Entanglement entropy at strong coupling

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Abstract: We analyze the entanglement entropy of a spherical region in $\mathcal{N}=4$ SYM theory in the strong coupling limit at zero and finite temperature. In this limit, we can apply Gauge/String duality so that the entropy is determined by the surface of minimal area in the dual gravity space such that its intersection with the boundary corresponds to the aforementioned sphere. We find the shape of the minimal surface in terms of the radius of the sphere, compute its area and examine its divergence and regularization. Finally, at nonzero temperature, we study how the regularized entanglement entropy compares to the thermal entropy as the volume of the sphere grows. We obtain that for spheres of radius $\sim 1/T$ the entanglement entropy is about half of the thermal entropy. As the radius grows, these two entropies approach one another and for radii as large as $10/\pi T$ this deviation is still at the 10% level.

I. INTRODUCTION

As is well known, thermodynamics successfully describes static systems in the limit of infinite volumes, considered as systems with a size much more larger than the microscopic scale. A natural question that arises is up to which extend these results hold for finite volumes and, in particular, what is the minimum size of a system in which thermodynamics applies. We aim to address this question in the strong coupling limit of Gauge theories. To do so, we use the so-called entanglement entropy [1], as it allows us to assign an entropy to a finite system.

Entanglement entropy provides a measure of how closely entangled are two complementary regions of a quantum system when only information of one of the subsystems is available. This quantity is hard to determine in general, but for the set of theories with a dual gravity system there exists a simple formula.

Gauge/String duality states that it is possible to describe a $(d+1)-$dimensional conformal field theory (CFT_{d+1}) as the boundary of an associated $(d+2)-$dimensional anti-de Sitter space (AdS_{d+2}) obtained introducing a new spatial coordinate z, such that the CFT is identified with the boundary. We work in $\mathcal{N}=4$ SYM theory in equilibrium, an scale-invariant theory. Therefore, we consider the setup of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence in the strong coupling limit. See [2] for a concrete geometric description via string theory in this scheme.

In this framework, we can follow the holographic procedure presented in [3] to obtain the entanglement entropy. Consider the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality where the quantum field theory lives on a $4-$dimensional Minkowski spacetime $\mathbb{R}\times\mathbb{R}^3$. Let us take a connected submanifold $A\subset\mathbb{R}^3$ at a fixed time $t=0$. We aim to compute the entanglement entropy of the region A, which we denote S_A. To this end, we extend A to a surface γ_A in the whole anti-de Sitter space so that its boundary corresponds to the boundary of the initial manifold A, i.e. such that $\delta\gamma_A=\delta A$. To find S_A, the surface γ_A with minimal area must be employed. With this minimal area surface, the entanglement entropy S_A in CFT_4 can be expressed as

$$S_A=\frac{\mathrm{Area}(\gamma_A)}{4G_N^{(5)}}, \tag{1}$$

where $G_N^{(5)}$ is the $5-$dimensional Newton constant. Note that the Gauge/String duality dictionary relates $G_N^{(5)}$ with the number of color in the Gauge theory (N_c) as $G_N^{(5)}=\pi R^3/2N_c^2$ [4], where R is the radius of AdS_5.

In this work, we use Eq. (1) to compute the entanglement entropy of an sphere A of a given radius in the Gauge theory. In section II, we examine the shape of the minimal area surface γ_A for different radii at null and finite temperature. Then, in section III, we analyze the regularization of the divergent area of γ_A, compute the entanglement entropy in terms of the volume of the sphere, and provide a detailed study of the asymptotic behaviours of the entropy for small and large volumes.

All the numerical calculations are computed with *Mathematica*.

II. MINIMAL AREA SURFACE

We choose region A to be an sphere in $\mathcal{N}=4$ SYM theory. We aim to find the minimal area surface γ_A in the dual gravity theory such that $\delta\gamma_A$ in the boundary corresponds to the boundary of the sphere δA. To do so, we consider the metric of AdS_5 given by

$$ds^2=\frac{R^2}{z^2}(-f(z)\,dt^2+d\vec{x}^2+\frac{dz^2}{f(z)}), \tag{2}$$

where z is the added spatial coordinate, $(t,\vec{x})$ the coordinates in the $4-$dimensional Minkowski spacetime, and

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$f(z)$ equals 1 in the vacuum case (at zero temperature) and $1-(z\cdot\pi T)^4$ in the thermal case (at finite temperature), where T is identified with the temperature in the boundary SYM theory.

With the given metric, the boundary is at $z=0$. Moreover, since we work in a conformal theory, all the quantities are expressed in terms of $1/T$. Then, we can choose the value of $T=1/\pi$ for simplicity in the thermal case. According to Eq. (5.33) in [4], this implies locating the horizon surface at $z=1/(\pi T)=1$.

We denote the radius of A as ρ_{max} and work with spherical coordinates (ρ,Ω) such that $d\vec{x}^2=d\rho^2+\rho^2 d\Omega^2$. To parametrize the surface in the gravity theory, we take $(t,\rho,\theta,\phi,z)=(0,\rho,\theta,\phi,z(\rho))$.

Now, to find the surface with minimal area, we make use of the area element $\sqrt{|\det g|}$, where g is the metric tensor so that $ds^2=g_{ij}dx^i dx^j$. With this, the integral corresponding to the area can be written as

$$\text{Area}(\gamma_A)=\int\sqrt{|\det g|}\,d\rho\,d\Omega= \\ =4\pi R^3\int_0^{\rho_{max}}\frac{\rho^2}{z^3}\sqrt{1+\frac{\dot{z}^2}{f(z)}}\,d\rho \quad (3)$$

where we used that $d\Omega=\sin\theta\,d\theta\,d\phi$, which implies $\int d\Omega=\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi=4\pi$.

Applying the Euler-Lagrange equations to the integrand of Eq. (3), we obtain the following ODE for $z(\rho)$:

$$4z\dot{z}^3+2[3\rho\dot{z}^2+\rho z\ddot{z}+2z\dot{z}]f(z)+ \\ +6\rho f(z)^2-\rho z\dot{z}^2\dot{f}(z)=0, \quad (4)$$

where we impose as boundary conditions $z(\rho=0)=z_0$, $\dot{z}(\rho=0)=0$. Notice that, instead of specifying the size of the sphere in CFT_4, we fix the depth z_0 of the minimal surface (z up to which it extends). At zero temperature, z_0 can take any positive value, whereas at nonzero temperature in our choice of coordinates it is restricted to $z_0\in(0,1)$, since the surface cannot cross the event-horizon.

For the vacuum case, there exists an analytical solution:

$$z^2+\rho^2=\rho_{max}^2 \quad (5)$$

where $\rho_{max}=z_0$. This means that, at zero temperature, γ_A corresponds to a 3−dimensional semi-sphere.

From Eq. (5) we observe that, regardless of the value of z_0, the minimal surface always meets the boundary $z=0$ perpendicularly. Furthermore, the shapes obtained for different radii are self similar, meaning that once the surface for a certain radius ρ_{max} is found, its expression for an arbitrary radius $\overline{\rho}_{max}$ can be easily obtained properly rescaling the coordinates (it suffices to set $(\overline{z},\overline{\rho})=(z\cdot{}^{\overline{\rho}_{max}}/_{\rho_{max}},\rho\cdot{}^{\overline{\rho}_{max}}/_{\rho_{max}})$).

On the other hand, at nonzero temperature, there is not known analytic solution to the surface profile. For this case, solving the differential equation numerically,

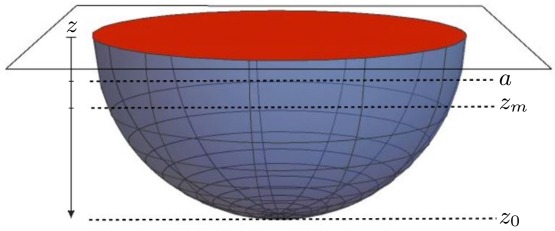

FIG. 1: Illustration of the minimal area surface in AdS_5 for a small radius where a represents the cut-off used to regularize the entropy, z_m the point at which the distinct numerical methods converge, and z_0 the deepest point of the surface. The red disk corresponds to the sphere A in the boundary SYM theory. See the text for more details.

we find that the silhouette of the solution is different for each radius. We observe that, as z_0 gets closer to the horizon, ρ_{max} gets bigger at a higher rate than z_0. Thus, the shape of the surface starts deforming and it flattens, becoming parallel to the horizon surface away from the boundary. This behaviour is represented in Fig. (2).

At this point, we face the following problem: with our methods, we are not able to obtain an exact value of the point ρ_{max} where the surface meets the boundary. In order to solve this issue, for each $z(\rho)$ obtained, we construct a solution as a power series of $\rho(z)$ around $z=0$ starting at an approximation of ρ_{max}. This allows us to be aware of the exact values that ρ takes when $z=0$. We check later that, from a certain order on, by considering an extra decimal value for ρ_{max} the resultant change in the area is imperceptible.

To find the aforementioned power series, we consider the parametrization $(t,\rho,\theta,\phi,z)=(0,\rho(z),\theta,\phi,z)$ and obtain the following equation for $\rho(z)$:

$$2z+2z\dot{\rho}^2\,(1-z^4)-z\rho\ddot{\rho}\,(1-z^4)+ \\ +\rho\dot{\rho}\,(3-z^4)+3\rho\dot{\rho}^3\,(1-z^4)^2=0. \quad (6)$$

Solving Eq. (6) for a general case, we get a power series of the form:

$$\rho(z)=\rho_{max}-\frac{z^2}{2\rho_{max}}+c_4 z^4+O(z^6) \quad (7)$$

where the rest of the odd coefficients are null and the even ones can be expressed in terms of c_4, which we will specify later. Moreover, computing the same expression for the case of null temperature, we obtain that both formulas coincide up to order 8 (not included) independently of the radius ρ_{max}.

For each value of ρ_{max}, we compute the corresponding coefficient c_4 taking a point (ρ_m,z_m) near the boundary and imposing $\rho(z)$ to provide the same result than $z(\rho)$ at that point. This intermediate point, as well as the other surface parameters, are illustrated in Fig. (1). To chose the intermediate point we adopt the following criteria:

i) If $z_0\geq 0.4$, we set $z_m\sim 0.1$ and, for each case, ρ_m is computed such that $z(\rho_m)=z_m$. We pick the value

0.1 because in this region, close to the boundary $z=0$, the derivative of $\rho(z)$ is small enough for the power series to be computed accurately, as we see in Fig. (2).

ii) If $z_0 < 0.4$, the surface starts rounding like a semi-sphere, as in the vacuum case. So we scale the intermediate point applying the relation derived for a zero temperature between the values of z for spheres with different radii.

To decide up to which point each criterion prevails, we plot z_0/ρ_{max} in terms of z_0. We obtain that up to $z_0 \sim 0.4$ the quotient is approximately equal to 1, i.e. the shape can be identified with a semi-sphere, and then it starts decreasing as the surface flattens.

However, we verify later that the results of the area integral do not depend on the choice of z_m as long as we take it close enough to the boundary $z=0$, so that the surface is still perpendicular to $z=0$. Furthermore, we confirm that the results do not depend on the order of the power series either.

Thus, we construct the minimal surface joining $z(\rho)$ up to z_m and $\rho(z)$ from ρ_m until ρ_{max} where the surface meets the boundary. We get that at the intermediate point (ρ_m, z_m) the difference between the derivatives of both functions is insignificant regardless of the value of z_0. This implies that, as expected, the transition is smooth, as is shown in Fig. (2).

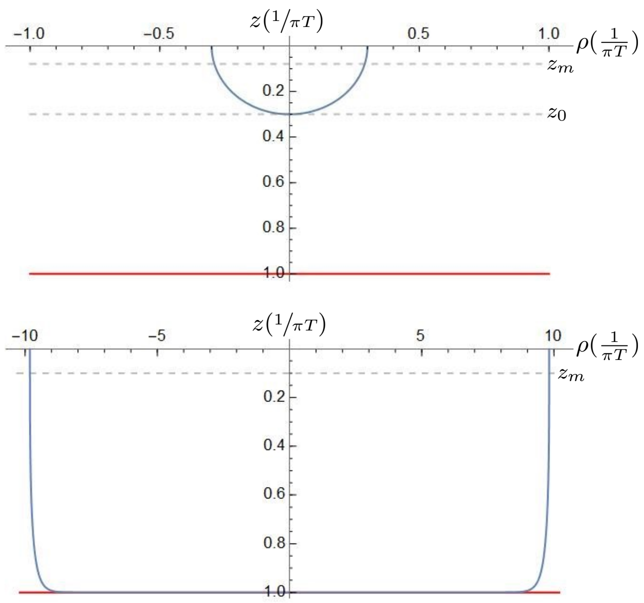

FIG. 2: Representation of sections of the minimal surface obtained joining $z(\rho)$ and the power series $\rho(z)$ for $z_0 = 0.3$ (top figure) and z_0 very close to 1 (bottom figure). The red lines correspond to the horizon surface.

III. DETERMINATION OF THE ENTANGLEMENT ENTROPY

To find the entanglement entropy of region A, we just have to compute integral (3) replacing $z(\rho)$ with the obtained solutions.

For the vacuum case, we use relation (5) and we compute the area of γ_A implementing a change of variables from $z(\rho)$ to $\rho(z)$ for simplicity. Besides, the area diverges as we approach the boundary $z=0$, so we integrate z from a cut-off $a>0$ to ρ_{max}.

Taking the limit when $a \to 0$, we get

$$\text{Area} = 4\pi R^3 \left[\frac{1}{2}\left(\frac{\rho_{max}}{a}\right)^2 - \frac{1}{2}\ln\left(\frac{\rho_{max}}{a}\right) - \frac{\ln 2}{2} - \frac{1}{4}\right], \quad (8)$$

where we denote Area = Area(γ_A).

Then, by Eq. (1), we obtain the following formula for the entanglement entropy:

$$S_A = \frac{\pi R^3}{G_N^{(5)}} \cdot \left[\frac{1}{2}\left(\frac{\rho_{max}}{a}\right)^2 - \frac{1}{2}\ln\left(\frac{\rho_{max}}{a}\right) - \frac{\ln 2}{2} - \frac{1}{4}\right]. \quad (9)$$

This expression matches the one presented in [5].

Let us provide some insight into the divergence of the entropy as a tends to 0. We can identify entanglement entropy as a measure of the amount of information lost when restricting ourselves to a certain region in the space. Since near the boundary there is a strong entanglement between the outer and the inner regions, a large amount of data is being lost. That is why, as we approach this limit, the area integral increases drastically. However, as these fluctuations in the edges of the surface do not affect the deeper points, the expression of this divergence does not depend on the volume of the sphere.

Notice that in the first divergent term $(\rho_{max}/a)^2$ there appears the area of the sphere of radius ρ_{max}. This is consistent with Eq. (2.9) in [3], and corresponds to the fact that, the bigger the area covered in the boundary, the more information is being lost.

In the next steps, to find the area for the thermal case, it is important to use an accurate upper limit for the integral to ensure that no significant contribution to it is lost, since near $z=0$ the slightest increment in ρ_{max} considerably affects the integral of the area. Then, we need both the solution $z(\rho)$ as well as the power series $\rho(z)$.

Therefore, we numerically compute the surface area splitting the integral in two parts: the first one in terms of $z(\rho)$ integrating with respect to ρ from 0 to ρ_m, and the second one in terms of $\rho(z)$ with respect to z from a cut-off $a>0$ to z_m.

We repeat the calculations for several cut-offs approaching 0, getting for all of them the same results. So we conclude that the found values hold in the limit $a \to 0$.

Although we cannot obtain an analytical expression for the entanglement entropy S_A in this case, we can employ the power series $\rho(z)$ given in Eq. (7) to estimate how it diverges.

Firstly, we observe that the minimal surface always approaches $z = 0$ perpendicularly, the same way than in the vacuum case, since it holds $\frac{d\rho}{dz}_{|z=0} = 0$. This means that the integral diverges analogously as for zero temperature. Indeed, by integrating the general power series obtained for $\rho(z)$ near the boundary, we verify that the divergent terms are equal to those in Eq. (9).

Then, we regularize the entropy subtracting the whole Eq. (9). We set $S_R = S_A(T \neq 0) - S_A(T = 0)$. So we get a null entropy for small spheres and finite values for bigger volumes, as we can see in Fig. (3). This way, all the nonzero contributions to S_R are due to the change in the metric caused by the addition of the event-horizon.

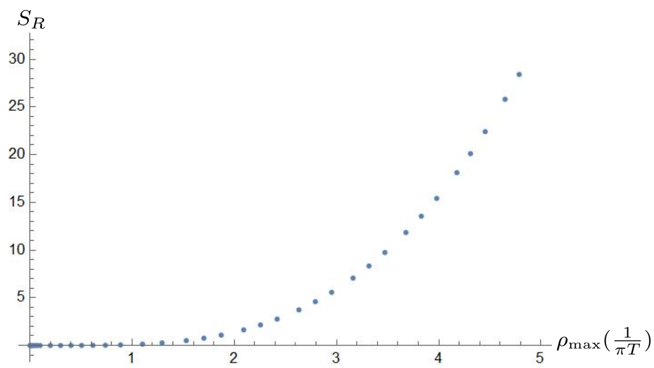

FIG. 3: Regularized entanglement entropy in terms of the radius of the boundary sphere for $a = 10^{-6}$ (which corresponds to the smallest cut-off used). The term $\pi R^3/G_N^{(5)}$ is not included in the calculations.

We saw that as ρ_{max} increases the minimal area surface can be identified with the event-horizon, except for the limits where it bends and touches the boundary $z = 0$. We expect the contribution of these limits to become negligible for sufficiently large radii, so that the asymptotic minimal surface can be assimilated to the horizon surface. Thus, the entanglement entropy will eventually match the AdS$_5$ black hole entropy as it is given by the area of the 3−dimensional event-horizon (A_3):

$$S_{SYM} = S_{BH} = \frac{A_3}{4G_N^{(5)}}, \qquad (10)$$

where $A_3 = \frac{R^3}{\overline{z}^3} \cdot \int d\vec{x}$, being $\int d\vec{x}$ the volume in the boundary conformal theory and $\overline{z}$ the position of the horizon surface. According to the duality description, this entropy corresponds to the thermal entropy in the SYM theory [4].

In our case, since we consider a sphere in the Gauge theory with volume $\frac{4}{3}\pi\rho_{max}^3$ and we take $\overline{z} = 1$, we can write Eq. (10) as

$$S_{BH} = \frac{\pi R^3}{3G_N^{(5)}} \cdot \rho_{max}^3 = \frac{2N_c^2}{3} \cdot \rho_{max}^3. \qquad (11)$$

This implies that for radii large enough the regularized entanglement entropy S_R must be proportional to ρ_{max}^3.

We observe that, as z_0 tends to 1, this dependence can be inferred, though with this procedure we cannot get close enough to the horizon surface to get a conclusive result due to numerical precision problems.

Conversely, for small radii we found that the surface turns into a semi-sphere, i.e. the minimal surface approximates the one at null temperature. This was the expected outcome, since near the boundary $z = 0$ the shape of the surface is not significantly affected by the change in the metric, i.e. the effects of the event-horizon can be neglected, and the surface behaves like in the vacuum case. Indeed, we obtain that the regularized entropy tends to 0, which implies that the entropy S_A equals the one at zero temperature. However, in this region the entropy does not behave like the third power of the radius, so other powers need to be taken into consideration.

Let us study both limits in more detail with an analytical approach to support the numerical results.

A. Small radii limit

We aim to find the dominant ρ−dependence corresponding to the behaviour of the regularized entropy for small radii, assuming that it is similar to the vacuum case. To do so, we make use of perturbative methods.

Given an sphere of a certain radius at $z = 0$, we need to measure the change in the entropy with respect to the case of zero temperature. Then, we fix the volume in the Gauge theory and we just have to determine how in the thermal case the minimum surface with this boundary changes. Hence, we set the value of ρ_{max} and apply perturbation theory on $z(\rho)$.

We consider the metric given in Eq. (2) with $f(z) = 1 - \epsilon z^4$ where $\epsilon \in [0, 1]$, so that $\epsilon = 0$ corresponds to the vacuum case and $\epsilon = 1$ to the thermal one. With this metric, we compute the ODE for $z(\rho)$ corresponding to the minimal surface. Then, we take a perturbative expansion $\tilde{z}(\rho) = z_{T=0}(\rho) + \epsilon \cdot \delta z(\rho) + O(\epsilon^2)$ and, by replacing this $\tilde{z}(\rho)$ in the new equation and assimilating each coefficient of ϵ to 0, we get an expression for the perturbative term:

$$\delta z(\rho) = \frac{(\rho^2 - 2\rho_{max}^2) \cdot (\rho_{max}^2 - \rho^2)^{3/2}}{10}. \qquad (12)$$

We observe that $z_{T=0} + \delta z$ matches the numerical values for $z(\rho)$ at nonzero temperature up to $z_0 \sim 0.6$ with a negligible error. Therefore, the perturbation accurately provides the change in the deepest point of the surface at a finite temperature, i.e. it successfully predicts the flattening of the semi-sphere for small radii.

Finally, we analyze the area integral with these new components and obtain the following dependence for the perturbative term:

$$\delta S_R \propto \frac{\pi R^3}{G_N^{(5)}} \cdot \rho_{max}^4, \qquad (13)$$

where $\tilde{S}_R = \epsilon \cdot \delta S_R + O(\epsilon^2)$. This relation holds for the numerical results as is shown in Fig. (4), where we get an asymptotic constant value after dividing by ρ_{max}^4.

B. Large radii limit

Now, let us consider a new variable z_p which allows us to move the horizon surface to $z_p = \infty$, so that we can get surfaces with considerable higher values of ρ_{max} without encountering the numerical precision issues that arose with the previous methods. We take:

$$z_p = \frac{z}{1-z}. \tag{14}$$

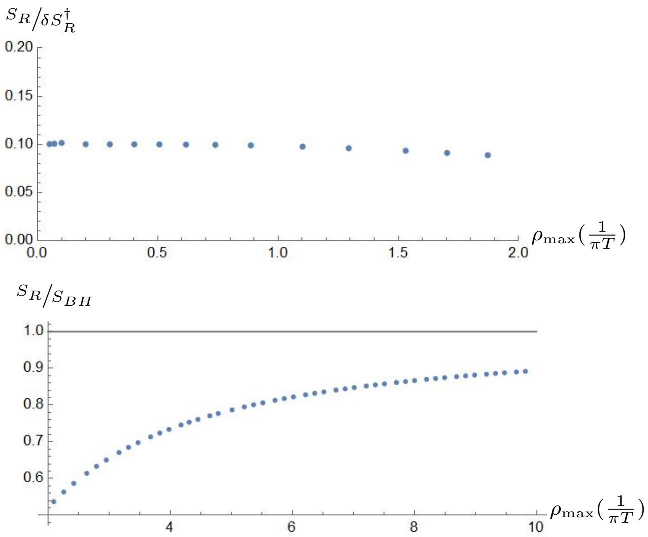

FIG. 4: Asymptotic behaviours of S_R compared with the analytical results: small radii limit where we take $\delta S_R^\dagger = \rho_{max}^4 \cdot \pi R^3/G_N^{(5)}$ (top figure) and large radii limit (bottom figure). All the numerical results are for $a = 10^{-6}$ (which corresponds to the smallest cut-off used).

With this new variable we are able to obtain spheres with radius up to $\rho_{max} \sim 10$. For larger values, the corresponding deepest point $(z_p)_0$ becomes too big for the surface to be computed with these techniques.

Again, we obtain the entropy for several values of a approaching 0 and the various results are indistinguishable. Hence, we state that these hold for the limit $a \to 0$. We provide a comparison of these values with the thermal entropy S_{BH} in Fig. (4).

IV. CONCLUSIONS

We successfully reproduced the entanglement entropy for strong coupling CFT_4 at zero temperature proceeding as in [5]. Besides, we went one step further and numerically found the correlation between the entanglement entropy and the volume of the subsystem considered for a finite temperature. Finally, we verified the obtained results with an analytical approach in two different limits.

Now, we are able to tackle our initial question about the thermal behaviour of the entanglement entropy. Even though Fig.(4) shows that the regularized entropy asymptotically approximates the entropy of the black hole, for a radius of the order $10/(\pi T)$ both quantities still differ about a 10%. Moreover, for $\rho_{max} \sim 1/T$ the entanglement entropy is only about a half of the thermal entropy

To give our results a wider perspective, it is interesting to put them into the context of other analysis of collective dynamics of small systems. In [6] it is shown that hydrodynamics starts applying for radii as small as $1/T$. This emphasizes even more the unexpected collective behaviour observed in small systems, in the sense that a hydrodynamic description holds even when there is still a significant deviation from the thermodynamic behaviour.

Although we have provided a complete overview of the entanglement entropy and its extreme behaviours for a finite temperature scheme, some issues still need to be addressed. Among them, finding a numerical method that would allow us to work with arbitrary large radii in the thermal case, or whether the procedure used is the most accurate to regularize the area integral.

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