FH06] this was proved using different arguments and, in [BD99], a first attempt to describe the boundary of the Baker domain was made, from a topological point of view. In [BFJK19], the escaping set is described in terms of harmonic measure, as an application of general theorems. Moreover, a conjecture is stated relating accessibility, periodic points and the escaping set.

Our careful analysis of this concrete example has multiple purposes. First of all, it helps us to better understand the theorems proved before. In the example, the sets of full (or zero) harmonic measure are no longer abstract and they become manageable, with explicit dynamics that can be studied. But this is not the main reason for considering an example. Examples are the perfect tool to experiment and make conjectures of what may happen in general or, at least, under a certain number of hypothesis. This is precisely what we do in Section 5. The reason for choosing this concrete example is because it is, to our knowledge, the only example of Baker domain of finite degree where the inner function can be computed explicitly. Moreover, both f and the inner function have an easy expression, which makes the example very manipulable.

In this thesis, we try to answer some of the unsolved questions regarding the function $f(z) = z + e^{-z}$. We prove the following.

Theorem A. *Let $f(z) = z + e^{-z}$ and U be the invariant Baker domain contained in the strip $S = \{-\pi \leq Im\ z \leq \pi\}$. Let $\varphi\colon \mathbb{D} \to U$ be the Riemann map such that $\varphi(0) = 0$ and $\varphi((-1, 1)) \subseteq \mathbb{R}$. Then, the following holds.*

(a) The inner function associated to U is $g(z) = \frac{3z^2+1}{z^2+3}$, with Denjoy-Wolff point 1 and of doubly-parabolic type.

(b) Infinity is accessible from U and accesses from U to infinity form a countable set. They consist precisely of the image by φ of the radial segments at points in $\partial\mathbb{D}$ which are preimages of 1.

(c) Periodic points of g in $\partial\mathbb{D}$ have well-defined radial limit under φ, which is a periodic point for f in ∂U. Moreover, all periodic points for f in ∂U are of this form.

(d) There are uncountably many curves in ∂U ending at ∞ consisting of non-accessible escaping points.

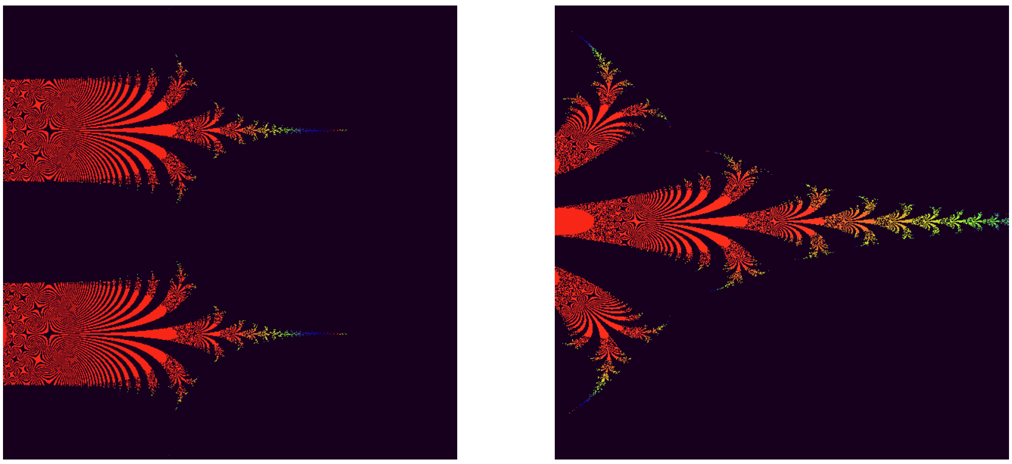


Figure 1: On the left, the dynamical plane of $f(z) = z + e^{-z}$. On the right, a zoom of it.

Structure of the thesis

As it is mentioned above, the goal of this project is to understand the dynamics on the boundary of Fatou components, both for rational and transcendental entire functions, taking advantage of working with the associated inner function in the unit disk. This thesis attempts to be a rigorous survey on the application of inner functions to the study of the boundary of Fatou components. Due to the wide extension of the topic, some results are stated without proof, giving the correspondent reference. Proofs that enlighten the reader may be included while more technical proofs will be omitted.

Moreover, although in some papers (e.g. [DG87; BD99; Bar08]) the motivation for using inner functions is to study topological properties of the Julia set such as accessible boundary points or local-connectivity, results of this kind are not discussed in the project. However, due to the clear connection with our topic, we included basic notes and complementary comments along the thesis, always with the correspondent references to enable the reader to explore this interesting area in between dynamics and topology.

Section 1 is devoted to introduce the reader to holomorphic dynamics and provide the basic tools of other areas of mathematics that are used in this project. Here one finds the formal definition and properties of the Fatou and Julia sets, as well as the classification of the dynamics inside the Fatou components by means of studying the dynamics inside the unit disk of the associated inner function. Most results are stated without proof since they are part of the basic background far from the goals of this thesis. On the other hand, basic notions on harmonic analysis, measure theory and ergodic theory are included. Although they play a crucial role in this project, these are not general standard tools on the field of complex dynamics.

In Section 2, we study inner functions from an analytic point of view, i.e. without considering their iteration. We are concerned with their extension to the boundary $\partial\mathbb{D}$ and especially the points where this extension is not possible, the so-called *singularities*. Moreover, we construct the *harmonic measure* of the unit circle $\partial\mathbb{D}$, a measure equivalent to the Lebesgue one which will be used to prove ergodic properties on the boundary, since it has better analytic properties.

Section 3 is basically devoted to prove the ergodic properties of inner functions, i.e. to prove the results stated in Table 1, following the approach of [DM91]. Moreover, the case of finite degree is studied separately. For finite degree inner functions of hyperbolic or simply-parabolic type, we give an alternative proof for the non-ergodicity and the non-recurrence. For finite degree inner functions of elliptic or doubly-parabolic type, we find a conjugacy between the inner function in the boundary and the function $x \mapsto dx \mod 1$. Although this cannot be use to deduce ergodic properties, it gives an accurate topological description of the dynamics on the boundary.

In Section 4, ergodic properties on the boundary of invariant simply-connected Fatou components, i.e. the results on Table 2, are proved. In the case of finite degree Fatou components with locally connected boundary, we give a more concrete characterization of the dynamics conjugating them to the function $x \mapsto dx \mod 1$. Moreover, open questions are also stated.

In Section 5, the function $f(z) = z + e^{-z}$ is studied, and Theorem A is proved, step by step. The proof of this theorem requires the use of wide-ranging tools, from Koebe's Distorsion Theorem to the use of the inner function. At the end of the section, the conclusions we arrive to and other further questions are gathered, as well as possible strategies to deal with them.

Finally, there is an appendix gathering more material which is not essential for the comprehension of the thesis but gives a deeper perspective on the topic. Due to the relevance of the Fatou, Riez and Riez theorem on radial limits (Thm. 2.1.2) we include its proof in Appendix A. Next, Appendix B is devoted to provide examples of all the types of inner functions described during the project. Most of them are due to J. Aaronson ([Aar81]), D. Bargmann ([Bar08]) and C. Doering and R. Mañé ([DM91]); however there are some examples made on purpose for this project. Due to the impossibility of finding an explicit reference for many of them, this section gathering all the examples is included. They illustrate the results proved in Section 3 and help to understand how iterates behave in practice. Finally, in Appendix C we collect some examples where the inner function has been computed explicitly. They show how theoretical results described in Section 4 are applied in practical examples.

1 Preliminaries

1.1 Iteration of rational and entire functions

Consider $f\colon \mathbb{C} \to \mathbb{C}$ an entire function, which is always assumed to be neither constant nor a linear transformation. Given any $z \in \mathbb{C}$, the goal is to understand the asymptotic behaviour of the sequence $\{f^n(z)\}_n$ when $n \to \infty$. To do so, there is a natural division of $\mathbb{C}$ into two invariant sets: the Fatou set (the set where iterates behave in a stable way) and the Julia set (the set of chaotic behaviour). A formal definition is as follows.

Definition 1.1.1. Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function and $z_0 \in \mathbb{C}$. If there is a neighbourhood U of z_0 so that the sequence of iterates $\left\{f^n_{|U}\right\}_n$ forms a normal family, then z_0 belongs to the **Fatou set** ($\mathcal{F}(f)$). Otherwise, if no such neighbourhood exists, then z_0 belongs to the **Julia set** ($\mathcal{J}(f)$).

This definition can be extended to rational functions, i.e. holomorphic functions in $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, just considering the iteration in $\widehat{\mathbb{C}}$. Observe that, for rational maps, ∞ plays no distinguished role, so it can be iterated as if it were any other point. For transcendental entire functions, i.e. when ∞ is an essential singularity, there is no reasonable way to define $f(\infty)$, so ∞ cannot be iterated and $\infty \in \mathcal{J}(f)$. This special nature of ∞ leads to the following definition.

Definition 1.1.2. Let $f\colon \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and let $z_0 \in \mathbb{C}$. If $f^n(z) \to \infty$, when $n \to \infty$, we say that z_0 belongs to the **escaping set** ($\mathcal{I}(f)$).

A useful criterion to know when a family $\left\{f^n_{|U}\right\}_n$ is normal (and hence, U belongs to the Fatou set) is the so-called Montel's theorem. We include also a criterion to know when a family is not normal.

Theorem 1.1.3. [Montel [Mil06, Thm. 3.7]] *Let $\{f_n\}_n$ be a family of holomorphic functions defined in Ω. If there exists $a, b, c \in \widehat{\mathbb{C}}$ such that $\{f_n\}_n$ omits this three values, then $\{f_n\}_n$ is normal in Ω.*

Theorem 1.1.4. (No bounded subsequences) [Mil06, Corollary 14.3] *Let f be a holomorphic map, let $z \in \mathcal{J}(f)$ and let U be a neighbourhood of z. Then, there is no subsequence $\{f^{n_k}\}_k$ such that $f^{n_k}(U)$ is uniformly bounded.*

The next proposition summarizes the main properties of the Fatou and the Julia sets, which will be used through the following sections.

Proposition 1.1.5. *Let f be either a rational function or a transcendental entire function. The following holds:*

1. *The Fatou set $\mathcal{F}(f)$ is open and the Julia set $\mathcal{J}(f)$ is closed.*
2. *For any integer $n \geq 1$, $\mathcal{F}(f^n) = \mathcal{F}(f)$ and $\mathcal{J}(f^n) = \mathcal{J}(f)$.*
3. *If f is rational, the Fatou and the Julia set are **completely invariant**, i.e. $f^{-1}(\mathcal{F}(f)) = \mathcal{F}(f) = f(\mathcal{F}(f))$ and $f^{-1}(\mathcal{J}(f)) = \mathcal{J}(f) = f(\mathcal{J}(f))$.*

   If f is a transcendental entire function, then $f(\mathcal{F}(f)) \subset \mathcal{F}(f)$ and $f(\mathcal{J}(f)) \subset \mathcal{J}(f)$.
4. *The Julia set $\mathcal{J}(f)$ is always non-empty.*
5. *Either $\mathcal{J}(f) = \widehat{\mathbb{C}}$ or $\mathcal{J}(f)$ has empty interior.*

6. *$\mathcal{J}(f)$ satisfies the* ***blow-up property****: for $z \in \mathcal{J}(f)$ and U a neighbourhood of z, $\left\{ f^n_{|U} \right\}_n$ omits at most two points in $\widehat{\mathbb{C}}$.*

7. *The Julia set is perfect (it has no isolated points).*

8. *Attracting fixed points and Siegel points belong to the Fatou set, whereas parabolic and repelling fixed points and Crémer points belong to the Julia set.*

For more properties of the iteration of rational functions, we refer to [Bea91; CG93; Mil06]; whereas [Ber93; HY98] are specific references for transcendental entire functions.

To study the Fatou set $\mathcal{F}(f)$, one shall consider the connected components of it, the so-called **Fatou components**. From the invariance of the Fatou set, one deduces that Fatou components are mapped among themselves, so they can be periodic, pre-periodic or wandering. Rational maps cannot have wandering domains ([Sul85]). For transcendental entire functions, wandering domains may appear ([Bak76]). On the other hand, for entire functions (either polynomial or transcendental), periodic Fatou components are always simply-connected ([Bak84]). Rational functions may have multiply-connected Fatou components (compare with Thm. 3.3.1).

As stated in the introduction, our goal is to study the boundary of the periodic simply-connected ones. To do so, the first step is to study the dynamics inside the Fatou component, and by conjugating with the Riemann map, it is enough to study the dynamics in the unit disk. Section 1.2 is devoted to present the Riemann Mapping Theorem and some important results on it, whereas the dynamics of self-maps of the unit disk are studied in Section 1.3.

Before moving on, we gather some results about the linearization around attracting and parabolic fixed points.

1.1.1 Attracting fixed points and linearizing coordinates

Recall that an **attracting fixed point** for f is a fixed point z_0 such that its **multiplier** $\lambda := f'(z_0)$ is smaller than 1 in modulus. Such a fixed point is called attracting because, in a small enough neighbourhood of it, points converge to it under iteration. When $\lambda \neq 0$, Koenigs' theorem ensures that, in a small enough neighbourhood of the attracting fixed point, orbits behave like the linear map $z \mapsto \lambda z$.

Theorem 1.1.6. (Linearizing coordinates) [Koenigs] *Let V be a domain in $\widehat{\mathbb{C}}$ and let $f\colon V \to f(V)$ be holomorphic. Let $z_0 \in V$ be an attracting fixed point and λ be its multiplier. Let $\mathcal{A}$ be the basin of attraction of z_0 (in V). Then:*

1. *There exists a conformal change of coordinates $w = \sigma(z)$, (the* ***linearizing coordinates****), defined in a neighbourhood V' of z_0 , with $\sigma(z_0) = 0$, so that $\sigma \circ f \circ \sigma^{-1}$ is the linear map $w \mapsto \lambda w$. Moreover, σ is unique up to multiplication by a non-zero constant.*

2. *The change of coordinates σ can be extended to a holomorphic map $\sigma\colon \mathcal{A} \to \mathbb{C}$ so that $\sigma f(z) = \lambda \sigma(z)$, for all $z \in \mathcal{A}$.*

A detailed exposition and a proof of Koenigs' theorem can be found, for instance, in [Ste93, Chapter 3.4] or [Mil06, Chapter 8].

When $\lambda = 0$ (the so-called super-attracting fixed points), although the fixed point remains being attracting, the map is no longer invertible around it. Koenig's coordinates are no longer valid to describe the map in a neighbourhood of it. However, there exists an alternative construction of coordinates due to Böttcher (e.g. [Ste93, Chapter 3.4] or [Mil06, Chapter 9]). We do not make use of them during the project, so we omit the details.

1.1.2 Parabolic fixed points and Fatou coordinates

A **parabolic fixed point** for f is a fixed point whose multiplier λ is a root of the unity, i.e. $\lambda = e^{i2\pi\frac{p}{q}}$, $p, q \in \mathbb{Z}$. The dynamics around such a fixed point are well-understood, and we present them next.

Definition 1.1.7. Let z_0 be a parabolic fixed point of a holomorphic map f. Consider a neighbourhood N of z_0 such that $f\colon N \to f(N)$ is a diffeomorphism. A connected, simply-connected open set $\mathcal{P}$, with compact closure $\overline{\mathcal{P}} \subset N \cap N'$ is called an **attracting petal** for f at z_0 if $f(\overline{\mathcal{P}}) \subset \mathcal{P} \cup \{z_0\}$ and $\bigcap_{k\geq 0} f^k(\overline{\mathcal{P}}) = \{z_0\}$.

An attracting petal for f^{-1} is called a **repelling petal**.

Theorem 1.1.8. (Leau-Fatou Flower Theorem) *Let V be a domain in $\widehat{\mathbb{C}}$ and $f\colon V \to f(V)$ holomorphic. Let z_0 be a parabolic fixed point of multiplicity $n+1$, i.e. there is $a \neq 0$ so that:*

$$f(z_0) = z_0 + (z - z_0) + a(z - z_0)^{n+1} \ldots$$

Then, there exists n disjoint attracting petals $\mathcal{P}_i$ and n disjoint repelling petals $\mathcal{P}'_i$, so that the union of these $2n$ petals and $\{z_0\}$ form a neighbourhood V' of z_0. Attracting and repelling petals alternate with each other, so that $\mathcal{P}_i$ only intersects $\mathcal{P}'_i$ and $\mathcal{P}'_{i-1}$.
Moreover, there exists n equally spaced attracting vectors $v_0, \ldots, v_n$ (that is, unitary vectors such that $nav_j^n = -1$). Then, if z converges to z_0 non-trivially, then $f^k(z)$ is asymptotic to $z_0 + \frac{v_j}{\sqrt[n]{k}}$, as $k \to \infty$, for some $0 \leq j \leq n$.

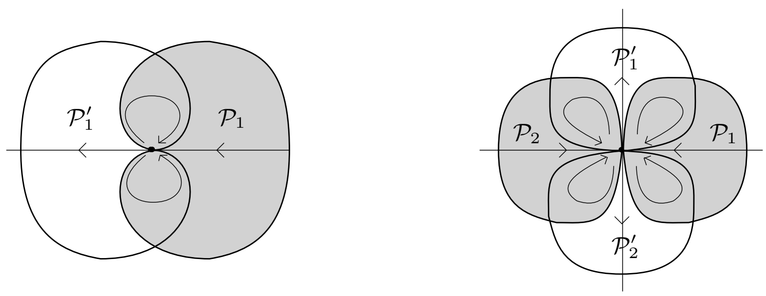


Figure 2: Arrangement of the attracting and repelling petals with $n = 1, 2$.

Therefore, each petal $\mathcal{P}_i$ determines a parabolic basin of attraction $\mathcal{A}_i$ consisting of all the points z such that its orbit eventually lands in the attracting petal $\mathcal{P}_i$ and hence it converges to the fixed point through $\mathcal{P}_i$. Clearly, $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are disjoint open sets, and they contain all the points $z \in \mathbb{C}$ that converge to z_0 is a non-trivial way.

The following theorem provides the existence of a conjugacy between f and the simpler linear map $w \mapsto w + 1$ in each of the petals, in a similar way to Koenigs' linearization theorem.

Theorem 1.1.9. (Fatou coordinates) *Let V be a domain in $\widehat{\mathbb{C}}$ and $f\colon V \to f(V)$ holomorphic. Let z_0 be a parabolic fixed point of f. Let $\mathcal{P}$ be an attracting petal at z_0 and $\mathcal{A}$ its corresponding parabolic basin. Then:*

1. *There exists a conformal change of coordinates $w = \sigma(z)$ defined in $\mathcal{P}$ (the* ***Fatou coordinates****), so that $\sigma(z_0) = \infty$ and $\sigma \circ f \circ \sigma^{-1}$ is the linear map $w \mapsto w + 1$ and $\sigma(U_i)$ contains a right half-plane. This change of variables is unique up to conformal conjugacy.*

2. *The map σ extends uniquely to a holomorphic map $\sigma\colon \mathcal{A} \to \mathbb{C}$, not necessarily injective, defined in the whole basin and satisfying $\sigma(f(z)) = \sigma(z) + 1$.*

For a complete exposition of the topic and the proofs of the theorems compare, for instance, [Ste93, Chapter 3.5] or [Mil06, Chapter 10].

1.2 Conformal mappings

Recall that conformal maps are the ones that are both holomorphic and univalent. Firstly, we deal with the conformal maps of the unit disk, the so-called Riemann maps, and its extension to the boundary. Secondly, we include some results on the distorsion induced by conformal maps.

1.2.1 Riemann maps and boundary behaviour

One of the key results in complex analysis asserts that any two simply-connected regions are conformally equivalent. This is known as the Riemann's Mapping Theorem.

Theorem 1.2.1. (Riemann's Mapping Theorem) [[Ahl94, Thm. 6.1]] *Let $U \subset \widehat{\mathbb{C}}$ be a non-empty simply-connected open set, such that $\widehat{\mathbb{C}} \setminus U$ consists of more than one point. Then, there exists a conformal map φ, called the* ***Riemann map****, from the unit disk $\mathbb{D}$ onto U. Moreover, given $z_0 \in U$, then there exists a unique Riemann map $\varphi\colon \mathbb{D} \to U$, with $\varphi(0) = z_0$, up to precomposition with rigid rotations of $\mathbb{D}$.*

Many properties of the Riemann map can be studied, but one of special interest is the extension of φ to the boundary $\partial\mathbb{D}$ and if this extension, when possible, is continuous. Indeed, this question was addressed by C. Carathéodory and it only depends on the topological properties of the boundary of U, more concretely, on its local connectivity.

Definition 1.2.2. A Hausdorff space X is said to be **locally connected** if every $x \in X$ has a arbitrarily small connected (but not necessarily open) neighbourhood.

Theorem 1.2.3. (Extension of the Riemann map) [Carathéodory [Pom92, Thm. 2.1]] *The Riemann map $\varphi\colon \mathbb{D} \to U$ has a continuous extension to $\overline{\mathbb{D}}$ if and only if ∂U is locally connected.*

In the case that the extension is not continuous, one can ask if at least the radial limits of the Riemann map exists. Recall that, given a conformal map $\varphi\colon \mathbb{D} \to U$, the radial limit φ^* at a point $e^{i\theta} \in \partial\mathbb{D}$ is the limit of the function φ taking along the radius (compare with Definition 2.1.1). It is clear that, if the boundary is locally connected, radial limits exist at every point, but it turns out that λ-almost every radial limit always exists, where λ denotes the (normalized) Lebesgue measure on $\partial\mathbb{D}$, even if ∂U is not locally connected. Here we give the statement of the theorem and its proof can be found in Appendix A.

Theorem 1.2.4. (Existence of radial limits) [Fatou, Riez and Riez] *Let $\varphi\colon \mathbb{D} \to U$ be conformal. Then, for λ-almost every θ, the radial limit*

$$\varphi^*(e^{i\theta}) \coloneqq \lim_{r\to 1^-} \varphi(re^{i\theta})$$

exists. Moreover, fixed $e^{i\theta}$ for which the radial limit exists, then $\varphi^*(e^{i\theta}) \neq \varphi^*(e^{i\xi})$ for λ-almost every ξ.

The fact that radial limits exist almost everywhere leads to consider the notion of accessible points, i.e. points that can be reached from inside U with a curve. The formal definition is as follows.

Definition 1.2.5. Given an open subset $U \subset \widehat{\mathbb{C}}$, a point $v \in \partial U$ is **accessible** from U if there is a path $\gamma\colon [0,1) \to U$ such that $\lim_{t\to 1} \gamma(t) = v$. We also say that γ **lands** at v, and write $\gamma(1) = v$.

Let $z_0 \in U$ and let $v \in \partial U$ be an accessible point. A homotopy class (with fixed endpoints) of curves $\gamma\colon [0,1] \to \widehat{\mathbb{C}}$ such that $\gamma([0,1)) \subset U$, $\gamma(0) = z_0$ and $\gamma(1) = v$ is called an **access** from U to v.

Apart from the trivial relation that if the radial limit exists at some point $e^{i\theta} \in \partial\mathbb{D}$ and equals $v \in U$ then v is accessible, more is true.

Theorem 1.2.6. (Correspondence Theorem) [[BFJK15]] *Let $U \subset \widehat{\mathbb{C}}$ be a simply-connected domain and let $v \in \partial U$. Then, there is a one-to-one correspondence between accesses from U to v and the points $e^{i\theta} \in \mathbb{D}$ such that $\varphi^*(e^{i\theta}) = v$. The correspondence is given as follows.*

- *If $\mathcal{A}$ is an access to $v \in \partial U$, then there is a point $e^{i\theta} \in \partial\mathbb{D}$ with $\varphi^*(e^{i\theta}) = v$. Moreover, different accesses correspond to different points in $\partial\mathbb{D}$.*
- *If, at a point $e^{i\theta} \in \partial\mathbb{D}$, the radial limit φ^* exists and it is equal to $v \in \partial U$, then there exists an access $\mathcal{A}$ to v. Moreover, for every curve $\eta \subset \mathbb{D}$ landing at $e^{i\theta}$, if $\varphi(\eta)$ lands at some point $w \in \widehat{\mathbb{C}}$, then $w = v$ and $\varphi(\eta) \in \mathcal{A}$.*

In the major part of this thesis, since we are interested in the ergodicity and the recurrence on the boundary of Fatou components, we do not really need deep results on the boundary behaviour of the Riemann map. Indeed, it suffices to put in ∂U the appropriate measure, as it is explained in Section 2.5.1. However, we have included this brief introduction to the topic for completeness and because it helps to understand the limitations of our study. In particular, with the considered measure, non-accessible points have measure zero, so we do not have any control on them.

For a general exposition on this wide field of research, including prime ends, cluster sets and accesses to infinity, we refer e.g. to [CL66; Mil06; Pom92]; whereas in [BD99; BFJK15; Bar08] there is a deep study on the connections of properties of the inner functions with the extension of the Riemann map and accesses to infinity from the Fatou component.

1.2.2 Distorsion of conformal maps

Next we include some results of distorsion of conformal maps. The first result is the statement of Koebe's Distorsion Theorem, which basically states that the distorsion of a conformal map is bounded. The second result is a consequence of the first one, often used in complex dynamics, and in particular, used here to prove accessibility in the example of Section 5.

Theorem 1.2.7. (Koebe's Distorsion Theorem) [[McM94, Thm. 2.7]] *Let $a \in \widehat{\mathbb{C}}$ and let $\varphi\colon D(a,1) \to U$ be conformal. For every $r \in (0,1)$ there exists a constant $k = k(r) \geq 1$ such that $k(r) \to 1$, as $r \to 0$ and for all $x, y \in D(a,r)$,*

$$\frac{1}{k}\left|\varphi'(a)\right| \leq \frac{\varphi(x)-\varphi(y)}{x-y} \leq k\left|\varphi'(a)\right|.$$

Observe the remarkable fact that k does not depend on the map φ but only on r.

Theorem 1.2.8. [[McM94, Thm. 2.9]] *Let U, U' be two topological disks in $\widehat{\mathbb{C}}$ such that $\overline{U'} \subset U$, let $a \in U'$ and let $\varphi\colon U \to V \subset \widehat{\mathbb{C}}$ be conformal. Then, there exists a constant $k > 0$ independent of φ such that*

$$D(\varphi(a), k \cdot \text{diam}\varphi(U')) \subset \varphi(U').$$

This last result asserts that, inside $\varphi(U')$ one can always find a disk of diameter proportional to the diameter of $\varphi(U')$, where the constant of proportionality k does not depend on φ. Indeed, it can be seen that k depends on the point a and the modulus of the annulus $U \smallsetminus \overline{U'}$. In particular, given a family of conformal maps $\varphi_n \colon U \to \widehat{\mathbb{C}}$ and $\overline{U'} \subset U$, this result claims that the sets $\varphi_n(U')$ cannot progressively degenerate to a segment: the limit of the sets $\varphi_n(U')$ must be a point, or contain a disk.

1.3 Dynamics of holomorphic functions in the unit disk $\mathbb{D}$

As we will see, the dynamics in the unit disk $\mathbb{D}$ are fully understood by the following results due to Denjoy, Wolff and Cowen. We remark that these results are valid for any holomorphic function in $\mathbb{D}$, not only for inner functions.

1.3.1 The Denjoy-Wolff theorem

The following result due to A. Denjoy ([Den26]) and J. Wolff ([Wol26a]) summarizes the behaviour of the iterates of a holomorphic function in $\mathbb{D}$.

Theorem 1.3.1. [Denjoy-Wolff] *Let g be an holomorphic self-map of the unit disk $\mathbb{D}$, not conjugate to a rotation. Then, there is $p \in \overline{\mathbb{D}}$, such that $\forall z \in \mathbb{D}$, $g^n(z) \to p$.*

*The point p is called the **Denjoy-Wolff point** of g.*

It is clear that if $p \in \mathbb{D}$, it is a fixed point of g. It is not that clear how to deal with it when it is in the boundary $\partial\mathbb{D}$. Now we state some results related with the radial limits and derivatives at the Denjoy-Wolff point that we will use later on. Recall that the radial limit g^* and the radial derivative $(g^*)'$ at a point $e^{i\theta}$ are defined to be the limit of the function and of the derivative, respectively, taken along the radial line (compare with Definition 2.1.1).

Theorem 1.3.2. (Limit at the Denjoy-Wolff point) [Wolff [Wol26b]] *Let $g \colon \mathbb{D} \to \mathbb{D}$ be holomorphic without fixed points in $\mathbb{D}$. Let $p \in \partial\mathbb{D}$ be the Denjoy-Wolff point of g. If $D \subset \mathbb{D}$ is a disk tangent to $\partial\mathbb{D}$ at p, then $g(D) \subset D$. Therefore, the radial limit $g^*(p)$ exists and it is equal to p.*

Theorem 1.3.3. (Derivative at the Denjoy-Wolff point) [Julia-Wolff [Pom92, p.82]] *Let $g \colon \mathbb{D} \to \mathbb{D}$ be holomorphic with Denjoy-Wolff point $p \in \partial\mathbb{D}$. Then, the radial derivative $(g^*)'(p)$ is real and $0 < (g^*)'(p) \leq 1$.*

Combining the previous theorems, one gets a stronger version of Theorem 1.3.2, as follows.

Theorem 1.3.4. [[BMS05, Section 2.3]] *Let $g \colon \mathbb{D} \to \mathbb{D}$ be holomorphic without fixed points in $\mathbb{D}$. Let $p \in \partial\mathbb{D}$ be the Denjoy-Wolff point of g. Consider $H(p,\lambda)$ to be the tangent disk at p with radius $\frac{\lambda}{2}$, i.e.*

$$H(p,\lambda) = \left\{ z \in \mathbb{D} \colon\ |p - z|^2 < \lambda(1 - |z|^2) \right\}.$$

Then, $g(H(p,\lambda)) \subset H(p, f'(p)\lambda)$.

Finally, other (radial) fixed points in the boundary are characterized for being (radially) repelling.

Theorem 1.3.5. (Fixed points are repelling) [[Cow81, lemma 2.4]] *Let $g \colon \mathbb{D} \to \mathbb{D}$ be holomorphic with Denjoy-Wolff point $p \in \partial\mathbb{D}$. Then, if there exists $q \in \partial\mathbb{D}$, with $q \neq p$, such that $g^*(q) = q$, then radial derivative at q satisfies to be real and $(g^*)'(q) > 1$.*

1.3.2 Absorbing domains and Cowen's classification

Now that we know where the orbits converge to, we can ask *how* the orbits converge. We want to find a 'linearization' of the function g around the Denjoy-Wolff point p, in some sense similar to what we did in Sections 1.1.1 and 1.1.2, but taking into account that g may not be defined at p. To do so, we start by defining the kind of neighbourhoods of p we are going to work with.

Definition 1.3.6. Let f be a map of a domain $\Delta \subset \mathbb{C}$ into itself. We say V is a **absorbing domain** for f on Δ if V is an open, connected, simply connected subset of Δ such that: $f(V) \subset V$ and for every compact set $K \subset \Delta$, there is a positive integer n so that $f^n(K) \subset V$.

Absorbing sets are often also called *fundamental sets*. Observe that petals defined in Section 1.1.2 are absorbing domains, but the definition of absorbing domain is weaker in the sense that we do not require $\overline{f(V)} \subset V \cup \{p\}$. The following theorem deals with the existence of absorbing domains, even in the case of $p \in \partial\mathbb{D}$ and g not defined at p.

Theorem 1.3.7. (Existence of absorbing domains)[Cowen [Cow81]] *Let g be a holomorphic self-map of the unit disk $\mathbb{D}$, not conjugate to a rotation, with Denjoy-Wolff point p and $g'(p) \neq 0$. Then there exist an absorbing domain V for g on $\mathbb{D}$, a domain Ω, which can be either the complex plane $\mathbb{C}$ or the upper half-plane $\mathbb{H}$, a Möbius transformation ϕ mapping Ω onto Ω and an analytic map σ from $\mathbb{D}$ to Ω such that σ and g are univalent in V, $\sigma(V)$ is an absorbing domain for ϕ on Ω and $\sigma \circ g = \phi \circ \sigma$. In this case, we write $g \sim \phi$.*

$$\begin{array}{ccc} V \subset \mathbb{D} & \xrightarrow{\quad g \quad} & V \subset \mathbb{D} \\ \downarrow \sigma & & \downarrow \sigma \\ \sigma(V) \subset \Omega & \xrightarrow{\quad \phi \quad} & \sigma(V) \subset \Omega \end{array}$$

Moreover, ϕ is unique up to conjugation by a Möbius transformation mapping Ω onto Ω, and ϕ and σ depend only on g, not on the particular absorbing domain V.

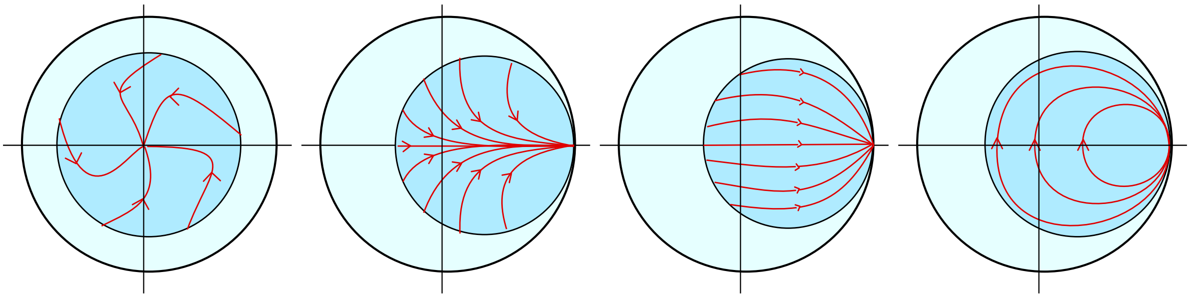
From this theorem it is easily deduced that, in the case that g is not conjugate to a rotation, only four cases can occur. Indeed, g is of one of the following types:

1. **Elliptic:** if $p \in \mathbb{D}$, with $0 < |g'(p)| < 1$ and $g \sim g'(p)\mathrm{id}_{\mathbb{C}}$.
2. **Doubly-parabolic:** if $p \in \partial\mathbb{D}$, with $g'(p) = 1$ and $g \sim \mathrm{id}_{\mathbb{C}} + 1$.
3. **Hyperbolic:** if $p \in \partial\mathbb{D}$, with $0 < g'(p) < 1$ and $g \sim g'(p)\mathrm{id}_{\mathbb{H}}$.
4. **Simply-parabolic:** if $p \in \partial\mathbb{D}$, with $g'(p) = 1$ and $g \sim \mathrm{id}_{\mathbb{H}} + 1$.

The elliptic type is clearly distinguished from the others, because it is the only one with $p \in \mathbb{D}$. To differentiate the other ones, we will use the following criterion, found in [Bar08; Kö99]. We denote by ρ the hyperbolic distance in the unit disk $\mathbb{D}$.

Theorem 1.3.8. *Let $g\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic self map of the unit disk, with Denjoy-Wolff point $p \in \partial\mathbb{D}$. Then, the following are equivalent:*

1. *The map g is of doubly-parabolic type.*
2. *There exists $z \in \mathbb{D}$ such that $\lim\limits_{n\to\infty} \rho(g^n(z), g^{n+1}(z)) = 0$.*
3. *For all $z \in \mathbb{D}$ there holds $\lim\limits_{n\to\infty} \rho(g^n(z), g^{n+1}(z)) = 0$.*



(a) *Elliptic* (b) *Doubly-parabolic* (c) *Hyperbolic* (d) *Simply-parabolic*

Figure 3: The different types of convergence to the Denjoy-Wolff point.

The following equivalence will also be useful.

Theorem 1.3.9. [[DM91, Theorem 3.1]] *Let $g\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic self map of the unit disk. Then, the following are equivalent:*

1. *There exists $z, w \in \mathbb{D}$ with $g^n(z) \neq g^n(w)$ for all $n \geq 0$ such that $\lim_{n\to\infty} \rho(g^n(z), g^n(w)) = 0$.*
2. *For all $z, w \in \mathbb{D}$, there holds $\lim_{n\to\infty} \rho(g^n(z), g^n(w)) = 0$.*

1.3.3 Application: dynamics in the interior of an invariant Fatou component

Given an invariant simply-connected Fatou component U of f, i.e. $f\colon U \to U$, then any Riemann map $\varphi\colon \mathbb{D} \to U$ conjugates the dynamics of f in U to those of a holomorphic self-map of $\mathbb{D}$. Therefore, applying the Denjoy-Wolff Theorem, one can classify the Fatou components according to its internal dynamics.

Theorem 1.3.10. (Classification of Fatou components) [Fatou, [Fat26]] *Let f be either a rational or an entire function and U be an invariant simply-connected Fatou component. Then exactly one of the following holds:*

1. *U contains an attracting fixed point z_0 and $f^n \to z_0$ uniformly on compact subsets of U. Then U is the* ***immediate basin of attraction*** *of z_0.*
2. *∂U contains a parabolic fixed point z_0 and $f^n \to z_0$ uniformly on compact subsets of U. Then U is the* ***immediate parabolic basin*** *of z_0.*
3. *There exists $z_0 \in U$, an irrationally neutral periodic point and $f_{|U}$ is conformally conjugate to an irrational rotation. Then, U is a* ***Siegel disk****.*
4. *If f is transcendental, U can also be a* ***Baker domain****. That is $f^n(z) \to \infty$ uniformly on compact subsets of U.*

Combining the results of Sections 1.1.1, 1.1.2 and 1.3.2, we get that the corresponding self-map g of the unit disk associated to an attracting basin U must be of elliptic type, and the one of a parabolic basin, of doubly-parabolic type. For Baker domains, each type can occur and this establishes a classification among them, as shows the following theorem.

Theorem 1.3.11. (Classification of Baker domains) [König, [Kö99]] *Let U be a Baker domain of f and $V \subset B$ an absorbing domain for f in U. Then, taking $\Omega = \mathbb{C}$ or $\Omega = \mathbb{H}$, there exists a map $\psi\colon U \to \Omega$, which is one-to-one in V, and a Möbius transformation $\phi\colon \Omega \to \Omega$, such that $\psi \circ f = \phi \circ \psi$. Moreover, Ω is unambiguously determined and ϕ is unique up to conjugacy, and they can be chosen among the following:*

1. $\Omega = \mathbb{C}$ *and* $\phi(z) = z + 1$. *In this case, we say that* U *is **doubly-parabolic**.*

2. $\Omega = \mathbb{H}$ *and* $\phi(z) = sz$, *with* $0 < s < 1$. *In this case, we say that* U *is **hyperbolic**.*

3. $\Omega = \mathbb{H}$ *and* $\phi(z) = z \pm 1$. *In this case, we say that* U *is **simply-parabolic**.*

1.4 Harmonic functions and the Dirichlet problem

One of the main tools that we will be using throughout this project is the harmonic measure. As we will see, one of the advantages of working with this measure is that it is a harmonic function and, therefore, all the results for harmonic functions apply. Next we state the definition of harmonic function and the properties needed. These results, and their proofs, can be found in [Con73; Rud87].

Definition 1.4.1. Let D be an open subset of $\mathbb{C}$. A function $h\colon D \to \mathbb{R}$ is **harmonic** if u has continuous second partial derivatives and

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0.$$

It is easy to see that, given a holomorphic function $f(z) = u(z) + iv(z)$ in a domain D, its real part $u(z)$, its imaginary part $v(z)$ and its modulus $|f|$ are harmonic in D. In particular, the function

$$P(z, w) = \operatorname{Re}\left(\frac{w+z}{w-z}\right) = \frac{1-|z|^2}{|w-z|^2}$$

is harmonic in $\mathbb{C} \setminus \{w\}$. This function is called the **Poisson kernel** of z at w and it will become relevant later.

For harmonic functions we have an analogous version of the maximum modulus principle for holomorphic functions.

Theorem 1.4.2. (Maximum principle for harmonic functions) *Let* D *be a domain and let* h *be harmonic in* D. *If* h *attains a local maximum (or minimum) in* D, *then* h *is constant. Moreover, assume* h *extends continuously to* $\overline{D}^{\infty}$ *(i.e., to the closure of* D *with respect to* $\widehat{\mathbb{C}}$*) and* $h \leq C$ *on* ∂D. *Then* $h \leq C$ *on* ∂D.

One may ask if, given a continuous real-valued function h defined in the boundary of some domain, there exists an harmonic function on this domain attaining the same values as h at the boundary. This is known as the Dirichlet problem, and a formal definition is as follows.

Definition 1.4.3. Let D be a domain and $h\colon \partial D \to \mathbb{R}$ be continuous. The **Dirichlet problem** for h in D consists of finding $\widehat{h}$ harmonic in D such that $\lim_{\xi \to z} \widehat{h}(\xi) = h(z)$, for all $z \in \partial D$.

Directly from the maximum principle, on deduces that, in case $\widehat{h}$ exists, it is unique. Observe that the Dirichlet problem may not be solvable. For instance, let $D = \mathbb{D} \setminus \{0\}$ and define $h(0) = 1$ and $h \equiv 0$ in $\partial\mathbb{D}$. By the maximum principle, it is clear that this Dirichlet problem is not solvable.

Theorem 1.4.4. (Dirichlet problem in the unit disk) *Let* $h\colon \partial\mathbb{D} \to \mathbb{R}$ *be continuous. Then, the Dirichlet problem for* h *in* $\mathbb{D}$ *is solvable and its solution is*

$$\widehat{h}(z) = \int_{\partial\mathbb{D}} h(w) P(z, w) d\lambda(w),$$

where $P(z, w)$ *is the Poisson kernel of* z *at* w *and* $d\lambda$ *is the normalized Lebesgue measure of* $\partial\mathbb{D}$.

By approximation of L^1-functions by continuous functions, it can be seen that the Poisson kernel also solves the Dirichlet problem in $\mathbb{D}$ for functions in $L^1(\partial\mathbb{D})$.

Before, we saw an example of a domain in which the Dirichlet problem is not solvable. In our case, we will be interested in simply-connected domains and, for those, it is always solvable.

Theorem 1.4.5. (Dirichlet problem in simply-connected domains) *If D is a simply-connected domain and $h \in L^1(\partial D)$, then the Dirichlet problem of h in D is solvable.*

Finally, we state Harnack's inequality, which gives control of harmonic functions in a given compact set.

Theorem 1.4.6. (Harnack's inequality) *Let $D \subset \mathbb{C}$ be open and let $K \subset D$ be compact. Assume h is a positive harmonic function in D. Then, there exists some constant C, depending only on D and K, such that, for all $z \in K$:*

$$\frac{1}{C} \leq h(z) \leq C.$$

1.5 Background on measure theory

Here we recall some basic definitions on measure theory that will be used throughout the project.

Definition 1.5.1. Consider a set X and a σ-algebra $\mathcal{A}$ on X. Then,

- the couple $(X, \mathcal{A})$ is a **measurable space**.
- a map $T\colon X \to X$ is **measurable** if for every $A \in \mathcal{A}$, there holds $T^{-1}(A) \in \mathcal{A}$.
- a **(positive) measure** on $(X, \mathcal{A})$ is a function $\mu\colon \mathcal{A} \to [0, +\infty)$ such that:
  1. $\mu(\emptyset) = 0$.
  2. if $\{A_i\}_i \subset \mathcal{A}$ is a countable collection of pairwise disjoint sets in $\mathcal{A}$, then $\mu(\cup_i A_i) = \sum_i \mu(A_i)$. This property is called **σ-additivity**.
- the triple $(X, \mathcal{A}, \mu)$, where μ is a measure on $(X, \mathcal{A})$ is a **measure space**.

Next we define some properties of the measure. Although all of them are stated for an arbitrary measure space, we will use them mainly for $(\partial\mathbb{D}, \mathcal{B}(\partial\mathbb{D}), \lambda)$, where $\mathcal{B}(\partial\mathbb{D})$ denotes the Borel σ-algebra of $\partial\mathbb{D}$ and λ its normalized Lebesgue measure.

Definition 1.5.2. Let $(X, \mathcal{A})$ be a measurable space and μ, μ_1, μ_2 measures in $(X, \mathcal{A})$. We say that:

- μ is **finite**, if $\mu(X) < \infty$.
- μ_1 is **absolutely continuous** with respect to μ_2, if for every set $A \in \mathcal{A}$ with $\mu_2(A) = 0$, there holds $\mu_1(A) = 0$.
- μ_1 and μ_2 are **equivalent**, if μ_1 is absolutely continuous with respect to μ_2 and μ_2 is absolutely continuous with respect to μ_1. That is, for every $A \in \mathcal{A}$:

$$\mu_1(A) = 0 \iff \mu_2(A) = 0.$$

Next we introduce the concept of Lebesgue density. In some sense, this formalizes the idea that, although a set of positive measure may not contain an interval, it must be thick, in some sense.

Definition 1.5.3. Let $X = \mathbb{R}^n$ and λ be the n-dimensional Lebesgue measure. Given any Borel subset $A \subset \mathbb{R}^n$, the **Lebesgue density** of A at x is defined as:

$$d_x(A) = \lim_{\varepsilon \to 0} \frac{\lambda(A \cap B(x, \varepsilon))}{\lambda(B(x, \varepsilon))}.$$

A point $x \in A$ is called a **Lebesgue density point** for A if $d_x(A) = 1$.

Theorem 1.5.4. (Positive measure sets are Lebesgue dense) *Let $X = \mathbb{R}^n$ and λ be the n-dimensional Lebesgue measure. Given any Borel subset $A \subset \mathbb{R}^n$ with $\lambda(A) > 0$, λ-almost every point in A is a Lebesgue density point for A.*

For a wider explanation and the proof of the stated result, see [Rud87, p. 138].

1.6 Ergodic properties of measurable maps

Here we give some basic definitions used in abstract ergodic theory and some results to deal with them. Obviously, this is only a small gathering from a wide area of knowledge. More details can be found in [Aar97; Haw21].

In some sense, ergodic theory tries to understand the dynamical systems with a measure point of view. Here it is not interesting the existence of certain types of orbits, but in which probability they take place. For instance, the existence of dense orbits, something which may be very interesting when studying chaos, is irrelevant if it only takes place for a set of measure zero. The goal of ergodic theory is to determine the most likely behaviour for a random point. The concepts of ergodicity, recurrence and invariance formalize this intuition.

Definition 1.6.1. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $T\colon X \to X$ be a measurable map. Then we say that T is:

- **ergodic** (with respect to μ), if for every $A \in \mathcal{A}$ such that $T^{-1}(A) = A$, there holds $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.
- **recurrent** (with respect to μ), if for every $A \in \mathcal{A}$ and μ-almost every $x \in A$, there exists an increasing infinite sequence of positive integers $\{n_k\}_k$ such that $T^{n_k}(x) \in A$.
- **μ-preserving**, if for every $A \in \mathcal{A}$, $\mu(T^{-1}(A)) = \mu(A)$. In this case we say that μ is **T-invariant**.

As stated above, being ergodic, recurrent or invariant are notions that depend on the measure that we are considering. If the measure we are using is the Lebesgue measure λ or it is clear the measure we are working with, we shall omit the dependence on the measure.

Since being ergodic or recurrent only involves sets of zero or full measure, if a map $T\colon X \to X$ is ergodic (or recurrent) with respect to some measure μ_1, it is also ergodic (or recurrent) with respect to μ_2, whenever μ_1 and μ_2 are equivalent. In particular, to prove that a map is ergodic (or recurrent) with respect to the Lebesgue measure, it is enough to see it for an equivalent measure. On the contrary, the property of being μ-preserving is not transfered by equivalent measures.

Moreover, these ergodic properties are preserved by absolutely continuous conjugacies. From the fact that a conjugacy is a homeomorphism, it preserved invariance, and assuming it is absolutely continuous, it sends zero measure sets to zero measure sets and sets of positive measure to sets of positive measure, so ergodicity and recurrence are preserved. However, one has to be careful because not every homeomorphism is absolutely continuous, so in general, topological conjugacies may not preserve the ergodic properties. In particular, conformal maps are absolutely continuous, because they are bi-Lipschitz, so they preserve ergodicity and recurrence.

Intuitively, one can relate ergodicity with the impossibility of reducing the dynamical system in independent parts and recurrence with the invariance of the dynamical system over time. In some sense, ergodicity is the equivalent of topological transivity, but from the measure point of view. In fact, since open subsets are Borel, topological transitivity is implied by ergodicity, but the converse may not be true.

The following is a criterion, due to Poincaré, that relates invariance and recurrence.

Theorem 1.6.2. (Poincaré Recurrence Theorem) *Let $(X, \mathcal{A}, \mu)$ be a measure space and $T\colon X \to X$ be a measurable function. Assume μ is finite and T-invariant. Then T is recurrent with respect to μ.*

Proof. Fixed $A \in \mathcal{A}$ we aim to see that the set

$$E \coloneqq \{x \in A \colon \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, T^n(x) \notin A\}$$

has measure zero. To do so, for every $n \in \mathbb{N}$ consider the set

$$A_n \coloneqq \bigcup_{k=n}^{\infty} T^{-k}(A).$$

It is clear from the definition that $A \subset A_0$ and $A_i \subset A_j$ if $j \leq i$. By assumption μ is T-invariant, so $\mu(A_i) = \mu(A_j)$, for all i, j. We have:

$$\mu(A \setminus A_n) \leq \mu(A_0 \setminus A_n) = \mu(A_0) - \mu(A_n) = 0.$$

Finally observe that $\bigcup_n A \setminus A_n$ is exactly E, so we are done. □

The result is not true if the measure we are considering is not finite. Consider, for instance, $X = \mathbb{R}$ with the Lebesgue measure λ. Let $T(x) = x + 1$ be the translation by one. Clearly, T is λ-preserving but not recurrent: all points tend to ∞ under iteration of T.

Note that in the previous proof we did not use the T-invariance of every set $A \in \mathcal{A}$, but only for the ones such that $T^{-1}(A) \subset A$. This allows us to deduce the following corollary, which is a characterization of recurrence when working with a finite measure.

Corollary 1.6.3. (Characterization of recurrence) *Let $(X, \mathcal{A}, \mu)$ be a measure space and $T\colon X \to X$ be a measurable function. Assume μ is finite. Then, T is recurrent if and only if, for every set $A \in \mathcal{A}$ such that $T^{-1}(A) \subset A$, it holds $\mu(A) = \mu(T^{-1}(A))$.*

Proof. By the reasoning above, it is only left to show the left-to-right implication. Observe that T is recurrent if, for any $A \in \mathcal{A}$, the set

$$\widetilde{A} \coloneqq \bigcap_{N \geq 0} \bigcup_{n \geq N} T^{-n}(A)$$

satisfies $\mu(\widetilde{A}) = \mu(A)$. Since we are assuming $T^{-1}(A) \subset A$, we have that $\widetilde{A} = \bigcap_{N \geq 0} T^{-n}(A)$. Finally, the fact that $\widetilde{A} \subset T^{-1}(A)$ implies that $\mu(A \setminus T^{-1}(A)) \leq \mu(A \setminus \widetilde{A}) = 0$, as desired. □

The Poincaré Recurrence Theorem gives us a strategy to prove that T is recurrent with respect to the Lebesgue measure λ, even in the case where T is not λ-preserving. Indeed, since invariance is not preserved by equivalent measures but so it is recurrence, we may find another measure μ which is T-invariant, finite and equivalent to λ. By the Poincaré Recurrence Theorem, T is recurrent with respect to μ and, by the equivalence between both measures, it is also recurrent with respect to λ.

From the dynamical point of view, the following consequence of ergodicity together with recurrence is remarkable, because it ensures the existence of dense orbits.

Theorem 1.6.4. (Dense orbits) *Let $(X, \mathcal{A}, \mu)$ be a measure space and $T\colon X \to X$ be a measurable function. The following are equivalent:*

1. *T is recurrent and ergodic.*
2. *For every $A \in \mathcal{A}$ of positive μ-measure, for μ-almost every $x \in X$, there exists an increasing infinite sequence of positive integers $\{n_k\}_k$ such that $T^{n_k}(x) \in A$.*

Note that, if open sets of X have positive μ-measure, the second condition implies that μ-almost every $x \in X$ has a dense orbit in X.

Proof. (1) ⇒ (2) : First observe that the set of points that satisfy condition (2) is the same as:

$$E := \left\{ x\colon \sum_{n=1}^{\infty} 1_A(T^n(x)) = \infty \right\}.$$

Clearly, $E \in \mathcal{A}$ and it is T-invariant. Since T is assumed to be recurrent, μ-almost every point in A is also in E, so E has positive measure. But T is also assumed to be ergodic, so this implies that $\mu(X \setminus E) = 0$. Therefore, (2) holds.

(2) ⇒ (1) : It is clear that, if T is non-recurrent condition (2) cannot hold. On the other hand, non-ergocity would imply the existence of a invariant set A of positive measure, with complement of positive measure. It is clear that condition (2) does not hold for A. □

Finally, observe that ergodicity and recurrence are independent notions, i.e. neither of them is implied by the other. Examples of the four cases are easy to find. Clearly, rational rotations on the unit circle are recurrent but not ergodic. In the following section we prove that the map $\theta \mapsto d\theta$ in the unit circle $\partial\mathbb{D}$ is ergodic and recurrent. The translation by one $x \mapsto x + 1$ on the extended real line $\mathbb{R} \cup \{\infty\}$ is non-ergodic and non-recurrent. For the non-ergodicity, notice that the set $\cup_{n\in\mathbb{Z}}(n, n + \frac{1}{2})$ is invariant. The same translation by one $x \mapsto x + 1$ but now on the set of integers endowed with the counting measure is ergodic but non-recurrent.

1.7 The map $x \mapsto dx$ in the unit circle $\partial\mathbb{D}$

In this section we study the map

$$\begin{aligned} T\colon \partial\mathbb{D} &\longrightarrow \partial\mathbb{D} \\ x &\longmapsto dx, \end{aligned}$$

where d is a positive integer. When $d = 2$, it is known as the **doubling map**. This is a very simple map: given a point in $\partial\mathbb{D}$, it multiplies the angle by d. However, it turns out to be a chaotic map (in Devaney's sense): it is topologically transitive, has sensitive dependence on initial conditions and periodic points are dense. This can be seen, for instance, semi-conjugating T to the shift on sequences of d symbols, where the map that takes a point to its expression in base d is the semi-conjugacy. For more details, we refer to [Dev89].

From the ergodic point of view, we are interested in more: our goal is to describe the behaviour of the *typical* orbit, with respect to the Lebesgue measure. Indeed, next we prove that λ-almost every point in $\partial\mathbb{D}$ has dense orbit. We remark that this is not trivial and it cannot be deduced directly from the fact that the map is chaotic. Even the fact that points with dense orbit are dense is not enough to deduce that λ-almost every orbit is dense.

Theorem 1.7.1. *The map* $T\colon x \mapsto dx \mod 1$ *is ergodic.*

Proof. Assume that A is a measurable set such that $T^{-1}(A) = A$. Our goal is to see that if $\lambda(A) < 1$, then $\lambda(A) = 0$. Clearly $\lambda(\partial\mathbb{D} \setminus A) > 0$, so by Theorem 1.5.4, there exists a Lebesgue density point in $\partial\mathbb{D} \setminus A$. Therefore, given $\varepsilon > 0$ one can choose n big enough so that

$$\frac{\lambda(I \setminus A)}{\lambda(I)} \geq 1 - \varepsilon.$$

Then we have:

$$\lambda(T^n(I \setminus A)) = d^n \lambda(I \setminus A) \geq d^n(1-\varepsilon)\lambda(I) = 1 - \varepsilon.$$

Finally, we shall see that $T^n(\partial\mathbb{D} \setminus A) \subset \partial\mathbb{D} \setminus A$, and letting $\varepsilon \to 0$, we deduce that $\lambda(\partial\mathbb{D} \setminus A) = 1$, as desired. Assume $x \in T^n(\partial\mathbb{D} \setminus A)$, so $x = T^n(y)$ for some $y \in \partial\mathbb{D} \setminus A$. This implies that $y \in T^{-n}(A) = A$, by invariance of A. This is a contradiction, so $T^n(\partial\mathbb{D} \setminus A) \subset \partial\mathbb{D} \setminus A$. □

Theorem 1.7.2. *The map* $T\colon x \mapsto dx \mod 1$ *is recurrent.*

Proof. It is easy to see that, for any measurable subset $I \subset \partial\mathbb{D}$, $T^{-1}(I)$ consist of d disjoint components of measure $\frac{\lambda(I)}{d}$. Therefore, T preserves the Lebesgue measure and, by the Poincaré Recurrence Theorem (1.6.2), T is recurrent. □

Finally, we observe that combining the last two properties, by Theorem 1.6.4, we get that λ-almost every point in $\partial\mathbb{D}$ has dense orbit. However, observe that there are infinitely many points with non-dense orbit (including the periodic and the preperiodic ones). This set should have λ-measure zero.

2 Inner functions

As stated in the introduction, inner functions are a powerful tool to study the dynamics inside and on the boundary of simply-connected Fatou components. In this chapter we will give a definition and some basic analytic results. Moreover, we will introduce the harmonic measure of $\partial\mathbb{D}$ and some results that relates it with the inner functions. For some of our purposes, it will be more convenient than the Lebesgue measure. Finally, we will discuss the relation between inner functions and invariant Fatou components.

2.1 Definition and first examples of inner functions

Recall that the reason we are considering self-maps of $\mathbb{D}$ is that the function f in an invariant Fatou component U is conjugate to some $g\colon \mathbb{D} \to \mathbb{D}$. Therefore, g has to mimic, in some sense, the behaviour of $f_{|U}$. From the invariance of the Fatou and the Julia sets, we have that $f(U) \subset U$ and $f(\partial U) \subset \partial U$. Therefore, it seems reasonable to ask that $g(\partial\mathbb{D}) \subset \partial\mathbb{D}$. However, the function g may not be defined in $\partial\mathbb{D}$, so the previous condition may not make sense. The way to relax it is by considering radial limits for the points in $\partial\mathbb{D}$.

Definition 2.1.1. Let $g\colon \mathbb{D} \to \mathbb{C}$ be holomorphic and $e^{i\theta} \in \partial\mathbb{D}$. The **radial limit** of g at $e^{i\theta}$ is defined as:

$$g^*(e^{i\theta}) = \lim_{r\to 1^-} g(re^{i\theta}).$$

The following theorem ensures the existence of radial limits λ-almost everywhere. Its proof is detailed in Appendix A.

Theorem 2.1.2. (Existence of radial limits) [Fatou, Riez and Riez] *Let g be a holomorphic bounded function of $\mathbb{D}$. Then, for λ-almost every θ, the radial limit $g^*(e^{i\theta})$ exists. Moreover, fixed $e^{i\theta}$ for which the radial limit exists, then $g^*(e^{i\theta}) \neq g^*(e^{i\xi})$ for λ-almost every ξ.*

Therefore, inner functions are defined to be the ones that satisfy $g^*(\partial\mathbb{D}) \subset \partial\mathbb{D}$ in the maximal set where we can assume g^* exists.

Definition 2.1.3. A holomorphic function $g\colon \mathbb{D} \to \mathbb{D}$ is an **inner function** if $\left|g^*(e^{i\theta})\right| = 1$, for λ-almost all θ.

A first observation is that an inner function cannot be constant. Indeed, if it were, we would have $g \equiv w \in \mathbb{D}$, so $\left|g^*(e^{i\theta})\right| = |w| < 1$ for all θ, so g cannot be an inner function.

The typical examples of inner functions are **Blaschke products**, i.e. functions of the form:

$$B(z) := e^{i\theta} \prod_{n=1}^{d} \frac{a_n - z}{1 - \overline{a_n} z},$$

where $\theta \in \mathbb{R}$, $d \in \mathbb{N} \cup \{\infty\}$ and $\{a_n\}_n$ is a sequence of points in $\mathbb{D}$, such that $\sum_n (1 - |a_n|) < \infty$. When $d < \infty$, we say that it is a **finite Blaschke product of degree** d; otherwise it is an **infinite Blaschke product**.

It is well-known that every Blaschke product is an inner function ([Gar07, Thm. II.2.2]). However, the converse is not true: there are inner functions that are not Blaschke products.

Example 2.1.4. (Inner function that is not a Blaschke product) Consider:

$$g(z) = \exp \frac{z+1}{z-1}.$$

This function is the composition of a Möbius transformation $M(z) = \frac{z+1}{z-1}$, which maps the unit disk $\mathbb{D}$ onto the left half-plane $\mathbb{H}^-$, with the exponential. Observe that it is an inner function, because all the points in $\partial\mathbb{D}$, except from 1, are mapped to a point in $\partial\mathbb{D}$. Clearly, it is not a Blaschke product because it has no zeros in $\mathbb{D}$. See Figure 4.

The function g is holomorphic in $\widehat{\mathbb{C}} \setminus \{1\}$. At 1, it is not defined, since $M(1) = \infty$ and ∞ is an essential singularity of the exponential. However, the radial limit at 1 can be easily computed, considering g restricted to the real line (which is invariant), so $g^*(1) = 0$.

Clearly g has no zeros in $\mathbb{D}$ but for all $w \in \mathbb{D}$, $w \neq 0$, and for all neighbourhood U of 1, there exists $z \in U$ with $f(z) = w$. That is, preimages of any point in $\mathbb{D} \setminus \{0\}$ accumulate at 1.

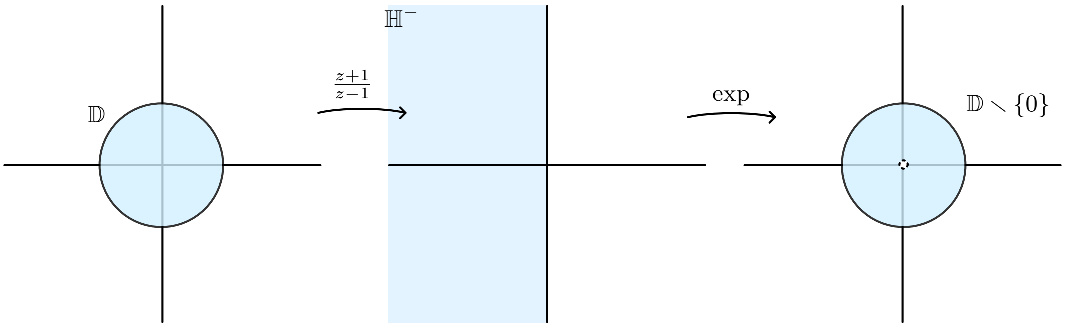


Figure 4: How g maps $\mathbb{D}$ onto $\mathbb{D} \setminus \{0\}$.

This kind of inner functions without zeros are called **singular inner functions**. Inner functions, including Blaschke products and singular inner functions, have been widely studied from an analytic point of view. We refer to [Con95] or [Gar07]. However, from a dynamical point of view, it is enough to study Blaschke products. Indeed, from the following theorem, one can deduce that every inner function is conjugate to a (finite or infinite) Blaschke product.

Theorem 2.1.5. **[Frostman** [Gar07, Thm. II.6.4]] *Let g be an inner function. Then, for all $w \in \mathbb{D}$, except for a set of zero logarithmic capacity,*

$$g_w(z) = \tau_w \circ g(z) = \frac{w - g(z)}{1 - \overline{w}g(z)}$$

is a Blaschke product, where $\tau_w(z) = \frac{w-z}{1-\overline{w}z}$ is a Möbius transformation of $\mathbb{D}$ sending w to 0.

We shall not discuss here the concept of logarithmic capacity, but we recall that sets of logarithmic capacity zero are extremely thin: they cannot contain connected sets of more than one point. In particular, zero logarithmic capacity implies zero λ-measure.

Corollary 2.1.6. (Every inner function is conjugate to a Blaschke product) *Let g be an inner function. Then $\tau_w \circ g \circ \tau_w$ is a Blaschke product, for all $w \in \mathbb{D}$, except for a set of zero logarithmic capacity.*

Proof. By Frostman's theorem, we know that $\tau_w \circ g$ is a Blaschke product. It is enough to see that, given a Blaschke product B, the function $B \circ \tau_w$ is always a Blaschke product. Since B is a product of automorphisms of $\mathbb{D}$, this follows immediately from the fact that automorphisms of $\mathbb{D}$ form a group with the composition. ☐

2.2 Singularities

It is clear that in Example 2.1.4 the point 1 plays a different role that the other points in $\partial\mathbb{D}$. Although the radial limit exists, it is the only point in $\widehat{\mathbb{C}}$ where the function is not holomorphic.

In this section we are going to describe precisely this kind of points, the ones where the inner function fails to be holomorphic in a neighbourhood of them.

From now on, whenever we refer to an inner function, we are considering it analytically extended to the maximal domain where this is possible. In particular, in $D' := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, the analytic extension is defined by the Schwarz reflection of g through the unit circle $\partial\mathbb{D}$, i.e. defining

$$g(z) := \frac{1}{\overline{g(\frac{1}{\overline{z}})}}, \quad \text{for } w \in D' := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

For a formal explanation of the Schwarz Reflection Principle, see [MH99, Section 6.1].

Definition 2.2.1. Let g be a inner function. A point $e^{i\theta} \in \partial\mathbb{D}$ is a **singularity** (or a **singular point**) for g if g fails to be holomorphic in any neighbourhood of $e^{i\theta}$.

By definition, in those points which are not singularities, g extends holomorphically. So, in particular, the radial limit g^* exists and belongs to $\partial\mathbb{D}$, coincides with g and everything works properly. On the other hand, the radial limit may fail to exist at singular points. However, the converse is not true: there are inner functions with singularities where the radial limit exists. For example, as we have seen before, the function $g(z) = \exp \frac{z+1}{z-1}$ has a singularity at 1 and $g^*(1) = 0$. Note that, in a singularity, the radial limit may belong to $\partial\mathbb{D}$, as in Examples B.4.2 and B.4.3, or not, as in the previous example.

In relation with the Denjoy-Wolff point, it may be a singularity (see Section B.4). However, by Theorem 1.3.2, the radial limit at the Denjoy-Wolff point always exists.

Finite Blaschke products are rational functions, so they are holomorphic in $\widehat{\mathbb{C}}$. In particular, they do not have singularities. Singularities of infinite Blaschke products are described by the following theorem.

Proposition 2.2.2. (Singularities for Blaschke products) *Let B be a Blaschke product and $\{z_n\}_n$ its sequence of zeros. Let $E \subset \partial\mathbb{D}$ be the accumulation set of $\{z_n\}_n$. Then B extends analytically to $\widehat{\mathbb{C}} \setminus E$. On the other hand, $|B|$ can not be extended continuously from $\mathbb{D}$ to any point in E.*

Proof. Clearly, B extends holomorphically to $\mathbb{D} \cup D'$. Let $\{B^n\}_n$ be the sequence of finite Blaschke products converging (uniformly on compact sets) towards B. Each B^n is a rational function, so it is holomorphic in $\widehat{\mathbb{C}}$. Observe that E is closed and it is the accumulation set of the zeros and the poles of $\{B^n\}_n$.

Now consider $z_0 \in \partial\mathbb{D} \setminus E$. For $\delta > 0$ small enough, $\{B^n\}_n$ is uniformly bounded in $D(z_0, \delta)$. By Cauchy's integral formula and since we can interchange the limit and the integral, we get

$$B^n(z_0) = \int_{\partial D(z_0,\delta)} B^n(z)dz \to \int_{\partial D(z_0,\delta)} B(z)dz = B(z_0),$$

so B is holomorphic at z_0

Finally, if $z_0 \in E$, it can be approximated by the sequence of zeros, so $\liminf_{z \to z_0} |B(z)| = 0$. But since $\left|B(e^{i\theta})\right| = 1$ λ-almost everywhere, we have $\limsup_{z \to z_0} |B(z)| = 1$. Therefore $|B|$ can not be extended continuously from $\mathbb{D}$ to any point in E. □

Note that, for an infinite Blaschke product, the sequence of zeros $\{z_n\}_n$ must be infinite and therefore, E is non-empty.

The following proposition explains that the accumulation of preimages around the singularity in Example 2.1.4 is not an isolated fact. Indeed, it happens for every singularity.

Proposition 2.2.3. *Let g be an inner function and $z_0 \in \partial\mathbb{D}$ a singularity of g. Then, for every $w \in \overline{\mathbb{D}}$, there exists $\{z_n\}_n \subset \mathbb{D}$ such that $z_n \to z_0$ and $g(z_n) \to w$.*

Proof. It is enough to prove that the set

$$R(g, z_0) = \{w \colon \exists \{z_n\}_n \subset \mathbb{D} \text{ such that } z_n \to z_0 \text{ and } g(z_n) = w\}$$

is equal to $\mathbb{D} \smallsetminus L$, where L is a set of zero logarithmic capacity. In the literature, the set $R(f, z_0)$ is called the **range set** of g at z_0.

First note that, if B is an infinite Blaschke product and z_0 is a singularity of it, z_0 is a accumulation point of the zeros of B, so $0 \in R(B, z_0)$.

Now, given the inner function g, consider $g_w(z) = \frac{w-g(z)}{1-\overline{w}g(z)}$, which is an infinite Blaschke product, except for $w \in L$, set of logarithmic capacity zero. The point z_0 must be a singularity for g_w and, by the previous remark, z_0 is a accumulation point of the zeros of g_w. Therefore, z_0 is a accumulation point of the preimages of w by g and $w \in R(g, z_0)$. □

As a remark, the **cluster set** of g at z_0, defined as

$$Cl(g, z_0) = \{w \colon \exists \{z_n\}_n \subset \mathbb{D} \text{ such that } z_n \to z_0 \text{ and } g(z_n) \to w\}$$

satisfies that $Cl(g, z_0) = \overline{\mathbb{D}}$, if z_0 is a singularity of g, by the previous proposition. In a more general setting, the cluster set of analytic functions of the unit disk forms an interesting topic in complex analysis and it has been widely studied, for instance in [CL66; Nos60].

The following proposition explains the boundary behaviour around the singularity.

Proposition 2.2.4. [Baker-Domínguez [BD99, Lemma 5]] *Let g be an inner function and $z_0 \in \partial\mathbb{D}$ a singularity of g. Fix $w \in \partial\mathbb{D}$. Then, there exists a sequence $\{z_n\}_n$, with $z_n \neq z_0$ for all n and $z_n \to z_0$, such that $g^*(z_n) = w$ for all n.*

2.3 Some analytical properties of finite Blaschke products

First recall that finite Blaschke products are of the form

$$B(z) = e^{i\theta} \prod_{n=1}^{d} \frac{a_n - z}{1 - \overline{a_n} z},$$

where $\theta \in \mathbb{R}$, $d \in \mathbb{N}$ and $\{a_n\}_n \subset \mathbb{D}$. From the formula, one directly deduces that they are, in fact, rational functions, so they are well-defined and holomorphic in $\widehat{\mathbb{C}}$. Clearly, $\mathbb{D}$, $\partial\mathbb{D}$ and D' are totally invariant under B.

Observe that, since the numerator and the denominator have no common factor, a Blachke product of degree d (in the sense that we are multiplying d factors) is actually a rational function of degree d (in the usual sense of degree of a rational function). Therefore, each point in $\widehat{\mathbb{C}}$ has d preimages. By the invariance of the sets $\mathbb{D}$, $\partial\mathbb{D}$ and D', these d preimages of the must belong to the same set as the point.

The following results gather information about the derivative of the Blaschke product and its critical points, which play an important role in the dynamics.

Proposition 2.3.1. (Derivative of finite Blaschke products) *Let B be a finite Blaschke product and $e^{i\theta} \in \partial\mathbb{D}$. Then, $B(e^{\imath\theta}) \neq 0$.*

Proof. Let $B(z) = \prod_{n=1}^{d} \frac{a_n - z}{1 - \overline{a_n} z}$. The derivative of each factor in the product is:

$$\left(\frac{a_n - z}{1 - \overline{a_n} z}\right)' = \frac{|a_n|^2 - 1}{(1 - \overline{a_n} z)^2}.$$

Now if we put B_n for the same Blaschke product as B but omitting the n-th term, we have:

$$B'(z) = -\sum_{n=1}^{d} \frac{1 - |a_n|^2}{(1 - \overline{a_n} z)^2} B_n(z).$$

Therefore:

$$\frac{B'(z)}{B(z)} = \sum_{n=1}^{d} \frac{1 - |a_n|^2}{(1 - \overline{a_n} z)(z - a_n)}.$$

Finally, it is clear that evaluating at $e^{i\theta}$ and taking the modulus, we get the desired result:

$$\left|B'(e^{i\theta})\right| = \sum_{n=1}^{d} \frac{1 - |a_n|^2}{|e^{i\theta} - z_k|^2} \geq 0.$$

□

Corollary 2.3.2. *If B is a finite Blaschke product of degree d, then for each $w \in \partial\mathbb{D}$, the equation $B(z) = w$ has exactly d distinct solutions on $\partial\mathbb{D}$.*

Proof. This is easily deduced from the previous proposition and the invariance of $\partial\mathbb{D}$. □

Finally, as a rational function of degree d, B must have $2d - 2$ critical points. By the last proposition, they must belong either to $\mathbb{D}$ or to D'. In fact, by the Schwarz Reflection Principle, they must be symmetric with respect to $\partial\mathbb{D}$.

2.4 The harmonic measure of the unit circle $\partial\mathbb{D}$

Recall that our goal is to prove some ergodic properties of the extension of the inner function to the boundary, with respect to the Lebesgue measure. However, as we commented in Section 1.6, other measures (equivalent to the Lebesgue one) may be considered in order to prove some ergodic properties. In this section we define the harmonic measure and give some interesting properties that will be used in order to prove ergodic properties of inner functions.

Harmonic measure is often constructed in three (equivalent) different ways, each of them with its advantages and its drawbacks. Now we present them briefly, although we are going to work with the second one. Assume in the sequel that $z \in \mathbb{D}$ is fixed (the base point) and $A \subset \partial\mathbb{D}$ is measurable (the set we want to measure).

1. First of all, from a **stochastic** point of view, harmonic measure can be seen as the boundary hitting distribution of Brownian motion. More concretely, starting at z, we run a random walk until the first time it hits the boundary. We define the measure of the set $A \subset \partial\mathbb{D}$ as the probability of being hit by the random walk.

   This is the most intuitive definition, although it is difficult to work with. It is clear from this point of view that the harmonic measure depends on the base point chosen.

2. From the point of view of **potential theory**, one can define the harmonic measure with base point $z \in \mathbb{D}$ as the harmonic extension of the indicator function 1_A evaluated at z. See Definition 2.4.1.

3. Finally, more **geometrically**, the harmonic measure of $\partial\mathbb{D}$ is the image of the normalized Lebesgue measure λ of $\partial\mathbb{D}$ under a conformal self-map T of $\mathbb{D}$, sending 0 to z. That is, given $A \subset \partial\mathbb{D}$ its harmonic measure is defined to be $\lambda(T^{-1}(A))$.

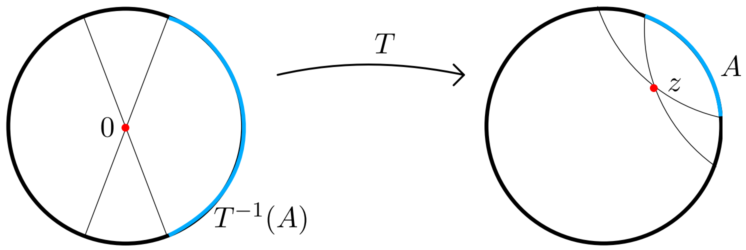


Figure 5: The harmonic measure of the set A on the right (with base point z) can be defined as the Lebesgue measure of its preimage under T, where T is a Möbius transformation of $\mathbb{D}$ bringing 0 to z.

Although the last two definitions may not be as intuitive as the first one, they allow us to use tools from complex analysis, so they seem more appropriate in our context. In contrast with other simply-connected domains, in the unit disk the Dirichlet problem can be solved explicitly by means of the Poisson kernel (Section 1.4), so the second definition is the one mainly used in our concrete setting ([DM91]) and more in general ([Con95, Chapter 21], [GM05]). Next we state the formal definition.

Definition 2.4.1. Let A be a Borel subset of $\partial\mathbb{D}$. The **harmonic measure** (with base point $z \in \mathbb{D}$) of A is:

$$\omega_z(A) = \omega(z, A, \mathbb{D}) := \frac{1}{2\pi}\int_A P(z,w)dw,$$

where $P(z,w) = \frac{1-|z|^2}{|w-z|^2}$ stands for the Poisson kernel of z at w.

From this definition it is clear that $\omega_z(A)$ is a measure. Moreover, since the integral of the Poison kernel on $\partial\mathbb{D}$ is 1, we have $0 \leq \omega_z(A) \leq 1$, for all $z \in \mathbb{D}$ and all $A \in \partial\mathbb{D}$. It is also clear that $\omega_z(A)$ is harmonic with respect to the variable z, and hence the name *harmonic measure*.

Observe that, by Harnack's inequality (Thm. 1.4.6), ω_p and ω_q are equivalent, for all $p, q \in \mathbb{D}$. In particular, since ω_0 is the Lebesgue measure of $\partial\mathbb{D}$, this implies that any harmonic measure is equivalent to the Lebesgue measure. This is precisely what we need to prove ergodic properties of the Lebesgue measure using the harmonic measure.

Observe that there is not only one harmonic measure in the disk, but a family of them, each with a different base point. The fact that all harmonic measures are equivalent justifies saying *the* harmonic measure if the choice of the base point is irrelevant.

Next we prove the following proposition, relating harmonic measure with Möbius transformations of the unit disk.

Proposition 2.4.2. *Let T be a Möbius transformation of $\mathbb{D}$. Then, $\omega_{T(z)}(T(A)) = \omega_z(A)$.*

Proof. If T is any Möbius transformation of $\mathbb{D}$, we have:

$$\frac{1-|z|^2}{|w-z|^2} = \frac{1-|T(z)|^2}{|T(w)-T(z)|^2}\left|T'(w)\right|$$

Then,

$$\frac{1}{2\pi}\int_{T(A)}\frac{1-|T(z)|^2}{|w-T(z)|^2}dw = \frac{1}{2\pi}\int_A \frac{1-|T(z)|^2}{|T(w)-T(z)|^2}\left|T'(w)\right|dw = \int_A \frac{1-|z|^2}{|w-z|^2}dw.$$

Thus, $\omega_{T(z)}(T(A)) = \omega_z(A)$. $\square$

Observe that this proposition proves the equivalence between the second and the third ways to define the harmonic measure. Indeed, letting z be the origin and T such that $T(z) = 0$, we have

$$\omega_z(A) = \omega(0, T^{-1}(A)) = \lambda(T^{-1}(A)).$$

The equivalence between the first and second definitions is more involved, and we refer to [GM05, Thm. F.6].

An interesting property relating inner functions and harmonic extension of functions defined in $\partial\mathbb{D}$ is that they commute. It is the same to compose the inner function with the harmonic extension than to compose both maps in $\partial\mathbb{D}$ and then consider the harmonic extension.

Proposition 2.4.3. [[DM91, Thm. 1.4]] *Let g be an inner function and let $h\colon \partial\mathbb{D} \to \mathbb{R}$, with $h \in L^1(\partial\mathbb{D})$. Then, $h \circ g^* \in L^1(\partial\mathbb{D})$ and*

$$\widehat{h} \circ g = \widehat{h \circ g^*},$$

where $\widehat{h}$ denotes the harmonic extension of h in $\mathbb{D}$.

Straightforward from this proposition we deduce the following.

Corollary 2.4.4. (Harmonic measures and inner functions) *Let g be an inner function and ω_z be the harmonic measure of $\partial\mathbb{D}$ with base point $z \in \mathbb{D}$. Then, for any measurable set $A \subset \partial\mathbb{D}$:*

$$\omega_z(g^{*-1}(A)) = \omega_{g(z)}(A).$$

Corollary 2.4.5. (Invariant harmonic measures) *Let g be an inner function. The harmonic measure ω_z is g-invariant if and only if $g(z) = z$.*

2.5 Inner function associated to an invariant Fatou component

Let U be an invariant Fatou component. If U is simply-connected, following the procedure described in Section 1.3.3, we may associate to $f\colon U \to U$ a holomorphic self-map g of the unit disk. Indeed, using a Riemann map $\varphi\colon \mathbb{D} \to U$, we define $g := \varphi^{-1} \circ f \circ \varphi$, that is:

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & U \\ \varphi \uparrow & & \uparrow \varphi \\ \mathbb{D} & \xrightarrow{\quad g \quad} & \mathbb{D} \end{array}$$

To study the dynamics inside the Fatou component, it is enough to know that g is a holomorphic self-map of $\mathbb{D}$. However, since our purposes are to study the dynamics on the boundary of Fatou components, we need control on $g^*(\partial\mathbb{D})$. The first step is to prove that g is an inner function.

Proposition 2.5.1. *Let U an invariant simply-connected Fatou component of f and $\varphi\colon \mathbb{D} \to U$ a Riemann map. Then, $g := \varphi^{-1} \circ f \circ \varphi$ is an inner function.*

Proof. It is clear that g is a holomorphic self-map of $\mathbb{D}$. Assume, that g is not inner. Because of the Fatou, Riez and Riez theorem on radial limits (2.1.2), there must be a set $E \subset \mathbb{D}$ of positive λ-measure with $\left|h^*(e^{i\theta})\right| < 1$, for all $e^{i\theta} \in E$. Since φ^* exists λ-almost everywhere, we can assume that φ^* exists for all the points in E. Moreover, $\varphi^*(E)$ must contain more than one point, so it cannot be equal to ∞.

On the one hand, since φ is a bijection from $\mathbb{D}$ to U, it follows that $\varphi^*(E) \subset \partial U \subset \mathcal{J}(f)$, so $f(\varphi^*(E)) \subset \mathcal{J}(f)$. On the other hand, $f(\varphi^*(E)) = \varphi^*(g^*(E)) \subset \varphi^*(\mathbb{D}) \subset U \subset \mathcal{F}(f)$. This is a contradiction, since the Fatou and the Julia sets are completely invariant. □

Observe that the contradiction in the proof comes from assuming that E has positive λ-measure. Indeed, if E had zero λ-measure, $\varphi*$ may not exist on it or, in the case of transcendental entire functions, it may be sent to ∞.

We say that g is the **inner function associated to** $f_{|U}$. Observe that g is not unique since it depends on the Riemann map chosen to pass from $\mathbb{D}$ to U. However, all possible associated functions are clearly conjugate, so they are dynamically equivalent. Because of this we can speak of *the* associated inner function.

By Frostman's theorem (2.1.5), we can assume that g is a Blaschke product, and this leads to two different cases:

- On the one hand, g may be a finite Blaschke product of degree d. In this case, every point in $\mathbb{D}$ has d preimages by g (counted with multiplicity), and therefore, each point in U has d preimages (in U) by f. Thus, $f_{|U}$ is a proper map of degree d.
- On the other hand, g may be an infinite Blaschke product. In this case, there are points with infinitely many preimages. In fact, more is ensured by the following theorem.

  Theorem 2.5.2. [Heins, [Hei57]] *Let f be a transcendental entire function and U an open set such that $f(U) \subset U$. Assume that there is a point $z \in U$ that has infinitely preimages by f. Then, every point must have infinitely preimages by f, with at most one exception.*

As a remark, the previous theorem is telling us that there are many inner functions that cannot be associated to any invariant Fatou component, in the sense described above. Indeed, for any closed set A of logarithmic capacity zero, then there exists an inner function omitting the set A ([Ste78]). But if A consists of more than one point, there cannot be a Fatou component having such an inner function associated. For more information about which inner functions can be associated to entire maps, we refer to [ERS20].

2.5.1 The harmonic measure of the boundary of a simply-connected domain

In Section 2.4 we have defined the harmonic measure of the unit circle. Now we are going to do so for any simply-connected domain. Observe that, in fact, the three constructions of the harmonic measure are extendible to an arbitrary simply-connected domain, as follows.

1. The definition of the harmonic measure as the probability distribution of hitting the boundary of a random walk, extends naturally to any simply-connected domain U, just considering the random walk in U.

   This definition is again the most intuitive, but difficult to work with. With this approach it is clear that harmonic measure depends on the chosen base point. Moreover, this is telling us that non-accessible points have zero harmonic measure, because they cannot be hit from the inside of U.

2. The second way to construct the harmonic measure on the unit circle was through the solutions to the Dirichlet problem. This approach was useful in the unit disk, because the Dirichlet problem has an explicit expression. For an arbitrary simply-connected domain this approach is also valid, because the Dirichlet problem is solvable (Thm. 1.4.5), but we do not have an explicit expression anymore.

3. Finally, the third possible way to define harmonic measure in the unit circle is as the image of the Lebesgue measure under an automorphism φ of $\mathbb{D}$. This extends naturally to any arbitrary simply-connected domain by the Riemann map.

   This is the approach we choose. Observe that it is very suitable to our setting because it gives a connection between the Lebesgue measure in the unit disk and the measure considered in the boundary of the Fatou component. More in general, this approach is often chosen because of the powerful tools from complex analysis one can work with (see [Bis18] and [Pom92, Chapter 4.4]).

Therefore, we shall define the harmonic measure of a simply-connected domain as follows.

Definition 2.5.3. Let $U \subset \widehat{\mathbb{C}}$ be a simply-connected domain and let $\varphi\colon \mathbb{D} \to U$ be a Riemann map, such that $\varphi(0) = z \in U$. The **harmonic measure** $\omega(z, U)$ of ∂U with base point $z \in U$ is the image under φ of the normalized Lebesgue measure of the unit circle $\partial\mathbb{D}$. That is, given a measurable subset $A \in \partial U$,

$$\omega(z, A, U) := \omega(0, (\varphi^*)^{-1}(A), \mathbb{D}) = \lambda((\varphi^*)^{-1}(A)).$$

The equivalence between this definition and the one given by the random walk can be found in [GM05, Thm. F.6], whereas in [DM91, p.21] there is the proof of the equivalence between this definition and the second one.

As before, by Harnack's inequality one gets the equivalence between any two harmonic measures in U, so it is non-relevant the base point chosen. We shall refer to it as *the* harmonic measure of ∂U and denote it ω_U.

Returning to our setting, we have just defined the measure that we are going to use in the boundary of a simply-connected Fatou component U. The special connection of this measure to the λ-measure of $\partial\mathbb{D}$ gives the following result, which justifies the use of inner functions to prove ergodic properties on the boundary of Fatou components.

Proposition 2.5.4. (Ergodic properties of the boundary map) *Let U be an invariant Fatou component of a holomorphic map f and let g its associated inner function. Then, g is ergodic (or recurrent) with respect to the Lebesgue measure λ if and only if $f_{|\partial U}$ is ergodic (or recurrent) with respect to the harmonic measure ω_U.*

Proof. It is straightforward from the definition of ω_U. □

Finally, we prove the following proposition, which relates the ergodic properties on the boundary of the Fatou components and the presence of dense orbits.

Proposition 2.5.5. (Dense orbits) *Let f be a holomorphic function and let U be a simply-connected invariant Fatou component for f. Assume $f_{|\partial U}$ is ergodic and recurrent. Then the orbit of ω_U-almost every point in ∂U is dense in ∂U.*

Proof. By Theorem 1.6.4, the forward orbit of ω_U-almost all points in ∂U intersects any set $B \subset \partial U$ with positive harmonic measure. Since open sets in ∂U have positive harmonic measure, we conclude that the forward orbit of ω_U-almost all points in ∂U is dense in ∂U. □

3 Iteration of inner functions

Recall that inner functions are the self-map of the unit disk $\mathbb{D}$ such that for λ-almost every point in $\partial\mathbb{D}$ the radial limit g^* exists and belongs to $\partial\mathbb{D}$. Therefore, one can consider the map $g^*\colon \partial\mathbb{D} \to \partial\mathbb{D}$, defined for λ-almost every point in $\partial\mathbb{D}$. This map g^* is called the **boundary map** of the inner function g.

Observe that the boundary map g^* defines a dynamical system in $\partial\mathbb{D}$. Indeed, g^* is defined and belongs to $\partial\mathbb{D}$ for λ-almost every point in $\partial\mathbb{D}$, so for λ-almost every point we can apply g^* again. Likewise, the second iterate g^{*2} is well-defined except on a set of λ-measure zero. Repeating this process inductively, and taking into account that a countable union of sets of measure zero has measure zero, we get that the (infinite) orbit under g^* for λ-almost every point in the boundary is well-defined.

However, this does not imply directly that the functions g^n are inner functions, for all n. Indeed, the definition of inner function relies on the concept of radial limit. *A priori*, although the existence of the radial limits for g^n is already known (Thm. 2.1.2), we cannot ensure they belong to $\partial\mathbb{D}$. Once g applied, radii are mapped into curves, and we have no control on curves anymore. Nevertheless, iterates of inner functions are inner functions. We state the result for completeness, but for our purposes it is enough to consider the dynamical system induced by g^* described in the previous paragraph.

Theorem 3.0.1. [Baker-Domínguez [BD99, Lemma 4]] *If g is an inner function then all iterates g^n, $n \in \mathbb{N}$, are inner functions.*

3.1 The Fatou and Julia sets for inner functions

Once we know that iterating boundary maps makes sense, a logical step would be trying to define the Fatou and the Julia sets for inner functions. If the inner function we are considering is a finite Blaschke product, since it extends as a rational function, the Fatou and the Julia sets can be defined in the usual way. However, in the case of inner functions of infinite degree, the definition should be modified to take into account singularities.

Definition 3.1.1. Let g be an inner function of the unit disk $\mathbb{D}$, extended to $\widehat{\mathbb{C}} \setminus \text{sing}(g)$ by analytic continuation. The **Fatou set** $\mathcal{F}(g)$ of g is the set of all points $z \in \widehat{\mathbb{C}}$ for which there is an open neighbourhood U such that $U \cap \text{sing}(g^n) = \emptyset$ and $\left\{g^n_{|U}\right\}_n$ is normal. The **Julia set** $\mathcal{J}(g)$ of g is the complement of $\mathcal{F}(g)$ in $\widehat{\mathbb{C}}$.

It follows from Montel's theorem that $\mathcal{J}(g) \subset \partial\mathbb{D}$. Moreover, if g is of elliptic or of doubly-parabolic type, $\mathcal{J}(g) = \partial\mathbb{D}$. Indeed, if the Denjoy-Wolff point is in $\mathbb{D}$, both $\mathbb{D}$ and D' are basins of attraction of two different fixed points, so its common boundary $\partial\mathbb{D}$ is the Julia set. In the doubly-parabolic case, the proof is more involved ([BD99, Lemma 10], [Bar08, Thm. 2.24]). For finite degree, we prove it in Theorem 3.3.1.

Some other typical properties of the Fatou and the Julia sets also holds for this special setting, but they require an *ad hoc* proof. For example, that the Julia set is perfect ([BD99, Lemma 8]) or that repelling periodic points are dense in the Julia set ([Bar08, Thm. 2.34]).

Invariance is a relevant property that is worth to mention. From the definition, one deduces that $g(\mathcal{F}(g)) \subset \mathcal{F}(g)$ and, therefore, $g^{-1}(\mathcal{J}(g)) \subset \mathcal{J}(g)$. The other inclusion does not hold in general when the image of a point in the boundary is considered to be its radial limit. For instance, in Example 2.1.4 the singularity (in the Julia set) is sent to the origin (in the Fatou set). This is solved, for example, discarding the orbits of the singularities, although, by doing that, we may be omitting a set of positive measure, and this is not convenient for our purposes.

Observe that the problematic points that do not respect the natural inclusions of the Fatou and the Julia set form a set of measure zero, so they do not represent a problem from the ergodic point of view. This explains in part the different approaches in [BD99] or [Bar08], which studies the Fatou and Julia set for inner functions, and the one in [DM91], which studies the iteration of inner functions from an ergodic point of view, so the Fatou-Julia approach is not useful anymore.

A detailed study of these sets for inner functions can be found in [BD99] and [Bar08]. For our purposes, the stated properties are enough. We must not forget that we are interested in ergodic properties and the description of the dynamics given by the Fatou and the Julia sets (purely topological) may fall short.

3.2 Iteration of rotations

We start iterating the simplest inner functions: rotations. Rotations are inner functions of degree 1 having a fixed point in the unit disk. Conjugating by an appropriate Möbius transformation, we shall assume that the fix point is the origin and the rotation takes the form: $g(z) = e^{i2\pi\theta}z$, for some $\theta \in \mathbb{R}$. When restricted to $\partial\mathbb{D}$, the rotation can be seen as $x \mapsto x + \theta \mod 1$. According to the value of θ (the angle of rotation), we distinguish between **rational rotations** (if $\theta \in \mathbb{Q}$) and **irrational rotations** (otherwise). In both cases ergodicity and recurrence is easy to study.

First, by Poincaré Recurrence Theorem (1.6.2) and the fact that rotations preserve Lebesgue measure, we deduce that rotations are recurrent. Note that this holds for rational as well for irrational rotations.

Regarding ergodicity, rational and irrational rotations behave differently. If $\theta = \frac{p}{q}$ is rational, all points are periodic of period q. Consider any interval I of length smaller than $\frac{1}{q}$ and the set $E = \left\{I, g(I), \dots, g^{q-1}(I)\right\}$. This set is completely invariant and with measure $0 < \lambda(E) < 1$. Therefore, rational rotations are not ergodic.

The irrational case is more complicated, and we rely on the following result.

Theorem 3.2.1. (Weyl's equidistribution theorem) [[SS03, p. 105–113]] *Let $\theta \in (0,1)$ be irrational and consider the dynamical system given by $x \mapsto x + \theta \mod 1$. Then, given any initial condition $x_0 \in [0,1)$, its orbit $\{x_0 + n\theta \mod 1\}_n$ is equidistributed in $[0,1)$ in the following sense: given a measurable set $A \subset [0,1)$,*

$$\lim_{N\to\infty} \frac{\#\left\{\{x_0 + n\theta \mod 1\}_{n=1}^{N} \cap A\right\}}{N} = \lambda(A).$$

Observe that this, by Theorem 1.6.4 implies ergodicity for irrational rotations. In particular, the orbit of any point in $[0,1)$ is dense. This is precisely Jacobi's theorem on irrational rotations ([Dev89, Thm. 3.13]). Observe that Jacobi's theorem is weaker in the sense that it does not imply ergodicity nor recurrence.

3.3 Iteration of finite Blaschke products

Our goal is to study finite Blaschke products and describe their dynamics. From an ergodic point of view, we prove some of the results in Table 1 for the finite degree case, without applying powerful tools of complex analysis. However, we do not manage to prove all the results, which are addressed in the next section, following the standard approach of [DM91]. Moreover, in the elliptic and the doubly-parabolic case we prove the existence of a topological conjugacy between the boundary map and the function $x \mapsto dx \mod 1$.

We start with this theorem that relates Cowen's classification with the dynamics of finite Blaschke products.

Theorem 3.3.1. (Characterization of finite Blaschke products) *Let g be a finite Blaschke product and $p \in \partial\mathbb{D}$ its Denjoy-Wolff point. Then:*

1. *The map g is of **hyperbolic type** if and only if $0 < g'(p) < 1$.*

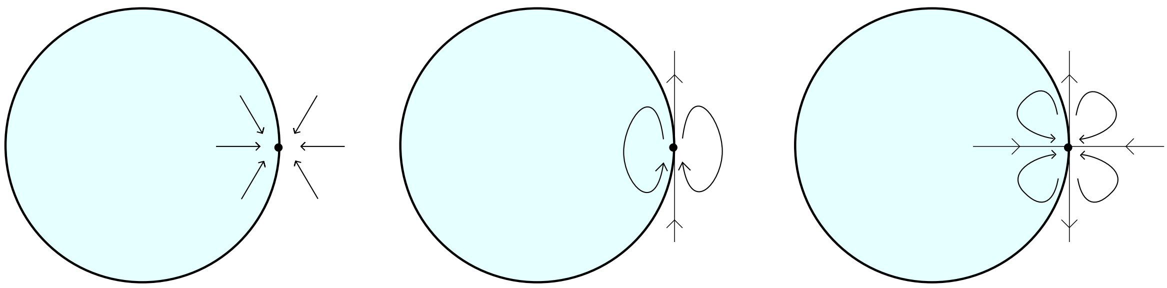
   In this case, p is an attracting fixed point and its basin of attraction is connected, non-simply connected and contains $\mathbb{D} \cup D'$, where $D' := \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. The Julia set $\mathcal{J}(g) \subsetneq \partial\mathbb{D}$ is a Cantor set.

2. *The map g is of **simply-parabolic type** if and only if $g'(p) = 1$ and $g''(p) \neq 0$.*

   In this case, p is a parabolic fixed point of multiplicity 1. The Fatou set $\mathcal{F}(g)$ is connected, non-simply connected and contains $\mathbb{D} \cup D'$. The Julia set $\mathcal{J}(g) \subsetneq \partial\mathbb{D}$ is a Cantor set.

3. *The map g is of **doubly-parabolic type** if and only if $g'(p) = 1$, $g''(p) = 0$ and $g'''(p) \neq 0$.*

   In this case, p is a parabolic fixed point of multiplicity 2. The Fatou set $\mathcal{F}(g)$ consists of two connected simply-connected components $\mathbb{D}$ and D'. The Julia set $\mathcal{J}(g)$ is $\partial\mathbb{D}$.



(a) *Hyperbolic* (b) *Simply-parabolic* (c) *Doubly-parabolic*

Figure 6: The three possibilities for the dynamics of a finite Blaschke product.

Proof. Recall that, as a direct consequence of Montel's theorem, $\mathbb{D} \cup D' \subset \mathcal{F}(g)$ and $\mathcal{J}(g) \subset \partial\mathbb{D}$. Observe also that, since finite Blaschke products cannot have singularities, the map is always holomorphic in a neighbourhood of the Denjoy-Wolff point. Therefore, the Denjoy-Wolff point is not only radially fixed, but fixed in the usual sense. According to its derivative, it may be attracting or parabolic, but in both cases the linearizing coordinates described in Sections 1.1.1 and 1.1.2 can be applied. We consider the three cases separately.

The first case is a direct consequence of the linearizing coordinates around an attracting fixed point (Thm. 1.1.6). Indeed, attracting fixed points are the ones with multiplier smaller than 1 (in modulus) and one deduces that there exists a neighbourhood of p belonging to $\mathcal{F}(g)$. Therefore, $\mathcal{J}(g) \subsetneq \partial\mathbb{D}$.

It is left to deal with the parabolic case. We must assume that, around the Denjoy-Wolff point we have the following Taylor expansion:

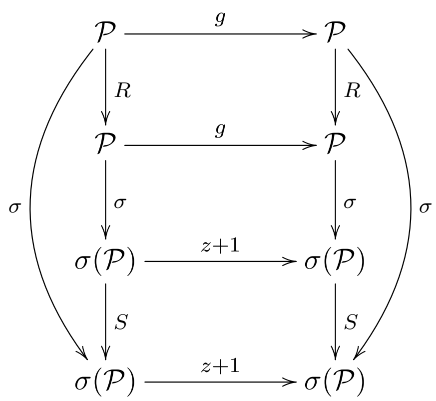
$$g(z) = p + (z - p) + a(z - p)^{n+1} + \dots,$$

where $n \geq 1$. With the notation of Section 1.1.2, there exist n attracting petals together with the corresponding (disjoint) attracting basins. Observe that $\mathbb{D}$ and D' are invariant components of the Fatou set in which iterates tend to p and no other such invariant components are possible. Therefore, $n \leq 2$. It is enough to see that $n = 2$ implies the doubly-parabolic case and $n = 1$ implies the simply-parabolic one.

Assume $n = 2$, then there exists two attracting petals $\mathcal{P}_1, \mathcal{P}_2$ and the corresponding parabolic basins Ω_1, Ω_2. We may assume $\Omega_1 = \mathbb{D}$ and $\Omega_2 = D'$. Then, P_1 is an absorbing domain for g in $\mathbb{D}$ and, by Theorem 1.1.9, $g_{|\mathcal{P}_1}$ is conjugate to $id_{\mathbb{C}} + 1$. By Cowen's theorem(1.3.7), any absorbing domain for g in $\mathbb{D}$ must have dynamics conjugate to $id_{\mathbb{C}} + 1$. We deduce that g is of doubly-parabolic type and $\mathcal{J}(g) = \partial\mathbb{D}$, because it is the common boundary between the parabolic basins.

Assume now $n = 1$. Then, there exists one attracting petal $\mathcal{P}$ and a parabolic basin Ω such that $\mathbb{D} \cup D' \subset \Omega$. Since all points in Ω must converge to p through the petal $\mathcal{P}$, we deduce that $\mathcal{P} \cap \mathbb{D}$ and $\mathcal{P} \cap D'$ are (non-empty) absorbing domains for $g_{|\mathbb{D}}$ and $g_{|D'}$, respectively. Since $\mathcal{P}$ is connected, it must intersect $\partial\mathbb{D}$, so $\mathcal{J} \subsetneq \partial\mathbb{D}$. We have to see that $\mathcal{P} \cap \mathbb{D}$ is conjugate to $id_{\mathbb{H}} + 1$. To see this, we are going to take advantage from that g is symmetric with respect to the unit circle $\partial\mathbb{D}$ in order to construct a petal such that, when conjugate to $id_{\mathbb{C}} + 1$, it is symmetric with respect to the real axis. From this we will deduce that $\mathcal{P} \cap \mathbb{D}$ is conjugate to $id_{\mathbb{H}} + 1$.

To develop the strategy above, consider $R(z) = \frac{1}{\bar{z}}$ to be the symmetry respect to the unit circle $\partial\mathbb{D}$. Clearly, R is an antiholomorphic involution. Without lost of generality, we may assume $\mathcal{P}$ to be symmetric with respect to R, i.e. if $z \in \mathcal{P}$, then $R(z) \in \mathcal{P}$. For instance, redefining $\mathcal{P}$ to be $(\mathcal{P} \cap \overline{\mathbb{D}}) \cup R(\mathcal{P} \cap \overline{\mathbb{D}})$. Recall that $g_{|\mathcal{P}}$ is conjugate (by σ) to $id_{\mathbb{C}} + 1$. Therefore, defining $S \coloneqq \sigma \circ R \circ \sigma^{-1}$, we have the following commutative diagram.



The function S is an antiholomorphic involution that commutes with $z \mapsto z + 1$ in $\sigma(\mathcal{P})$. Up to now, S is only defined in $\sigma(\mathcal{P})$, but in can be easily extended to $\mathbb{C}$. Indeed, since $\sigma(\mathcal{P})$ is an absorbing domain for $id_{\mathbb{C}} + 1$ (Thm. 1.3.7), for any $z \in \mathbb{C}$ there exists n such that $z + n \in \sigma(\mathcal{P})$. Therefore, let us define $S(z) \coloneqq S(z + n) - n$, where n is such that $z + n \in \sigma(\mathcal{P})$. This definition does not depend on the choice of n. It is easy to check that, extended in this way, S is an antiholomorphic involution on $\mathbb{C}$, $S(z + 1) = S(z) + 1$ and $\sigma(R(z)) = S(\sigma(z))$ for all $z \in \mathbb{C}$. Moreover, points that are fixed by R are mapped by σ to points that are fixed by S.

Since S is an antiholomorphic and bijective in $\mathbb{C}$, it must be of the form $S(z) = a\bar{z} + b$, with $a, b \in \mathbb{C}$. Imposing $S(z + 1) = S(z) + 1$, we get that $a = 1$. Forcing S to be an involution, i.e. $S(S(z)) = 1$, we get that $b \in i\mathbb{R}$. Therefore, $S(z) = z + ib'$, with $b' \in \mathbb{R}$. Observe that the fixed points of S are the ones with imaginary part $\frac{b'}{2}$. Therefore, points in $\Omega \cap \partial\mathbb{D}$ are mapped by σ to the horizontal line $\left\{ z = x + iy \colon y = \frac{b'}{2} \right\}$.

Now, define $\widetilde{\sigma}(z) \coloneqq \sigma(z) - \frac{b'}{2}$. Observe that $\widetilde{\sigma}$ satisfies the conditions of Cowen's theorem (1.3.7) to be a conjugacy, because σ was chosen to be so. Moreover, points in $\Omega \cap \partial\mathbb{D}$ are mapped by σ to the real axis. Therefore, $\widetilde{\sigma}(\mathbb{D})$ is mapped to $\mathbb{H}$. Hence $\widetilde{\sigma}$ is a conjugacy in Cowen's sense between $g_{|\mathbb{D}}$ and $id_{\mathbb{H}} + 1$. We deduce that g is of simply-parabolic type.

Finally, it is left to prove that in cases (2) and (3), the Julia set is a Cantor set. We follow the ideas of [CG93, p. 58]. Since the Julia set is closed and perfect (Prop. 1.1.5), to prove that it is a Cantor set, it is enough to see that it is totally disconnected. We already know that, in the cases we are considering, $\mathcal{J}(g) \subsetneq \partial\mathbb{D}$. Therefore, there exists a point $z_0 \in \mathcal{J}(g)$ such that it can be approximated by points of $\partial\mathbb{D}$ in the Fatou set, i.e there exists a sequence $\{w_n\}_n \subset \mathcal{F}(g) \cap \partial\mathbb{D}$ such that $w_n \to z_0$. Since preimages of z_0 are dense in the Julia set, it is enough to show that preimages of z_0 can be approximated by points in $\mathcal{F}(g) \cap \partial\mathbb{D}$. Taking $z_1 = g^{-n}(z_0)$, for some branch of the inverse, z_1 can be approximated by $\{g^{-n}(w_n)\}_n$. By the invariance of $\partial\mathbb{D}$ and the Fatou set under g, this last sequence lies in $\mathcal{F}(g) \cap \partial\mathbb{D}$. This ends the proof of the theorem. □

In some cases the Cantor set can be computed explicitly. A concrete example can be found in [Bea91, p. 21].

Although it does not belong to this section, let us remark that in the case of infinite degree, one can find examples (in the hyperbolic and simply-parabolic cases) of Julia sets that are strictly contained in $\partial\mathbb{D}$ but they are not Cantors set (see Examples B.4.2 and B.4.3). The previous argument cannot be applied with infinite degree because one cannot ensure that preimages of points in the Fatou set are again in the Fatou set.

For the hyperbolic and the simply-parabolic cases, the dynamics in $\partial\mathbb{D}$ are well-understood: points in $\mathcal{F} \cap \partial\mathbb{D}$ are the only ones that converge to the Denjoy-Wolff point, and they form a set of positive measure. With this we deduce the following ergodic properties.

Corollary 3.3.2. (Ergodic properties of finite Blaschke products) *Let g be a finite Blaschke product and let p be its Denjoy-Wolff point. Assume g is of hyperbolic or simply-parabolic type. Then, g is non-ergodic and non-recurrent.*

Proof. From the fact that the Fatou set $\mathcal{F}(g) \cap \partial\mathbb{D}$ is non-empty, one deduces that there is a set of positive measure such that every point on it converges to p. This proves non-recurrence.

To prove non-ergodicity one may be tempted to define the two invariant sets to be $\mathcal{F}(g)$ and $\mathcal{J}(g)$. However this does not work because $\mathcal{J}(g)$ is a Cantor set and may have zero measure. To fix this, one may consider an open interval $I \subset \partial\mathbb{D}$, containing the Denjoy-Wolff point, the dynamics is conjugate either to $z \mapsto \lambda z$ or to $z \mapsto z+1$. Taking any $z_0 \in I$, consider the interval $J = [z_0, g(z_0)]$. In a more general context, this is called a *fundamental set*. All trajectories converging to the Denjoy-Wolff point in this direction have a unique point in this interval (observe that in the hyperbolic case, orbits can converge from the other side). Take J_1 and J_2 disjoint open intervals in J and consider $I_1 = \bigcup_{n\in\mathbb{Z}} g^n(J_1)$ and $I_2 = \bigcup_{n\in\mathbb{Z}} g^n(J_2)$. Then, I_1 and I_2 are disjoint invariant sets of positive measure, so the map is not ergodic. □

On the other hand, for the elliptic and the doubly-parabolic cases we only know that $\mathcal{J} = \partial\mathbb{D}$, but we do not have an explicit description of the dynamics yet. Next we will prove that the dynamics on the boundary are conjugate to $x \mapsto dx \mod 1$.

To do so, we deal with circle maps and 1-dimensional dynamics. A basic introduction to this topic can be found in [Dev89, Chapter 1], and a more detailed exposition in [MS93]. Recall that a self-map of $\partial\mathbb{D}$ is said to be a covering if it has no critical points. A basic fact is that every covering of degree $d < \infty$ is at least semiconjugate to multiplication by d.

Proposition 3.3.3. [Shub [MS93, Thm. II.2.1]] *Let $f\colon \partial\mathbb{D} \to \partial\mathbb{D}$ be such that $f(x) = dx$ and let $g\colon \partial\mathbb{D} \to \partial\mathbb{D}$ be a covering map of degree d. Then, there exists $h\colon \partial\mathbb{D} \to \partial\mathbb{D}$ continuous, monotone (but not necessarily strictly monotone) and surjective such that $h \circ g = f \circ h$.*

Proof. Take $\widehat{f}$ and $\widehat{g}$ lifts of f and g respectively. Then, $\widehat{f}$ and $\widehat{g}$ are diffeomorphisms of $\mathbb{R}$ and, since both have degree d, $\widehat{f}(x+1) = \widehat{f}(x) + d$ and $\widehat{g}(x+1) = \widehat{g}(x) + d$. Moreover, $(\widehat{f}^{-1})^{(n)}(x) = \frac{1}{d^n}$.

Consider the functional space:

$$\varepsilon := \{\phi\colon \mathbb{R} \to \mathbb{R}\colon \phi \text{ continuous, monotone and } \phi(x+1) = \phi(x) + 1\},$$

equipped with the following distance:

$$d(\phi_1, \phi_2) = \sup_x |\phi_1(x) - \phi_2(x)|.$$

We remark that non-strictly monotone functions are included in ε and that the supremum defining the distance is finite, because $\phi_1 - id$ and $\phi_2 - id$ are 1-periodic. Clearly, the space (ε, d) is complete.

Now consider the following functional:

$$\begin{aligned} T\colon \varepsilon &\to \varepsilon \\ \phi &\mapsto T\phi = \widehat{f}^{-1} \circ \phi \circ \widehat{g}. \end{aligned}$$

Observe that T is well-defined, i.e. $T\phi \in \varepsilon$. Clearly, $T\phi$ is monotone and continuous and: $T\phi(x+1) = T\phi(x) + 1$. Moreover, T is a contraction. Indeed,

$$d(T^n\phi_1, T^n\phi_2) = \sup_x \left| \widehat{f}^{-n}(\phi_1(\widehat{g}^n(x))) - \widehat{f}^{-n}(\phi_2(\widehat{g}^n(x))) \right| \leq \frac{1}{d^n} d(\phi_1, \phi_2).$$

Finally, applying Banach fixed-point theorem, there exists a unique $\widehat{h}$ satisfying $h \circ g = f \circ h$. Then, $\widehat{h}$ is the lift of a function h satisfying $h \circ g = f \circ h$, as required. □

As a consequence of the previous proposition, we will see that, given a finite Blaschke product g of degree d of elliptic or doubly-parabolic type, $g_{|\partial\mathbb{D}}$ is conjugate to $x \mapsto dx \mod 1$.

Corollary 3.3.4. *Let $g\colon \mathbb{D} \to \mathbb{D}$ be a finite Blaschke product of degree d of elliptic or doubly-parabolic type. Then, $g_{|\partial\mathbb{D}}$ is (topologically) conjugate to $x \mapsto dx \mod 1$.*

Proof. Recall that Blaschke products cannot have critical points in $\partial\mathbb{D}$ (Prop. 2.3.1), so Proposition 3.3.3 applies. We have to see that the map h of Proposition 3.3.3 is a conjugacy. It is enough to show that it is strictly monotone. Assume, on the contrary, that there exists $z \in \partial\mathbb{D}$ such that $I := h^{-1}(z)$ is an interval. Two possibilities arise: z being an eventually periodic point for the map $x \mapsto dx \mod 1$ or not.

In the case that z is an eventually periodic point for $x \mapsto dx \mod 1$, I must be an eventually periodic interval. Since h is surjective and monotone, $\cup_n g^n(I) \neq \partial\mathbb{D}$, and in fact omits more than two points. This would imply that $I \subset \mathcal{F}(g)$, but this is not possible since in the elliptic and in the doubly-parabolic case $\mathcal{J}(g) = \partial\mathbb{D}$.

If z is not an eventually periodic point for $x \mapsto dx \mod 1$, then all images of z are disjoint, and they cannot tend to a periodic cycle, because periodic points are repelling. Therefore, the same happens for the orbit of I under $g_{\partial\mathbb{D}}$: all images of the interval are disjoint, and they cannot tend to a periodic cycle. Then, I is a wandering domain for $g_{\partial\mathbb{D}}$. This cannot happen because $g_{\partial\mathbb{D}}$ is analytic and $\mathcal{C}^2(\partial\mathbb{D})$-functions do not have wandering domains (Denjoy's theorem, [MS93, Thm. I.2.1]). Therefore this possibility is also excluded, so h must be monotone. □

Therefore, in the elliptic and in the doubly-parabolic cases, the boundary map is topologically conjugate to $x \mapsto dx \mod 1$. This gives us, in particular, a concrete description of the boundary map in terms of fixed points, periodic orbits, and other topological invariants. However, it is not enough to prove ergodicity nor recurrence. Recall that, to do so, we need the conjugacy to be absolutely continuous and, *a priori*, we do not know that. Moreover, it is proved in [SS85;

Ham96] that, for inner functions of doubly-parabolic type, it is never the case. In the elliptic case, the conjugacy may be absolutely continuous or it may not. If it is, it implies, by results in Section 1.7, ergodicity and recurrence. Nevertheless, it is true that such maps are ergodic and recurrent, but another approach is needed, such as the one we present in the next section.

3.4 Ergodicity and recurrence of inner functions

Next, we study the ergodicity and recurrence for inner functions. Apart from the concrete cases of finite degree whose ergodic properties can be proved by hand, the idea is now to develop a general theory that includes all possible inner functions, even the ones of infinite degree. We aim to prove the ergodic properties of inner functions described in Table 1.

Due to the difficulties of studying the boundary map, the approach of [DM91] consists of using the harmonic measure to move the problem from the boundary to the interior of the disk, where the function is holomorphic. In this direction, they proved an equivalence between ergodicity and a fast decrease of the hyperbolic distance between consecutive points on the orbits (Thm. 3.4.1), and also between recurrence and a slow convergence of the orbits to the boundary (Thm. 3.4.4). The next step is to connect these properties of the inner map with the different types of inner functions according to Cowen's classification. This can be done perfectly for ergodicity, and we shall prove that the same result that holds for finite Blaschke products holds for general inner functions. However, in the case of recurrence this is no longer true: there exist inner functions of doubly-parabolic which are non-recurrent. Recall that this is impossible for finite Blaschke products.

Notice that there is a concrete case when the problem is easier: when the Denjoy-Wolff point is not a singularity. Indeed, if the Denjoy-Wolff point is not a singularity (and it is in the boundary), one can apply the linearization coordinates described in Sections 1.1.1 and 1.1.2. As it is detailed next, in the cases when we can ensure the existence of points of the Fatou set on $\partial\mathbb{D}$, it follows that the map is non-ergodic and non-recurrent. However, although it is true that the Julia set for inner functions of elliptic and doubly-parabolic type is $\partial\mathbb{D}$, this cannot be used to deduce any ergodic property. Recall that ergodic properties are stronger than topological properties, since they must hold for λ-almost every point. In what follows, we make all these ideas precise.

3.4.1 Ergodicity

We shall start by studying ergodicity. First, it is clear that, for the case when the Denjoy-Wolff point is non-singular and the map is of hyperbolic or simply-parabolic type, then $\mathcal{F}(g) \cap \partial\mathbb{D}$ is non-empty, so the boundary map cannot be ergodic. Even in the case when the Denjoy-Wolff point is singular, but $\mathcal{F}(g) \cap \partial\mathbb{D}$ is non-empty, non-ergodicity holds.

The general result concerning the ergodicity of the boundary map relate it with the hyperbolic distance ρ in the unit disk.

Theorem 3.4.1. (Characterization of ergodicity) [Doering-Mañé [DM91, Section 3]] *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function. Then, g^* is ergodic with respect to the Lebesgue measure if and only if, for all $z, w \in \mathbb{D}$, $\lim\limits_{n\to\infty} \rho(f^n(z), f^n(w)) = 0$.*

Proof. We are going to prove that, if the previous limit tends to zero, the map is ergodic. The other implication is much more involved, and we refer to [DM91, Thm. 3.1].

Let $A \subset \partial\mathbb{D}$ be a Borel set such that $A = (g^*)^{-1}(A)$. We want to prove that $\lambda(A) = 0$ or $\lambda(A) = 1$. Equivalently, if we denote by $\psi\colon \partial\mathbb{D} \to \partial\mathbb{D}$ the characteristic function of A, we have to see that ψ is constant. It is enough to see that $\widehat{\psi}$, the harmonic extension of ψ to $\mathbb{D}$, is constant.

First observe that, since A is invariant by f, $\psi \circ g^* = \psi$ and $\psi \circ (g^*)^n = \psi$. By Theorem 2.4.3, we have $\widehat{\psi} \circ g^n = \widehat{\psi \circ (g^*)^n} = \widehat{\psi}$. Therefore, for any $xz, w \in \mathbb{D}$, we have:

$$\widehat{\psi}(z) - \widehat{\psi}(w) = \widehat{\psi}(g^n(z)) - \widehat{\psi}(g^n(w)).$$

Fix $n \geq 0$ and let $T_n \colon \mathbb{D} \to \mathbb{D}$ be a Möbius transformation with $T_n(0) = g^n(z)$. Then,

$$\widehat{\psi}(g^n(z)) - \widehat{\psi}(g^n(w)) = \widehat{\psi \circ T_n}(0) - \widehat{\psi \circ T_n}(T_n^{-1} g^n(w)) = \int_{\partial\mathbb{D}} \psi(T_n(u))(1 - P_{T_n^{-1} g^n(w)}(u)) d\lambda(u),$$

where in the last equality we used Theorem 2.4.3 again.

Now, we claim that $\lim_{n\to\infty} T_n^{-1} g^n(w) = 0$. Indeed, it is enough to see that $\rho(0, T_n^{-1} g^n(w)) \to 0$, as $n \to \infty$. Applying that the hyperbolic metric is invariant under Möbius transformations, we have:

$$\rho(0, T_n^{-1} g^n(w)) = \rho(T_n(0), g^n(w)) = \rho(g^n(z), g^n(w)),$$

and this last expression tends to zero by hypothesis. This implies that

$$\lim_{n\to\infty} \sup_{|w|=1} \left|1 - P(T_n^{-1} g^n(w))\right| = 0,$$

and therefore, $\lim_{n\to\infty} \widehat{\psi}(g^n(z)) - \widehat{\psi}(g^n(w)) = 0$. Since for all n, $\widehat{\psi}(z) - \widehat{\psi}(w) = \widehat{\psi}(g^n(z)) - \widehat{\psi}(g^n(w))$, we deduce that $\widehat{\psi}(z) - \widehat{\psi}(w) = 0$. But z, w are arbitrary, so we deduce that $\widehat{\psi}$ is constant, as desired. □

In view of the last theorem, one can determine precisely which types of inner functions, according to Cowen's classification, are ergodic and which are not. Indeed, it is clear that, when the Denjoy-Wolff point lies in $\mathbb{D}$, the limit of the hyperbolic distance between any two orbits must tend to zero. In case the Denjoy-Wolff point is in the boundary, the previous limit tends to zero if and only if the map is of doubly-parabolic type, according to Theorems 1.3.8 and 1.3.9. This is stated in the following corollary.

Corollary 3.4.2. (Ergodic inner functions) *Let $g \colon \mathbb{D} \to \mathbb{D}$ be an inner function.*

(a) If g is of elliptic or doubly-parabolic type, then g^* is ergodic with respect to the Lebesgue measure.

(b) If g is of hyperbolic or simply-parabolic type, then g^* is non-ergodic with respect to the Lebesgue measure.

3.4.2 Recurrence

As before, proving non-recurrence is easier than proving recurrence. Indeed, for the hyperbolic and simply-parabolic cases, if the Fatou set in the unit circle $\mathcal{F}(g) \cap \partial\mathbb{D}$ is non-empty, the map cannot be recurrent. Points in the Fatou set must converge to the Denjoy-Wolff point and, since the Fatou set is open, it has positive λ-measure. Another easy case is when the Denjoy-Wolff point lies inside $\mathbb{D}$.

Proposition 3.4.3. (Recurrence in the elliptic case) *Let g be an inner function (of finite or infinite degree) with Denjoy-Wolff point $p \in \mathbb{D}$. Then, g is recurrent.*

Proof. The proof is straightforward combining the Poincaré Recurrence Theorem (1.6.2) with the fact that the harmonic measure ω_p is invariant by g (Corollary 2.4.5). ☐

One may be tempted to try to find a finite invariant measure for the case of the Denjoy-Wolff point in $\partial\mathbb{D}$, in order to prove recurrence. However, this does not work: in [DM91, Thm. A] it is proved that there does not exist a finite invariant measure when the Denjoy-Wolff point lies in $\partial\mathbb{D}$. Another approach is needed. Indeed, this problem was solved by J. Aaronson ([Aar78]). The proof can also be found in [DM91, Thms. 4.1, 4.2].

Theorem 3.4.4. (Recurrence dichotomy) [Aaronson] *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function with Denjoy-Wolff point p. Then the following hold:*

1. *If $\sum_{n=1}^{\infty}(1-|g^n(z)|) < \infty$ for some $z \in \mathbb{D}$, then $p \in \partial\mathbb{D}$ and $(g^*)^n(z)$ converges to p for λ-almost every $z \in \mathbb{D}$.*
2. *If $\sum_{n=1}^{\infty}(1-|g^n(z)|) = \infty$ for some $z \in \mathbb{D}$, then g^* is recurrent in $\partial\mathbb{D}$ with respect to the Lebesgue measure.*

We refer to the sum $\sum_{n=1}^{\infty}(1-|g^n(z)|)$ as **Aaronson's sum** (at the point z). Observe that this sum being convergent is the same as asking that the points in the orbit of z satisfy the Blaschke condition, i.e. there exists a bounded holomorphic function in $\mathbb{D}$ with this sequence of zeros.

Observe that both possibilities in Theorem 3.4.4 are mutually exclusive. Indeed, it is not possible that λ-almost every point converges to the Denjoy-Wolff point and that λ-almost every point returns infinitely many times to any Borel set containing it. Therefore, the condition that Aaronson's sum converges or diverges at one point can be replaced by the same condition holding for all points in $\mathbb{D}$.

Proof of theorem 3.4.4. 1. We start by proving that, if the sum converges, then λ-almost every point in the boundary must converge to the Denjoy-Wolff point. It is enough to see that, for all $S \subset \partial\mathbb{D}$ neighbourhood of p, if we consider

$$\widetilde{S} \coloneqq \bigcup_{N\geq 0}\bigcap_{n\geq N}(g^*)^{-n}(S) = \{x \in \partial\mathbb{D}\colon \exists N \geq 0 \text{ such that } \forall n \geq N,\ (g^*)^n(x) \in S\},$$

it holds $\lambda(\widetilde{S}) = 1$. Once this is proved, taking a nested sequence of neighbourhoods of p, with diameter tending to zero, we get the desired result.

Let $S \subset \partial\mathbb{D}$ be a neighbourhood of p and consider ψ to be the characteristic function of S^c. Let $z \in \mathbb{D}$ be the point for which the Aaronson's sum converges. By the definition of harmonic measure and Poison kernel (compare with Section 2.4), we have the following:

$$\omega_z((g^*)^{-n}(S^c)) = \widehat{\psi}(g^n(z)),$$

$$\frac{\widehat{\psi}(g^n(z))}{1-|g^n(z)|} = (1-|g^n(z)|)\int_{S^c}\frac{1}{|w-g^n(z)|^2}d\lambda(w).$$

From the second equality, letting $n \to \infty$, we get:

$$\frac{\widehat{\psi}(g^n(z))}{1-|g^n(z)|} \to 2\int_{S^c}\frac{1}{|w-p|^2}d\lambda(w),$$

where this last integral converges because $p \in S$. Therefore, there exists $C \geq 0$ such that $\widehat{\psi}(g^n(z)) \leq C(1-|g^n(z)|)$, for all $n \geq 0$. Therefore,

$$\sum_n \widehat{\psi}(g^n(z)) \leq C\sum_n(1-|g^n(z)|).$$

In particular, since the right hand-side sum is assumed to be convergent, we have

$$\inf_{N\geq 0}\sum_{n\geq N}\widehat{\psi}(g^n(z))=0.$$

But precisely from the definition of $\widetilde{S}$, we have

$$\widetilde{S}^c=\bigcup_{N\geq 0}\bigcap_{n\geq N}(g^*)^{-n}(S^c),$$

so $\omega_z(\widetilde{S}^c)\leq\inf_{N\geq 0}\sum_{n\geq N}\widehat{\psi}(g^n(z))$. We conclude $\omega_z(\widetilde{S}^c)=0$, and therefore $\lambda(\widetilde{S})=1$, as desired.

2. Now we need to prove that, if the sum diverges for a fixed $z\in\mathbb{D}$, the map is recurrent. By Corollary 1.6.3, it is enough to see that, for any Borel set $S\subset\partial\mathbb{D}$ such that $(g^*)^{-1}(S)\subset S$, it holds $\omega_z((g^*)^{-1}(S))=\omega_z(S)$. Equivalently, we prove that, for any Borel set $A\subset S\smallsetminus(g^*)^{-1}(S)$, it holds $\omega_z(A)=0$. On the contrary, assume $\omega_z(A)>0$ and denote by ψ the characteristic function of A. As before, we have:

$$\omega_z((g^*)^{-n}(A))=\widehat{\psi}(g^n(z)),$$

$$\frac{\widehat{\psi}(g^n(z))}{1-|g^n(z)|}=(1-|g^n(z)|)\int_A\frac{1}{|w-g^n(z)|^2}d\lambda(w).$$

From the second equality, letting $n\to\infty$, we get:

$$\frac{\widehat{\psi}(g^n(z))}{1-|g^n(z)|}\to(1+|p|)\int_A\frac{1}{|w-p|^2}d\lambda(w),$$

where this last integral converges because p cannot be in A, since p is fixed and A is not invariant. Moreover, since $\omega_z(A)>0$, the integral is positive. Therefore, there exists $C>0$ such that $\widehat{\psi}(g^n(z))\geq C(1-|g^n(z)|)$, for all $n\geq 0$.

On the other hand, observe that since $A\subset S\smallsetminus(g^*)^{-1}(S)$, we have $(g^*)^{-i}(A)\cap(g^*)^{-j}(A)\neq\emptyset$, for all $i\neq j$. Therefore, we have:

$$1\geq\omega_z\bigcup_{n\geq 0}(g^*)^{-n}(A)=\sum_{n\geq 0}\omega_z((g^*)^{-n}(A))=\sum_{n\geq 0}\widehat{\psi}(g^n(z)),$$

but we know that this last term is bounded from below by $C(1-|g^n(z)|)$, whose sum, for $n\geq 0$ diverges. This is a contradiction, so necessarily $\omega_z(A)=0$, as desired. □

Next, we want to relate the condition of the convergence or divergence of the Aaronson's sum with the type of inner functions in Cowen's classification. For the elliptic case, it is clear that the sum is infinite, so the map is recurrent. Recall that in this case there is no need to apply Aaronson's theorem and it can be deduced directly from Poincaré Recurrence Theorem (Prop. 3.4.3). For the hyperbolic and simply-parabolic cases, we prove next that Aaronson's sum always converges, so the boundary map is non-recurrent.

Proposition 3.4.5. (Non-recurrent inner functions) *Let $g\colon\mathbb{D}\to\mathbb{D}$ be an inner function of hyperbolic or simply-parabolic type. Then, g^* is non-recurrent with respect to the Lebesgue measure. Moreover, $(g^*)^n(z)$ converges to the Denjoy-Wolff point for λ-almost every $z\in\mathbb{D}$.*

Proof. In both cases we are going to see that the Aaronson's sum at 0 is convergent. We start with the hyperbolic case. Assume $p\in\partial\mathbb{D}$ is the Denjoy-Wolff point. Applying the stronger version of Julia-Wolff lemma (1.3.4), we have:

$$|p-g^n(0)|^2\leq(g'(p))^n(1-|g^n(0)|^2)\leq(g'(p))^n.$$

On the other hand, by the triangle inequality,

$$1 - |g^n(0)| \leq |p - g^n(0)| .$$

Putting together both inequalities and applying that $\sum_{n=1}^{\infty}(g'(p))^n$ converges, because it is a geometric series with ratio $|g'(p)| < 1$, we get the desired result.

Now consider the simply-parabolic case. Applying that in the simply-parabolic the inner function g is semi-conjugate to $id_{\mathbb{H}} + 1$ by σ, we have $w_n := \sigma(g^n(0)) = \sigma(0) + n$, for all $n \geq 0$. Now observe that the Blaschke condition $\sum_{n=1}^{\infty}(1 - |z_n|) < \infty$ for sequences $\{z_n\}_n$ in the unit disk is equivalent to $\sum_{n=1}^{\infty} \frac{\operatorname{Im}(w_n)}{|w_n+i|^2} < \infty$ for sequences $\{w_n\}_n$ in the upper half-plane.

Clearly, the sequence $\{\sigma(0) + n\}_n$ satisfies the Blaschke condition in $\mathbb{H}$. Therefore, there exists a bounded holomorphic function F in the upper half-plane whose zeros are precisely the points w_n. Then, $G = F \circ \sigma$ is a non-constant bounded holomorphic function. Observe that the points in the orbit of 0 are zeros of G:

$$G(f^n(0)) = F(\sigma(g^n(0))) = F(\sigma(0) + n) = 0.$$

Therefore, $\{g^n(0)\}_n$ is the sequence of zeros of a bounded holomorphic function in $\mathbb{D}$, so $\sum_{n=1}^{\infty}(1 - |g^n(0)|) < \infty$. □

For the case of a doubly-parabolic inner function, it is not clear whether the map is recurrent or not. In fact, there are examples of inner functions of doubly-parabolic type where the boundary map is recurrent (see Examples B.3.1, B.4.5) and examples where it is not (see Example B.4.6). However, it can be proved that, if the Denjoy-Wolff point is not a singularity, then the map is not recurrent, using a general argument on parabolic basins that we develop next. In particular, this holds for doubly-parabolic inner functions of finite degree because, at the end, they are rational functions for which the unit disk is a parabolic basin.

Keeping in mind the dichotomy of Aaronson's theorem (3.4.4), we are going to give conditions for the sum $\sum_{n=1}^{\infty}(1 - |g^n(z)|)$ for the corresponding inner function g to diverge and therefore the map to be recurrent.

Theorem 3.4.6. (Recurrence for parabolic basins) [Doering-Mañé [DM91, Thm. 6.1]] *Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and let U be a simply-connected invariant parabolic basin for f. Let g be the inner function associated to f. Then, there exists $z_0 \in \mathbb{D}$ such that*

$$\sum_{n=1}^{\infty}(1 - |g^n(z_0)|) = \infty.$$

Before starting the proof, notice that the result is quite expectable in the sense that in an attracting petal iterates converge to the attracting direction at a slow rate and the sum in Aaronson's theorem quantifies the rate of convergence to the boundary.

Proof. Without lost of generality, we shall assume that the parabolic fixed point is the origin, and we have the following Taylor series around it:

$$f(z) = z - z^{n+1} + \sum_{k>n+1} a_k z^k,$$

with $n \geq 1$. Therefore, we are assuming that one attracting direction is the positive real axis. Conjugating with a rotation if needed, we shall assume that this attracting direction is the one corresponding to the parabolic basin U. Then the following set is a petal for U:

$$S := \left\{ z \in \mathbb{C} \colon \ |z| < \delta, \ |\operatorname{Arg}(z)| < \frac{\pi(1-\varepsilon)}{n} \right\},$$

taking ε small enough and $\delta = \frac{1-\varepsilon}{n}$. For any $z \in S$, we have $\lim_{k\to\infty} f^k(z)\sqrt[n]{k} = \frac{1}{\sqrt[n]{n}} \in \mathbb{R}_+$ (compare with Theorem 1.1.8). Therefore, for all k big enough, $\left|f^k(z)\right| \geq C\left(\frac{1}{k}\right)^{\frac{1}{n}}$.

Now consider $u\colon S \to \mathbb{C}$ to be the branch of $z^{\frac{1}{2\delta}}$ that fixes $\mathbb{R}_+$. This is well-defined and holomorphic in S. Geometrically, u expands the circular sector S to a half-circle, whose straight boundary coincides with the imaginary axis. Consider the harmonic function $h(z) := \operatorname{Re}(u(z))$. Clearly, for all $z \in \partial S \cap \{|z| < \delta\}$, $h(z) = 0$.

Next consider φ to be a Riemann map from $\mathbb{D}$ to U and $\widehat{S} := \varphi^{-1}(S)$. The function $h \circ \varphi$ is harmonic and for some $K > 0$ satisfies $-K \log|z| \geq h \circ \varphi(z)$. Indeed, since the function is harmonic it is enough to prove the inequality in $\partial\widehat{S}$. For points in $\partial\widehat{S} \cap \{|z| = \delta\}$, assuming δ small enough, $\log|z| < 0$ and $h \circ \varphi(z) = 0$. The set $\partial\widehat{S} \cap \{|z| < \delta\}$ is compact, so $\frac{h\circ\varphi(z)}{\log|z|}$ must attain a maximum, and this gives the desired inequality. Compare with Figure 7.

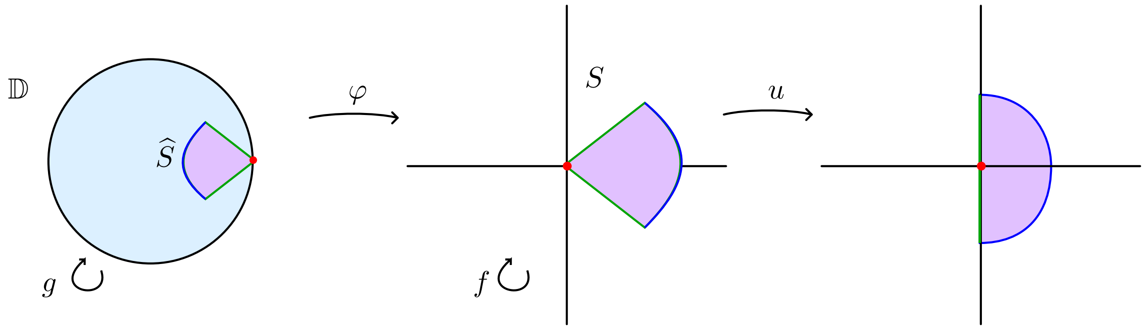


Figure 7: The absorbing domain $\widehat{S}$ for the associated inner function g is mapped to the petal S for f. The map u acts on S multiplying the argument by $\frac{1}{2\delta}$, so part of the boundary is mapped to the imaginary axis.

Choose $z_0 \in \widehat{S}$ and let g be the inner function associated to the parabolic basin. Since $\widehat{S}$ is an absorbing domain, $g^k(z_0) \in \widehat{S}$ for all k and

$$-K\log\left|g^k(z_0)\right| \geq h \circ \varphi(g^k(z_0)) = h(f^k(\varphi(z_0))) = \operatorname{Re}(u(f^k(\varphi(z_0)))).$$

Applying that u fixes $\mathbb{R}_+$ and $\mathbb{R}_+$ is precisely an attracting direction for f at the origin, we deduce that, for k big enough,

$$\operatorname{Re}(u(f^k(\varphi(z_0)))) \geq \frac{1}{2}\left|u(f^k(\varphi(z_0)))\right| = \frac{1}{2}\left|f^k(\varphi(z_0))\right|^{\frac{1}{2\delta}}.$$

Joining the previous inequalities, for k big enough,

$$-K\log\left|g^k(z_0)\right| \geq \frac{1}{2}\left|f^k(\varphi(z_0))\right|^{\frac{1}{2\delta}} \geq \frac{1}{2}C^{\frac{1}{2\delta}}\left(\frac{1}{k}\right)^{\frac{n}{2\delta}} = \frac{1}{2}C^{\frac{1}{2\delta}}\left(\frac{1}{k}\right)^{\frac{1}{2(1-\varepsilon)}},$$

where K and C are constants, and ε is arbitrarily small. Therefore the series $\sum_{n=1}^{\infty}(1-|g^n(z_0)|)$ diverges, as desired. $\square$

Corollary 3.4.7. (Recurrence for parabolic basins) *Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function and let U be an invariant parabolic basin for f. Then, $f_{|\partial U}$ is recurrent with respect to the harmonic measure ω_U.*

Proof. The proof is straightforward from the previous theorem (3.4.6) and Aaronson's dichotomy (Thm. 3.4.4). $\square$

Corollary 3.4.8. (Inner functions of doubly-parabolic type) *Let $g\colon \mathbb{D} \to \mathbb{D}$ be an inner function of doubly-parabolic type with Denjoy-Wolff point $p \in \partial\mathbb{D}$. If p is not a singularity for g, then g^* is recurrent with respect to the Lebesgue measure.*

Proof. For the proof of Theorem 3.4.6, only the local development near the parabolic fixed point and the simply-connectivity of the parabolic basin are needed. Both hypothesis are satisfied when g is an inner function of doubly parabolic type and its Denjoy-Wolff point is non-singular. Therefore, the corollary is deduced directly from the theorem (taking $f = g$ and the Riemann map to be the identity). □

As a remark, a different proof of the last corollary can be found in [BFJK19, Thm. B].

4 Dynamics on the boundary of Fatou components

In the last section, we described the dynamics of the boundary map of an inner function and its ergodic properties. As an application, we wish to describe the dynamics on the boundary of a simply-connected invariant Fatou component of a holomorphic map f, either rational or transcendental entire. As it was detailed in Section 2.5, given an invariant simply-connected Fatou component U for a holomorphic function f, one can conjugate the dynamics of $f_{|U}$ with those of the associated inner function g. Moreover, the ergodicity and the recurrence of g^* pass to $f_{|\partial U}$ by means of considering the harmonic measure ω_U on ∂U.

In this section, we describe some results concerning the dynamics of f on the boundary of its periodic Fatou components, distinguishing between finite and infinite degree. Moreover, we include some brief notes on the local-connectivity of the boundary of such components. Although this is not the main interest of the project, it connects with the problems we address. Indeed, if the boundary is locally connected, then the conjugacy provided by the Riemann map extends to the boundary.

4.1 Dynamics on the boundary of periodic Fatou components of finite degree

First we consider invariant simply-connected Fatou components of finite degree. In this case, the associated inner function is a finite Blaschke product. Their dynamical and ergodic properties were studied in Section 3.3, and from them we deduce the following results.

4.1.1 Attracting basins and parabolic basins of finite degree

The associated inner function for attracting basins and parabolic basins of finite degree are finite Blaschke products of elliptic and doubly-parabolic type. We know that these maps are ergodic (Thm. 3.4.1) and recurrent (Prop. 3.4.3, Corol. 3.4.8).

Theorem 4.1.1. *Let f be a holomorphic map either of $\mathbb{C}$ or of $\widehat{\mathbb{C}}$ and let U be an invariant attracting basin or parabolic basin for f of finite degree. Assume U is simply-connected. Then, $f_{|\partial U}$ is ergodic and recurrent with respect to the harmonic measure ω_U.*

Observe that this implies, by Proposition 2.5.5, that ω_U-almost every point in ∂U has dense orbit in ∂U.

In the case of an attracting basin with locally connected boundary, it can be deduced from the results in Section 3.3 that the boundary map is conjugate to $x \mapsto dx \mod 1$, where d is the degree of f inside the basin. The condition of the boundary being locally connected holds in many cases, for example when the iterated map is rational and geometrically finite, i.e. when the orbit of every critical point in the Julia set is finite. More information about locally connected boundaries can be found in [Mil06, Section 19] and [Ste93, Section 5].

4.1.2 Siegel disks

Recall that a Siegel disk U is a simply-connected Fatou component of f whose associated inner function is an irrational rotation. Therefore, simply applying the results on Section 3.2, we get that $f_{|\partial U}$ is ergodic and recurrent with respect to the harmonic measure ω_U.

Theorem 4.1.2. *Let f be a holomorphic map either of $\mathbb{C}$ or of $\widehat{\mathbb{C}}$ and let U be an invariant Siegel disk for f. Then, $f_{|\partial U}$ is ergodic and recurrent with respect to the harmonic measure ω_U.*

By Proposition 2.5.5, ω_U-almost every point in ∂U has dense orbit in ∂U.

Before going on, let us make some remarks. First, it is not possible that an associated inner function is a rational rotation. Otherwise, there would exist some integer, say k, such that f restricted to this Fatou component is the identity. By analytic continuation, $f^k \equiv id$, and this contradicts the invariance of the Julia and the Fatou sets.

Second, assume that the boundary of the Siegel disk U is locally connected. In this case, $f_{|\partial U}$ is conjugate to $x \mapsto x + \theta \mod 1$. Therefore, all orbits are dense in ∂U and there are no periodic points. However, repelling periodic points are dense in the Julia set. Therefore, although there are no periodic points in the boundary of the Siegel disk, they accumulate infinitely close to it. Examples and conditions for the boundary of a Siegel disk to be locally connected can be found in [Mil06, p. 192-193], whereas in [Che11] there is an example of a Siegel disk for a transcendental entire function with non-locally connected boundary. Moreover, for a transcendental entire function, if the Siegel disk is unbounded and ∞ is accessible from it, then the boundary is non-locally connected ([BD99, Thm. 1.1]).

4.1.3 Baker domains of finite degree

Finally, we study the ergodic properties on the boundary of Baker domains of finite degree. Recall that, in this case, the three types of Cowen's classification (with Denjoy-Wolff point on the boundary) are realizable, and they present different dynamics. Indeed, taking into account Theorem 3.4.1 and Corollary 3.4.8, one deduces that the boundary map of a doubly-parabolic Baker domain is ergodic and recurrent, whereas, by Theorem 3.3.2, for a hyperbolic or simply-parabolic Baker domain it is non-ergodic and non-recurrent.

Theorem 4.1.3. *Let f be a transcendental entire function and let U be a Baker domain for f of finite degree. If U is of doubly-parabolic type, then $f_{|\partial U}$ is ergodic and recurrent with respect to the harmonic measure ω_U. On the other hand, if U is either of hyperbolic or of simply-parabolic type, then $f_{|\partial U}$ is non-ergodic and non-recurrent with respect to the harmonic measure ω_U.*

When studying the iteration of Baker domains, the escaping set $\mathcal{I}(f)$ plays a significant role. Recall that the escaping set consist on all the points that converge to ∞ under iteration (compare with Definition 1.1.2). It is clear that Baker domains are included in the escaping set. It is natural to ask whether there are points in their boundary that escape to infinity and if so, how large is the set of such points.

We first focus on the doubly-parabolic case. For these Baker domains, the boundary map is ergodic and recurrent, so ω_U-almost every point in ∂U has dense orbit in ∂U. Escaping points cannot have dense orbit, since they converge to ∞, so they have zero harmonic measure.

For a hyperbolic or simply-parabolic Baker domain, its boundary map is non-recurrent and non-ergodic, so points with dense orbit have harmonic measure zero. Moreover, the escaping set has full harmonic measure. This was proved first in [RS18] in the univalent case, and then generalized for finite degree in [BFJK19, Theorem A].

Theorem 4.1.4. (Escaping points have full measure) *Let $f\colon \mathbb{C} \to \mathbb{C}$ be holomorphic and U be an invariant Baker domain for f of finite degree and of hyperbolic or simply-parabolic type. Then, $\mathcal{I}(f) \cap \partial U$ (the set of escaping points in the boundary of U) has full harmonic measure.*

Since the proof is quite involved, we shall omit it. However, let us comment on why the theorem is so difficult to prove. Under the hypothesis of the theorem, the corresponding associated inner function is a finite Blaschke product of hyperbolic or simply-parabolic type, so λ-almost every point in $\partial \mathbb{D}$ converge to the Denjoy-Wolff point. However, this does not imply directly the theorem.

Although ergodicity or recurrence for the inner function imply ergodicity or recurrence on the boundary of the Fatou component, we cannot extrapolate the convergence to the Denjoy-Wolff point to the boundary of the Fatou component. This is because ergodicity and recurrence are properties of the measure, but convergence is not. Recall that the Riemann map can be highly discontinuous, so convergence to ∞ cannot be granted even for the points whose preimage in $\partial\mathbb{D}$ converge to the Denjoy-Wolff point.

Regarding the question of the local-connectivity of the boundary, I. Baker and P. Domínguez showed that the boundary may be locally connected only in the case of univalence ([BD99, Corollary 1.3]). In the paper [BF01], K. Barański and N. Fagella give examples of univalent Baker domains both with and without locally connected boundary.

4.2 Dynamics on the boundary of periodic Fatou components of infinite degree

Recall that Fatou components with infinite degree can only appear for transcendental functions and must be unbounded. Rational functions always have finite degree in $\widehat{\mathbb{C}}$, so it must be also the case when restricted to the Fatou components. On the other hand, in a Fatou component of infinite degree, preimages of the same point accumulate at the boundary, so they must accumulate at ∞ (the essential singularity) and the Fatou component is therefore unbounded.

As before, ergodic properties on the boundary of a Fatou component are deduced from the ergodic properties of the boundary map of the associated inner function. Therefore, by results 3.4.2, 3.4.3 and 3.4.7, one deduces the ergodicity and the recurrence for attracting and parabolic basins, as before.

Theorem 4.2.1. *Let f be a holomorphic map either of $\mathbb{C}$ or of $\widehat{\mathbb{C}}$ and let U be an invariant basin of attraction or parabolic basin (either of finite or infinite degree). Then, $f_{|\partial U}$ is ergodic and recurrent with respect to the harmonic measure ω_U.*

The case Baker domains is not that obvious, and we develop it next.

As in the case of invariant Fatou components of finite degree, next we give some basic notions on the local-connectivity of the boundary. In the case of invariant attracting and parabolic basins of infinite degree of transcendental entire functions, for which ∞ is accessible from them, their boundary is always non-locally connected ([BD99, Thm. 1.1]). The same holds for Baker domains in which the map is not univalent ([BD99, Corollary 1.3]), so in particular in the case of infinite degree. For transcendental meromorphic functions, there exists such periodic Fatou components with locally connected boundary (see Examples B.4.5, B.4.6).

4.2.1 Baker domains of infinite degree

Concerning the dynamics on the boundary, recall Baker domains of doubly-parabolic type behave distinctly from the simply-parabolic and the hyperbolic ones. Directly from the properties of the inner functions, we get the following result.

Theorem 4.2.2. *Let f be a transcendental entire function and let U be a Baker domain for f. If U is of doubly-parabolic type, then $f_{|\partial U}$ is ergodic with respect to the harmonic measure ω_U. On the other hand, if U is either of hyperbolic or of simply-parabolic type, then $f_{|\partial U}$ is non-ergodic and non-recurrent with respect to the harmonic measure ω_U.*

We focus first on Baker domains of doubly-parabolic type. We cannot deduce directly recurrence for its boundary map. In the case that the associated inner function has non-singular

Denjoy-Wolff point, Corollary 3.4.8 applies, implying recurrence. However, there are examples of Baker domains where the associated inner function has singular Denjoy-Wolff point, so the map may be recurrent or not. Examples of both cases can be found (see Appendix C.2). An interesting question may be to find a characterization for recurrence and non-recurrence, but up to now it remains as an open question. However there are partial results giving sufficient conditions for the map to be recurrent, for example the following one, relating the recurrence with the hyperbolic metric on the Baker domain.

Theorem 4.2.3. (Recurrence for doubly-parabolic Baker domains)[[BFJK19, Thm. C]] *Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic map and let U be an invariant Baker domain for f, such that*

$$\rho_U(f^{n+1}(z), f^n(z)) \leq \frac{1}{n} + O\left(\frac{1}{n^r}\right)$$

as $n \to \infty$ and $r > 1$. Then, $f_{|\partial U}$ is recurrent with respect to the harmonic measure ω_U.

Now consider the case of invariant Baker domains of hyperbolic or simply-parabolic type. It is clear that the boundary map is non-ergodic and non-recurrent. However, one may ask if an analogous result to Theorem 4.1.4, i.e. whether escaping points have full harmonic measure or not. Although it can be seen directly from the proof that Theorem 4.1.4 holds in the case that the Denjoy-Wolff point is non-singular, it is an open question if escaping points are dense in the boundary of an arbitrary invariant Baker domain of hyperbolic or simply-parabolic type.

5 Study of the function $f(z) = z + e^{-z}$

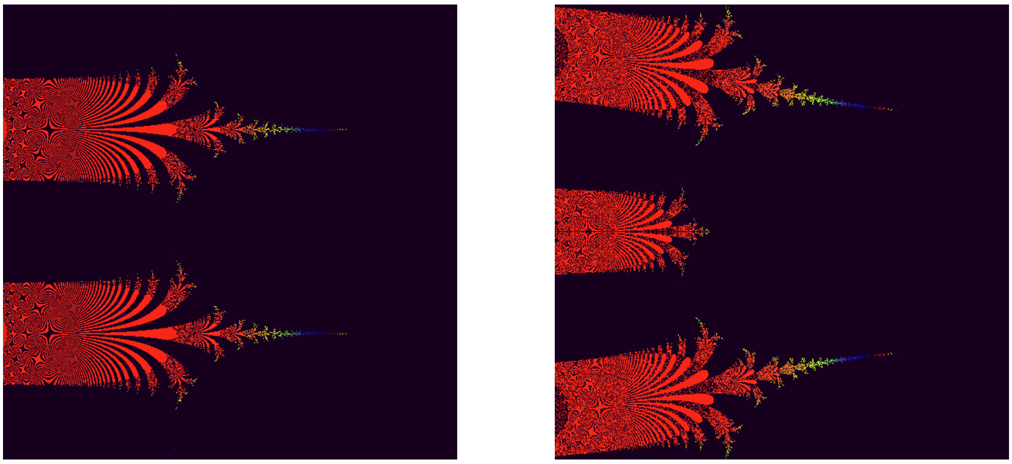
In this section we shall study closely the example

$$f(z) = z + e^{-z},$$

which was also considered in [BD99; FH06; BFJK19].

In [BD99, Section 5] and [FH06, Example 3], it is proved the existence of infinitely many invariant Baker domains for f. Namely, there exists exactly one invariant Baker domain U_k in each strip $\{(2k-1)\pi < \text{Im } z < (2k+1)\pi\}$, $k \in \mathbb{Z}$, being the dynamics on each one the same, since $f(z + 2k\pi i) = f(z) + 2k\pi i$, for all $z \in \mathbb{C}$. To prove this, it is convenient to work with the map $h(w) = we^{-w}$, which is semiconjugate to f by $w = e^{-z}$. Moreover, the Fatou set $\mathcal{F}(f)$ consists precisely on the invariant Baker domains and their preimages.

We study the dynamics of h in Section 5.1, and we prove the existence of the invariant Baker domains in Section 5.2. It can also be shown that they are of doubly-parabolic type and, moreover, the associated inner function can be computed explicitly. Some of the proofs presented in the previously mentioned papers are now simplified by the use of some tools developed throughout the project.



$z \mapsto f(z) = z + e^{-z}$ $\qquad$ $w \mapsto h(w) = we^{-w}$

Figure 8: On the left, the dynamical plane of $f(z) = z + e^{-z}$. On the right, the dynamical plane of $h(w) = we^{-w}$. The map f is semiconjugate to g by $w = e^{-z}$.

Additionally, in [BD99, Section 6] they describe the topology of ∂U_k. In [BFJK19, Example 1.2], they describe some dynamical sets on ∂U_k in terms of measure, as applications of some very general theorems. They also conjectured that all escaping points in ∂U_k are non-accessible from U_k, while accessible repelling points are dense in ∂U_k. Our goal in this section is to work towards proving the conjecture. In particular, in Section 5.4, we prove that all repelling periodic points in ∂U_k are accessible, and we give a complete characterization of them by means of the inner function. In Section 5.5, we construct uncountably many curves of non-accessible escaping points. Although this does not prove the conjecture, it is a first step on its proof.

Moreover, we aim to give stronger results for this example using it as a toy model for doubly-parabolic domains of finite degree. From these results we will try to state some conjectures for the general case.

5.1 The parabolic basin of $h(w) = we^{-w}$

We start by studying the map $h(w) = we^{-w}$. Solving the equation $h(w) = w$, one finds that the fixed points of h are $w_k = 2k\pi i$, with $k \in \mathbb{Z}$. Computing the multiplier, one shows that $w_0 = 0$ is a parabolic fixed point, while the other ones are repelling. The Taylor development of h near 0 yields:

$$h(w) = w - w^2 + \mathcal{O}(w^3).$$

From the Leau-Fatou Flower Theorem (1.1.8) we know that there are exactly one attracting and one repelling direction of h at 0. More precisely, they are contained in the positive and in the negative real axis, respectively. Let us denote by $\mathcal{A}$, the parabolic basin of 0; by $\mathcal{A}_0$, the immediate parabolic basin; and by $\mathcal{P}$, any petal in $\mathcal{A}_0$. Recall that the petal $\mathcal{P}$ can be chosen, without loss of generality, such that its boundary is a Jordan curve which is tangent to $\mathbb{R}_-$. By studying the map restricted to the real line (Figure 9), one deduces that $\mathbb{R}_+ \subset \mathcal{A}_0$ and $\mathbb{R}_- \subset \mathcal{J}(h)$.

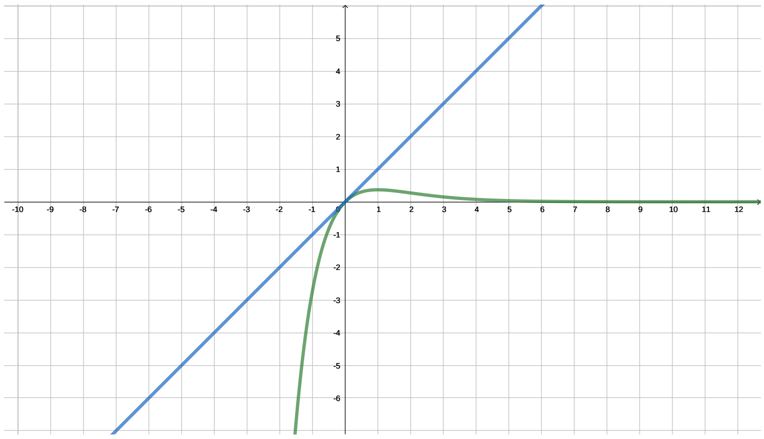


Figure 9: Plot of the real function $h(x) = xe^{-x}$ (green), together with the diagonal $y = x$ (blue). It is clear that 0 is a parabolic fixed point. Points in $\mathbb{R}_+$ are attracted to 0, while points in $\mathbb{R}_-$ converge to $-\infty$ exponentially fast. Therefore, $\mathbb{R}_+$ is included in the immediate basin of attraction $\mathcal{A}_0$ of 0, while $\mathbb{R}_-$ is contained in the Julia set $\mathcal{J}(h)$.

Next we find the singular values for h. Since $h'(w) = (1 - w)e^{-w}$, it is clear that the only critical point is 1, with critical value $\frac{1}{e}$. The only finite asymptotical value is 0, which is precisely the parabolic fixed point. Therefore, the function is critically finite, so neither Baker nor wandering domains are possible ([EL84; GK86]). Moreover, since the free singular value is included in the immediate basin of attraction $\mathcal{A}_0$ of the parabolic fixed point, we deduce that $\mathcal{F}(h) = \mathcal{A}$. We remark that, differently from the example considered by Devaney and Golberg (see C.1), the Fatou set $\mathcal{F}(h)$ consists of infinitely many connected components, being $\mathcal{A}_0$ the only Fatou component forward invariant while the others are preimages of it. The following lemma gives a more detailed description of the situation.

Lemma 5.1.1. *Let h, $\mathcal{A}$, $\mathcal{A}_0$ and $\mathcal{P}$ be as above. Then,*

1. *$\mathcal{A}_0$ is unbounded but contained in the horizontal strip $\{|Im\ w| < 2\pi\}$.*
2. *As $Re\ w \to \infty$, the basin $\mathcal{A}_0$ approaches asymptotically the strip $\{|Im\ w| < \pi\}$.*
3. *The map h has degree two on $\mathcal{A}_0$.*

Proof. 1. Clearly, $\mathcal{A}_0$ is unbounded because it contains $\mathbb{R}_+$. To see that $\mathcal{A}_0$ has bounded imaginary part, we compute the preimages of $\mathbb{R}$ by h. Observe that the imaginary part of $h(w)$ can be written as

$$\text{Im } h(x+iy) = \text{Im } (x+iy)e^{-x-iy} = y\cos y - x\sin y.$$

Equating this to zero, we get that either $y=0$ or $x = \dfrac{y}{\tan y}$. This last equation is satisfied by infinitely many curves whose real part is unbounded both to the right and to the left, and one bended curve going through 1. These curves are alternatively preimages of $\mathbb{R}_+$ and of $\mathbb{R}_-$, as can be seen in Figure 10. By the invariance of the Fatou and the Julia sets, preimages of $\mathbb{R}_+$ lie in the Fatou set and preimages of $\mathbb{R}_-$ lie in the Julia set. Therefore, the immediate parabolic basin $\mathcal{A}_0$ must lie in between the lines $|\text{Im } w| = 2\pi$.

As a remark, from Figure 10, we also deduce that the real part of $\mathcal{A}_0$ is unbounded both from the left and from the right.

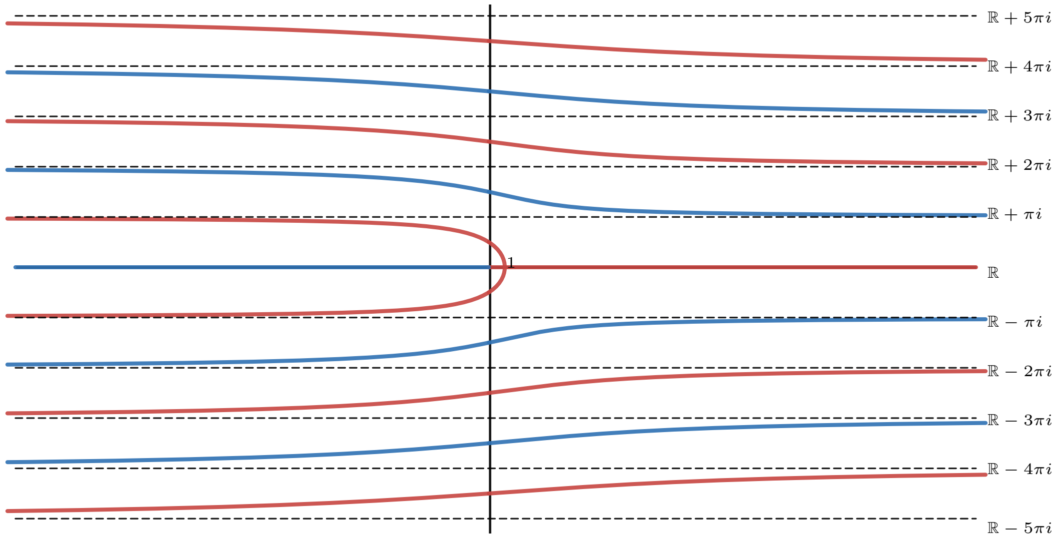


Figure 10: In red, the preimages of the positive real line $\mathbb{R}_+$ and, in blue, the preimages of the negative real line $\mathbb{R}_-$. By the invariance of the Fatou and the Julia sets, all red lines are contained in the Fatou set, while the blue ones are in the Julia set. One deduces that the immediate parabolic basin $\mathcal{A}_0$ is contained in the region bounded by the two blue lines lying in the strips $\{\pi < y < 2\pi\}$ and $\{-2\pi < y < -\pi\}$ respectively. Observe also that in the strip $\{-2\pi < y < 2\pi\}$, each positive real value has exactly two preimages.

2. To see that the basin $\mathcal{A}_0$ approaches asymptotically the strip $\{|\text{Im } w| < \pi\}$, as $\text{Re } w \to \infty$, one shall consider the vertical segments $\gamma_x = \{x + it \colon t \in [-\pi, \pi]\}$. These segments are mapped under h to the curves

$$h(\gamma_x) = \left(1 + \frac{it}{x}\right) xe^{-x-it}, \quad -\pi \leq t \leq \pi.$$

It can be checked that, as $x \to +\infty$, $h(\gamma_x)$ approaches a round circle of radius tending to 0. Recall that the petal $\mathcal{P}$ can be chosen to be tangent to the negative real axis. Hence, as $x \to +\infty$, an increasing part of $h(\gamma_x)$ is contained in the petal and, therefore, in the Fatou set. Therefore, the part of γ_x that is not contained in $\mathcal{A}_0$ shrinks to nothing, as $x \to +\infty$, proving the desired result.

3. Finally, to see that h has degree two in $\mathcal{A}_0$, observe that any positive real point has exactly two preimages in $\{|\text{Im } w| < 2\pi\}$. By the connectivity of $\mathcal{A}_0$, it is clear that these two preimages actually lie in $\mathcal{A}_0$. This is enough to ensure that h restricted to $\mathcal{A}_0$ has degree 2, according to Section 2.5.

□

Observe that ∞ is accessible from $\mathcal{A}_0$, since $\mathbb{R}_+$ is contained in $\mathcal{A}_0$ and therefore it is an access to ∞.

5.2 General description of the dynamics of $f(z) = z + e^{-z}$

Now we return to the dynamical plane of f by lifting the results obtained for h. According to [Ber95], $z \in \mathcal{J}(f)$ if and only if $w = e^{-z} \in \mathcal{J}(h)$. Hence, the Fatou and the Julia sets for f are the preimages under $z \mapsto w = e^{-z}$ of their analogues for h. Observe that the preimages of the negative real line $\mathbb{R}_-$ are the horizontal lines $\{\text{Im } z = (2k+1)\pi,\ k \in \mathbb{Z}\}$. All of them are invariant by f and their points have orbits whose real part tends to $-\infty$ exponentially fast. All of them lie in the Julia set $\mathcal{J}(f)$.

Each horizontal strip contained in between two consecutive preimages of $\mathbb{R}_-$ is mapped under $z \mapsto w = e^{-z}$ to the whole dynamical plane of g in a one-to-one fashion. Therefore, each strip contains a preimage of $\mathcal{A}_0$, say $\ldots, U_{-1}, U_0, U_1, \ldots$. However, this does not imply directly the existence of an invariant Baker domain in each strip. Indeed, since $z \mapsto w = e^{-z}$ is not a conjugacy but a semiconjugacy, one can only deduce that each Fatou component for f is mapped by f to another Fatou component, in any strip, so that their projections in the w-plane are mapped one to the other by h. However, due to the particularities of our concrete map f, we know that $\mathbb{R}$ is forward invariant and, since $f(z + 2\pi i) = f(z) + 2\pi i$, so are all horizontal lines of the form $\{\text{Im } z = 2k\pi,\ k \in \mathbb{Z}\}$. These lines are precisely the preimages of $\mathbb{R}_+$ under $z \mapsto w = e^{-z}$, so each for them is contained in some U_k. This implies the invariance of each Baker domain U_k.

Since $f(z + 2\pi i) = f(z) + 2\pi i$ holds for all $z \in \mathbb{C}$, it is enough to study the dynamics on the strip containing the real line, and in its corresponding invariant Baker domain. We set the following notation.

$$S := \{z\colon\ -\pi \leq \text{Im } z \leq \pi\}$$

$$L^{\pm} := \{z\colon \text{Im } z = \pm\pi\}$$

We denote by U the invariant Baker domain contained in S.

Since the Baker domain U comes from the lift of an immediate parabolic basin, U is of doubly-parabolic type. In addition, by studying the dynamics on the real line (Figure 11), one deduces that, in the Baker domain, points converge to infinity to the right, i.e. Re $f^n(z) \to +\infty$.

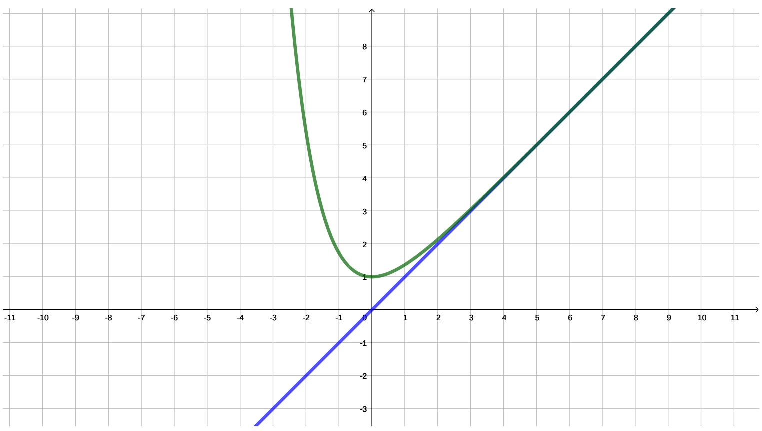


Figure 11: Plot of the real function $f(x) = x + e^{-x}$ (green), together with the diagonal $y = x$ (blue). It is clear that all points in $\mathbb{R}$ converge to $+\infty$, so the Baker domain U contains the real line $\mathbb{R}$. In particular, it is unbounded both on the left and on the right.

Figure 12 presents an outline of the dynamics of f studied so far.

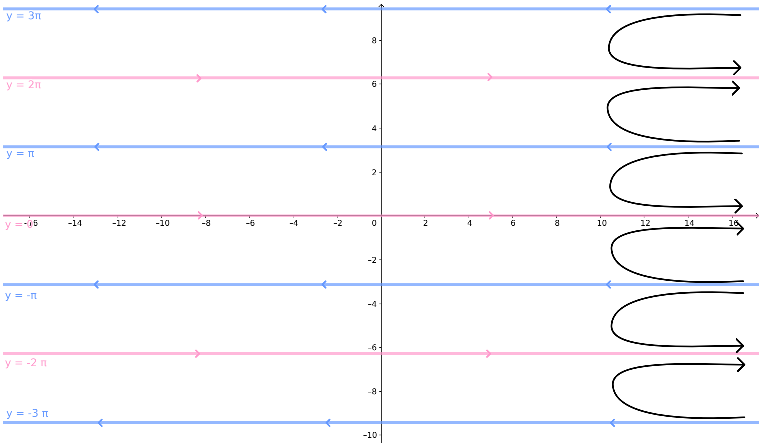


Figure 12: Sketch of the dynamical plane of f, compare with Figure 8. There is an invariant Baker domain U_k in each strip $\{x+iy\colon (2k-1)\pi < y < (2k+1)\pi\}$, $k \in \mathbb{Z}$. Each line $\{y = 2k\pi\}$ (pink) is contained in the Baker domain U_k, while lines $\{y = (2k+1)\pi\}$ (blue) are contained in the Julia set $\mathcal{J}(f)$. The Baker domains U_k are of doubly-parabolic type and its respective absorbing domain (or petal) $\mathcal{P}_k$ approaches asymptotically the strip $\{x+iy\colon (2k-1)\pi < y < (2k+1)\pi\}$, as $\text{Re}z \to +\infty$.

One of the peculiarities of f is that the inner function associated to the Baker domain U can be computed explicitly. This is done in the following result.

Proposition 5.2.1. (Associated inner function) [Baker-Domínguez [BD99, Thm. 5.2]] *The inner function associated to the Baker domain U is*

$$g(z) = \frac{3z^2+1}{z^2+3}.$$

Proof. One shall consider the Riemann map $\varphi\colon \mathbb{D} \to U$ with $\varphi(0) = 0$ and g to be $g := \varphi \circ f \circ \varphi^{-1}$. Since $f_{|U}$ has degree 2, g is a Blaschke product of degree 2. Since U is symmetric with respect to the real line, one shall assume $\varphi(\mathbb{D} \cap \mathbb{R}) = \mathbb{R}$. Therefore $\varphi^*(1) = +\infty$ and $\varphi^*(-1) = -\infty$. We have $\lim_{x\to-\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = +\infty$, g must satisfy $g(1) = 1$ and $g(-1) = 1$. In particular, 1 is the Denjoy-Wolff point of g.

Now choose $\alpha \in (0,1)$ such that $\varphi(\alpha) = 1$. Since 1 is the critical value of f, α must be the critical value of g, so $g(z) = \alpha \Leftrightarrow z = 0$. Consider $T\colon \mathbb{D} \to \mathbb{D}$ to be defined as

$$T(z) = \frac{g(z) - \alpha}{1 - \alpha g(z)}.$$

Clearly, T is real in $(-1,1)$, has degree 2, its only zero is the point 0 and $T(1) = 1$. Therefore $T(z) = z^2$. Therefore,

$$g(z) = \frac{\alpha + z^2}{1 + \alpha z^2}.$$

To determine the concrete value of α, just impose it to be of doubly-parabolic type, i.e. $g'(1) = 1$. Doing so, one gets $\alpha = \frac{1}{3}$, as desired. $\square$

In particular, $g_{|\partial\mathbb{D}}$ is conjugate to $\theta \mapsto 2\theta \mod 1$. This is precisely the doubling map studied in Section 1.7. Consider the space of infinite sequences of 2 elements,

$$\Sigma_2 = \left\{k = \{k_j\}_j : k_j = 0 \text{ or } k_j = 1, \text{ for all } j \geq 0\right\},$$

and the closed circular intervals $I_0 := \overline{\partial\mathbb{D} \cap \mathbb{H}^+}$ and $I_1 := \overline{\partial\mathbb{D} \cap \mathbb{H}^-}$, where $\mathbb{H}^+$ and $\mathbb{H}^-$ denote the upper and the lower half plane, respectively.

To any point $e^{i\theta} \in \partial\mathbb{D}$, we associate a sequence $k = \{k_n\}_n \in \Sigma_2$ such that $g^n(e^{i\theta}) \in I_j$ if and only if $k_n = j$, with $j = 0$ or 1. We say k is the itinerary of $e^{i\theta}$ and write $I_{\partial\mathbb{D}}(e^{i\theta})$. The map $I_{\partial\mathbb{D}} \colon \Sigma_2 \to \partial\mathbb{D}$ is a semiconjugacy between the Bernoulli's shift in Σ_2 and $g_{|\partial\mathbb{D}}$. Moreover, it is a conjugacy if we restrict to non-eventually constant sequences in Σ_2 and points which are not preimages of 1 in $\partial\mathbb{D}$. Compare with [Dev89].

Observe that it is also possible to associate an itinerary to points z in $\mathbb{D}$ describing the dynamics of g, depending on if $g^n(z)$ lies in $D_0 := \mathbb{D} \cap \mathbb{H}^+$ or in $D_1 := \mathbb{D} \cap \mathbb{H}^-$.

Therefore, for any $z \in \overline{\mathbb{D}}$, its itinerary $I_{\overline{\mathbb{D}}}$ is a sequence $k \in \Sigma_2$ depending on the position of $g^n(z)$: in $\overline{D_0}$ or in $\overline{D_1}$. Observe that this is not well-defined at the preimages of $[-1, 1]$. However, we already know the dynamics of these points. We are interested in the remaining ones.

Since the Baker domain is of doubly-parabolic type and finite degree, by Theorem 4.1.3, we know that $f_{|\partial U}$ is ergodic and recurrent, and the escaping set has zero harmonic measure. The goal now is to characterize in a more explicit way the dynamics in the boundary of U, having a detailed description of the periodic points, the accesses to infinity and the escaping set.

To do so, we start by studying how f behaves in the strip S. Observe that, to the left, f behaves like the exponential and, to the right, like the identity. Moreover, if one writes f as

$$f(x + iy) = x + e^{-x}\cos y + i(y - e^{-x}\sin y),$$

preimages of the real axis and of $L^{\pm}$ can be computed explicitly. Indeed, preimages of $\mathbb{R}$ are curves of the form $\{y - e^{-x}\sin y = 0\}$, while preimages of $L^{\pm}$ are curves of the form $\{y - e^{-x}\sin y = \pm\pi\}$. This can be represented graphically to show that there are exactly two preimages of $\mathbb{R}$ in S, one being itself and the other a bended curve going through 0. The preimages of $L^{\pm}$ are themselves and two bended curves converging to ∞ to the left, being asymptotic to $\mathbb{R}$ and to $L^{\mp}$. The interior of these curves is mapped outside S. Therefore, the map $f\colon f^{-1}(S) \cap S \to S$ is a proper map of degree 2 and each point in $\mathbb{C} \setminus S$ has exactly one preimage in S. Compare with Figures 13 and 14.

Next we define the set

$$V := \left\{x + iy\colon\ x > 0,\ \frac{-\pi}{2} < y < \frac{\pi}{2}\right\},$$

which is an absorbing domain for f. Indeed, all points in U converge to ∞ eventually approaching the positive real line, so they enter the absorbing domain. Since the unique critical point 0 lies outside V, the map restricted to V is one-to-one. Moreover, it can be seen that $f(\overline{V}) \subset V$, by computing f in the boundary, which can be done explicitly due to its simplicity. However, observe that this absorbing domain can be improved in the sense that it can be taken larger. Indeed, it can be taken to be the preimage of the petal $\mathcal{P}$ of the parabolic basin of the previous section. Taking $\mathcal{P}$ tangent to the negative real axis, one gets that the absorbing domain for f could be chosen so that it approaches asymptotically $L^{\pm}$ to the right.

Next, define the set

$$\widehat{S} := \{z \in S\colon f^n(z) \in S, \text{ for all } n\}.$$

Clearly, $U \subset \widehat{S}$, since U is forward invariant under f. Moreover, since $f\colon f^{-1}(S) \cap S \to S$ has degree 2, and the Baker domain has already degree 2, there cannot be preimages of the Baker domain U in S. Therefore, $\mathcal{F}(f) \cap \widehat{S} = U$.

One may be tempted to say, regarding the last afirmation, that $\mathcal{J}(f) \cap \widehat{S}$ is precisely the boundary of U. Although it is true that $\partial U \subset \mathcal{J}(f) \cap \widehat{S}$, the other inclusion might be false.

It is clear that, if $z \in \mathcal{J}(f) \cap S$ is in the boundary of a Fatou component, but is not in ∂U, it must leave S at one iteration. However, points in the Julia set which are not in the boundary of any Fatou component, the so-called **buried points**, may occur. It is known that such points exist for f ([Qia95; BD00a]). However, the existence of buried points in $\widehat{S}$, and therefore if $\mathcal{J}(f) \cap \widehat{S} = \partial U$, is left as an open question.

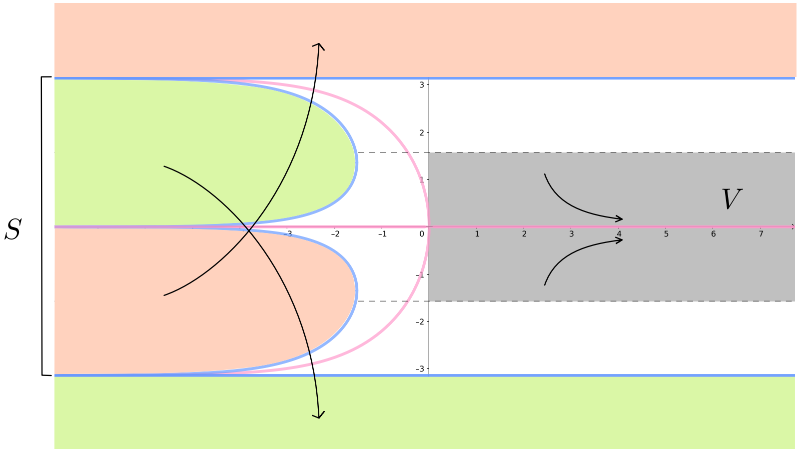


Figure 13: Sketch of the action of f on the strip S. In grey, there is the absorbing domain $V = \{x + iy \colon x > 0, -\frac{\pi}{2} < y < \frac{\pi}{2}\}$, hence $V \subset U$ and $f(\overline{V}) \subset V$. In pink, there are the preimages of $\mathbb{R}$. Observe that each point in $\mathbb{R}$ has exactly two preimages in S. In blue, we see $L^{\pm}$ and their preimages. Notice that points in the green and orange regions are mapped outside S. Any other point in S is mapped inside S.

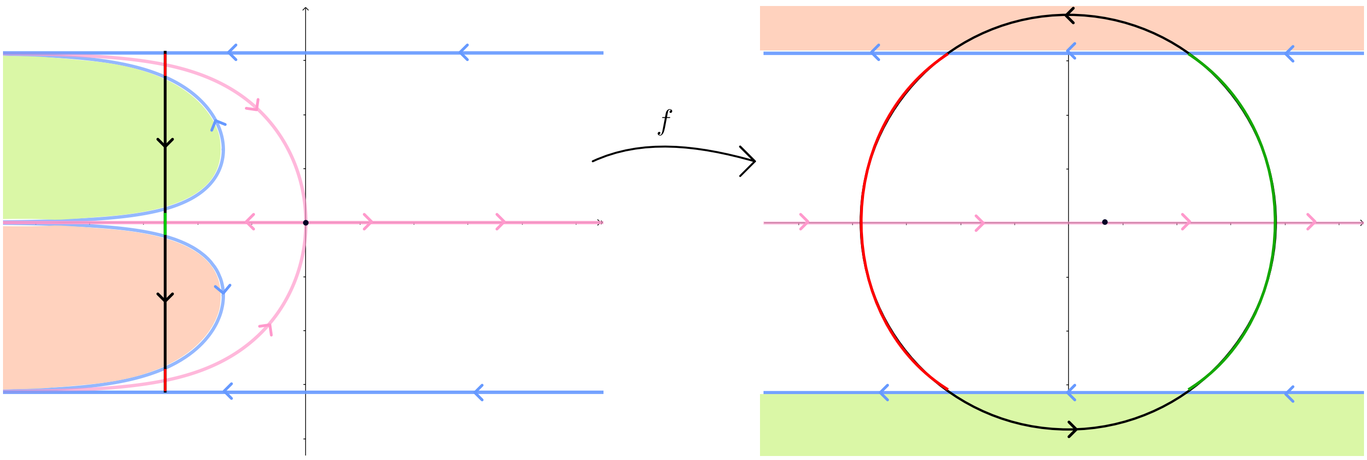


Figure 14: A more detailed sketch of the action of f on the strip S. It is emphasized how f acts on points in the left-hand size of the strip. The arrows on the curves do not indicate the dynamical behaviour of the points but the orientation in which curves are mapped.

5.3 Accesses to infinity

Following the notation in [BD99; Bar08], we define

$$\Theta := \left\{ e^{i\theta} \in \partial\mathbb{D} \colon \varphi^*(e^{i\theta}) = \infty \right\}.$$

By the Correspondence Theorem (1.2.6), there is a bijection between points in Θ and accesses to infinity from U. Denoting the radial segments in $\mathbb{D}$ by

$$r_\theta = \left\{ re^{i\theta} \colon r \in [0, 1) \right\},$$

each curve $\varphi(r_\theta)$, with $e^{i\theta} \in \Theta$, defines an access from U to infinity, and all access can be defined this way. An access is represented by any curve in its homotopy class, preferably the image under φ of a radial segment, but not always. We can also refer to an access by specifying the point $e^{i\theta} \in \Theta$ which corresponds to it.

The goal in this section is to characterize the accesses to infinity from U or, equivalently, to know which points in $\partial\mathbb{D}$ have radial limit φ^* equal to infinity. In particular, we are going to prove that both sets are countable.

Although the set Θ depends on the chosen Riemann map, it is well-defined up to a rotation and the set of accesses to infinity does not depend on the chosen Riemann map. To fix notation, we consider φ to be the Riemann map defined in the proof of Theorem 5.2.1, that is the one fixing 0 and sending $\mathbb{D} \cap \mathbb{R}_+$ to $\mathbb{R}_+$. Observe that Θ is non-empty, since $\varphi^*(-1) = \infty$ and $\varphi^*(1) = \infty$.

Recall that, when working with a transcendental entire function, any preimage of a curve γ such that $\gamma(t) \to \infty$ is another curve σ satisfying that $\sigma(t) \to \infty$. In the concrete case of f, since there are no finite asymptotical values, it is also true that, given a curve $\gamma(t) \to \infty$, then $f(\gamma(t)) \to \infty$. Moreover, we are restricting ourselves to the strip S, where there are only two different (and exclusive) ways to converge to ∞ for a curve. Indeed, the real part of γ must tend to $-\infty$ or to $+\infty$. We say that γ converges to $-\infty$ or to $+\infty$, or converging to ∞ from the left or from the right, respectively.

According to the action of f on S described above (recall that f behave like the identity when Re $z \to +\infty$), one deduces that a curve $\gamma \subset \widehat{S}$ converging to $+\infty$ is always mapped to a curve $f(\gamma)$ converging to $+\infty$. On the other hand, a curve $\gamma \subset \widehat{S}$ converging to $-\infty$ is mapped to a curve $f(\gamma)$ converging either to $-\infty$ or to $+\infty$. Furthermore, if $\gamma(t) \subset \widehat{S}$ is a curve converging to $-\infty$, it must approach asymptotically $L^{\pm}$ or $\mathbb{R}_-$. Hence, on the one hand, if γ approaches asymptotically $L^{\pm}$, then $f(\gamma)$ converges to $-\infty$. On the other hand, if γ approaches asymptotically $\mathbb{R}_-$, then $f(\gamma)$ converges to $+\infty$. Compare with Figures 13 and 14.

Proposition 5.3.1. *Let $e^{i\theta} \in \partial\mathbb{D}$ be such that $g^n(e^{i\theta}) = 1$, for some n. Then, $\varphi^*(e^{i\theta}) = \infty$. In particular, Θ dense in $\partial\mathbb{D}$.*

Proof. Observe that $\varphi(r_0) = \mathbb{R}_+$, where $r_0 = \{r \in [0, 1)\}$, which defines an access to infinity. By the previous remark on the preimages of curves landing at infinite, $f^{-n}(\mathbb{R}_+)$ is a set of curves landing to infinity, for $n \geq 1$. In $\mathbb{D}$, this corresponds to the preimages under g of r_0. This curves land at points $e^{i\theta} \in \partial\mathbb{D}$ with $g^n(e^{i\theta}) = 1$, for some n. By the Correspondence Theorem (1.2.6), $\varphi^*(e^{i\theta}) = \infty$.

Finally, observe that the backward orbit of 1 is dense in $\partial\mathbb{D}$, since $\mathcal{J}(g) = \partial\mathbb{D}$, for being of doubly-parabolic type. □

The argument can be extended to any Baker domain of doubly-parabolic type, even in the infinite degree case. Compare with [BD99, Thm. 1.1] and [Bar08, Thm. 3.1].

Clearly, 1 defines an access to infinity to the right. This access, the one through which orbits converge to infinity, is known as the **dynamical access** for a Baker domain. In the concrete example of f this is the unique access from U to ∞ to the right. Curves in any other access converge to $-\infty$.

Proposition 5.3.2. *There is only one access $\mathcal{A}$ to infinity from the Baker domain U satisfying that, for $\gamma \in \mathcal{A}$, Re $\gamma(t) \to +\infty$. This access coincides with the one defined by $\varphi(r_0)$. For γ belonging to any other access to infinity, it must hold Re $\gamma(t) \to -\infty$.*

Proof. From the discussion above, we see that the only access to infinity from the right corresponding to a point in the backward orbit of 1, is precisely the one corresponding to 1.

Indeed, the only preimage of 1 different from itself is -1. The access to ∞ corresponding to -1 is the one defined by $\mathbb{R}_-$. Hence, any curve γ belonging to this access satisfies Re $\gamma(t) \to -\infty$. Moreover, preimages of these curves also converge to $-\infty$. Therefore, any curve accesses to ∞ corresponding to points in the backward orbit of 1 converges to $-\infty$, as desired.

It is left to see that the same holds for any other access to infinity. Such an access should correspond to a $e^{i\alpha}$, with $g^n(e^{i\alpha}) \neq 1$, for all n. Since the backward orbit of 1 is dense in $\partial\mathbb{D}$, one can find θ_1, θ_2 preimages of 1, such that $\theta_1 < \alpha < \theta_2$ and they all lie either above or below the real line. Both $\varphi(r_{\theta_1})$ and $\varphi(r_{\theta_2})$ are curves converging to $-\infty$. Since the Riemann map is a bijection, $\varphi(r_\alpha)$ also converges to $-\infty$. Compare with Figure 15. □

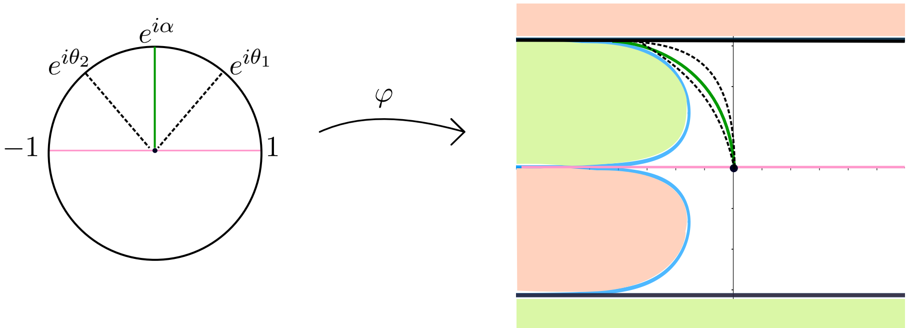


Figure 15: Assume $e^{i\theta} \in \partial\mathbb{D}$. The figure shows how $e^{i\theta_1}$ and $e^{i\theta_2}$ are taken in the unit circle, and how this forces $\varphi(r_\alpha)$ to converge to $-\infty$.

Finally, we give a complete characterization of the accesses to infinity. The result was already proved in [BD99, Thm. 6.3], but our proof is simpler in the sense that we do not need to compute the prime end corresponding to 1.

Proposition 5.3.3. (Accesses to infinity) *The set Θ consists precisely of the backward orbit of 1, i.e. points $e^{i\theta} \in \partial\mathbb{D}$ such that $g^n(e^{i\theta}) = 1$. Equivalently, accesses from U to ∞ are defined exactly by the preimages of $\mathbb{R}_+$ under f.*

Proof. By the previous proposition, any curve landing at $+\infty$ corresponds to the access defined by 1. Any curve $\gamma \subset \widehat{S}$ converging to $-\infty$ approaches asymptotically $\mathbb{R}_-$ or $L^\pm$. In the first case, $f(\gamma)$ is a curve landing at $+\infty$, so it corresponds to the access defined by 1. Then, γ must correspond to the access defined by -1, which is in the backward orbit of 1.

If $\gamma \subset \widehat{S}$ converges to $-\infty$ approaching asymptotically $L^\pm$, and it does not belong to an access defined by a preimage of 1, $f^n(\gamma)$ must converge to $-\infty$ approaching asymptotically $L^\pm$, for all n. Take $z \in \gamma$ enough in the left so it satisfies Re $z + e^{-z} < 2$ Re $z < 0$. Since $f^n(\gamma)$ is also asymptotic to $L^\pm$, Re $f^n(z) < 2^n$ Re $z < 0$. But, since $z \in U$, this is a contradiction with the fact that Re $f^n(z) \to +\infty$ in U. □

As a remark, observe that, in this case, Θ is countable, while in the example studied by Devaney and Golberg it is uncountable (C.1). However, in both cases, this set has zero λ-measure, by the Fatou, Riez and Riez Theorem (1.2.4).

5.4 Accessibility of periodic points

The goal now is to prove accessibility of periodic points in ∂U. Observe that this does not imply directly their existence, which will be proved later in Theorem 5.4.2. The proof is based in ideas of [BD00b, Thm. H] in a different setting, adapted to our problem.

Theorem 5.4.1. (Periodic points are accessible) *Let $z_0 \in \partial U$ be periodic under f, i.e. $f^p(z_0) = z_0$, for some p. Then z_0 is accessible.*

Proof. First observe that f has no asymptotical values and its critical points are $w_k = 2k\pi i$, for $k \in \mathbb{Z}$. Since the lines $\{\text{Im } z = 2k\pi,\ k \in \mathbb{Z}\}$ are forward invariant under f, the only critical point in S is $w_0 = 0$. Moreover, if we consider the same absorbing domain as before,

$$V := \left\{ x + iy\colon\ x > 0,\ \frac{-\pi}{2} < y < \frac{\pi}{2} \right\},$$

then the orbit of $w_0 = 0$ is completely contained in $\overline{V}$. Therefore, $P(f) \cap S \subset V$, where $P(f)$ denotes the postcritical set of f.

Now consider $z_0 \in \partial U$ periodic. Since there are no periodic points in $L^{\pm}$, we have $z_0 \in \text{Int}(S)$. It is repelling, so $|(f^p)'(z_0)| > 1$. We can choose $\varepsilon > 0$ such that $D(z_0, \varepsilon) \subset S$ and $D(z_0, \varepsilon) \cap \overline{V} = \emptyset$. Consider ϕ to be the branch of f^{-p} fixing z_0, defined and analytic in $S \smallsetminus \overline{V}$. Taking ε smaller if needed, we shall assume $\phi(D(z_0, \varepsilon)) \subset D(z_0, \varepsilon)$ and $|\phi'| < 1$ in $D(z_0, \varepsilon)$. Since z_0 is an attracting fixed point for ϕ, Koenigs' linearization (Thm. 1.1.6) ensures that this construction can be done.

Since $z_0 \in \partial U$, one can take $w_0 \in D(z_0, \varepsilon) \cap U$. Then, $w_1 := \phi(w_0) \in D(z_0, \varepsilon) \cap U$. Indeed, it is clear that $w_1 \in D(z_0, \varepsilon)$. Since U is forward invariant by f and $D(z_0, \varepsilon)$ does not intersect the postcritical set, it also belongs to U. Inductively, let us define $w_n := \phi^n(w_0)$. It is clear that $w_n \in D(z_0, \varepsilon) \cap U$ and $w_n \to z_0$, as $n \to \infty$.

Then choose γ_0 to be a Jordan arc joining z_0 and z_1, such that $\gamma_0 \subset U$ and $\gamma_0 \cap \overline{V} = \emptyset$. This can be done because $U \smallsetminus \overline{V}$ is simply-connected. Observe that we can not ensure $\gamma_0 \subset D(z, \varepsilon) \cap U$, but it is enough to have $\gamma_0 \subset U$.

Recall that ϕ is well-defined and analytic in $S \smallsetminus \overline{V}$. Since V is an absorbing domain for f, and hence for f^p, such that $f(\overline{V}) \subset V$, we have that $\phi(S \smallsetminus \overline{V}) \subset S \smallsetminus \overline{V}$. Since z is an attracting fixed point for ϕ in $S \smallsetminus \overline{V}$, by the Denjoy-Wolff Theorem (1.3.1), $\phi^n \to z$ uniformly on compact sets of $S \smallsetminus \overline{V}$. Therefore, $\gamma_n := \phi(\gamma_0)$ is a curve in U connecting z_n with z_{n+1}, and $\phi^n \to z$ uniformly on γ_0. Then, $\cup_n \gamma_n$ is a curve in U landing at z. Thus, z is accessible from U, as desired. $\square$

Now we prove the existence of periodic points in ∂U. Those points are obtained as the (convergent) radial limit of a periodic point for g in $\partial\mathbb{D}$. The proof consists precisely of ensuring the convergence of the previous limit and discarding the possibility of it being infinity.

Theorem 5.4.2. (Radial limit for periodic points) *Let $e^{i\theta} \in \partial\mathbb{D}$ be periodic under g, i.e. $g^p(e^{i\theta}) = e^{i\theta}$ for some $p > 1$. Then, $\varphi^*(e^{i\theta})$ exists and it is a periodic point of period p.*

Proof. Assume $e^{i\theta} \in \partial\mathbb{D}$ is periodic. Then, it must be repelling, so $\left|(g^p)'(e^{i\theta})\right| > 1$.

The first step in the proof is to construct a curve $\gamma \subset \mathbb{D}$ landing at $e^{i\theta}$ and being invariant under g. Since $\left|(g^p)'(e^{i\theta})\right| > 1$, there exists a branch ϕ of the inverse of g^p satisfying that $\phi(e^{i\theta}) = e^{i\theta}$ and $\left|\phi'(e^{i\theta})\right| < 1$. Moreover, ϕ is defined and analytic in some disk $D(e^{i\theta}, \varepsilon)$ and $\phi(D(e^{i\theta}, \varepsilon)) \subset D(e^{i\theta}, \varepsilon)$, for $\varepsilon > 0$ small enough. Therefore, ϕ^n is analytic in $D(e^{i\theta}, \varepsilon)$ and $\phi^n(D(e^{i\theta}, \varepsilon)) \subset D(e^{i\theta}, \varepsilon)$, for all n. By the Denjoy-Wolff Theorem (1.3.1), $\phi^n \to e^{i\theta}$ uniformly on compact sets of $D(e^{i\theta}, \varepsilon)$. See Figure 16.

Now choose $z_0 \in D(e^{i\theta}, \varepsilon) \cap \mathbb{D}$ and $z_1 := \phi(z_0) \in D(e^{i\theta}, \varepsilon) \cap \mathbb{D}$. Join them by a curve $\gamma_0 \subset D(e^{i\theta}, \varepsilon) \cap \mathbb{D}$. For all n, define $z_n := \phi^n(z_0)$ and $\gamma_n := \phi^n(\gamma_0)$. Clearly, length$\gamma_n \to 0$. Therefore $\gamma = \cup_n \gamma_n$ is a curve in $\mathbb{D}$ ending at $e^{i\theta}$ and invariant under g.

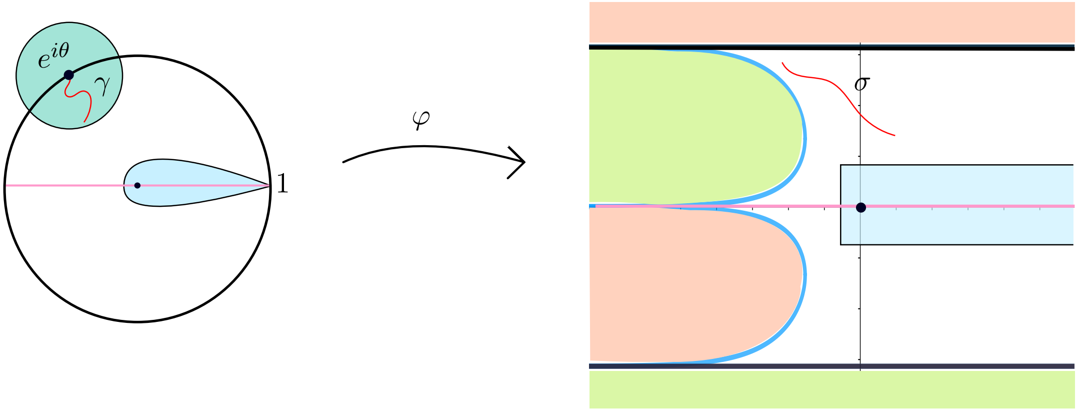


Figure 16: The point $e^{i\theta} \in \partial\mathbb{D}$ is a repelling point in $\partial\mathbb{D}$. One can choose a disk centered at this point with radius small enough so that it does not intersect the postcritical set, which lies in the blue region. The curve γ is constructed to be invariant under g and land at $e^{i\theta}$. The goal of the theorem is to show that $\sigma = \varphi(\gamma)$ converges to a finite periodic point.

The next step is to see that $\sigma(t) = \varphi(\gamma(t)) \subset U$ converges to a unique point in ∂U. To do so, choose V_0', V_0 in $D(e^{i\theta}, \varepsilon) \cap \mathbb{D}$ such that $\overline{V_0'} \subset V_0$ and $\gamma_0 \subset V_0'$. Define $V_n' := \phi^n(V_0')$ and $V_n := \phi^n(V_0)$, for all $n \geq 0$. Observe that $z_n, z_{n+1} \in V_n'$. Now, define $U_n' := \varphi(V_n')$, $U_n := \varphi(V_n)$ and $w_n := \varphi(z_n)$. It is satisfied that $w_n, w_{n+1} \in \overline{U_n'} \subset U_n$.

The points $\{w_n\}_n$ must have an accumulation point w_∞ in ∂U (*a priori*, w_∞ may be ∞). Therefore, there exists a subsequence $\{w_{n_k}\}_k$ such that $w_{n_k} \to w_\infty$. Two options are possible.

- <u>If $\operatorname{diam} U_{n_k}' \to 0$</u>, by the triangle inequality,

  $$\left|w_{n_{k+1}} - w_\infty\right| \leq \left|w_{n_{k+1}} - w_{n_k}\right| + \left|w_{n_k} - w_\infty\right| \to 0.$$

  Then, if w_∞ is a finite point, it must satisfy

  $$f^p(w_\infty) = f^p(\lim_{k\to\infty} w_{n_k}) = \lim_{k\to\infty} w_{n_{k+1}} = w_\infty.$$

  For a holomorphic map, periodic points form a discrete set. Therefore, in the case that $\operatorname{diam} U_{n_k}' \to 0$, the curve σ either converges to ∞ or to a finite periodic point.

- <u>If $\operatorname{diam} U_{n_k}' \not\to 0$</u>, then, taking a subsequence if needed, one can assume $\operatorname{diam} U_{n_k}' > \varepsilon$, for some $\varepsilon > 0$. Since $V_n := \phi^n(V_0)$, with $\phi^n_{|V_0}$ conformal, we have $U_n = \varphi_n(U_0) := \varphi \circ \phi^n \circ \varphi^{-1}(U_0)$, with $\varphi_{n|U_0}$ conformal, for being the composition of conformal maps. Applying Koebe's Theorem (1.2.8), there exists $k > 0$, independent of n, such that

  $$D(w_{n_k}, k \cdot \operatorname{diam} U_{n_k}') \subset U_{n_k}'.$$

  Since we are assuming $\operatorname{diam} U_{n_k}' \geq \varepsilon$, we have $D(w_{n_k}, k \cdot \varepsilon) \subset U_{n_k}'$, for all k. Therefore, for k_0 big enough, there exists $\varepsilon' > 0$ such that

  $$D(w_\infty, \varepsilon') \subset \bigcap_{k\geq k_0} U_{n_k}' = \bigcap_{k\geq k_0} \varphi_{n_k}(U_0').$$

  By definition φ_{n_k} coincides with a branch of the inverse of f^p. Therefore, $f^{p\cdot n_k}(D(w_\infty, \varepsilon')) \subset V_0'$, for all $k \geq k_0$. Therefore, $\left\{f^n_{|D(w_\infty,\varepsilon')}\right\}_n$ has a bounded subsequence, hence it is normal, by Theorem 1.1.4. This is a contradiction because $w_0 \in \partial U \subset \mathcal{J}(f)$.

  In conclusion, this second option cannot hold, and we are always in the previous case.

Finally, observe we proved that $\sigma(t) = \varphi(\gamma(t)) \subset U$ converges to a unique point in ∂U, which is either periodic or infinity. This second possibility must be discarded. Indeed, if $\sigma(t)$ converges to ∞, by the Correspondence Theorem, this implies that $\varphi^*(e^{i\theta}) = \infty$. This is impossible because $e^{i\theta}$ was assumed to be periodic, and points in Θ are precisely the preimages of 1 (Thm. 5.3.3). □

As a remark, observe that the main construction of the previous proof is valid in general, whenever the periodic point $e^{i\theta}$ is not a singularity. The only place where we are using the specific features of our particular function is when discarding the possibility of the radial limit being infinity. Therefore, in general, the radial limit φ^* of a periodic point $e^{i\theta}$ for the associated inner function, is either a (finite) periodic point for the original function or infinite.

Joining the results of the previous theorems, we give a complete characterization of the periodic points in ∂U, as follows.

Theorem 5.4.3. (Periodic points in ∂U) *A point $z \in \partial U$ satisfies $f^p(z) = z$ for some $p \geq 1$ if, and only if, $z = \varphi^*(e^{i\theta})$ for some $e^{i\theta} \in \partial\mathbb{D}$ satisfying $g^p(e^{i\theta}) = e^{i\theta}$.*

Proof. The right-to-left implication has already been proved in the previous theorem (5.4.2). It remains the left-to-right implication. By Theorem 5.4.1, if z is a periodic point in ∂U it is accessible. By the Correspondence Theorem (1.2.6), this implies the existence of a point $e^{i\theta}$ for which the radial limit exists and it is equal to z, i.e. $\varphi^*(e^{i\theta}) = z$. Since φ^* satisfies $\varphi^* \circ g = f \circ \varphi^*$, if it is defined, the point $e^{i\theta}$ should be periodic of the same period as z. □

As a remark, observe that this theorem does not answer the question if all periodic points in $\widehat{S}$ are in the boundary of U. This is left as an open question.

5.5 The escaping set

Here we focus on the question of the existence of escaping points in the boundary of the Baker domain U. For a general doubly-parabolic Baker domain, even of finite degree, the question remains open. In our particular example, it is possible to prove the existence of infinitely many curves of escaping points in the boundary of U.

First of all, we start by observing that in $\overline{U}$, as well in S, there are two distinguished ways to escape to infinity. Indeed, points can escape to infinity to the left, to the right or oscillating from left to right. This leads us to define the following sets:

$$\mathcal{I}_S^+ := \left\{ z \in \mathcal{I}(f) \cap \widehat{S} \colon \text{ there exists } \{n_k\}_k \text{ such that Re } f^{n_k}(z) \to +\infty \right\}$$

$$\mathcal{I}_S^- := \left\{ z \in \mathcal{I}(f) \cap \widehat{S} \colon \text{ there exists } \{n_k\}_k \text{ such that Re } f^{n_k}(z) \to -\infty \right\}$$

Observe that, *a priori*, this two sets are not disjoint: points which escape to ∞ oscillating from left to right belong to both sets. However, this possibility is excluded, as it is shown in the next result.

Proposition 5.5.1. (No oscillating escaping points) *The sets $\mathcal{I}_S^+$ and $\mathcal{I}_S^-$ are disjoint, i.e.*

$$\mathcal{I}(f) \cap \widehat{S} = \mathcal{I}_S^+ \sqcup \mathcal{I}_S^- .$$

Proof. Assume $z \in \mathcal{I}(f) \cap \widehat{S}$. For any $r > 0$, there exists n_0 such that, for all $n \geq n_0$, $f^n(z) \in S$ and $|f^n(z)| > r$. In particular, taking $r > \pi$, there exists $R > 0$ such that Re $f^n(z) > R$ or

Re $f^n(z) < -R$, for all $n \geq n_0$. Assuming that Re $z > R$, we are going to see that it is not possible to have Re$f(z) < -R$, so oscillating escaping orbits are not possible. Indeed,

$$\text{Re } f(x+iy) = x + e^{-x}\cos y \geq x - e^{-x} \geq R - e^{-R}.$$

Assuming $R > 1$, the right-hand side of the inequality is greater than 0, so it does not hold Re $f(z) < -R$, as desired. □

We already know that points in U escape to ∞ with real part tending to $+\infty$. No other points in $\widehat{S}$ exhibit the same behaviour.

Proposition 5.5.2. *The set $\mathcal{I}_S^+$ is equal to the Baker domain U.*

Proof. It is left to show that the orbit of no point z in $\widehat{S} \setminus U$ is such that Re $f^n(z) \to +\infty$. Indeed, such a point never enters the absorbing domain, so, when Re $f^n(z) > 0$, either Im $f^n(z) > \frac{\pi}{2}$ or Im $f^n(z) < -\frac{\pi}{2}$. In both cases, Re $f^{n+1}(z) <$ Re $f^n(z)$. Therefore, it is impossible that a point which is not in the Baker domain belongs to $\mathcal{I}_S^+$. □

Therefore, all escaping points in $\widehat{S} \setminus U$ belong to $\mathcal{I}_S^-$. Next, we construct explicitly curves of escaping points. To do so, we endow $\widehat{S}$ with an appropriate symbolic dynamics.

To every point z in $\widehat{S}$, since it is never mapped outside S, one can associate a sequence in Σ_2 to it (the **itinerary**, $I_S(z)$) describing its dynamics. More precisely, let

$$S_0 := S \cap \mathbb{H}^+ \qquad S_1 := S \cap \mathbb{H}^-,$$

where $\mathbb{H}^+$ and $\mathbb{H}^-$ denote the upper and the lower half plane, respectively.

To any point $z \in \widehat{S}$, we associate a sequence $k = \{k_n\}_n \in \Sigma_2$ such that $f^n(z) \in S_j$ if and only if $k_n = j$, with $j = 0$ or 1. Observe that this construction is not possible when $z \in f^{-n}(\mathbb{R})$, for some $n \geq 0$, but this is not a problem since then its dynamics are already understood: it belongs to the Baker domain and converge to $+\infty$.

Observe that points in the Baker domain eventually fall in the absorbing domain V described before. The dynamics on it are conjugate to $\text{id}_{\mathbb{C}} + 1$ so, in particular, $V \cap \mathbb{H}^+$ and $V \cap \mathbb{H}^-$ are invariant by f. Hence, sequences on the Baker domain are eventually constant. Therefore, sequences which are not eventually constant must correspond to points in the Julia set $\mathcal{J}(f)$. On the other hand, observe that, since $L^{\pm}$ are invariant, points on them have itinerary $\overline{0}$ or $\overline{1}$ respectively, so eventually constant sequences are not exclusive of the Baker domain.

Next we prove the existence of curves of escaping points with itinerary given by any sequence in Σ_2. *A priori*, such curves belong to $\widehat{S} \cap \mathcal{J}(f)$, but they can be proved to belong to ∂U.

Theorem 5.5.3. (Curves of escaping points) *For every sequence $k = \{k_j\}_j \in \Sigma_2$ there exists a curve $\gamma_k \subset S$ whose points belong to $\mathcal{I}(f) \cap \widehat{S}$ and have the itinerary prescribed by k. Moreover, $\gamma_k \subset \partial U$.*

Proof. First observe that in $S \setminus \mathbb{R}_+$ two branches of f^{-1} can be defined, namely $\phi_0 \colon S \to S_0$ and $\phi_1 \colon S \to S_1$. Compare with Figure 17.

A simple computation yields

$$f'(x+iy) = 1 - e^{-x}\cos y + ie^{-x}\sin y,$$

$$\left|f'(x+iy)\right| = \sqrt{1 + e^{-2x} - 2e^{-x}}.$$

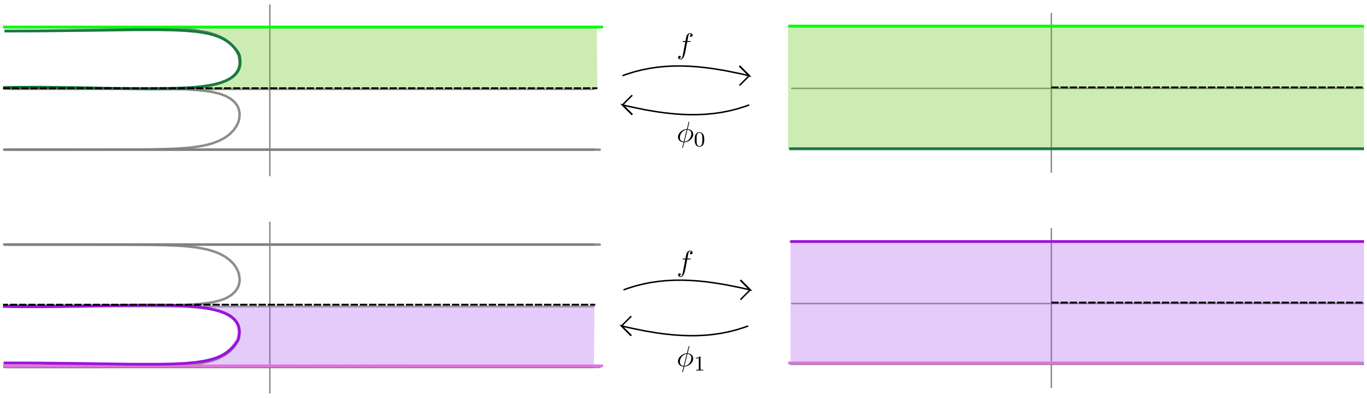


Figure 17: In the strip $S = \{-\pi \le \text{Im } z \le \pi\}$, points have two preimages in S. Therefore, there are two branches of f^{-1} defined in $S \setminus \mathbb{R}_+$, namely ϕ_0 and ϕ_1. The figure shows the range of both branches of the inverse.

Therefore, for $x < -\ln 2$, we have $|f'| > 1$. Moreover, fixing $l < -\ln 2$, then $|f'| \ge M > 1$ uniformly on $\{\text{Re } z < l\}$.

Now, choose $l < -\ln 2$ and let D_0 be the square of side length π located in S_{k_0} and left-hand side at $l_0 = \{x = l\}$. We construct inductively a sequence of squares $\{D_j\}_j$ side length π, located in S_{k_j} and left-hand side at $l_j = \left\{x = l_{j-1} - e^{-l_{j-1}}\right\}$. Observe that $l_j \to \infty$, as $j \to \infty$.

The squares $\{D_j\}_j$ satisfy $D_j \subset f(D_{j-1})$. Indeed, D_{j-1} is mapped inside an annulus of inner boundary given by $\{f(l_{j-1} + iy) \colon -\pi \le y \le \pi\}$. Since

$$\text{Re } f(l_{j-1} + iy) = l_{j-1} + e^{-l_{j-1}} \cos y \ge l_{j-1} - e^{-l_{j-1}},$$

and the distance between the inner and the outer boundary of the annulus is bigger than π, we have $D_j \subset f(D_{j-1})$. See Figure 18.

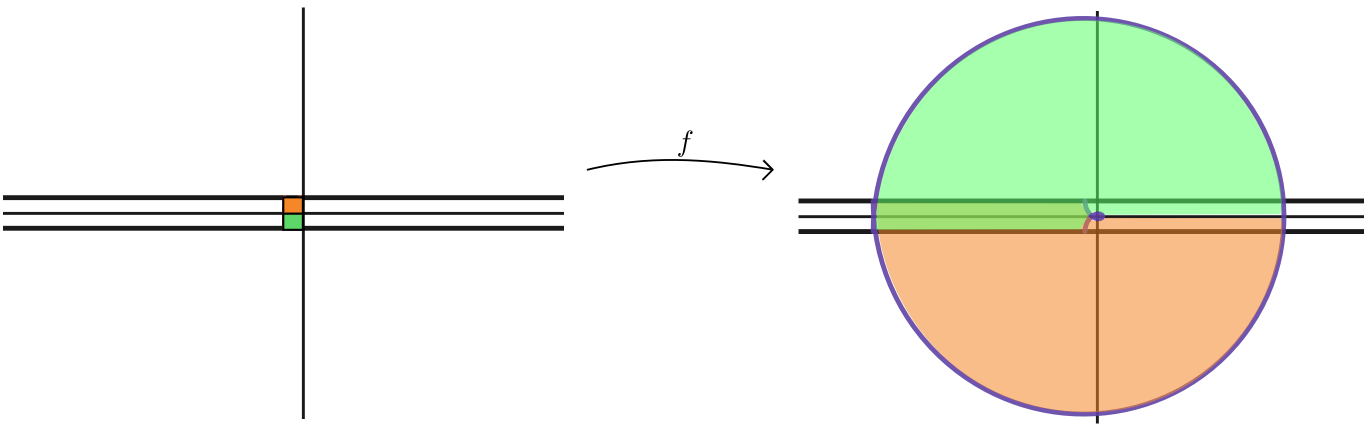


Figure 18: On the left, two squares D_j^0 and D_j^1 of side length π are chosen, one in S_0 and the other in S_1. Then, f is applied and $f(D_j^0)$ and $f(D_j^1)$ are represented on the right. Observe that both $f(D_j^0)$ and $f(D_j^1)$ intersect S and a square of side π can be chosen in $f(D_j^i) \cap S_k$, for $i, k = 1, 2$, following the procedure described in the proof of Theorem 5.5.3. The figure is a scale drawing, so in particular it shows that f is expansive.

Now, let us define:

$$Q_j = \phi_{k_0} \circ \cdots \circ \phi_{k_j}(\overline{D_{j+1}}),$$

$$z_{k,l} = \bigcap_{j \ge 0} Q_j.$$

Notice that z_k is a unique point. Indeed, $\{Q_j\}_j$ is a sequence of nested compact sets contained in D_0. Its intersection is a connected compact set, whose diameter tends to zero because

$|\phi_1'|, |\phi_2'| \leq \frac{1}{M} < 1$. The point $z_{k,l}$ follows the itinerary prescribed by k and it escapes to $-\infty$. In particular, it cannot belong to U. Observe that the dependence of $z_{k,l}$ on l is continuous. Hence, letting $l \to \infty$, the points $z_{k,l}$ describe a curve, which we call γ_k, of escaping points of itinerary k.

Finally, to see that $\gamma_k \subset \partial U$, it is enough to show that each $z_{k,l}$ can be approximated by points in U. Following the notation used above, if $\overline{D_{j+1}}$ contains points of U, so does Q_j (recall that U has no preimage in S different from itself). Indeed, $\overline{D_{j+1}}$ contains points of U, because $f(\overline{D_{j+1}})$ intersects U (compare with Figure 18). Since the diameter of the squares in $\{D_j\}_j$ shrinks to 0, this gives a sequence of points in U approximating γ_k. Thus, the proof is finished. □

Observe that the preimages in S of the curves constructed in the previous theorem (obtained by ϕ_0^n and ϕ_1^n, for $n \geq 0$) contain also escaping points of ∂U. An interesting question is if these curves (the ones constructed in the theorem together with their preimages) are precisely $\mathcal{I}(f) \cap \partial U$, or even $\mathcal{I}(f) \cap \widehat{S}$.

Now, recall that for points in $\overline{\mathbb{D}}$, we can associate a sequence $k \in \Sigma_2$ describing its itinerary $I_{\overline{\mathbb{D}}}$. For points in $\widehat{S}$, we proceed similarly to get the itinerary I_S, depending on the position of $f^n(z)$: in S_0 or in S_1. Since $\varphi(D_0) \subset S_0$ and $\varphi(D_1) \subset S_1$, we have that $I_{\overline{\mathbb{D}}}(z) = I_S(\varphi(z))$, for $z \in \mathbb{D}$. The same holds in the boundary, making use of radial limits: if $\varphi(r_\theta)$ accumulates at a finite point $w \in \partial U$, then $I_{\overline{\mathbb{D}}}(e^{i\theta}) = I_S(w)$.

From that, one deduces that, given a sequence $k \in \Sigma_2$, there is at most an accessible point from U in ∂U with itinerary k. In the case of a periodic sequence k, there is an accessible point, which is periodic (Thm. 5.4). Therefore, all the escaping points in the curve γ_k of Theorem 5.5 are not accessible.

Another remarkable observation is that $L^{\pm}$ correspond to the curves γ_k with $k = \overline{0}$ and $k = \overline{1}$, respectively. Indeed, points in $\mathcal{I}^+ \setminus L^+$ eventually exit S_0 under the iteration, and points in $\mathcal{I}^- \setminus L^-$ eventually exit S_1. In particular, $L^{\pm} \subset \partial U$ and, by the correspondence between itineraries in $\partial\mathbb{D}$ and in ∂U, $L^{\pm}$ must be contained in the accumulation set of the unrestricted limit $\lim\limits_{r \to 1} \varphi(r)$. In the language of cluster sets, this is the cluster set of φ at 1, and we shall denote it by Q. Since Q is compact in $\widehat{\mathbb{C}}$, so it is $Q \cap S_0$. If $L^+ \cup \{\infty\} \subsetneq Q \cap S_0$, there should exists a point z in $(Q \cap S_0) \setminus (L^+ \cup \{\infty\})$ with minimal imaginary part. Then, $\text{Im } f(z) < \text{Im } z$, which is a contradiction with the fact that z has minimal imaginary part. Therefore, $L^+ \cup \{\infty\} = Q \cap S_0$ and, by symmetry, $L^- \cup \{\infty\} = Q \cap S_1$. This implies that the cluster set of φ at 1 is precisely $L^+ \cup \{\infty\} \cup L^-$.

Rewriting this last affirmation about the cluster set of φ at 1 in terms of prime ends (see [Pom92, Chapter 2], [Mil06, Chapter 17]), one gets that the prime end corresponding to 1 by φ is precisely $L^+ \cup \{\infty\} \cup L^-$. This gives an alternative proof to [BD99, Thm. 6.1].

Finally, we observe that there are no accessible points in $L^{\pm}$, since the sequences $k = \overline{0}$ and $k = \overline{1}$ correspond to 1 in $\partial\mathbb{D}$, and $\varphi^*(1) = +\infty$. Moreover, any curve γ_k with k eventually constant corresponds to a preimage of $L^{\pm}$. Since accessible points are mapped to accessible points, there is no accessible escaping point with eventually constant sequence.

5.6 Conclusions and further questions

First of all, observe that the points in ∂U we have been able to describe, have harmonic measure zero. Indeed, we describe the periodic poins in ∂U, which are countable, and the escaping set, which we know beforehand (Thm. 4.1.3) to be of harmonic measure zero. Therefore, a further study of this map should be aimed at finding oscillating points, i.e. points that neither have bounded orbit nor escape, which are the typical points in terms of the harmonic measure.

In relation with the conjecture stated in [BFJK19, Example 1.2], we have completely characterized the periodic points in ∂U. This may be used to prove the density of periodic points in ∂U. On the other hand, concerning escaping points, we constructed infinitely many curves of escaping points and for many of them, we proved non-accessibility. This points at the conjecture being true, but more work is needed.

Many of the theorems we proved may be generalized for doubly-parabolic Baker domains of finite degree. The fact that periodic points are accessible (Thm. 5.4.1) seems to be generalizable whenever we have some control on the postcritical set. On the other hand, the proof of convergence for the radial limit of a periodic boundary point seems to work whenever the periodic point is not a singularity. However, this radial limit may converge to infinity and this does not ensure the existence of periodic points in the boundary of the Baker domain. Recall that, in our concrete setting, we used specific features of our inner function to discard that such limit is equal to infinity.

Moreover, it may be interesting to generalize the fact that no escaping point in the Julia set escapes to infinity in the same direction that points in the Baker domain (Prop. 5.5.2). This may be related with the fact that, in the inner function, no point in the boundary converges non-trivially to the Denjoy-Wolff point. However, in a general setting, we may not have two directions of convergence to infinity defined, so we may start by defining what does it mean to converge to infinity in different directions.

Another possible way to continue the study of this function may be to use the symbolic dynamics described in Section 5.5 to prove that ∂U is a Cantor Bouquet. To do so, one may try to pull back the curves constructed in Theorem 5.5.3 and find that either they have a finite endpoint or both endpoints are infinity. It seems reasonable that, for periodic sequences, one endpoint is the periodic point found in Theorem 5.4.1 whereas, for eventually constant sequences, both endpoints are infinity. In the case of periodic sequences, a similar argument to the one used in the proof of Theorem 5.4.1, based on Koebe's Distorsion Theorem, may work.

More questions that arise in relation with the previous study concern the interplay among the Cantor Bouquet, the boundary of the Baker domain and the points in $\mathcal{J}(f) \cap \widehat{S}$. Recall that the existence of buried points in $\widehat{S}$ was left as an open question. The construction of the Cantor Bouquet mentioned above may help to answer it. Indeed, if it can be proved that it is precisely $\mathcal{J}(f) \cap \widehat{S}$, this will block the existence of buried points in $\widehat{S}$.

References

[Aar78] Jon Aaronson. “Ergodic theory for inner functions of the upper half plane”. In: *Annales de l’Institut Henri Poincaré. Section B. Calcul des Probabilités et Statistique. Nouvelle Série* 14.3 (1978), 233–253.

[Aar81] Jon Aaronson. “A remark on the exactness of inner functions”. In: *Journal of the London Mathematical Society* 2 (1981), pp. 469–474.

[Aar97] Jon Aaronson. “An Introduction to Infinite Ergodic Theory”. In: *American Mathematical Society, Providende, RI* (1997).

[Ahl94] Lars V. Ahlfors. *Complex Analysis.* Mc Graw Hill, Inc., 1994. ISBN: 9780070006560.

[Bak76] Irvine N. Baker. “An entire function which has wandering domains”. In: *Journal of the Australian Mathematical Society. Series A* 22 (1976), pp. 173–176.

[Bak84] Irvine N. Baker. “Wandering domains in the iteration of entire functions”. In: *Proceedings of the Journal Mathematical Society* 47 (1984), pp. 337–360.

[BD99] Irvine N. Baker and Patricia Domínguez. “Boundaries of unbounded Fatou components of entire functions”. In: *Annales Academiae Scientiarum Fennicae Mathematica* 24 (1999), pp. 437–464.

[BD00a] Irvine N. Baker and Patricia Domínguez. “Residual Julia sets”. In: *The Journal of Analysis* 8 (2000), 121–137.

[BD00b] Irvine N. Baker and Patricia Domínguez. “Some connectedness properties of Julia sets”. In: *Complex Variables. Theory and Application.* 24.4 (2000), 371–389.

[BF01] Krzysztof Barański and Núria Fagella. “Univalent Baker Domains”. In: *Nonlinearity* 14 (2001), pp. 411–429.

[BFJK15] Krzysztof Barański, Núria Fagella, Xavier Jarque, and Bogusława Karpińska. “Accesses to infinity from Fatou components”. In: *Transactions of the American Mathematical Society* 369.3 (2017), 1835–1867.

[BFJK19] Krzysztof Barański, Núria Fagella, Xavier Jarque, and Bogusława Karpińska. “Escaping points in the boundaries of Baker domains”. In: *Journal d’Analyse Mathématique* 137 (Nov. 2019), pp. 679–706.

[Bar08] Detlef Bargmann. “Iteration of inner functions and boundaries of components of the Fatou set”. In: *Transcendental Dynamics and Complex Analysis. Cambridge University Press* (2008), pp. 1–36.

[Bea91] Alan F. Beardon. *Iteration of rational functions. Complex analytic dynamical systems.* Vol. 1. Springer-Verlag, New York, 1991. ISBN: 0387975896.

[Ben+19] Anna Miriam Benini, Vasiliki Evdoridou, Núria Fagella, Philip J. Rippon, and Gwyneth M. Stallard. *Classifying simply connected wandering domains.* 2019. arXiv: `1910.04802 [math.DS]`.

[Ber93] Walter Bergweiler. “Iteration of meromorphic functions”. In: *Bulletin of the American Mathematical Society* 29.2 (Oct. 1993), pp. 151–188.

[Ber95] Walter Bergweiler. “On the Julia set of analytic self-maps of the punctured plane”. In: *Analysis* 15 (Apr. 1995), pp. 251–256.

[Bis18] Christopher J. Bishop. “Harmonic measure: algorithms and applications”. In: *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. III. Invited lectures* (2018), 1511–1537.

[BMS05] Paul S. Bourdon, Valentin Matache, and Joel H. Shapiro. “On the convergence to the Denjoy-Wolff point”. In: *Illinois Journal of Mathematics* 49.2 (2005), 405–430.

[CG93] Lennart Carleson and Theodore W. Gamelin. *Complex dynamics*. Springer-Verlag, 1993. ISBN: 0387979425.

[Che11] Arnaud Cheritat. "Relatively compact Siegel disks with non-locally connected boundaries". In: *Mathematische Annalen* 349.3 (2011), 529–542.

[CL66] Edward F. Collingwood and Arthur J. Lohwater. "The theory of cluster sets". In: *Cambridge Tracts in Mathematics and Mathematical Physics* 56 (1966).

[Con73] John B. Conway. *Functions of One Complex Variable I*. Graduate Texts in Mathematics. Springer, 1973. ISBN: 3540903283.

[Con95] John B. Conway. *Functions of One Complex Variable II*. Graduate Texts in Mathematics. Springer, 1995. ISBN: 387903283.

[Cow81] Carl C. Cowen. "Iteration and the solution of functional equations for functions in the unit disk". In: *Transactions of the American Mathematical Society* 265.1 (May 1981), pp. 69–95.

[Den26] Arnaud Denjoy. "Sur l'itération des fonctions analytiques". In: *Comptes rendus hebdomadaires des séances de l'Académie des sciences* 182 (1926), pp. 255–257.

[DG87] Robert Devaney and Lisa Goldberg. "Uniformization of attracting basins for exponential maps". In: *Duke Mathematical Journal* 55.2 (1987), pp. 253–266.

[Dev89] Robert L. Devaney. *An Introduction to Chaotic Dynamical Systems*. Addison-Wesley Publishing Company, Inc., 1989. ISBN: 0201130467.

[DM91] Claus Doering and Ricardo Mañé. "The Dynamics of Inner Functions". In: *Ensaios de Matemática (SBM)* Volume 3 (Jan. 1991), pp. 1–79.

[EL84] Alexandre Eremenko and Mikhail Lyubich. "Iterations of entire functions". In: *Doklady Akademii Nauk SSSR* 279.1 (1984), pp. 25–27.

[ERS20] Vasiliki Evdoridou, Lasse Rempe, and David J. Sixsmith. "Fatou's associates". In: *Arnold Mathematical Journal* 6.3-4 (2020), pp. 91–100.

[FH06] Núria Fagella and Christian Henriksen. "Deformation of Entire Functions with Baker Domains". In: *Discrete and Continuous Dynamical Systems* 15 (June 2006), pp. 379–394.

[Fat26] Pierre Fatou. "Sur l'itération des fonctions transcendantes entières". In: *Acta Mathematica* 47 (1926), pp. 337–360.

[Gar07] John B. Garnett. *Bounded Analtyic Functions*. Graduate Texts in Mathematics. Springer, 2007. ISBN: 387336214.

[GM05] John B. Garnett and Donald E. Marshall. *Harmonic Measure*. New Mathematical Monographs. Cambridge University Press, 2005. ISBN: 9780511546617.

[GK86] Lisa R. Goldberg and Linda Keen. "A finiteness theorem for a dynamical class of entire functions". In: *Ergodic Theory and Dynamical Systems* 6.2 (1986), 183–192.

[Ham96] David H. Hamilton. "Absolutely continuous conjugacies of Blaschke products". In: *Advances in Mathematics* 121 (1996), pp. 1–20.

[Haw21] Jane Hawkins. *Ergodic Dynamics. From Basic Theory to Applications*. Graduate Texts in Mathematics. Springer, 2021. ISBN: 9783030592417.

[Hei57] Maurice Heins. "Asymptotic spots of entire and meromorphic functions". In: *Annals of Mathematics*. 2nd ser. 66 (1957), 430–439.

[HY98] Xin-Hou Hua and Chung-Chun Yang. *Dynamics of Transcendental Functions*. Vol. 1. Asian Mathematics Series, 1998. ISBN: 9056991612.

[Kö99] Harald König. “Conformal conjugacies in Baker domains”. In: *Journal of the London Mathematical Society* 59.1 (Feb. 1999), pp. 153–170.

[MH99] Jerrold Marsden and Michael J. Hoffman. *Basic Complex Analysis.* WH Freeman. New York, 1999. ISBN: 1464152195.

[McM94] Curtis T. McMullen. *Complex Dynamics and Renormalization.* Princeton University Press, 1994. ISBN: 0691029822.

[MS93] Welington de Melo and Sebastian van Strien. *One Dimensional Dynamics.* Springer-Verlag, Berlin-Heidelberg, 1993. ISBN: 9783642780455.

[Mil06] John Milnor. *Dynamics in One Complex Variable. Third Edition.* Princeton University Press, 2006. ISBN: 9780691124889.

[Nos60] Kiyoshi Noshiro. *Cluster Sets.* Vol. 28. Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge. Springer, 1960. ISBN: 9783642859281.

[Pom92] Christian Pommerenke. *Boundary behaviour of conformal maps.* Springer-Verlag, Berlin, 1992. ISBN: 3540547517.

[Qia95] Jianyong Qiao. “The buried points on the Julia sets of rational and entire functions”. In: *Science in China (Scientia Sinica). Series A.* 38.12 (1995), pp. 1409–1419.

[RS11] Philip J. Rippon and Gwyneth M. Stallard. “Boundaries of escaping Fatou components”. In: *Proceedings of the American Mathematical Society* (2011), pp. 2807–2820.

[RS18] Philip J. Rippon and Gwyneth M. Stallard. “Boundaries of univalent Baker domains.” In: *Journal d’Analyse Mathématique* 134.2 (2018), 801–810.

[Rud87] Walter Rudin. *Real and Complex Analysis, 3rd Ed.* USA: McGraw-Hill, Inc., 1987. ISBN: 0070542341.

[SS85] Michael Schub and Dennis Sullivan. “Expanding endomorphisms of the circle revisited”. In: *Ergodic Theory and Dynamical Systems* 5 (1985), pp. 285–289.

[SS03] Elias M. Stein and Rami Shakarchi. *Fourier Analysis. An Introduction.* Princeton University Press, 2003. ISBN: 069111384.

[Ste93] Norbert Steinmetz. *Rational iteration. Complex analytic dynamical systems.* Walter de Gruyter, Berlin, 1993. ISBN: 3110137658.

[Ste78] Kenneth Stephenson. “Omitted values of singular inner functions”. In: *Michigan Mathematical Journal* 25.1 (1978), pp. 91–100.

[Sul85] Dennis Sullivan. “Quasiconformal homeomorphisms and dynamics, I, Solution of the Fatou-Julia problem on wandering domains”. In: *Annals of Mathematics* 122 (1985), pp. 401–418.

[Wol26a] Julius Wolff. “Sur l’itération des fonctions bornées”. In: *Comptes rendus hebdomadaires des séances de l’Académie des sciences* 182 (1926), pp. 200–201.

[Wol26b] Julius Wolff. “Sur une géneralisation d’un théorème de Schwarz”. In: *Comptes rendus hebdomadaires des séances de l’Académie des sciences* 182 (1926), pp. 918–919.

Appendices

A The Fatou, Riez and Riez theorem on radial limits

Here we give a proof of the Fatou, Riez and Riez theorem on the existence of radial limits, for a bounded holomorphic function of the unit disk $\mathbb{D}$ as well for a conformal one. Compare with Theorems 1.2.4 and 2.1.2. The statement we want to prove is the following.

Theorem A. [Fatou, Riez and Riez] *Let f be a holomorphic function of $\mathbb{D}$. Assume either f is bounded or f is conformal. Then, for λ-almost every θ, the radial limit $f^*(e^{i\theta})$ exists. Moreover, fixed $e^{i\theta}$ for which the radial limit exists, then $f^*(e^{i\theta}) \neq f^*(e^{i\xi})$ for λ-almost every ξ.*

We structure the proof as follows. First, we deal with the existence of radial limits, distinguishing between the bounded case (A.1) and the conformal one (A.2). Finally, we prove that radial limits are different almost surely (A.3). The proof for the bounded case has been taken from [CL66, Thm. II.2.1] and from [Mil06, Thm. A.3], for the almost surely uniqueness. For the conformal case, we follow the proof of [Mil06, Thm. 17.4].

Finally, it is important to say that this theorem is as good as possible, in the sense that, for any set $E \subset \partial\mathbb{D}$ of zero λ-measure, there is a holomorphic self-map of $\mathbb{D}$ with no radial limits in E. The construction of such function can be found in [CL66, Section II.10].

A.1 Existence of radial limits in the bounded case

The proof we present is based on Fourier analysis. For a basic background on it and the statements of the results used, we refer to [SS03].

Since f is a holomorphic function of the unit disk $\mathbb{D}$, it can be written as a power series centered at the origin: $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Assume $|f(z)| < M$, for $z \in \mathbb{D}$.

First we prove that $\sum_{n=0}^{\infty} |a_n|^2$ is bounded. Indeed, for $r < 1$, we have:

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \int_0^{2\pi} \left| f(re^{i\theta}) \right|^2 \frac{d\theta}{2\pi} \leq M^2.$$

Since M is independent of r, letting $r \to 1$, we get that $\sum_{n=0}^{\infty} |a_n|^2$ is bounded.

Now, applying the Riez-Fischer representation theorem, since the sequence $\{a_n\}_n \in \ell^2$, there exists a function $g \in L^2[0, 2\pi]$ such that its Fourier series have this sequence of coefficients:

$$Sf(g)(t) = \sum_{n=0}^{\infty} a_n e^{int} = f(e^{it}).$$

By Carleson's theorem, for functions in $L^2[0, 2\pi]$, the Fourier series approximates pointwise the function almost everywhere. Therefore, radial limits for f exists almost everywhere.

A.2 Existence of radial limits in the conformal case

We want to prove that, if $f : \mathbb{D} \to U$ is conformal, radial limits $f^*(e^{i\theta})$ exists for λ-almost every θ. Since f is assumed to be conformal, the radial limit f^* must accumulate in the boundary, either on a continuum or on a single point. Therefore, considering the radius $r_\theta = \left\{ re^{i\theta} : r \in (0, 1) \right\}$, if $f(r_\theta)$ is a curve of finite length, then $f^*(e^{i\theta})$ is a single point. Therefore, the strategy is to prove that, under a conformal map, almost every segment of finite spherical length is mapped into a segment of finite spherical length.

Consider $I = (0, \delta)$ to be an open interval of real numbers, $\delta > 0$, and consider $I^2 \subset \mathbb{C}$ to be the open square of points $z = x + iy$, with $x, y \in I$. Assume $\rho\colon I^2 \to (0, \infty)$ is a metric on I^2. Then, the area of I^2 and the length of a horizontal segment $\{z = x + ic\}$, with c constant, are defined as follows.

$$A(I^2) = \text{Area}(I^2) = \int\int_{I^2} \rho(x + iy)^2 dxdy.$$

$$L(c) = \text{Length}(\{z = x + ic\}) = \int_I \rho(x + ic)dx.$$

Lemma A.2.1. (Length-area inequality) *If the area A of I^2 is finite, then the length $L(c)$ is finite for almost every height $c \in I$.*

Proof. We use Cauchy-Schwarz inequality:

$$\left(\int_I f(x)g(x)dx\right)^2 \leq \left(\int_I f(x)^2 dx\right)\left(\int_I g(x)^2 dx\right),$$

with $f \equiv 1$ and $g(x) = \rho(x + ic)$. Therefore,

$$L(c)^2 = \left(\int_I \rho(x + ic)dx\right)^2 \leq \delta\left(\int_I g(x)^2 dx\right) \leq \delta A,$$

which gives the desired result. □

Then, given a univalent embedding $\eta\colon I^2 \to U \subset \widehat{\mathbb{C}}$, consider the spherical metric σ in U and the pull-back metric ρ in I^2, i.e. $\rho(z) = \sigma(\eta(z))$, for $z \in I^2$. Then,

$$L_\rho(\{y = c\}) = \int_I \rho(x + ic)dx = \int_I \sigma(\eta(x + ic))dx = L_\sigma(\eta(\{y = c\})).$$

Since we are considering the spherical metric in U, the area of U is finite, so previous the lemma applies. Therefore, almost every horizontal line segment $\{y = c\}$ in I^2 is mapped by η to a curve of finite spherical length.

Finally, to prove the existence of almost every radial limit of a conformal map $f\colon \mathbb{D} \to U$, consider $\mathbb{H}^-$ to be the left half-plane and the square $I^2 \coloneqq \{x + iy\colon -2\pi < x < 0, 0 < y < 2\pi\}$. Consider the univalent embedding $f \circ \exp\colon I^2 \to U$. By the previous remark, almost every horizontal segment is mapped to a curve of finite spherical length. Since horizontal segments in I^2 are mapped by the exponential to the end part of the radii of the disk, this finishes the proof of the theorem.

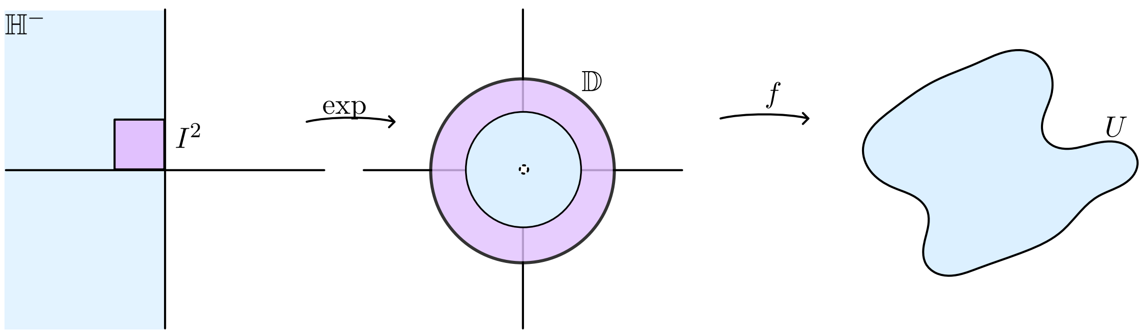


Figure 19: How the square $I^2 \subset \mathbb{H}^-$ is mapped first to $\mathbb{D}$ and then to U.

A.3 Different radial limits λ-almost everywhere

We start proving the result in the case when $f(\mathbb{D})$ is bounded, but not necessarily conformal. The key tool on the proof is the following average

$$A(f,r) := \int_0^{2\pi} \log\left|f(re^{i\theta})\right| \frac{d\theta}{2\pi}$$

and its relation with the zeros of f by the well-known Jensen's formula.

Theorem A.3.1. (Jensen's formula) [[Rud87, Thm. 15.18]] *Let $f\colon \mathbb{D} \to \mathbb{C}$ be bounded and holomorphic on $\mathbb{D}$. Assume $f(0) \neq 0$. Fix $r \in (0,1)$. Let $z_1, \ldots, z_n$ be the zeros of f in $\overline{D(0,r)}$. Then,*

$$A(f,r) := \int_0^{2\pi} \log\left|f(re^{i\theta})\right| \frac{d\theta}{2\pi} = \log|f(0)| + \sum_{k=1}^{n} \log\left(\frac{r}{|z_k|}\right).$$

From this one deduces that, if $f \not\equiv 0$, then $A(f,r)$ is increasing, as a function of r. Therefore, as $r \to 1^-$, $A(f,r)$ converges to a finite limit or diverges to $+\infty$.

We are going to prove that if there exists a measurable set E, with $\lambda(E) > 0$, and $f^*(e^{i\theta}) = c_0$, for all $\theta \in E$, then $f \equiv c_0$. Without lost of generality, we shall assume $c_0 = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$.

Consider, for $\varepsilon > 0$ and $0 < \delta < 1$, the set:

$$E(\varepsilon,\delta) := \left\{\theta \in E\colon \left|f(re^{i\theta})\right| < \varepsilon, \text{ if } \delta < r < 1\right\}.$$

Clearly, if $\delta_1 \leq \delta_2$, we have $E(\varepsilon,\delta_1) \subset E(\varepsilon,\delta_2)$ and $\cup_{0<\delta<1} E(\varepsilon,\delta) = E$. Moreover, as $\delta \to 1$, $\lambda(E(\varepsilon,\delta)) \to \lambda(E)$. In particular, if we fix $\varepsilon > 0$, one can choose $\delta \in (0,1)$ such that $\lambda(E(\varepsilon,\delta)) \geq \frac{\lambda(E)}{2} > 0$. Take r such that $\delta < r < 1$. Since $|f(z)| < 1$ for $z \in \mathbb{D}$, we have $\log\left|f(re^{i\theta})\right| < 0$ for all θ. For $\theta \in E(\varepsilon,\delta)$, $\log\left|f(re^{i\theta})\right| < \log\varepsilon$. Therefore, we have:

$$2\pi A(f,r) = \int_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta < \log\varepsilon \cdot \frac{\lambda(E)}{2}.$$

Since ε is arbitrarily small, we have $\lim\limits_{r\to 1^-} A(f,r) = -\infty$, so $f \equiv 0$. Thus, the case when f is bounded is proved.

The conformal case can be deduced from the previous one. Indeed, if $f(\mathbb{D})$ is bounded, we are done. Assume now that $f(\mathbb{D})$ is not bounded, but it omits an open neighbourhood of some point $z_0 \in \widehat{\mathbb{C}}$. Consider the Möbius transformation $M(z) = \frac{1}{z-z_0}$, which maps z_0 to ∞. It is clear that $M(f(\mathbb{D}))$ is bounded, so the radial limits of $M \circ f$ are different almost surely. Since M is a Möbius transformation, the radial limits of f should also be different almost surely.

Now we deal with the general case. Since f is conformal, $f(\mathbb{D})$ omits at least two different values. Composing with an appropriate Möbius transformation if needed, we shall assume that this values are 0 and ∞. Then, $f(\mathbb{D}) \subset \mathbb{C}$ is simply-connected and omits the value 0, one can define $\sqrt{f}$ as a single-valued function. Since $\sqrt{f(\mathbb{D})}$ omits an open set, by the previous case, radial limits of $\sqrt{f}$ must be different almost everywhere. The inverse of the square root (the squaring function) is defined in the whole plane $\mathbb{C}$, so it takes curves of finite spherical length to curves of finite spherical length. Moreover, since it has degree 2, it sends sets of measure zero to sets of measure zero. Therefore, since $\sqrt{f}$ has different radial limits almost everywhere, also has f. This finishes the proof of the theorem.

B Examples of inner functions

In this section we are going to give examples of inner functions, which will represent all the dynamic behaviours described along this thesis.

Examples of inner functions of elliptic type are not mentioned because of its simplicity. In general, given any Blaschke product B, the map zB is a Blaschke product having 0 as Denjoy-Wolff point, i.e. of elliptic type. Moreover, the map is always holomorphic at the Denjoy-Wolff point and the dynamics around it are trivial in the following sense. If it is attracting, but not super-attracting, the Koenigs coordinates give a linearization around it. In case it is super-attracting, one shall use Böttcher coordinates. Therefore, the difficult and interesting examples appear when the Denjoy-Wolff point lies in the boundary.

The simplest examples will be given as functions of $\mathbb{D}$ but, as the difficulty increases, it is convenient to work in the upper half-plane $\mathbb{H}$. From the theoretical point of view, there is no difference between considering $\mathbb{H}$ or $\mathbb{D}$, since both are conformally equivalent. However, this change allows us to build simpler examples and to make the computations explicitly. There is a complete characterization of inner functions of $\mathbb{H}$ ([DM91, Section 5]), but for our purposes is not needed, it is enough the definition and the result given next.

B.1 Inner functions of the upper half-plane $\mathbb{H}$

We say that h is an **inner function of the upper half-plane** $\mathbb{H}$ if, given any conformal transformation $\varphi\colon \mathbb{D} \to \mathbb{H}$, the function $g := \varphi^{-1} \circ h \circ \varphi$ is an inner function of the unit disk $\mathbb{D}$. Equivalently, if h is holomorphic in $\mathbb{H}$, $h(\mathbb{H}) \subset \mathbb{H}$ and the radial limits satisfy:

$$h^*(x) = \lim_{y \to 0^+} h(x+iy) \in \mathbb{R} \cup \{\infty\},$$

for λ-almost every $x \in \mathbb{R} \cup \{\infty\}$.

We start with a theorem ([Bar08, Lemma 2.33]) that, given an inner function h of $\mathbb{H}$, characterizes when ∞ is the Denjoy-Wolff point of h.

Proposition B.1.1. *Given a non-Möbius inner function h of $\mathbb{H}$, the following are equivalent:*

1. *∞ is the Denjoy-Wolff point of h.*
2. *$Im(h(z)) \geq Im(z)$, for each $z \in \mathbb{H}$.*
3. *$H(z) := h(z) - z$ is an inner function of $\mathbb{H}$.*
4. *$Im(h(z)) > Im(z)$, for each $z \in \mathbb{H}$.*

Proof. (1) ⇒ (2): It is the reformulation of Wolff lemma (1.3.2) for the upper half-plane.

(2) ⇒ (3): Clearly, H is holomorphic in $\mathbb{H}$, its radial limits exist λ-everywhere in $\mathbb{R} \cup \{\infty\}$ and are real. Therefore, it only has to be checked that $H(\mathbb{H}) \subset \mathbb{H}$. By the hypothesis, $\text{Im}(H(z)) = \text{Im}(h(z)) - \text{Im}(z) \geq 0$, so $H(\mathbb{H}) \subset \overline{\mathbb{H}}$. By the Open Mapping Theorem, either H is constant or $H(\mathbb{H}) \subset \mathbb{H}$. Finally, observe that H cannot be constant, otherwise h must be a Möbius transformation.

(3) ⇒ (4): Trivial.

(4) ⇒ (1): Since, by hypothesis, $\text{Im}(h(z)) > \text{Im}(z)$ for all $z \in \mathbb{H}$, there cannot be fixed points in $\mathbb{H}$. Therefore, the Denjoy-Wolff must be in $\mathbb{R} \cup \{\infty\}$. The hypothesis also implies that the iterates cannot converge to any point in $\mathbb{R}$, so ∞ must be the Denjoy-Wolff point. □

B.2 Examples of inner functions of finite degree

The first example we are giving is the typical and simplest example of doubly-parabolic type, given first by Cowen as an example for his classification [Cow81].

Example B.2.1. (Degree 2, doubly-parabolic type) Consider the function

$$g(z) = \frac{3z^2+1}{z^2+3}.$$

First observe that this is indeed an inner function, because can be written as a Blaschke product:

$$g(z) = \frac{3z^2+1}{z^2+3} = \frac{z - \frac{1}{\sqrt{3}}i}{1+\frac{1}{\sqrt{3}}iz} \cdot \frac{z+\frac{1}{\sqrt{3}}i}{1-\frac{1}{\sqrt{3}}iz}.$$

We claim that 1 is the Denjoy-Wolff point of g and it has multiplicity 2. Indeed, $g(1) = 1$ and

$$g'(z) = \frac{16z}{(z^2+3)^2} \quad \text{and} \quad g''(z) = \frac{48-48z^2}{(z^2+3)^3},$$

so $g'(1) = 1$ and $g''(1) = 0$. Therefore, by Theorem 3.3.1, g is of doubly-parabolic type.

For the hyperbolic and simply-parabolic type, the examples of [Cow81] are of Möbius transformations. Here we include examples of degree greater than 1. Observe that doubly-parabolic Möbius transformations cannot exist, otherwise $\mathbb{D}$ could be taken itself as an absorbing domain and Theorem 1.3.7 would imply the existence of a conformal mapping $\mathbb{D} \to \mathbb{C}$, which is impossible by Liouville.

In a similar way of Example B.2.1, we can construct an easy example of the hyperbolic type.

Example B.2.2. (Degree 2, hyperbolic type) We claim that the function

$$g(z) = \frac{5z^2+1}{z^2+5}$$

is an inner function of degree two and hyperbolic type. As in the previous example, it can be written as a degree two Blaschke product. Clearly 1 is a fixed point of g and

$$g'(z) = \frac{32z}{(3z^2+5)^2},$$

so $g'(1) = \frac{1}{2}$ and we deduce that 1 is the Denjoy-Wolff point of g, which is of hyperbolic type.

It is left to find an example of the simply-parabolic type. First observe that its expression cannot be as simple as the two previous examples, because it cannot leave the real line invariant (compare with Figure 3). To find easier expressions it is better to work in the upper half-plane $\mathbb{H}$. In the sequel, we consider rational maps of the form:

$$h(z) = az + b - \sum_{j=1}^{n} \frac{1}{z-a_j},$$

with $a > 0$, $b, a_j \in \mathbb{R}$, for all j. Although the function has poles, this is not a problem because they are on the real line, so they are preimages of infinity, which is a point in the boundary of $\mathbb{H}$. It is easy to see that h is an inner function of $\mathbb{H}$ of degree $n+1$. Observe that, by Proposition B.1.1, ∞ is the Denjoy-Wolff point of h if and only if $a \geq 1$. We restrict ourselves to this case since, conjugating by a rotation, we can always assume that ∞ is the Denjoy-Wolff point. Notice that the multiplier of ∞ as a fixed point is given by $\frac{1}{a}$.

D. Bargmann ([Bar08, Example 2.39]) gives the following example of inner function of finite degree of simply-parabolic type.

Example B.2.3. (Finite degree, simply-parabolic type) Let $a_1, \dots, a_n$ be positive real numbers and consider the following function:

$$h(z) = z - 1 - \sum_{j=1}^{n} \frac{1}{z - a_j}.$$

It is clear that it is an inner function of $\mathbb{H}$ of degree $n+1$ and ∞ is its Denjoy-Wolff point (so the elliptic case is discarded). The multiplier of ∞ as a fixed point is equal to 1, so the function is of parabolic type. Near ∞, all the terms $\frac{1}{z-a_j}$ are as small as wanted. Therefore, near ∞, the map essentially acts as the translation $z \mapsto z - 1$. Hence, there is a forward invariant interval in the real line $I = (-\infty, c)$, for some $c \in \mathbb{R}$. This is telling us that $\mathcal{F}(h) \cap \mathbb{R} \neq \emptyset$, so the function is of simply-parabolic type. Notice that, with this procedure, we get examples of inner functions of simply-parabolic type of any (finite) degree.

In a similar way, we can make an analogous construction to get examples of inner functions of hyperbolic and doubly-parabolic type of any (finite) degree.

Example B.2.4. (Finite degree, hyperbolic type) We proceed as in the previous example, but forcing the multiplier to be smaller than 1 in modulus. Hence, let $a_1, \dots, a_n$ be positive real numbers and consider the following function:

$$h(z) = az + b - \sum_{j=1}^{n} \frac{1}{z - a_j},$$

with $a > 1$ and $b \in \mathbb{R}$. This is an inner function of $\mathbb{H}$ having ∞ as Denjoy-Wolff point. Observe that now the multiplier of ∞ as fixed point is $\frac{1}{a}$, so the function is clearly of hyperbolic type.

Example B.2.5. (Finite degree, doubly-parabolic type) To build an example of function of doubly-parabolic type with ∞ as the Denjoy-Wolff point, we need $a = 1$ and $b = 0$. Indeed, $\frac{1}{a}$ is the multiplier of ∞ as a fixed point of the inner function, so it must be 1 to be of parabolic type. On the other hand, if $b > 0$, since the right-hand side sum is as small as wanted near infinity, there must exist an interval of the form (c, ∞) where iterates converge uniformly to ∞. Analogously, if $b < 0$, one finds an interval of the form $(-\infty, c)$ where iterates converge uniformly to ∞. In both cases, this implies that the Fatou set intersects $\mathbb{R} \cup \{\infty\}$, so the function cannot be of doubly-parabolic type. Therefore, we need $b = 0$.

A way to ensure that the function we are constructing is of doubly-parabolic type is to force the imaginary axis to be invariant. To do so, we place the poles in a symmetric way, as follows. Let $a_1, \dots, a_n$ be positive real numbers and consider

$$h(z) = z - \sum_{j=1}^{n} \frac{2z}{z^2 - a_j^2} = z - \sum_{j=1}^{n} \left(\frac{1}{z - a_j} + \frac{1}{z + a_j} \right).$$

It is easy to check that the imaginary axis is invariant, so the function is of doubly-parabolic type. As a remark, notice that the degree of the function is $2n+1$, because for each pole a_j we have also the pole $-a_j$. To build an example of doubly-parabolic type of even degree, simply add the pole 0 as follows:

$$h(z) = z - \frac{1}{z} - \sum_{j=1}^{n} \frac{2z}{z^2 - a_j^2} = z - \frac{1}{z} - \sum_{j=1}^{n} \left(\frac{1}{z - a_j} + \frac{1}{z + a_j} \right).$$

B.3 Examples of inner functions of infinite degree with non-singular Denjoy-Wolff point

Now, we start to look for examples of inner functions of infinite degree, that is functions that have at least one singularity in $\partial\mathbb{D}$. The easiest kind of functions to work with are singular inner functions with only one singularity, as the one in the example of Section 2.1. Playing with it we will find examples of the elliptic, the doubly-parabolic and the hyperbolic cases, always with the Denjoy-Wolff point being non-singular. This kind of functions are so simple that we cannot find examples of them with singular Denjoy-Wolff point nor of simply-parabolic type. As we will see, the radial limit at the singularity is always 0, so the singularity cannot be the Denjoy-Wolff point (it needs to be radially fixed, Thm. 1.3.2). Moreover, the real line is always invariant, so the simply-parabolic case cannot appear. An example of simply-parabolic type can be found in the next section (Example B.4.3).

Example B.3.1. (Infinite degree, non-singular DW point) Consider the following family of functions, depending on $\lambda > 0$,

$$g_\lambda(z) = \exp\left(\lambda \frac{z-1}{z+1}\right).$$

It is easy to see that this functions are inner functions. Indeed, $z \mapsto \lambda \dfrac{z-1}{z+1}$ is a conformal map from the unit disk to the left half-plane and then the exponential sends it back to $\mathbb{D} \setminus \{0\}$.

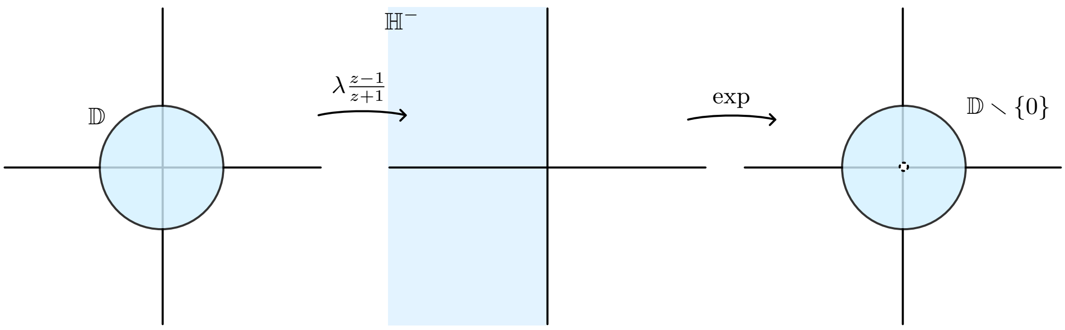


Figure 20: How g_λ maps $\mathbb{D}$ onto $\mathbb{D} \setminus \{0\}$.

All functions of the family have -1 as a singularity since it is mapped to ∞ by the conformal map, which is the essential singularity of the exponential. The radial limit at -1 is 0 (it can be computed easily since the real axis is invariant). The different values for λ change the position of the Denjoy-Wolff point and the value of its multiplier.

Observe that $z = 1$ is always a fixed point of the function. The derivative of the function is:

$$g'_\lambda(z) = \lambda \frac{2}{(z+1)^2} \exp\left(\lambda \frac{z+1}{z-1}\right).$$

Then, $g'_\lambda(1) = \frac{1}{2}\lambda$. For $\lambda \in (0, 2]$, 1 is the Denjoy-Wolff point and, for $\lambda > 2$ it is a repelling fixed point (so it cannot be the Denjoy-Wolff point). For $\lambda \in (0, 2)$ the map is hyperbolic and for $\lambda = 2$ it is parabolic. Since the real axis is invariant, one deduces that it is doubly-parabolic. Finally, for $\lambda > 2$, there is a real fixed point in $(-1, 1)$, which is the Denjoy-Wolff point and the map is elliptic. Indeed, since 1 is a repelling fixed point (the derivative is larger than 1) and at -1 the radial derivative is equal to 0, the map must intersect the diagonal in some point in between. Compare with Figures 21, 22 and 23.

On the concrete case of elliptic or doubly-parabolic type, the Julia set is $\partial\mathbb{D}$. We can study the dynamics on it generalizing the arguments of [DG87, Section 2]. First observe that -1 is

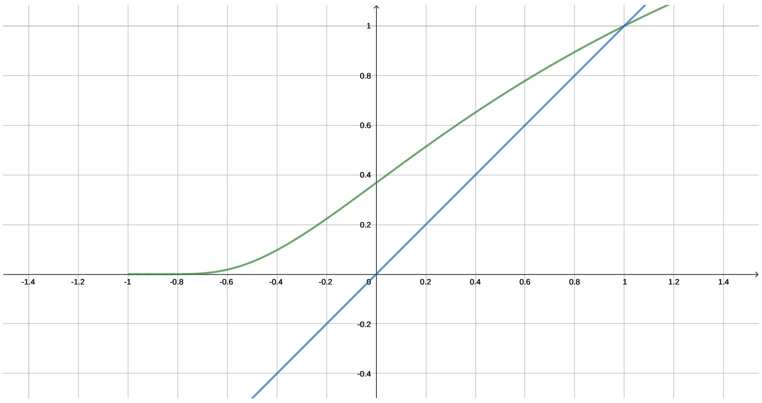


Figure 21: Plot of the real function $g(x) = \exp\left(\frac{x+1}{x-1}\right)$ (green), together with the diagonal $y = x$ (blue). It is clear that 1 is an attracting fixed point. The function is of hyperbolic type.

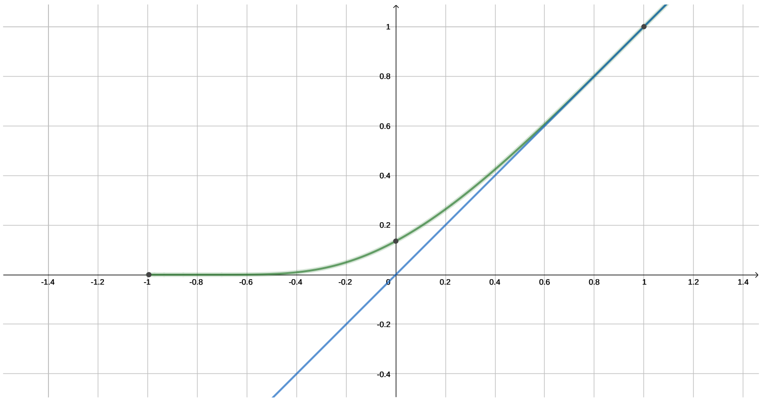


Figure 22: Plot of the real function $g(x) = \exp\left(2\frac{x+1}{x-1}\right)$ (green), together with the diagonal $y = x$ (blue). One observes that 1 is an parabolic fixed point and points in $(-1, 1)$ are attracted to it. Therefore, the map is doubly-parabolic.

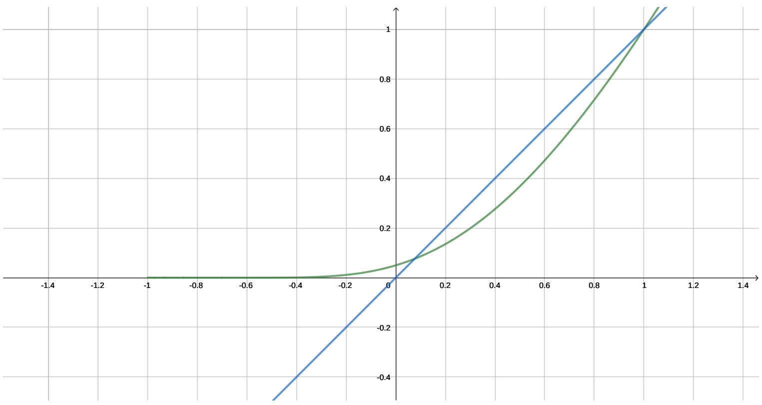


Figure 23: Plot of the real function $g(x) = \exp\left(3\frac{x+1}{x-1}\right)$ (green), together with the diagonal $y = x$ (blue). Observe that, although 1 is again a fixed point, now is repelling. An attracting fixed point appears in between -1 and 1. The function is of elliptic type.

the unique singularity of g_λ and, since $\mathcal{J}(g) = \partial\mathbb{D}$, points z such that $f^n(z) = -1$, for some n, are dense in $\partial\mathbb{D}$. On the other hand, if we only consider the preimages of -1, i.e. points z such that $f(z) = -1$, these points form a countable set and accumulate to -1 both clockwise and counterclockwise. We shall denote them by z_k, with $k \in \mathbb{Z}$, so that they are ordered in a circular order, and denote by I_k the closed circular interval $[z_k, z_{k+1}]$. Since g'_λ is strictly positive in $\partial\mathbb{D}$, it follows that each interval I_k is mapped bijectively onto $\partial\mathbb{D}$. Therefore, given a point $z \in \partial\mathbb{D}$ which is not eventually mapped to -1, we can assign to it a sequence $\mathcal{I}(z) = s_0 s_1 s_2 ...$, so $s_j = k$ if and only if $T_\mu^j(z) \in I_k$. We say that $\mathcal{I}(z)$ is the **itinerary** of z. If we denote by Σ the space of infinite sequence of integers and $\Lambda = \{z \in \partial\mathbb{D} \colon g_\lambda^n \neq -1, \text{ for all } n \geq 0\}$, the itinerary function $\mathcal{I}$ gives a map $\Lambda \to \Sigma$.

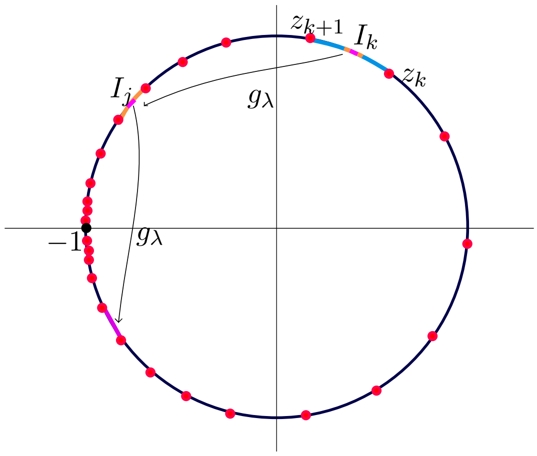


Figure 24: Schematic representation of the z_k's in $\partial\mathbb{D}$, and how the intervals I_k's are mapped.

Proposition B.3.2. *$\mathcal{I} \colon \Lambda \to \Sigma$ is a bijection.*

Proof. Consider any $s = s_0 s_1 s_2 ... \in \Sigma$ and define $z_s = \bigcap_{j=0}^{\infty} T_\mu^{-j}\left(I_{s_j}\right)$. Therefore, z_s is not empty because it is the intersection of nested closed intervals. Any point in z_s has itinerary s. Note that z_s is connected, closed and cannot contain an interval. Otherwise, it would contain preimages of -1, which are not in the required intersection. Thus, z_s is a unique point. □

On Σ one can consider the **shift map**, denoted by σ and defined as:

$$\sigma(s_0 s_1 s_2 ...) = (s_1 s_2 ...).$$

Therefore, by choosing an appropriate metric in Σ, we have that the itinerary map conjugates $g_{\lambda|\Lambda}$ to $\sigma_{|\Sigma}$, i.e. $\mathcal{I} \circ g_\lambda = \sigma \circ \mathcal{I}$.

On the opposite case, when the map is of hyperbolic type, one can see that the Julia set is a Cantor set, applying the arguments of Theorem 3.3.1, although we are in the case of infinite degree. Indeed, observe that the map is well-defined and continuous in $\partial\mathbb{D}$, except at -1. A neighbourhood of the Denjoy-Wolff point is contained in the Fatou set, so there exists a point in the Julia set that can be approximated from points in the Fatou set. Taking as many preimages as needed and applying that backward orbits of points in the Julia set are dense, one gets that any point in the Julia set that can be approximated from points in the Fatou set. Therefore, the Julia set must be a Cantor set.

B.4 Examples of inner functions of infinite degree with singular Denjoy-Wolff point

In this section we present examples of inner functions with singular Denjoy-Wolff point. The first examples we are considering here are of the same type and are due to D. Bargmann ([Bar08, Section 2.5]). In some sense, this is a generalization of the formula we used to get inner functions of finite degree, but taking infinitely many poles so that the sum is infinite. Indeed, let $A = \{a_n\}_n \subset \mathbb{R}$ and $g\colon A \to (0, +\infty)$ be such that $\sum_{n=1}^{\infty} g(a_n)$ converges. Let $a \in (0, +\infty)$ and $b \in \mathbb{R}$. Define:

$$h(z) := az + b - \sum_{n=1}^{\infty} \frac{g(a_n)}{z - a_n}. \tag{B.4.1}$$

Theorem B.4.1. *The function h defined above is an inner function of the upper half-plane $\mathbb{H}$.*

Proof. First we have to check that h is holomorphic in $\mathbb{H}$. If A is finite, this is trivial because it is the finite sum of holomorphic functions. If A is infinite it is enough to see that the series $\sum_{n=1}^{\infty} \frac{g(a_n)}{z-a_n}$ converges uniformly on compact subsets of $\mathbb{H}$. Consider $K \subset \mathbb{H}$ compact. Since $A \subset \mathbb{R}$ and $d(K, \mathbb{R}) = k > 0$, $\left|\frac{g(a_n)}{z-a_n}\right| < \frac{g(a_n)}{k}$ and $\sum_{n=1}^{\infty} g(a_n) < \infty$ by hypothesis. By the Weierstrass M-test, the series converges uniformly on K, so h is holomorphic in $\mathbb{H}$.

To see that $h(\mathbb{H}) \subset \mathbb{H}$ simply observe that each addend leaves $\mathbb{H}$ invariant, so $h(\mathbb{H}) \subset \overline{\mathbb{H}}$. From the Open Mapping Theorem and the fact that clearly h is not a Möbius Transformation, we deduce that $h(\mathbb{H}) \subset \mathbb{H}$.

Finally, we observe that radial limits exists for all point in $\mathbb{R} \cup \{\infty\}$ and that $\mathbb{R} \cup \{\infty\}$ is invariant by h, directly from the formula. □

An alternative way to define the inner function h is as follows. This kind of examples are introduced in [DM91, p. 18]. Let $A = \{a_n\}_n \subset \mathbb{R}$ be such that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges. Let $a \in (0, +\infty)$ and $b \in \mathbb{R}$. Define:

$$h(z) := az + b - \sum_{n=1}^{\infty} \frac{1}{z - a_n}.$$

Checking that this is indeed an inner function of the upper half-plane is done similarly as in the previous proof. In this case, the holomorphy is ensured by the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$. Note that, in Bargmann's construction, any set of points is valid, in contrast of the last example, but in compensation a correction function for the numerator has to be introduced.

Observe that, if $|A|$ is finite, h is a rational function and there are no singularities. These are precisely the inner functions we considered for examples of finite degree. On the other hand, if $|A| = \infty$, the accumulation points of A are the singularities of h and h has infinite degree.

The following are examples of inner functions with singular Denjoy-Wolff point. Recall that in these points the radial limit exists (Thm. 1.3.2) and, in fact, they are radially fixed.

Example B.4.2. (Infinite degree, singular DW point, hyperbolic type) Let $\{a_n\}_n \subset (0, +\infty)$ be dense in $(0, +\infty)$ and consider h as in B.4.1, with $a > 1$ and $b \in \mathbb{R}$, i.e.

$$h(z) := az + b - \sum_{n=1}^{\infty} \frac{g(a_n)}{z - a_n}.$$

Since $a > 1$, the Denjoy-Wolff point is ∞. Since ∞ is an accumulation point of A, it is a singularity.

Observe that $F(g) \cap \mathbb{R}$ is non-empty. Indeed, since A is bounded from the left, one can find an interval $I = (-\infty, c)$, for some $c \in \mathbb{R}$, invariant under g. It is clear that points in I converge uniformly to ∞, so it is a subset of the Fatou set.

Since the Julia set is not the whole extended real line, it cannot be of doubly-parabolic type. It is concretely of hyperbolic type. An intuitive idea of why is that, for points in $\mathbb{R}_-$ close to

∞ enough, the map behaves like $z \mapsto az$, with $a > 1$. For a detailed proof, we refer to [Bar08, Example 2.36].

Finally, it is worth to mention that the Julia set contains $(0, +\infty)$, for being accumulation points of the singularities. For finite degree inner functions of hyperbolic type, the Julia set must be a Cantor set. This is an example that, for infinite degree, the situation is different. However, although the dynamics in $(0, +\infty)$ is extremely chaotic (in any neighbourhood of a singularity all real values are attained, Thm. 2.2.4), Aaronson's theorem (3.4.4) applies and almost every point in $\mathbb{R}$ converges to infinity.

Example B.4.3. (Infinite degree, singular or non-singular DW point, simply-parabolic type) Let $A = \{a_n\}_n \subset (0, +\infty)$ and consider h as in B.4.1, with $a = 1$ and $b = -1$, i.e.

$$h(z) \coloneqq z - 1 - \sum_{n=1}^{\infty} \frac{g(a_n)}{z - a_n}.$$

It is clear that ∞ is the Denjoy-Wolff point. Observe also that, since A is bounded from the left, one can find an interval I of the form $(-\infty, c)$, for some $c \in \mathbb{R}$, where iterates converge to ∞. Therefore, $\mathcal{F}(g) \cap \mathbb{R}$ is non-empty, and the function cannot be of doubly-parabolic type.

If A is bounded, i.e. ∞ is not an accumulation point of A, then ∞ is non-singular. In this case, the map is holomorphic in a neighbourhood of ∞ and, taking derivatives, one can check that the map is of simply-parabolic type. In the case that A is unbounded, then ∞ is a accumulation point of A and it is a singularity. In this case the map is also simply-parabolic, but the proof is more involved and it can be found in [Bar08, Example 2.39]. An intuitive idea of why is that, for points in $\mathbb{R}_-$ close to ∞ enough, the map behaves like $z \mapsto z - 1$.

Example B.4.4. (Infinite degree, singular DW point, doubly-parabolic type) Let $\{a_n\}_n$ be an increasing sequence of positive real values such that $\sum_{n=1}^{\infty} \frac{1}{a_n^2} < \infty$ and let:

$$h(z) = z - \sum_{n=1}^{\infty} \frac{2z}{z^2 - a_n^2} = z - \sum_{n=1}^{\infty} \left(\frac{1}{z - a_n} + \frac{1}{z + a_n} \right).$$

It is clear that h is an inner function of the upper half-plane $\mathbb{H}$ and ∞ is its Denjoy-Wolff point. An easy computation shows that $h'(x) > 1$ for $x \in \mathbb{R}$, so h maps every interval $(-a_{n+1}, -a_n)$, $(-a_1, a_1)$ and (a_n, a_{n+1}) bijectively onto $\mathbb{R}$. Following the ideas we used to study the dynamics on $\partial\mathbb{D}$ (Section B.3), one can see the existence of dense orbits. On the other hand, one can find a dense set of points converging to the Denjoy-Wolff point (∞). Recall this does not imply recurrence nor non-recurrence, because we only know this property for a set of λ-measure zero. By standard arguments on the hyperbolic metric, the map can be seen to be of doubly-parabolic type ([DM91, p. 18]), so it is ergodic. However, we cannot ensure recurrence.

The following examples are of inner functions of doubly-parabolic type, the first recurrent and the second non-recurrent. To do so, we particularize Example B.4.4 for a concrete sequence of a_n. Indeed, take $a_n = n^{\frac{\gamma}{2}}$, with $\gamma > 1$ to ensure the convergence of the series. In [DM91, p. 18], it is proved that different values of γ lead to either recurrence or non-recurrence. We summarize their results in the following examples.

Example B.4.5. (Infinite degree, singular DW point, doubly-parabolic type, recurrent) Consider the inner function:

$$h(z) = z - \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^{\gamma}},$$

with $\gamma \geq 2$. This function is of doubly-parabolic type and recurrent.

Example B.4.6. (Infinite degree, singular DW point, doubly-parabolic type, non-recurrent) [Aaronson] Consider the inner function:

$$h(z) = z - \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^\gamma},$$

with $1 < \gamma < 2$. This function is of doubly-parabolic type and non-recurrent.

We remark that this example was given by Aaronson ([Aar81]) and, as far as we are aware, this is the only example in the literature of an inner function that it is both non-recurrent and ergodic.

Finally we include another example of recurrent inner function of doubly-parabolic type, due to the simplicity of its analytic expression.

Example B.4.7. (Infinite degree, singular DW point, doubly-parabolic type, recurrent) Consider:

$$h(z) = z + \tan z.$$

This example is described in [DM91, p. 15], [BFJK15, Example 7.1] and [BFJK19, Example 1.6].

Given the expression

$$\tan(x + iy) = \frac{\sin 2x}{\cos 2x + \cosh 2y} + i \frac{\sinh 2y}{\cos 2x + \cosh 2y}$$

for $x, y \in \mathbb{R}$, we observe that the upper half-plane $\mathbb{H}$ is invariant and the map is symmetric with respect to $\mathbb{R}$. Therefore, h is an inner function of the upper half-plane. Observe that the imaginary axis is also invariant and the dynamics on it are given by

$$y \mapsto y + \frac{\sinh 2y}{1 + \cosh 2y} = y + \frac{\sinh(y)}{\cosh(y)}.$$

From this expression, one deduces that ∞ is the Denjoy-Wolff point (see Figure 25). In [DM91] it is checked that it is doubly-parabolic type and recurrent.

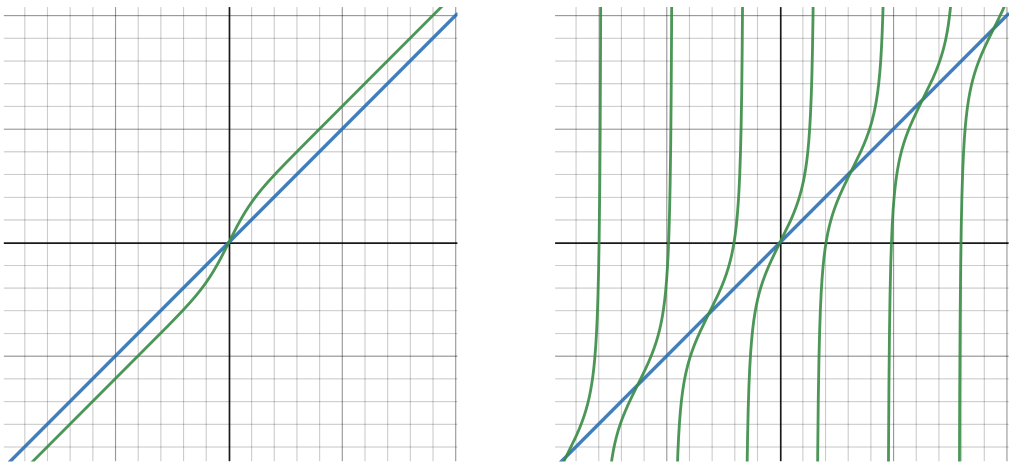


Figure 25: On the left, a plot of the function h restricted to the imaginary axis: $x \mapsto x + \frac{\sinh(x)}{\cosh(x)}$. On the right, a plot of the function h restricted to the real axis: $x \mapsto x + \tan x$.

Let us study the function on the boundary, i.e. on $\mathbb{R} \cup \{\infty\}$. Clearly, ∞ is the only singularity and its preimages are the points of the form $\left\{\frac{\pi}{2} + n\pi\right\}_{n\in\mathbb{Z}}$. Any interval delimited by two consecutive preimages of ∞ is mapped to $\mathbb{R}$ in a one-to-one fashion, as one can appreciate in Figure 25.

C Examples of inner functions associated to Fatou components

Few examples of inner functions associated to Fatou components have been computed explicitly. They are compiled in [BD99] and [ERS20]. Appart from the example studied in Section 5, we present examples of attracting and parabolic basins, by Devaney and Golberg (C.1), and of doubly-parabolic Baker domains (C.2).

C.1 The exponential map λe^z, with $0 < \lambda < \frac{1}{e}$

This first example was given by Devaney and Goldberg ([DG87]). In it, they study the iteration of the function $f_\lambda(z) = \lambda e^z$, with $0 < \lambda < \frac{1}{e}$. For this values of λ, there exists a real attracting fixed point, and they proved that the Fatou set consists precisely of the immediate attracting basin of this point. We denote by Ω_λ its basin of attraction, so $\mathcal{F}(f_\lambda) = \Omega_\lambda$.

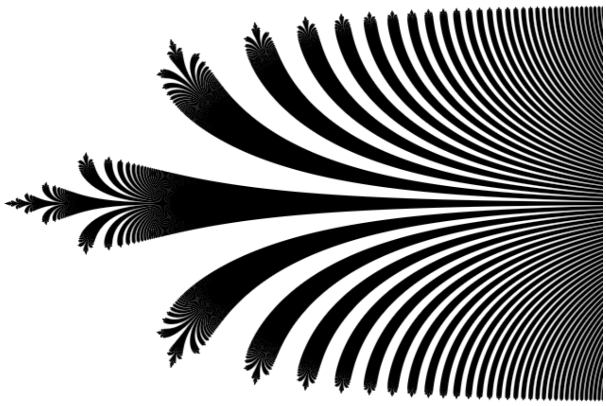


Figure 26: The dynamical plane of E_λ, with the Julia set in black, which is a Cantor Bouquet. Image courtesy of Arnaud Chéritat.

If one conjugates f_λ restricted to Ω_λ by a Riemann map sending 0 to 0, one deduces that the associated inner function is an infinite covering of $\mathbb{D}$. Therefore, the inner function is an exponential inner function of the form described in B.3.1, of elliptic type. Recall that this kind of functions have only one singularity on the boundary, and, for the points which are not preimages of a singularity, its behaviour is described via symbolic dynamics. Concretely, the boundary map is conjugate to the shift in the space of sequences of infinite symbols. Also recall that, in this case, the boundary map is ergodic and recurrent, and so is f_λ in $\partial\Omega_\lambda$.

On the other hand, they describe the dynamics on the Julia set $\mathcal{J}(f)$, which is precisely $\partial\Omega_\lambda$. It consists of an infinite set of disjoint curve with a finite endpoint and the other endpoint being ∞. These curves are called **hairs** and all the structure is called a **Cantor bouquet**. The dynamics on $\partial\Omega_\lambda$ are completely understood: hairs are mapped among them following a certain symbolic dynamics and all points but the endpoints escape to ∞ exponentially fast. Endpoints are mapped among them following the same symbolic dynamics. Here, the symbolic dynamics is again a shift in the space of sequences of infinite symbols, but with the difference that not all sequences are possible: only the ones that grow slowlier than $\{e^n\}_n$ (the **allowable sequences**).

The culminating point of their work is when they use the extension of the Riemann map to the boundary to deduce which points in $\partial\Omega_\lambda$ are accessible. Indeed, they proved that the singularity and all its preimages have radial limit equal to ∞, and the same for the points whose itinerary is a non-allowable sequence. The points whose itinerary is an allowable sequence correspond to endpoints of curves with the same itinerary. This implies that the only accessible points in $\partial\Omega_\lambda$ are the endpoints of the curves, whereas all other points are non-accessible. Since accessible points are dense, any point in the Julia set $\mathcal{J}(f_\lambda)$ can be approximated by endpoints of curves.

Recall that, by Fatou, Riez and Riez Theorem, the set of points in $\partial\mathbb{D}$ for which the radial limit is ∞ has zero Lebesgue-measure. However, as it is pointed out in [BD99], this set is non-countable. On the other hand, the end-points have full harmonic measure. Moreover, since the boundary map is ergodic and recurrent, the end-points with dense orbit must have full harmonic measure. Therefore, the set of end-points with bounded orbit (for example, periodic ones) has zero harmonic measure as well as the set of end-points that converge to ∞. In this sense, the orbit of the typical point (with respect to the harmonic measure) is oscillating in the sense that it goes as close to ∞ as wanted but it always comes back.

Finally observe that, for the limit case $\lambda = \frac{1}{e}$, the function f_λ has a parabolic fixed point at 1. As before, one can consider the associated inner function g_λ, which is again an infinite covering of the unit circle, but now has the Denjoy-Wolff point in the boundary and it is of doubly-parabolic type. Observe that the Denjoy-Wolff point is non-singular, because the map has only one singularity which is not radially fixed. Then, the inner function is again ergodic and recurrent and the dynamics can be described via symbolic dynamics, and one shall do the same topological construction of the Julia set as before. Observe that now the parabolic fixed point is in $\partial\Omega_\lambda$ and it is an endpoint of a curve, so $\partial\Omega_\lambda$ is not locally connected in the parabolic fixed point. Then, the Riemann map does not extend continuously around the Denjoy-Wolff point, but this is not an obstruction for the inner function to be holomorphic at it.

C.2 Baker domains

Examples of doubly-parabolic Baker domains are given by the examples of doubly-parabolic inner functions B.4.5, B.4.6 and B.4.7. Observe that the inner function is well-defined in $\mathbb{C}$, with a unique essential singularity at ∞, so it can be viewed as the iteration of a transcendental meromorphic function. In all cases, the upper half-plane and the lower half-plane are invariant Baker domains and its common boundary, the extended real line, is the Julia set. In particular, in the second example the boundary map is non-recurrent. In particular, this is telling us that there exists Baker domains of doubly-parabolic type and infinite degree where the set of escaping points as harmonic measure zero (when the map is recurrent) and Baker domains where the set of escaping points has full harmonic measure (when the map is non-recurrent).

Note that in this examples, the iterated function is not entire but meromorphic, because it has poles. A logical question may be if for entire functions also recurrence and non-recurrence are possible.